Mechanical Vibrations for Engineers

Michel Lalanne, Patrick Berthier, Johan Der Hagopian

Mechanical Vibrations

for

ENGINEERS

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MESSAGE

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Veuillez recevoir, Cher Professeur Lalanne, mes sentiments les meilleurs.

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Mechanical Vibrations for Engineers

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Mechanical Vibrations for Engineers

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Preface

This book is a revised and extended version of *Mécanique des Vibrations Linéaires* published by Masson in 1980. This version has been translated and adapted for English use by F. C. Nelson.

The book is a basis for the study of linear, mechanical vibrations. It is intended primarily for the use of students and practising mechanical engineers. Its purpose is to provide engineering students and practitioners with:

- an understanding of vibration phenomena and concepts
- the ability to formulate and solve the equations of motion of vibrating systems
- an appreciation of the role and technique of vibration measurement.

The authors have adopted a systematic but practical approach in order to make the book both brief and useful. Special care has been taken to make the presentation easy to read and to understand. It is, however, necessary to have some background in rigid-body mechanics, strength of materials, and matrix calculation.

This book treats the vibrations of structures with a sequence of structural models: first, discrete elements such as springs and masses; then, continuous elements such as beams and plates; and finally, simple structures using finite elements and the method of substructures.

Chapter 1 presents spring–mass systems which have one degree of freedom. In Chapters 2 and 3, these are extended, respectively, to 2 and N degrees of freedom. Chapter 4 presents continuous systems and emphasizes energy methods which then provide an introduction to the widely used finite element method which is described in Chapter 5. Chapter 6 discusses simple measurement devices and introduces modern measurement systems. Chapter 7 presents 12 computer programs written in BASIC and suitable for use with a desk-top computer and graphical display. These programs allow students and practising engineers to consider applications which cannot be considered with hand calculations.

The book also contains over 100 exercises, many of which consider problems of practical interest. All have answers and most have their solutions given in some detail. These exercises are an important part of the book and the reader is encouraged to work as many as possible.

When calculations are performed in an exercise, both with or without the aid of a computer program, at least 8 digits are used for the entire set of calculations. The results are in most cases rounded-off to 4 significant figures. When the results of one exercise are used in another, the rounded-off results of the original exercise are used.

At the end there is an Appendix on Lagrange's equations, a short Bibliography, and an Index.

We wish to thank particularly Mrs. J. Aiello for the careful typing of the manuscript.

1
Single Degree-of-Freedom Systems

The study of single degree-of-freedom systems is a good introduction to the basic phenomena of mechanical vibrations such as resonance, damping, and forced response. It also facilitates an understanding of the behavior of complex systems having a large number of degrees of freedom. In addition, single degree-of-freedom systems can often be used as a convenient first approximation to a real structure. Finally, they are helpful in understanding the behavior of widely used measuring devices such as piezoelectric accelerometers.

The contents of the chapter are as follows:

1.1 Free Vibration

One degree-of-freedom systems are illustrated by the system shown in Figure 1 for which the motion is assumed to be only vertical. Let x be the displacement of the mass m from the equilibrium position established by the action of gravity; k be the stiffness of the spring; c be the viscous damping coefficient of the damper. The force exerted by the spring on the mass is $-kx$; the force exerted by the damper on the mass is $-cx^{\circ}$. The time-dependent external force acting on the mass is $F(t)$.

From Newtonian mechanics, the differential equation of motion is

$$mx^{\infty} = -cx^{\circ} - kx + F(t) \tag{1}$$

which for free vibration, that is vibration in the absence of an external force, becomes

$$mx^{\infty} + cx^{\circ} + kx = 0 \tag{2}$$

The solutions of this linear differential equation with constant coefficients

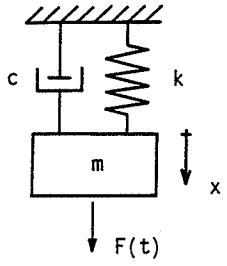


Figure 1 Single degree-of-freedom system with viscous damping

have the exponential form:

$$x = Ae^{rt} \tag{3}$$

Substituting (3) into (2) gives

$$mr^2 + cr + k = 0 \tag{4}$$

This characteristic equation has two roots r_1 and r_2 given by:

$$r_{1,2} = \frac{1}{2}\left[-\frac{c}{m} \pm \sqrt{\left(\frac{c}{m}\right)^2 - \frac{4k}{m}}\right] \tag{5}$$

hence the solutions of (2) can be written as:

$$x = A_1 e^{r_1 t} + A_2 e^{r_2 t} \tag{6}$$

The expression for r_1 and r_2 is best put into a form in which the parameters are much easier to measure. Define the parameters ω and α such as:

$$\omega^2 = \frac{k}{m} \tag{7}$$

$$\alpha = \frac{c}{c_c} \tag{8}$$

where ω is the angular frequency in rad/sec and α is the viscous damping ratio. The critical viscous damping coefficient c_c is defined by the vanishing of the discriminant in (5):

$$\left(\frac{c_c}{m}\right)^2 - \frac{4k}{m} = 0 \tag{9}$$

from which

$$c_c = 2\sqrt{km} = 2m\omega \tag{10}$$

then

$$c = 2m\alpha\omega \tag{11}$$

and

$$\alpha = \frac{c}{2\sqrt{km}} \tag{12}$$

Using (7) and (11), the expression (5) becomes

$$r_{1,2} = -\alpha\omega \pm \omega\sqrt{\alpha^2 - 1} \tag{13}$$

This clearly shows that the behavior of the system of Figure 1 is completely characterized by the two parameters α and ω. The form of the solution of the differential equation (2) depends on the value of the parameter α.

Case 1: $\alpha < 1$

In practice, this is the most important case. Equation (13) gives

$$r_{1,2} = -\alpha\omega \pm j\omega\sqrt{1 - \alpha^2} \tag{14}$$

with

$$j = \sqrt{-1} \tag{15}$$

and the general solution (6) is written

$$x = A_1 \exp\left[-\alpha\omega t + j\omega\sqrt{1-\alpha^2}\, t\right] + A_2 \exp\left[-\alpha\omega t - j\omega\sqrt{1-\alpha^2}\, t\right] \tag{16}$$

This can be put into the more convenient form:

$$\begin{aligned} x &= Ae^{-\alpha\omega t} \sin\left(\omega\sqrt{1-\alpha^2}\, t + \psi\right) \\ &= Ae^{-\alpha\omega t} \sin\left(\omega_d t + \psi\right) \end{aligned} \tag{17}$$

where

$$\omega_d = \omega\sqrt{1-\alpha^2} \tag{18}$$

is the angular frequency of the damped system. A similar analysis shows that ω is the angular frequency of the undamped system. The two constants A and ψ are determined from the two initial conditions: displacement x_0 and velocity x_0° at the initial time t_0.

Also, one can define the frequency f, in hertz (Hz), which is related to the angular frequency ω by

$$\omega = 2\pi f \tag{19}$$

and to the period of oscillation by

$$T = \frac{1}{f} \tag{20}$$

The period is not as widely used in practice as are the frequencies ω and f.

In this first chapter of the text, we are careful to call ω the angular frequency and f the frequency. As has been shown, they are the frequencies with which the undamped system undergoes free vibration. As such, some texts call them the natural frequencies. In subsequent chapters, this distinction in terminology will be dropped and both ω and f will be called the

$$* \qquad r_{1,2} = -\frac{\alpha\omega_a}{\sqrt{1-\alpha^2}} \pm j\omega_a$$

frequency of the system. Which frequency is intended will be clear from the symbol used or the units required.

The experimental determination of α can be achieved by obtaining the logarithmic decrement δ, the natural logarithm of the ratio of two successive maxima of response:

$$\begin{aligned}\delta &= \ln \frac{x_p}{x_{p+1}} \\ &\simeq \ln \frac{\exp[-\alpha\omega t_p]}{\exp[-\alpha\omega(t_p+T)]}\end{aligned} \tag{21}$$

This latter expression is an approximation because the points of contact with the exponential envelope curve do not coincide exactly with the maximum response points. From (21),

$$\begin{aligned}\delta &\simeq \ln e^{\alpha\omega T} \\ &\simeq \alpha\omega T \\ &\simeq \frac{2\pi\alpha}{\sqrt{1-\alpha^2}}\end{aligned} \tag{22}$$

For α small, the usual case in practical situations, this simplifies to

$$\delta \simeq 2\pi\alpha \tag{23}$$

The smaller α, the more difficult it will be to measure the ratio of two successive maxima accurately since this ratio will approach unity. It is then better to measure response maxima which are separated by an integral number of periods and

$$\begin{aligned}\ln \frac{x_p}{x_{p+q}} &= q\alpha\omega T \\ &\simeq 2\pi\alpha q\end{aligned} \tag{24}$$

hence

$$\alpha \simeq \frac{1}{2\pi q} \ln \frac{x_p}{x_{p+q}} \tag{25}$$

Case 2: $\alpha = 1$

This case is seldom encountered in mechanical systems. The characteristic equation has a double root:

$$r_{1,2} = -\omega \tag{26}$$

The response is aperiodic with critical damping and is given by

$$\begin{aligned}x &= A_1 e^{-\omega t} + A_2 t e^{-\omega t} \\ &= e^{-\omega t}(A_1 + A_2 t)\end{aligned} \tag{27}$$

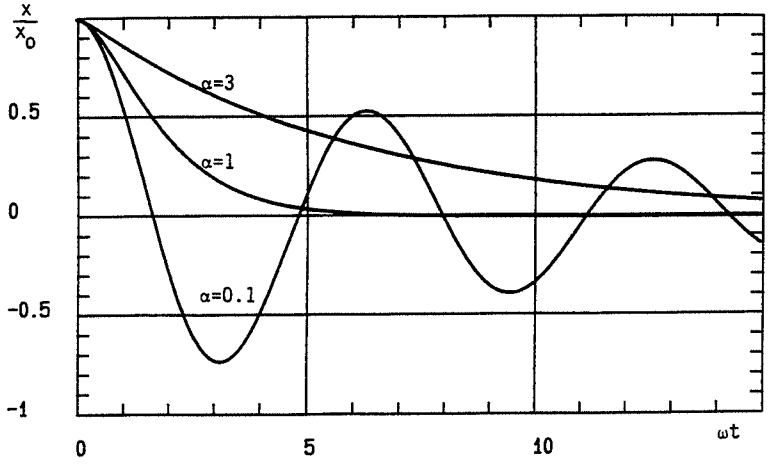


Figure 2 Free motion of the system of Figure 1 for various amounts of damping

Case 3: $\alpha > 1$

This case is also rare in mechanical systems. The solution has the form

$$x = A_1 \exp\left[-\alpha\omega t + \omega\sqrt{\alpha^2-1}\, t\right] + A_2 \exp\left[-\alpha\omega t - \omega\sqrt{\alpha^2-1}\, t\right] \qquad (28)$$

Figure 2 shows the free vibration of a system with initial conditions $x = x_0$, $x^\circ = 0$ for $t_0 = 0$ and for damping ratios of $\alpha = 0.1$, 1, and 3. The calculations give, for:

$$\begin{aligned} \alpha = 0.1: &\quad x = 1.005 x_0 e^{-0.1\omega t} \sin(0.995\omega t + 1.471) \\ \alpha = 1: &\quad x = x_0(1+\omega t)e^{-\omega t} \\ \alpha = 3: &\quad x = 1.030 x_0 e^{-0.1716\omega t} - 0.030 x_0 e^{-5.828\omega t} \end{aligned} \qquad (29)$$

In these three types of free response, x tends toward zero and this fact will justify the subsequent decision to ignore the transient part of the forced response for large time. Also notice that the case $\alpha = 1$ returns the system to rest in the minimum time.

1.2 Forced Vibration

The general solution of equation (1) is equal to the sum of the solution to the homogeneous equation, (2), and a particular solution of (1). In the most frequent case of $\alpha < 1$, one has

$$\begin{aligned} x = {} & A e^{-\alpha\omega t} \sin(\omega\sqrt{1-\alpha^2}\, t + \psi) \\ & + \text{a particular solution of equation (1)} \end{aligned} \qquad (30)$$

Frequently only the steady-state motion is of interest; that is, the system motion existing after a sufficient length of time that the initial transient motion associated with the free vibration has become negligible. Three cases of excitation will be considered: harmonic; periodic; and general function of time. The solutions to harmonic and periodic excitation will be limited to steady-state motion.

1.2.1 Harmonic excitation

Let

$$F(t) = F \sin \Omega t \tag{31}$$

where F is the amplitude of the exciting force and Ω is the angular forcing frequency in rad/sec. The symbol Ω will be used for the angular forcing frequency to distinguish it from ω, the angular frequency of the system. In subsequent chapters, Ω will simply be called the forcing frequency.

Equation (1) becomes

$$mx^{\infty} + cx^{\circ} + kx = F \sin \Omega t \tag{32}$$

The steady-state solution has the form

$$x = X \sin(\Omega t - \phi) \tag{33}$$

where X is the amplitude of the steady-state solution and ϕ is the phase angle. Substituting (33) in (32) gives

$$(k - m\Omega^2)X \sin(\Omega t - \phi) + c\Omega X \cos(\Omega t - \phi) = F \sin \Omega t \tag{34}$$

which can be written as

$$[c\Omega \cos \phi - (k - m\Omega^2) \sin \phi]X \cos \Omega t + [(k - m\Omega^2)X \cos \phi + c\Omega X \sin \phi - F] \sin \Omega t = 0 \tag{35}$$

Equation (35) holds for any time t, hence

$$c\Omega \cos \phi - (k - m\Omega^2) \sin \phi = 0 \tag{36}$$

$$[(k - m\Omega^2) \cos \phi + c\Omega \sin \phi]X - F = 0 \tag{37}$$

Equation (36) permits $\sin \phi$ to be obtained in terms of $\cos \phi$:

$$\sin \phi = \frac{c\Omega \cos \phi}{k - m\Omega^2} \tag{38}$$

On substituting this expression into (37) one obtains

$$\cos \phi = \frac{F}{X} \frac{k - m\Omega^2}{(k - m\Omega^2)^2 + c^2\Omega^2} \tag{39}$$

in which F, X, and $(k - m\Omega^2)^2 + c^2\Omega^2$ are positive quantities.

Using (38) and (39), it can be shown that:

$$\text{for} \quad \Omega < \sqrt{\frac{k}{m}} = \omega$$

$$\sin\phi > 0$$

$$\cos\phi > 0$$

$$\text{hence} \quad 0 < \phi < \frac{\pi}{2} \tag{40}$$

$$\text{for} \quad \Omega > \sqrt{\frac{k}{m}} = \omega$$

$$\sin\phi > 0$$

$$\cos\phi < 0$$

$$\text{hence} \quad \frac{\pi}{2} < \phi < \pi \tag{41}$$

Since $0 < \phi < \pi$ the phase can be uniquely defined by its tangent taken from (38):

$$\tan\phi = \frac{c\Omega}{k - m\Omega^2} = \frac{2\alpha(\Omega/\omega)}{1-(\Omega/\omega)^2} \tag{42}$$

Using the identity

$$\cos^2\phi + \sin^2\phi = 1$$

and (38) and (39),

$$X = \frac{F}{\sqrt{(k-m\Omega^2)^2 + c^2\Omega^2}} = \frac{X_{st}}{\sqrt{[1-(\Omega/\omega)^2]^2 + [2\alpha(\Omega/\omega)]^2}} \tag{43}$$

with

$$X_{st} = \frac{F}{k} \tag{44}$$

which is the displacement of the system subjected to a static force F.

It can be shown that the amplitude X has a maximum for

$$\frac{\Omega}{\omega} = \sqrt{1-2\alpha^2} \tag{45}$$

and that the corresponding value of X is

$$X_r = \frac{X_{st}}{2\alpha\sqrt{1-\alpha^2}} \tag{46}$$

At this maximum, the tangent of the phase angle is

$$\tan\phi = \frac{\sqrt{1-2\alpha^2}}{\alpha} \tag{47}$$

This is the definition of amplitude resonance. Note that if $\alpha > 1/\sqrt{2}$, the maximum amplitude occurs at $\Omega = 0$.

If, on the other hand, one takes

$$\frac{\Omega}{\omega} = 1$$

$\tan\phi$ becomes infinite at $\phi = \pi/2$ and $X_r = X_{st}/2\alpha$. This is the definition of phase resonance.

For most practical systems the damping is small, i.e. $\alpha < 0.1$, and (45), (46), and (47) become

$$\frac{\Omega}{\omega} \simeq 1-\alpha^2 \simeq 1 \tag{48}$$

$$X_r = \frac{X_{st}}{2\alpha}\left(1+\frac{\alpha^2}{2}\right)$$

$$\simeq \frac{X_{st}}{2\alpha} = Q \cdot X_{st} \tag{49}$$

$$\tan\phi \simeq \frac{1}{\alpha} \tag{50}$$

where Q is the so-called Q-factor. This definition of Q can be used for any form of damping.

Thus for small damping the angular forcing frequency associated with amplitude resonance and that associated with phase resonance are essentially equal and it is not necessary to distinguish between them. Under these restrictions, resonance occurs when Ω matches the frequency of the system for free vibration, ω. For this reason, ω will often be referred to as the resonant frequency.

At resonance, the forces in the system elements can be very large if the damping is small. In the spring the force amplitude is

$$F_r = kX_r$$

$$= \frac{kX_{st}}{2\alpha} = \frac{F}{2\alpha} = Q \cdot F \tag{51}$$

which can become very large because of the term $1/2\alpha$. This, of course, is the reason why the determination of resonance is important in structures.

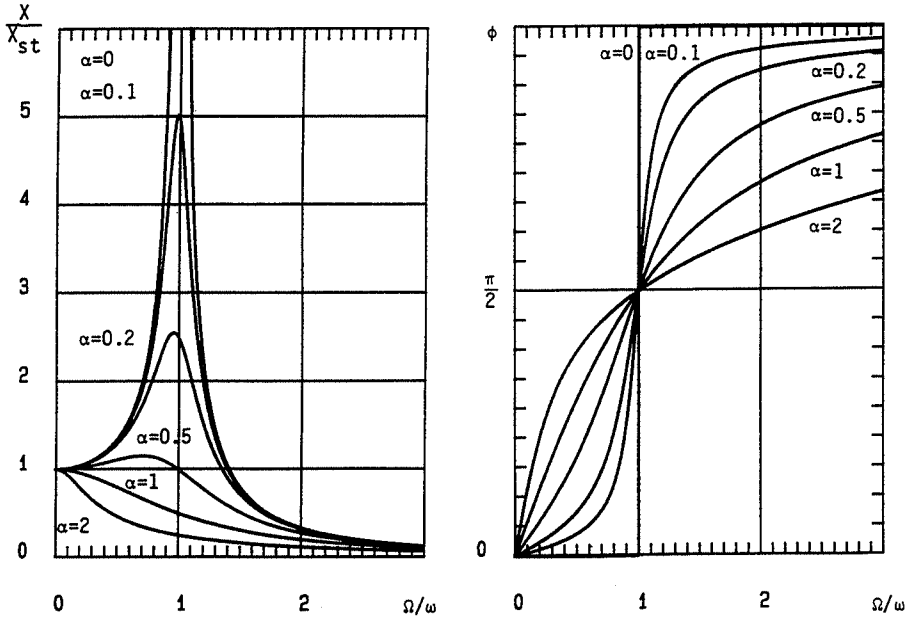


Figure 3 Steady-state amplitude and phase response of the system of Figure 1 for a sinusoidal forcing function

The curves of Figure 3 show X/X_{st} and ϕ as a function of Ω/ω for several values of the damping factor α.

For the case of damping so small that we may take $\alpha = 0$, equations (42) and (43) give:

$$\Omega < \omega: \quad \phi = 0 \quad \text{and} \quad x = \frac{X_{st} \sin \Omega t}{1-(\Omega/\omega)^2} \tag{52}$$

$$\Omega = \omega: \quad \phi = \frac{\pi}{2} \quad \text{and} \quad x \to \infty$$

$$\Omega > \omega: \quad \phi = \pi \quad \text{and} \quad x = \frac{X_{st} \sin (\Omega t - \pi)}{(\Omega/\omega)^2 - 1} \tag{53}$$

These are the equations for the steady-state response of the undamped system.

The results (52) and (53) can also be obtained by considering the solution of

$$mx^{\infty} + kx = F \sin \Omega t \tag{54}$$

in the form

$$x = X \sin \Omega t \tag{55}$$

from which

$$x = \frac{F}{k} \frac{\sin \Omega t}{1-(\Omega/\omega)^2} \tag{56}$$

which is the same expression as (52). If $\omega \gg \Omega$, equation (56) will have the same form as (53).

Energy dissipation per cycle

The energy supplied by the external force $F(t)$ during one cycle of vibration is

$$E = \int_0^T F(t) \frac{dx}{dt} dt \tag{57}$$

$$= \int_0^{2\pi/\Omega} \Omega X F(t) \cos(\Omega t - \phi)\, dt$$

$$= \Omega X F \int_0^{2\pi/\Omega} \sin \Omega t \cos(\Omega t - \phi)\, dt$$

$$= \Omega X F \left[\int_0^{2\pi/\Omega} \sin \Omega t \cos \Omega t \cos \phi\, dt + \int_0^{2\pi/\Omega} \sin^2 \Omega t \sin \phi\, dt \right]$$

$$= \pi X F \sin \phi \tag{58}$$

Using (38) and (39), one has

$$F \sin \phi = \frac{c\Omega \cos \phi}{k - m\Omega^2} F$$

$$= \frac{c\Omega F^2}{X[(k-m\Omega^2)^2 + c^2\Omega^2]} \tag{59}$$

$$= c\Omega X$$

and combining (58) and (59), the expression for energy dissipation in one cycle of motion becomes

$$E = \pi c \Omega X^2 \tag{60}$$

One could also obtain this result by considering the energy dissipated per cycle in the damper:

$$E = \int_0^{2\pi/\Omega} c\dot{x} \frac{dx}{dt} dt \tag{61}$$

Bandwidth

For sufficiently small values of damping, the bandwidth is the frequency interval $(f_2 - f_1)$ situated around the resonance frequency such that the

amplitude of response at frequencies f_1 and f_2 is equal to the resonant amplitude divided by $\sqrt{2}$. Equation (60) shows that the energies dissipated at the frequencies f_1 and f_2 are then equal to one-half of the energy dissipated at resonance. For this reason, the above bandwidth is sometimes called the half-power bandwidth. Then if α is small,

$$X(\Omega_1) = X(\Omega_2) = \frac{X_r}{\sqrt{2}} = \frac{1}{\sqrt{2}} \frac{X_{st}}{2\alpha} \tag{62}$$

Ω_1 and Ω_2 must be solutions of the following equation:

$$\frac{1}{2\alpha\sqrt{2}} = \frac{1}{\sqrt{[1-(\Omega/\omega)^2]^2 + [2\alpha\Omega/\omega]^2}} \tag{63}$$

Rewriting equation (63) gives

$$\left(\frac{\Omega}{\omega}\right)^4 + (4\alpha^2 - 2)\left(\frac{\Omega}{\omega}\right)^2 + 1 - 8\alpha^2 = 0 \tag{64}$$

hence

$$\begin{aligned} \left(\frac{\Omega}{\omega}\right)^2 &= 1 - 2\alpha^2 \pm \sqrt{(1-2\alpha^2)^2 - (1-8\alpha^2)} \\ &= 1 - 2\alpha^2 \pm 2\alpha\sqrt{1+\alpha^2} \\ &\simeq 1 \pm 2\alpha \end{aligned} \tag{65}$$

then

$$\left(\frac{\Omega_2}{\omega}\right)^2 - \left(\frac{\Omega_1}{\omega}\right)^2 \simeq 4\alpha \tag{66}$$

or:

$$\left(\frac{\Omega_2}{\omega}\right)^2 - \left(\frac{\Omega_1}{\omega}\right)^2 = \frac{\Omega_2 - \Omega_1}{\omega} \frac{\Omega_2 + \Omega_1}{\omega} \tag{67}$$

$$\simeq \frac{\Delta\Omega\, 2\omega}{\omega^2} \tag{68}$$

From the relations (66) and (68), one finds

$$\frac{\Delta\Omega}{\omega} = \frac{\Delta f}{f} = 2\alpha = \frac{1}{Q} \tag{69}$$

with $\Delta f = f_2 - f_1$, the bandwidth, and f the resonant frequency. A very useful way of determining α is to measure the bandwidth. Note that

$$20 \log_{10} \frac{X(\Omega_1)}{X_r} = 20 \log_{10} \frac{X(\Omega_2)}{X_r} \simeq -3 \text{ dB} \tag{70}$$

which justifies the frequently used technique of determining the half power

bandwidth by locating the frequencies on either side of resonance for which the response has decreased by 3 decibels (dB).

1.2.2 Periodic excitation

Consider next an excitation force which can be developed in a Fourier series. If the forcing frequency is given by $\Omega = 2\pi/T$, this series has the form

$$F(t) = \frac{a_0}{2} + \sum_{p=1}^{\infty} (a_p \cos p\Omega t + b_p \sin p\Omega t) \tag{71}$$

with

$$a_p = \frac{2}{T} \int_0^T F(t) \cos p\Omega t \, dt \qquad p = 0, 1, 2, \ldots \tag{72}$$

$$b_p = \frac{2}{T} \int_0^T F(t) \sin p\Omega t \, dt \qquad p = 1, 2, \ldots \tag{73}$$

In steady-state motion, the response to each harmonic component is calculated separately and these responses are then added to obtain the complete solution.

Consider, for example, a spring–mass system with small damping under the

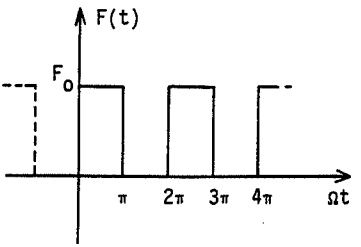


Figure 4 Periodic square-wave forcing function

action of the force shown in Figure 4. Using equations (71), (72), and (73):

$$F(t) = \frac{F_0}{2} + \frac{2F_0}{\pi} \sum_{p=1,3,\ldots}^{\infty} \frac{\sin p\Omega t}{p} \tag{74}$$

The system equation is then

$$mx^{\infty} + kx = \frac{F_0}{2} + \frac{2F_0}{\pi} \sum_{p=1,3,\ldots}^{\infty} \frac{\sin p\Omega t}{p} \tag{75}$$

The solutions to (75) for each term of $F(t)$ given in (74) is obtained in the

same manner as used for (56). It follows that

$$x = \frac{F_0}{2k} + \frac{2F_0}{\pi} \sum_{p=1,3,\ldots}^{\infty} \frac{\sin p\Omega t}{p[k - m(p\Omega)^2]} \tag{76}$$

Resonance will occur whenever

$$k - m(p\Omega)^2 = 0 \tag{77}$$

which is equivalent to

$$\Omega = \frac{1}{p}\sqrt{\frac{k}{m}} = \frac{\omega}{p} \tag{78}$$

1.2.3 General function of time

In this case, the system response is required starting from the initial time $t_0 = 0$. In practice, a numerical step-by-step method is usually used (see Chapter 3). However, one can also solve this problem with the Laplace transform applied to equation (1). This method will be briefly reviewed without a discussion on the question of convergence.

For a function $f(t)$ defined for $t > 0$ and zero otherwise, the Laplace transform, $\mathcal{F}(p)$, is given by:

$$L[f(t)] = \mathcal{F}(p) = \int_0^{\infty} e^{-pt} f(t)\, dt \tag{79}$$

The Laplace transform of the first and second derivatives of $f(t)$ are obtained from (79). If $f(0)$ and $f^{\circ}(0)$ are respectively the values of $f(t)$ and of its first derivative at $t = 0$, integration by parts gives

$$L\left[\frac{df(t)}{dt}\right] = -f(0) + p\mathcal{F}(p) \tag{80}$$

$$L\left[\frac{d^2 f(t)}{dt^2}\right] = -f^{\circ}(0) - pf(0) + p^2\mathcal{F}(p) \tag{81}$$

From equations (1), (80), and (81), the Laplace transform of $x(t)$ is found to be $\chi(p)$, where

$$\chi(p) = \frac{\mathcal{F}(p)}{m(p^2 + 2\alpha\omega p + \omega^2)} + \frac{(p + 2\alpha\omega)x(0)}{p^2 + 2\alpha\omega p + \omega^2} + \frac{x^{\circ}(0)}{p^2 + 2\alpha\omega p + \omega^2} \tag{82}$$

and $\mathcal{F}(p)$ is the Laplace transform of the exciting force. The last two terms of (82) are well-known Laplace transforms of simple functions and hence can be easily inverted. But to invert the first term, Borel's theorem must be used. This theorem shows that if $\mathcal{F}_1(p)$, $\mathcal{F}_2(p)$ are the transforms of $f_1(t)$, $f_2(t)$, the transform

$$\mathcal{F}(p) = \mathcal{F}_1(p) \cdot \mathcal{F}_2(p) \tag{83}$$

originates from

$$f(t) = \int_0^t f_1(\tau) \cdot f_2(t-\tau)\, d\tau \tag{84}$$

The inverse of (82) is then

$$x(t) = \frac{1}{m\omega_d} \int_0^t F(\tau) e^{-\alpha\omega(t-\tau)} \sin \omega_d (t-\tau)\, d\tau$$
$$+ x(0) e^{-\alpha\omega t} \left(\cos \omega_d t + \frac{\alpha}{(1-\alpha^2)^{1/2}} \sin \omega_d t \right) + \frac{x^\circ(0)}{\omega_d} e^{-\alpha\omega t} \sin \omega_d t \tag{85}$$

where the terms containing $x(0)$ and $x^\circ(0)$ correspond to the transient response. This form shows very clearly that the transient response is necessary to satisfy the initial conditions at $t = 0$ but as time grows large it decays to zero, leaving only the steady-state solution. Also notice that one must carefully distinguish between ω and ω_d in (85).

Thus the steady-state motion of a damped system is given by

$$x(t) = \frac{1}{m\omega_d} \int_0^t F(\tau) e^{-\alpha\omega(t-\tau)} \sin \omega_d (t-\tau)\, d\tau \tag{86}$$

and for an undamped or very lightly damped system by

$$x(t) = \frac{1}{m\omega} \int_0^t F(\tau) \sin \omega (t-\tau)\, d\tau \tag{87}$$

1.3 Damping in Real Systems

Actual systems always have some damping but rarely is this damping viscous. Among the most common forms of damping are structural damping and Coulomb damping. Structural damping is a material characteristic whose value can be strongly dependent on both temperature and forcing frequency. Coulomb damping arises from the relative motion between dry surfaces in contact; it is quite difficult to quantify this phenomenon because it depends on so many parameters.

An equivalent viscous damping coefficient can be defined for the case of harmonic excitation by using the previous expression for energy dissipated per cycle (60).

For structural damping it has been observed that the energy dissipated per cycle has the form

$$E = aX^2 \tag{88}$$

over a limited range of frequency and temperature. X is the displacement amplitude and a is a constant of proportionality. The coefficient of equivalent viscous damping is found from (60) and (88):

$$aX^2 = \pi c_{eq} \Omega X^2 \tag{89}$$

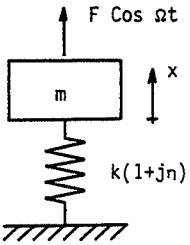


Figure 5 Single degree-of-freedom system with structural damping

hence

$$c_{eq} = \frac{a}{\pi\Omega} \tag{90}$$

The calculation of systems with structural damping subjected to harmonic excitation is more conveniently achieved with the use of complex notation. A single degree-of-freedom system with structural damping and excited by the force $F\cos\Omega t$ is shown in Figure 5. It has the equation

$$mx^{\infty} + \frac{a}{\pi\Omega}x^{\circ} + kx = F\cos\Omega t \tag{91}$$

In complex notation, this becomes

$$mz^{\infty} + \frac{a}{\pi\Omega}z^{\circ} + kz = Fe^{j\Omega t} \tag{92}$$

where $x = \mathrm{Re}[z]$, the real part of the complex quantity z. Solutions are sought in the form

$$z = Ze^{j\Omega t} \tag{93}$$

which, when substituted in (92), gives

$$(k - m\Omega^2)Z + j\frac{a}{\pi}Z = F \tag{94}$$

Equation (94) is conveniently written as:

$$-m\Omega^2 Z + k(1 + j\eta)Z = F \tag{95}$$

with

$$\eta = \frac{a}{\pi k} \qquad \text{the structural damping factor} \tag{96}$$

and, in addition, one can define

$$k^* = k(1 + j\eta) \qquad \text{the complex stiffness} \tag{97}$$

The term η in equations (96) and (97) is often referred to as the loss factor.

Equation (95) gives

$$Z = \frac{F}{k - m\Omega^2 + j\eta k} \tag{98}$$

which can be put into the form

$$Z = |Z| e^{-j\phi} \tag{99}$$

with

$$|Z| = \frac{F}{\sqrt{(k - m\Omega^2)^2 + \eta^2 k^2}} \tag{100}$$

and where

$$\begin{aligned} \sin \phi &= \frac{\eta k}{\sqrt{(k - m\Omega^2)^2 + \eta^2 k^2}} \\ \cos \phi &= \frac{k - m\Omega^2}{\sqrt{(k - m\Omega^2)^2 + \eta^2 k^2}} \end{aligned} \tag{101}$$

Since

$$x = \mathrm{Re}\,[Ze^{j\Omega t}] \tag{102}$$

one has

$$x = \frac{F}{\sqrt{(k - m\Omega^2)^2 + \eta^2 k^2}} \cos(\Omega t - \phi) \tag{103}$$

with

$$\tan \phi = \frac{\eta k}{k - m\Omega^2} \qquad 0 < \phi < \pi \tag{104}$$

Equations (103) and (104) are more conveniently written in the form

$$x = \frac{F/k}{\sqrt{[1 - (\Omega/\omega)^2]^2 + \eta^2}} \cos(\Omega t - \phi) \tag{105}$$

$$\tan \phi = \frac{\eta}{1 - (\Omega/\omega)^2} \tag{106}$$

The determination of η is easy because

$$\eta = \frac{\Delta f}{f} \tag{107}$$

where Δf is the half-power bandwidth. This then provides a means of measuring damping during steady-state vibration (see exercise 15).

It is important to note that the use of complex notation results in a major reduction in the algebra required to solve the equation of motion of a

harmonically forced oscillator: compare the number of steps required to get (43) with the number of steps to get (95). For this reason, complex notation will be used in subsequent analysis.

1.4 Rayleigh's Method

This method was proposed by Rayleigh to obtain a close approximation to the lowest frequency of free vibration for an undamped system. In this chapter, Rayleigh's method will be applied to single degree-of-freedom systems only.

The method proceeds as follows: from a reasonable hypothesis about the motion of the system, calculate the approximate kinetic and strain energies and then use the theorem of conservation of mechanical energy or the equations of Lagrange to obtain the approximate frequency.

Let us apply the method to estimate the effect of the mass of the spring on the dynamic behavior of a spring–mass system, as shown in Figure 6, where: $m_s = \rho L$, mass of the spring; L, spring length; ρ, mass per unit length of the spring.

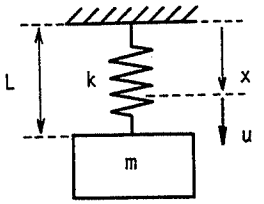


Figure 6 Influence of spring mass on the frequency of a spring–mass system

The motion of the system is assumed to be separable; that is, if $u(x, t)$ is the displacement of a point on the spring located at a distance x from the fixed end, then it is assumed that $u(x, t)$ can be separated into the product

$$u(x, t) = \phi(x)p(t) \tag{108}$$

By analogy with the static case, a reasonable hypothesis for $\phi(x)$ is

$$\phi(x) = ax \tag{109}$$

where a is a constant of proportionality. The expression for kinetic energy is

$$T = T_{\text{mass}} + T_{\text{spring}} \tag{110}$$

$$= \tfrac{1}{2}ma^2L^2p^{\circ 2}(t) + \frac{1}{2}\int_0^L \rho a^2x^2\,dx\,p^{\circ 2}(t)$$

$$= \frac{1}{2}\left(m + \frac{m_s}{3}\right)a^2L^2p^{\circ 2}(t) \tag{111}$$

and the expression for strain energy is

$$U = \tfrac{1}{2}ka^2L^2p^2(t) \tag{112}$$

Application of Lagrange's equation gives

$$\left(m + \frac{m_s}{3}\right)p^{\infty}(t) + kp(t) = 0 \tag{113}$$

and then the approximate angular frequency is

$$\omega = \sqrt{\frac{k}{m + m_s/3}} \tag{114}$$

As stated above, this result can also be obtained by writing that the sum of (111) and (112) is constant in time.

1.5 Applications

The two applications described below are important because they are first-order models for systems such as vehicle suspensions, accelerometers, and vibration isolators. Only the expressions for response amplitudes are developed since the phase responses are usually of less interest.

1.5.1 System on a moving foundation

The system represented in Figure 7 models a machine mounted to a foundation by a spring and viscous damper. The foundation has a displace-

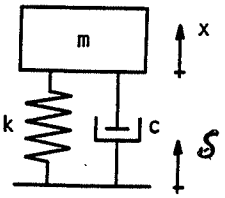


Figure 7 Single degree-of-freedom system on a moving foundation

ment of the form

$$\delta = \Delta \cos \Omega t \tag{115}$$

and it is desired to keep the machine motion, that is the motion of the mass, to the smallest possible value. This situation also arises for a vehicle going over a rough road or for a container of delicate electronics attached to a vibrating surface.

The movement of the mass can be deduced from the equation of motion:

$$mx^{\infty} = k(\delta - x) + c(\delta^{\circ} - x^{\circ}) \tag{116}$$

Substituting (115) into (116) gives

$$mx^{\infty}+cx^{\circ}+kx=\Delta(k\cos\Omega t-c\Omega\sin\Omega t) \tag{117}$$

In order to use complex notation, we associate (118) with (117):

$$j(my^{\infty}+cy^{\circ}+ky)=j\Delta(k\sin\Omega t+c\Omega\cos\Omega t) \tag{118}$$

and obtain the equation

$$mz^{\infty}+cz^{\circ}+kz=\Delta(k+jc\Omega)e^{j\Omega t} \tag{119}$$

The real part of z is x. The amplitude of the mass displacement X is

$$X=\Delta\sqrt{\frac{k^2+c^2\Omega^2}{(k-m\Omega^2)^2+c^2\Omega^2}} \tag{120}$$

$$=\Delta\sqrt{\frac{1+[2\alpha(\Omega/\omega)]^2}{[1-(\Omega/\omega)^2]^2+[2\alpha(\Omega/\omega)]^2}} \tag{121}$$

The ratio X/Δ is plotted in Figure 8 as a function of Ω/ω with α as a parameter. In order to have a small motion of the mass, that is, good isolation, it is required that $\Omega/\omega \gg 1$. In other words, the resonant frequency of the system, ω, must be as low as possible. In practice, this is limited by the

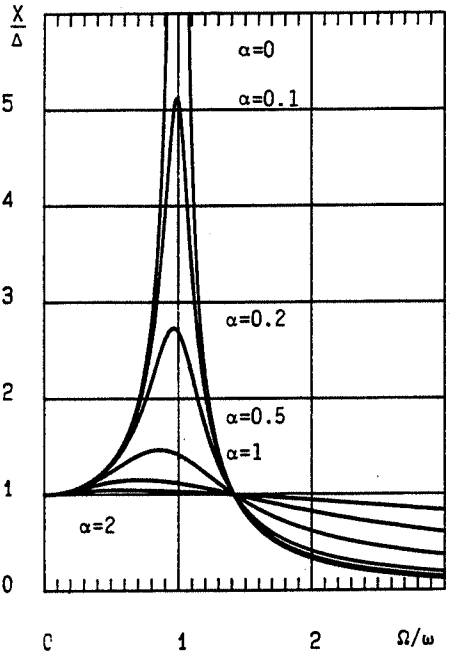


Figure 8 Transmissibility curves for a single degree-of-freedom system

increasing static displacement of the mass due to gravity:

$$kx_{st} = mg \tag{122}$$

1.5.2 Transmissibility

Now let the mass of the system just considered be subjected to a force $F \sin \Omega t$ and require that the force transmitted to the foundation, which is now fixed, be as small as possible.

The force transmitted is

$$F_t = kx + cx^{\circ} \tag{123}$$

or if

$$x = X \sin (\Omega t - \phi) \tag{124}$$

$$F_t = X[k \sin (\Omega t - \phi) + c\Omega \cos (\Omega t - \phi)] \tag{125}$$

The amplitude of F_t can be shown to be

$$|F_t| = X\sqrt{k^2 + c^2\Omega^2}$$

$$= kX\sqrt{1 + [2\alpha(\Omega/\omega)]^2} \tag{126}$$

and after using (43),

$$|F_t| = F\sqrt{\frac{1 + [2\alpha(\Omega/\omega)]^2}{[1 - (\Omega/\omega)^2]^2 + [2\alpha(\Omega/\omega)]^2}} \tag{127}$$

The ratio $|F_t|/F$ is identical to X/Δ and the conclusion is therefore: to limit the transmitted force, it is necessary that $\Omega/\omega \gg 1$.

1.6 Exercises

Exercise 1: *Calculate the equivalent stiffness k of two springs of stiffness k_1 and k_2 which are in parallel.*

The forces and displacements are shown in Figure 9 in their positive directions. Then

$$F = F_1 + F_2$$

$$F_1 = k_1(x_2 - x_1)$$

$$F_2 = k_2(x_2 - x_1)$$

hence

$$F = (k_1 + k_2)(x_2 - x_1) = k(x_2 - x_1)$$

and

$$k = k_1 + k_2$$

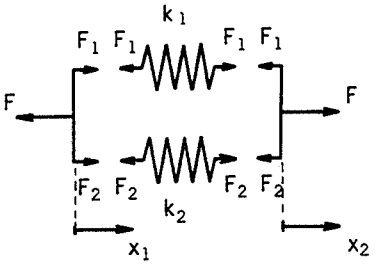


Figure 9

Exercise 2: *Repeat exercise* 1 *for two springs in series.*

The answer is

$$\frac{1}{k}=\frac{1}{k_1}+\frac{1}{k_2}$$

Exercise 3: *The influence of gravity. Consider the vertical spring–damper–mass system shown in Figure* 10. *Derive the equation of motion using as origin for X the unextended length of the spring.*

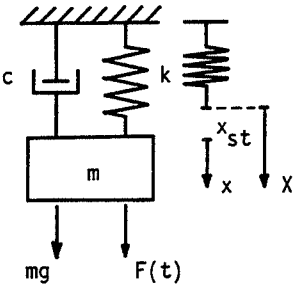


Figure 10

Newton's law gives

$$mX^{\infty}+cX^{\circ}+kX=F(t)+mg$$

In introducing

$$X=x_{st}+x$$

where

$$x_{st}=\frac{mg}{k}$$

the equation of motion becomes

$$mx^{\infty} + cx^{\circ} + kx = F(t)$$

The influence of gravity is therefore just a change of origin and does not affect the dynamic response.

Exercise 4: *Find the free vibration of the system shown in Figure* 11 *for*

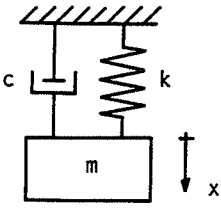


Figure 11

$\alpha < 1.0$ *with initial condition* $x(0) = x_0$ *and* $x^{\circ}(0) = x_0^{\circ}$.

The displacement $x(t)$ has the form:

$$x = Ae^{-\alpha\omega t} \sin(\omega\sqrt{1-\alpha^2}\, t + \psi)$$

Applying the initial conditions gives

$$x_0 = A \sin \psi$$

$$x_0^{\circ} = -\alpha\omega A \sin \psi + A\omega\sqrt{1-\alpha^2} \cos \psi$$

$$\tan \psi = \frac{x_0\omega\sqrt{1-\alpha^2}}{x_0^{\circ} + \alpha\omega x_0}$$

When ψ is known, one can find $\sin \psi$ and then A.

Exercise 5: *Consider the same system as in exercise* 4. *Find the free vibration of the system for* $\alpha = 0.05$ *and the initial conditions* $x(0) = 0$, $x^{\circ}(0) = x_0^{\circ}$.

The answer is

$$x(t) = \frac{1.00125}{\omega} x_0^{\circ} e^{-0.05\omega t} \sin 0.99875\, \omega t$$

Exercise 6: *The base of the system shown in Figure* 12 *has a prescribed displacement* $\delta = \Delta \sin \Omega t$. *Calculate the steady-state response and the base force necessary to impose this displacement.*

The equation of motion is

$$mx^{\infty} + kx = k\Delta \sin \Omega t$$

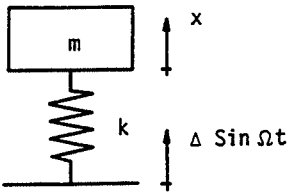


Figure 12

If the motion of the mass has the form

$$x = X \sin \Omega t$$

the response is

$$x = \frac{\Delta}{1-(\Omega/\omega)^2} \sin \Omega t$$

The force necessary to produce this displacement is

$$\begin{aligned} F &= k(\delta - x) \\ &= -\frac{k\Delta(\Omega/\omega)^2}{1-(\Omega/\omega)^2} \sin \Omega t \end{aligned}$$

At resonance of the undamped system, this force becomes infinite. In practical testing situations, since it is impossible to produce a force of this magnitude, it is necessary to significantly reduce the displacement amplitude whenever a system becomes resonant.

Exercise 7: *Consider the same system as shown in exercise* 4. *Find the steady-state motion of the system subjected to* $F(t) = F \sin \Omega t$ *by using complex notation.*

An equation

$$my^{\infty} + cy^{\circ} + ky = F \cos \Omega t$$

can be combined with the standard equation (32):

$$mx^{\infty} + cx^{\circ} + kx = F \sin \Omega t$$

so that x is the imaginary part of z, which is the solution of

$$mz^{\infty} + cz^{\circ} + kz = Fe^{i\Omega t}$$

where

$$\begin{aligned} z &= Ze^{i\Omega t} \\ &= |Z|\, e^{i(\Omega t - \phi)} \end{aligned}$$

It can be shown that

$$|Z| = \frac{F}{\sqrt{(k - m\Omega^2)^2 + c^2\Omega^2}}$$

$$\cos\phi = \frac{k - m\Omega^2}{\sqrt{(k - m\Omega^2)^2 + c^2\Omega^2}}$$

$$\sin\phi = \frac{c\Omega}{\sqrt{(k - m\Omega^2)^2 + c^2\Omega^2}}$$

Equation (42) for $\tan\phi$ can then be obtained and it can be seen that $0 < \phi < \pi$. Finally,

$$\begin{aligned} x &= \mathrm{Im}\,[z(t)] \\ &= \mathrm{Im}\,[|Z|\, e^{j(\Omega t - \phi)}] \\ &= \frac{F}{\sqrt{(k - m\Omega^2)^2 + (c\Omega)^2}} \sin(\Omega t - \phi) \end{aligned}$$

Exercise 8: *Write the equation of motion in terms of θ for the system shown in Figure 13 for the case of small θ. The beam ADB is rigid and is pinned at*

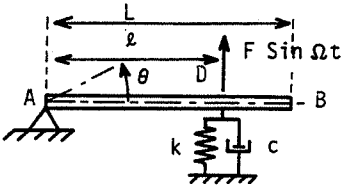


Figure 13

point A and free at point B. The mass moment of inertia of the beam with respect to A is I_A. Find the value of the damping factor α if the values of the various system parameters are as follows:

	I_A	k	c	l	L
SI	0.065 $m^2 \cdot kg$	3×10^4 N/m	1 Ns/m	0.4 m	0.5 m
English units	0.048 ft^2 slug	2056 lbf/ft	0.0685 lbf · sec/ft	1.31 ft	1.64 ft

The equation of motion in SI units is

$$0.065\theta^{\infty} + 0.16\theta^{\circ} + 4800\theta = 0.4F \sin\Omega t$$

and therefore

$$\alpha = \frac{0.16}{2\sqrt{0.065 \times 4800}} = 0.45\%$$

The equation of motion in English units is

$$0.048\theta^{\infty} + 0.118\theta^{\circ} + 3528\theta = 1.31F \sin \Omega t$$

Exercise 9: *For the system shown in exercise* 8, *calculate the steady-state response in terms of the vertical displacement* $x(t)$ *at point B. Draw* X/F *as a function of* Ω *using both log-log and linear-linear scales. Discuss the approximations which can be obtained for frequencies much less than the resonant frequency and for frequencies much more than the resonant frequency.*

The equation of motion (in SI units) is

$$0.13x^{\infty} + 0.32x^{\circ} + 9600x = 0.4F \sin \Omega t$$

which is a specific form of:

$$m^*x^{\infty} + c^*x^{\circ} + k^*x = F^* \sin \Omega t$$

The solution is

$$x = X \sin (\Omega t - \phi)$$

with

$$X = \frac{F^*}{\sqrt{(k^* - m^*\Omega^2)^2 + (c^*\Omega)^2}}$$

At low frequency (i.e. $\Omega \ll \sqrt{k/m}$):

$$X \simeq \frac{F^*}{k^*}$$

At high frequency (i.e. $\Omega \gg \sqrt{k/m}$):

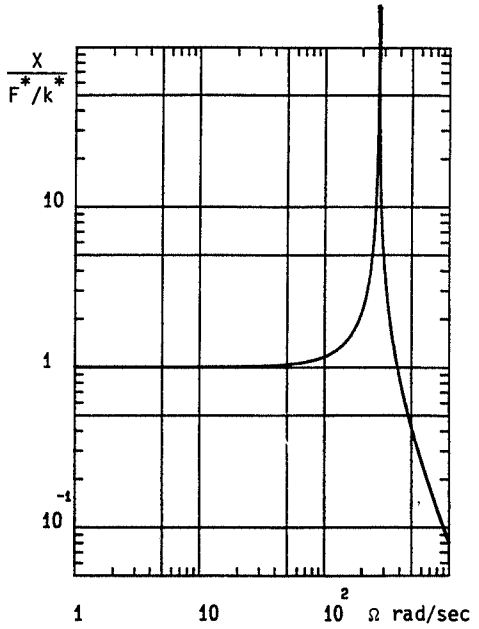
$$X \simeq \frac{F^*}{m^*\Omega^2}$$

In this latter case, on the logarithm scales:

$$\log \frac{X}{F^*} = -\log (m^*\Omega^2) = -\log m^* - 2 \log \Omega$$

which corresponds to a straight line of slope -2.

The graphs are shown in Figure 14. From this exercise one is able to see that logarithm scales are more appropriate for showing system response at low and high frequencies than are linear scales.


X
F*/k*
10
1
-1
10
1
10
2
10
Ω rad/sec

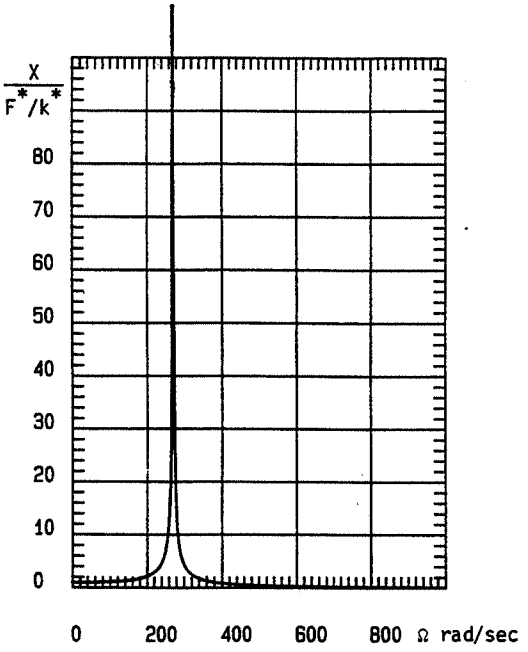

X
F*/k*
80
70
60
50
40
30
20
10
0
0
200
400
600
800
Ω rad/sec

Figure 14

Exercise 10: *The mass shown in Figure 15 is subjected to a force step of magnitude F at $t=0$. The initial conditions are $x(0)=0$, $x^{\circ}(0)=0$. Calculate*

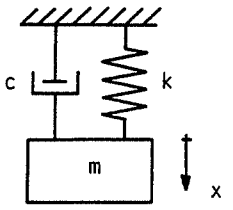


Figure 15

$x(t)$ for $\alpha=0$ and for $\alpha<1$

The equation is

$$mx^{\infty}+cx^{\circ}+kx=F \qquad (t>0)$$

For $\alpha=0$, the general solution is

$$x=A\sin\omega t+B\cos\omega t+\frac{F}{k}$$

and after applying the initial conditions this becomes

$$x=\frac{F}{k}(1-\cos\omega t)$$

For $\alpha<1$ the general solution is

$$x=e^{-\alpha\omega t}(A\sin\omega\sqrt{1-\alpha^2}\,t+B\cos\omega\sqrt{1-\alpha^2}\,t)+\frac{F}{k}$$

where

$$A=\frac{-\alpha F}{k\sqrt{1-\alpha^2}}, \qquad B=-\frac{F}{k}$$

Hence

$$x=-\frac{F}{k}e^{-\alpha\omega t}\left(\frac{\alpha}{\sqrt{1-\alpha^2}}\sin\omega\sqrt{1-\alpha^2}\,t+\cos\omega\sqrt{1-\alpha^2}\,t\right)+\frac{F}{k}$$

Exercise 11: *Consider the same system as in exercise 10 for the case $\alpha<1$ but now consider the force step of magnitude F to be a general function of time. Calculate $x(t)$ for the same initial conditions.*

Due to the initial conditions, equation (86) is used:

$$x(t)=\frac{1}{m\omega_d}\int_0^t F(\tau)e^{-\alpha\omega(t-\tau)}\sin\omega_d(t-\tau)\,d\tau$$

In this case

$$F(\tau) = F$$

and, if one lets

$$t - \tau = u$$

$$\mathrm{d}\tau = -\mathrm{d}u$$

$x(t)$ can be written in the simpler form:

$$x(t) = \frac{F}{m\omega_d} \int_0^t e^{-\alpha\omega u} \sin \omega_d u \, \mathrm{d}u$$

from which:

$$x(t) = \frac{F}{m\omega\sqrt{1-\alpha^2}} \int_0^t e^{-\alpha\omega u} \sin \omega_d u \, \mathrm{d}u$$

or since:

$$\int_0^t e^{-\alpha\omega u} \sin \omega_d u \, \mathrm{d}u = \frac{[e^{-\alpha\omega u}(-\alpha\omega \sin \omega_d u - \omega_d \cos \omega_d u)]_0^t}{\alpha^2\omega^2 + \omega_d^2}$$

$$x(t) = \frac{Fe^{-\alpha\omega t}}{m\omega\sqrt{1-\alpha^2}} \cdot \frac{[-\alpha\omega \sin \omega\sqrt{1-\alpha^2}\, t - \omega\sqrt{1-\alpha^2} \cos \omega\sqrt{1-\alpha^2}\, t] + \omega\sqrt{1-\alpha^2}}{\omega^2}$$

As $\omega^2 = k/m$, the corresponding result of exercise 10 is obtained.

Exercise 12: *The system shown in Figure* 16 *represents an elementary model of a rotating machine of mass* $(M-m)$ *which is attached to a foundation by a spring* k *and damper* c. *The machine has a rotating unbalance of mass* m

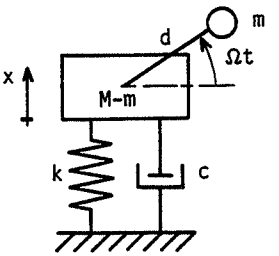


Figure 16

which is located at radial distance d *from an axis of rotation about which the radius has a constant angular velocity* Ω. *Assume that only motion in the* x *direction is possible. Determine the displacement amplitude* X *and the amplitude of the force transmitted to the foundation* F_t.

The answers are

equation of motion:

$$Mx^{\infty} + cx^{\circ} + kx = md\Omega^2 \sin \Omega t$$

amplitude of displacement:

$$X = \frac{md\Omega^2}{\sqrt{(k - M\Omega^2)^2 + (c\Omega)^2}}$$

amplitude of transmitted force:

$$F_t = md\Omega^2 \sqrt{\frac{k^2 + c^2\Omega^2}{(k - M\Omega^2)^2 + c^2\Omega^2}}$$

Exercise 13: *Develop the force shown in Figure* 17 *in a Fourier series. The force is applied to the mass of a spring–mass–damper system. Calculate the steady-state motion* $x(t)$.

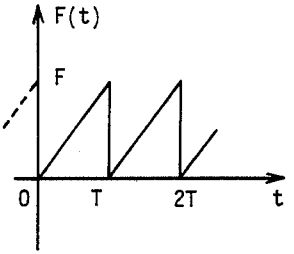


Figure 17

The force expansion is

$$F(t) = F\left(\frac{1}{2} - \frac{1}{\pi} \sum_{p=1}^{\infty} \frac{\sin p\Omega t}{p}\right)$$

and

$$x(t) = \frac{F}{2k} + \sum_{p=1}^{\infty} X_p \sin(p\Omega t - \phi_p)$$

with

$$X_p = \frac{-F/p\pi}{\sqrt{[k - m(p\Omega)^2]^2 + (cp\Omega)^2}}$$

$$\tan \phi_p = \frac{2\alpha(p\Omega/\omega)}{1 - (p\Omega/\omega)^2} \qquad 0 < \phi_p < \pi$$

Exercise 14: *Figure* 18 *shows a spring–mass system with structural damping. Show that the conditions of amplitude resonance and phase resonance are identical.*

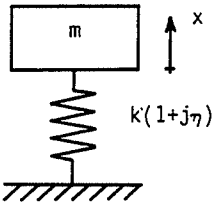


Figure 18

The phase angle equal to $\pi/2$ defines the phase resonance, see equation (106), and it occurs for $\Omega = \omega$.

The amplitude resonance occurs for maximum amplitude, see equation (105), and that can also be shown to occur for $\Omega = \omega$ at which the displacement amplitude is $X_r = F/\eta k$.

Exercise 15: *Calculate the half-power frequency bandwidth for the system in the previous exercise.*

The limits of the bandwidth correspond to one-half of the energy dissipated at resonance. Using equations (88), (105), and the result obtained in exercise 14:

$$\frac{1}{\eta\sqrt{2}} = \frac{1}{\sqrt{[1-(\Omega/\omega)^2]^2+\eta^2}}$$

from which

$$\left(\frac{\Omega_2}{\omega}\right)^2 = 1+\eta$$

$$\left(\frac{\Omega_1}{\omega}\right)^2 = 1-\eta$$

Hence if $\eta \ll 1$,

$$\eta \simeq \frac{\Omega_2-\Omega_1}{\omega} = \frac{\Delta\Omega}{\omega} = \frac{1}{Q}$$

Exercise 16: *The Nyquist diagram. For the system of exercise* 14 *draw the locus of values of* $Z/F = u + jv$ *in the complex plane. Show that this locus is a circle passing through the origin and having its center on the imaginary axis. Explain how to obtain* ω, η, k, *and* m *from this locus.*

Let

$$\frac{Z}{F} = \frac{1}{k-m\Omega^2+j\eta k} = u + jv$$

and identify

$$u=\frac{1}{k}\frac{1-(\Omega/\omega)^2}{[1-(\Omega/\omega)^2]^2+\eta^2}$$

$$v=\frac{1}{k}\frac{-\eta}{[1-(\Omega/\omega)^2]^2+\eta^2}$$

These two relations yield the equation of a circle:

$$u^2+\left(v+\frac{1}{2\eta k}\right)^2=\frac{1}{4\eta^2k^2}$$

where Ω increases in a clockwise sense (see Figure 19).

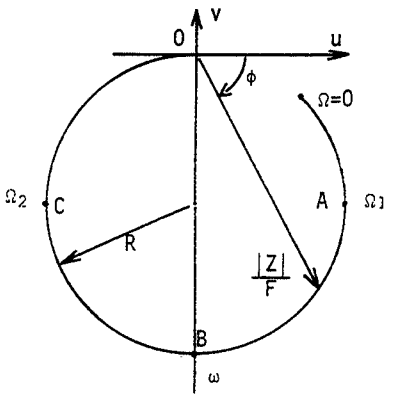


Figure 19

Point B corresponds to the maximum amplitude and therefore at point B, $\Omega=\omega$. Since points A and C are such that $OA=OC=OB/\sqrt{2}$, at point A, $\Omega=\Omega_1$ and at point B, $\Omega=\Omega_2$. Then if, as in exercise 15, η is small,

$$\eta\simeq\frac{\Omega_2-\Omega_1}{\omega}$$

Finally, if R is the radius of the circle:

$$k=\frac{1}{2\eta R}$$

and it follows that the mass of the system is

$$m=\frac{k}{\omega^2}$$

Exercise 17: *Coulomb friction damping. A system consists of a spring and mass supported on a horizontal plane. Let F be the friction force opposing the system motion. At the initial time* $t=0$ *the system starts from rest with* $x_0=aF/k>0$. *Find the motion of the system and then plot* $x(t)$ *for* $a=15.5$.

As the initial position is positive, the equation of motion has the form

$$mx^{\infty} + kx = F$$

with the solution

$$x = \frac{F}{k} + (a-1)\frac{F}{k}\cos\omega t$$

The velocity is

$$x^{\circ} = -(a-1)\frac{F}{k}\omega \sin \omega t$$

For this solution to be valid, the velocity must be negative; that is, $a > 1$. This motion will exist until the velocity becomes zero at:

$$\omega t = \pi$$

and at this instant,

$$x = \frac{F}{k}(2-a)$$

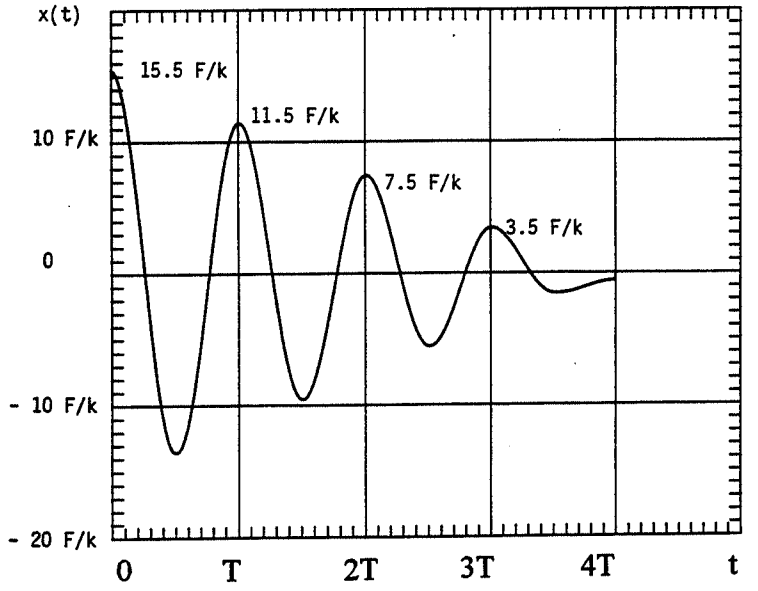
If $1 < a < 2$, the mass stops at a position between 0 and F/k.

If $2 < a < 3$, the mass stops at a position between $-F/k$ and 0.

If $a > 3$ the motion continues in the positive x-direction and is governed by the equation

$$mx^{\infty} + kx = -F$$

which can be solved as before.



The system is piecewise linear with pieces of length π/ω. The motion of the mass is shown in Figure 20 for $a = 15.5$.

It can be shown that the envelope curves of the maximum values of response are straight lines.

Exercise 18: *Transfer matrices. The transfer matrix links force and displacement at two different points of a system. Determine the transfer matrix of a spring and a mass. Then use these matrices to obtain the frequency of free vibration of a spring–mass system.*

For a system undergoing undamped free vibration, the forces and displacements will be sinusoidal. If F and X are the force and displacement

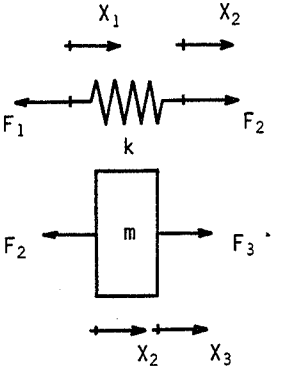


Figure 21

amplitudes at a point in the system (see Figure 21), one has for the spring:

$$F_2 - F_1 = 0$$
$$F_2 = k(X_2 - X_1)$$

for the mass:

$$F_3 - F_2 = -m\omega^2 X_2$$
$$X_3 = X_2$$

The transfer matrices are obtained from these equations and have the following form

for the spring:

$$\begin{bmatrix} F_2 \\ X_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{k} & 1 \end{bmatrix} \begin{bmatrix} F_1 \\ X_1 \end{bmatrix} = T_k \begin{bmatrix} F_1 \\ X_1 \end{bmatrix}$$

for the mass:

$$\begin{bmatrix} F_3 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1 & -m\omega^2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} F_2 \\ X_2 \end{bmatrix} = T_m \begin{bmatrix} F_2 \\ X_2 \end{bmatrix}$$

hence for the system, spring and mass in series,

$$\begin{bmatrix} F_3 \\ X_3 \end{bmatrix} = T_m \cdot T_k \begin{bmatrix} F_1 \\ X_1 \end{bmatrix} = \begin{bmatrix} 1 - \dfrac{m\omega^2}{k} & -m\omega^2 \\ \dfrac{1}{k} & 1 \end{bmatrix} \begin{bmatrix} F_1 \\ X_1 \end{bmatrix}$$

If the spring–mass system is attached at point 1 and free at point 3.

$$F_3 = 0 = \left(1 - \frac{m\omega^2}{k}\right) F_1$$

and since $F_1 \neq 0$, the frequency is

$$\omega = \sqrt{\frac{k}{m}}$$

Exercise 19: *Find the spring and mass transfer matrices for sinusoidal forced vibration. Then use these matrices to obtain the forced response of the spring–mass system subjected to a sinusoidal excitation of amplitude F and frequency Ω applied to the mass.*

It has been shown that in sinusoidal forced vibration, the force and displacement are sinusoidal with the same frequency. Using the same notation as in exercise 18 with F the amplitude of force excitation, one has

for the mass:

$$F_3 - F_2 + F = -m\Omega^2 X_2$$

$$X_3 = X_2$$

and the transfer matrix can be written as:

$$\begin{bmatrix} F_3 \\ X_3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -m\Omega^2 & -F \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F_2 \\ X_2 \\ 1 \end{bmatrix} = T_m^* \begin{bmatrix} F_2 \\ X_2 \\ 1 \end{bmatrix}$$

for the spring:

$$\begin{bmatrix} F_2 \\ X_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \dfrac{1}{k} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F_1 \\ X_1 \\ 1 \end{bmatrix} = T_k^* \begin{bmatrix} F_1 \\ X_1 \\ 1 \end{bmatrix}$$

then

$$\begin{bmatrix} F_3 \\ X_3 \\ 1 \end{bmatrix} = T_m^* \cdot T_k^* \begin{bmatrix} F_1 \\ X_1 \\ 1 \end{bmatrix}$$

and

$$\begin{bmatrix} F_3 \\ X_3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-\dfrac{m\Omega^2}{k} & -m\Omega^2 & -F \\ \dfrac{1}{k} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F_1 \\ X_1 \\ 1 \end{bmatrix}$$

Because of the boundary conditions

$$F_3 = 0 = \left(1-\frac{m\Omega^2}{k}\right)F_1 - F$$

and

$$X_3 = \frac{F_1}{k}$$

Finally,

$$X_3 = \frac{F}{k(1-m\Omega^2/k)}$$

and

$$x_3(t) = \frac{F\sin\Omega t}{(k-m\Omega^2)}$$

Exercise 20: *Damped transfer matrices for sinusoidal excitation. Determine the transfer matrices for the parallel spring–viscous damper system shown in Figure 22 by separating the real and imaginary parts of the forces and displacements. Then, for this system attached to a mass m, derive the displacement amplitude of the steady-state motion.*

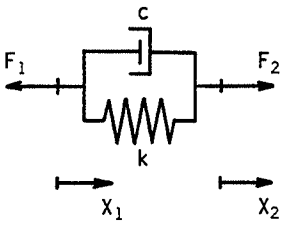


Figure 22

For the parallel spring–viscous damper system shown:

$$F_2 e^{j\Omega t} = k(X_2 e^{j\Omega t} - X_1 e^{j\Omega t}) + jc\Omega(X_2 e^{j\Omega t} - X_1 e^{j\Omega t})$$

Then

$$F_2 = (k + jc\Omega)(X_2 - X_1)$$

Since $F_2 = F_1$,

$$\begin{bmatrix} F_2 \\ X_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \dfrac{1}{k+jc\Omega} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F_1 \\ X_1 \\ 1 \end{bmatrix}$$

and separating the real and imaginary parts,

$$\begin{bmatrix} F_{2r} \\ X_{2r} \\ F_{2i} \\ X_{2i} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ \dfrac{k}{k^2+c^2\Omega^2} & 1 & \dfrac{c\Omega}{k^2+c^2\Omega^2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \dfrac{-c\Omega}{k^2+c^2\Omega^2} & 0 & \dfrac{k}{k^2+c^2\Omega^2} & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F_{1r} \\ X_{1r} \\ F_{1i} \\ X_{1i} \\ 1 \end{bmatrix}$$

The transfer matrix for the mass is found in a similar manner, and is equal to

$$\begin{bmatrix} F_{3r} \\ X_{3r} \\ F_{3i} \\ X_{3i} \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -m\Omega^2 & 0 & 0 & -F_r \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -m\Omega^2 & -F_i \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} F_{2r} \\ X_{2r} \\ F_{2i} \\ X_{2i} \\ 1 \end{bmatrix}$$

Because of the boundary conditions, F_{1r} and F_{1i} are obtained as a function of the components of the excitation force F_r and F_i. Then X_{3r} and X_{3i} can be obtained.

2
Two Degree-of-Freedom Systems

Two degree-of-freedom systems, even though included in N degree-of-freedom systems, are treated separately. This is because their small size allows analytical solution, understanding of more general methods, and an introduction to the concept of coupling. In addition, they provide an explanation of useful applications such as the dynamic vibration absorber.

Certain properties of a vibrating sysem used here will not be proven until the next chapter and, as a preparation for the next chapter, the modal method is used even though direct calculations are simpler.

The contents of the chapter are as follows:

2.1 Undamped Systems

These are illustrated by the example of Figure 1. The two masses rest on a frictionless horizontal plane.

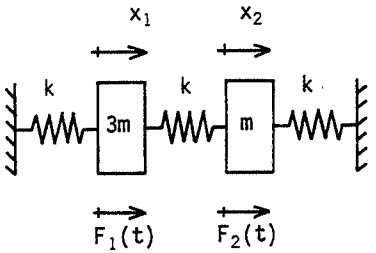


Figure 1 Two degree-of-freedom undamped system

From Newtonian mechanics, the two differential equations of motion are

$$\begin{aligned} 3mx_1^{\infty} + 2kx_1 - kx_2 &= F_1(t) \\ mx_2^{\infty} + 2kx_2 - kx_1 &= F_2(t) \end{aligned} \tag{1}$$

where $F_1(t)$ and $F_2(t)$ are the forcing functions acting respectively on masses $3m$ and m.

The equations (1) can also be obtained from Lagrange's equations (see the Appendix). For this purpose, the kinetic energy T, strain energy U, and virtual work of the external forces δW are

$$T = \frac{3mx_1^{o2}}{2} + \frac{mx_2^{o2}}{2} \tag{2}$$

$$U = \frac{kx_1^2}{2} + \frac{k(x_2 - x_1)^2}{2} + \frac{kx_2^2}{2} \tag{3}$$

$$\delta W = F_1(t)\,\delta x_1 + F_2(t)\,\delta x_2 \tag{4}$$

Equations (1) can be written more compactly in the following form:

$$Mx^{\infty} + Kx = F(t) \tag{5}$$

with

$$M = \begin{bmatrix} 3m & 0 \\ 0 & m \end{bmatrix} \qquad K = \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \tag{6}$$

being the mass matrix and stiffness matrix and

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad F(t) = \begin{bmatrix} F_1(t) \\ F_2(t) \end{bmatrix} \tag{7}$$

being the displacement and external force vectors.

2.1.1 Free vibration

In the case of $F(t) = 0$, equations (1) become

$$\begin{aligned} 3mx_1^{\infty} + 2kx_1 - kx_2 &= 0 \\ mx_2^{\infty} + 2kx_2 - kx_1 &= 0 \end{aligned} \tag{8}$$

Solutions are sought in the form

$$\begin{aligned} x_1 &= X_1 e^{rt} \\ x_2 &= X_2 e^{rt} \end{aligned} \tag{9}$$

Substituting (9) in (8) gives the homogeneous equations

$$\begin{aligned} (3mr^2 + 2k)X_1 - kX_2 &= 0 \\ (mr^2 + 2k)X_2 - kX_1 &= 0 \end{aligned} \tag{10}$$

In matrix form, these equations are

$$[r^2M+K]\begin{bmatrix}X_1\\X_2\end{bmatrix}=\Delta(r^2)\begin{bmatrix}X_1\\X_2\end{bmatrix}=0 \tag{11}$$

with

$$\Delta(r^2)=\begin{bmatrix}3mr^2+2k & -k\\ -k & mr^2+2k\end{bmatrix} \tag{12}$$

The trivial solution $X_1=X_2=0$ is not of interest. The nontrivial solution is associated with the condition $\Delta(r^2)=0$. The expansion of the determinant gives

$$(3mr^2+2k)(mr^2+2k)-k^2=0 \tag{13}$$

and the two roots are

$$\begin{aligned} r_1^2&=-0.4514\frac{k}{m}\\ r_2^2&=-2.215\frac{k}{m}\end{aligned} \tag{14}$$

From (14),

$$\begin{aligned} r_1&=\pm j0.6719\sqrt{\frac{k}{m}}\\ &=\pm j\omega_1\\ r_2&=\pm j1.488\sqrt{\frac{k}{m}}\\ &=\pm j\omega_2\end{aligned} \tag{15}$$

where ω_1 and ω_2 are the frequencies of the system.

Substituting ω_1 and ω_2 into (11) gives

$$\begin{aligned} \Delta(-\omega_1^2)\begin{bmatrix}X_1\\X_2\end{bmatrix}&=\Delta(-\omega_1^2)\phi_1=0\\ \Delta(-\omega_2^2)\begin{bmatrix}X_1\\X_2\end{bmatrix}&=\Delta(-\omega_2^2)\phi_2=0\end{aligned} \tag{16}$$

where ϕ_1 and ϕ_2 are the mode shapes of the system.

In a broad sense ω_1, ϕ_1 and ω_2, ϕ_2 characterize respectively the two modes of vibration of the system. The word 'mode' is used throughout this text to describe either the mode of vibration ω_i, ϕ_i or the mode shape ϕ_i.

Relationships between X_1 and X_2 can be obtained from each of equations (16)

$$\begin{aligned} &\text{for } \omega_1: && X_2=0.6458X_1\\ &\text{for } \omega_2: && X_2=-4.646X_1\end{aligned} \tag{17}$$

It is important to realize that X_1 and X_2 are not independent because (13) implies that the two equations (10) are no longer independent. The two components X_1 and X_2 can therefore only be determined to within a multiplicative constant. The process of choosing this constant is called normalization and there is no unique way to proceed. For systems with only a few degrees of freedom several choices are possible, for example:

(a) Set one of the components equal to unity, then

$$\phi_1 = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.6458 \end{bmatrix} \qquad \phi_2 = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -4.646 \end{bmatrix} \tag{18}$$

(b) Set the magnitude of the vector equal to unity, then

$$X_1^2 + X_2^2 = 1 \tag{19}$$

and hence

$$\phi_1 = \begin{bmatrix} 0.8401 \\ 0.5425 \end{bmatrix} \qquad \phi_2 = \begin{bmatrix} 0.2104 \\ -0.9776 \end{bmatrix} \tag{20}$$

(c) Set the matrix product $\phi_i^t M \phi_i$ equal to the total mass of the system, then

$$(3X_1^2 + X_2^2)m = 4m \tag{21}$$

and hence

$$\phi_1 = \begin{bmatrix} 1.082 \\ 0.6986 \end{bmatrix} \qquad \phi_2 = \begin{bmatrix} 0.4034 \\ -1.874 \end{bmatrix} \tag{22}$$

From (9) and (18), the motion in free vibration of the system is

$$\begin{aligned} x_1(t) &= \alpha_1 e^{j\omega_1 t} + \beta_1 e^{-j\omega_1 t} + \alpha_2 e^{j\omega_2 t} + \beta_2 e^{-j\omega_2 t} \\ x_2(t) &= 0.6458(\alpha_1 e^{j\omega_1 t} + \beta_1 e^{-j\omega_1 t}) - 4.646(\alpha_2 e^{j\omega_2 t} + \beta_2 e^{-j\omega_2 t}) \end{aligned} \tag{23}$$

Equation (23) can be rewritten in the more convenient form:

$$\begin{aligned} x_1(t) &= a_1 \sin \omega_1 t + b_1 \cos \omega_1 t + a_2 \sin \omega_2 t + b_2 \cos \omega_2 t \\ x_2(t) &= 0.6458(a_1 \sin \omega_1 t + b_1 \cos \omega_1 t) - 4.646(a_2 \sin \omega_2 t + b_2 \cos \omega_2 t) \end{aligned} \tag{24}$$

where the four constants are determined from the initial conditions. If, at the instant $t = 0$, one takes the initial conditions to be

$$\begin{aligned} x_1(0) &= x_0 \\ x_2(0) &= 0 \\ x_1^{\circ}(0) &= 0 \\ x_2^{\circ}(0) &= 0 \end{aligned} \tag{25}$$

the four constants are solutions to the set of equations

$$\begin{aligned} b_1 + b_2 &= x_0 \\ 0.6458b_1 - 4.646b_2 &= 0 \\ a_1\omega_1 + a_2\omega_2 &= 0 \\ 0.6458a_1\omega_1 - 4.646a_2\omega_2 &= 0 \end{aligned} \tag{26}$$

After completing this solution,

$$\begin{aligned} x_1(t) &= x_0(0.8780\cos\omega_1 t + 0.1220\cos\omega_2 t) \\ x_2(t) &= x_0(0.5669\cos\omega_1 t - 0.5669\cos\omega_2 t) \end{aligned} \tag{27}$$

Figure 2 shows $x_2(t)/x_0$ as a function of $t\sqrt{k/m}$. Note that the solution is not sinusoidal.

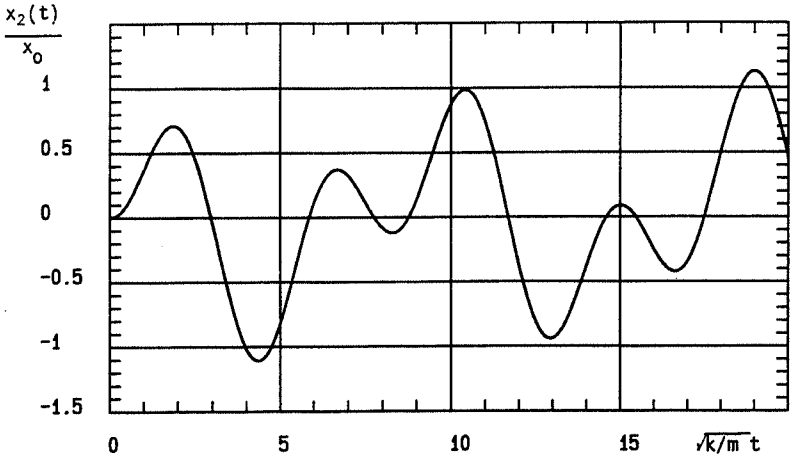


Figure 2 Free motion of mass no. 2 in Figure 1

For the example choosen, it is possible to verify that the modes ϕ_i are related to the mass matrix and the stiffness matrix as follows:

$$\begin{aligned} \phi_1^t M\phi_2 &= 0 \\ \phi_1^t K\phi_2 &= 0 \end{aligned} \tag{28}$$

By analogy with the scalar vector product, the modes are said to be orthogonal with respect to M and K. This orthogonality property will be proven for general systems in the next chapter. In addition, from (16):

$$\begin{aligned} \omega_1^2 M\phi_1 &= K\phi_1 \\ \omega_2^2 M\phi_2 &= K\phi_2 \end{aligned} \tag{29}$$

and after premultiplying respectively by ϕ_1^t and ϕ_2^t,

$$\begin{aligned}\omega_1^2\phi_1^tM\phi_1 &= \phi_1^tK\phi_1\\ \omega_2^2\phi_2^tM\phi_2 &= \phi_2^tK\phi_2\end{aligned} \tag{30}$$

The expressions $\phi_1^tM\phi_1$ and $\phi_2^tM\phi_2$ are called the modal masses, and the expressions $\phi_1^tK\phi_1$ and $\phi_2^tK\phi_2$ the modal stiffnesses. If the first method of normalization is used in the above example, one obtains

$$\begin{aligned}\phi_1^tM\phi_1 &= 3.417m\\ \phi_2^tM\phi_2 &= 24.58m\\ \phi_1^tK\phi_1 &= 1.542k\\ \phi_2^tK\phi_2 &= 54.46k\end{aligned} \tag{31}$$

2.1.2 Forced vibration

The transient part of the solution of equation (1) is the general solution of (8). Since engineering systems always possess some damping, this transient solution decays to zero with time. In this section the steady-state solution of (1) is presented. Two cases are discussed: harmonic excitation and excitation which is a general function of time.

Harmonic excitation: direct method

Let

$$\begin{aligned}F_1(t) &= F\sin\Omega t\\ F_2(t) &= 0\end{aligned} \tag{32}$$

then (1) becomes

$$\begin{aligned}3mx_1^{\infty} + 2kx_1 - kx_2 &= F\sin\Omega t\\ mx_2^{\infty} + 2kx_2 - kx_1 &= 0\end{aligned} \tag{33}$$

Solutions are sought in the form:

$$\begin{aligned}x_1(t) &= A_1\sin\Omega t + B_1\cos\Omega t\\ x_2(t) &= A_2\sin\Omega t + B_2\cos\Omega t\end{aligned} \tag{34}$$

These are then substituted into (33). Since equations (33) must hold for all time, each equation gives two relations corresponding to the vanishing of the coefficients of $\sin\Omega t$ and $\cos\Omega t$. The results are

$$B_1 = B_2 = 0$$

$$A_1 = \frac{F(2k - m\Omega^2)}{(2k - 3m\Omega^2)(2k - m\Omega^2) - k^2}\quad,\quad A_2 = \frac{Fk}{(2k - 3m\Omega^2)(2k - m\Omega^2) - k^2} \tag{35}$$

hence

$$x_1(t) = \frac{F(2k - m\Omega^2)\sin \Omega t}{(2k - 3m\Omega^2)(2k - m\Omega^2) - k^2} \quad , \quad x_2(t) = \frac{Fk \sin \Omega t}{(2k - 3m\Omega^2)(2k - m\Omega^2) - k^2} \tag{36}$$

The values of Ω for which the denominators of $x_1(t)$ and $x_2(t)$ vanish correspond to the frequencies ω_1 and ω_2 found for free vibration. If the system is subjected to a harmonic force whose frequency is equal to ω_1 or ω_2, the amplitude of the response will approach infinity. This is the phenomenon of undamped resonance. However, if some damping is included, the constants B_1 and B_2 will be nonzero, and the amplitude of response at resonance will be finite.

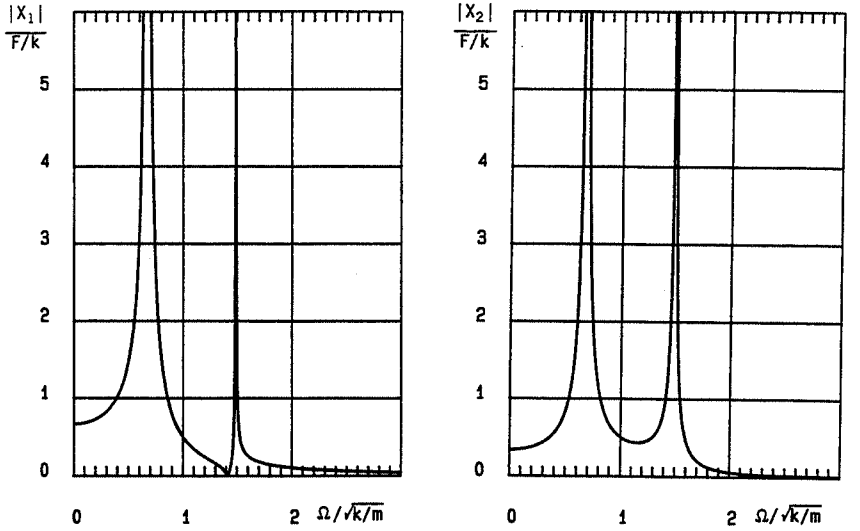


Figure 3 Steady-state amplitude responses of the system of Figure 1 for a sinusoidal forcing function applied to mass no. 1

Figure 3 shows $|X_1|/(F/k)$ and $|X_2|/(F/k)$ as a function of $\Omega/\sqrt{k/m}$ in linear coordinates. Figure 4 shows the phases of $x_1(t)$ and $x_2(t)$ as a function of $\Omega/\sqrt{k/m}$. The phases are $\pi/2$ for $\Omega = \omega_1$ and $\Omega = \omega_2$, but the phase of $x_1(t)$ is also $\pi/2$ at the antiresonance ($|X_1| = 0$).

General function of time: modal method

The main advantage of the modal method for two degree-of-freedom systems is to uncouple the equations of motion. This can be accomplished by

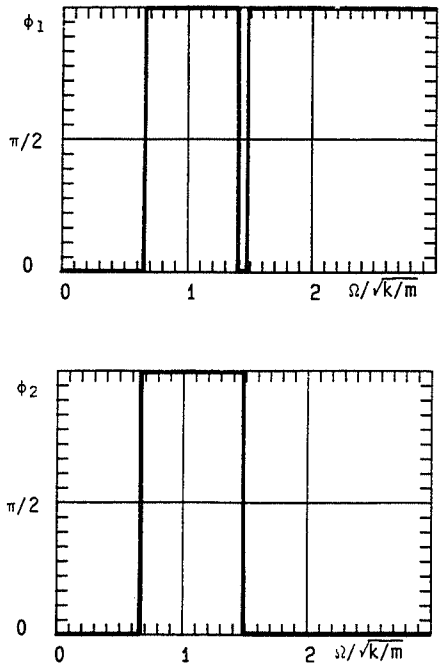


Figure 4 Steady-state phase responses of the system of Figure 1 for a sinusoidal forcing function applied to mass no. 1

using the following change of variable:

$$x = [\phi_1, \phi_2]\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \phi q \tag{37}$$

where ϕ_1 and ϕ_2 are the modes of the system and q_1 and q_2 are the new variables. Substituting (37) into (1) and premultiplying by ϕ^t gives

$$\phi^t M \phi q^{\infty} + \phi^t K \phi q = \phi^t F(t) \tag{38}$$

Because of the orthogonality conditions (28), the system (38) consists of two uncoupled differential equations. At this point it is straightforward to obtain $q_1 = q_1(t)$ and $q_2 = q_2(t)$ by proceeding as in section 1.2.3. Finally, $x_1(t)$ and $x_2(t)$ can be recovered by using (37).

The previous example solved by the direct method will now be solved by the modal method. Equation (37) gives

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0.6458 & -4.646 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \tag{39}$$

and (38) becomes

$$\begin{bmatrix} 3.417m & 0 \\ 0 & 24.58m \end{bmatrix}\begin{bmatrix} q_1^{\infty} \\ q_2^{\infty} \end{bmatrix}+\begin{bmatrix} 1.542k & 0 \\ 0 & 54.46k \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}=\begin{bmatrix} F\sin\Omega t \\ F\sin\Omega t \end{bmatrix} \tag{40}$$

The steady-state solutions of the two uncoupled equation in (40) are

$$q_1=\frac{F\sin\Omega t}{1.542k-3.417m\Omega^2}, \qquad q_2=\frac{F\sin\Omega t}{54.46k-24.58m\Omega^2} \tag{41}$$

and the physical displacements become

$$x_1(t)=q_1+q_2, \qquad x_2(t)=0.6458q_1-4.646q_2 \tag{42}$$

If Ω is near ω_1 then (42) becomes

$$x_1(t)\simeq q_1$$
$$x_2(t)\simeq 0.6458q_1$$

The system response is then a function of q_1 only. A comparison of x_1 and x_2 shows that only the mode associated with ω_1 appears. A similar conclusion is reached when Ω is near ω_2. This provides a means of measuring the system modes.

For the case of a periodic force represented by a Fourier series, the procedure is the same as above. Let $F_2(t)=0$ and $F_1(t)$ the force presented in Figure 4 of Chapter 1, then (38) can be written as:

$$\begin{aligned} 3.417mq_1^{\infty}+1.542kq_1&=\frac{F_0}{2}+\frac{2F_0}{\pi}\sum_{p=1,3,\ldots}^{\infty}\frac{\sin p\Omega t}{p} \\ 24.58mq_2^{\infty}+54.46kq_2&=\frac{F_0}{2}+\frac{2F_0}{\pi}\sum_{p=1,3,\ldots}^{\infty}\frac{\sin p\Omega t}{p} \end{aligned} \tag{43}$$

It is easy to obtain the steady-state solutions, which are

$$\begin{aligned} q_1(t)&=\frac{F_0}{3.086k}+\frac{2F_0}{\pi}\sum_{p=1,3,\ldots}^{\infty}\frac{\sin p\Omega t}{p[1.542k-3.417m(p\Omega)^2]} \\ q_2(t)&=\frac{F_0}{108.9k}+\frac{2F_0}{\pi}\sum_{p=1,3,\ldots}^{\infty}\frac{\sin p\Omega t}{p[54.46k-24.58m(p\Omega)^2]} \end{aligned} \tag{44}$$

Then $x_1(t)$ and $x_2(t)$ can be recovered with the aid of (39).

2.2 Damped Systems

Only viscous damping is considered here; a damper of this type is shown in Figure 5. Structural damping is reserved for the exercises. The differential equations of motion of the system can be obtained as in the case of undamped system by using either Newtonian or Lagrangian mechanics. In

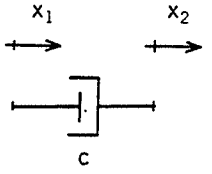


Figure 5 Viscous damper

the latter case, the dissipation function R has the form

$$R = \tfrac{1}{2}c(x_2^\circ - x_1^\circ)^2 \tag{45}$$

The general form of the equation of motion is

$$Mx^{\infty} + Cx^\circ + Kx = F(t) \tag{46}$$

where C is the viscous damping matrix.

2.2.1 Free vibration

In this case,

$$Mx^{\infty} + Cx^\circ + Kx = 0 \tag{47}$$

and solutions are sought in the form (9). One finds

$$\Delta(r)\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0 \tag{48}$$

The nontrivial solutions are associated with the condition $\Delta(r) = 0$. The expansion of the determinant gives

$$a_1 r^4 + a_2 r^3 + a_3 r^2 + a_4 r + a_5 = 0 \tag{49}$$

where, in general, the coefficients are nonzero. As these coefficients are real, the roots of (49) are real or in complex conjugate pairs. In the case of small damping, two pairs of complex conjugate roots are obtained:

$$\begin{aligned} r_1 &= -\alpha_1 + j\beta_1 \\ r_2 &= -\alpha_1 - j\beta_1 \\ r_3 &= -\alpha_2 + j\beta_2 \\ r_4 &= -\alpha_2 - j\beta_2 \end{aligned} \tag{50}$$

where α_1 and α_2 are positive. Then $x_1(t)$ may be written as:

$$\begin{aligned} x_1(t) = {} & e^{-\alpha_1 t}(A_1 \cos \beta_1 t + B_1 \sin \beta_1 t) \\ & + e^{-\alpha_2 t}(A_2 \cos \beta_2 t + B_2 \sin \beta_2 t) \end{aligned} \tag{51}$$

and $x_2(t)$ can be obtained by using the relationship between modes expressed by (48). The quantities α_1 and α_2 characterize the damping and the quantities β_1 and β_2 the frequencies.

In practice, Bairstow's method is often used to find the roots of equations such as (49).

2.2.2 Forced vibration

If the matrix C can be put in the form

$$C = aM + bK \tag{52}$$

where a and b are constants, the damping is said to be proportional. An orthogonality relation identical to (28) can then be shown to hold; that is,

$$\phi_1^t C \phi_2 = a\phi_1^t M \phi_2 + b\phi_1^t K \phi_2 = 0 \tag{53}$$

Hence the modal method can be used as in the case of undamped systems because the differential equations of motion are uncoupled.

In practice, the matrix C is most often nonproportional and therefore $\phi_1^t C \phi_2 \neq 0$.

Harmonic excitation: direct method

The steady-state solutions are sought in the form:

$$\begin{aligned} x_1(t) &= A_1 \sin \Omega t + B_1 \cos \Omega t \\ x_2(t) &= A_2 \sin \Omega t + B_2 \cos \Omega t \end{aligned} \tag{54}$$

These are substituted into (46) and the constants A_1, B_1, A_2, B_2, are determined as in the undamped case.

The method has been applied to the system shown in Figure 6, which has the equations of motion:

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} x_1^{\infty} \\ x_2^{\infty} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} x_1^{\circ} \\ x_2^{\circ} \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} F \sin \Omega t \\ 0 \end{bmatrix} \tag{55}$$

Plots of $|X_1|/(F/k) = \sqrt{A_1^2 + B_1^2}/(F/k)$ are shown in Figure 7 for three values of the damping coefficient $c = a\sqrt{km}$.

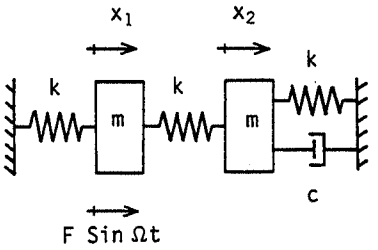


Figure 6 Two degree-of-freedom system with viscous damping

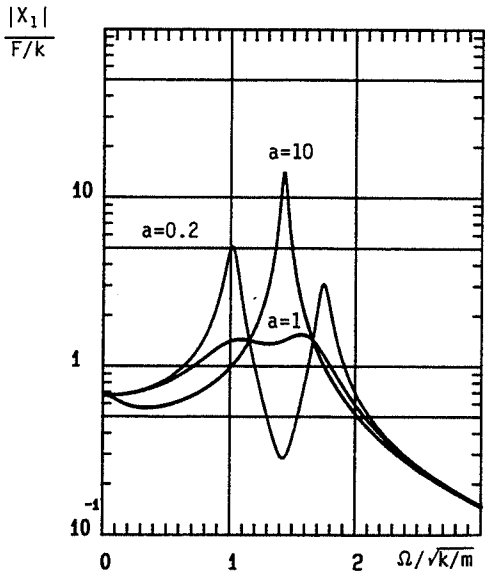


Figure 7 Steady-state amplitude response of mass no. 1 in Figure 6 for various amounts of damping

It can be seen from these curves that, for a reasonable value of damping ($a = 0.2$), the response shows two resonant peaks. For a higher value of damping ($a = 1.0$), the two peaks are barely discernible. Finally, for very high damping ($a = 10$), mass 2 is nearly motionless and the system is reduced to that of mass 1 between the two springs of stiffness k which is a single degree-of-freedom system with small damping. As a result, only one resonant peak is observed.

2.3 Vibration Absorber

The principle of a vibration absorber is simple and this device is frequently used to reduce the amplitude of a vibrating system. Let a single degree-of-freedom system (k_1, m_1) be subjected to a force $F \cos \Omega t$. In steady-state motion,

$$x_1(t) = \frac{F}{k_1 - m_1\Omega^2} \cos \Omega t \tag{56}$$

Suppose now that one adds to the original system a second spring–mass system (k_2, m_2) (see Figure 8). The equations of this combined system are then

$$\begin{aligned} m_1 x_1^{\infty} + (k_1 + k_2)x_1 - k_2 x_2 &= F \cos \Omega t \\ m_2 x_2^{\infty} + k_2 x_2 - k_2 x_1 &= 0 \end{aligned} \tag{57}$$

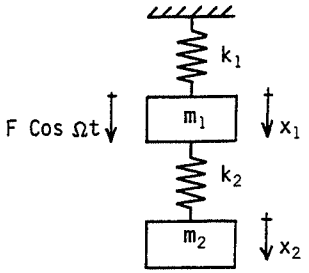


Figure 8 Vibration absorber (k_2, m_2) attached to the original spring–mass system (k_1, m_1)

The steady-state solutions for this undamped system have the form

$$x_1(t) = X_1 \cos \Omega t$$
$$x_2(t) = X_2 \cos \Omega t \tag{58}$$

where

$$X_1 = \frac{F(k_2 - m_2\Omega^2)}{(k_1 + k_2 - m_1\Omega^2)(k_2 - m_2\Omega^2) - k_2^2} \tag{59}$$

$$X_2 = \frac{Fk_2}{(k_1 + k_2 - m_1\Omega^2)(k_2 - m_2\Omega^2) - k_2^2} \tag{60}$$

In particular, for $\sqrt{k_2/m_2}$ chosen equal to Ω, the forcing frequency,

$$X_1 = 0 \qquad \text{and} \qquad X_2 = \frac{-F}{k_2} \tag{61}$$

Equation (61) shows that for $\sqrt{k_2/m_2} = \Omega$ the motion of the original spring–mass system is completely suppressed. This is the principle of the vibration absorber. In practice, it is more usual to reduce the vibration amplitude of the original system when $\Omega = \sqrt{k_1/m_1}$. Then the vibration of mass m_1 is absorbed if

$$\Omega = \sqrt{\frac{k_1}{m_1}} = \sqrt{\frac{k_2}{m_2}} \tag{62}$$

In using these results for designing vibration absorbers it is necessary to fulfil three requirements:

1. The frequency Ω must be constant or varying only over a small range because the attachment of the absorber splits ω_1 into two resonant frequencies, one on either side of ω_1. Thus, if Ω is too far above or below its design value of ω_1, one will get resonance instead of absorption of its motion.
2. The addition of an auxiliary system to the original system must be technically feasible.
3. The absorber spring (k_2) must be capable of withstanding the force of excitation $F \cos \Omega t$; see equation (61).

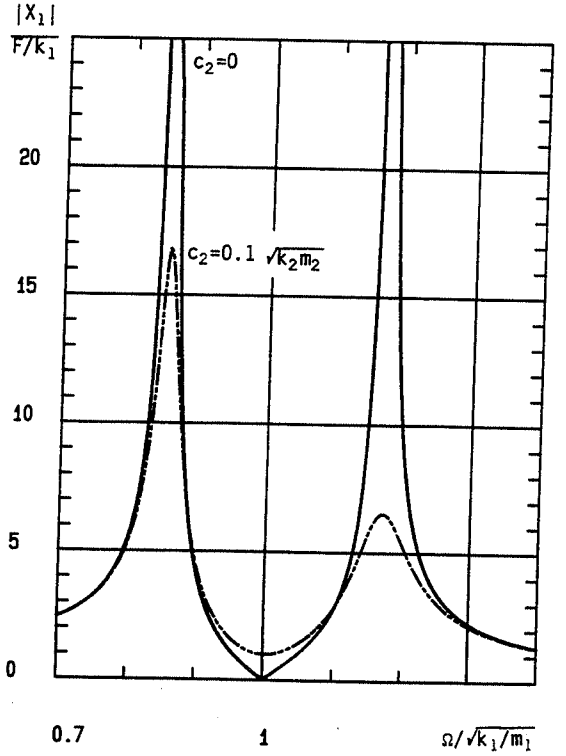


Figure 9 Steady-state amplitude response of the original spring–mass system (k_1, m_1) in Figure 8 for two different values of absorber damping (c_2)

Vibration absorbers are sometimes incorporated at the design stage. They are commonly used to reduce torsional vibration in engines and shafts.

Figure 9 shows the response $|X_1|/(F/k_1)$ as a function of $\Omega/\sqrt{k_1/m_1}$ for the case where

$$\sqrt{\frac{k_1}{m_1}} = \sqrt{\frac{k_2}{m_2}} \qquad \text{and} \qquad m_1 = 10m_2 \tag{63}$$

A viscous damper of coefficient $c_2 = 0.1\sqrt{k_2 m_2}$ has also been included to show a more realistic situation. Notice that damping results in some motion of the original system (k_1, m_1) at the design conditions.

2.4 Exercises

Exercise 1: *Calculate the frequencies, the associated modes, the modal masses, and the modal stiffnesses for the system shown in Figure* 10. *Normalize the modes by setting the first modal component equal to unity.*

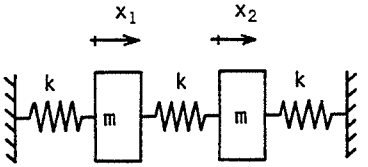


Figure 10

The answers are

$$\omega_1 = \sqrt{\frac{k}{m}}, \qquad \omega_2 = \sqrt{\frac{3k}{m}}$$

$$\phi_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \phi_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\phi_1^t M \phi_1 = 2m, \qquad \phi_2^t M \phi_2 = 2m$$

$$\phi_1^t K \phi_1 = 2k, \qquad \phi_2^t K \phi_2 = 6k$$

Exercise 2: *Find the free motion $x_1(t)$ and $x_2(t)$ for the system of exercise* 1. *The initial conditions at $t = 0$ are $x_1 = x_2 = 0$; $x_1^\circ = v_0$; $x_2^\circ = 0$. Also find the force in the spring connecting the two masses.*

The general solutions are

$$x_1(t) = a_1 \sin \omega_1 t + b_1 \cos \omega_1 t + a_2 \sin \omega_2 t + b_2 \cos \omega_2 t$$

$$x_2(t) = a_1 \sin \omega_1 t + b_1 \cos \omega_1 t - a_2 \sin \omega_2 t - b_2 \cos \omega_2 t$$

Applying the initial conditions gives

$$0 = b_1 + b_2$$

$$0 = b_1 - b_2$$

$$v_0 = a_1 \omega_1 + a_2 \omega_2$$

$$0 = a_1 \omega_1 - a_2 \omega_2$$

then

$$b_1 = b_2 = 0$$

$$a_1 \omega_1 = a_2 \omega_2 = \frac{v_0}{2}$$

Hence

$$x_1 = \frac{v_0}{2}\left[\frac{1}{\omega_1} \sin \omega_1 t + \frac{1}{\omega_2} \sin \omega_2 t\right]$$

$$x_2 = \frac{v_0}{2}\left[\frac{1}{\omega_1} \sin \omega_1 t - \frac{1}{\omega_2} \sin \omega_2 t\right]$$

and

$$F = k(x_2 - x_1) = -\frac{kv_0}{\omega_2}\sin \omega_2 t$$

Exercise 3: *Find the steady-state motion of the system in exercise 1. The force of excitation is harmonic of the form $F \cos \Omega t$ and it is applied to mass 2. Use the direct method.*

The equations are

$$m_1 x_1^{\infty} + 2kx_1 - kx_2 = 0$$
$$mx_2^{\infty} + 2kx_2 - kx_1 = F \cos \Omega t$$

As there is no damping, solutions are sought in the form

$$x_1 = X_1 \cos \Omega t$$
$$x_2 = X_2 \cos \Omega t$$

X_1 and X_2 are solutions of

$$(2k - m\Omega^2)X_1 - kX_2 = 0$$
$$(2k - m\Omega^2)X_2 - kX_1 = F$$

After completing the calculations,

$$x_1 = \frac{Fk \cos \Omega t}{(3k - m\Omega^2)(k - m\Omega^2)}$$
$$x_2 = \frac{F(2k - m\Omega^2) \cos \Omega t}{(3k - m\Omega^2)(k - m\Omega^2)}$$

x_1 and x_2 become infinite for

$$\Omega = \omega_1 = \sqrt{\frac{k}{m}}$$
$$\Omega = \omega_2 = \sqrt{\frac{3k}{m}}$$

Exercise 4: *Repeat exercise 3 but use the modal method.*

The modes, the modal stiffnesses and the modal masses have been calculated in exercise 1. Therefore:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$$

After completing the calculation,

$$2mq_1^{\infty} + 2kq_1 = F \cos \Omega t$$
$$2mq_2^{\infty} + 6kq_2 = -F \cos \Omega t$$

For steady-state motion:

$$q_1 = Q_1 \cos \Omega t$$
$$q_2 = Q_2 \cos \Omega t$$

then

$$q_1 = \frac{F \cos \Omega t}{2k - 2m\Omega^2}$$
$$q_2 = \frac{-F \cos \Omega t}{6k - 2m\Omega^2}$$

and finally

$$x_1 = q_1 + q_2$$
$$x_2 = q_1 - q_2$$

Exercise 5: *Calculate the frequencies, the associated modes, the modal masses and the modal stiffnesses for the system shown in Figure* 11. *Normalize the mode so that the modulus of each mode is equal to unity.*

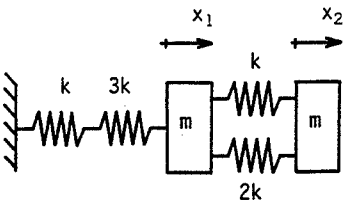


Figure 11

The springs in parallel have an equivalent stiffness of $3k$; those in series have an equivalent stiffness of $3k/4$.

The equations of motion are

$$m\ddot{x}_1 + 3.75kx_1 - 3kx_2 = 0$$
$$m\ddot{x}_2 + 3kx_2 - 3kx_1 = 0$$

After completing the calculations, one finds

$$\omega_1 = 0.593\sqrt{\frac{k}{m}} \qquad \phi_1 = \begin{bmatrix} 0.6618 \\ 0.7497 \end{bmatrix}$$

$$\omega_2 = 2.529\sqrt{\frac{k}{m}} \qquad \phi_2 = \begin{bmatrix} 0.7497 \\ -0.6618 \end{bmatrix}$$

$$\phi_1^t M \phi_1 = m \qquad \phi_2^t M \phi_2 = m$$
$$\phi_1^t K \phi_1 = 0.3517k \qquad \phi_2^t K \phi_2 = 6.398k$$

Exercise 6: *Rigid-body modes. Calculate the frequencies, the associated modes, the modal stiffnesses, and the modal masses for the system shown in*

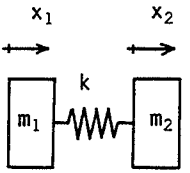


Figure 12

Figure 12. *Normalize the modes by setting the first modal component equal to unity.*

The equations of motion are

$$m_1 x_1^{\infty} + kx_1 - kx_2 = 0$$
$$m_2 x_2^{\infty} + kx_2 - kx_1 = 0$$

The solutions are found from (9), which gives

$$\begin{bmatrix} m_1 r^2 + k & -k \\ -k & m_2 r^2 + k \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = 0$$

The expansion of the determinant is as follows:

$$m_1 m_2 r^4 + k(m_1 + m_2) r^2 = 0$$

and the roots are

$$r_1^2 = 0$$
$$r_2^2 = -\frac{k(m_1 + m_2)}{m_1 m_2}$$

Then

$$\omega_1 = 0, \qquad \phi_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\omega_2 = \sqrt{\frac{k(m_1 + m_2)}{m_1 m_2}}, \qquad \phi_2 = \begin{bmatrix} 1 \\ -m_1/m_2 \end{bmatrix}$$

The mode associated with ω_1 is the rigid-body mode. Finally, it can be shown that:

$$\phi_1^t M \phi_1 = m_1 + m_2 \qquad \phi_1^t K \phi_1 = 0$$
$$\phi_2^t M \phi_2 = m_1 + \frac{m_1^2}{m_2} \qquad \phi_2^t K \phi_2 = k\left(1 + \frac{m_1}{m_2}\right)^2$$

Exercise 7: *Consider the same system as in exercise* 6 *with* $m_1 = m_2 = m$. *Find* $x_1(t)$ *and* $x_2(t)$ *for free vibration using the modal method. The initial conditions at* $t = 0$ *are* $x_1 = x_2 = 0$; $x_1^{\circ} = v_0$; $x_2^{\circ} = 0$.

The uncoupled modal equations are

$$2mq_1^{\infty} = 0$$
$$2mq_2^{\infty} + 4kq_2 = 0$$

The solutions for q_1 and q_2 are

$$q_1 = \alpha t + \beta$$
$$q_2 = \gamma \sin \sqrt{\frac{2k}{m}}\, t + \delta \cos \sqrt{\frac{2k}{m}}\, t$$

and

$$x_1 = q_1 + q_2$$
$$x_2 = q_1 - q_2$$

Using the initial conditions:

$$0 = \beta + \delta$$
$$0 = \beta - \delta$$
$$v_0 = \alpha + \gamma \sqrt{\frac{2k}{m}}$$
$$0 = \alpha - \gamma \sqrt{\frac{2k}{m}}$$

Then

$$\beta = \delta = 0; \qquad \alpha = \frac{v_0}{2}; \qquad \gamma = \frac{v_0}{2}\sqrt{\frac{m}{2k}}$$

and finally,

$$x_1 = \frac{v_0}{2}\left[t + \sqrt{\frac{m}{2k}} \sin \sqrt{\frac{2k}{m}}\, t\right]$$
$$x_2 = \frac{v_0}{2}\left[t - \sqrt{\frac{m}{2k}} \sin \sqrt{\frac{2k}{m}}\, t\right]$$

Exercise 8: *Consider the same system as in exercise 7. Find the steady-state motion $x_1(t)$ and $x_2(t)$ by the direct method. The forcing function is $F \sin \Omega t$ and is applied to mass m_2.*

The equations of motion are

$$mx_1^{\infty} + kx_1 - kx_2 = 0$$
$$mx_2^{\infty} + kx_2 - kx_1 = F \sin \Omega t$$

The solutions have the form

$$x_1 = X_1 \sin \Omega t$$
$$x_2 = X_2 \sin \Omega t$$

Substituting into the equations of motion gives

$$(k-m\Omega^2)X_1-kX_2=0$$
$$(k-m\Omega^2)X_2-kX_1=F$$

After completing the necessary calculations,

$$x_1=\frac{-Fk}{m\Omega^2(2k-m\Omega^2)}\sin\Omega t$$
$$x_2=\frac{-F(k-m\Omega^2)}{m\Omega^2(2k-m\Omega^2)}\sin\Omega t$$

Note that $x_1(t)$ and $x_2(t)$ become infinite for $\Omega=0$ and $\Omega=\sqrt{2k/m}$.

Exercise 9: *The behavior of a two degree-of-freedom system is described by the following equations of motion:*

$$\begin{bmatrix} m & 0 \\ 0 & 3m \end{bmatrix}\begin{bmatrix} x_1^{\infty} \\ x_2^{\infty} \end{bmatrix}+\begin{bmatrix} 3k & -k \\ -k & 3k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=\begin{bmatrix} 0 \\ F\sin\Omega t \end{bmatrix}$$

where the frequencies and modes can be shown to be

$$\omega_1=0.9194\sqrt{\frac{k}{m}} \qquad \phi_1=\begin{bmatrix} 1 \\ 2.155 \end{bmatrix}, \qquad \omega_2=1.776\sqrt{\frac{k}{m}} \qquad \phi_2=\begin{bmatrix} 1 \\ -0.1547 \end{bmatrix}$$

Find the steady-state response if $\Omega=\omega_2$.

It is sufficient to write the equation corresponding to q_2:

$$\phi_2^t M\phi_2 q_2^{\infty}+\phi_2^t K\phi_2 q_2=\phi_2^t F(t)$$
$$1.072mq_2^{\infty}+3.381kq_2=-0.1547F\sin\Omega t$$

which implies

$$q_2=\frac{-0.1547F\sin\Omega t}{3.381k-1.072m\Omega^2}$$

Then

$$x_1=\frac{-0.1547F\sin\Omega t}{3.381k-1.072m\Omega^2}$$
$$x_2=\frac{0.0239F\sin\Omega t}{3.381k-1.072m\Omega^2}$$

Exercise 10; *Calculate the frequencies and modes of the system shown in Figure* 13 *using the method of transfer matrices.*

The transfer matrix between points 6 and 1 is

$$\begin{bmatrix} F_6 \\ X_6 \end{bmatrix}=T_k\cdot T_m\cdot T_k\cdot T_m\cdot T_k\begin{bmatrix} F_1 \\ X_1 \end{bmatrix}=T\begin{bmatrix} F_1 \\ X_1 \end{bmatrix}=\begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix}\begin{bmatrix} F_1 \\ X_1 \end{bmatrix}$$

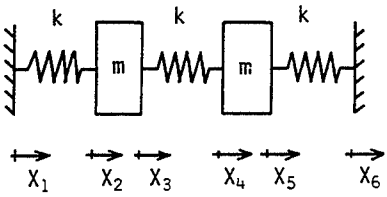


Figure 13

The frequencies are obtained by applying the appropriate boundary conditions. As $X_6 = X_1 = 0$,

$$t_{21} F_1 = 0$$

and ω_1 and ω_2 are the values of ω which make t_{21} vanish. Since

$$t_{21} = \frac{1}{k}\left(1 - \frac{m\omega^2}{k}\right)\left(3 - \frac{m\omega^2}{k}\right)$$

one finds

$$\omega_1 = \sqrt{\frac{k}{m}} \qquad \omega_2 = \sqrt{\frac{3k}{m}}$$

The ratio between X_2 and X_4 establishes the mode:

$$\begin{bmatrix} F_2 \\ X_2 \end{bmatrix} = T_k \begin{bmatrix} F_1 \\ X_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{k} & 1 \end{bmatrix} \begin{bmatrix} F_1 \\ 0 \end{bmatrix}$$

hence

$$F_2 = F_1 \qquad \text{and} \qquad X_2 = \frac{F_1}{k}$$

Also,

$$\begin{bmatrix} F_4 \\ X_4 \end{bmatrix} = T_k \cdot T_m \begin{bmatrix} F_2 \\ X_2 \end{bmatrix} = \begin{bmatrix} 1 & -m\omega^2 \\ \frac{1}{k} & 1 - \frac{m\omega^2}{k} \end{bmatrix} \begin{bmatrix} F_2 \\ X_2 \end{bmatrix}$$

and then

$$X_4 = \frac{F_2}{k} + \left(1 - \frac{m\omega^2}{k}\right) X_2 = F_1 \left[\frac{1}{k} + \frac{1}{k}\left(1 - \frac{m\omega^2}{k}\right)\right]$$

$$= \frac{F_1}{k}\left(2 - \frac{m\omega^2}{k}\right)$$

For

$$\omega = \omega_1 = \sqrt{\frac{k}{m}}$$

$$X_4 = \frac{F_1}{k} = X_2 \qquad \phi_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and for

$$\omega = \omega_2 = \sqrt{\frac{3k}{m}}$$

$$X_4 = -\frac{F_1}{k} = -X_2 \qquad \phi_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Exercise 11: *Find the steady-state response of the system shown in exercise* 10 *using the method of transfer matrices. The mass whose motion is* $x_2(t) = x_3(t)$ *is subjected to a force* $F \sin \Omega t$.

The transfer matrix between points 6 and 1 is

$$\begin{bmatrix} F_6 \\ X_6 \\ 1 \end{bmatrix} = \begin{bmatrix} t_{11}^* & t_{12}^* & t_{13}^* \\ t_{21}^* & t_{22}^* & t_{23}^* \\ t_{31}^* & t_{32}^* & t_{33}^* \end{bmatrix} \begin{bmatrix} F_1 \\ X_1 \\ 1 \end{bmatrix}$$

From the boundary conditions given before, one can obtain F_1:

$$X_6 = t_{21}^* F_1 + t_{23}^* = 0$$

from which

$$F_1 = -\frac{t_{23}^*}{t_{21}^*}$$

Knowing F_1, X_1, one can calculate, successively, $F_2, X_2; F_3, X_3; \ldots$. After all the calculations have been completed,

$$t_{23}^* = -\frac{F}{k}\left(2 - \frac{m\Omega^2}{k}\right)$$

$$t_{21}^* = t_{21} \qquad \text{(see exercise 10 where } \Omega = \omega\text{)}.$$

and

$$F_1 = \frac{F(2 - m\Omega^2/k)}{(1 - m\Omega^2/k)(3 - m\Omega^2/k)}$$

$$X_2 = \frac{F_1}{k} = \frac{F}{k} \frac{2 - m\Omega^2/k}{(1 - m\Omega^2/k)(3 - m\Omega^2/k)}$$

$$X_4 = \frac{F}{k} \frac{1}{(1 - m\Omega^2/k)(3 - m\Omega^2/k)}$$

Finally,

$$x_2(t) = X_2 \sin \Omega t$$

$$x_4(t) = X_4 \sin \Omega t$$

Exercise 12: *Find the response for the vibration absorber using complex notation.*

The equations of motion

$$m_1 y_1^{\infty} + (k_1 + k_2) y_1 - k_2 y_2 = F \sin \Omega t$$
$$m_2 y_2^{\infty} + k_2 y_2 - k_2 y_1 = 0$$

can be combined with equations (57) so that

$$m_1 z_1^{\infty} + (k_1 + k_2) z_1 - k_2 z_2 = F e^{j\Omega t}$$
$$m_2 z_2^{\infty} + k_2 z_2 - k_2 z_1 = 0$$

then x_1 and x_2 are the real parts of z_1 and z_2.

The solutions have the form:

$$z_1 = Z_1 e^{j\Omega t}$$
$$z_2 = Z_2 e^{j\Omega t}$$

and by elimination,

$$Z_1 = \frac{F(k_2 - m_2\Omega^2)}{(k_1 + k_2 - m_1\Omega^2)(k_2 - m_2\Omega^2) - k_2^2}$$

which is a real quantity.

Then the motion of the base mass is

$$x_1 = Z_1 \cos \Omega t$$

Exercise 13: *A two degree-of-freedom system* $(2m, k, 2m)$ *is mounted on a rigid support by a spring of stiffness* ak; *see Figure* 14. *The two frequencies of the complete system are* ω_1, ω_2, *the nonzero frequency of the initial system is* ω. *Use the computer program no.* 5 (*see Chapter* 7) *to calculate* ω_1/ω *and* ω_2/ω *for the parameter a in the range* $10^{-2} < a < 1$.

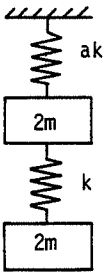


Figure 14

From the results of exercise 6,

$$\omega = \sqrt{\frac{k}{m}}$$

The results from the computer analysis are as tabulated below:

a	1	0.5	0.2	0.1	0.05	0.02	0.01
ω_1/ω	0.44	0.33	0.22	0.16	0.11	0.07	0.05
ω_2/ω	1.14	1.068	1.026	1.013	1.006	1.003	1.001

Note that for $a<0.1$, $\omega_2 \approx \omega$. Therefore, a vibration experiment can be performed on a free system by suspending it on a weak spring without the support appreciably affecting the results.

Exercise 14: *A single degree-of-freedom system with frequency* $\omega_2=\sqrt{k_2/m_2}$ *is mounted on another single degree-of-freedom system having frequency* $\omega_1=\sqrt{k_1/m_1}$, *as illustrated in Figure* 15. *The resulting two degree-of-freedom system has frequencies* ω_1^*, ω_2^*. *Using computer program no.* 5, *calculate* $(\omega_2^*-\omega_2)/\omega_2$ *as a function of* ω_2/ω_1 *and* m_2/m_1.

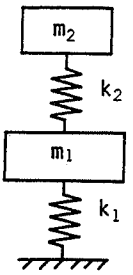


Figure 15

The results of the computer calculation for $(\omega_2^*-\omega_2)/\omega_2$ are as tabulated below:

m_2/m_1 \ ω_2/ω_1	1	1.5	2	3	10	100
0.001	0.016	0.0009	0.0007	0.0006	0.0005	0.0005
0.003	0.029	0.003	0.002	0.0017	0.0015	0.0015
0.01	0.05	0.009	0.007	0.006	0.005	0.005
0.03	0.09	0.025	0.02	0.016	0.015	0.015
0.1	0.17	0.08	0.06	0.05	0.05	0.05
0.3	0.33	0.20	0.17	0.15	0.14	0.14
1	0.62	0.50	0.46	0.43	0.42	0.41

Values of ω_2/ω_1 and m_2/m_1 (or values of k_1, m_1 for k_2, m_2 constant) which are in the upper right-hand corner of this table result in $\omega_2^*=\omega_2$. This

implies that with judiciously chosen values of k_1 and m_1, it is possible to 'soft' mount a system k_2, m_2 and still measure its 'hard' mounted, or fixed-base, frequency.

Exercise 15: *Torsional vibration absorber. In Figure* 16 *a hollow disk of inertia* I_1 *is coupled to a solid disk of inertia* I_2 *by a liquid. The damping coefficient of the resulting viscous coupling element is c. The torsional stiffness of the elastic shaft is k. The hollow disk is subjected to a harmonic couple of amplitude* Γ. *Write the equations of motion for the system and find the amplitude of response,* Θ_1. *Use complex notation. For* $I_1/I_2 = 2$, *plot* $|\Theta_1|/(\Gamma/k)$ *as a function of* $\Omega/\sqrt{k/I_1}$ *for values of viscous damping factor equal to* $a = c/\sqrt{kI_1} = 0.05$, 0.5 and 10.

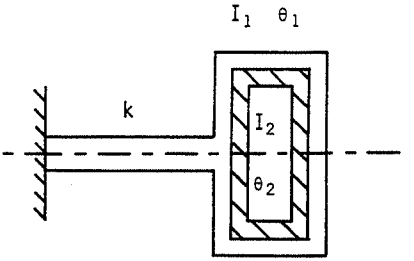


Figure 16

The equations of motion are

$$I_1\theta_1^{\infty} + c\theta_1^{\circ} + k\theta_1 - c\theta_2^{\circ} = \Gamma e^{i\Omega t}$$

$$I_2\theta_2^{\infty} + c\theta_2^{\circ} - c\theta_1^{\circ} = 0$$

Solutions are sought in the form

$$\theta_1 = \Theta_1 e^{i\Omega t}$$

$$\theta_2 = \Theta_2 e^{i\Omega t}$$

from which

$$\Theta_1 = \frac{\Gamma(jc\Omega - I_2\Omega^2)}{I_2\Omega^2(I_1\Omega^2 - k) + jc\Omega(k - I_1\Omega^2 - I_2\Omega^2)}$$

$$= |\Theta_1| e^{-j\phi}$$

and:

$$|\Theta_1| = \Gamma\sqrt{\frac{I_2^2\Omega^4 + c^2\Omega^2}{I_2^2\Omega^4(I_1\Omega^2 - k)^2 + c^2\Omega^2(k - I_1\Omega^2 - I_2\Omega^2)^2}}$$

One can obtain $|\Theta_1|/(\Gamma/k)$ for different values of a, as illustrated in Figure 17.

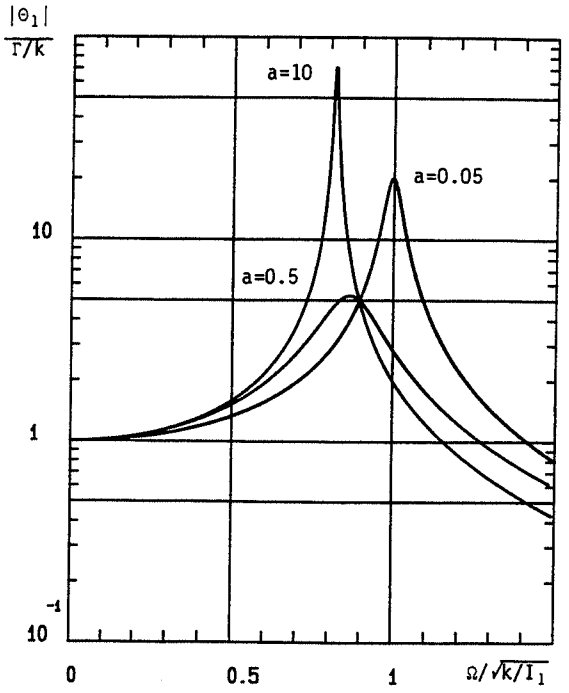


Figure 17

Notice the difference between these response curves and those of Figure 9 in the text. In particular, a viscously coupled absorber does not have a point at which the response of the primary system is brought almost to rest. On the other hand, it does not introduce any new resonances and this allows operation over a wider speed range.

Exercise 16: *Structural damping. Find expressions for the displacement amplitudes of the system shown in Figure* 18 *by using the direct method with complex notation.*

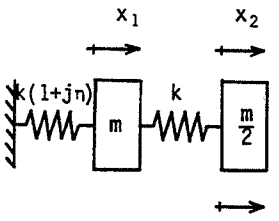


Figure 18

Proceeding as in exercise 12:

$$x_1 = \mathrm{Re}\,[z_1]$$
$$x_2 = \mathrm{Re}\,[z_2]$$

and

$$z_1 = Z_1 e^{j\Omega t}$$
$$z_2 = Z_2 e^{j\Omega t}$$

with

$$F = (F_r + jF_i)e^{j\Omega t} = F_r e^{j\Omega t}$$

Then the equations of motion are

$$-\Omega^2 \begin{bmatrix} m & 0 \\ 0 & \frac{m}{2} \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} + \begin{bmatrix} 2k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} + j\eta \begin{bmatrix} k & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ F_r \end{bmatrix}$$

Separating the real parts (Z_{1r}, Z_{2r}) and imaginary parts (Z_{1i}, Z_{2i}) of Z_1 and Z_2 gives

$$\begin{bmatrix} 2k - m\Omega^2 & -k & -\eta k & 0 \\ -k & k - \frac{m\Omega^2}{2} & 0 & 0 \\ \eta k & 0 & 2k - m\Omega^2 & -k \\ 0 & 0 & -k & k - \frac{m\Omega^2}{2} \end{bmatrix} \begin{bmatrix} Z_{1r} \\ Z_{2r} \\ Z_{1i} \\ Z_{2i} \end{bmatrix} = \begin{bmatrix} 0 \\ F_r \\ 0 \\ 0 \end{bmatrix}$$

For each value of Ω, this system gives $Z_{1r}(\Omega)$, $Z_{2r}(\Omega)$, $Z_{1i}(\Omega)$, and $Z_{2i}(\Omega)$. Then

$$x_1(t) = Z_{1r}(\Omega)\cos\Omega t - Z_{1i}(\Omega)\sin\Omega t$$
$$x_2(t) = Z_{2r}(\Omega)\cos\Omega t - Z_{2i}(\Omega)\sin\Omega t$$

and the response amplitudes are

$$|X_1| = \sqrt{Z_{1r}^2(\Omega) + Z_{1i}^2(\Omega)}$$
$$|X_2| = \sqrt{Z_{2r}^2(\Omega) + Z_{2i}^2(\Omega)}$$

It should be clear that it is a very tedious process to perform these calculations by hand.

Exercise 17: *Using program no.* 6 *of Chapter* 7, *plot* $|X_2|/(F/k)$ *as a function of* $\Omega/\sqrt{k/m}$ *for the results of exercise* 16. *Consider the cases* $\eta = 0.1$, 1.0, *and* 10.

The results are shown in Figure 19.

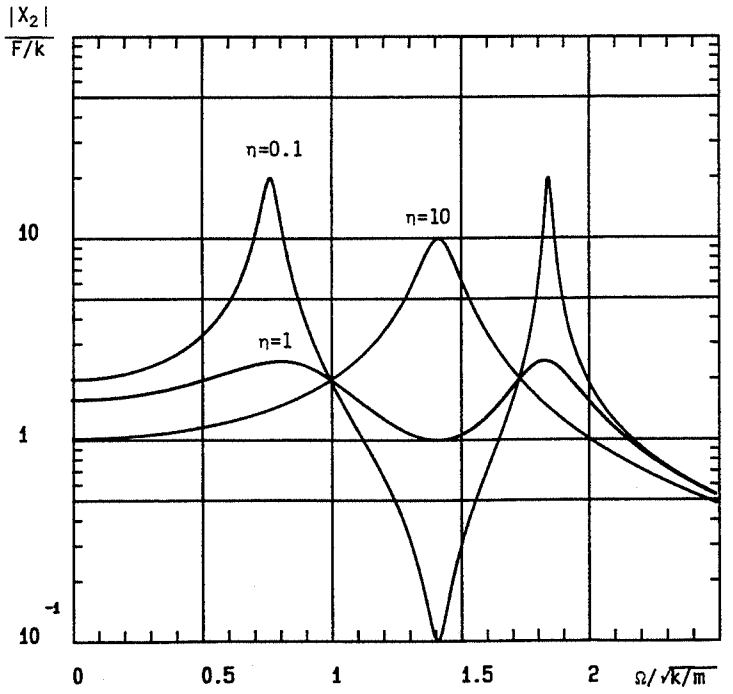


Figure 19

3
N Degree-of-Freedom Systems

The equations of motion for these systems can be obtained from either Newtonian or Lagrangian mechanics. In this chapter, only viscous damping will be considered; however, some of the system properties demonstrated are general and will be used in subsequent chapters. In contrast, some of the numerical methods employed here cannot be used for predicting the vibration behavior of systems having a very large number of degrees of freedom, such as engineering structures which are modeled by the finite element method.

The contents of the chapter are as follows:

3.1 Matrix Properties

The equations of motion of an N degree-of-freedom system have the form

$$Mx^{\circ\circ} + Cx^{\circ} + Kx = F(t) \tag{1}$$

where M, C, K are, respectively, the mass matrix, the viscous damping matrix, and the stiffness matrix. $F(t)$ and x are the force and displacement vectors.

3.1.1 Symmetry

To simplify the presentation, suppose that each mass of the system has only one degree of freedom.

Mass matrix

The kinetic energy has the form

$$T = \tfrac{1}{2} \sum_{i=1}^{N} m_i x_i^{\circ 2} \tag{2}$$

and can be written as

$$T = \tfrac{1}{2} x^{\circ t} M x^{\circ} \tag{3}$$

with

$$x^{\circ} = (x_1^{\circ}, \ldots, x_N^{\circ})^t$$

$$M = \begin{bmatrix} m_1 & & \text{zero} \\ & \ddots & \\ \text{zero} & & m_N \end{bmatrix} \tag{4}$$

where it is noted that the mass matrix M is diagonal. Systems which are modeled by having their mass lumped at discrete points will have diagonal mass matrices. If the linear transformation

$$x = Ay \tag{5}$$

is substituted into (3), one obtains

$$\begin{aligned} T &= \tfrac{1}{2}(Ay^{\circ})^t M A y^{\circ} \\ &= \tfrac{1}{2} y^{\circ t} A^t M A y^{\circ} \\ &= \tfrac{1}{2} y^{\circ t} M^* y^{\circ} \end{aligned} \tag{6}$$

where

$$M^* = A^t M A \tag{7}$$

Symmetry of M^* is guaranteed by the fact that M is premultiplied by A^t and postmultiplied by A.

Stiffness matrix

Let k_{ij} be the stiffness of a spring connecting the x_i to the x_j degree-of-freedom. The corresponding strain energy is

$$U_{ij} = \tfrac{1}{2} k_{ij} (x_i - x_j)^2 \tag{8}$$

From equation (8), one constructs a symmetric submatrix. When this submatrix is inserted into the system stiffness matrix, it appears in a symmetric position. Therefore, the stiffness matrix of the system K is also symmetric. This process of building the matrix of a system from the submatrices of its elements is called assembly. Thus, the system strain energy is given by

$$U = \tfrac{1}{2} x^t K x \tag{9}$$

If the linear transformation (5) is used, one finds

$$U = \tfrac{1}{2} y^t A^t K A y \tag{10}$$

and

$$K^* = A^t K A \tag{11}$$

which is also a symmetric matrix.

Viscous damping matrix

Let c_{ij} be the coefficient of a viscous damper connecting the x_i to the x_j degree of freedom. The dissipation function R_{ij} is

$$R_{ij} = \tfrac{1}{2} c_{ij} (x_i^{\circ} - x_j^{\circ})^2 \tag{12}$$

As before, it can be shown that the two matrices C and C^* are symmetric, where

$$C^* = A^t C A \tag{13}$$

3.1.2 Positive definite and positive semi-definite matrices

The kinetic energy of the system is always positive unless all the velocities are zero. The mass matrix is then said to be positive definite. The dissipation function R and the strain energy U can be zero with one or more displacements not equal to zero; for example, for rigid-body motion of the system. The matrices C and K are then said to be positive semi-definite.

3.1.3 Properties of the modes

Consider the modes or the mode shapes of the conservative, i.e. undamped, system:

$$M x^{\infty} + K x = 0 \tag{14}$$

Orthogonality relations

Solutions to (14) are sought in the form

$$x = X e^{rt} \tag{15}$$

which gives

$$r^2 M X + K X = 0 \tag{16}$$

or, alternatively,

$$\omega^2 M X = K X \tag{17}$$

in which

$$r = \pm j\omega \tag{18}$$

It can be shown that the solutions ω of (17) are real since the matrix M is symmetric, positive definite and the matrix K is symmetric, positive semi-definite.

Let ω_i, ϕ_i and ω_j, ϕ_j be two solutions of (17); that is,

$$\omega_i^2 M \phi_i = K \phi_i \tag{19}$$

$$\omega_j^2 M \phi_j = K \phi_j \tag{20}$$

Premultiply (19) by ϕ_j^t and (20) by ϕ_i^t. This yields

$$\omega_i^2 \phi_j^t M \phi_i = \phi_j^t K \phi_i \tag{21}$$

$$\omega_j^2 \phi_i^t M \phi_j = \phi_i^t K \phi_j \tag{22}$$

Since M and K are symmetric, equation (22) can be transposed to give

$$\omega_j^2 \phi_j^t M \phi_i = \phi_j^t K \phi_i \tag{23}$$

and combining (21) with (23) gives

$$(\omega_j^2 - \omega_i^2) \phi_j^t M \phi_i = 0 \tag{24}$$

If $\omega_i \neq \omega_j$, which is the most general case, it follows that

$$\phi_j^t M \phi_i = 0 \tag{25}$$

Similarly, from (23), one finds

$$\phi_j^t K \phi_i = 0 \tag{26}$$

Equations (25) and (26) are the orthogonality relations of the modes associated with (14). It follows that if ϕ is the square modal matrix such that

$$\phi = [\phi_1, \ldots, \phi_N] \tag{27}$$

then the matrices $\phi^t M \phi$ and $\phi^t K \phi$ are diagonal.

Consider now the important case of a system having several rigid-body modes for which, of course, the frequencies are zero. For the case of two rigid-body modes, i.e.

$$\omega_1 = 0, \phi_1; \qquad \omega_2 = 0, \phi_2; \qquad \omega_i, \phi_i \qquad i = 3, \ldots, N$$

it follows that:

$$\begin{aligned} K\phi_1 &= 0 \\ K\phi_2 &= 0 \end{aligned} \tag{28}$$

$$\omega_i^2 M \phi_i = K \phi_i \qquad i = 3, \ldots, N \tag{29}$$

As a result of (28):

$$\phi_i^t K \phi_1 = \phi_i^t K \phi_2 = 0 \tag{30}$$

and postmultiplying by ϕ_1 then ϕ_2, the transpose of (29) becomes

$$\phi_i^t M \phi_1 = \phi_i^t M \phi_2 = 0 \tag{31}$$

The classical orthogonality relations among modes, i.e. (25) and (26), are thereby verified between the rigid body modes and the others. In contrast, equation (28) implies that between the two rigid-body modes, one has the relation

$$\phi_1^t K \phi_2 = 0 \tag{32}$$

but, in general

$$\phi_1^t M \phi_2 \neq 0 \tag{33}$$

Thus rigid-body modes can give rise to off-diagonal terms in the matrix product $\phi^t M \phi$.

Modal mass and modal stiffness

Premultiplying (19) by ϕ_i^t gives

$$\omega_i^2 \phi_i^t M \phi_i = \phi_i^t K \phi_i \tag{34}$$

or

$$\omega_i^2 = \frac{\phi_i^t K \phi_i}{\phi_i^t M \phi_i} = \frac{k_i}{m_i} \tag{35}$$

where k_i and m_i are the modal stiffness and modal mass associated with the mode ω_i, ϕ_i.

Viscous damping

The matrix $\phi^t C \phi$ is symmetric. If, in addition, C can be written in the form

$$C = aM + bK \tag{36}$$

where a and b are constants, the matrix C is said to be proportional. Then $\phi^t C \phi$ is diagonal and

$$\phi_i^t C \phi_i = a\phi_i^t M \phi_i + b\phi_i^t K \phi_i \tag{37}$$

$$= c_i$$

$$= am_i + bk_i \tag{38}$$

and by analogy with the single degree-of-freedom system:

$$\alpha_i = \frac{c_i}{c_{ci}} = \frac{c_i}{2m_i\omega_i}$$

which can also be written

$$\alpha_i = \frac{a}{2\omega_i} + \frac{b\omega_i}{2} \tag{39}$$

This property allows an uncoupled solution for system response and permits the definition of the modal damping factors. Notice that this last equation allows any two modal damping factors to be chosen at will, but then all the others are automatically determined.

3.2 Calculation of Frequencies and Modes

Recall from Chapter 1 that small damping has only a second-order effect on frequency. The effect of small damping on modes could also be shown to be

small. Therefore, little error is introduced by neglecting damping when it is small.

This section describes methods of finding the frequencies ω_i and modes ϕ_i of an undamped system of N degrees of freedom where the frequencies are arranged so that $\omega_1 < \omega_2 < \ldots < \omega_N$.

3.2.1 Direct method

Solutions are sought in the form

$$x = Xe^{rt} = Xe^{j\omega t} \tag{40}$$

This yields

$$(K - \omega^2 M)X = 0 \tag{41}$$

The procedure is the same as in Chapter 2: the frequencies are obtained by setting the determinant of the matrix $K - \omega^2 M$ equal to zero; then for each ω_i, the components of X are determined and normalized to give ϕ_i. This direct method is efficient only for a few degrees of freedom.

3.2.2 The Rayleigh–Ritz method

This is a generalization by Ritz of Rayleigh's method described in Chapter 1. It is used to reduce the number of degrees of freedom of the system and to estimate the lowest frequencies. As before, one begins by making a reasonable hypothesis about the displacement of the system. Ritz suggested that this hypothesis be taken in the form of an expansion; for example, in the form:

$$\begin{aligned} x &= [\gamma_1, \ldots, \gamma_n]\begin{bmatrix} p_1 \\ \vdots \\ p_n \end{bmatrix} \\ &= \gamma p \end{aligned} \tag{42}$$

where γ_i are N-dimensional vectors with $n \ll N$. Equation (42) must satisfy the geometry boundary conditions of the system, i.e. boundary conditions on displacements and slopes. On substituting (42) into (3) and (9),

$$\begin{aligned} T &= \tfrac{1}{2} p^{\circ t} \gamma^t M \gamma p^{\circ} \\ U &= \tfrac{1}{2} p^t \gamma^t K \gamma p \end{aligned} \tag{43}$$

and using Lagrange's equations, one finds

$$\gamma^t M \gamma p^{\infty} + \gamma^t K \gamma p = 0 \tag{44}$$

The order of the system (44) is n, which is much lower than that of the system (1). This reduction in order simplifies the process of finding the

frequencies and modes but the solution will not be exact. After solving equations (44) by the direct method, or any other method, the physical displacements are recovered from (42). In fact, it is not obvious how to select the vectors γ_i. A reasonable choice for γ_1 is the static solution of the system subjected to forces equal to the weight of the masses.

In Chapter 5, the Rayleigh–Ritz method will be shown to be very useful in predicting the effects of small changes in structural parameters.

Finally, notice that for $n = 1$ this method reduces to Rayleigh's method.

3.2.3 Iterative method

Iterative methods are very useful. In the iterative method illustrated below, which is only one of many, the frequencies and modes are calculated in succession. Recall equation (17):

$$\omega^2 MX = KX \tag{45}$$

and rewrite it in the form

$$AX = \frac{1}{\omega^2} X \tag{46}$$

with

$$A = K^{-1}M$$

Choose a so-called trial vector $\phi_{(1)}$ which is sufficiently general that it can be written as an expansion in terms of all the modes of the system; that is,

$$\phi_{(1)} = a_1\phi_1 + a_2\phi_2 + \ldots \tag{47}$$

with

$$a_1, a_2, \ldots, \neq 0$$

Note the similarity of (47) to an expansion of a n-dimensional cartesian vector in terms of scalar components and unit base vectors. Now the product

$$A\phi_{(1)} = a_1 A\phi_1 + a_2 A\phi_2 + \ldots \tag{48}$$

when account is taken of (46), that is,

$$A\phi_1 = \frac{\phi_1}{\omega_1^2}; \qquad A\phi_2 = \frac{\phi_2}{\omega_2^2}; \tag{49}$$

becomes

$$A\phi_{(1)} = \frac{a_1\phi_1}{\omega_1^2} + \frac{a_2\phi_2}{\omega_2^2} + \tag{50}$$

Repeating this process with the new trial vector

$$\phi_{(2)} = A\phi_{(1)} \tag{51}$$

gives

$$A\phi_{(2)} = a_1 \frac{A\phi_1}{\omega_1^2} + a_2 \frac{A\phi_2}{\omega_2^2} + \dots$$

$$= a_1 \frac{\phi_1}{\omega_1^4} + a_2 \frac{\phi_2}{\omega_2^4} + \dots \tag{52}$$

After r iterations, the result is

$$A\phi_{(r)} \simeq \frac{a_1 \phi_1}{\omega_1^{2r}} \tag{53}$$

and after $(r+1)$ iterations,

$$A\phi_{(r+1)} \simeq \frac{a_1 \phi_1}{\omega_1^{2(r+1)}} \tag{54}$$

Combining (53) and (54) gives

$$A\phi_{(r+1)} \simeq \frac{1}{\omega_1^2} A\phi(r) \tag{55}$$

Equations (54) and (55) allow the determination of ω_1 and ϕ_1. The convergence of this process will be rapid, i.e. r will be small, if $\omega_1 \ll \omega_2$.

The calculation of ω_2, ϕ_2 proceeds in the same manner except that the new trial vector $\phi_{(1)}$ must satisfy

$$\phi_1^t M \phi_{(1)} = 0 \tag{56}$$

Combining (56) and (47) gives

$$\phi_1^t M(a_1\phi_1 + a_2\phi_2 + \dots) = a_1\phi_1^t M\phi_1 + a_2\phi_1^t M\phi_2 + \dots = 0 \tag{57}$$

After using the orthogonality relations (25), equation (57) becomes

$$a_1 \phi_1^t M \phi_1 = 0 \tag{58}$$

since $\phi_1^t M \phi_1 \neq 0$, it follows that $a_1 = 0$. Therefore condition (56) ensures that $\phi_{(1)}$ contains no component of ϕ_1. The use of this $\phi_{(1)}$ will then lead to a sequence which converges to ω_2 and ϕ_2.

The process of eliminating the component ϕ_1 from the trial vector is called 'sweeping' ϕ_1 out of $\phi_{(1)}$. To determine higher frequencies, it is necessary to use a trial function which has been swept of all previous modes.

The above iterative procedure can also be done using $M^{-1}K$ instead of $K^{-1}M$. The advantage of this is that when M is diagonal, M^{-1} is easy to obtain. The disadvantage is that the procedure will converge to the frequencies in the order $\omega_N, \omega_{N-1}, \dots, \omega_1$. This alternative is used only for systems having a few degrees of freedom.

The problem of finding the solutions of equations (17) or (46) is an algebraic eigenvalue problem. The eigenvalues and eigenvectors correspond to the frequencies and modes.

3.3 Response to Excitation

Modern methods for calculating the response of N degree-of-freedom systems to excitation when N is large are usually based on a step-by-step method (numerical integration in time) and frequently are preceded by a modal transformation to reduce the number of degrees of freedom. If the motion of interest is the steady-state response to a harmonic force and if N is not too large, the direct method can also be used.

3.3.1 Steady-state response

Direct method for harmonic excitation

For brevity, complex notation is used. Let

$$\begin{aligned} F(t) &= (F_r + jF_i)e^{j\Omega t} \\ x(t) &= (X_r + jX_i)e^{j\Omega t} \end{aligned} \tag{59}$$

Using (59) in (1) gives.

$$-\Omega^2 M(X_r + jX_i) + j\Omega C(X_r + jX_i) + K(X_r + jX_i) = F_r + jF_i \tag{60}$$

Separation of the real and imaginary parts results in

$$\begin{bmatrix} K - \Omega^2 M & -\Omega C \\ \Omega C & K - \Omega^2 M \end{bmatrix} \begin{bmatrix} X_r \\ X_i \end{bmatrix} = \begin{bmatrix} F_r \\ F_i \end{bmatrix} \tag{61}$$

These equations must be solved for each value of Ω and the response constructed using the amplitude and phase of each component of $x(t)$.

This method has disadvantages when N is large; the order of equation (61) is $2N$ and the matrix is no longer symmetric, which causes difficulties with computer calculations. Also, if the damping is small, which is the usual case, it is difficult to obtain the response in the vicinity of resonances because the resonant frequencies are not known.

Modal method

Suppose that the damping is either zero or is proportional. Find the first n frequencies and modes for the undamped system; that is, solve the eigenvalue problem of equation (14) for $\omega_1, \phi_1, \ldots, \omega_n, \phi_n$, with $n \ll N$. This allows the following transformation from physical to modal coordinates:

$$x = [\phi_1, \ldots, \phi_n] \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} \tag{62}$$

$$= \phi q \tag{63}$$

The modal matrix in (62) is a truncated form of the modal matrix defined in (27).

Using (63) in (1) and premultiplying by ϕ^t gives

$$\phi^t M\phi q^{\infty} + \phi^t C\phi q^{\circ} + \phi^t K\phi q = \phi^t F(t) \tag{64}$$

According to the orthogonality conditions (25), (26), and (37), equation (64) is a set of n uncoupled differential equations. The first of these equations is

$$m_1 q_1^{\infty} + c_1 q_1^{\circ} + k_1 q_1 = F_1(t) \tag{65}$$

with

$$\begin{aligned} m_1 &= \phi_1^t M\phi_1 & k_1 &= \phi_1^t K\phi_1 \\ c_1 &= \phi_1^t C\phi_1 = am_1 + bk_1 & F_1(t) &= \phi_1^t F(t) \end{aligned} \tag{66}$$

The solution of equations of the type (65) is now straightforward because we can use the results of Chapter 1. Equation (62) is used to recover the physical degrees of freedom. This ability to uncouple the equations of motion is the value of the modal method, and it is a value in both theory and practice.

In many cases of practical interest, the damping is small but is not proportional. In this case, it is possible to use a 'pseudo-modal' method. The coupled modal equations of motion for harmonic excitation have the form:

$$\begin{bmatrix} m_1 & & \text{zero} \\ & \ddots & \\ \text{zero} & & m_n \end{bmatrix} \begin{bmatrix} q_1^{\infty} \\ \vdots \\ q_n^{\infty} \end{bmatrix} + \phi^t C\phi \begin{bmatrix} q_1^{\circ} \\ \vdots \\ q_n^{\circ} \end{bmatrix} + \begin{bmatrix} k_1 & & \text{zero} \\ & \ddots & \\ \text{zero} & & k_n \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_n \end{bmatrix} = \begin{bmatrix} F_1(t) \\ \vdots \\ F_n(t) \end{bmatrix} \tag{67}$$

and they can be solved using the direct method described above because the disadvantages of the direct method identified previously do not exist in this case; that is, $n \ll N$ and the frequencies $\omega_1, \ldots, \omega_n$, at which resonance will occur, have already been found. Of course, the response obtained is valid only for $\Omega < \omega_n$.

For the case of structural damping and harmonic excitation, equation (67) assumes the form

$$-\Omega^2 \phi^t M\phi(Q_r + jQ_i) + \phi^t(K_1 + jK_2)\phi(Q_r + jQ_i) = \phi^t(F_r + jF_i) \tag{68}$$

If the damping is constant throughout the structure, the orthogonality relations will uncouple (68). If this is not the case, the pseudo-modal method is used.

3.3.2 General response

The general response can be obtained by a step-by-step method which gives the response of a system at time t in terms of its response at times prior to t. The method given here is not the most convenient for computation nor does it always converge but it has the advantage of being easy to understand. The step-by-step computer program given in Chapter 7 (program no. 8) uses a related method which can converge.

Develop $x(t)$ in a Taylor series and keep only terms up to the third order, then:

$$x(t)=x(t-\Delta t)+\frac{\Delta t}{1!}x^{\circ}(t-\Delta t)+\frac{\Delta t^2}{2!}x^{\circ\circ}(t-\Delta t)+\frac{\Delta t^3}{3!}x^{\circ\circ\circ}(t-\Delta t) \tag{69}$$

Similarly, for $x^{\circ}(t)$ and $x^{\circ\circ}(t)$:

$$x^{\circ}(t)=x^{\circ}(t-\Delta t)+\frac{\Delta t}{1!}x^{\circ\circ}(t-\Delta t)+\frac{\Delta t^2}{2!}x^{\circ\circ\circ}(t-\Delta t) \tag{70}$$

$$x^{\circ\circ}(t)=x^{\circ\circ}(t-\Delta t)+\frac{\Delta t}{1!}x^{\circ\circ\circ}(t-\Delta t) \tag{71}$$

This results in an acceleration which is a linear function of the time step Δt. From (69),

$$x^{\circ\circ\circ}(t-\Delta t)=\frac{6}{\Delta t^3}\left[x(t)-x(t-\Delta t)-\Delta t x^{\circ}(t-\Delta t)-\frac{\Delta t^2}{2}x^{\circ\circ}(t-\Delta t)\right] \tag{72}$$

Substituting (72) in (70) and (71) gives

$$\begin{aligned}x^{\circ}(t)&=\frac{3}{\Delta t}x(t)-\frac{3}{\Delta t}x(t-\Delta t)-2x^{\circ}(t-\Delta t)-\frac{\Delta t}{2}x^{\circ\circ}(t-\Delta t)\\&=\frac{3x(t)}{\Delta t}-A_1(t)\end{aligned} \tag{73}$$

$$\begin{aligned}x^{\circ\circ}(t)&=\frac{6}{\Delta t^2}x(t)-\frac{6}{\Delta t^2}x(t-\Delta t)-\frac{6}{\Delta t}x^{\circ}(t-\Delta t)-2x^{\circ\circ}(t-\Delta t)\\&=\frac{6}{\Delta t^2}x(t)-A_2(t)\end{aligned} \tag{74}$$

where

$$A_1(t)=\frac{3}{\Delta t}x(t-\Delta t)+2x^{\circ}(t-\Delta t)+\frac{\Delta t}{2}x^{\circ\circ}(t-\Delta t) \tag{75}$$

$$A_2(t)=\frac{6}{\Delta t^2}x(t-\Delta t)+\frac{6}{\Delta t}x^{\circ}(t-\Delta t)+2x^{\circ\circ}(t-\Delta t) \tag{76}$$

Putting (73) and (74) in (1) gives

$$\left(\frac{6M}{\Delta t^2}+\frac{3C}{\Delta t}+K\right)x(t)=F(t)+MA_2(t)+CA_1(t) \tag{77}$$

If x, x°, $x^{\circ\circ}$ are known at $(t-\Delta t)$, then $A_1(t)$ and $A_2(t)$ can be determined from (75) and (76) and $x(t)$ can be determined from (77). Finally, (73) and (74) are used to determine $x^{\circ}(t)$ and $x^{\circ\circ}(t)$. The process is then repeated for $t=t+\Delta t$.

3.4 Exercises

Exercise 1: *Write the equations of motion of the system shown in Figure 1 and calculate the frequencies and modes by the direct method. Normalize the modes by setting the first modal component equal to unity.*

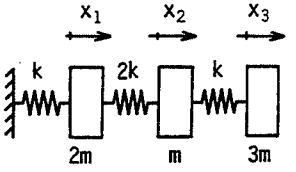


Figure 1

The equations of motion are

$$\begin{bmatrix} 2m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 3m \end{bmatrix} \begin{bmatrix} x_1^{\infty} \\ x_2^{\infty} \\ x_3^{\infty} \end{bmatrix} + \begin{bmatrix} 3k & -2k & 0 \\ -2k & 3k & -k \\ 0 & -k & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If solutions are sought in the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} e^{j\omega t}$$

the system of equations becomes

$$\begin{bmatrix} 3k-2m\omega^2 & -2k & 0 \\ -2k & 3k-m\omega^2 & -k \\ 0 & -k & k-3m\omega^2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = 0$$

The determinant of the square matrix vanishes for

$$\omega_1^2 = 0.1052 \frac{k}{m}$$

$$\omega_2^2 = 0.8086 \frac{k}{m}$$

$$\omega_3^2 = 3.920 \frac{k}{m}$$

For which it can be shown that:

$$\omega_1 = 0.3243 \sqrt{\frac{k}{m}}; \qquad \phi_1 = \begin{bmatrix} 1 \\ 1.395 \\ 2.038 \end{bmatrix}$$

$$\omega_2 = 0.8992\sqrt{\frac{k}{m}}; \qquad \phi_2 = \begin{bmatrix} 1 \\ 0.6914 \\ -0.4849 \end{bmatrix}$$

$$\omega_3 = 1.980\sqrt{\frac{k}{m}}; \qquad \phi_3 = \begin{bmatrix} 1 \\ -2.420 \\ 0.2249 \end{bmatrix}$$

Exercise 2: *Consider the same system as in exercise 1. Calculate ω_1 by Rayleigh's method. Assume the displacement pattern obtained by subjecting each mass to a static force proportional to its weight. Normalize the first component of this pattern to unity.*

The deformation pattern is obtained from

$$\begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

and is

$$x_1 = 6; \qquad x_2 = 8; \qquad x_3 = 11$$

Normalizing the x-vector gives

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1.333 \\ 1.833 \end{bmatrix} p$$

The expressions for kinetic and strain energy become

$$T = 6.928mp^{\circ 2}$$

$$U = 0.7359kp^2$$

and therefore:

$$\omega_1 = 0.3259\sqrt{\frac{k}{m}}$$

This estimate is high by only 0.5% (see exercise 1).

Exercise 3: *Consider the same problem posed in exercise 2 but now obtain the displacement pattern by subjecting only mass 3 to a static force of unit magnitude.*

This leads to

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1.5 \\ 2.5 \end{bmatrix} p$$

$$T = 11.5mp^{\circ 2}$$

$$U = 1.25kp^2$$

and

$$\omega_1 = 0.3297\sqrt{\frac{k}{m}}$$

This estimate is high by 2%.

Exercise 4: *Consider the same system as in exercise 1. Calculate* ω_1 *and* ω_2 *by the Rayleigh–Ritz method. The first displacement pattern,* γ_1 *in equation* (42), *is the same as in exercise* 2. *The second pattern,* γ_2, *is arbitrarily chosen to be* $(1, 2, 3)^t$.

In this case:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1.333 & 2 \\ 1.833 & 3 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

and

$$T = \frac{1}{2}\begin{bmatrix} p_1^\circ \\ p_2^\circ \end{bmatrix}^t \begin{bmatrix} 13.86m & 21.16m \\ 21.16m & 33m \end{bmatrix} \begin{bmatrix} p_1^\circ \\ p_2^\circ \end{bmatrix}$$

$$U = \frac{1}{2}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix}^t \begin{bmatrix} 1.472k & 2.166k \\ 2.166k & 4k \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

The differential equations of motion become

$$m\begin{bmatrix} 13.86 & 21.16 \\ 21.16 & 33 \end{bmatrix} \begin{bmatrix} p_1^\infty \\ p_2^\infty \end{bmatrix} + k\begin{bmatrix} 1.472 & 2.166 \\ 2.166 & 4 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = 0$$

Solving by the direct method gives the estimates

$$\omega_1 = 0.3249\sqrt{\frac{k}{m}}$$

$$\omega_2 = 1.098\sqrt{\frac{k}{m}}$$

Comparison to the exact solutions in exercise 1 shows that the estimate for ω_1 is high by 0.2% and the estimate for ω_2 is high by 20%.

The modes can be calculated as follows.

for ω_1:

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.1319 \end{bmatrix}$$

then

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1.333 \\ 1.833 \end{bmatrix} + 0.1319\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1.132 \\ 1.597 \\ 2.229 \end{bmatrix}$$

Making the first component unity gives

$$\phi_1 = \begin{bmatrix} 1 \\ 1.411 \\ 1.969 \end{bmatrix}$$

which is very near the exact result.

for ω_2:

$$\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.6525 \end{bmatrix}$$

then

$$\begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1.333 \\ 1.833 \end{bmatrix} - 0.6526 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0.3475 \\ 0.02808 \\ -0.1244 \end{bmatrix}$$

Making the first component unity gives

$$\phi_2 = \begin{bmatrix} 1 \\ 0.0808 \\ -0.3579 \end{bmatrix}$$

This last result is quite different from the exact result. This is to be expected since the value of ω_2 is a distant estimate.

Exercise 5: *Consider the same problem as posed in exercise 4. Use only* $\gamma_1 = (1, 1, 1)^t$ *and* $\gamma_2 = (1, 1.333, 1.833)^t$.

The answers are

$$\omega_1 = 0.3244 \sqrt{\frac{k}{m}}; \qquad \phi_1 = \begin{bmatrix} 1 \\ 1.412 \\ 2.030 \end{bmatrix}$$

$$\omega_2 = 0.9343 \sqrt{\frac{k}{m}}; \qquad \phi_2 = \begin{bmatrix} 1 \\ 0.4292 \\ -0.4279 \end{bmatrix}$$

Exercise 6: *Find the expressions for kinetic energy and strain energy for the system shown in Figure 2. Find the equations of motion. Calculate the frequencies, modes, modal masses, and modal stiffnesses. Normalize the modes by setting the first modal component equal to unity.*

The energies are

$$2T = m\dot{x}_1^2 + m\dot{x}_2^2 + m\dot{x}_3^2$$

$$2U = k(x_1 - x_2)^2 + 5k(x_2 - x_3)^2 + k(x_1 - x_3)^2$$

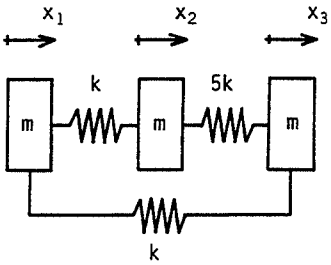


Figure 2

and the equations of motion are

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} x_1^{\infty} \\ x_2^{\infty} \\ x_3^{\infty} \end{bmatrix} + \begin{bmatrix} 2k & -k & -k \\ -k & 6k & -5k \\ -k & -5k & 6k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Solutions are sought in the form

$$x = Xe^{i\omega t}$$

and result in

$$m\omega^2(m^2\omega^4 - 14mk\omega^2 + 33k^2) = 0$$

which gives solutions

$$\omega_1 = 0; \qquad \phi_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \text{with} \qquad \begin{aligned} \phi_1^t M \phi_1 &= 3m \\ \phi_1^t K \phi_1 &= 0 \end{aligned}$$

$$\omega_2 = 1.732\sqrt{\frac{k}{m}}; \qquad \phi_2 = \begin{bmatrix} 1 \\ -0.5 \\ -0.5 \end{bmatrix} \qquad \text{with} \qquad \begin{aligned} \phi_2^t M \phi_2 &= 1.5m \\ \phi_2^t K \phi_2 &= 4.5k \end{aligned}$$

$$\omega_3 = 3.317\sqrt{\frac{k}{m}}; \qquad \phi_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \qquad \text{with} \qquad \begin{aligned} \phi_3^t M \phi_3 &= 2m \\ \phi_3^t K \phi_3 &= 22k \end{aligned}$$

Exercise 7: *Consider the same system as in exercise 1. Calculate, using the results of exercise 1, the matrix products* $\phi_i^t K \phi_j$, $\phi_i^t M \phi_j$. *Take the first component of the modes equal to unity.*

The results are

$$\phi_1^t K\phi_1 = 1.725k \qquad \phi_1^t M\phi_1 = 16.41m$$
$$\phi_2^t K\phi_2 = 2.574k \qquad \phi_2^t M\phi_2 = 3.183m$$
$$\phi_3^t K\phi_3 = 31.39k \qquad \phi_3^t M\phi_3 = 8.008m$$

All the other products are zero.

Exercise 8: *Consider the same system as in exercise 1. Calculate* ω_1 *and* ϕ_1 *by iteration using the trial vector* $(1, 1, 1)^t$.

$$K^{-1} = \frac{1}{k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{3}{2} & \frac{3}{2} \\ 1 & \frac{3}{2} & \frac{5}{2} \end{bmatrix}$$

$$A = K^{-1}M = \frac{m}{k}\begin{bmatrix} 2 & 1 & 3 \\ 2 & \frac{3}{2} & \frac{9}{2} \\ 2 & \frac{3}{2} & \frac{15}{2} \end{bmatrix} = \frac{m}{k}B$$

Then

$$BX = \frac{k}{m\omega^2}X$$

In what follows, the iteration vector is normalized at each step by making its first component unity.

$$B\phi_{(1)} = \begin{bmatrix} 2 & 1 & 3 \\ 2 & \frac{3}{2} & \frac{9}{2} \\ 2 & \frac{3}{2} & \frac{15}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 11 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 1.333 \\ 1.833 \end{bmatrix} = 6\phi_{(2)}$$

$$B\phi_{(2)} = 8.832\begin{bmatrix} 1 \\ 1.385 \\ 2.008 \end{bmatrix} \qquad B\phi_{(3)} = 9.409\begin{bmatrix} 1 \\ 1.393 \\ 2.033 \end{bmatrix}$$

$$B\phi_{(4)} = 9.492\begin{bmatrix} 1 \\ 1.393 \\ 2.036 \end{bmatrix} \qquad B\phi_{(5)} = 9.501\begin{bmatrix} 1 \\ 1.394 \\ 2.036 \end{bmatrix}$$

The mode calculated from $B\phi_{(5)}$ is very near that calculated from $B\phi_{(4)}$ and the number of iterations can therefore be considered sufficient. The frequency is obtained from

$$9.501 = \frac{k}{m\omega_1^2}$$

or

$$\omega_1 = 0.3244\sqrt{\frac{k}{m}}$$

and the mode is

$$\phi_1 = \begin{bmatrix} 1 \\ 1.394 \\ 2.036 \end{bmatrix}$$

The results for ω_1 and ϕ_1 have been deduced rapidly because, as exercise 1 shows, ω_1 is much smaller than ω_2.

Exercise 9: *Consider the same system as in exercise 1. Calculate ω_2 and ϕ_2 by iteration using the trial vector* $(1, 1, 1)^t$.

The successive vectors $\phi_{(i)}$ are such that

$$[1 \quad 1.394 \quad 2.036]\begin{bmatrix} 2m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 3m \end{bmatrix}\begin{bmatrix} \phi_{(i)1} \\ \phi_{(i)2} \\ \phi_{(i)3} \end{bmatrix} = 0$$

where $\phi_{(i)j}$ is the jth element of $\phi_{(i)}$ at the ith iteration. Expanding the expression gives

$$2\phi_{(i)1} + 1.394\phi_{(i)2} + 6.108\phi_{(i)3} = 0$$

This can be considered in the following form:

$$\begin{bmatrix} \phi^*_{(i)1} \\ \phi^*_{(i)2} \\ \phi^*_{(i)3} \end{bmatrix} = \begin{bmatrix} 0 & -0.6970 & -3.054 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \phi_{(i)1} \\ \phi_{(i)2} \\ \phi_{(i)3} \end{bmatrix}$$

where $\phi^*_{(i)}$ are now the successive trial vectors. The iteration is then performed using the matrix C:

$$C = B\begin{bmatrix} 0 & -0.6970 & -3.054 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -0.394 & -3.108 \\ 0 & 0.106 & -1.608 \\ 0 & 0.106 & 1.392 \end{bmatrix}$$

Then

$$C\phi_{(1)} = C\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = -3.502\begin{bmatrix} 1 \\ 0.4288 \\ -0.4277 \end{bmatrix}$$

$$C\phi_{(2)} = 1.160\begin{bmatrix} 1 \\ 0.6319 \\ -0.4740 \end{bmatrix}; \qquad C\phi_{(3)} = 1.224\begin{bmatrix} 1 \\ 0.6773 \\ -0.4843 \end{bmatrix}$$

$$C\phi_{(4)} = 1.238\begin{bmatrix} 1 \\ 0.6869 \\ -0.4865 \end{bmatrix}; \qquad C\phi_{(5)} = 1.241\begin{bmatrix} 1 \\ 0.6890 \\ -0.4869 \end{bmatrix}$$

$$C\phi_{(6)} = 1.241\begin{bmatrix} 1 \\ 0.6896 \\ -0.4872 \end{bmatrix}$$

The iteration can now be stopped; the frequency is obtained from

$$1.241 = \frac{k}{m\omega_2^2}$$

and

$$\omega_2 = 0.8977\sqrt{\frac{k}{m}}; \qquad \phi_2 = \begin{bmatrix} 1 \\ 0.6896 \\ -0.4872 \end{bmatrix}$$

Exercise 10: *Structural modification. Consider the same system as in exercise 1 except that mass 2 is modified to have a mass of* $1.3m$. *Using Rayleigh's method, calculate the third frequency of the modified system,* ω_3^*.

This calculation can be done easily because the exact mode corresponding to ω_3 is known from exercise 1. The additional kinetic energy due to the mass addition $0.3m$ is

$$\Delta T = \tfrac{1}{2}0.3m(-2.42)^2 p^{\circ 2} = 0.8785mp^{\circ 2}$$

Then using the results of exercise 7, the kinetic energy of the modified system is

$$\begin{aligned} T^* &= 4.004mp^{\circ 2} + 0.8785mp^{\circ 2} \\ &= 4.882mp^{\circ 2} \end{aligned}$$

Since the strain energy is constant

$$\omega_3^* = \omega_3\sqrt{\frac{T}{T^*}} = 1.793\sqrt{\frac{k}{m}}$$

The exact values ω_3^{**} and ϕ_3^{**} of the modified system can be shown to be:

$$\omega_3^{**} = 1.807\sqrt{\frac{k}{m}}; \qquad \phi_3^{**} = \begin{bmatrix} 1 \\ -1.766 \\ 0.2007 \end{bmatrix}$$

so it can be seen that this technique gives convenient estimates of the effect on frequency of small changes of a structure.

Exercise 11: *Consider the same system as in exercise* 1. *Find the steady-state response to a force* $F\sin\Omega t$ *acting on mass m. Use the modal method and plot* $|X_1|/(F/k)$ *as a function of* $\Omega/\sqrt{k/m}$.

The results of exercises 1 and 7 are used to obtain the three uncoupled equations:

$$16.41mq_1^{\infty} + 1.725kq_1 = \quad 1.395F \sin \Omega t$$
$$3.183mq_2^{\infty} + 2.574kq_2 = \quad 0.6914F \sin \Omega t$$
$$8.008mq_3^{\infty} + 31.39kq_3 = -2.420F \sin \Omega t$$

The steady-state solution for q_1 is

$$q_1 = \frac{1.395F \sin \Omega t}{1.725k - 16.41m\Omega^2}$$

or

$$q_1 = \frac{0.8087F \sin \Omega t}{k[1-(\Omega/\omega_1)^2]}$$

Similarly,

$$q_2 = \frac{0.2686F \sin \Omega t}{k[1-(\Omega/\omega_2)^2]}$$

$$q_3 = \frac{-0.07709F \sin \Omega t}{k[1-(\Omega/\omega_3)]^2}$$

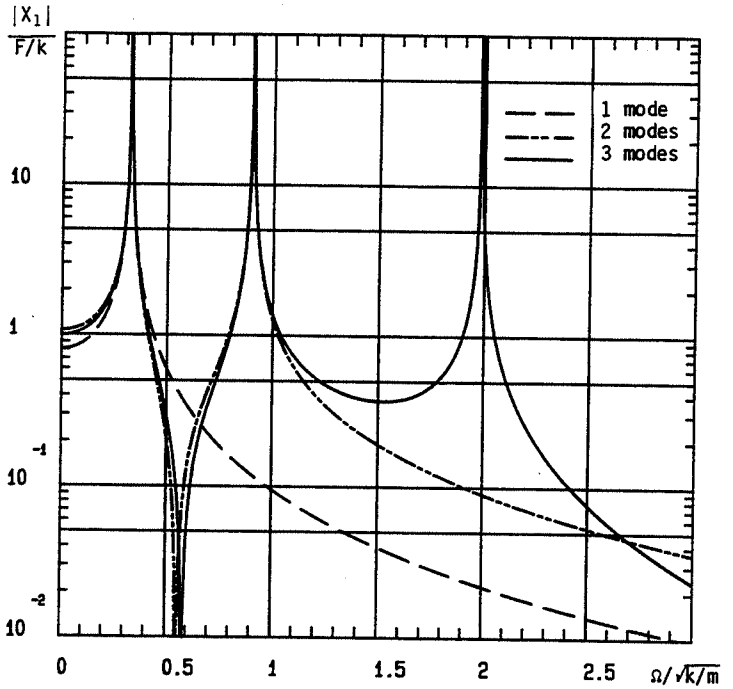


Figure 3

Then

$$x_1 = \left[\frac{0.8087}{1-(\Omega/\omega_1)^2} + \frac{0.2686}{1-(\Omega/\omega_2)^2} - \frac{0.07709}{1-(\Omega/\omega_3)^2}\right]\frac{F\sin\Omega t}{k}$$
$$= X_1 \sin\Omega t$$

Also

$$x_3 = 2.038q_1 - 0.4849q_2 + 0.2249q_3$$
$$x_2 = 1.395q_1 + 0.6914q_2 - 2.420q_3$$

The results are plotted in Figure 3.

Exercise 12: *The system shown in Figure 4 is in the vertical plane. The three rods of length L have negligible mass but have a concentrated mass of magnitude m at their tip. The rods are pinned at* A, B, *and* C; *their positions are defined by the angular coordinates* θ_1, θ_2, *and* θ_3. *The coordinates* θ_1, θ_2, θ_3 *all remain small during the motion of the system. Take the acceleration of gravity to be g. Write the equations of motion for free vibration. Calculate the frequencies and modes.*

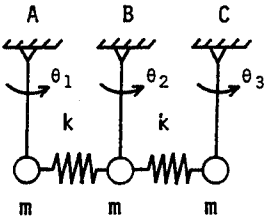


Figure 4

The equations of motion are

$$\begin{bmatrix} mL^2 & 0 & 0 \\ 0 & mL^2 & 0 \\ 0 & 0 & mL^2 \end{bmatrix}\begin{bmatrix} \theta_1^{\infty} \\ \theta_2^{\infty} \\ \theta_3^{\infty} \end{bmatrix}$$
$$+\begin{bmatrix} mgL+kL^2 & -kL^2 & 0 \\ -kL^2 & mgL+2kL^2 & -kL^2 \\ 0 & -kL^2 & mgL+kL^2 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = 0$$

The frequencies and modes are

$$\omega_1 = \sqrt{\frac{g}{L}}; \qquad \phi_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \omega_2 = \sqrt{\frac{g}{L}+\frac{k}{m}}; \qquad \phi_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\omega_3 = \sqrt{\frac{g}{L}+\frac{3k}{m}}; \qquad \phi_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

Exercise 13: *Imposed displacement. When a displacement is imposed on a mass point of a spring–mass system, that degree of freedom is lost and the frequencies and modes of the system are changed. Consider the system of exercise 1 with a displacement* $x_3 = X_3 \sin \Omega t$ *imposed on mass 3. Calculate the frequencies and modes and the steady-state response.*

The displacement x_3 is known and the three unknowns are x_1, x_2, and $F(t)$, the force required to generate the displacement x_3 at mass 3. The equations of motion are

$$\begin{bmatrix} 2m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & 3m \end{bmatrix} \begin{bmatrix} x_1^{\infty} \\ x_2^{\infty} \\ -\Omega^2 X_3 \sin \Omega t \end{bmatrix} + \begin{bmatrix} 3k & -2k & 0 \\ -2k & 3k & -k \\ 0 & -k & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ X_3 \sin \Omega t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ F(t) \end{bmatrix}$$

Consider just the first two equations:

$$\begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} x_1^{\infty} \\ x_2^{\infty} \end{bmatrix} + \begin{bmatrix} 3k & -2k \\ -2k & 3k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ kX_3 \sin \Omega t \end{bmatrix}$$

The frequencies and modes are deduced from the homogeneous form of these equations, i.e.

$$\begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} x_1^{\infty} \\ x_2^{\infty} \end{bmatrix} + \begin{bmatrix} 3k & -2k \\ -2k & 3k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

from which

$$\omega_1 = 0.8057 \sqrt{\frac{k}{m}}, \quad \phi_1 = \begin{bmatrix} 1 \\ 0.8508 \end{bmatrix}; \quad \omega_2 = 1.962 \sqrt{\frac{k}{m}}, \quad \phi_2 = \begin{bmatrix} 1 \\ -2.351 \end{bmatrix}$$

The steady-state response can be obtained by seeking solutions of the form

$$x_1 = X_1 \sin \Omega t$$
$$x_2 = X_2 \sin \Omega t$$

One finds

$$\frac{X_1}{X_3} = \frac{2k^2}{(3k - m\Omega^2)(3k - 2m\Omega^2) - 4k^2}$$
$$\frac{X_2}{X_3} = \frac{(3k - 2m\Omega^2)k}{(3k - m\Omega^2)(3k - 2m\Omega^2) - 4k^2}$$

where it is observed that X_1 and X_2 approach infinity for $\Omega = \omega_1$ or $\Omega = \omega_2$.

The force on mass 3 is found from the third equation of motion:

$$F(t) = [-kX_2 + (k - 3m\Omega^2)X_3] \sin \Omega t$$

Some thought will show that if a displacement is imposed on the *base* of a system, the frequencies and modes do not change. For example, if

the system of exercise 1 has a displacement $\delta = \Delta \sin \Omega t$ imposed on its base, the equations of motion are the same as for that system with a fixed base and with a *force* $k\Delta \sin \Omega t$ imposed on mass 1.

Exercise 14: *Step-by-step response. The system shown in Figure 5 has a viscous damping coefficient of* $c = 0.2\sqrt{km}$. *It is subjected to a constant force F at* $t = 0$. *The initial conditions are* $x(0) = 0$, $x^{\circ}(0) = 0$. *Find* $x(t)$ *in an exact manner and then by the step-by-step method using* $\Delta t = T/10$ *where T is the period of undamped frequency.*

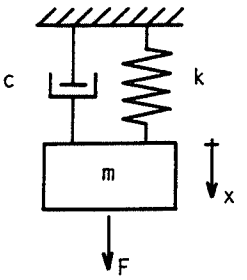


Figure 5

The equation of motion is

$$mx^{\circ\circ} + 0.2\sqrt{km}\, x^{\circ} + kx = F \qquad (t > 0)$$

Taking account of the initial conditions, the exact solution of this equation is

$$x(t) = \frac{F}{k}[1 - 1.005e^{-0.1\omega t} \sin(0.995\omega t + 1.471)]$$

where the phase angle is in radians.

The expressions utilized in the step-by-step calculation are the following:

$$\frac{6M}{\Delta t^2} + \frac{3C}{\Delta t} + K = 17.15k$$

$$A_1(t) = 4.775\sqrt{\frac{k}{m}}\, x(t - \Delta t) + 2x^{\circ}(t - \Delta t) + 0.3142\sqrt{\frac{m}{k}}\, x^{\circ\circ}(t - \Delta t)$$

$$A_2(t) = 15.20\frac{k}{m}\, x(t - \Delta t) + 9.549\sqrt{\frac{k}{m}}\, x^{\circ}(t - \Delta t) + 2x^{\circ\circ}(t - \Delta t)$$

$$x(t) = 0.05830\frac{F}{k} + 0.05830\frac{m}{k} A_2(t) + 0.01166\sqrt{\frac{m}{k}} A_1(t)$$

$$x^{\circ}(t) = 4.775\sqrt{\frac{k}{m}}\, x(t) - A_1(t)$$

$$x^{\circ\circ}(t) = 15.20\frac{k}{m}\, x(t) - A_2(t)$$

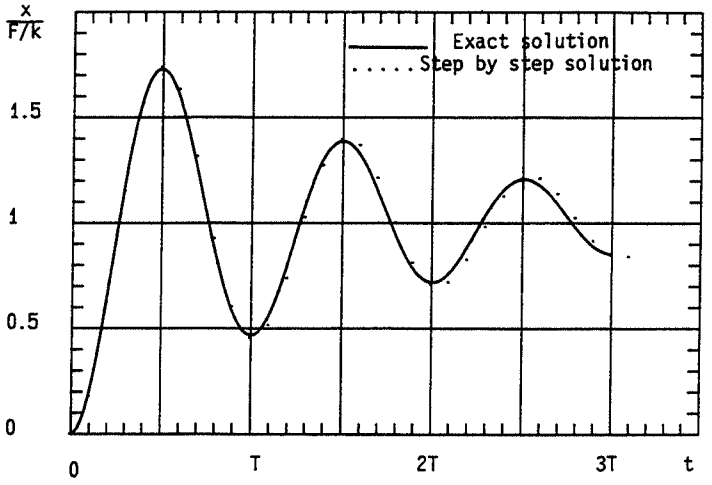


Figure 6

The resultant shown in Figure 6 for step-by-step solution has been calculated with the aid of program no. 8 of Chapter 7 with $\theta = 1$. As the size of the time step Δt is reduced, the step-by-step solution will approach the exact solution.

Exercise 15: *Consider a system of eight degrees of freedom consisting of a chain of identical springs and masses; see Figure 7. It can be shown that the*

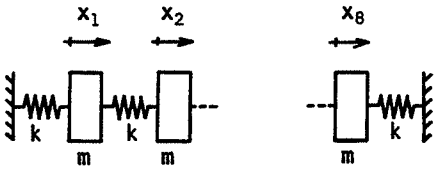


Figure 7

frequencies of this system are $\omega_i = 2\sqrt{k/m}\cos(9-i)\pi/18$, $i = 1, 2, \ldots, 8$.

1. *Calculate the modes* ϕ_1 *and* ϕ_2 *corresponding to* ω_1 *and* ω_2.
2. *The first two springs, beginning from the left, are modified so as to have stiffnesses* $1.5k$. *Calculate the two lowest frequencies* ω_1^* *and* ω_2^* *and their corresponding modes* ϕ_1^* *and* ϕ_2^*. *Use program no. 5.*
3. *Recalculate* ω_1^*, ω_2^*; ϕ_1^*, ϕ_2^* *by the Rayleigh–Ritz method using as displacement patterns the first two modes of the unmodified system. Designate these estimates as* ω_1^{**}, ω_2^{**}; ϕ_1^{**}, ϕ_2^{**}.

For part 1:

$$\omega_1 = 0.3473\sqrt{\frac{k}{m}};$$

$$\phi_1 = [1, 1.879, 2.532, 2.879, 2.879, 2.532, 1.879, 1]^t$$

$$\omega_2 = 0.6840\sqrt{\frac{k}{m}};$$

$$\phi_2 = [1, 1.532, 1.347, 0.5321, -0.5321, -1.347, -1.532, -1]^t$$

For part 2:

$$\omega_1^* = 0.3728\sqrt{\frac{k}{m}};$$

$$\phi_1^* = [1, 1.907, 3.003, 3.682, 3.849, 3.482, 2.630, 1.413]^t$$

$$\omega_2^* = 0.7219\sqrt{\frac{k}{m}};$$

$$\phi_2^* = [1, 1.653, 1.770, 0.9651, -0.3429, -1.472, -1.834, -1.240]^t$$

For part 3:
For the modified system, one has

$$[\phi_1, \phi_2]^t[M^*][\phi_1, \phi_2] = m\begin{bmatrix} 38.46 & 0 \\ 0 & 10.89 \end{bmatrix}$$

$$[\phi_1, \phi_2]^t[K^*][\phi_1, \phi_2] = k\begin{bmatrix} 5.525 & 0.7338 \\ 0.7338 & 5.737 \end{bmatrix}$$

from which ω_1^{**} and ω_2^{**} and the associated modes of this two degree-of-freedom system can be deduced.

$$\omega_1^{**} = 0.3746\sqrt{\frac{k}{m}} \quad \text{with} \quad \begin{bmatrix} 1 \\ -0.1744 \end{bmatrix};$$

$$\omega_2^{**} = 0.7281\sqrt{\frac{k}{m}} \quad \text{with} \quad \begin{bmatrix} 1 \\ 20.26 \end{bmatrix}$$

Then the modes of the modified system are:

$$\phi_1^{**} = [1, 1.952, 2.782, 3.375, 3.599, 3.351, 2.599, 1.422]^t$$

and:

$$\phi_2^{**} = [1, 1.548, 1.403, 0.6425, -0.3716, -1.165, -1.372, -0.9060]^t$$

Exercise 16: *A rotating shaft subjected to torsion is modeled by three disks of inertia I, I, I/2 and three rods of identical torsional stiffness k; see Figure 8.*

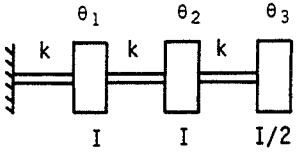


Figure 8

The shaft rotates with a constant angular velocity Ω *and* θ_1, θ_2, θ_3 *are measured with respect to the instantaneous shaft position. Write the kinetic and strain energies of this system in terms of* θ_1, θ_2, θ_3. *Calculate the frequencies and modes.*

The kinetic energy associated with shaft rotation is a constant. As such, it makes no contribution to the equations of motion and will, therefore, be ignored in what follows. The kinetic and strain energies are

$$2T = I\theta_1^{\circ 2} + I\theta_2^{\circ 2} + \tfrac{1}{2}I\theta_3^{\circ 2}$$

$$2U = k\theta_1^2 + k(\theta_2 - \theta_1)^2 + k(\theta_3 - \theta_2)^2$$

The equations of motion are

$$I\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}\theta_1^{\infty}\\ \theta_2^{\infty}\\ \theta_3^{\infty}\end{bmatrix} + k\begin{bmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix}\begin{bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{bmatrix} = 0$$

from which

$$\omega_1 = 0.5176\sqrt{\frac{k}{I}}; \qquad \phi_1 = \begin{bmatrix}1\\ 1.732\\ 2\end{bmatrix}$$

$$\omega_2 = 1.414\sqrt{\frac{k}{I}}; \qquad \phi_2 = \begin{bmatrix}1\\ 0\\ -1\end{bmatrix}$$

$$\omega_3 = 1.932\sqrt{\frac{k}{I}}; \qquad \phi_3 = \begin{bmatrix}1\\ -1.732\\ 2\end{bmatrix}$$

Exercise 17: *Two torsion shafts are connected by a coupling of ratio n,* $\theta_3 = -n\theta_2$; *see Figure 9. The shafts rotate at constant speed. The model is of*

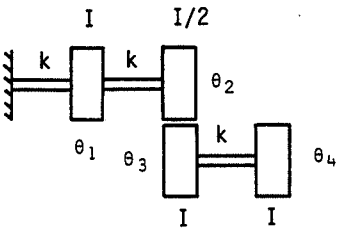


Figure 9

the same kind as used in exercise 16. Write the equations of motion and calculate the frequencies and modes for $n = 2$.

The kinetic and strain energies are

$$\begin{aligned} 2T &= I\theta_1^{\circ 2} + \tfrac{1}{2}I\theta_2^{\circ 2} + I\theta_3^{\circ 2} + I\theta_4^{\circ 2} \\ &= I\theta_1^{\circ 2} + I(\tfrac{1}{2} + n^2)\theta_2^{\circ 2} + I\theta_4^{\circ 2} \\ 2U &= k\theta_1^2 + k(\theta_2 - \theta_1)^2 + k(\theta_4 - \theta_3)^2 \\ &= k\theta_1^2 + k(\theta_2 - \theta_1)^2 + k(\theta_4 + n\theta_2)^2 \end{aligned}$$

The resulting equations of motion are

$$I\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2}+n^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \theta_1^{\circ\circ} \\ \theta_2^{\circ\circ} \\ \theta_4^{\circ\circ} \end{bmatrix} + k\begin{bmatrix} 2 & -1 & 0 \\ -1 & 1+n^2 & n \\ 0 & n & 1 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_4 \end{bmatrix} = 0$$

and if $n = 2$:

$$\omega_1 = 0.2358\sqrt{\frac{k}{I}}; \qquad \phi_1 = \begin{bmatrix} 1 \\ 1.944 \\ -4.118 \end{bmatrix}$$

$$\omega_2 = 1.299\sqrt{\frac{k}{I}}; \qquad \phi_2 = \begin{bmatrix} 1 \\ 0.3114 \\ 0.9046 \end{bmatrix}$$

$$\omega_3 = 1.538\sqrt{\frac{k}{I}}; \qquad \phi_3 = \begin{bmatrix} 1 \\ -0.3670 \\ -0.5369 \end{bmatrix}$$

Exercise 18: *The system in exercise 17 is subjected to a couple* $\Gamma \sin \Omega t$ *applied to disk 1. In order to suppress the resonance at* ω_2*, a torsional vibration absorber of inertia* $I/2$ *and stiffness* $0.844k$ *is attached to disk 4; see Figure 10.*

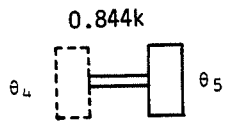


Figure 10

Plot $|\Theta_4|/(\Gamma/k)$ *with and without the absorber as a function of* $\Omega/\sqrt{k/I}$*. Use program no. 6.*

The response plot is shown in Figure 11.

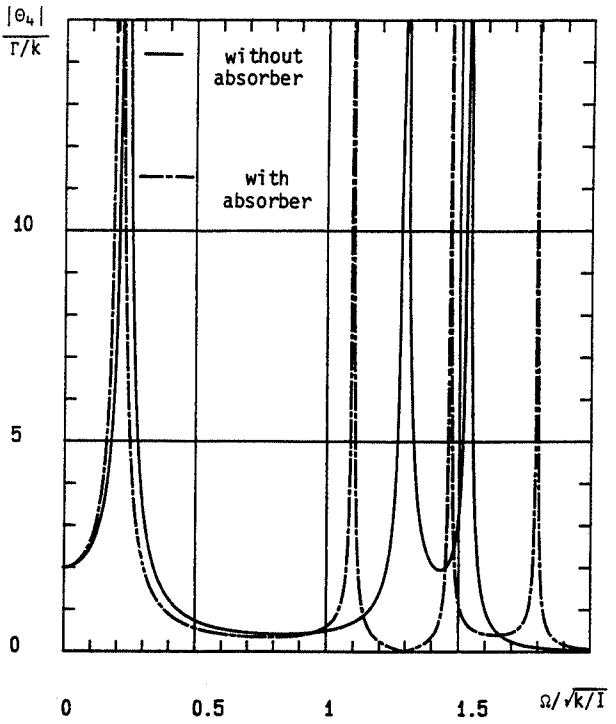


Figure 11

Exercise 19: *Figure* 12 *shows a five degree-of-freedom system which has a chain of identical masses m coupled by identical springs k and dampers*

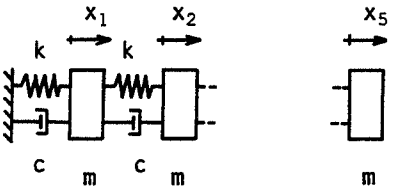


Figure 12

$c = 10^{-2}\sqrt{km}$. *Mass* 3 *is subjected to a forcing function* $F\cos\Omega t$. *Plot* $|X_5|/(F/k)$ *versus* $\Omega/\sqrt{k/m}$. *Use the modal method together with program no. 7. First, obtain the plot by using only the first two modes and then by using the first four modes.*

The response plot is shown in Figure 13.

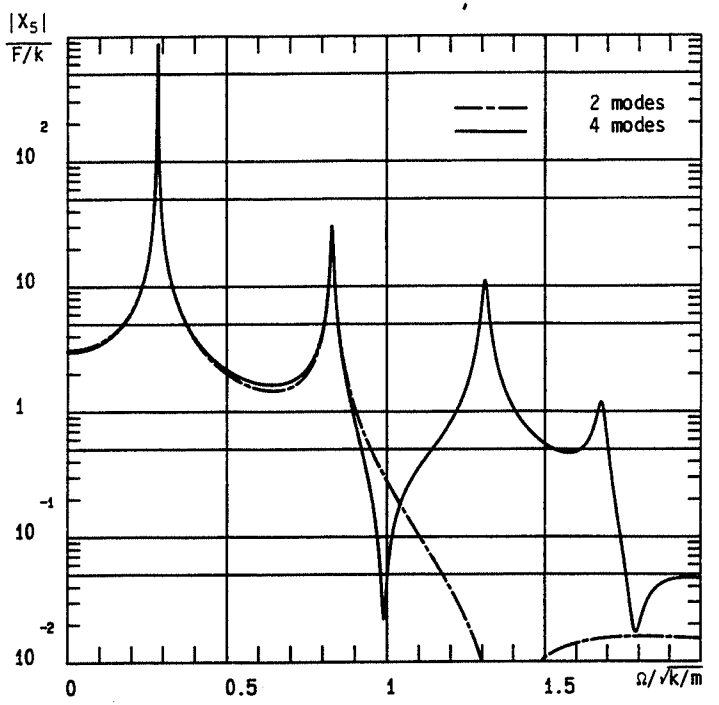


Figure 13

4
Continuous Systems

Many practical structures are built up using simple elements like beams, plates, and shells which have continuous distributions of mass and elasticity. Except possibly for the important case of straight beams of constant cross-section, analytical solutions for the dynamic behavior of these basic structural elements are limited to simple geometries and boundary conditions.

In cases where analytical approaches are laborious, it is better to go directly to numerical methods. In this chapter, the Rayleigh–Ritz method is usually chosen; in the following chapter, the finite element method is utilized. Also in this chapter, damping is not taken into account and only steady-state response is considered.

The contents of the chapter are as follows:

4.1 Equations of Motion for Bars, Rods, and Beams

In the following sections, the classical equations for straight members with continuous distributions of mass and elasticity are derived for longitudinal, torsional, and bending motion. Continuous systems are the next logical step in this text, since they may be considered as N degree-of-freedom systems with $N \rightarrow \infty$.

4.1.1 Longitudinal motion of a bar

The motion is defined by:

u, axial displacement

P, axial force acting on the cross-section

P_{ex}, external axial force per unit length
S, cross-sectional area
E, Young's modulus
ρ, mass density.

The application of Newton's laws along the longitudinal axis of the element

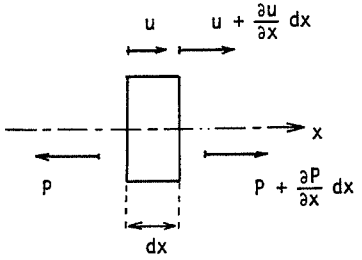


Figure 1 Element of a bar in longitudinal motion

in Figure 1 gives

$$\rho S\,\mathrm{d}x\frac{\partial^2 u}{\partial t^2}=P+\frac{\partial P}{\partial x}\mathrm{d}x-P+P_{ex}\,\mathrm{d}x \tag{1}$$

which becomes

$$\rho S\frac{\partial^2 u}{\partial t^2}=\frac{\partial P}{\partial x}+P_{ex} \tag{2}$$

The relation between force and displacement is

$$\frac{P}{S}=E\frac{\partial u}{\partial x} \tag{3}$$

Combining (2) and (3) gives the classical partial differential equation of a bar in longitudinal motion:

$$\rho S\frac{\partial^2 u}{\partial t^2}=\frac{\partial}{\partial x}\left(ES\frac{\partial u}{\partial x}\right)+P_{ex} \tag{4}$$

which for a constant cross-section becomes

$$\rho S\frac{\partial^2 u}{\partial t^2}=ES\frac{\partial^2 u}{\partial x^2}+P_{ex} \tag{5}$$

4.1.2 Torsional motion of a rod

The motion is defined by:
θ, angle of twist
T, torsional couple

T_{ex}, external torsional couple per unit length
I_θ, mass moment of inertia about longitudinal axis, per unit length
J, area moment of inertia about the longitudinal axis
G, shear modulus of the rod material.

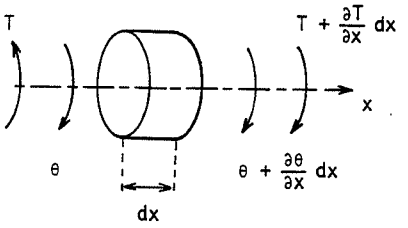


Figure 2 Element of a rod in torsion

The application of Newton's laws about the x direction of the element in Figure 2 gives

$$I_\theta \, dx \frac{\partial^2 \theta}{\partial t^2} = T + \frac{\partial T}{\partial x} dx - T + T_{ex} \, dx \tag{6}$$

which becomes

$$I_\theta \frac{\partial^2 \theta}{\partial t^2} = \frac{\partial T}{\partial x} + T_{ex} \tag{7}$$

The relation between torsional couple and the angle of twist is

$$T = GJ \frac{\partial \theta}{\partial x} \tag{8}$$

Combining (7) and (8) gives the classical partial differential equation of a rod in torsion:

$$I_\theta \frac{\partial^2 \theta}{\partial t^2} = \frac{\partial}{\partial x}\left(GJ \frac{\partial \theta}{\partial x}\right) + T_{ex} \tag{9}$$

which for a constant cross-section becomes

$$I_\theta \frac{\partial^2 \theta}{\partial t^2} = GJ \frac{\partial^2 \theta}{\partial x^2} + T_{ex} \tag{10}$$

Note that equations (5) and (10) are mathematically the same. It can be shown that $I_\theta = J\rho$ is valid only for rods of circular cross-section. Using this relationship for other cross-sections, e.g. rectangular, can lead to serious error.

4.1.3 Bending motion of a beam

The motion is defined by:

v, lateral deflection
ψ, slope of neutral axis
Q, lateral shear force
M, bending moment
Q_{ex}, lateral external force per unit length
I, area moment of inertia of beam cross-section about the neutral axis
a, shear factor which appears in the expression for lateral deflection due to shear; it is of the order of unity for beam cross-sections usually found in practice.

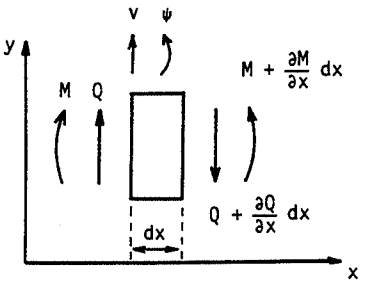


Figure 3 Element of a beam in bending

The application of Newton's laws to the element in Figure 3 in the lateral y-direction and about the z-direction gives

$$\rho S\,\mathrm{d}x\frac{\partial^2 v}{\partial t^2} = Q - Q - \frac{\partial Q}{\partial x}\mathrm{d}x + Q_{ex}\,\mathrm{d}x \tag{11}$$

$$\rho I\,\mathrm{d}x\frac{\partial^2 \psi}{\partial t^2} = -M + M + \frac{\partial M}{\partial x}\mathrm{d}x - Q\,\mathrm{d}x \tag{12}$$

which simplify to

$$\rho S\frac{\partial^2 v}{\partial t^2} = -\frac{\partial Q}{\partial x} + Q_{ex} \tag{13}$$

$$\rho I\frac{\partial^2 \psi}{\partial t^2} = \frac{\partial M}{\partial x} - Q \tag{14}$$

From the theory of strength of materials, the relations among Q, M, v, ψ are

$$\frac{\partial \psi}{\partial x} = \frac{M}{EI} \tag{15}$$

$$\psi - \frac{Q}{aSG} = \frac{\partial v}{\partial x} \tag{16}$$

The terms $\rho I\,\partial^2\psi/\partial t^2$ and Q/aSG are usually second-order effects. The first is the rotatory inertia introduced by Rayleigh and the second is the shear deformation introduced by Timoshenko. In so-called Timoshenko beams these effects are considered; in Bernoulli–Euler beams they are not.

In the following, we shall ignore these secondary effects and obtain the equations of motion for the Bernoulli–Euler beam. Equations (13), (14), (15), (16) become

$$\begin{aligned} \rho S\frac{\partial^2 v}{\partial t^2} &= -\frac{\partial Q}{\partial x}+Q_{ex} \\ \frac{\partial M}{\partial x}-Q &= 0 \\ \frac{\partial \psi}{\partial x} &= \frac{M}{EI} \\ \psi &= \frac{\partial v}{\partial x} \end{aligned} \tag{17}$$

Eliminating Q, M, ψ among these four equations gives the classical partial differential equation of a beam in bending:

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 v}{\partial x^2}\right)+\rho S\frac{\partial^2 v}{\partial t^2}-Q_{ex}=0 \tag{18}$$

which for a constant cross section becomes

$$EI\frac{\partial^4 v}{\partial x^4}+\rho S\frac{\partial^2 v}{\partial t^2}-Q_{ex}=0 \tag{19}$$

Note that (19) is mathematically different from both (5) and (10).

4.2 Frequencies, Modes, and Orthogonality

Consider the case for which equations (5) and (19) are homogeneous, i.e. $P_{ex}=0$, $Q_{ex}=0$. The free-vibration solutions to these equations will be obtained by the method of separation of variables. Since the differential equation of motion for torsion is mathematically identical to that for longitudinal motion, the case of torsion will not be treated.

4.2.1 Longitudinal motion

Let

$$u(x,t)=\phi(x)f(t) \tag{20}$$

and substitute into (5). This gives

$$\rho S\phi(x)\frac{d^2 f(t)}{dt^2}-ESf(t)\frac{d^2\phi(x)}{dx^2}=0 \tag{21}$$

which can be separated into two ordinary differential equations of motion, one in space and one in time:

$$\frac{E}{\rho}\frac{1}{\phi(x)}\frac{d^2\phi(x)}{dx^2}=\frac{1}{f(t)}\frac{d^2f(t)}{dt^2}=\text{const.}=-\omega^2 \tag{22}$$

The separation constant has been set equal to $-\omega^2$ so that the solutions will be bounded in time. It follows that

$$\frac{d^2f(t)}{dt^2}+\omega^2 f(t)=0 \tag{23}$$

$$\frac{d^2\phi(x)}{dt^2}+\omega^2\frac{\rho}{E}\phi(x)=0 \tag{24}$$

These have the solutions

$$f(t)=A\sin\omega t+B\cos\omega t \tag{25}$$

$$\phi(x)=C\sin\omega\sqrt{\frac{\rho}{E}}x+D\cos\omega\sqrt{\frac{\rho}{E}}x \tag{26}$$

hence for each value of ω, one has a solution of the form

$$u(x,t)=(A\sin\omega t+B\cos\omega t)\left(C\sin\omega\sqrt{\frac{\rho}{E}}x+D\cos\omega\sqrt{\frac{\rho}{E}}x\right) \tag{27}$$

The frequencies ω are obtained by application of the boundary conditions. For example, in the case of the clamped–free (C–F) bar shown in

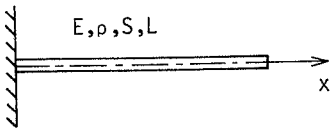


Figure 4 Clamped–free bar in longitudinal motion

Figure 4, one must have for all instants of time

$$\phi(0)=0$$

and

$$P(L,t)=ES\,f(t)\left.\frac{d\phi(x)}{dx}\right|_{(x=L)}=0 \tag{28}$$

The first condition requires that $D=0$ and the second that

$$C\cos\omega\sqrt{\frac{\rho}{E}}L=0 \tag{29}$$

After discarding solutions which are identically zero, this requires that

$$\cos \omega \sqrt{\frac{\rho}{E}} L = 0 \tag{30}$$

and therefore the frequencies are given by

$$\omega_n = (2n-1)\frac{\pi}{2L}\sqrt{\frac{E}{\rho}} \qquad n = 1, 2, \ldots . \tag{31}$$

To within an arbitrary constant, the modes associated with ω_n are then

$$\phi_n(x) = \sin (2n-1)\frac{\pi x}{2L} \tag{32}$$

and the general solution for free vibration has the form

$$u(x, t) = \sum_{n=1}^{\infty} (A_n \sin \omega_n t + B_n \cos \omega_n t) \sin (2n-1)\frac{\pi x}{2L} \tag{33}$$

The constants A_n and B_n are determined by the initial conditions; see exercise 12.

In other cases, such as clamped–clamped bars (C–C) or free–free bars (F–F), the calculation proceeds in the same fashion. All the above results are summarized in Table 1.

Table 1

C–F	$\omega_n = \frac{(2n-1)\pi}{2L}\sqrt{\frac{E}{\rho}}$	$\phi_n(x) = \sin (2n-1)\frac{\pi x}{2L}$	with	$n = 1, 2, \ldots$
C–C	$\omega_n = \frac{n\pi}{L}\sqrt{\frac{E}{\rho}}$	$\phi_n(x) = \sin n\frac{\pi x}{L}$	with	$n = 1, 2, \ldots$
F–F	$\omega_n = \frac{n\pi}{L}\sqrt{\frac{E}{\rho}}$	$\phi_n(x) = \cos n\frac{\pi x}{L}$	with	$n = 0, 1, 2, \ldots$

Orthogonality relations

Using the preceeding results, equation (4) for free vibration can be written

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(ES\frac{\mathrm{d}\phi}{\mathrm{d}x}\right) = -\rho S\omega^2\phi \tag{34}$$

where, for simplicity, ϕ is written for $\phi(x)$. Equation (34) is true for each of the solution pairs: ω_i, ϕ_i; ω_j, ϕ_j. Hence,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(ES\frac{\mathrm{d}\phi_i}{\mathrm{d}x}\right) = -\rho S\omega_i^2\phi_i \tag{35}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(ES\frac{\mathrm{d}\phi_j}{\mathrm{d}x}\right) = -\rho S\omega_j^2\phi_j \tag{36}$$

If the coordinates of the ends of the bar are 0 and L then after multiplying (35) by ϕ_j and (36) by ϕ_i, one obtains

$$\int_0^L \phi_j \frac{d}{dx}\left(ES\frac{d\phi_i}{dx}\right)dx = -\omega_i^2 \int_0^L \rho S \phi_i \phi_j \, dx \tag{37}$$

$$\int_0^L \phi_i \frac{d}{dx}\left(ES\frac{d\phi_j}{dx}\right)dx = -\omega_j^2 \int_0^L \rho S \phi_i \phi_j \, dx \tag{38}$$

Integrating (37) and (38) by parts and assuming free ($ES\, d\phi/dx = 0$) or clamped ($\phi = 0$) boundary conditions yields the results:

$$-\int_0^L ES\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}dx = -\omega_i^2 \int_0^L \rho S \phi_i \phi_j \, dx \tag{39}$$

$$-\int_0^L ES\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}dx = -\omega_j^2 \int_0^L \rho S \phi_i \phi_j \, dx \tag{40}$$

Subtracting (40) from (39) gives

$$(\omega_j^2 - \omega_i^2)\int_0^L \rho S \phi_i \phi_j \, dx = 0 \tag{41}$$

and since $\omega_i \neq \omega_j$

$$\int_0^L \rho S \phi_i \phi_j \, dx = 0 \tag{42}$$

and from (39)

$$\int_0^L ES\frac{d\phi_i}{dx}\frac{d\phi_j}{dx}dx = 0 \tag{43}$$

Equations (42) and (43) are the orthogonality conditions for a continuous system deforming as a bar. Note that (42) and (43) are generalizations of equations (25) and (26) in Chapter 3.

Also note that multiplying both sides of (35) by ϕ_i and integrating from 0 to L gives

$$\int_0^L \phi_i \frac{d}{dx}\left(ES\frac{d\phi_i}{dx}\right)dx = -\omega_i^2 \int_0^L \rho S \phi_i^2 \, dx \tag{44}$$

which, on integration by parts and use of the boundary conditions, becomes

$$\omega_i^2 = \frac{\displaystyle\int_0^L ES\left(\frac{d\phi_i}{dx}\right)^2 dx}{\displaystyle\int_0^L \rho S \phi_i^2 \, dx} = \frac{k_i}{m_i} \tag{45}$$

with

$$k_i = \int_0^L ES\left(\frac{d\phi_i}{dx}\right)^2 dx \tag{46}$$

$$m_i = \int_0^L \rho S \phi_i^2 \, dx \tag{47}$$

where k_i and m_i are the ith modal stiffness and modal mass of this continuous system.

4.2.2 Bending motion

Let

$$v(x, t) = \phi(x) f(t) \tag{48}$$

and substitute in (19). This gives

$$EI\frac{d^4\phi(x)}{dx^4} f(t) + \rho S \phi(x) \frac{d^2 f(t)}{dt^2} = 0 \tag{49}$$

which can be separated into

$$\frac{EI}{\rho S}\frac{1}{\phi(x)}\frac{d^4\phi(x)}{dx^4} = -\frac{1}{f(t)}\frac{d^2 f(t)}{dt^2} = \text{const.} = \omega^2 \tag{50}$$

where, in this case, the separation constant must be set equal to $+\omega^2$. It follows that

$$\frac{d^2 f(t)}{dt^2} + \omega^2 f(t) = 0 \tag{51}$$

$$\frac{d^4\phi(x)}{dx^4} - \frac{\rho S}{EI}\omega^2 \phi(x) = 0 \tag{52}$$

The solution to (51) is

$$f(t) = A \sin \omega t + B \cos \omega t \tag{53}$$

Solutions to (52) are sought in the form e^{rx} resulting in the characteristic equations:

$$r^4 - \frac{\rho S \omega^2}{EI} = 0 \tag{54}$$

which has the roots

$$r = \beta, -\beta, j\beta, -j\beta$$

with

$$\beta = \sqrt[4]{\frac{\rho S \omega^2}{EI}} \tag{55}$$

Hence for each value of β, one has a solution of the form

$$\phi(x) = C \sin \beta x + D \cos \beta x + E \sinh \beta x + F \cosh \beta x \tag{56}$$

The frequencies ω associated with each β are determined by application of the boundary conditions. The most frequent boundary conditions for beams are

$$\begin{aligned} \text{Free (F):} \quad & M = 0, \quad Q = 0 \\ \text{Clamped (C):} \quad & v = 0, \quad \psi = 0 \\ \text{Simply-supported (S):} \quad & v = 0, \quad M = 0 \end{aligned} \tag{57}$$

In the case of a beam which is clamped at $x = 0$ and free at $x = L$, equations (17), (56), and (57) give

$$\begin{aligned} D + F &= 0 \\ C + E &= 0 \\ -C \sin \beta L - D \cos \beta L + E \sinh \beta L + F \cosh \beta L &= 0 \\ -C \cos \beta L + D \sin \beta L + E \cosh \beta L + F \sinh \beta L &= 0 \end{aligned} \tag{58}$$

After discarding solutions which are identically zero, these equations require that the determinant of their coefficients must vanish. Completing this calculation gives

$$1 + \cos \beta L \cosh \beta L = 0 \tag{59}$$

The solutions to this equation have the form

$$\beta_n L = X_n \tag{60}$$

Accordingly, the frequencies become

$$\omega_n = \frac{X_n^2}{L^2} \sqrt{\frac{EI}{\rho S}} \tag{61}$$

The five lowest values of X_n^2 are given in Table 2 for the most common boundary conditions. Except for the case of X_1^2 for the C–F beam, all these values of X_n^2 can be easily obtained; see exercises 7 and 8. In addition, there are two rigid-body modes for the F–F beam and one rigid-body mode for the S-F beam; these zero frequencies have not been included in Table 2.

Table 2

		X_1^2	X_2^2	X_3^2	X_4^2	X_5^2
C–F	$1 + \cosh X \cos X = 0$	3.516	22.03	61.69	120.9	199.8
S–S	$\sin X = 0$	9.869	39.47	88.82	157.9	246.7
C–C F–F	$1 - \cosh X \cos X = 0$	22.37	61.67	120.9	199.8	298.5
C–S F–S	$\tan X = \tanh X$	15.41	49.96	104.2	178.2	272.0

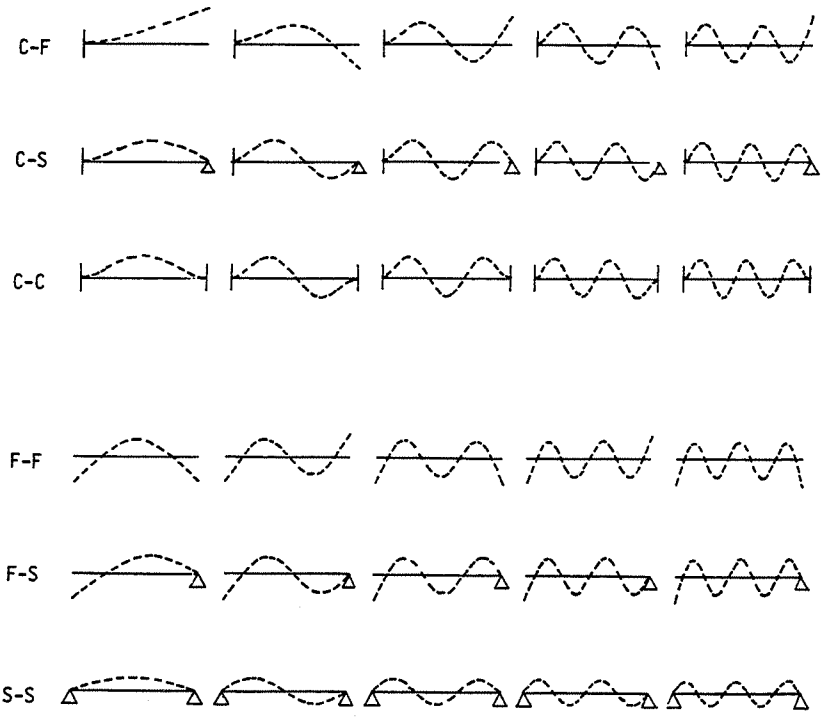


Figure 5 Five lowest bending modes of beams with various boundary conditions

The relations (56) together with (58) and (61) allow the determination of the modes. Their shapes are depicted in Figure 5.

Orthogonality relations

Starting with (18) for free vibration and again using ϕ for $\phi(x)$, one finds

$$\frac{d^2}{dx^2}\left(EI\frac{d^2\phi}{dx^2}\right)=\rho S\omega^2\phi \tag{62}$$

Equation (62) is true for each of the solutions pairs: ω_i, ϕ_i; ω_j, ϕ_j. Hence, one has

$$\frac{d^2}{dx^2}\left(EI\frac{d^2\phi_i}{dx^2}\right)=\rho S\omega_i^2\phi_i \tag{63}$$

$$\frac{d^2}{dx^2}\left(EI\frac{d^2\phi_j}{dx^2}\right)=\rho S\omega_j^2\phi_j \tag{64}$$

Using the same procedure as was used for longitudinal motion and integrating twice by parts, one finds for each of the three boundary conditions F, C, S:

$$\int_0^L EI\frac{d^2\phi_i}{dx^2}\frac{d^2\phi_j}{dx^2}\,dx = \omega_i^2\int_0^L \rho S\phi_i\phi_j\,dx \tag{65}$$

$$\int_0^L EI\frac{d^2\phi_i}{dx^2}\frac{d^2\phi_j}{dx^2}\,dx = \omega_j^2\int_0^L \rho S\phi_i\phi_j\,dx \tag{66}$$

As $\omega_i \neq \omega_j$, subtracting (65) and (66) yields the orthogonality relation

$$\int_0^L \rho S\phi_i\phi_j\,dx = 0 \tag{67}$$

and from (65),

$$\int_0^L EI\frac{d^2\phi_i}{dx^2}\frac{d^2\phi_j}{dx^2}\,dx = 0 \tag{68}$$

In the same manner as before,

$$\omega_i^2 = \frac{k_i}{m_i} \tag{69}$$

with

$$k_i = \int_0^L EI\left(\frac{d^2\phi_i}{dx^2}\right)^2 dx \tag{70}$$

$$m_i = \int_0^L \rho S\phi_i^2\,dx \tag{71}$$

where k_i and m_i are the ith modal stiffness and modal mass.

4.3 Approximate Methods

This section will show how the Rayleigh–Ritz method can be used to obtain estimates of frequencies and modes for continuous systems. First, the kinetic and strain energies are calculated based on a reasonable hypothesis for the system displacement. In making this hypothesis it is essential that the assumed displacement pattern satisfy the geometric boundary conditions. Then the equations of motion are deduced by application of Lagrange's equations. Finally, the frequencies and modes are obtained by using the methods presented in Chapter 3. When a single parameter function is used as a displacement hypothesis, as in Chapter 1, this procedure is Rayleigh's method.

The expressions for the kinetic energy T and the strain energy U for straight member of length L are given by the following:

1. Longitudinal motion of a bar

$$T=\frac{1}{2}\int_0^L \rho S\left(\frac{\partial u(x,t)}{\partial t}\right)^2 dx \tag{72}$$

$$U=\frac{1}{2}\int_0^L ES\left(\frac{\partial u(x,t)}{\partial x}\right)^2 dx \tag{73}$$

2. Torsional motion of a rod

$$T=\frac{1}{2}\int_0^L I_\theta\left(\frac{\partial \theta(x,t)}{\partial t}\right)^2 dx \tag{74}$$

$$U=\frac{1}{2}\int_0^L GJ\left(\frac{\partial \theta(x,t)}{\partial x}\right)^2 dx \tag{75}$$

3. Bending motion of a beam

$$T=\frac{1}{2}\int_0^L \rho S\left(\frac{\partial v(x,t)}{\partial t}\right)^2 dx \tag{76}$$

$$U=\frac{1}{2}\int_0^L EI\left(\frac{\partial^2 v(x,t)}{\partial x^2}\right)^2 dx \tag{77}$$

4.3.1 Rayleigh's method

Consider for example a clamped–free beam and assume that its lateral deflection is given by

$$v(x,t)=\left[3\left(\frac{x}{L}\right)^2-\left(\frac{x}{L}\right)^3\right]p \tag{78}$$

This expression satisfies the geometric boundary conditions $v(0,t)=0$ and $\partial v(0,t)/\partial x=\psi(0,t)=0$, or equivalently $\phi(0)=0$ and $\partial\phi(0)/\partial x=0$. It also satisfies the moment boundary condition $M(L,t)=0$, or equivalently $\partial^2\phi(L)/\partial x^2=0$ The latter condition is useful but not essential. Substituting (78) in (76) and (77) gives

$$\begin{aligned} T&=\frac{1}{2}\int_0^L \rho S\left[3\left(\frac{x}{L}\right)^2-\left(\frac{x}{L}\right)^3\right]^2 dx\dot{p}^2 \\ &=0.4714\rho SL\dot{p}^2 \end{aligned} \tag{79}$$

$$\begin{aligned} U&=\frac{1}{2}\int_0^L EI\left[\frac{6}{L^2}-\frac{6x}{L^3}\right]^2 dx\, p^2 \\ &=6\frac{EI}{L^3}p^2 \end{aligned} \tag{80}$$

Hence the equation of motion is

$$0.9428\rho SLp^{\infty} + 12\frac{EI}{L^3}p = 0 \tag{81}$$

and the frequency is

$$\omega_1 = \frac{3.567}{L^2}\sqrt{\frac{EI}{\rho S}} \tag{82}$$

This result is 2% greater than the exact value.

4.3.2 The Rayleigh–Ritz method

Consider for example the longitudinal vibration of a bar which is clamped at $x = 0$ and free at $x = L$. It is required to estimate the first two frequencies and their corresponding modes. A four-parameter displacement hypothesis which satisfies the geometric boundary conditions is utilized; namely,

$$u(x, t) = \frac{x}{L}p_1 + \left(\frac{x}{L}\right)^2 p_2 + \left(\frac{x}{L}\right)^3 p_3 + \left(\frac{x}{L}\right)^4 p_4 \tag{83}$$

This is substituted into (72) and (73) and then Lagrange's equations are used to give the equations of motion:

$$\rho SL\begin{bmatrix} \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & \frac{1}{6} \\ \frac{1}{4} & \frac{1}{5} & \frac{1}{6} & \frac{1}{7} \\ \frac{1}{5} & \frac{1}{6} & \frac{1}{7} & \frac{1}{8} \\ \frac{1}{6} & \frac{1}{7} & \frac{1}{8} & \frac{1}{9} \end{bmatrix}\begin{bmatrix} p_1^{\infty} \\ p_2^{\infty} \\ p_3^{\infty} \\ p_4^{\infty} \end{bmatrix} + \frac{ES}{L}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \frac{4}{3} & \frac{3}{2} & \frac{8}{5} \\ 1 & \frac{3}{2} & \frac{9}{5} & 2 \\ 1 & \frac{8}{5} & 2 & \frac{16}{7} \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} = 0 \tag{84}$$

The frequencies and modes are obtained from (84):

$$\omega_1 = \frac{1.571}{L}\sqrt{\frac{E}{\rho}} \quad \text{with} \quad \phi_1 = \frac{x}{L} + 0.02782\left(\frac{x}{L}\right)^2 - 0.5001\left(\frac{x}{L}\right)^3 + 0.1106\left(\frac{x}{L}\right)^4 \tag{85}$$

$$\omega_2 = \frac{4.724}{L}\sqrt{\frac{E}{\rho}} \quad \text{with} \quad \phi_2 = \frac{x}{L} - 0.6898\left(\frac{x}{L}\right)^2 - 2.561\left(\frac{x}{L}\right)^3 + 2.056\left(\frac{x}{L}\right)^4 \tag{86}$$

The frequencies are accurate to within less than 1% and the modes are good estimates of the exact modes.

4.4 Response to excitation

In this section, the Rayleigh–Ritz method is used to determine the steady-state response of a continuous system to an excitating force. If, in addition,

the exact modes of free vibration of the system are known, the method can be extended to permit a modal calculation of the response.

The kinetic energy and strain energy are calculated as before. The generalized forces are deduced from the expression for virtual work of external forces.

4.4.1 Rayleigh–Ritz method

Consider, as in section 4.3, the longitudinal motion of a clampled–free bar. In the present case the bar is subjected to a longitudinal force $F(t)$ at its midpoint. From (83) the displacement at the midpoint is

$$u\left(\frac{L}{2},t\right)=\tfrac{1}{2}p_1+\tfrac{1}{4}p_2+\tfrac{1}{8}p_3+\tfrac{1}{16}p_4 \tag{87}$$

and the virtual work of external forces is therefore :

$$\delta W=F(t)\left(\frac{\delta p_1}{2}+\frac{\delta p_2}{4}+\frac{\delta p_3}{8}+\frac{\delta p_4}{16}\right) \tag{88}$$

The right-hand side of (88) gives the generalized force vector

$$[\tfrac{1}{2}F(t),\tfrac{1}{4}F(t),\tfrac{1}{8}F(t),\tfrac{1}{16}F(t)]^{\mathrm{t}} \tag{89}$$

The system response can now be obtained by the methods described in Chapter 3.

4.4.2 Modal method

A bar which is clamped at each end is subjected to a longitudinal force $F(t)=F\sin\Omega t$ at its midpoint. The exact modes are used, then:

$$u(x,t)=\sin\frac{\pi x}{L}p_1+\sin\frac{2\pi x}{L}p_2+\sin\frac{3\pi x}{L}p_3+\ldots \tag{90}$$

Substituting (90) into (72) and (73) and using the orthogonality relations (42) and (43), gives

$$T=\tfrac{1}{2}\sum_{n=1}^{\infty}\int_0^L \rho S\sin^2\frac{n\pi x}{L}\,\mathrm{d}x\,p_n{}^{\circ 2}=\frac{\rho SL}{4}\sum_{n=1}^{\infty}p_n{}^{\circ 2} \tag{91}$$

$$U=\tfrac{1}{2}\sum_{n=1}^{\infty}\int_0^L ES\left(\frac{n\pi}{L}\right)^2\cos^2\frac{n\pi x}{L}\,\mathrm{d}x\,p_n^2=\frac{\pi^2}{4}\frac{ES}{L}\sum_{n=1}^{\infty}n^2p_n^2 \tag{92}$$

The virtual work of external force $F(t)$ is

$$\delta W=F\sin\Omega t\left(\sin\frac{\pi}{2}\delta p_1+\sin\pi\,\delta p_2+\sin\frac{3\pi}{2}\delta p_3+\ldots\right) \tag{93}$$

The use of (91), (92), and (93) in Lagrange's equations gives a set of

uncoupled modal equations for the parameters $p_i(t)$; namely:

$$\begin{aligned}
&\frac{\rho SL}{2} p_1^{\infty} + \frac{\pi^2}{2}\frac{ES}{L} p_1 = F \sin \Omega t \\
&\frac{\rho SL}{2} p_2^{\infty} + \frac{4\pi^2}{2}\frac{ES}{L} p_2 = 0 \\
&\frac{\rho SL}{2} p_3^{\infty} + \frac{9\pi^2}{2}\frac{ES}{L} p_3 = -F \sin \Omega t \\
&\vdots
\end{aligned} \tag{94}$$

For i even, the generalized forces are zero, hence the steady-state solutions for $i = 2, 4, \ldots$ are $p_i = 0$. Using the steady-state solutions to (94), for i odd, and substituting the results into (90), gives

$$u(x,t) = \frac{\sin\dfrac{\pi x}{L} F \sin \Omega t}{\dfrac{\pi^2 ES}{2L} - \Omega^2 \dfrac{\rho SL}{2}} - \frac{\sin\dfrac{3\pi x}{L} F \sin \Omega t}{\dfrac{9\pi^2}{2}\dfrac{ES}{L} - \Omega^2 \dfrac{\rho SL}{2}} - - - \tag{95}$$

It is evident that for $\Omega = (\pi/L)\sqrt{(E/\rho)}$, $(3\pi/L)\sqrt{(E/\rho)}, \ldots$, the system is resonant. These resonant frequencies correspond to the frequencies of free vibration identified in Table 1 for a C–C bar.

The longitudinal force in the bar can be obtained by using (95) and (3).

4.5 Kinetic and Strain Energies of Plates

The formulation of the dynamic behavior of a plate in bending can be done in an analogous manner to that used for the beam in bending. However, since a plate is thin in only one dimension while a beam is thin in two dimensions, the plate variables are a function of two space coordinates as well as time. This section will present only a brief introduction to the dynamic theory of plates and will be limited to a consideration of their kinetic and strain energies.

Let the xy plane be the middle plane of the plate so that z is the normal distance from the middle plane to a point in the plate. Also the plate is assumed thin and second order effects, i.e. rotatory inertia and shear deformation, are neglected. The classical strain–displacement relations for a plate are

$$\begin{aligned}
\varepsilon_x &= -z\frac{\partial^2 w}{\partial x^2} \\
\varepsilon_y &= -z\frac{\partial^2 w}{\partial y^2} \\
\gamma_{xy} &= -2z\frac{\partial^2 w}{\partial x\, \partial y}
\end{aligned} \tag{96}$$

The stress–strain relations are

$$\sigma = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = D\varepsilon \tag{97}$$

where ν is Poisson's ratio. The general expression for strain energy is

$$U = \tfrac{1}{2} \int_\tau \varepsilon^t \sigma \, d\tau \tag{98}$$

where τ is the volume. Substituting (96) and (97) into (98) gives

$$U = \int_\tau \frac{E}{2(1-\nu^2)} \begin{bmatrix} z\frac{\partial^2 w}{\partial x^2} \\ z\frac{\partial^2 w}{\partial y^2} \\ 2z\frac{\partial^2 w}{\partial x\, \partial y} \end{bmatrix}^t \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} \begin{bmatrix} z\frac{\partial^2 w}{\partial x^2} \\ z\frac{\partial^2 w}{\partial y^2} \\ 2z\frac{\partial^2 w}{\partial x\, \partial y} \end{bmatrix} d\tau \tag{99}$$

If the thickness of the plate, h, is constant, this becomes

$$U = \frac{D}{2} \int_S \left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2 + 2\nu \left(\frac{\partial^2 w}{\partial x^2}\right)\left(\frac{\partial^2 w}{\partial y^2}\right) + 2(1-\nu)\left(\frac{\partial^2 w}{\partial x\, \partial y}\right)^2 \right] dx\, dy \tag{100}$$

with S being the surface and D being the bending stiffness of the plate:

$$D = \frac{Eh^3}{12(1-\nu^2)} \tag{101}$$

The expression for kinetic energy is

$$T = \frac{1}{2} \int_\tau \rho \left(\frac{\partial w}{\partial t}\right)^2 d\tau \tag{102}$$

which for constant thickness becomes

$$T = \frac{\rho h}{2} \int_S \left(\frac{\partial w}{\partial t}\right)^2 dx\, dy \tag{103}$$

For circular plates, U and T are better expressed in polar coordinates. If the plate thickness is constant, these expressions are

$$U = \frac{D}{2} \int_S \left[\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)^2 - 2(1-\nu)\frac{\partial^2 w}{\partial r^2}\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) + 2(1-\nu)\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial w}{\partial \theta}\right)\right)^2 \right] r\, d\theta\, dr \tag{104}$$

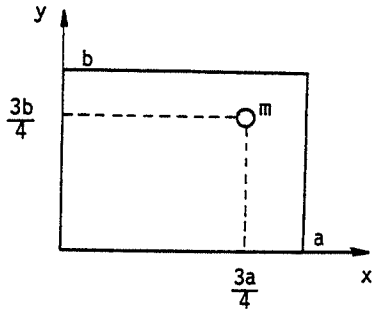


Figure 6 Rectangular plate simply supported along all four edges with a concentrated mass at (3a/4, 3b/4)

concentrated mass of magnitude $m = \rho abh/10$ at the point $x = 3a/4$, $y = 3b/4$; see Figure 6. The displacement hypothesis is

$$w(x, y, t) = \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \sin \frac{r\pi x}{a} \sin \frac{s\pi y}{b} p_{rs} \tag{112}$$

which satisfies the geometric boundary conditions at the plate edges (i.e. $w = 0$) as well as the force boundary conditions at the edge: $\partial^2 w/\partial x^2 = 0$ for $x = 0,\ a$; $\partial^2 w/\partial y^2 = 0$ for $y = 0, b$. The latter conditions are, not essential.

Substituting (112) into (100) and (103) and taking advantage of the orthogonality of sine functions, one finds

$$U = \frac{\pi^4 abD}{8} \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} \left(\frac{r^2}{a^2} + \frac{s^2}{b^2}\right)^2 p_{rs}^2 \tag{113}$$

$$T = \frac{abh\rho}{8} \sum_{r=1}^{\infty} \sum_{s=1}^{\infty} p_{rs}{}^{\circ 2} \tag{114}$$

Expanding (114) for $r = 1, 2$ and $s = 1, 2$ gives

$$U = \frac{\pi^4 abD}{8} \left[\left(\frac{1}{a^2} + \frac{1}{b^2}\right)^2 p_{11}^2 + \left(\frac{4}{a^2} + \frac{1}{b^2}\right)^2 p_{21}^2 + \left(\frac{1}{a^2} + \frac{4}{b^2}\right)^2 p_{12}^2 + \left(\frac{4}{a^2} + \frac{4}{b^2}\right)^2 p_{22}^2\right] \tag{115}$$

$$T = \frac{abh\rho}{8} (p_{11}{}^{\circ 2} + p_{21}{}^{\circ 2} + p_{12}{}^{\circ 2} + p_{22}{}^{\circ 2}) \tag{116}$$

The kinetic energy of the concentrated mass must be added to (116). This increment in kinetic energy is given by:

$$T_m = \frac{abh\rho}{20} \left(0.5 p_{11}{}^{\circ} - \frac{\sqrt{2}}{2} p_{21}{}^{\circ} - \frac{\sqrt{2}}{2} p_{12}{}^{\circ} + p_{22}{}^{\circ}\right)^2 \tag{117}$$

$$T=\frac{\rho h}{2}\int_S \left(\frac{\partial w}{\partial t}\right)^2 r\,d\theta\,dr \tag{105}$$

4.6 Frequencies and Modes for Plates, Response to Excitation

The procedures are the same as used for bars and beams. They will be illustrated by three examples: two dealing with the calculation of frequency, one by Rayleigh's method and the other by the Rayleigh–Ritz method, and one dealing with the determination of steady-state response to a concentrated force.

4.6.1 Rayleigh's method

The problem is to calculate the lowest frequency of a circular plate undergoing axisymmetric vibration. The plate is fixed at its center point and free at its circumference. Consider the displacement hypothesis:

$$w(r,t)=\left[3\left(\frac{r}{R}\right)^2-\left(\frac{r}{R}\right)^3\right]p \tag{106}$$

where symmetry requires that w be a spatial function of r only. Equation (106) satisfies the geometric boundary conditions at the central support point. Since the displacement is not a function of θ, equations (104) and (105) become

$$U=\frac{D}{2}\int_0^R\left[\left(\frac{\partial^2 w}{\partial r^2}\right)^2+\frac{1}{r^2}\left(\frac{\partial w}{\partial r}\right)^2+\frac{2\nu}{r}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2}\right]2\pi r\,dr \tag{107}$$

$$T=\frac{\rho h}{2}\int_0^R\left(\frac{\partial w}{\partial t}\right)^2 2\pi r\,dr \tag{108}$$

Substituting (106) in (107) and (108) gives

$$U=\frac{\pi D}{R^2}(11.25+9\nu)p^2 \tag{109}$$

$$T=2.412\rho h R^2 p^{\circ 2} \tag{110}$$

from which the frequency is found to be

$$\omega=\frac{1}{R^2}\sqrt{\frac{(11.25+9\nu)D\pi}{2.412\rho h}}$$

$$=\frac{h}{R^2}\sqrt{\frac{(11.25+9\nu)E\pi}{28.94(1-\nu^2)\rho}} \tag{111}$$

This result is 15% higher than the exact value.

4.6.2 Rayleigh–Ritz method

A rectangular plate of constant thickness is simply supported, i.e. no lateral displacement and no bending moment, along all its edges and has a

the above energy expressions will be simplified by considering the special case of $b = a/2$. The total kinetic energy, i.e. the sum of (116) and (117), is

$$T = \frac{a^2 h\rho}{16}(1.1p_{11}^{\circ 2} + 1.2p_{21}^{\circ 2} + 1.2p_{12}^{\circ 2} + 1.4p_{22}^{\circ 2} - 0.2828p_{11}^{\circ}p_{21}^{\circ} - 0.2828p_{11}^{\circ}p_{12}^{\circ} + 0.4p_{11}^{\circ}p_{22}^{\circ} + 0.4p_{21}^{\circ}p_{12}^{\circ} - 0.5657p_{21}^{\circ}p_{22}^{\circ} - 0.5657p_{12}^{\circ}p_{22}^{\circ}) \tag{118}$$

and the strain energy is

$$U = \frac{\pi^4 D}{16a^2}(25p_{11}^2 + 64p_{21}^2 + 289p_{12}^2 + 400p_{22}^2) \tag{119}$$

Lagrange's equations result in the equations of motion:

$$a^2\rho h\begin{bmatrix} 1.1 & -0.1414 & -0.1414 & 0.2 \\ -0.1414 & 1.2 & 0.2 & -0.2828 \\ -0.1414 & 0.2 & 1.2 & -0.2828 \\ 0.2 & -0.2828 & -0.2828 & 1.4 \end{bmatrix}\begin{bmatrix} p_{11}^{\infty} \\ p_{21}^{\infty} \\ p_{12}^{\infty} \\ p_{22}^{\infty} \end{bmatrix} + \frac{\pi^4 D}{a^2}\begin{bmatrix} 25 & & & \text{zero} \\ & 64 & & \\ & & 289 & \\ \text{zero} & & & 400 \end{bmatrix}\begin{bmatrix} p_{11} \\ p_{21} \\ p_{12} \\ p_{22} \end{bmatrix} = 0 \tag{120}$$

and the frequencies and modes can be determined with the aid of program no. 5 of Chapter 7. The results are

$$\omega_1 = 4.730\frac{\pi^2 h}{a^2}\sqrt{\frac{E}{12(1-\nu^2)\rho}} \tag{121}$$

$$\phi_1(x, y) = \sin\frac{\pi x}{a}\sin\frac{2\pi y}{a} - 0.0892\sin\frac{2\pi x}{a}\sin\frac{2\pi y}{a} - 0.0139\sin\frac{\pi x}{a}\sin\frac{4\pi y}{a} + 0.0139\sin\frac{2\pi x}{a}\sin\frac{4\pi y}{a} \tag{122}$$

$$\omega_2 = 7.347\frac{\pi^2 h}{a^2}\sqrt{\frac{E}{12(1-\nu^2)\rho}} \tag{123}$$

$$\phi_2(x, y) = \sin\frac{\pi x}{a}\sin\frac{2\pi y}{a} + 4.092\sin\frac{2\pi x}{a}\sin\frac{2\pi y}{a} + 0.1744\sin\frac{\pi x}{a}\sin\frac{4\pi y}{a} - 0.1675\sin\frac{2\pi x}{a}\sin\frac{4\pi y}{a} \tag{124}$$

4.6.3 Steady-state response

The steady-state response of the plate just studied will be determined for an excitation force $F(t)$ at the point $x = 3a/4$, $y = 3a/8$. In order to simplify

the calculation, the concentrated mass will be removed. The displacement function will then give uncoupled equations of motion.

The expressions for kinetic and strain energies can be obtained by setting $b = a/2$ in (115) and (116). The generalized forces are deduced from the expression for the external virtual work of $F(t)$:

$$\delta W = F(t)\left[\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\left(\sin\frac{3\pi r}{4}\sin\frac{3\pi s}{4}\right)\delta p_{rs}\right] \tag{125}$$

which implies that the generalized force corresponding to each coordinate p_{rs} is

$$\sin\frac{3\pi r}{4}\sin\frac{3\pi s}{4}F(t) \tag{126}$$

Application of Lagrange's equation gives the uncoupled equations of motion:

$$\frac{a^2h\rho}{8}p_{rs}^{\infty}+\frac{\pi^4a^2D}{8}\left(\frac{r^2}{a^2}+\frac{4s^2}{a^2}\right)p_{rs}=\sin\frac{3\pi r}{4}\sin\frac{3\pi s}{4}F(t) \tag{127}$$

from which p_{rs} can be obtained and then $w(x, y, t)$ can be found from (112).

4.7 Kinetic and Strain Energies of Rotor Elements

The major elements which constitute rotating machinery are disks, shafts, and bearings. This section will present a derivation of the kinetic energy of a rigid rotating disk and the strain energy of a flexible rotating shaft. The kinetic energy of the shaft is not considered but its expression could be obtained from an extension of the kinetic energy expression of the disk. Also in the applications which are presented in the exercises, the bearings are assumed rigid.

Figure 7 shows the frames of reference which are used in the study of rotors. The axes XYZ are an inertial frame while the axes xyz are fixed to the disk. The xyz system is related to the XYZ system through a set of the three angles θ, ψ, ϕ. To achieve the orientation of the disk, one first rotates an amount ψ around the Z axis; then an amount θ around the new x-axis, denoted by x_1; and lastly an amount ϕ around the final y-axis.

4.7.1 Kinetic energy of the disk

With the aid of Figure 7, one can write the instantaneous angular velocity vector of the xyz frame as

$$\vec{\omega} = \dot{\psi}\vec{Z} + \dot{\theta}\vec{x}_1 + \dot{\phi}\vec{y} \tag{128}$$

where $\vec{Z}$, $\vec{x}_1$, $\vec{y}$ are unit vectors. The angular velocity of the disk and shaft is $\dot{\phi}$ After some algebra, the components of $\vec{\omega}$ in the xyz coordinate

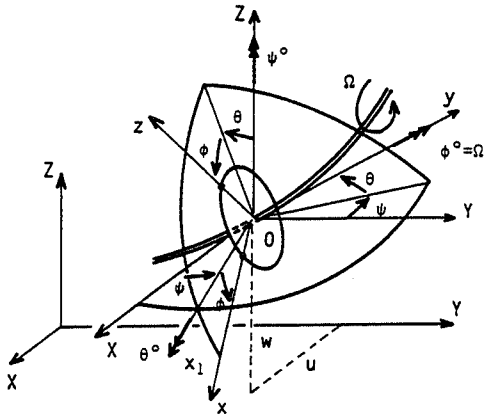


Figure 7 Reference frames for a disk on a rotating, flexible shaft

system can be written as

$$\vec{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = \begin{bmatrix} -\psi^\circ \cos\theta \sin\phi + \theta^\circ \cos\phi \\ \phi^\circ + \psi^\circ \sin\theta \\ \psi^\circ \cos\theta \cos\phi + \theta^\circ \sin\phi \end{bmatrix} \tag{129}$$

Let u and w in Figure 7 designate the coordinates of the disk center of mass, which is assumed to be on the axis of the shaft. Then the kinetic energy of the disk is given by

$$T_D = \tfrac{1}{2}M_D(u^{\circ 2} + w^{\circ 2}) + \tfrac{1}{2}(I_{Dx}\omega_x^2 + I_{Dy}\omega_y^2 + I_{Dz}\omega_z^2) \tag{130}$$

where I_{Dx}, I_{Dy}, and I_{Dz} are the mass moments of inertia of the disk with respect to the xyz axes, and M_D is the mass of the disk. Equation (130) is general but in our case $\phi^\circ = \Omega$, and θ and ψ are small. If, in addition, $I_{Dx} = I_{Dz}$, (129) and (130) give

$$T_D = \tfrac{1}{2}M_D(u^{\circ 2} + w^{\circ 2}) + \tfrac{1}{2}I_{Dx}(\theta^{\circ 2} + \psi^{\circ 2}) + \tfrac{1}{2}I_{Dy}(\Omega^2 + 2\Omega\psi^\circ\theta) \tag{131}$$

This provides a set of generalized coordinates

$$q = (u, w, \theta, \psi)^t \tag{132}$$

which, when used in Lagrange's equations, give

$$\begin{bmatrix} M & 0 & 0 & 0 \\ 0 & M & 0 & 0 \\ 0 & 0 & I_{Dx} & 0 \\ 0 & 0 & 0 & I_{Dx} \end{bmatrix} \begin{bmatrix} u^{\circ\circ} \\ w^{\circ\circ} \\ \theta^{\circ\circ} \\ \psi^{\circ\circ} \end{bmatrix} + \Omega \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_{Dy} \\ 0 & 0 & I_{Dy} & 0 \end{bmatrix} \begin{bmatrix} u^\circ \\ w^\circ \\ \theta^\circ \\ \psi^\circ \end{bmatrix} \tag{133}$$

The two matrices in (133) are the mass matrix and the so-called Coriolis matrix.

4.7.2 Strain energy of the shaft

We use the following notation:

E, Young's modulus of the shaft material

ε,σ, the strain and stress in the shaft

u^*,v^*,w^*, the displacements of the geometric center of the shaft with respect of the xyz axes (see Figure 8).

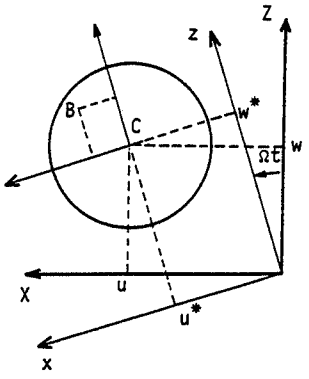


Figure 8 Coordinates of geometric center (C) and arbitrary point (B) of the shaft

The longitudinal displacement v^* is assumed to vanish. If the second-order terms are included, the longitudinal strain of a point B can be shown to be

$$\varepsilon = -z\frac{\partial^2 w^*}{\partial y^2} - x\frac{\partial^2 u^*}{\partial y^2} + \frac{1}{2}\left(\frac{\partial u^*}{\partial y}\right)^2 + \frac{1}{2}\left(\frac{\partial w^*}{\partial y}\right)^2 \tag{134}$$

or

$$\varepsilon = \varepsilon_l + \varepsilon_{nl} \tag{135}$$

where ε_l contains the linear terms and ε_{nl} the nonlinear terms of (134). The strain energy is

$$U_1 = \frac{1}{2}\int_\tau \varepsilon^t\sigma \, d\tau = \tfrac{1}{2}E\int_\tau (\varepsilon_l^2 + 2\varepsilon_{nl}\varepsilon_l + \varepsilon_{nl}^2)\, d\tau \tag{136}$$

The symmetry of the shaft cross-section with respect to x and z result in

$$\int_\tau \varepsilon_{nl}\varepsilon_l \, d\tau = 0$$

The term $\int_\tau \varepsilon_{nl}^2 \, d\tau$ is of the second order and neglected. The strain energy is then

$$U_i = \tfrac{1}{2}E\int_\tau \left[-z\frac{\partial^2 w^*}{\partial y^2} - x\frac{\partial^2 u^*}{\partial y^2}\right]^2 d\tau \tag{137}$$

or

$$U_1 = \tfrac{1}{2}E\int_0^L I_x\left(\frac{\partial^2 w^*}{\partial y^2}\right)^2 dy + \tfrac{1}{2}E\int_0^L I_z\left(\frac{\partial^2 u^*}{\partial y^2}\right) dy \tag{138}$$

where

$$\begin{aligned} I_x &= \int_S z^2\, dx\, dz \\ I_z &= \int_S x^2\, dx\, dz \end{aligned} \tag{139}$$

are the area moments of inertia of the shaft.

If the shaft is subjected to a constant axial force F_0, there is a second contribution to the strain energy of the shaft given by

$$U_2 = \int_\tau \left(\frac{F_0}{S}\right)[\varepsilon_l + \varepsilon_{nl}]\, d\tau \tag{140}$$

Because of symmetry, the integral of ε_l over the shaft cross-sectional area will vanish, and (140) becomes

$$U_2 = \tfrac{1}{2}F_0\int_0^L \left[\left(\frac{\partial w^*}{\partial y}\right)^2 + \left(\frac{\partial u^*}{\partial y}\right)^2\right] dy \tag{141}$$

The combined strain energy $U_1 + U_2$ is then

$$\begin{aligned} U = \frac{1}{2}\int_0^L EI_x\left(\frac{\partial^2 w^*}{\partial y^2}\right)^2 dy + \frac{1}{2}\int_0^L EI_z\left(\frac{\partial^2 u^*}{\partial y^2}\right)^2 dy + \tfrac{1}{2}F_0\int_0^L \left(\frac{\partial w^*}{\partial y}\right)^2 dy \\ + \tfrac{1}{2}F_0\int_0^L \left(\frac{\partial u^*}{\partial y}\right)^2 dy \end{aligned} \tag{142}$$

Using

$$\begin{aligned} w^* &= w\cos\Omega t + u\sin\Omega t \\ u^* &= -w\sin\Omega t + u\cos\Omega t \end{aligned} \tag{143}$$

which can be deduced from Figure 8, equation (142) is written

$$\begin{aligned} U = \tfrac{1}{2}E\int_0^L \left[I_x\left(\cos\Omega t\frac{\partial^2 w}{\partial y^2} + \sin\Omega t\frac{\partial^2 u}{\partial y^2}\right)^2\right. \\ \left. + I_z\left(-\sin\Omega t\frac{\partial^2 w}{\partial y^2} + \cos\Omega t\frac{\partial^2 u}{\partial y^2}\right)^2\right] dy \\ + \tfrac{1}{2}F_0\int_0^L \left[\left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] dy \end{aligned} \tag{144}$$

Finally, for the most common case of a symmetric shaft, i.e. $I_x = I_z = I$, the

strain energy becomes

$$U=\frac{EI}{2}\int_0^L\left[\left(\frac{\partial^2 w}{\partial y^2}\right)^2+\left(\frac{\partial^2 u}{\partial y^2}\right)^2\right]dy+\frac{F_0}{2}\int_0^L\left[\left(\frac{\partial w}{\partial y}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]dy \quad (145)$$

Equations (131) and (145) allow the derivation of equations of motion for the system of Figure 7 via Lagrange's equations. Previous methods can be used to solve these equations of motion and to assist in the design of a rotating rotor–disk system. The determination of system frequencies and critical speeds due to unbalance is left to the exercises.

4.8 Exercises

Exercise 1: *Bar in longitudinal motion. The bar is fixed at one end and has a spring of stiffness k at the other end; see Figure 9. Calculate the first two frequencies and their corresponding modes for k = ES/L.*

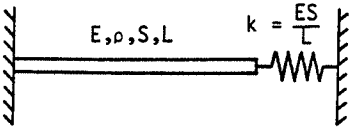


Figure 9

Since the bar is fixed at the origin, equation (26) implies

$$\phi(x)=\sin\omega\sqrt{\frac{\rho}{E}}\,x$$

and as

$$P(L)=ES\left.\frac{d\phi(x)}{dx}\right|_{x=L}=-k\phi(L)=-\frac{ES}{L}\phi(L)$$

it follows that

$$ES\omega\sqrt{\frac{\rho}{E}}\cos\omega\sqrt{\frac{\rho}{E}}L=-\frac{ES}{L}\sin\omega\sqrt{\frac{\rho}{E}}L$$

Let

$$X=\omega\sqrt{\frac{\rho}{E}}L$$

then the above becomes

$$\tan X=-X$$

The first two lowest values of X which satisfy this transcendental equation

are

$$\omega_1 = \frac{2.029}{L}\sqrt{\frac{E}{\rho}}; \qquad \omega_2 = \frac{4.913}{L}\sqrt{\frac{E}{\rho}}$$

and the modes are

$$\phi_1(x) = \sin 2.029\frac{x}{L}; \qquad \phi_2(x) = \sin 4.913\frac{x}{L}$$

Exercise 2: *Consider a bar which is fixed at one end and free at the other. At the free end it has a concentrated mass M. Find the equation for the frequencies.*

Proceeding as in exercise 1, one finds

$$\tan\omega\sqrt{\frac{\rho}{E}}L = \frac{ES}{M\omega}\sqrt{\frac{\rho}{E}}$$

Exercise 3: *Consider the bar shown in Figure* 10 *which is fixed at one end and free at the other and undergoes longitudinal motion. Find the relation which allows the calculation of the frequencies for this system.*

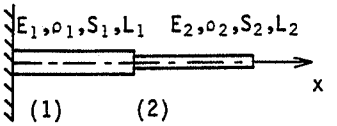


Figure 10

To simplify the equation the origin for bar (1) is taken at $x = 0$ and the origin for bar (2) is at $x = L_1$. The motion of each bar is defined by equation (27):

$$u_1(x, t) = (A_1 \sin\omega_{(1)}t + B_1\cos\omega_{(1)}t)\left(C_1\sin\omega_{(1)}\sqrt{\frac{\rho_1}{E_1}}x + D_1\cos\omega_{(1)}\sqrt{\frac{\rho_1}{E_1}}x\right)$$

$$u_2(x, t) = (A_2 \sin\omega_{(2)}t + B_2\cos\omega_{(2)}t)\left(C_2\sin\omega_{(2)}\sqrt{\frac{\rho_2}{E_2}}x + D_2\cos\omega_{(2)}\sqrt{\frac{\rho_2}{E_2}}x\right)$$

Since bar (1) is fixed at $x = 0$, $u_1(0, t) = 0$ and $D_1 = 0$. Continuity of displacement at the junction of the two bars is assumed for all t and hence requires

$$A_1 = A_2 = A; \qquad B_1 = B_2 = B; \qquad \omega_{(1)} = \omega_{(2)} = \omega$$

$$D_2 = C_1\sin\omega\sqrt{\frac{\rho_1}{E_1}}L_1$$

Equality of the forces at the junction gives

$$C_1\left[E_1S_1\omega\sqrt{\frac{\rho_1}{E_1}}\cos\omega\sqrt{\frac{\rho_1}{E_1}}L_1\right]=C_2\left[E_2S_2\omega\sqrt{\frac{\rho_2}{E_2}}\right]$$

The force must vanish at the free end, which implies

$$C_2\cos\omega\sqrt{\frac{\rho_2}{E_2}}L_2-D_2\sin\omega\sqrt{\frac{\rho_2}{E_2}}L_2=0$$

Putting these three equations in matrix form gives

$$\begin{bmatrix}\sin\gamma_1L_1 & 0 & -1\\ E_1S_1\gamma_1\cos\gamma_1L_1 & -E_2S_2\gamma_2 & 0\\ 0 & \cos\gamma_2L_2 & -\sin\gamma_2L_2\end{bmatrix}\begin{bmatrix}C_1\\C_2\\D_2\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

with

$$\gamma_1=\omega\sqrt{\frac{\rho_1}{E_1}};\qquad \gamma_2=\omega\sqrt{\frac{\rho_2}{E_2}}$$

The frequencies are then determined by setting the determinant of the coefficients equal to zero. This gives

$$E_2S_2\gamma_2\sin\gamma_1L_1\sin\gamma_2L_2-E_1S_1\gamma_1\cos\gamma_1L_1\cos\gamma_2L_2=0$$

Exercise 4: *Find the exact transfer matrix for the longitudinal motion of a uniform bar; see Figure* 11.

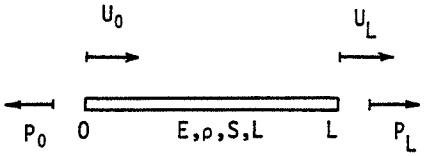


Figure 11

Referring to (26) and (3) and using $u(x, t)=U(x)\,f(t)$ gives

$$U(x)=C\sin\omega\sqrt{\frac{\rho}{E}}x+D\cos\omega\sqrt{\frac{\rho}{E}}x$$

$$P(x)=ES\frac{\mathrm{d}U(x)}{\mathrm{d}x}$$

$$=\omega\sqrt{\frac{\rho}{E}}ES\left(C\cos\omega\sqrt{\frac{\rho}{E}}x-D\sin\omega\sqrt{\frac{\rho}{E}}x\right)$$

Since at

$$x=0,\qquad U(x)=U_0\qquad\text{and}\qquad P(x)=P_0$$
$$x=L,\qquad U(L)=U_L\qquad\text{and}\qquad P(L)=P_L$$

Exercise 6: *Find the orthogonality relationships for the system of exercise* 2.

The solutions are:

$$\int_0^L ES \frac{d\phi_i(x)}{dx} \frac{d\phi_j(x)}{dx}\, dx = 0$$

$$\int_0^L \rho S \phi_i(x)\phi_j(x)dx + M\phi_i(L)\phi_j(L) = 0$$

Exercise 7: *Plot the two functions* $\cos X$ *and* $-1/\cosh X$ *and determine their points of intersection. Relate these to the three lowest frequencies of a clamped free beam.*

Let X_1, X_2, X_3 be the three lowest points of intersection as shown in the Figure 13. Notice that the values of $X_2, X_3, \ldots,$ rapidly approach the zeros

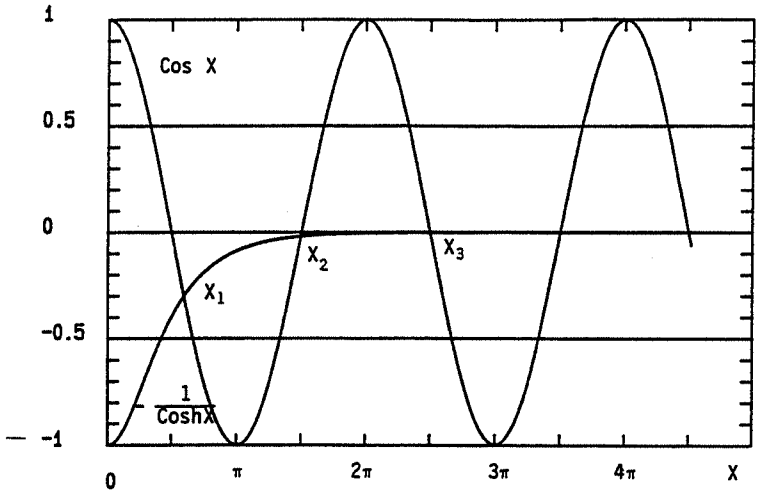


Figure 13

of the function $\cos X$ so that

$$X_2 \simeq \frac{3\pi}{2}, \qquad X_3 \simeq \frac{5\pi}{2}$$

and

$$X_2^2 \simeq 22.21, \qquad X_3^2 \simeq 61.69$$

Therefore only X_1 has to be calculated from the transcendental equation for frequency.

then

$$U_0 = D \qquad \text{and} \qquad P_0 = \omega\sqrt{\frac{\rho}{E}}\,ESC$$

Let

$$\gamma = \omega\sqrt{\frac{\rho}{E}}$$

then

$$U(x) = \frac{\sin \gamma x}{ES\gamma} P_0 + \cos \gamma x U_0$$

and

$$\begin{bmatrix} P_L \\ U_L \end{bmatrix} = \begin{bmatrix} \cos \gamma L & -ES\gamma \sin \gamma L \\ \dfrac{\sin \gamma L}{ES\gamma} & \cos \gamma L \end{bmatrix} \begin{bmatrix} P_0 \\ U_0 \end{bmatrix}$$

Exercise 5: *Consider the same system as in exercise* 1. *With the aid of the resultant of exercise* 4, *rederive the frequency equation for the system*; *see Figure* 12.

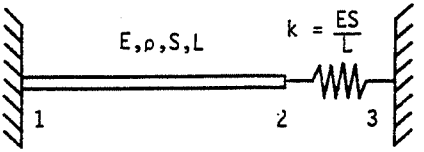


Figure 12

$$\begin{bmatrix} P_3 \\ U_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \dfrac{1}{k} & 1 \end{bmatrix} \begin{bmatrix} \cos \gamma L & -ES\gamma \sin \gamma L \\ \dfrac{\sin \gamma L}{ES\gamma} & \cos \gamma L \end{bmatrix} \begin{bmatrix} P_1 \\ U_1 \end{bmatrix}$$

$$U_3 = \left(\frac{\cos \gamma L}{k} + \frac{\sin \gamma L}{ES\gamma}\right) P_1 + \left(\cos \gamma L - \frac{ES\gamma}{k} \sin \gamma L\right) U_1$$

The frequency equation is obtained from the boundary conditions $U_3 = 0$ and $U_1 = 0$ which result in

$$\frac{\cos \gamma L}{k} + \frac{\sin \gamma L}{ES\gamma} = 0$$

Since $k = ES/L$ and setting $X = \gamma L = \omega\sqrt{\rho/E}\,L$, one obtains

$$\tan X = -X$$

Exercise 8: *Show that the transcendental frequency formulas for* C–C, F–F, C–S, F–S *beams given in Table* 2 *of section* 4.2.2 *can be simplified as in exercise* 7.

Neglecting the case of zero frequency, which is mentioned in the text, one can show that seeking graphical solutions to the transcendental equations

$$\cos X = \frac{1}{\cosh X}$$

and

$$\tan X = \tanh X$$

yields the results that:

(1) for C–C and F–F beams the values X_i^2 can be found from $\cos X = 0$ and these values are

$$X_1^2 = \left(\frac{3\pi}{2}\right)^2 = 22.21, \qquad X_2^2 = \left(\frac{5\pi}{2}\right)^2 = 61.69, \ldots$$

(2) for C–S and F–S beams, the values X_i^2 can be found from $\tan X = 1$ with $X > \pi/4$ and these values are

$$X_1^2 = \left(\frac{\pi}{4} + \pi\right)^2 = 15.42, \qquad X_2^2 = \left(\frac{\pi}{4} + 2\pi\right)^2 = 49.96, \ldots$$

Exercise 9: *Consider a beam which is simply supported at each end. Find the expression for the modes and determine the corresponding kinetic and strain energies, modal masses and stiffnesses.*

The solution is

$$\phi_n(x) = \sin\frac{n\pi x}{L}$$

Using (76) and (77),

$$T_n = \frac{\rho SL}{4}\, p_n^{o2}(t); \qquad U_n = \frac{n^4\pi^4}{4L^3} EI p_n^2(t)$$

and from (70) and (71),

$$m_n = \frac{\rho SL}{2}; \qquad k_n = \frac{n^4\pi^4 EI}{2L^3}$$

Exercise 10: *Find the exact transfer matrix for a beam in bending using the same procedure as in exercise 4.*

The solution is

$$\begin{bmatrix} Q_L \\ M_L \\ \psi_L \\ V_L \end{bmatrix} = \begin{bmatrix} \frac{\operatorname{ch}\beta L+\cos\beta L}{2} & \frac{\beta(\operatorname{sh}\beta L-\sin\beta L)}{2} & \frac{EI\beta^2(\operatorname{ch}\beta L-\cos\beta L)}{2} & \frac{EI\beta^3(\operatorname{sh}\beta L+\sin\beta L)}{2} \\ \frac{\operatorname{sh}\beta L+\sin\beta L}{2\beta} & \frac{\operatorname{ch}\beta L+\cos\beta L}{2} & \frac{EI\beta(\operatorname{sh}\beta L-\sin\beta L)}{2} & \frac{EI\beta^2(\operatorname{ch}\beta L-\cos\beta L)}{2} \\ \frac{\operatorname{ch}\beta L-\cos\beta L}{2EI\beta^2} & \frac{\operatorname{sh}\beta L+\sin\beta L}{2EI\beta} & \frac{\operatorname{ch}\beta L+\cos\beta L}{2} & \frac{\beta(\operatorname{sh}\beta L-\sin\beta L)}{2} \\ \frac{\operatorname{sh}\beta L-\sin\beta L}{2EI\beta^3} & \frac{\operatorname{ch}\beta L-\cos\beta L}{2EI\beta^2} & \frac{\operatorname{sh}\beta L+\sin\beta L}{2\beta} & \frac{\operatorname{ch}\beta L+\cos\beta L}{2} \end{bmatrix} \begin{bmatrix} Q_0 \\ M_0 \\ \psi_0 \\ V_0 \end{bmatrix}$$

where ch = cosh, sh = sinh.

Exercise 11: *Using the exact transfer matrix for a beam, rederive the relation for the frequencies of a clamped–free beam.*

The boundary conditions require

$$Q_L = \frac{\cosh\beta L+\cos\beta L}{2} Q_0 + \beta\frac{\sinh\beta L-\sin\beta L}{2} M_0 = 0$$

$$M_L = \frac{\sinh\beta L+\sin\beta L}{2\beta} Q_0 + \frac{\cosh\beta L+\cos\beta L}{2} M_0 = 0$$

Hence

$$(\cosh\beta L+\cos\beta L)(\cosh\beta L+\cos\beta L) - (\sinh\beta L+\sin\beta L)(\sinh\beta L-\sin\beta L) = 0$$

The expansion of this expression gives

$$1+\cos\beta L\cosh\beta L = 0$$

Exercise 12: *Consider the bar in longitudinal motion, see Figure* 14, *subjected*

to a step force which is constant at F for $t \leq 0$ and equal to zero for $t > 0$. Find $u(x, t)$ and $P(x, t)$.

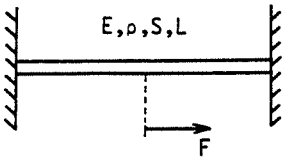


Figure 14

Since the bar is fixed at both ends,

$$u(x, t) = \sum_{n=1,2,\ldots} (A_n \cos \omega_n t + B_n \sin \omega_n t) \sin \frac{n\pi x}{L}$$

with

$$\omega_n = \frac{n\pi}{L} \sqrt{\frac{E}{\rho}}$$

The initial velocity distribution is given by

$$\left. \frac{\partial u(x, t)}{\partial t} \right|_{t=0} = 0$$

Since

$$\frac{\partial u(x, t)}{\partial t} = \sum_{n=1,2,\ldots} (-\omega_n A_n \sin \omega_n t + \omega_n B_n \cos \omega_n t) \sin \frac{n\pi x}{L}$$

then at $t = 0$

$$\sum_{n=1,2,\ldots} \omega_n B_n \sin \frac{n\pi x}{L} = 0$$

Due to the orthogonality of the modes,

$$\int_0^L \omega_n B_n \sin^2 \frac{n\pi x}{L} \, dx = \frac{L\omega_n B_n}{2} = 0$$

hence $B_n = 0$.

The value of A_n is found from

$$u(x, t) = \sum_{n=1,2,\ldots} A_n \cos \omega_n t \sin \frac{n\pi x}{L}$$

which at $t = 0$ becomes

$$u(x, 0) = \sum_{n=1,2,\ldots} A_n \sin \frac{n\pi x}{L}$$

At the initial instant $t = 0$:

$$u(x, 0) = \frac{Fx}{2ES} \qquad \text{for} \qquad 0 < x < \frac{L}{2}$$

$$u(x, 0) = \frac{F(L-x)}{2ES} \qquad \text{for} \qquad \frac{L}{2} < x < L$$

which, taking account of orthogonality, implies

$$\frac{F}{2ES}\left[\int_0^{L/2} x \sin\frac{n\pi x}{L}\,dx + \int_{L/2}^{L} (L-x)\sin\frac{n\pi x}{L}\,dx\right] = A_n \int_0^L \sin^2\frac{n\pi x}{L}\,dx$$

$$= \frac{A_n L}{2}$$

Since

$$\int_{L/2}^{L} (L-x)\sin\frac{n\pi x}{L}\,dx = \int_0^{L/2} x \sin\left(n\pi - \frac{n\pi x}{L}\right)dx$$

then

$$A_n = \frac{F}{ESL}\int_0^{L/2} x(1-\cos n\pi)\sin\frac{n\pi x}{L}\,dx$$

Completing the calculations gives

$$A_n = \frac{2FL}{n^2\pi^2 ES}(-1)^{(n-1)/2} \qquad \text{with } n = 1, 3, \ldots$$

$$u(x, t) = \frac{2FL}{\pi^2 ES}\sum_{n=1,3,\ldots}(-1)^{(n-1)/2}\frac{1}{n^2}\cos\omega_n t \sin\frac{n\pi x}{L}$$

and

$$P(x, t) = ES\frac{\partial u(x, t)}{\partial x}$$

$$= \frac{2F}{\pi}\sum_{n=1,3,\ldots}(-1)^{(n-1)/2}\frac{1}{n}\cos\omega_n t \cos\frac{n\pi x}{L}$$

The series representing $u(x, t)$ converges like $1/n^2$ and that for $P(x, t)$ like $1/n$. It is necessary to use more terms in calculating P than in calculating u. This observation has general applicability.

Exercise 13: *Beam on elastic foundation. A beam rests on an elastic foundation whose stiffness per unit length is given by* $k = K/L$. *Find the equation of motion, the general expression for the frequencies, and the specific expression for the frequencies of an* S–S *beam*; *see Figure* 15.

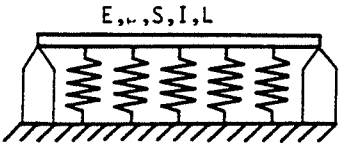


Figure 15

The first equation of (17) becomes

$$\rho S\frac{\partial^2 v}{\partial t^2}=-\frac{\partial Q}{\partial x}+Q_{ex}-kv$$

the others are unchanged. By elimination among these equations,

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 v}{\partial x^2}\right)+\rho S\frac{\partial^2 v}{\partial t^2}+kv-Q_{ex}=0$$

For free vibration and a constant cross-section, this becomes

$$EI\frac{\partial^4 v}{\partial x^4}+\rho S\frac{\partial^2 v}{\partial t^2}+kv=0$$

Solutions are sought in the form

$$v(x,t)=\phi(x)\cdot f(t)$$

which gives

$$\frac{\mathrm{d}^2 f(t)}{\mathrm{d}t^2}+\omega^2 f(t)=0$$

and

$$\frac{\mathrm{d}^4\phi(x)}{\mathrm{d}x^4}+\frac{k-\omega^2\rho S}{EI}\phi(x)=0$$

Setting

$$\beta^*=\sqrt[4]{\frac{\rho S\omega^2-k}{EI}}$$

one gets the same general solution as in the beam; namely:

$$\phi(x)=C\sin\beta^* x+D\cos\beta^* x+E\sinh\beta^* x+F\cosh\beta^* x$$

Using the boundary conditions, one obtains, as in section 4.2,

$$\beta_n^* L=X_n$$

from which

$$\omega_n=\frac{1}{L^2}\sqrt{\frac{EI}{\rho S}}\cdot\sqrt{X_n^4+\frac{KL^3}{EI}}$$

For the S–S case, ϕ_n reduces to

$$\phi_n(x) = C_n \sin \beta_n^* x$$

and

$$\beta_n^* L = n\pi$$

so that

$$\omega_n = \frac{1}{L^2}\sqrt{\frac{EI}{\rho S}}\sqrt{(n\pi)^4 + \frac{KL^3}{EI}}$$

Exercise 14: *Consider the same system as in exercise* 1. *Use the Rayleigh–Ritz method to calculate the first two frequencies and their corresponding modes. Use a displacement function containing the terms* (x/L), $(x/L)^2$, $(x/L)^3$, $(x/L)^4$, *and program no.* 5 *of Chapter* 7.

The fixed–free bar has been previously analyzed with the same displacement functions; see (84). The spring gives the following supplementary stiffness matrix:

$$\frac{ES}{L}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Use of the computer program on the modified equation of motion gives

$$\omega_1 = \frac{2.029}{L}\sqrt{\frac{E}{\rho}}$$

$$\phi_1(x) = \frac{x}{L} + 0.06346\left(\frac{x}{L}\right)^2 - 0.8993\left(\frac{x}{L}\right)^3 + 0.2805\left(\frac{x}{L}\right)^4$$

$$\omega_2 = \frac{4.932}{L}\sqrt{\frac{E}{\rho}}$$

$$\phi_2(x) = \frac{x}{L} - 0.9260\left(\frac{x}{L}\right)^2 - 2.231\left(\frac{x}{L}\right)^3 + 1.978\left(\frac{x}{L}\right)^4$$

Exercise 15: *Consider the same system as in exercise* 1. *Use the Rayleigh–Ritz method to calculate the first two frequencies and their corresponding modes. Use a displacement function containing the terms* $\sin(\pi x/2L)$ *and* $\sin(3\pi x/2L)$.

The displacement is

$$u(x, t) = \sin\frac{\pi x}{2L}\, p_1 + \sin\frac{3\pi x}{2L}\, p_2$$

The kinetic energy is

$$T=\frac{1}{2}\int_0^L \rho S\left(\sin\frac{\pi x}{2L}\,p_1^{\circ}+\sin\frac{3\pi x}{2L}\,p_2^{\circ}\right)^2 dx$$

$$=\frac{\rho SL}{4}\,p_1^{\circ 2}+\frac{\rho SL}{4}\,p_2^{\circ 2}$$

The strain energy is

$$U=U_{\text{beam}}+U_{\text{spring}}$$

$$U=\frac{1}{2}\int_0^L ES\left(\frac{\pi}{2L}\cos\frac{\pi x}{2L}\,p_1+\frac{3\pi}{2L}\cos\frac{3\pi x}{2L}\,p_2\right)^2 dx+\frac{ES}{2L}(p_1-p_2)^2$$

$$=\frac{ES}{L}\left(\frac{\pi^2}{16}+\frac{1}{2}\right)p_1^2+\frac{ES}{L}\left(\frac{9\pi^2}{16}+\frac{1}{2}\right)p_2^2-\frac{ES}{L}\,p_1p_2$$

The application of Lagrange's equations gives

$$\rho SL\begin{bmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}p_1^{\infty}\\ p_2^{\infty}\end{bmatrix}+\frac{ES}{L}\begin{bmatrix}\frac{\pi^2}{8}+1 & -1\\ -1 & \frac{9\pi^2}{8}+1\end{bmatrix}\begin{bmatrix}p_1\\ p_2\end{bmatrix}=0$$

from which the frequencies and modes are

$$\omega_1=\frac{2.066}{L}\sqrt{\frac{E}{\rho}}; \qquad \phi_1(x)=\sin\frac{\pi x}{2L}+0.1003\sin\frac{3\pi x}{2L}$$

$$\omega_2=\frac{4.940}{L}\sqrt{\frac{E}{\rho}}; \qquad \phi_2(x)=\sin\frac{\pi x}{2L}-9.970\sin\frac{3\pi x}{2L}$$

Exercise 16: *Consider a fixed–free bar in longitudinal motion which has a concentrated tip mass of* $m=\rho SL/10$; *see Figure* 16. *Use the Rayleigh–Ritz method and the same displacement function as in exercise* 15 *to calculate the frequencies and modes.*

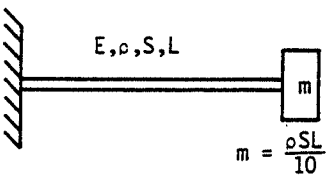


Figure 16

The strain energy is

$$U=\frac{ES}{2}\left[\frac{\pi^2}{8L}\,p_1^2+\frac{9\pi^2}{8L}\,p_2^2\right]$$

and the kinetic energy is

$$T = T_{\text{beam}} + T_{\text{mass}}$$
$$= \frac{\rho SL}{4}(p_1^{\circ 2} + p_2^{\circ 2}) + \frac{\rho SL}{20}(p_1^{\circ 2} + p_2^{\circ 2} - 2p_1^{\circ} p_2^{\circ})$$

Applying Lagrange's equations gives

$$\rho SL\begin{bmatrix} 0.6 & -0.1 \\ -0.1 & 0.6 \end{bmatrix}\begin{bmatrix} p_1^{\infty} \\ p_2^{\infty} \end{bmatrix} + \frac{ES}{L}\begin{bmatrix} 1.234 & 0 \\ 0 & 11.10 \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = 0$$

from which

$$\omega_1 = \frac{1.431}{L}\sqrt{\frac{E}{\rho}} \qquad \text{with the mode} \qquad \begin{bmatrix} 1 \\ -0.0208 \end{bmatrix}$$

Hence

$$\phi_1(x) = \sin\frac{\pi x}{2L} - 0.0208 \sin\frac{3\pi x}{2L}$$

and also

$$\omega_2 = \frac{4.370}{L}\sqrt{\frac{E}{\rho}} \qquad \text{with the mode} \qquad \begin{bmatrix} 1 \\ 5.354 \end{bmatrix}$$

Hence

$$\phi_2(x) = \sin\frac{\pi x}{2L} + 5.354 \sin\frac{3\pi x}{2L}$$

Exercise 17: *Use Rayleigh's method to calculate the frequency for the beam shown in Figure* 17. *Use the displacement function* $3(x/L)^2 - (x/L)^3$.

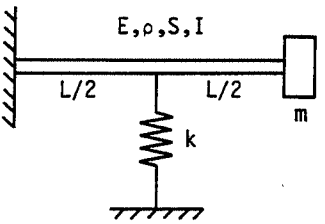


Figure 17

As suggested,

$$v(x, t) = \left[3\left(\frac{x}{L}\right)^2 - \left(\frac{x}{L}\right)^3\right]p$$

The kinetic and strain energy of the beam have been calculated in section 4.3. Then

$$T = 0.4714\rho SLp^{\circ 2} + 2mp^{\circ 2}$$

$$U = \frac{6EI}{L^3}p^2 + 0.1953kp^2$$

Then the equation of motion is

$$(4m + 0.9428\rho SL)p^{\infty} + \left(\frac{12EI}{L^3} + 0.3906k\right)p = 0$$

and

$$\omega = \sqrt{\frac{12EI/L^3 + 0.3906k}{4m + 0.9428\rho SL}}$$

Exercise 18: *Effect of axial force. A clamped–free beam is subjected to a constant axial force F_0. Use Rayleigh's method to calculate the frequency with the same displacement function as in exercise* 17.

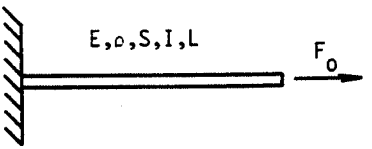


Figure 18

From (79) and (80),

$$T = 0.4714\rho SLp^{\circ 2}$$

$$U = \frac{6EI}{L^3}p^2$$

It is necessary to add the influence of axial force to the strain energy U. From (141), the supplementary strain energy is

$$U_{F_0} = \frac{F_0}{2}\int_0^L \left(\frac{\partial v}{\partial x}\right)^2 dx = \frac{F_0}{2}\int_0^L \left(\frac{6x}{L^2} - \frac{3x^2}{L^3}\right)^2 p^2\, dx$$

$$= \frac{2.4F_0}{L}p^2$$

The system equation then becomes

$$0.9428\rho SLp^{\infty} + \left(\frac{12EI}{L^3} + \frac{4.8F_0}{L}\right)p = 0$$

and the frequency is

$$\omega = \sqrt{\frac{12EI/L^3 + 4.8F_0/L}{0.9428\rho SL}}$$

The numerator of ω vanishes for $F_0 = -2.5EI/L^2$. This gives a good approximate value of the first Euler buckling load, since the exact value is

$$-\frac{\pi^2}{4}\frac{EI}{L^2} = -2.46\frac{EI}{L^2}$$

The vibration mode becomes the buckling mode.

Exercise 19: *A beam simply supported at each end has a spring of stiffness* $k = 40EI/L^3$ *at its quarter-point; see Figure* 19. *Use Rayleigh's method to calculate the lowest frequency.*

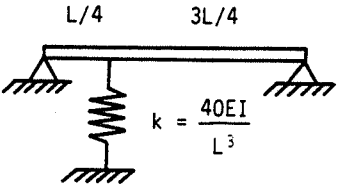


Figure 19

Since the beam is S–S, an appropriate displacement function is $\sin(\pi x/L)$. From the results of exercise 9:

$$T_{\text{beam}} = \frac{\rho SL}{4}p^{\circ 2}$$

$$U_{\text{beam}} = \frac{\pi^4}{4}\frac{EI}{L^3}p^2$$

The strain energy for the spring is

$$U_{\text{spring}} = \frac{20EI}{L^3}\left(\sin\frac{\pi}{4}\right)^2 p^2 = \frac{10EI}{L^3}p^2$$

The strain energy of the system is then

$$U = U_{\text{beam}} + U_{\text{spring}} = \frac{EI}{L^3}\left(\frac{\pi^4}{4} + 10\right)p^2$$

hence the frequency is

$$\omega = \frac{1}{L^2}\sqrt{40 + \pi^4}\sqrt{\frac{EI}{\rho S}} = \frac{11.72}{L^2}\sqrt{\frac{EI}{\rho S}}$$

The exact solution obtained from the transfer matrix program no. 10 is $11.62/L^2\sqrt{EI/\rho S}$.

Exercise 20: *Consider the same system as in exercise* 19. *Use the Rayleigh–Ritz method to calculate the first frequency and its corresponding mode. Use the displacement function* $v(x, t) = \sin(\pi x/L)p_1 + \sin(2\pi x/L)p_2$.

For the beam: see exercise 9, from which

$$T_{\text{beam}} = \frac{\rho SL}{4} p_1^{\circ 2} + \frac{\rho SL}{4} p_2^{\circ 2}$$

$$U_{\text{beam}} = \frac{\pi^4}{4}\frac{EI}{L^3}(p_1^2 + 16p_2^2)$$

For the spring:

$$U_{\text{spring}} = \frac{20EI}{L^3}\left[\frac{\sqrt{2}}{2}p_1 + p_2\right]^2$$

$$= \frac{20EI}{L^3}\left(\frac{p_1^2}{2} + p_2^2 + \sqrt{2}\,p_1 p_2\right)$$

Application of Lagrange' equations gives

$$\frac{\rho SL}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}p_1^{\infty}\\ p_2^{\infty}\end{bmatrix} + \frac{EI}{L^3}\begin{bmatrix}\frac{\pi^4}{2}+20 & 20\sqrt{2}\\ 20\sqrt{2} & 8\pi^4+40\end{bmatrix}\begin{bmatrix}p_1\\ p_2\end{bmatrix} = 0$$

from which

$$\omega = \frac{11.63}{L^2}\sqrt{\frac{EI}{\rho S}} \qquad \text{with the mode} \qquad \begin{bmatrix}1\\ -0.0376\end{bmatrix}$$

Hence

$$\phi(x) = \frac{\sin \pi x}{L} - 0.0376 \sin\frac{2\pi x}{L}$$

Note that this result for frequency is more accurate than that of exercise 19.

Exercise 21: *Consider the clamped–free beam shown in Figure* 20. *Use the Rayleigh–Ritz method to calculate the first two frequencies and modes. Use the displacement function*: $v(x, t) = (x/L)^2 p_1 + (x/L)^3 p_2$.

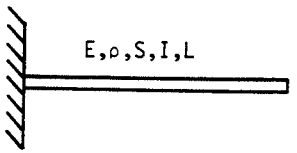


Figure 20

The kinetic and strain energies are respectively

$$T=\frac{1}{2}\int_0^L \rho S\left[\left(\frac{x}{L}\right)^2 p_1^{\circ}+\left(\frac{x}{L}\right)^3 p_2^{\circ}\right]^2 dx$$

$$=\frac{\rho SL}{10}p_1^{\circ 2}+\frac{\rho SL}{6}p_1^{\circ}p_2^{\circ}+\frac{\rho SL}{14}p_2^{\circ 2}$$

$$U=\frac{1}{2}\int_0^L EI\left(\frac{2}{L^2}p_1+\frac{6x}{L^3}p_2\right)^2 dx$$

$$=\frac{2EI}{L^3}p_1^2+\frac{6EI}{L^3}p_1p_2+\frac{6EI}{L^3}p_2^2$$

Application of Lagrange's equations gives

$$\rho SL\begin{bmatrix}\frac{1}{5} & \frac{1}{6}\\ \frac{1}{6} & \frac{1}{7}\end{bmatrix}\begin{bmatrix}p_1^{\infty}\\ p_2^{\infty}\end{bmatrix}+\frac{EI}{L^3}\begin{bmatrix}4 & 6\\ 6 & 12\end{bmatrix}\begin{bmatrix}p_1\\ p_2\end{bmatrix}=0$$

from which

$$\omega_1=\frac{3.533}{L^2}\sqrt{\frac{EI}{\rho S}} \quad \text{with the mode} \quad \begin{bmatrix}1\\ -0.3837\end{bmatrix}$$

and

$$\omega_2=\frac{34.81}{L^2}\sqrt{\frac{EI}{\rho S}} \quad \text{with the mode} \quad \begin{bmatrix}-0.8221\\ 1\end{bmatrix}$$

Hence

$$\phi_1(x)=\left(\frac{x}{L}\right)^2-0.3837\left(\frac{x}{L}\right)^3$$

$$\phi_2(x)=-0.8221\left(\frac{x}{L}\right)^2+\left(\frac{x}{L}\right)^3$$

Exercise 22: *Consider a beam which is simply supported at its two ends and which has a concentrated mass of* $m=\rho SL/5$ *at its quarter-point; see Figure* 21. *Use the Rayleigh–Ritz method to calculate its first two frequencies and their corresponding modes. Use a displacement function containing the terms* $\sin(\pi x/L)$, $\sin(2\pi x/L)$, $\sin(3\pi x/L)$.

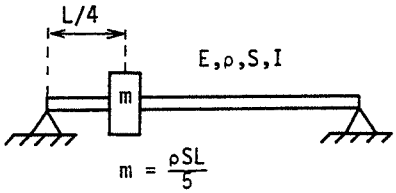


Figure 21

The results of exercise 9 give

$$U_{\text{beam}} = \frac{\pi^4 EI}{4L^3}(p_1^2 + 16p_2^2 + 81p_3^2)$$

$$T_{\text{beam}} = \frac{\rho SL}{4}(p_1^{\circ 2} + p_2^{\circ 2} + p_3^{\circ 2})$$

To the kinetic energy of the beam must be added the kinetic energy of the concentrated mass:

$$T_{\text{mass}} = \frac{\rho SL}{10}\left(\frac{\sqrt{2}}{2}p_1^{\circ} + p_2^{\circ} + \frac{\sqrt{2}}{2}p_3^{\circ}\right)^2$$

Using these energy expressions in Lagrange's equations gives

$$\rho SL\begin{bmatrix}0.6 & 0.1414 & 0.1\\ 0.1414 & 0.7 & 0.1414\\ 0.1 & 0.1414 & 0.6\end{bmatrix}\begin{bmatrix}p_1^{\infty}\\ p_2^{\infty}\\ p_3^{\infty}\end{bmatrix} + \frac{\pi^4 EI}{2L^3}\begin{bmatrix}1 & 0 & 0\\ 0 & 16 & 0\\ 0 & 0 & 81\end{bmatrix}\begin{bmatrix}p_1\\ p_2\\ p_3\end{bmatrix} = 0$$

from which

$$\omega_1 = \frac{8.991}{L^2}\sqrt{\frac{EI}{\rho S}} \quad \text{with the mode} \quad \begin{bmatrix}1\\ 0.0159\\ 0.0021\end{bmatrix}$$

and

$$\omega_2 = \frac{34.13}{L^2}\sqrt{\frac{EI}{\rho S}} \quad \text{with the mode} \quad \begin{bmatrix}1\\ -3.835\\ -0.1587\end{bmatrix}$$

Hence

$$\phi_1(x) = \sin\frac{\pi x}{L} + 0.0159\sin\frac{2\pi x}{L} + 0.0021\sin\frac{3\pi x}{L}$$

$$\phi_2(x) = \sin\frac{\pi x}{L} - 3.835\sin\frac{2\pi x}{L} - 0.1587\sin\frac{3\pi x}{L}$$

Exercise 23: *The system shown in Figure 22 consists of a clamped–free beam with a spring–mass system suspended from its free end. Use the Rayleigh–Ritz method to calculate the first three frequencies and modes of this system. For the beam use the displacement function:* $v(x,t)=(x/L)^2p_1+(x/L)^3p_2$ *and let* p_3 *be the displacement of the mass. Also,* $k=EI/L^3$ *and* $m=\rho SL/7$.

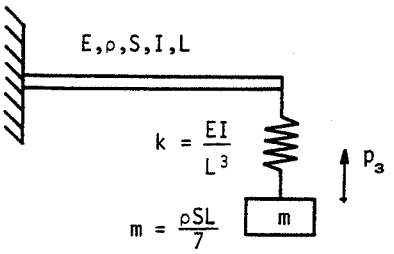


Figure 22

The kinetic energy of the mass m and the strain energy of the spring k must be added to the energies expressions found in exercise 21. These additions are

$$U_{\text{spring}}=\tfrac{1}{2}k(p_1+p_2-p_3)^2$$
$$=\frac{EI}{2L^3}(p_1^2+p_2^2+p_3^2+2p_1p_2-2p_2p_3-2p_1p_3)$$
$$T_{\text{mass}}=\tfrac{1}{2}mp_3{}^{\circ 2}=\frac{\rho SL}{14}p_3{}^{\circ 2}$$

Application of Lagrange's equations gives

$$\rho SL\begin{bmatrix}\frac{1}{5} & \frac{1}{6} & 0\\ \frac{1}{6} & \frac{1}{7} & 0\\ 0 & 0 & \frac{1}{7}\end{bmatrix}\begin{bmatrix}p_1^{\infty}\\ p_2^{\infty}\\ p_3^{\infty}\end{bmatrix}+\frac{EI}{L^3}\begin{bmatrix}5 & 7 & -1\\ 7 & 13 & -1\\ -1 & -1 & 1\end{bmatrix}\begin{bmatrix}p_1\\ p_2\\ p_3\end{bmatrix}=0$$

from which

$$\omega_1=\frac{2.143}{L^2}\sqrt{\frac{EI}{\rho S}};\qquad \phi_1=\begin{bmatrix}0.5308\\ -0.1871\\ 1\end{bmatrix}$$

$$\omega_2=\frac{4.346}{L^2}\sqrt{\frac{EI}{\rho S}};\qquad \phi_2=\begin{bmatrix}1\\ -0.4077\\ -0.3486\end{bmatrix}$$

$$\omega_3 = \frac{34.92}{L^2}\sqrt{\frac{EI}{\rho S}}; \qquad \phi_3 = \begin{bmatrix} -0.8215 \\ 1 \\ -0.0010 \end{bmatrix}$$

The exact frequencies calculated from program no. 10 are

$$\omega_1 = \frac{2.143}{L^2}\sqrt{\frac{EI}{\rho S}}; \qquad \omega_2 = \frac{4.320}{L^2}\sqrt{\frac{EI}{\rho S}}; \qquad \omega_3 = \frac{22.13}{L^2}\sqrt{\frac{EI}{\rho S}}$$

Exercise 24: *A clamped–free bar is subjected to a longitudinal sinusoidal exciting force $F \sin \Omega t$ at its free end as shown in Figure 23. Find the analytic solution for steady-state motion.*

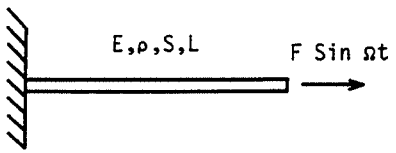


Figure 23

The solution to the equation of motion is

$$u(x, t) = (A \sin \Omega t + B \cos \Omega t)\left(C \sin \Omega \sqrt{\frac{\rho}{E}}\, x + D \cos \Omega \sqrt{\frac{\rho}{E}}\, x\right)$$

since $u(0, t) = 0$, $D = 0$ and

$$u(x, t) = (A^* \sin \Omega t + B^* \cos \Omega t) \sin \Omega \sqrt{\frac{\rho}{E}}\, x$$

At $x = L$,

$$P(L, t) = ES \left.\frac{\partial u}{\partial x}\right|_{x=L} = F \sin \Omega t$$

thus

$$ES\Omega \sqrt{\frac{\rho}{E}} \cos \Omega \sqrt{\frac{\rho}{E}}\, L (A^* \sin \Omega t + B^* \cos \Omega t) = F \sin \Omega t$$

which requires

$$B^* = 0, \qquad \text{and} \qquad A^* = \frac{F}{ES\Omega \sqrt{\frac{\rho}{E}} \cos \Omega \sqrt{\frac{\rho}{E}}\, L}$$

Finally,

$$u(x, t) = \frac{F}{ES\Omega \sqrt{\frac{\rho}{E}} \cos \Omega \sqrt{\frac{\rho}{E}}\, L} \sin \Omega \sqrt{\frac{\rho}{E}}\, x \sin \Omega t$$

Exercise 25: *A beam simply supported at each end is subjected to a force $F \sin \Omega t$ at its midpoint, as in Figure 24. Using the results of exercise 9, find*

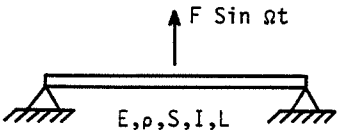


Figure 24

the steady-state motion. Plot

$$\left|\frac{V(L/2)}{F}\right|\frac{EI}{L^3} \quad \textit{as a function of} \quad f^* = \frac{f}{\frac{1}{L^2}\sqrt{\frac{EI}{\rho S}}}.$$

From exercise 9:

$$T = \sum_{n=1}^{\infty} T_n = \frac{\rho SL}{4}\sum_{n=1}^{\infty} p_n{}^{o2} = \frac{\rho SL}{4}(p_1{}^{o2} + p_2{}^{o2} + \ldots)$$

and

$$U = \sum_{n=1}^{\infty} U_n = \frac{\pi^4 EI}{4L^3}\sum_{n=1}^{\infty} n^4 p_n^2 = \frac{\pi^4 EI}{4L^3}(p_1^2 + 16p_2^2 + \ldots)$$

The generalized forces are deduced from the virtual work of the external force $F \sin \Omega t$.

$$\delta W = F \sin \Omega t \sum_{n=1}^{\infty} \sin\frac{n\pi}{2}\,\delta p_n = F \sin \Omega t \sum_{1,3,5,\ldots} (-1)^{(n-1)/2}\,\delta p_n$$

Application of Lagrange's equations gives a set of uncoupled equations of motion:

$$\frac{\rho SL}{2} p_n{}^{oo} + \frac{\pi^4 EIn^4}{2L^3} p_n = (-1)^{(n-1)/2} F \sin \Omega t$$

for $n = 1, 3, 5, \ldots$.

The equations of motion for n even do not contribute to the steady-state solutions since the generalized forces are zero. The steady-state solution is

$$p_n = \frac{2L^3}{\pi^4 EIn^4} \cdot \frac{(-1)^{(n-1)/2} F \sin \Omega t}{1 - (\Omega/\omega_n)^2}$$

Hence

$$v(x, t) = \sum_{n=1,3,\ldots} \phi_n(x) p_n = \sum_{n=1,3,\ldots} \sin\frac{n\pi x}{L}\, p_n = V(x) \sin \Omega t$$

The quantities M and $Q = \partial M/\partial x$ correspond to the second and third derivatives of the variable $v(x, t)$ with respect to space, x. It is then necessary to sum more modes to achieve accurate values of M and Q than it is for the lateral deflection v. The results are plotted in Figure 25.

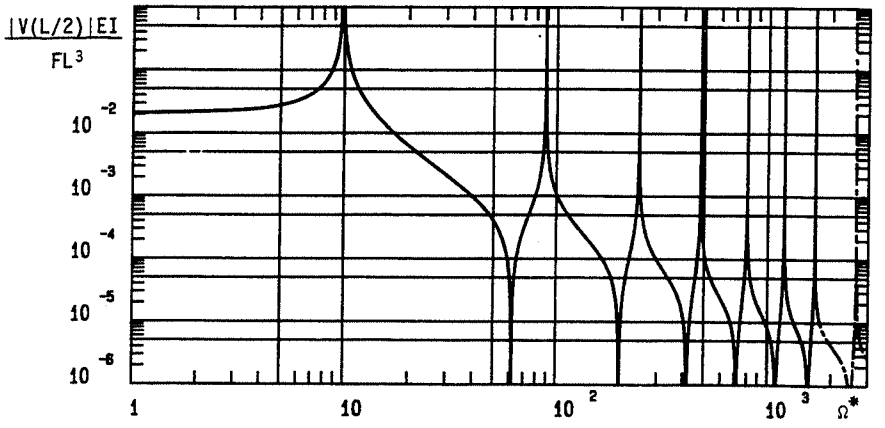


Figure 25

Exercise 26: *For the case of a rectangular plate of sides a, b which is simply supported on all four edges, calculate* ω_{rs} *using the displacement function given in equation* (112). *Plot the nodal lines of the first six modes for the case* $b = 2a/3$.

Using the expressions (113), (114) in Lagrange's equations gives

$$\omega_{rs} = \pi^2 \sqrt{\frac{D}{\rho h}} \left(\frac{r^2}{a^2} + \frac{s^2}{b^2} \right)$$

For $b = 2a/3$:

$$\omega_{rs} = \frac{\pi^2 h}{a^2} \sqrt{\frac{E}{12(1-\nu^2)\rho}} \cdot \left(r^2 + \frac{9s^2}{4} \right)$$

$$= A\omega_{rs}^*$$

where

$$A = \frac{\pi^2 h}{a^2} \sqrt{\frac{E}{12(1-\nu^2)\rho}} \quad \text{and} \quad \omega_{rs}^* = r^2 + \frac{9s^2}{4}$$

The nodal lines are plotted in Figure 26.

Exercise 27: *Consider the case of a circular plate of radius R and constant thickness h which is clamped at its center and free at its circumference. Use Rayleigh's method and the displacement function:*

$$w(r, \theta, t) = p_n[3(r/R)^2 - (r/R)^3] \cos n\theta$$

to calculate the frequencies f_n.

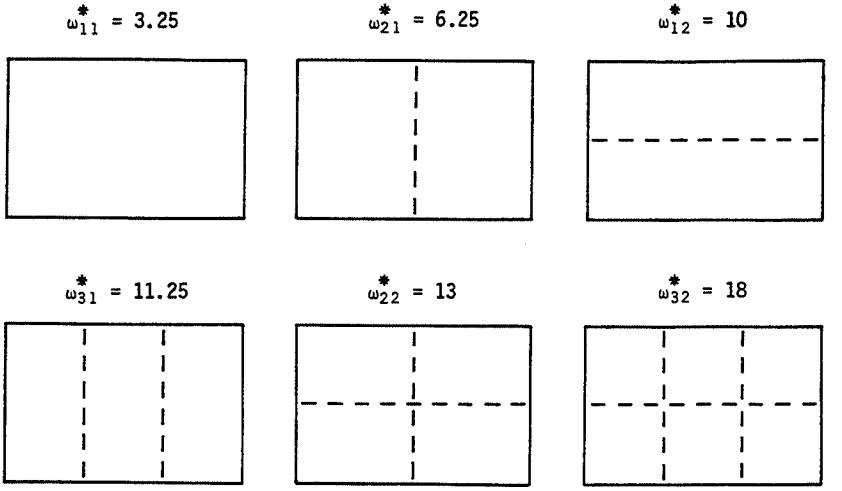


Figure 26

Substituting $w(r, \theta, t)$ in the expressions for strain energies (104) and kinetic energy (105) and completing all the calculations gives, for $n \neq 0$:

$$T_n = 1.206\rho h R p_n^{\circ 2}$$

$$U_n = 0.326 \frac{D}{R^2}[11n^4 - n^2(26+32\nu)+(45+36\nu)]p_n^2$$

The frequencies are then

$$f_n = \frac{0.026}{\sqrt{1-\nu^2}} \frac{h}{R^2} \sqrt{\frac{E}{\rho}} \sqrt{11n^4 - n^2(26+32\nu)+(45+36\nu)}$$

The formula for frequency is valid for $n = 0$ but in this case the kinetic and strain energies have their coefficients multiplied by a factor of two.

Exercise 28: *The system shown in Figure 27 represents a rotor simply supported at its ends on rigid bearings. The disk is rigid with properties* $I_{Dx} = 0.1225$ kg·m^2 (13 slug·in^2), $I_{Dy} = 0.2450$ kg·m^2 (26 slug·in^2), $M_D =$ 7.85 kg (0.538 slug). *The shaft has a length* $L = 0.4$ m, (15.75 in) *and area moment of inertia of its cross-section is:* $I_x = I_z = 0.49 \times 10^{-9}$ m^4 (1.177×10^{-3} in^4). *Young's modulus is* $E = 2 \times 10^{11}$ N/m^2 (29×10^6 psi). *The mass of the shaft is neglected. Taking the exact modes of a simply supported beam as displacement functions use the Rayleigh–Ritz method to calculate the first two frequencies of the rotor as a function of the angular velocity of rotation* $\Omega_r = 2\pi N$ *where* N *is the shaft speed in rev/sec.*

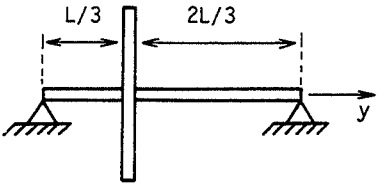


Figure 27

The displacement functions are

$$u(y, t) = \sin\frac{\pi y}{L} p_1$$

$$w(y, t) = \sin\frac{\pi y}{L} p_2$$

where

$$\theta = \frac{\partial w}{\partial y} = \frac{\pi}{L}\cos\frac{\pi y}{L} p_2$$

$$\psi = -\frac{\partial u}{\partial y} = -\frac{\pi}{L}\cos\frac{\pi y}{L} p_1$$

Using equations (131) the kinetic energy of the disk in SI units is

$$T_D = 3.888(p_1^{\circ 2} + p_2^{\circ 2}) - 3.778\Omega_r p_1^{\circ} p_2 + 0.1225\Omega_r^2$$

Using equation (145), the strain energy of the shaft in SI units is

$$U_s = 3.729 \times 10^4 (p_1^2 + p_2^2)$$

Applying Lagrange's equations gives

$$7.777 p_1^{\infty} - 3.778\Omega_r p_2^{\circ} + 7.458 \times 10^4 p_1 = 0$$

$$7.777 p_2^{\infty} + 3.778\Omega_r p_1^{\circ} + 7.458 \times 10^4 p_2 = 0$$

Solutions are sought in the form $p_1 = P_1 e^{rt}$ and $p_2 = P_2 e^{rt}$ which gives the characteristic equation

$$r^4 + (1.918 \times 10^4 + 0.2360\Omega_r^2) r^2 + 9.197 \times 10^7 = 0$$

Using $r = \pm j\omega$ in this gives $\omega = \omega(\Omega_r)$ and the frequency $f(\Omega_r) = \omega(\Omega_r)/2\pi$ (see Figure 28) for curves A and B which are the desired solution. Points (1) and (2) correspond to the equality between the speed of rotation of the rotor and its frequency. They are 12.8 Hz and 21.7 Hz. The results shown in the figure are the same in SI and English units.

Exercise 29: *Critical speed due to unbalance. The data are the same as in exercise* 28. *The unblance mass is* $m \ll M_D$. *This mass situated on z (Figure*

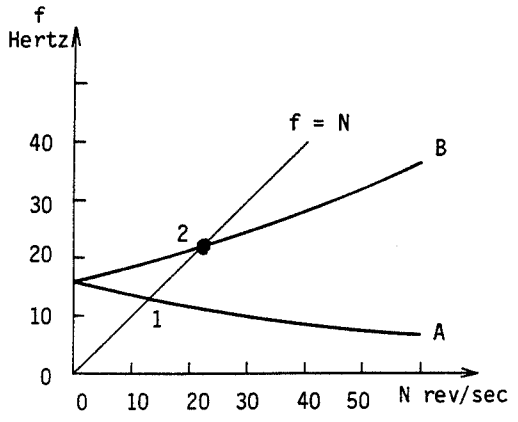


Figure 28

8), *is located on the disk at a radial distance d from its geometric center. Calculate the kinetic energy of this mass, find the equation of motion of the rotor system and its steady-state response.*

In the fixed coordinate system, the coordinates of the unbalanced mass are

$$\sin\frac{\pi}{3}p_1 + d\sin\Omega_r t = 0.8660p_1 + d\sin\Omega_r t$$

$$\sin\frac{\pi}{3}p_2 + d\cos\Omega_r t = 0.8660p_2 + d\cos\Omega_r t$$

The kinetic energy of the unbalance mass is

$$T_m = \frac{m}{2}(0.75p_1^{\circ 2} + 0.75p_2^{\circ 2} + 1.732d\Omega_r\cos\Omega_r t p_1^{\circ} - 1.732d\Omega_r\sin\Omega_r t p_2^{\circ} + d^2\Omega_r^2)$$

Hence

$$\frac{d}{dt}\left(\frac{\partial T_m}{\partial p_1^{\circ}}\right) - \frac{\partial T_m}{\partial p_1} = 0.75\, mp_1^{\circ\circ} - 0.8660d\Omega_r^2\sin\Omega_r t$$

$$\frac{d}{dt}\left(\frac{\partial T_m}{\partial p_2^{\circ}}\right) - \frac{\partial T_m}{\partial p_2} = 0.75mp_2^{\circ\circ} - 0.8660d\Omega_r^2\cos\Omega_r t$$

which are to be added to the equations of the motion found in exercise 28. However, since $m \ll M_D$ the terms of $0.75mp_1^{\circ\circ}$ and $0.75mp_2^{\circ\circ}$ can be neglected. The equations of motion for the rotor system with a mass

unbalance of m are then, in SI units:

$$7.777p_1^{\infty} - 3.778\Omega_r p_2^{\circ} + 7.458\times 10^4 p_1 = 0.8660 md\Omega_r^2 \sin \Omega_r t$$
$$7.777p_2^{\infty} + 3.778\Omega_r p_1^{\circ} + 7.458\times 10^4 p_2 = 0.8660 md\Omega_r^2 \cos \Omega_r t$$

If $\alpha = 0.1114 md\Omega_r^2$, these can be written

$$p_1^{\infty} - 0.4858\Omega_r p_2^{\circ} + 9.590\times 10^3 p_1 = \alpha \sin \Omega_r t$$
$$p_2^{\infty} + 0.4858\Omega_r p_1^{\circ} + 9.590\times 10^3 p_2 = \alpha \cos \Omega_r t$$

For steady-state response,

$$p_1 = P_1 \sin \Omega_r t$$
$$p_2 = P_2 \cos \Omega_r t$$

which after some algebra yields

$$P_1 = P_2 = \frac{\alpha}{-0.5142\Omega_r^2 + 9.590\times 10^3}$$

The critical speed is the value of Ω_r for which the steady-state response grows without bound. This corresponds to the value of Ω_r which makes the denominator of the above expression vanish, i.e. $\Omega_r = 136.6$ rad/sec or $N = 136.6/2\pi = 21.7$ rev/sec $= 21.7$ Hz. Note that the frequency 12.8 Hz found in exercise 28 is not a critical speed.

Exercise 30: *Consider the system of exercise* 28 *with a constant axial force* F_0 *applied to the shaft. Using the same method as in exercise* 28, *find the equations of motion of the system and plot the dependence of the first two frequencies on* N *for the values* $F_0 = 10^4$ *newtons* (2248 *lbf*).

One has

$$\frac{\partial w}{\partial y} = \frac{\pi}{L}\cos\frac{\pi y}{L} p_2; \qquad \frac{\partial u}{\partial y} = \frac{\pi}{L}\cos\frac{\pi y}{L} p_1$$

hence from equation (145) the additional strain energy due to the axial force is

$$U = \frac{F_0\pi^2}{4L}(p_1^2 + p_2^2)$$
$$= 2.4167\frac{F_0}{L}(p_1^2 + p_2^2)$$

The equations of motion of the system are then modified to:

$$7.777p_1^{\infty} - 3.778p_2^{\circ}\Omega_r + (12.34F_0 + 7.458\times 10^4)p_1 = 0$$
$$7.777p_2^{\infty} + 3.778p_1^{\circ}\Omega_r + (12.34F_0 + 7.458\times 10^4)p_2 = 0$$

in SI units. The frequencies are found as in exercise 28 and are plotted in Figure 29.

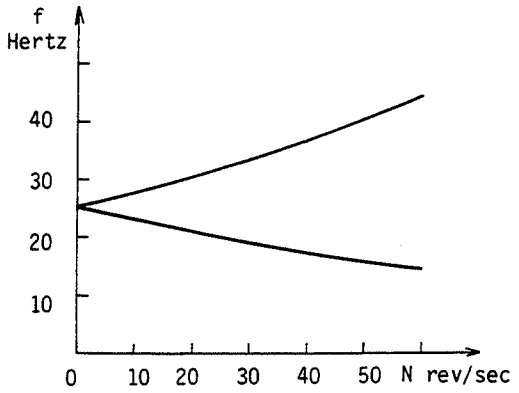


Figure 29

Exercise 31: *Consider a clamped–free beam whose deflection is given by equation* (78). *Use program no.* 12 *of Chapter* 7 *to calculate the kinetic and strain energies for the cases of* 2, 3, *and* 4 *Gauss points.*

The results of the computation are:

n	2	3	4
$\frac{L^3U}{EIp^2}$	6	6	6
$\frac{T}{\rho SLp^{\circ 2}}$	0.4769	0.4713	0.4714

In using n Gauss points, one gets an exact value of the integral of a polynomial of degree $2n-1$. U and T correspond, respectively, to the integration of a polynomial of degree 2 and degree 6. Therefore, it is sufficient to take $n=2$ to calculate U, and $n=4$ to calculate T.

Exercise 32: *Consider a clamped–free beam having the displacement function* $v(x, t)=(1-\cos \pi x/2L)p$. *Calculate the kinetic energy, the strain energy, and the frequency* ω *using program no.* 12 *with* 4 *Gauss points.*

The solution is

$$T=0.1134\rho SLp^{\circ 2}$$

$$U=1.522\frac{EI}{L^3}p^2$$

$$\omega=\frac{3.664}{L^2}\sqrt{\frac{EI}{\rho S}}$$

Exercise 33: *Discretization. Consider the same system as in exercise* 1. *The bar is modeled by* 5 *elements of length* $L/5$. *Each element is modeled by a mass* $(\rho SL/10)$*–spring* $(5ES/L)$*–mass* $(\rho SL/10)$ *system. Write the overall system mass and stiffness matrices. Using program no.* 5 *calculate the first two frequencies and modes.*

$$M=\frac{\rho SL}{5}\begin{bmatrix} 1 & & & & \text{zero} \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ \text{zero} & & & & \frac{1}{2} \end{bmatrix} \qquad K=\frac{5ES}{L}\begin{bmatrix} 2 & -1 & & & \text{Zero} \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ \text{zero} & & & -1 & \frac{6}{5} \end{bmatrix}$$

$$\omega_1=\frac{2.024}{L}\sqrt{\frac{E}{\rho}}, \qquad \phi_1=[1,\ 1.836,\ 2.371,\ 2.518,\ 2.252]^t$$

$$\omega_2=\frac{4.748}{L}\sqrt{\frac{E}{\rho}}, \qquad \phi_2=[1,\ 1.098,\ 0.2062,\ -0.8718,\ -1.164]^t$$

5
Calculation Using Finite Elements

In most cases, the complex geometries and nonclassical boundary conditions of real structures require that numerical methods must be used to determine their static or dynamic behavior. For such cases, the finite element method is widely used and some of its aspects are presented in this chapter.

The contents of the chapter are as follows:

5.1 Derivation of Equations

The finite element method can be presented simply in terms of the following steps:

(a) the structure is divided into elements of finite size, called finite elements, which are connected at certain points, called nodal points or nodes, situated on the boundary of the element;

(b) after making a reasonable hypothesis for the displacement field of element i, the kinetic energy T_i, the strain energy U_i, and the dissipation function R_i are calculated for that element as a function of the nodal point displacements

(c) if the structure is composed of N elements, then:

$$T = \sum_{i=1}^{N} T_i$$

$$U = \sum_{i=1}^{N} U_i \tag{1}$$

$$R = \sum_{i=1}^{N} R_i$$

The generalized forces are determined by writing the virtual work of the external forces.

The application of Lagrange's equations then allows the differential equations of motion of the whole structure to be obtained. In order to simplify the presentation of the method in what follows, it will be assumed that there is no dissipation of energy, i.e. $R_i = 0$ for all i.

5.1.1 Discretization: the division of the structure into finite elements

First, the type and distribution of finite elements must be chosen. This choice must take into account the geometry and the behavior of the structure. An effective and efficient discretization of a structure requires a great deal of experience. In particular, the analyst must take into account geometric discontinuities, material discontinuities, boundary conditions, and the forces applied to the structure.

For the accurate calculations of frequencies and modes or the calculation of dynamic response, the distribution of the finite elements, the so-called mesh, can be relatively coarse and regular. The mesh must be finer to calculate the stresses accurately and must be transformed into an even finer mesh in regions of stress concentation.

5.1.2 Strain energy – The stiffness matrix

In order to simplify the notation, the index i which designates quantities associated with the ith element is omitted in the following derivation.

The general expression for strain energy of an element is

$$U = \frac{1}{2} \int_{\tau} \varepsilon^{t} \sigma \, d\tau \tag{2}$$

The displacement vector d of an arbitrary point of the element is related to the nodal displacement vector δ of the element by means of a matrix N. The matrix N is generated from the hypothesis about the displacement field inside the element. The relation between d and δ is

$$d = N\delta \tag{3}$$

By differentiating (3), the relation between strains within the element and the nodal displacements is obtained:

$$\varepsilon = B\delta \tag{4}$$

In the case where there are no initial stresses, the stresses and strains are related by

$$\sigma = D\varepsilon \tag{5}$$

where D is a square symmetric matrix whose elements depend on the mechanical characteristics of the material, usually Young's modulus E and

Poisson's ratio ν. Using (5) and (4), equation (2) becomes

$$U = \frac{1}{2}\int_{\tau} (B\delta)^{t} DB\delta \, d\tau \tag{6}$$

$$= \tfrac{1}{2}\delta^{t}\left[\int_{\tau} B^{t}DB \, d\tau\right]\delta \tag{7}$$

which can be written

$$U = \tfrac{1}{2}\delta^{t} k\delta \tag{8}$$

with

$$k = \int_{\tau} B^{t}DB \, d\tau \tag{9}$$

being the stiffness matrix of the element. Matrix k is symmetric because if D is symmetric, the matrix product $B^{t}DB$ is also symmetric.

The determination of the strain energy U and stiffness matrix k will be illustrated for the cases of a bar in longitudinal motion and a beam in bending.

Bar in longitudinal motion

An element of the bar is shown in Figure 1. It has two nodes with one degree of freedom at each node; namely u, the longitudinal displacement.

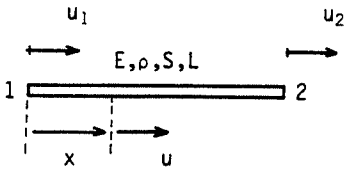


Figure 1 A bar in longitudinal motion

The nodal displacement vector is

$$\delta = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{10}$$

Since there are two nodal displacements, the displacement hypothesis is chosen to have two constants:

$$u(x) = a_1 + a_2 x \tag{11}$$

The constants a_1 and a_2 are obtained from the values of $u(x)$ at the two nodes; that is,

$$\begin{aligned} &\text{at } x = 0, \quad && u_1 = a_1 \\ &\text{at } x = L, \quad && u_2 = a_1 + a_2 L \end{aligned} \tag{12}$$

Equations (12) allow a_1 and a_2 to be determined in terms of u_1 and u_2 and substituted into (11) to give

$$u(x) = \left[1 - \frac{x}{L}, \frac{x}{L}\right]\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{13}$$

This relation corresponds to (3). The polynomials in the N matrix, in this case $1 - x/L$ and x/L, are called the shape functions.

Since

$$\sigma = \sigma_x = E\varepsilon_x = E\varepsilon \tag{14}$$

with

$$\varepsilon = \frac{\partial u}{\partial x} \tag{15}$$

the matrix D reduces to the scalar E. Using (13), (15) becomes

$$\varepsilon = \left[-\frac{1}{L}, \frac{1}{L}\right]\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{16}$$

which corresponds to (4) and identifies B as

$$B = \left[-\frac{1}{L}, \frac{1}{L}\right] \tag{17}$$

Substituting (17) into (7) gives

$$U = \frac{1}{2}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}^t \int_0^L ES \begin{bmatrix} \frac{1}{L^2} & -\frac{1}{L^2} \\ -\frac{1}{L^2} & \frac{1}{L^2} \end{bmatrix} dx \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{18}$$

$$= \frac{1}{2}\frac{ES}{L}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}^t \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{19}$$

and the stiffness matrix of the element is then

$$k = \frac{ES}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{20}$$

This matrix is singular, i.e. its determinant is zero. This is due to the fact that the displacement function (11) allows rigid-body translation.

Beam in bending

An element of the beam is shown in Figure 2. It has two nodes with two degrees of freedom at each node; namely v, the lateral displacement, and ψ, the slope. In what follows, only the case of a Bernoulli–Euler beam is considered. This means that the secondary effects of transverse shear and rotatory inertia are neglected.

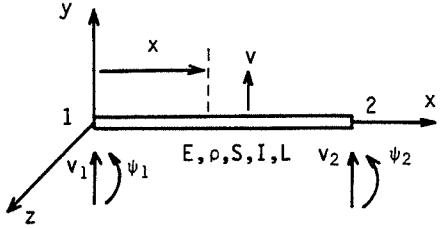


Figure 2 A beam in bending

The nodal displacement vector is

$$\delta = [v_1, \psi_1, v_2, \psi_2]^t \tag{21}$$

with

$$\psi = \frac{\partial v}{\partial x} \tag{22}$$

Since there are four nodal displacements, the displacement hypothesis is chosen to have four constants:

$$v = a_1 + a_2 x + a_3 x^2 + a_4 x^3 \tag{23}$$

Then from (22), the slope is

$$\psi = a_2 + 2a_3 x + 3a_4 x^2 \tag{24}$$

The constants a_1, a_2, a_3, a_4 are obtained from the values of v and ψ at the two nodes; that is,

at $x = 0$:

$$\begin{aligned} v_1 &= a_1 \\ \psi_1 &= a_2 \end{aligned} \tag{25}$$

at $x = L$:

$$\begin{aligned} v_2 &= a_1 + a_2 L + a_3 L^2 + a_4 L^3 \\ \psi_2 &= a_2 + 2a_3 L + 3a_4 L^2 \end{aligned} \tag{26}$$

The constants a_1, a_2, a_3, a_4 can be determined from (25) and (26) in terms of v_1, ψ_1, v_2, ψ_2. Substituting these constants into (23) gives

$$v(x) = \left[1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3}, \quad x - \frac{2x^2}{L} + \frac{x^3}{L^2}, \frac{3x^2}{L^2} - \frac{2x^3}{L^3}, \quad -\frac{x^2}{L} + \frac{x^3}{L^2}\right] \begin{bmatrix} v_1 \\ \psi_1 \\ v_2 \\ \psi_2 \end{bmatrix} \tag{27}$$

This equation corresponds to (3). For this case, the matrix D also reduces to a scalar equal to E. The bending stress and strain are related by (14), where

now

$$\varepsilon = \varepsilon_x = -y\frac{\partial^2 v}{\partial x^2} \tag{28}$$

Using (27), ε becomes

$$\varepsilon = -y\left[-\frac{6}{L^2}+\frac{12x}{L^3}, \quad -\frac{4}{L}+\frac{6x}{L^2}, \frac{6}{L^2}-\frac{12x}{L^3}, \quad -\frac{2}{L}+\frac{6x}{L^2}\right]\begin{bmatrix} v_1 \\ \psi_1 \\ v_2 \\ \psi_2 \end{bmatrix} \tag{29}$$

This equation corresponds to (4) and identifies B. Using this expression for B in (7) and completing all the calculations gives

$$U = \frac{1}{2}\frac{EI}{L^3}\begin{bmatrix} v_1 \\ \psi_1 \\ v_2 \\ \psi_2 \end{bmatrix}^t \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{bmatrix} v_1 \\ \psi_1 \\ v_2 \\ \psi_2 \end{bmatrix} \tag{30}$$

and the stiffness matrix of the element is then

$$k = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix} \tag{31}$$

This matrix is singular and of rank two; this means that only two of the vectors which compose k are linearly independent. This is because the displacement function (23) allows two rigid-body motions, one in translation and one in rotation.

5.1.3 Kinetic energy – The mass matrix

The general expression for kinetic energy of an element is

$$T = \frac{1}{2}\int_\tau \rho V^2 \, d\tau \tag{32}$$

where $V = d^\circ$. Using (3), the velocity V becomes

$$V = N\delta^\circ \tag{33}$$

Substituting this into (32) gives

$$T = \frac{1}{2}\int_\tau \rho(N\delta^\circ)^t N\delta^\circ \, d\tau \tag{34}$$

which can be written in the form

$$T = \tfrac{1}{2}\delta^{\circ t} m \delta^{\circ} \tag{35}$$

where

$$m = \int_{\tau} \rho N^t N \, d\tau \tag{36}$$

is a symmetric mass matrix. The mass matrix defined by (36) is called the consistent mass matrix because it is calculated using the same shape functions as the stiffness matrix; in this sense, the calculations of k and m are consistent.

Instead of a consistent mass matrix, a lumped mass matrix m_l is often used. This matrix is obtained by assuming that the mass of the element is concentrated, or lumped, at the nodes of the element. The advantage of lumping the mass is that the mass matrix becomes easy to construct and is diagonal. A diagonal mass matrix is convenient because it conserves computer memory and simplifies the algorithms used to compute frequencies and modes. For several elements, such as a bar in longitudinal motion, it is observed that the lumped mass matrix m_l is as efficient as the consistent mass matrix m.

Bar in longitudinal motion

From (13):

$$N = \left[1 - \frac{x}{L}, \frac{x}{L}\right] \tag{37}$$

and combining this with (34) gives

$$T = \tfrac{1}{2}\rho S \begin{bmatrix} u_1^{\circ} \\ u_2^{\circ} \end{bmatrix}^t \int_0^L \begin{bmatrix} \left(1-\frac{x}{L}\right)^2 & \frac{x}{L}\left(1-\frac{x}{L}\right) \\ \frac{x}{L}\left(1-\frac{x}{L}\right) & \left(\frac{x}{L}\right)^2 \end{bmatrix} dx \begin{bmatrix} u_1^{\circ} \\ u_2^{\circ} \end{bmatrix} \tag{38}$$

$$= \frac{1}{2}\frac{\rho SL}{6} \begin{bmatrix} u_1^{\circ} \\ u_2^{\circ} \end{bmatrix}^t \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} u_1^{\circ} \\ u_2^{\circ} \end{bmatrix} \tag{39}$$

Hence the consistent mass matrix is

$$m = \frac{\rho SL}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{40}$$

The lumped mass matrix m_l is obtained as follows:

$$T = \frac{1}{2}\frac{\rho SL}{2}(u_1^{\circ 2} + u_2^{\circ 2}) \tag{41}$$

$$= \frac{1}{2}\frac{\rho SL}{2} \begin{bmatrix} u_1^{\circ 2} \\ u_2^{\circ 2} \end{bmatrix}^t \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1^{\circ} \\ u_2^{\circ} \end{bmatrix} \tag{42}$$

Hence

$$m_1 = \frac{\rho SL}{2}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \tag{43}$$

Beam in bending

Using the shape functions given by (27) in expression (34) and completing all the calculations, results in:

$$T = \frac{1}{2}\frac{\rho SL}{420}\begin{bmatrix}v_1^\circ\\ \psi_1^\circ\\ v_2^\circ\\ \psi_2^\circ\end{bmatrix}^t\begin{bmatrix}156 & 22L & 54 & -13L\\ 22L & 4L^2 & 13L & -3L^2\\ 54 & 13L & 156 & -22L\\ -13L & -3L^2 & -22L & 4L^2\end{bmatrix}\begin{bmatrix}v_1^\circ\\ \psi_1^\circ\\ v_2^\circ\\ \psi_2^\circ\end{bmatrix} \tag{44}$$

and, by inspection, the consistent mass matrix is

$$m = \frac{\rho SL}{420}\begin{bmatrix}156 & 22L & 54 & -13L\\ 22L & 4L^2 & 13L & -3L^2\\ 54 & 13L & 156 & -22L\\ -13L & -3L^2 & -22L & 4L^2\end{bmatrix} \tag{45}$$

The lumped mass matrix is obtained from

$$T = \frac{1}{2}\frac{\rho SL}{2}(v_1^{\circ 2} + v_2^{\circ 2}) \tag{46}$$

Hence

$$m_1 = \frac{\rho SL}{2}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\end{bmatrix} \tag{47}$$

In bending, the consistent mass matrix is more efficient than the lumped mass matrix because the latter does not take into account the effect of beam element rotation.

5.1.4 Assembly

To assemble element properties into structure properties it is necessary to write that the kinetic energy and the strain energy of the structure are the sum of the kinetic energy and strain energy of each of the elements. The generalized forces are obtained from virtual work. Lagrange's equations are then used to give the classical form of the differential equations of motion for the whole structure; that is,

$$M\delta^{\circ\circ} + K\delta = F(t) \tag{48}$$

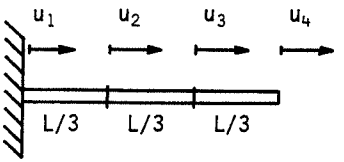


Figure 3 Clamped–free bar modeled by three finite elements

where the structure mass matrix M is symmetric, positive definite, and the structure stiffness matrix K is symmetric, positive semi-definite. The vectors δ and $F(t)$ contain all the nodal displacements and generalized forces. In general, the mass and stiffness matrices have a banded form.

The process of assembly is illustrated by considering a fixed–free bar of length L undergoing longitudinal motion and which has been discretized into three finite elements of the type shown in Figure 1. As shown in Figure 3, the bar has a force $F(t)$ applied to node 3. Summing the elements strain energies (19), and accounting for the boundary condition $u_1 = 0$, gives

$$U = \frac{1}{2}\frac{3ES}{L}\begin{bmatrix}0\\u_2\end{bmatrix}^t\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}0\\u_2\end{bmatrix} + \frac{1}{2}\frac{3ES}{L}\begin{bmatrix}u_2\\u_3\end{bmatrix}^t\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}u_2\\u_3\end{bmatrix} + \frac{1}{2}\frac{3ES}{L}\begin{bmatrix}u_3\\u_4\end{bmatrix}^t\begin{bmatrix}1 & -1\\-1 & 1\end{bmatrix}\begin{bmatrix}u_3\\u_4\end{bmatrix} \tag{49}$$

Applying Lagrange's equations gives

$$\begin{bmatrix}\frac{\partial U}{\partial u_2}\\ \frac{\partial U}{\partial u_3}\\ \frac{\partial U}{\partial u_4}\end{bmatrix} = \frac{3ES}{L}\begin{bmatrix}2 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix}\begin{bmatrix}u_2\\u_3\\u_4\end{bmatrix} = K\delta \tag{50}$$

It can be observed that the structure stiffness matrix K in (50) can be obtained by a superposition of the three element stiffness matrices k given by (20) provided L is replaced by $L/3$, and the rows and columns corresponding to $u_1 = 0$ are deleted.

Similarly, using the consistent mass matrix formulation of (39) gives

$$T = \frac{1}{2}\frac{\rho SL}{18}\begin{bmatrix}0\\u_2^\circ\end{bmatrix}^t\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}0\\u_2^\circ\end{bmatrix} + \frac{1}{2}\frac{\rho SL}{18}\begin{bmatrix}u_2^\circ\\u_3^\circ\end{bmatrix}^t\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}u_2^\circ\\u_3^\circ\end{bmatrix} + \frac{1}{2}\frac{\rho SL}{18}\begin{bmatrix}u_3^\circ\\u_4^\circ\end{bmatrix}^t\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}\begin{bmatrix}u_3^\circ\\u_4^\circ\end{bmatrix} \tag{51}$$

and application of Lagrange's equations gives

$$\begin{bmatrix} \frac{d}{dt}\left(\frac{\partial T}{\partial u_2^{\circ}}\right) \\ \frac{d}{dt}\left(\frac{\partial T}{\partial u_3^{\circ}}\right) \\ \frac{d}{dt}\left(\frac{\partial T}{\partial u_4^{\circ}}\right) \end{bmatrix} = \frac{\rho SL}{18}\begin{bmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}\begin{bmatrix} u_2^{\infty} \\ u_3^{\infty} \\ u_4^{\infty} \end{bmatrix} = M\delta^{\infty} \tag{52}$$

where, again, M can be obtained by a superposition of the element mass matrices m.

Using (42) and recalculating (51) for the lumped mass matrix formulation gives

$$T = \frac{\rho SL}{6} u_2^{\circ 2} + \frac{\rho SL}{6} u_3^{\circ 2} + \frac{\rho SL}{12} u_4^{\circ 2} \tag{53}$$

from which the structure lumped mass matrix M_1 can be shown to be

$$M_1 = \frac{\rho SL}{6}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{54}$$

The force $F(t)$ acting at **node** 3 results in the external force vector

$$[0, F(t), 0]^{\mathrm{T}} \tag{55}$$

In summary, this section has presented systematic ways of deriving and assembling the stiffness and mass matrices of simple elements. The library of finite elements available in many structural dynamics computer programs is much more extensive than can be presented here. It is also worth pointing out the connection between the Rayleigh–Ritz method which was used in Chapters 3 and 4 and the finite element method used in this chapter. The Rayleigh–Ritz method uses a single displacement hypothesis which spans the entire structure; the finite element method uses many similar displacements functions, each being defined over a different element. In a broad sense, the finite element method is a piecewise version of the Rayleigh–Ritz method.

5.2 Frequencies and Modes; Response to an Excitation

The frequencies and modes are obtained from the homogeneous equation of motion:

$$M\delta^{\infty} + K\delta = 0 \tag{56}$$

Let

$$\delta = \delta_0 e^{j\omega t} \tag{57}$$

then (56) becomes

$$\omega^2 M\delta_0 = K\delta_0 \tag{58}$$

This is the classical algebraic eigenvalue–eigenvector problem discussed in Chapter 3.

For a finite element formulation of (56), the matrices M and K can have several hundreds to several thousands degrees of freedom. Extracting the frequencies and modes from such a large system requires computer methods which are specifically adapted to minimize the amount of execution time and the size of the required core memory. The details of such specific eigenvalue algorithms are too specialized for this text. However, a discussion will be given of two aspects which are particularly useful in the numerical calculation of large systems.

Rigid-body modes

In order to avoid problems due to zero values of frequency, it is useful to associate with equation (58) the identity

$$aM\delta_0 = aM\delta_0 \tag{59}$$

Combining (58) and (59) gives

$$(\omega^2 + a)M\delta_0 = (K + aM)\delta_0 \tag{60}$$

which can be written

$$\omega^{*2}M\delta_0 = K^*\delta_0 \tag{61}$$

The frequencies found from (61) are not zero, since

$$\begin{aligned} \omega_1^{*2} &= \omega_1^2 + a = a \\ &\vdots \\ \omega_i^{*2} &= \omega_i^2 + a \\ &\vdots \end{aligned} \tag{62}$$

The modes associated with ω_i^* are the same as those associated with ω_i of the initial system. The constant a is positive and must be choosen with equation (62) in mind: that is, $\omega_i^2 + a$ must not be $\simeq \omega_i^2$ or $\simeq a$.

Diagonal mass matrix

Let k_{ij} be the elements of K, and m_{ii} the elements of M where M is diagonal (i.e. $m_{ij} = 0$, $i \neq j$). The standard form of (58) is

$$\omega^2\delta_0 = A\delta_0 \tag{63}$$

where $A = M^{-1}K$. The matrix A is not symmetric and this complicates the determination of the eigenvalues and eigenvectors. In addition, A cannot be represented by its semi-bandwidth and this increases the size of core memory which must be used. These difficulties can be avoided by adopting the change of variable

$$\delta_i^* = \sqrt{m_{ii}}\,\delta_i \tag{64}$$

which allows (56) to be rewritten as

$$I\delta^{*\infty} + K^*\delta^* = 0 \tag{65}$$

where I is the unit matrix and

$$k_{ij}^* = \frac{k_{ij}}{\sqrt{m_{ii}m_{jj}}} \tag{66}$$

Clearly, $A^* = I^{-1}K^* = K^*$ and is symmetric.

Response to an excitation

In determining the response for a system which has been discretized by finite elements, one proceeds as in Chapter 3; that is, one uses a modal or pseudo-modal method or, in the case of an arbitrary excitation, a step-by-step method which can, if necessary, be combined with the modal or pseudo-modal method.

5.3 Structural Modifications

In practice it is useful to have a method of calculating how minor changes in the components of a structure affect the frequencies of the whole structure without being obliged to recalculate the new frequencies. The finite element method can provide such a method and thereby allows economies in manpower and computer time.

5.3.1 Influence of a small structural modification

Let ω_i, ϕ_i, $\phi_i^t M\phi_i$, $\phi_i^t K\phi_i$ be the ith frequency, mode, modal mass, and modal stiffness of the unmodified structure. Then one has

$$\omega_i^2 \phi_i^t M\phi_i = \phi_i^t K\phi_i \tag{67}$$

After modification, the mass matrix becomes $M+\Delta M$ and the stiffness matrix becomes $K+\Delta K$. The frequencies and modes become $\omega_i+\Delta\omega_i$ and $\phi_i+\Delta\phi_i$ and, instead of (67), one has

$$(\omega_i+\Delta\omega_i)^2(\phi_i+\Delta\phi_i)^t(M+\Delta M)(\phi_i+\Delta\phi_i) = (\phi_i+\Delta\phi_i)^t(K+\Delta K)(\phi_i+\Delta\phi_i) \tag{68}$$

Expanding (68) and keeping only first-order terms gives

$$\Delta\omega_i = \frac{\omega_i}{2}\frac{\phi_i^t \Delta K\phi_i}{\phi_i^t K\phi_i} - \frac{\omega_i}{2}\frac{\phi_i^t \Delta M\phi_i}{\phi_i^t M\phi_i} \tag{69}$$

The same expression may be found by using Rayleigh's method rather than starting with (67). To see this, suppose that the modes are not

significantly changed when M and K are modified. Then for mode ϕ_i:

$$T_i = \tfrac{1}{2}\phi_i^t(M+\Delta M)\phi_i p_i^{\circ 2} \tag{70}$$

$$U_i = \tfrac{1}{2}\phi_i^t(K+\Delta K)\phi_i p_i^2 \tag{71}$$

and Rayleigh's method leads to

$$\omega_i^{*2} = \frac{\phi_i^t K\phi_i + \phi_i^t \Delta K\phi_i}{\phi_i^t M\phi_i + \phi_i^t \Delta M\phi_i} \tag{72}$$

which can be written

$$\omega_i^{*2} = \omega_i^2 \frac{1+\phi_i^t \Delta K\phi_i/\phi_i^t K\phi_i}{1+\phi_i^t \Delta M\phi_i/\phi_i^t M\phi_i} \tag{73}$$

or

$$\omega_i^* \simeq \omega_i + \frac{\omega_i}{2}\frac{\phi_i^t \Delta K\phi_i}{\phi_i^t K\phi_i} - \frac{\omega_i}{2}\frac{\phi_i^t \Delta M\phi_i}{\phi_i^t M\phi_i} \tag{74}$$

and, as $\omega_i^* - \omega_i = \Delta\omega_i$, equation (69) is exactly recovered.

In using (69) or (74), it is only necessary to know ΔK and ΔM and to compute the matrix products $\phi_i^t \Delta K\phi_i$ and $\phi_i^t \Delta M\phi_i$. These operations can be performed for many different structural modifications without the expense of calculating the frequencies of the modified structure.

5.3.2 How to modify a structure

It is very difficult to anticipate the influence of a particular modification on the dynamic behavior of a complex structure. In the process of analyzing a structure for its frequencies and modes, it is easy to determine the quantities $\phi_i^t M\phi_i$ and $\phi_i^t K\phi_i$ for each finite element. The kinetic and strain energy corresponding to a mode ω_i, ϕ_i can then be written as a sum over the N elements, i.e.

$$\begin{aligned} T_i &= \tfrac{1}{2}\phi_i^t M\phi_i p_i^{\circ 2} \\ &= \frac{1}{2}\sum_{r=1}^{N} (\phi_{ir}^t m_r \phi_{ir}) p_i^{\circ 2} \end{aligned} \tag{75}$$

$$\begin{aligned} U_i &= \tfrac{1}{2}\phi_i^t K\phi_i p_i^2 \\ &= \frac{1}{2}\sum_{r=1}^{N} (\phi_{ir}^t k_r \phi_{ir}) p_i^2 \end{aligned} \tag{76}$$

where k_r and m_r are the stiffness and mass matrices of element r and ϕ_{ir} is the portion of the vector ϕ_i associated with the motion of the rth element.

It is sufficient to examine the relative importance of these terms for a mode to establish the influence of a particular finite element or group of finite elements on the behavior of the whole structure. It is then possible to choose a good location for a structural modification such as local mass or stiffness, or even a layer of damping material. Another important part of

finite element analysis of structures which have a large number of degrees of freedom is the judgment as to whether the answers for frequencies and modes are correct. With the vast quantity of computer output in such cases, it is very difficult to use physical intuition to validate the answers. Ways have to be found to condense the output data so that its validity can be established by intuition or by simplified analysis. One of the most effective ways of doing this is to calculate and display the distribution of element kinetic and strain energy. Such displays provide concise information about motion and stress distribution and facilitate judgments about correctness.

5.4 The Method of Substructures

This method is used to predict the dynamic behavior of a complex structure by dividing the structure into a series of smaller structures, called substructures, and studying the dynamic characteristics of these components. Substructuring allows a considerable reduction in the number of degrees of freedom which would otherwise be required to model the entire structure. This method can also be extended so that some substructures are studied analytically and some experimentally; however, this refinement will not be discussed here. In order to simplify the following presentation, it is supposed that the structure is divided into just two substructures; see Figure 4. The first substructure SS1 is connected to the second substructure SS2 by common degrees of freedom denoted by δ_c. Each substructure has, in addition, its own internal degrees of freedom denoted by $\delta_i^{(1)}$ and $\delta_i^{(2)}$. The same approach as described in section 5.1 is used to obtain the equations of motion. It is first necessary to write the kinetic energy and strain energy of the substructures and to write the virtual work of their external forces. Let $F^{(1)}$, $F^{(2)}$ be the external forces acting on the internal degrees of freedom $\delta_i^{(1)}$, $\delta_i^{(2)}$ and assume, for simplification, that there are no external forces

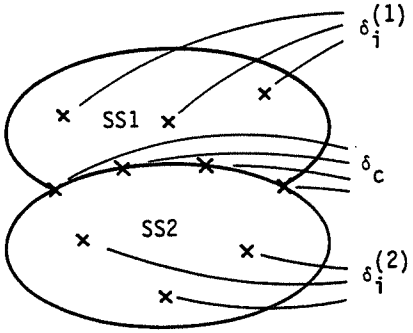


Figure 4 Structure decomposed into two substructures

associated with δ_c. Using the additive property of the energies:

$$T = T^{(1)} + T^{(2)}$$
$$U = U^{(1)} + U^{(2)} \tag{77}$$

Adopting the finite element formulation of T and U and partitioning the matrices to separate the internal degrees of freedom $\delta_i^{(1)}$, $\delta_i^{(2)}$ from the connection degrees of freedom δ_c gives

$$T^{(1)} = \frac{1}{2}\begin{bmatrix}\delta_i^{\circ(1)} \\ \delta_c^{\circ}\end{bmatrix}^t \begin{bmatrix}m_{ii}^{(1)} & m_{ic}^{(1)} \\ m_{ci}^{(1)} & m_{cc}^{(1)}\end{bmatrix}\begin{bmatrix}\delta_i^{\circ(1)} \\ \delta_c^{\circ}\end{bmatrix} \tag{78}$$

$$U^{(1)} = \frac{1}{2}\begin{bmatrix}\delta_i^{(1)} \\ \delta_c\end{bmatrix}^t \begin{bmatrix}k_{ii}^{(1)} & k_{ic}^{(1)} \\ k_{ci}^{(1)} & k_{cc}^{(1)}\end{bmatrix}\begin{bmatrix}\delta_i^{(1)} \\ \delta_c\end{bmatrix} \tag{79}$$

$$T^{(2)} = \frac{1}{2}\begin{bmatrix}\delta_i^{\circ(2)} \\ \delta_c^{\circ}\end{bmatrix}^t \begin{bmatrix}m_{ii}^{(2)} & m_{ic}^{(2)} \\ m_{ci}^{(2)} & m_{cc}^{(2)}\end{bmatrix}\begin{bmatrix}\delta_i^{\circ(2)} \\ \delta_c^{\circ}\end{bmatrix} \tag{80}$$

$$U^{(2)} = \frac{1}{2}\begin{bmatrix}\delta_i^{(2)} \\ \delta_c\end{bmatrix}^t \begin{bmatrix}k_{ii}^{(2)} & k_{ic}^{(2)} \\ k_{ci}^{(2)} & k_{cc}^{(2)}\end{bmatrix}\begin{bmatrix}\delta_i^{(2)} \\ \delta_c\end{bmatrix} \tag{81}$$

Different variations of the method of substructures are characterized by the changes in variable introduced into (78), (79), (80), (81), and the virtual work expression. Two frequently used methods are presented below.

5.4.1 Method using free modes

In this method, the vibration modes ϕ of each substructure are calculated supposing the connection nodes are free. The kinetic and strain energies and the virtual work of the external forces are expressed in terms of these free modes. The corresponding variables are:

- $q^{(1)}$ SS1 generalized coordinates
- $q^{(2)}$ SS2 generalized coordinates
- δ_c the physical coordinates connecting SS1 to SS2.

Using (78) and (79), substructure 1, which is free at its boundaries and without external forces, has a dynamic behavior defined by

$$\begin{bmatrix}m_{ii}^{(1)} & m_{ic}^{(1)} \\ m_{ci}^{(1)} & m_{cc}^{(1)}\end{bmatrix}\begin{bmatrix}\delta_i^{\infty(1)} \\ \delta_c^{\infty}\end{bmatrix} + \begin{bmatrix}k_{ii}^{(1)} & k_{ic}^{(1)} \\ k_{ci}^{(1)} & k_{cc}^{(1)}\end{bmatrix}\begin{bmatrix}\delta_i^{(1)} \\ \delta_c\end{bmatrix} = 0 \tag{82}$$

The solution of this system supplies a set of modes which can be assembled into the modal matrix $\phi^{(1)}$ and this then gives

$$\begin{bmatrix}\delta_i^{(1)} \\ \delta_c\end{bmatrix} = \begin{bmatrix}\phi_i^{(1)} \\ \phi_c^{(1)}\end{bmatrix} q^{(1)} \tag{83}$$

According to (83),

$$\delta_i^{(1)} = \phi_i^{(1)} q^{(1)} \tag{84}$$

and substituting this into (78) and (79) gives

$$T^{(1)} = \frac{1}{2}\begin{bmatrix} q^{o(1)} \\ \delta_c^{o} \end{bmatrix}^t \begin{bmatrix} \phi_i^{(1)t} m_{ii}^{(1)} \phi_i^{(1)} & \phi_i^{(1)t} m_{ic}^{(1)} \\ m_{ci}^{(1)} \phi_i^{(1)} & m_{cc}^{(1)} \end{bmatrix} \begin{bmatrix} q^{o(1)} \\ \delta_c^{o} \end{bmatrix} \tag{85}$$

$$U^{(1)} = \frac{1}{2}\begin{bmatrix} q^{(1)} \\ \delta_c \end{bmatrix}^t \begin{bmatrix} \phi_i^{(1)t} k_{ii}^{(1)} \phi_i^{(1)} & \phi_i^{(1)t} k_{ic}^{(1)} \\ k_{ci}^{(1)} \phi_i^{(1)} & k_{cc}^{(1)} \end{bmatrix} \begin{bmatrix} q^{(1)} \\ \delta_c \end{bmatrix} \tag{86}$$

The energies $T^{(2)}$ and $U^{(2)}$ will have expressions identical to (85) and (86), except superscript (1) is replaced by (2).

Since here $F_c = 0$, the force vector corresponding to $\delta_i^{(1)}$, $\delta_i^{(2)}$, δ_c is

$$[F^{(1)}, F^{(2)}, 0]^t \tag{87}$$

In the light of (84) and of a similar expression for SS2, the generalized force vector corresponding to $q^{(1)}$, $q^{(2)}$, δ_c is

$$[\phi_i^{(1)t} F^{(1)}, \phi_i^{(2)t} F^{(2)}, 0]^t \tag{88}$$

Applying Lagrange's equations to the structure then gives

$$\begin{bmatrix} \phi_i^{(1)t} m_{ii}^{(1)} \phi_i^{(1)} & 0 & \phi_i^{(1)t} m_{ic}^{(1)} \\ 0 & \phi_i^{(2)t} m_{ii}^{(2)} \phi_i^{(2)} & \phi_i^{(2)t} m_{ic}^{(2)} \\ m_{ci}^{(1)} \phi_i^{(1)} & m_{ci}^{(2)} \phi_i^{(2)} & m_{cc}^{(1)} + m_{cc}^{(2)} \end{bmatrix} \begin{bmatrix} q^{oo(1)} \\ q^{oo(2)} \\ \delta_c^{oo} \end{bmatrix} +$$

$$\begin{bmatrix} \phi_i^{(1)t} k_{ii}^{(1)} \phi_i^{(1)} & 0 & \phi_i^{(1)t} k_{ic}^{(1)} \\ 0 & \phi_i^{(2)t} k_{ii}^{(2)} \phi_i^{(2)} & \phi_i^{(2)t} k_{ic}^{(2)} \\ k_{ci}^{(1)} \phi_i^{(1)} & k_{ci}^{(2)} \phi_i^{(2)} & k_{cc}^{(1)} + k_{cc}^{(2)} \end{bmatrix} \begin{bmatrix} q^{(1)} \\ q^{(2)} \\ \delta_c \end{bmatrix} = \begin{bmatrix} \phi_i^{(1)t} F^{(1)} \\ \phi_i^{(2)t} F^{(2)} \\ 0 \end{bmatrix} \tag{89}$$

The system (89) is now the analytical model of the entire structure. Notice that all the necessary quantities to form (89) are obtained from dynamic analysis of the two substructures. In other words, the behavior of the entire structure has been obtained in terms of the behavior of each substructure. The solution of (89) is obtained by the usual methods for determining system frequencies or system response. The recovery of the physical variable is accomplished via (84) and its analog for SS2. The number of degrees of freedom can be reduced only in one substructure; see exercise 14.

5.4.2 Method using constrained modes

This method is only presented briefly in order to avoid a too lengthy discussion of the substructure method. The connection degrees of freedom δ_c are retained as variables and the internal displacement variables are determined following two steps.

1. Static displacement: Let $F^{(1)} = 0$, then for SS1 under static conditions:

$$\begin{bmatrix} k_{ii}^{(1)} & k_{ic}^{(1)} \\ k_{ci}^{(1)} & k_{cc}^{(1)} \end{bmatrix} \begin{bmatrix} \delta_i^{(1)} \\ \delta_c \end{bmatrix} = 0 \tag{90}$$

from which one can obtain the relation

$$\delta_i^{(1)} = -(k_{ii}^{(1)})^{-1} \cdot k_{ic}^{(1)} \cdot \delta_c \tag{91}$$

2. Dynamic displacements: In this method the connection nodes are constrained by clamping (i.e. $\delta_c = 0$); also it is assumed there are no external forces ($F^{(1)} = F^{(2)} = 0$). The dynamic behavior of SS1 is then found from

$$m_{ii}^{(1)} \delta_i^{(1)\infty} + k_{ii}^{(1)} \delta_i^{(1)} = 0 \tag{92}$$

The modes of (92) give

$$\delta_i^{(1)} = \phi_i^{(1)} q^{(1)} \tag{93}$$

Combining equations (91) and (93) gives the matrix relation

$$\begin{bmatrix} \delta_i^{(1)} \\ \delta_c \end{bmatrix} = \begin{bmatrix} \phi_i^{(1)} & -(k_{ii}^{(1)})^{-1} k_{ic}^{(1)} \\ 0 & I \end{bmatrix} \begin{bmatrix} q^{(1)} \\ \delta_c \end{bmatrix} \tag{94}$$

where I is the identity matrix.

It is now possible to substitute (94) into (78) and (79) to obtain $T^{(1)}$ and $U^{(1)}$ and one can proceed in a similar fashion to obtain $T^{(2)}$ and $U^{(2)}$.

The generalized forces vector becomes

$$\begin{bmatrix} \phi_i^{(1)t} F^{(1)} \\ \phi_i^{(2)t} F^{(2)} \\ -[(k_{ii}^{(1)})^{-1} k_{ic}^{(1)}]^t F^{(1)} - [(k_{ii}^{(2)})^{-1} k_{ic}^{(2)}]^t F^{(2)} \end{bmatrix} \tag{95}$$

All the necessary terms to formulate the equations of motion in terms of $q^{(1)}$, $q^{(2)}$, δ_c are now available. After solution, the physical coordinates for SS1 can be recovered from (94) and those for SS2 from (94) with superscript (2) exchanged for (1).

5.5 Exercises

Exercise 1: *A beam is simply supported at each end; see Figure 5. Using only a single finite element, calculate its lowest frequency and the corresponding mode.*

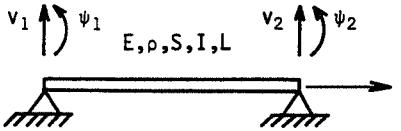


Figure 5

The consistent mass matrix has to be used, otherwise the kinetic energy would be zero. The equation of motion is

$$\frac{\rho SL}{420}\begin{bmatrix} 4L^2 & -3L^2 \\ -3L^2 & 4L^2 \end{bmatrix}\begin{bmatrix} \psi_1^{\infty} \\ \psi_2^{\infty} \end{bmatrix}+\frac{EI}{L^3}\begin{bmatrix} 4L^2 & 2L^2 \\ 2L^2 & 4L^2 \end{bmatrix}\begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}=0$$

from which one finds

$$\omega=\frac{10.95}{L^2}\sqrt{\frac{EI}{\rho S}} \quad \text{and} \quad \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}=\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Using equation (27):

$$v(x)=\left(x-\frac{2x^2}{L}+\frac{x^3}{L^2}\right)(1)+\left(-\frac{x^2}{L}+\frac{x^3}{L^2}\right)(-1)=x-\frac{x^2}{L}$$

The above value of ω is within 11% of the exact fundamental frequency.

Exercise 2: *Repeat exercise* 1 *for a clamped–free cantilever beam. Use a consistent mass matrix.*

The frequency is

$$\omega=\frac{3.533}{L^2}\sqrt{\frac{EI}{\rho S}}$$

and the corresponding mode is

$$\begin{bmatrix} v_2 \\ \psi_2 \end{bmatrix}=\begin{bmatrix} 1 \\ \dfrac{1.378}{L} \end{bmatrix}$$

from which

$$v(x)=1.622\frac{x^2}{L^2}-0.622\frac{x^3}{L^3}$$

The above frequency is within 1% of the exact value.

Exercise 3: *Consider the same system as in exercise* 1 *of Chapter* 4. *Model the beam by three finite elements; see Figure* 6. *Calculate T and U using the consistent mass matrix. Also calculate the two lowest frequencies and modes.*

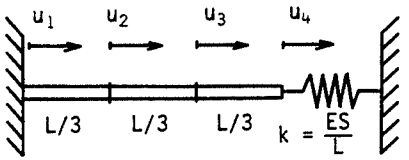


Figure 6

T is the same as given in this chapter in section 5.1.4. To obtain U it is necessary to add the strain energy of the spring; namely

$$U_s = \tfrac{1}{2}ku_4^2$$

Completing all the calculations results in the stiffness matrix

$$K = \frac{3ES}{L}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & \frac{4}{3} \end{bmatrix}$$

The frequencies and modes are

$$\omega_1 = \frac{2.068}{L}\sqrt{\frac{E}{\rho}}; \qquad \phi_1 = \begin{bmatrix} 1 \\ 1.560 \\ 1.433 \end{bmatrix}$$

$$\omega_2 = \frac{5.474}{L}\sqrt{\frac{E}{\rho}}; \qquad \phi_2 = \begin{bmatrix} 1 \\ -0.1410 \\ -0.9801 \end{bmatrix}$$

Exercise 4: *A cantilever beam has a length of 3L. Model the beam with three finite elements (see Figure 7) and write the system stiffness matrix and consistent mass matrix. Using program no. 5 of Chapter 7, calculate the first two frequencies, the corresponding modes, and the associated modal stiffnesses and masses.*

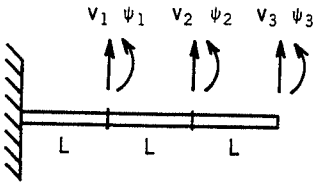


Figure 7

The solutions are:

$$K = \frac{EI}{L^3}\begin{bmatrix} 24 & 0 & -12 & 6L & 0 & 0 \\ 0 & 8L^2 & -6L & 2L^2 & 0 & 0 \\ -12 & -6L & 24 & 0 & -12 & 6L \\ 6L & 2L^2 & 0 & 8L^2 & -6L & 2L^2 \\ 0 & 0 & -12 & -6L & 12 & -6L \\ 0 & 0 & 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

$$M=\frac{\rho SL}{420}\begin{bmatrix} 312 & 0 & 54 & -13L & 0 & 0 \\ 0 & 8L^2 & 13L & -3L^2 & 0 & 0 \\ 54 & 13L & 312 & 0 & 54 & -13L \\ -13L & -3L^2 & 0 & 8L^2 & 13L & -3L^2 \\ 0 & 0 & 54 & 13L & 156 & -22L \\ 0 & 0 & -13L & -3L^2 & -22L & 4L^2 \end{bmatrix}$$

from which

$$\omega_1=\frac{3.516}{(3L)^2}\sqrt{\frac{EI}{\rho S}}$$

$$\phi_1=\left[1,\frac{1.821}{L},3.304,\frac{2.636}{L},6.041,\frac{2.772}{L}\right]^t$$

with

$$\phi_1^t M\phi_1=27.36\rho SL$$

$$\phi_1^t K\phi_1=4.176\frac{EI}{L^3}$$

and

$$\omega_2=\frac{22.11}{(3L)^2}\sqrt{\frac{EI}{\rho S}}$$

$$\phi_2=\left[1,\frac{0.9966}{L},0.7179,-\frac{1.672}{L},-1.695,-\frac{2.704}{L}\right]^t$$

with

$$\phi_2^t M\phi_2=2.129\rho SL$$

$$\phi_2^t K\phi_2=12.85\frac{EI}{L^3}$$

The above frequencies are slightly above the exact values and their accuracy is very good.

Exercise 5: *Consider the same system as in exercise 4. In the present case, form the lumped mass matrix and use it to calculate the first two frequencies and modes. Use program no. 5,*

The solutions are:

$$M=\frac{\rho SL}{2}\begin{bmatrix} 2 & & & & & \text{zero} \\ & 0 & & & & \\ & & 2 & & & \\ & & & 0 & & \\ & & & & 1 & \\ \text{zero} & & & & & 0 \end{bmatrix}$$

from which

$$\omega_1 = \frac{3.346}{(3L)^2}\sqrt{\frac{EI}{\rho S}}$$

$$\phi_1 = \left[1, \frac{1.829}{L}, 3.339, \frac{2.700}{L}, 6.181, \frac{2.914}{L}\right]^t$$

and

$$\omega_2 = \frac{18.89}{(3L)^2}\sqrt{\frac{EI}{\rho S}}$$

$$\phi_2 = \left[1, \frac{1.058}{L}, 0.9673, -\frac{1.331}{L}, -1.368, -\frac{2.838}{L}\right]^t$$

These frequencies are lower than the exact value and their accuracy is not very good.

Exercise 6: *Consider the same system as in exercise 23 of Chapter 4. Model the beam by a single finite element (see Figure 8) and calculate the three frequencies and modes. Use the consistent mass matrix.*

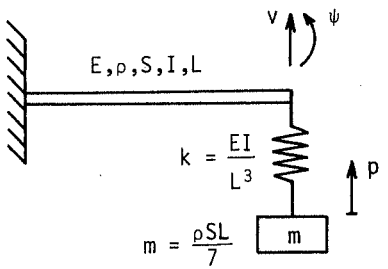


Figure 8

In this case, the kinetic and strain energies become

$$T = \frac{1}{2}\frac{\rho SL}{420}\begin{bmatrix} v^\circ \\ \psi^\circ \end{bmatrix}^t \begin{bmatrix} 156 & -22L \\ -22L & 4L^2 \end{bmatrix}\begin{bmatrix} v^\circ \\ \psi^\circ \end{bmatrix} + \frac{1}{2}\frac{\rho SL}{7}p^{\circ 2}$$

$$U = \frac{1}{2}\frac{EI}{L^3}\begin{bmatrix} v \\ \psi \end{bmatrix}^t \begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix}\begin{bmatrix} v \\ \psi \end{bmatrix} + \frac{1}{2}\frac{EI}{L^3}(v-p)^2$$

and application of Lagrange's equations gives

$$\rho SL\begin{bmatrix} 156/420 & -22L/420 & 0 \\ -22L/420 & 4L^2/420 & 0 \\ 0 & 0 & 1/7 \end{bmatrix}\begin{bmatrix} v^{\circ\circ} \\ \psi^{\circ\circ} \\ p^{\circ\circ} \end{bmatrix} + \frac{EI}{L^3}\begin{bmatrix} 13 & -6L & -1 \\ -6L & 4L^2 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} v \\ \psi \\ p \end{bmatrix} = 0$$

from which the following results are obtained:

$$\omega_1 = \frac{2.143}{L^2}\sqrt{\frac{EI}{\rho S}}; \qquad \phi_1 = \begin{bmatrix} 0.3437 \\ 0.5004/L \\ 1 \end{bmatrix}$$

$$\omega_2 = \frac{4.346}{L^2}\sqrt{\frac{EI}{\rho S}}; \qquad \phi_2 = \begin{bmatrix} 0.7624 \\ 1/L \\ -0.4488 \end{bmatrix}$$

$$\omega_3 = \frac{34.92}{L^2}\sqrt{\frac{EI}{\rho S}}; \qquad \phi_3 = \begin{bmatrix} 0.1316 \\ 1/L \\ -0.0007 \end{bmatrix}$$

Exercise 7: *Use the system as in the preceding exercise. Prepare the data required by program no.* 11 *of Chapter* 7. *Calculate the first three frequencies and modes and determine the percentage distribution of the kinetic and strain energies for each element in each of the three modes. Use SI units with* $L = 2\ m$; $S = 10^{-2}\ m^2$; $I = 10^{-6}\ m^4$; $\rho = 7800\ kg/m^3$; $E = 2 \times 10^{11}\ N/m^2$.

Number of elements: 2
Number of nodes:3
Number of materials: 1
Number of sections: 1
Number of constrained nodes: 3

Kind of element 1: B
Kind of element 2: MSM

Beam element no. 1 – Number of 1st node: 1
Number of 2nd node: 2
Number of material: 1
Number of section: 1

Material no. 1 – Young's modulus: 2.E+11
Mass per unit volume: 7800

Section no. 1 – Area: 1.E–2
Inertia: 1.E–6

MSM Element no. 1 – Number of 1st node: 2
Mass in X: 0
Mass in Y: 0
Inertia in PSI: 0

System MSM no. 1 – Number of 2nd node: 3
Mass in X: 22.285714
Mass in Y: 0
Inertia in PSI: 0
Stiffness in X: 2.5E+4
Stiffness in Y: 0
Stiffness in PSI: 0

Node 1 – Absissa: 0
Ordinate: 0
Node 2 – Absissa: 2
Ordinate: 0
Node 3 – Absissa: 2
Ordinate: −1 (This figure is only to establish a reference frame for the spring–mass system).

Number of 1st constrained node: 1
For this node condition in $u = 1$
For this node condition in $v = 1$
For this node condition in PSI = 1

Number of 2nd constrained node: 2
For this node condition in $u = 1$
For this node condition in $v = 0$
For this node condition in PSI = 0

Number of 3rd constrained node: 3
For this node condition in $u = 1$
For this node condition in $v = 0$
For this node condition in PSI = 1

Results:

$\omega_1 = 27.13$ rad/sec; $\phi_1 = [0.3437, 0.2502, 1]^t$

$\omega_3 = 55.02$ rad/sec; $\phi_2 = [1, 0.6558, -0.5886]^t$

$\omega_3 = 442.1$ rad/sec; $\phi_3 = [0.2631, 1, -0.00152]^t$

Mode 1: Element 1 SE 45.2%
Element 2 SE 54.8%
Element 1 KE 16.5%
Element 2 KE 83.5%

Mode 2: Element 1 SE 55.5%
Element 2 SE 44.5%
Element 1 KE 83.5%
Element 2 KE 16.5%

Mode 3: Element 1 SE 99.3%
Element 2 SE 0.7%
Element 1 KE 100%
Element 2 KE 0%

Exercise 8: *Repeat exercise* 6 *with the beam modeled by two finite elements; see Figure* 9.

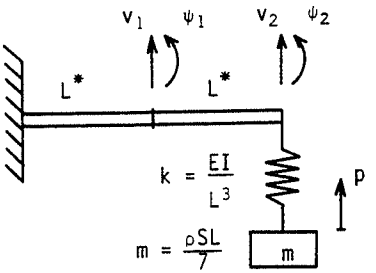


Figure 9

To keep the same matrix expressions, let $L^* = L/2$. Then

$$T = \frac{1}{2}\frac{\rho SL^*}{420}\begin{bmatrix} v_1^\circ \\ \psi_1^\circ \end{bmatrix}^t \begin{bmatrix} 156 & -22L^* \\ -22L^* & 4L^{*2} \end{bmatrix} \begin{bmatrix} v_1^\circ \\ \psi_1^\circ \end{bmatrix}$$

$$+\frac{1}{2}\frac{\rho SL^*}{420}\begin{bmatrix} v_1^\circ \\ \psi_1^\circ \\ v_2^\circ \\ \psi_2^\circ \end{bmatrix}^t \begin{bmatrix} 156 & 22L^* & 54 & -13L^* \\ 22L^* & 4L^{*2} & 13L^* & -3L^{*2} \\ 54 & 13L^* & 156 & -22L^* \\ -13L^* & -3L^{*2} & -22L^* & 4L^{*2} \end{bmatrix} \begin{bmatrix} v_1^\circ \\ \psi_1^\circ \\ v_2^\circ \\ \psi_2^\circ \end{bmatrix}$$

$$+\frac{1}{2}\frac{2\rho SL^*}{7}p^{\circ 2}$$

$$U = \frac{1}{2}\frac{EI}{L^{*3}}\begin{bmatrix} v_1 \\ \psi_1 \end{bmatrix}^t \begin{bmatrix} 12 & -6L^* \\ -6L^* & 4L^{*2} \end{bmatrix} \begin{bmatrix} v_1 \\ \psi_1 \end{bmatrix}$$

$$+\frac{1}{2}\frac{EI}{L^{*3}}\begin{bmatrix} v_1 \\ \psi_1 \\ v_2 \\ \psi_2 \end{bmatrix} \begin{bmatrix} 12 & 6L^* & -12 & 6L^* \\ 6L^* & 4L^{*2} & -6L^* & 2L^{*2} \\ -12 & -6L^* & 12 & -6L^* \\ 6L^* & 2L^{*2} & -6L^* & 4L^{*2} \end{bmatrix} \begin{bmatrix} v_1 \\ \psi_1 \\ v_2 \\ \psi_2 \end{bmatrix}$$

$$+\frac{1}{2}\frac{EI}{8L^{*3}}(v_2 - p)^2$$

and applying Lagrange's equations gives

$$\rho SL^* \begin{bmatrix} 312/420 & 0 & 54/420 & -13L^*/420 & 0 \\ 0 & 8L^{*2}/420 & 13L^*/420 & -3L^{*2}/420 & 0 \\ 54/420 & 13L^*/420 & 156/420 & -22L^*/420 & 0 \\ -13L^*/420 & -3L^{*2}/420 & -22L^*/420 & 4L^{*2}/420 & 0 \\ 0 & 0 & 0 & 0 & 2/7 \end{bmatrix} \begin{bmatrix} v_1^{\infty} \\ \psi_1^{\infty} \\ v_2^{\infty} \\ \psi_2^{\infty} \\ p^{\infty} \end{bmatrix}$$

$$+\frac{EI}{L^{*3}} \begin{bmatrix} 24 & 0 & -12 & 6L^* & 0 \\ 0 & 8L^{*2} & -6L^* & 2L^{*2} & 0 \\ -12 & -6L^* & 12.125 & -6L^* & -0.125 \\ 6L^* & 2L^{*2} & -6L^* & 4L^{*2} & 0 \\ 0 & 0 & -0.125 & 0 & 0.125 \end{bmatrix} \begin{bmatrix} v_1 \\ \psi_1 \\ v_2 \\ \psi_2 \\ p \end{bmatrix} = 0$$

from which one obtains with the aid of program no. 5:

$$\omega_1 = \frac{2.143}{L^2}\sqrt{\frac{EI}{\rho S}}; \qquad \phi_1 = \left[0.1108, \frac{0.3918}{L}, 0.3440, \frac{0.5008}{L}, 1\right]^t$$

$$\omega_2 = \frac{4.323}{L^2}\sqrt{\frac{EI}{\rho S}}; \qquad \phi_2 = \left[0.3545, \frac{1.184}{L}, 1, \frac{1.309}{L}, -0.5991\right]^t$$

$$\omega_3 = \frac{22.32}{L^2}\sqrt{\frac{EI}{\rho S}}; \qquad \phi_3 = \left[-0.2990, \frac{0.1872}{L}, 0.4187, \frac{2}{L}, -0.0060\right]^t$$

Exercise 9: *Consider the system of exercise 7. Calculate the first three frequencies and their modes using computer program no. 11 and with three elements for the beam.*

$\omega_1 = 27.13$ rad/sec; $\phi_1 = [0.0531, 0.1478, 0.1819, 0.2279, 0.3441, 0.2504, 1]^t$

$\omega_2 = 54.70$ rad/sec; $\phi_2 = [0.1752, 0.4719, 0.5626, 0.6475, 1, 0.6544, -0.6000]^t$

$\omega_3 = 281.0$ rad/sec; $\phi_3 = [-0.2466, -0.3672, -0.1749, 0.6223, 0.4213, 1, -0.0061]^t$

Exercise 10: *Consider a clamped–clamped beam. Calculate the six lowest frequencies using first the exact theory and then a finite element model consisting of four elements (see Figure 10). Use the consistent mass matrix. In the latter case, use program no. 11. Use SI units with* $L = 1\ m$; $E = 2 \times 10^{11}\ N/m^2$; $\rho = 7800\ kg/m^3$; $I = 10^{-6}\ m^4$; $S = 10^{-2}\ m^2$.

The results for ω in rad/sec are as follows:

Exact	70.81	195.2	382.6	632.5	944.9	1320
FEM	70.90	197.0	390.8	739.4	1223	1970

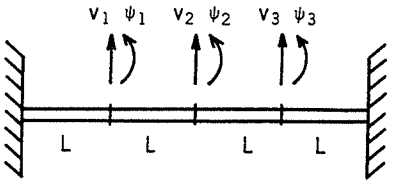


Figure 10

Exercise 11: *Consider the same system as in exercise* 4. *Add a mass of* $\rho SL/10$ *at the free end. Using program no.* 5, *calculate the values of the two lowest frequencies* ω_1^*, ω_2^*. *Then considering the added mass to be a small modification of the structure, calculate the values* ω_1^{**}, ω_2^{**}.

In the mass matrix of exercise 4, it is necessary to increase m_{55} by $\rho SL/10$; then:

$$\omega_1^* = \frac{3.302}{(3L)^2}\sqrt{\frac{EI}{\rho S}}$$

$$\omega_2^* = \frac{20.88}{(3L)^2}\sqrt{\frac{EI}{\rho S}}$$

and from the structural modification method of section 5.3:

$$\omega_1^{**} = \frac{3.303}{(3L)^2}\sqrt{\frac{EI}{\rho S}}$$

$$\omega_2^{**} = \frac{20.75}{(3L)^2}\sqrt{\frac{EI}{\rho S}}$$

Exercise 12: *Consider the same system as in exercise* 6. *However, in this case the thickness of the beam h is multiplied by* 1.1. *Using the results of exercise* 7, *find the influence of this structural modification on the first three frequencies. Compare these results to the results obtained using program no.* 11.

The strain energy of a beam in bending is proportional to h^3 and the kinetic energy to h. Hence for the first frequency

$$U_1 \text{ becomes } 0.548 + 0.452 \times 1.331 = 1.150$$

$$T_1 \text{ becomes } 0.835 + 0.165 \times 1.1 = 1.017$$

from which

$$\omega_1^* = 27.13\sqrt{\frac{1.150}{1.017}} = 28.85 \text{ rad/sec}$$

For the second frequency

$$U_2 \text{ becomes } 0.445+0.555\times 1.331=1.184$$
$$T_2 \text{ becomes } 0.165+0.835\times 1.1=1.084$$
$$\omega_2^*=55.62\sqrt{\frac{1.184}{1.084}}=58.13 \text{ rad/sec}$$

For the third frequency

$$U_3 \text{ becomes } 0.007+0.993\times 1.331=1.329$$
$$T_3 \text{ becomes } 0 \quad +1.0 \quad \times 1.1 \quad =1.1$$
$$\omega_3^*=442.1\sqrt{\frac{1.329}{1.1}}=485.9 \text{ rad/sec}$$

The results obtained using program no. 11 are

$\omega_1^{**}=28.56$ rad/sec $\quad \omega_2^{**}=57.55$ rad/sec $\quad \omega_3^{**}=485.9$ rad/sec

The result $\omega_3^*=\omega_3^{**}$ is to be expected because in this mode the mass has no kinetic energy and the spring has a very low strain energy.

Exercise 13: *The plane frame shown in Figure* 11 *is clamped at* A *and* C *and has a rigid joint at* B. *With the notation of Figure* 12, *express* u_2, v_2, ψ_2 *at*

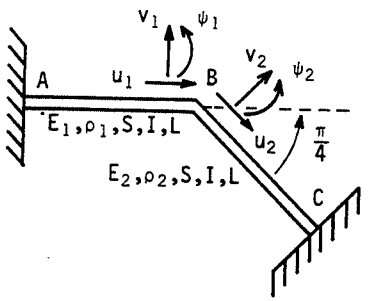


Figure 11

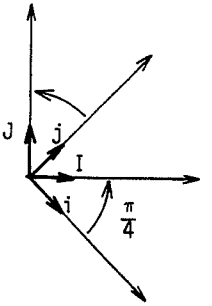


Figure 12

B *as functions of* u_1, v_1, ψ_1 *at* B *and obtain the kinetic energy* T *and strain energy* U *in terms of these latter variables. Use the consistent mass matrix and SI units with* $L=1\,m$, $S=10^{-2}\,m^2$, $I=10^{-6}\,m^4$, $E_1=2\times 10^{11}\,N/m^2$, $\rho_1=7800\,kg/m^3$, $E_2=7\times 10^{10}\,N/m^2$, $\rho_2=2700\,kg/m$.

$$u_2\vec{i}+v_2\vec{j}=u_1\vec{I}+v_1\vec{J}$$

from which

$$u_2=u_1\vec{I}\cdot\vec{i}+v_1\vec{J}\cdot\vec{i}$$
$$v_2=u_1\vec{I}\cdot\vec{j}+v_1\vec{J}\cdot\vec{j}$$

Then

$$\begin{bmatrix} u_2 \\ v_2 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ \psi_1 \end{bmatrix}$$

Using a consistent mass matrix,

$$T(1) = \frac{1}{2} \begin{bmatrix} u_1^\circ \\ v_1^\circ \\ \psi_1^\circ \end{bmatrix}^t \begin{bmatrix} 26 & 0 & 0 \\ 0 & 28.97 & -4.086 \\ 0 & -4.086 & 0.7429 \end{bmatrix} \begin{bmatrix} u_1^\circ \\ v_1^\circ \\ \psi_1^\circ \end{bmatrix}$$

$$T(2) = \frac{1}{2} \begin{bmatrix} u_1^\circ \\ v_1^\circ \\ \psi_1^\circ \end{bmatrix}^t \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}^t \begin{bmatrix} 9 & 0 & 0 \\ 0 & 10.03 & 1.414 \\ 0 & 1.414 & 0.2571 \end{bmatrix}$$

$$\times \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1^\circ \\ v_1^\circ \\ \psi_1^\circ \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} u_1^\circ \\ v_1^\circ \\ \psi_1^\circ \end{bmatrix}^t \begin{bmatrix} 9.514 & 0.5143 & 1 \\ 0.5143 & 9.514 & 1 \\ 0 & 1 & 0.2571 \end{bmatrix} \begin{bmatrix} u_1^\circ \\ v_1^\circ \\ \psi_1^\circ \end{bmatrix}$$

from which

$$T = \frac{1}{2} \begin{bmatrix} u_1^\circ \\ v_1^\circ \\ \psi_1^\circ \end{bmatrix}^t \begin{bmatrix} 35.51 & 0.5143 & 1 \\ 0.5143 & 38.49 & -3.086 \\ 1 & -3.086 & 1 \end{bmatrix} \begin{bmatrix} u_1^\circ \\ v_1^\circ \\ \psi_1^\circ \end{bmatrix}$$

Similarly,

$$U(1) = \frac{1}{2} \begin{bmatrix} u_1 \\ v_1 \\ \psi_1 \end{bmatrix}^t \begin{bmatrix} 2\times10^9 & 0 & 0 \\ 0 & 2.4\times10^6 & -1.2\times10^6 \\ 0 & -1.2\times10^6 & 0.8\times10^6 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ \psi_1 \end{bmatrix}$$

$$U(2) = \frac{1}{2} \begin{bmatrix} u_1 \\ v_1 \\ \psi_1 \end{bmatrix}^t \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}^t \begin{bmatrix} 7\times10^8 & 0 & 0 \\ 0 & 8.4\times10^5 & 4.2\times10^5 \\ 0 & 4.2\times10^5 & 2.8\times10^5 \end{bmatrix}$$

$$\times \begin{bmatrix} \sqrt{2}/2 & -\sqrt{2}/2 & 0 \\ \sqrt{2}/2 & \sqrt{2}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ \psi_1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} u_1 \\ v_1 \\ \psi_1 \end{bmatrix}^t \begin{bmatrix} 3.504\times10^8 & -3.496\times10^8 & 2.97\times10^5 \\ -3.496\times10^8 & 3.504\times10^8 & 2.97\times10^5 \\ 2.97\times10^5 & 2.97\times10^5 & 2.8\times10^5 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ \psi_1 \end{bmatrix}$$

and

$$U = \frac{1}{2} \begin{bmatrix} u_1 \\ v_1 \\ \psi_1 \end{bmatrix}^t \begin{bmatrix} 2.35\times10^9 & -3.496\times10^8 & 2.97\times10^5 \\ -3.496\times10^8 & 3.528\times10^8 & -9.03\times10^5 \\ 2.97\times10^5 & -9.03\times10^5 & 1.08\times10^6 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ \psi_1 \end{bmatrix}$$

Using computer program no. 5, one finds

$$\omega_1 = 1030 \text{ rad/sec}; \qquad \phi_1 = [0.0010, -0.0087, 1]^t$$
$$\omega_2 = 3131 \text{ rad/sec}; \qquad \phi_2 = [0.0609, 0.3170, 1]^t$$
$$\omega_3 = 8619 \text{ rad/sec}; \qquad \phi_3 = [-0.4794, 0.1653, 1]^t$$

These results can be obtained directly using computer program no. 12.

Exercise 14: *Consider the clamped–clamped beam of exercise* 10. *Calculate the two lowest frequencies by using the method of substructures which employs free modes; see Figure* 13. *Use only the first mode of* SS1, *but retain all the degrees of freedom of* SS2. *Use program no.* 5 *to calculate frequencies and modes.*

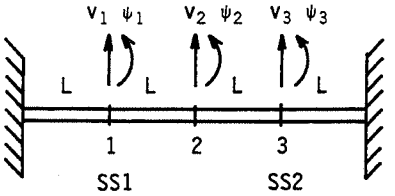


Figure 13

It is necessary to find the first mode of SS1 with node 2 free; see

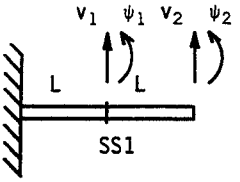


Figure 14

Figure 14. The consistent mass matrix and stiffness matrix are

$$M = \begin{bmatrix} 57.94 & 0 & 10.03 & -2.414 \\ 0 & 1.486 & 2.414 & -0.5571 \\ 10.03 & 2.414 & 28.97 & -4.086 \\ -0.2414 & -0.5571 & -4.086 & 0.7429 \end{bmatrix}$$

and

$$K = 2 \times 10^5 \begin{bmatrix} 24 & 0 & -12 & 6 \\ 0 & 8 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix}$$

Computer program no. 5 gives

$$\omega_1 = 44.53 \text{ rad/sec}; \qquad \phi_1 = \begin{bmatrix} 0.3395 \\ 0.5815 \\ 1 \\ 0.6882 \end{bmatrix}$$

and permits the change of variable

$$\begin{bmatrix} v_1 \\ \psi_1 \end{bmatrix} = \begin{bmatrix} 0.3395 \\ 0.5815 \end{bmatrix} q_1$$

Using the variables q_1, v_2, ψ_2 and completing all calculations gives

$$T_{SS1} = \frac{1}{2} \begin{bmatrix} q_1^\circ \\ v_2^\circ \\ \psi_2^\circ \end{bmatrix}^t \begin{bmatrix} 7.182 & 4.809 & -1.144 \\ 4.809 & 28.97 & -4.086 \\ -1.144 & -4.086 & 0.7429 \end{bmatrix} \begin{bmatrix} q_1^\circ \\ v_2^\circ \\ \psi_2^\circ \end{bmatrix}$$

$$U_{SS1} = \frac{10^5}{2} \begin{bmatrix} q_1 \\ v_2 \\ \psi_2 \end{bmatrix}^t \begin{bmatrix} 10.94 & -15.13 & 6.4 \\ -15.13 & 24 & -12 \\ 6.4 & -12 & 8 \end{bmatrix} \begin{bmatrix} q_1 \\ v_2 \\ \psi_2 \end{bmatrix}$$

For SS2 one obtains

$$T_{SS2} = \frac{1}{2} \begin{bmatrix} v_2^\circ \\ \psi_2^\circ \\ v_3^\circ \\ \psi_3^\circ \end{bmatrix}^t \begin{bmatrix} 28.97 & 4.086 & 10.03 & -2.414 \\ 4.086 & 0.7429 & 2.414 & -0.5571 \\ 10.03 & 2.414 & 57.94 & 0 \\ -2.414 & -0.5571 & 0 & 1.486 \end{bmatrix} \begin{bmatrix} v_2^\circ \\ \psi_2^\circ \\ v_3^\circ \\ \psi_3^\circ \end{bmatrix}$$

$$U_{SS2} = \frac{10^5}{2} \begin{bmatrix} v_2 \\ \psi_2 \\ v_3 \\ \psi_3 \end{bmatrix}^t \begin{bmatrix} 24 & 12 & -24 & 12 \\ 12 & 8 & -12 & 4 \\ -24 & -12 & 48 & 0 \\ 12 & 4 & 0 & 16 \end{bmatrix} \begin{bmatrix} v_2 \\ \psi_2 \\ v_3 \\ \psi_3 \end{bmatrix}$$

The mass and stiffness matrices for the total structure can now be assembled.

$$M = \begin{bmatrix} 7.182 & 4.809 & -1.144 & 0 & 0 \\ 4.809 & 57.94 & 0 & 10.03 & -2.414 \\ -1.144 & 0 & 1.486 & 2.414 & -0.5571 \\ 0 & 10.03 & 2.414 & 57.94 & 0 \\ 0 & -2.414 & -0.5571 & 0 & 1.486 \end{bmatrix}$$

$$K = 10^5 \begin{bmatrix} 10.94 & -15.13 & 6.4 & 0 & 0 \\ -15.13 & 48 & 0 & -24 & 12 \\ 6.4 & 0 & 16 & -12 & 4 \\ 0 & -24 & -12 & 48 & 0 \\ 0 & 12 & 4 & 0 & 16 \end{bmatrix}$$

Program no. 5 gives

$$\omega_1 = 72.16 \text{ rad/sec}; \qquad \phi_1 = [1, 0.6981, 0.0255, 0.3874, -0.5381]^t$$

$$\omega_2 = 216.4 \text{ rad/sec}; \qquad \phi_2 = [1, 0.0754, -0.9043, -0.4652, -0.1871]^t$$

In order to recover the modes in terms of the initial variables, one uses the relation among v_1, ψ_1, and q_1. For example, for the lowest frequency one finds the following mode in terms of the degrees of freedom v_1, ψ_1, v_2, ψ_2, v_3, ψ_3:

$$[0.4864, 0.8330, 1, 0.0365, 0.5550, -0.7708]^t$$

where the mode has been normalized so that its largest component is unity.

Exercise 15: *Consider the same system as in exercise* 14 *but use constrained modes for* SS1.

It is necessary to find the first mode of SS1 with node 2 clamped; see

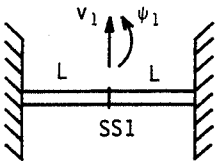


Figure 15

Figure 15. The consistent mass matrix and stiffness matrix are

$$M = \begin{bmatrix} 57.94 & 0 \\ 0 & 1.486 \end{bmatrix}; \qquad k = 10^5 \begin{bmatrix} 48 & 0 \\ 0 & 16 \end{bmatrix}$$

from which one finds

$$\omega_1 = 287.8 \text{ rad/sec}; \qquad \phi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The static displacement is found from

$$2 \times 10^5 \begin{bmatrix} 24 & 0 & -12 & 6 \\ 0 & 8 & -6 & 2 \\ -12 & -6 & 12 & -6 \\ 6 & 2 & -6 & 4 \end{bmatrix} \begin{bmatrix} v_1 \\ \psi_1 \\ v_2 \\ \psi_2 \end{bmatrix} = 0$$

From the first two lines one obtains

$$\begin{bmatrix} v_1 \\ \psi_1 \end{bmatrix} = \begin{bmatrix} 0.5 & -0.25 \\ 0.75 & -0.25 \end{bmatrix} \begin{bmatrix} v_2 \\ \psi_2 \end{bmatrix}$$

The transformation matrix becomes

$$\begin{bmatrix} v_1 \\ \psi_1 \\ v_2 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} 1 & 0.5 & -0.25 \\ 0 & 0.75 & -0.25 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ v_2 \\ \psi_2 \end{bmatrix}$$

and this leads to

$$T_{SS1} = \frac{1}{2} \begin{bmatrix} q_1^\circ \\ v_2^\circ \\ \psi_2^\circ \end{bmatrix}^t \begin{bmatrix} 1 & 0.5 & -0.25 \\ 0 & 0.75 & -0.25 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^t$$

$$\times \begin{bmatrix} 57.94 & 0 & 10.03 & -2.414 \\ 0 & 1.486 & 2.414 & -0.5571 \\ 10.03 & 2.414 & 28.97 & -4.086 \\ -2.414 & -0.5571 & -4.086 & 0.7429 \end{bmatrix} \begin{bmatrix} 1 & 0.5 & -0.25 \\ 0 & 0.75 & -0.25 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} q_1^\circ \\ v_2^\circ \\ \psi_2^\circ \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} q_1^\circ \\ v_2^\circ \\ \psi_2^\circ \end{bmatrix}^t \begin{bmatrix} 57.94 & 39 & -16.90 \\ 39 & 57.94 & -16.34 \\ -16.90 & -16.34 & 5.943 \end{bmatrix} \begin{bmatrix} q_1^\circ \\ v_2^\circ \\ \psi_2^\circ \end{bmatrix}$$

Proceeding in a similar manner:

$$U_{SS1} = \frac{10^5}{2} \begin{bmatrix} q_1 \\ v_2 \\ \psi_2 \end{bmatrix}^t \begin{bmatrix} 48 & 0 & 0 \\ 0 & 3 & -3 \\ 0 & -3 & 4 \end{bmatrix} \begin{bmatrix} q_1 \\ v_2 \\ \psi_2 \end{bmatrix}$$

Using the expression for T_{SS2} and U_{SS2} from exercise 14, the mass and stiffness matrices for the total structure can be assembled.

$$M = \begin{bmatrix} 57.94 & 39 & -16.90 & 0 & 0 \\ 39 & 86.91 & -12.26 & 10.03 & -2.414 \\ -16.90 & -12.26 & 6.686 & 2.414 & -0.5571 \\ 0 & 10.03 & 2.414 & 57.94 & 0 \\ 0 & -2.414 & -0.5571 & 0 & 1.486 \end{bmatrix}$$

$$K = 10^5 \begin{bmatrix} 48 & 0 & 0 & 0 & 0 \\ 0 & 27 & 9 & -24 & 12 \\ 0 & 9 & 12 & -12 & 4 \\ 0 & -24 & -12 & 48 & 0 \\ 0 & 12 & 4 & 0 & 16 \end{bmatrix}$$

Computer program no. 5 gives

$$\omega_1 = 70.91 \text{ rad/sec}; \qquad \phi_1 = [0.0435, 1, 0, 0.5435, -0.7612]^t$$

$$\omega_2 = 197.0 \text{ rad/sec}; \qquad \phi_2 = [-0.2576, 0, 1, 0.5067, -0.2729]^t$$

In order to recover the modes in terms of the initial variable, one uses the above transformation matrix. For example, for the lowest frequency one finds the following mode in terms of the degrees of freedom v_1, ψ_1, v_2, ψ_2, v_3, ψ_3:

$$[0.5435, 0.7500, 1, 0, 0.5435, -0.7612]^t$$

where the mode has been normalized so that its largest component is unity.

Exercise 16; *Consider the same system and same modeling procedure as in exercise* 14. *In this case use just the first mode of both* SS1 *and* SS2; *see Figure* 16.

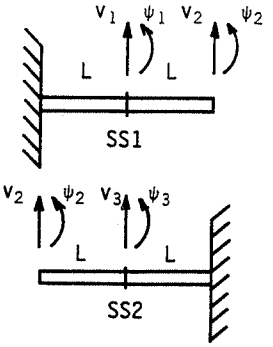


Figure 16

The appropriate expressions for T_{SS1} and U_{SS1} are given in exercise 14. The first frequency and mode of SS2 are

$$\omega_1 = 44.53 \text{ rad/sec}; \qquad \phi_1 = \begin{bmatrix} 1 \\ -0.6882 \\ 0.3395 \\ -0.5815 \end{bmatrix}$$

which permits the change of variable:

$$\begin{bmatrix} v_3 \\ \psi_3 \end{bmatrix} = \begin{bmatrix} 0.3395 \\ -0.5815 \end{bmatrix} q_2$$

Using this in the expressions for T_{SS2} and U_{SS2} gives

$$T_{SS2} = \frac{1}{2} \begin{bmatrix} v_2^\circ \\ \psi_2^\circ \\ q_2^\circ \end{bmatrix}^t \begin{bmatrix} 28.97 & 4.086 & 4.809 \\ 4.086 & 0.7429 & 1.144 \\ 4.809 & 1.144 & 7.182 \end{bmatrix} \begin{bmatrix} v_2^\circ \\ \psi_2^\circ \\ q_2^\circ \end{bmatrix}$$

$$U_{SS2} = \frac{10^5}{2} \begin{bmatrix} v_2 \\ \psi_2 \\ q_2 \end{bmatrix}^t \begin{bmatrix} 24 & 12 & -15.13 \\ 12 & 8 & -6.4 \\ -15.13 & -6.4 & 10.94 \end{bmatrix} \begin{bmatrix} v_2 \\ \psi_2 \\ q_2 \end{bmatrix}$$

The mass and stiffness matrices for the total structure can then be assembled.

$$M = \begin{bmatrix} 7.182 & 4.809 & -1.144 & 0 \\ 4.809 & 57.94 & 0 & 4.809 \\ -1.144 & 0 & 1.486 & 1.144 \\ 0 & 4.809 & 1.144 & 7.182 \end{bmatrix}$$

$$K = 10^5 \begin{bmatrix} 10.94 & -15.13 & 6.4 & 0 \\ -15.13 & 48 & 0 & -15.13 \\ 6.4 & 0 & 16 & -6.4 \\ 0 & -15.13 & -6.4 & 10.94 \end{bmatrix}$$

where the variables are, in order, q_1, v_2, ψ_2, q_2. Computer program no. 5 gives

$$\omega_1 = 73.58 \text{ rad/sec}; \qquad \phi_1 = [1, 0.6860, 0, 1]^t$$
$$\omega_2 = 244.7 \text{ rad/sec}; \qquad \phi_2 = [1, 0, -0.9378, -1]^t$$

Using the transformations between v_1, ψ_1, and q_1 and v_2, ψ_2, and q_2, the mode associated with the lowest frequency in terms of the degrees of freedom $v_1, \psi_1, v_2, \psi_2, v_3, \psi_3$ is

$$[0.4950, 0.8477, 1, 0, 0.4950, 0.8477]^t$$

where the mode has been normalized so that its largest component is unity.

6
Experimental Methods

This chapter is addressed to readers who are not expert in the measurement of mechanical vibrations. It is intended primarily as an introduction to current equipment and techniques but it should also enable the reader to perform simple tests and to understand more advanced treatments.

The experimental study of a structure is necessary for a number of reasons: to determine its performance under actual operating conditions; to establish the accuracy of analytical predictions; or to obtain its dynamic characteristics when calculation is too uncertain or too expensive. This last case can be encountered when the structure is so complex that it is not amenable to economical calculation by the computer which is available or when the values of damping or the boundary conditions are inadequately known.

Among the most common measurements are the vibration response due to a sinusoidal excitation which is swept over a wide range of frequencies, the vibration response due to broadband noise, or the vibration response due to impact. These last two types of measurements are becoming more frequent because they are closely related to modern techniques of signal analysis and to recent developments in equipment for real-time signal display.

In what follows, common kinds of vibration transducers and exciters are presented as well as a brief introduction to the measurement of the dynamic characteristics of a structure subjected to a known excitation or under operating conditions.

The contents of the chapter are as follows:

6.1 Transducers

The basic principles of the most common vibration transducers are presented without a discussion of the technical details of their construction. Vibration transducers fall into one of two categories – contact or noncontact – and each is discussed in turn.

6.1.1 Transducers in contact with the structure

Piezoelectric transducers

These transducers use a property of piezoelectric quartz or ceramic materials which have asymmetric internal charge distributions. If a force is applied to such materials, the lattice is deformed and equal external charges of opposite polarity appear on the opposite sides of the crystal. To permit measurement, this charge, which is proportional to the applied force F, must be converted into a voltage by one of the following devices:

1. A charge-sensitive preamplifier. This device converts the charge into a voltage in such a way that the output voltage is essentially independent of the length of cable between the transducer and the preamplifier.
2. A voltage-sensitive preamplifier. The output of this device is directly proportional to the voltage difference across the transducer terminals. However, the signal is affected by cable length and often a correction for this effect must be applied.

Transducers which incorporate the sensor and the signal preamplifier within a single unit have been available for several years. The most common types of piezoelectric transducers are accelerometers and force gages.

Accelerometer. A slice of piezoelectric quartz crystal Q is attached to a mass M on one face and to a rigid base B on the other. The resulting transducer is attached to the structure by its base, as shown in Figure 1.

As a first approximation, this transducer is modeled as a mass m in series with a spring of stiffness k; see Figure 2. Such devices are given the generic name of seismic pickups.

The displacements x_1 and x_2 are related by the differential equation of motion for mass m, i.e.

$$mx_2^{\infty} + k(x_2 - x_1) = 0 \tag{1}$$

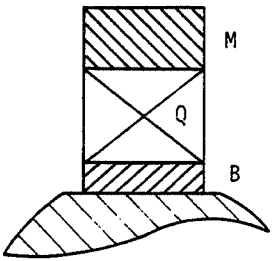


Figure 1 Schematic of an accelerometer

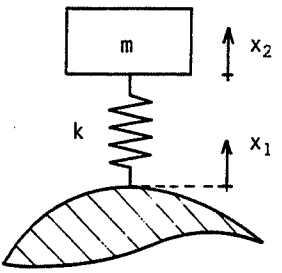


Figure 2 Single degree-of-freedom model of the accelerometer in Figure 1

and the force acting on the quartz crystal is

$$f = k(x_1 - x_2) \tag{2}$$

Let $x_1 = X_1 \sin \Omega t$, then in the steady-state motion

$$x_2 = X_2 \sin \Omega t \tag{3}$$

and (1) becomes

$$-m\Omega^2 X_2 + k(X_2 - X_1) = 0 \tag{4}$$

from which

$$X_2 = \frac{X_1}{1 - (\Omega/\omega)^2} \tag{5}$$

where

$$\omega^2 = \frac{k}{m}$$

Also, (2) becomes

$$f = -kX_1 \frac{(\Omega/\omega)^2}{1 - (\Omega/\omega)^2} \sin \Omega t = F \sin \Omega t \tag{6}$$

For $\Omega \ll \omega$, (6) becomes

$$f \simeq -k\left(\frac{\Omega}{\omega}\right)^2 X_1 \simeq -m\Omega^2 X_1 \tag{7}$$

and thus the force f and hence the surface electric charge produced is proportional to the acceleration of the base. The transducer therefore acts as an accelerometer.

Most accelerometers are supplied with a calibration chart similar to that shown in Figure 3. If s_f is the transducer sensitivity at an arbitrary frequency f and s_0 is the sensitivity at a low frequency, equations (6) and (7) give

$$\frac{s_f}{s_0} = \frac{1}{1 - (\Omega/\omega)^2} \tag{8}$$

Equation (8) is plotted in Figure 3 and shows that if $\Omega/\omega < 0.3$, $s_f = s_0$ to within 10 percent.

The measurements from an accelerometer can contain errors from several sources. Transverse acceleration will induce a small spurious indication of axial acceleration and stresses in the structure can be transferred to the base and then to the piezoelectric material to give a signal unrelated to vibration. Also, errors can arise from harsh environments such as high temperature, strong electromagnetic fields, or intense acoustic radiation.

Force transducer. A force transducer is constructed from piezoelectric material which is compressed between two disks of mass m_1 and m_2; see the

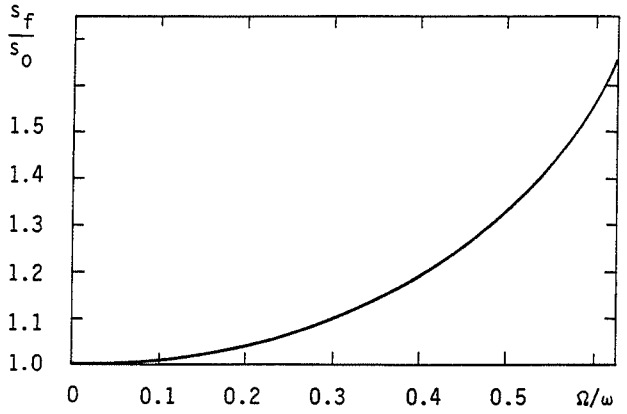


Figure 3 Calibration chart of an accelerometer

schematic representation in Figure 4. An axial force causes the production of surface charge on the faces of the piezoelectric material. This is transformed into a voltage difference via a charge-sensitive or voltage-sensitive preamplifier.

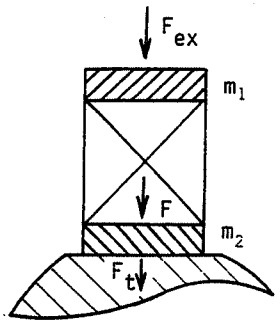


Figure 4 Schematic of a force transducer

The signal generated is not solely proportional to the force F_t transmitted to the structure. As a first approximation, it is proportional to the force F which is transmitted to mass m_2. The difference between F and F_t is equal to the inertia force of mass m_2. If the structure is small, it is necessary to compensate the output signal for the inertia force of m_2 in order to obtain an accurate measure of F_t. To cancel this mass effect, the corresponding inertia force is subtracted by calculation or with the aid of an analog circuit.

Like piezoelectric accelerometers, piezoelectric force transducers are sensitive to their conditions of use and to their environment. When they are subjected to base bending moments they will give measurement errors and can even be destroyed. These undesirable effects can be controlled by the simple means shown in Figure 5.

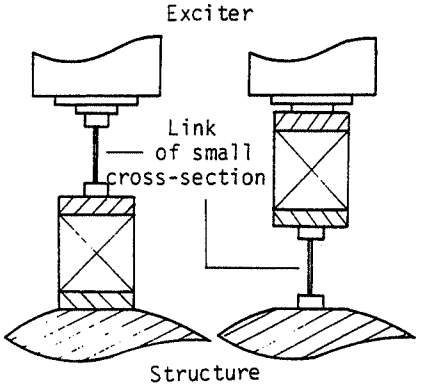


Figure 5 Mounting of a force transducer to minimize moment transfer

The mechanical link between the vibration exciter and the force transducer or between the force transducer and the structure is a beam of small cross-sectional dimension. Such a beam can have a high ratio of axial stiffness to bending stiffness and thereby limits the moment transferred to the transducer.

Electric transducers

Strain gage. This device is based on the fact that mechanical strain affects electrical properties such as the resistance R; these small changes in resistance can be easily measured with the aid of a Wheatstone bridge. The gage is bonded directly on the structure so that its strain is identical to the strain of the structure.

Strain gages are of two types. The most common type is the electrical resistance gage based on the variation of resistance of a wire with its elongation; the other type is based on the piezoresistive effect exhibited by certain semiconductor materials.

The relation between the percentage change in resistance and the longitudinal strain of a wire of length L is given by:

$$\frac{\Delta R}{R} = \alpha \frac{\Delta L}{L} \tag{9}$$

For the first type of gage, the so-called gage-factor α is in the neighborhood of 2 for most wire materials. This type of gage can measure static as well as dynamic strain. For the second type, the gage factor is of the order of several hundred and the measuring equipment can therefore be less sensitive; however, only dynamic strains can be measured. If either of these types is bonded to a metal structure, the surface strain can be measured without

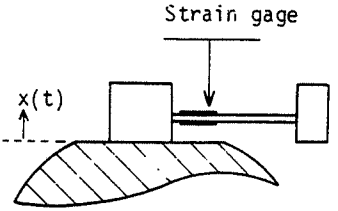


Figure 6 Strain gage vibration transducer

introducing significant perturbation to the structural behavior. If they are made part of a transducer, they can be used to measure strains which are proportional either to acceleration – a vibration transducer – or to the force applied to the base – a force transducer. Figure 6 shows a strain gage being used as a vibration transducer. Note that the gage is mounted near the root of the cantilever where the strain is high.

6.1.2 Transducers not in contact with the structure

It is obvious that proximity pickups, i.e. noncontacting transducers, do not affect the behavior of a structure while the preceding contact transducers add local mass and stiffness. These local effects are important if the structure is light or flexible. On the other hand, noncontacting transducers will generate significant harmonics of the forcing frequency if the displacement of the structure is large.

Capacitive transducer

The structure is at a DC potential V_S. The transducer, the plate A, located at rest at a distance h_0 from the surface of the structure, is at a DC potential V_A; see Figure 7. The potential difference $V = V_A - V_S$ is maintained constant.

The transducer is assumed fixed in space. When the structure vibrates the capacitance of the capacitor formed by the structure and the plate varies. The variation in charge thus produced is transformed into a voltage $v(t)$ by a charge-sensitive preamplifier. Let the time-varying distance between the

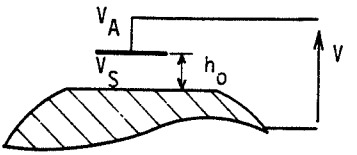


Figure 7 Capacitive transducer

structure and the transducer be

$$h(t) = h_0 + d \sin \Omega t \tag{10}$$

The charge $Q(t)$ and the potential difference V are related by

$$Q(t) = C(t) \cdot V \tag{11}$$

where V is a constant.

Reference to the formulas for the capacitance of plate capacitors shows that to first order

$$Q(t) = \frac{\alpha}{h(t)} \tag{12}$$

where α is a constant. The voltage output of a charge-sensitive preamplifier is then

$$v(t) = \frac{\beta}{h(t)} = \frac{\beta}{h_0 + d \sin \Omega t} \tag{13}$$

with β = constant. If d is small compared to h_0, (13) becomes

$$v(t) \simeq \frac{\beta}{h_0}\left(1 - \frac{d}{h_0} \sin \Omega t\right) \tag{14}$$

The alternating part of $v(t)$ is therefore proportional to the vibration displacement of the structure. If d is not small compared to h_0 and if one supposes that $v(t)$ is proportional to d in accordance with (14), one introduces a percentage error having a peak-to-peak value given by

$$\delta = \frac{(d/h_0)^2}{1 - (d/h_0)^2} \tag{15}$$

This is plotted in Figure 8.

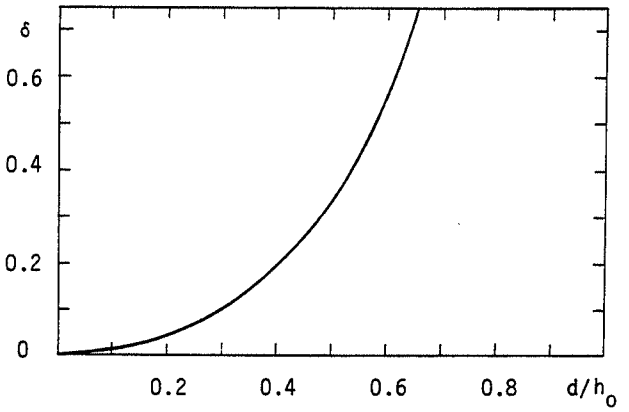


Figure 8 Peak-to-peak percentage error of a capacitive transducer

Magnetic transducer

A schematic diagram of a magnetic transducer is given in Figure 9. It shows a magnetic circuit composed of a coil wound on a permanent magnet and a structure made of a ferromagnetic material. If the structural material is nonmagnetic, such as aluminium or austenitic steel, a thin ferromagnetic disk can be bonded to the structure just beneath the transducer.

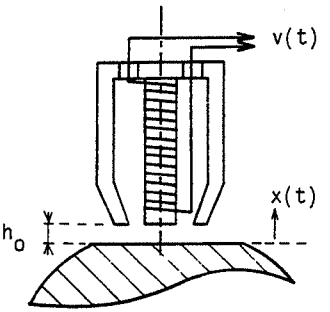


Figure 9 Magnetic transducer

At rest, the transducer and the structure are separated by a small distance h_0. The magnetic flux across h_0 is ϕ.

When the structure vibrates relative to the fixed transducer, the flux varies and a voltage is induced in the coil. This voltage, $v(t)$, is proportional to the product of the rate of change of ϕ with t and to the velocity of the structure. If the terminals of the electric circuit are connected to a high impedance and if the amplitude of sinusoidal displacement d is small compared to h_0, it can be shown that a first approximation is

$$v(t) = \alpha d\Omega \sin \Omega t \tag{16}$$

The magnetic transducer is therefore a velocity pickup. Its sensitivity depends on h_0 but an increase in sensitivity is accompanied by an increase in nonlinearity.

Eddy-current transducer

This transducer consists of a coil in which flows a high-frequency current, usually in the megahertz range. The coil produces a magnetic field and if a structure made of a conductive material is placed in this field, there is a dissipation of energy which is a function of the material and the distance separating the transducer from the structure. This dissipation of energy affects the inductance of the coil and this is reflected in the amplitude variation of the high-frequency voltage across the coil. After demodulation, the coil voltage is proportional, under certain conditions, to the distance

between the transducer and the structure. The transducer therefore acts as a displacement pickup. This transducer can be used for static as well as dynamic measurements. Its sensitivity is function of the nature of material.

6.2 Exciters

As was the case for transducers, the basic principles of vibration exciters are presented without a discussion of the technical details of their construction. Also as for transducers, exciters are classified into contacting and non-contacting types.

6.2.1 Exciters in contact with the structure

Electrodynamic shaker

This shaker generates a force by passing a current I through a coil mounted in a magnetic field of flux density B; see Figure 10. The force is transmitted to the structure by means of a fixture. The force generated in the exciter coil is

$$F = 2\pi RNBI \tag{17}$$

where N is the number of turns in the coil and R is the coil radius. If the current variation is sinusoidal of frequency Ω, equation (17) becomes

$$F = 2\pi RNBI_0 \sin \Omega t \tag{18}$$

Since the exciter is attached to the structure, it can modify the dynamic behavior of the structure by introducing supplementary springs and masses. Electrodynamic exciters are capable of generating a wide variety of force shapes; in particular, harmonic forces can be generated in a very pure form. The useful operating range of frequency is usually from a few hertz to about

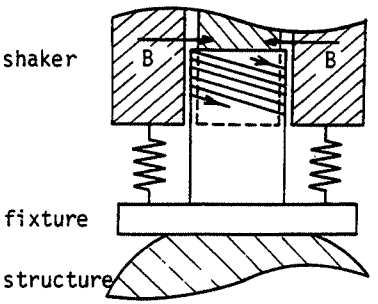


Figure 10 Electrodynamic shaker mounted to a structure by means of a fixture

10 kHz. The coil displacements are usually small, the maximum being of the order of a few centimeters, peak-to-peak.

Electrohydraulic shaker

This consists of a hydraulic cylinder and piston together with a servo-control system. The electronic control system allows the vibration signal to be generated easily and the hydraulic drive system can be designed to give large displacements and high forces. The useful operating frequency range is usually from fractions of a hertz to a few hundred hertz.

6.2.2 Exciters not in contact with the structure

These have the advantage of not affecting the dynamic behavior of the structure.

Magnetic exciter

This is essentially a magnetic transducer used as a driver rather than as a pickup. It is a convenient way of delivering low force levels with a device of simple design. Usually, the coil is wound on a permanent magnet whose magnetic field is much stronger than that due to the current from the power amplifier; the permanent magnet is sometimes replaced by a DC electro-magnet. If the amplitude of motion of the structure is much less than the static separation distance, the excitation force has the form

$$F = F_0 + F_1 \cos \Omega t \tag{19}$$

where F_0 is the constant force of attraction.

In the case when the coil has a core of ferromagnetic material rather than a magnetic core, it can be shown that

$$F = F_0 + F_1 \cos 2\Omega t \tag{20}$$

If the amplitude of motion is large, the excitation force contains non-negligible harmonics and it is necessary to use a tracking filter.

When the structure is nonmagnetic, direct excitation can be obtained by fixing a thin disk of ferromagnetic material to the surface of the structure.

6.3 Measurements

Under laboratory conditions, the experimenter can impose a known excitation on a structure in order to determine its dynamic characteristics; that is, its resonant frequencies, modes, and modal damping. The imposed excitation can be a sinusoidal frequency sweep, a white noise, or an impact.

Under field conditions, the experimenter cannot always impose an excitation of his choice. It is necessary to measure and analyze the operating

behavior of the structure in order to detect problems and assess performance.

6.3.1 Measurement system for sinusoïdal excitation

If the exciter is in contact with the structure, one can eliminate the influence of the fixture and exciter mass and stiffness by using transducer 1, depicted in Figure 11, as a reference. If this transducer is a force transducer, the measurement is the response to an imposed force; if it is a displacement transducer, the response is due to imposed displacement.

If transducers 1 and 2 measure the same quantities, either forces or displacements, the ratio of these like quantities is the force or displacement transmissibility. If transducers 1 and 2 measure different quantities, the ratios of these quantities can, be x_2^{∞}/F_1, acceleration mobility, x_2°/F_1, velocity mobility, or x_2/F_1, displacement mobility. The latter quantity is sometimes called the dynamic compliance or the receptance. If the reciprocals of these quantities are measured, i.e., F_1/x_2^{∞}, F_1/x_2°, F_1/x_2, one obtains, respectively, the dynamic mass, the mechanical impedance, or the dynamic stiffness. If points 1 and 2 coincide, these ratios are driving point quantities; if points 1 and 2 are different, they are transfer quantities.

As shown in Figure 11, an oscilloscope is used to monitor the quality of the electrical signals. The reasons for this are:

(a) the sinusoïdal excitation may not be sufficiently pure

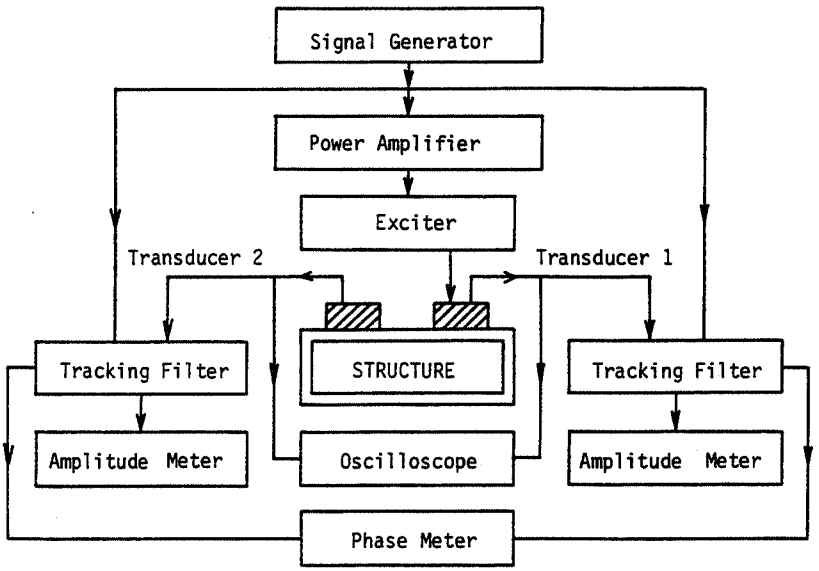


Figure 11 Measurement system for sinusoidal excitation

(b) the signal level can be reduced if it is too large or increased if it is below the noise level
(c) system nonlinearities can be detected.

The experimenter is thereby made aware when it is necessary to adjust the excitation level or to be particularly careful in the processing of the response. The tracking filter guarantees that the response is measured only at the excitation frequency. The combination of a shaker which can deliver a very pure sine wave and a tracking filter to minimize harmonics results in a response with very little extraneous noise. This is a distinct advantage of the measuring system. The transducer signal amplitudes and phase difference are the most important quantities to be measured.

Determination of dynamic characteristics

The first step is to plot the modulus and phase of the response over the full frequency range of interest. Such a frequency response plot allows estimation of the dynamic characteristics of the structure and thereby the amount of damping and position of resonant frequencies. If the structure is lightly damped and the modes well separated, each peak of the modulus plot corresponds to a resonant frequency. The corresponding mode can then be determined by dwelling at each resonant frequency and making a spatial survey of the structure. Lastly, a narrow sweep can be made around each resonant frequency to establish the half-power bandwidth from which the modal damping or Q-factor can be calculated.

If the resonant frequencies are strongly coupled, either because of large damping or close spacing, it is difficult to extract the individual modal characteristics of the structure accurately. In these circumstances, specialized computer techniques must be used, an explanation of which is beyond the scope of this text.

A polar plot of the modulus and phase of the response is known as a Nyquist diagram. This diagram is better suited for extracting modal information because the effects of frequency, damping, stiffness, and mass are magnified. If the damping is of the structural type, the response variable in the Nyquist diagram is usually displacement mobility (see exercise 16 of Chapter 1), and if the damping is of the viscous type, the response variable is usually velocity mobility.

As mentioned previously, the determination of the mode shape is done by exciting the structure at one of its resonant frequencies and surveying the resulting distribution of displacement amplitudes. Of particular interest are the nodal points or lines associated with each mode shape. A nodal line is a locus of points which has zero displacement amplitude. Nodal lines can be identified by a number of methods which depend on the geometry of the structure, the magnitude of the displacements, and the frequency of resonance. For example:

Stroboscopy. This method is used when the displacements are large

enough to be seen by the human eye and the frequency is higher than about twenty hertz.

Powder. In this method a thin layer of dry powder is sprinked on a structure whose geometry is a horizontal, planar surface. Under vibration, the grains of powder migrate to the nodal lines. It is necessary that the acceleration exceeds that of gravity. Sand is often used for this purpose.

Phase measurement. A nodal line separates two regions of opposite phase and this can be detected by comparing the phases of two transducers. One transducer is used as a fixed reference and the other is moved around the structure. At each point where the phase shifts by 180°, a nodal line has been crossed.

Amplitude minima. The usual procedure is to use an accelerometer attached to the top of a hand-held probe. The probe can then be used to search the structure surface for zones of minimum acceleration amplitude.

Holographic interferometry. This method can either be done in a time-averaged mode or in a real-time mode. The latter allows the complete pattern of nodal lines to be depicted at a given instant of time.

Of course, the shape and distribution of nodal lines is only a qualitative indication of the mode. A quantitative determination is usually done by using an array of piezoelectric accelerometers. Alternatively, one can use one fixed reference accelerometer and one moving accelerometer; this requires only two accelerometers, but more test time.

6.3.2 Spectral analysis

Spectral analysis is the process of representing a general function of time in terms of the amplitude and phase of a series of harmonic functions. In vibration, the general function is an analog signal which can be, for example, a force, an acceleration, or a ratio of a force over an acceleration. This spectral representation is a frequency-domain description of the time-dependent general function $x(t)$.

If $x(t)$ is the given time function, the Fourier transform establishes the relationship between $x(t)$ and its frequency-domain representation $X(f)$, i.e.

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft}\, dt \tag{21}$$

In (21), f is the frequency in hertz and $X(f)$ is a complex function of f. A schematic of the relationship between $x(t)$ and $X(f)$ is presented in Figure 12.

In practical situations, $x(t)$ is given in the form of an analog signal and not as an analytical formula. Furthermore, it is known only in a finite time interval (t_1, t_2); therefore, instead of using (21), it is necessary to use the discrete form of the Fourier transform; namely:

$$X\left(\frac{n}{NT}\right) = T \sum_{k=0}^{N-1} x(kT) e^{-j2\pi n(k/N)} \tag{22}$$

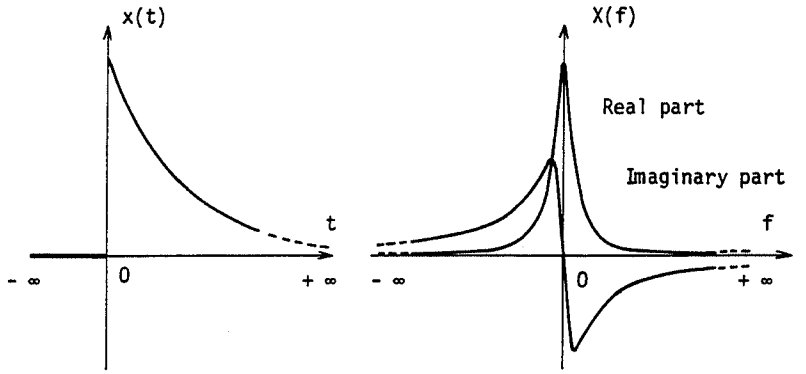


Figure 12 A function and its Fourier transform

where $n = 0, 1, \ldots, N-1$, and:

N, total number samples of x in the time domain, which also equals the total number of spectral lines of X in the frequency domain
T, the time interval between two successive samples of x
n, the numbers assigned to the spectral lines of X.

More information on the discrete Fourier transform (DFT) and on the fast Fourier transform (FFT) can be found for example in *The Fast Fourier Transform*, E. O. Brigham, Prentice-Hall, 1974. The relationship between a time-sampled function and its discrete frequency spectrum is shown schematically in Figure 13.

For a finite interval of time NT, it is first necessary to sample $x(t)$ in order to obtain N numerical values of $x(t)$ at the discrete times $t_1, t_1+T, t_1+2T, \ldots, t_2$. Then the DFT can be used to obtain the corresponding discrete spectrum. This spectrum is the frequency-domain representation of a periodic function of period NT which is an approximation to $x(t)$ in the interval (t_1, t_2).

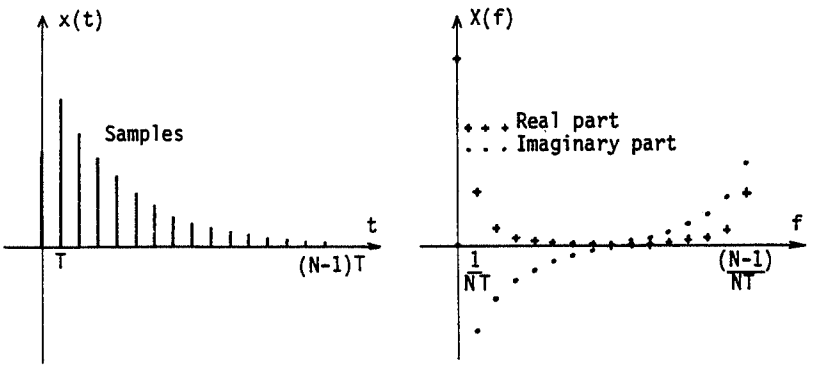


Figure 13 A sampled function and its discrete Fourier transform

A source of error in this process is the fact that sampling truncates the function at t_1 and t_2 and this can introduce discontinuities in the periodic representation of the function. Such discontinuities introduce error into the frequency spectrum. This error, which is called leakage, can be minimized by tapering the sampled amplitudes $x(nT)$ with a data window such as a triangular function (Bartlett window) or a shifted cosine function (Hanning window).

The sampling theorem due to Shannon states that a time function can be uniquely reconstructed from its spectrum, if the frequency $f_n = 1/T$, known as the Nyquist rate, is such that $f_n \geq 2f_c$. If $f_n < 2f_c$ the frequency spectrum is not acceptable. This error, which is called aliasing, can be minimized by sending $x(t)$ through a lowpass filter whose cut-off frequency is set $\leq f_c$ before the signal is sampled. These are called anti-aliasing filters. In practice, one usually selects a sampling rate so that $f_n = \alpha f_c$, where α is in the range of 3 to 4.

If the signal represents a displacement or a force at a point of a structure which is subjected to its actual operating conditions, real-time spectral analysis allows an immediate determination of the frequencies at which the structure has large response amplitudes. This then provides insight into either excitation frequencies which must be avoided, or frequencies which can be the resonant frequencies of the structure.

White noise is often used as a convenient form of broadband excitation. White noise is a random analog signal whose spectrum is continuous and constant over a range of frequencies; if the range of frequencies is broad, the white noise is broadband. The response of a linear structure to a white noise excitation is the superposition of the responses to all the frequencies in the excitation bandwidth. Spectral analysis of this response will give the same information as sinusoïdal sweeping but in this case all the frequencies are excited simultaneously and the data is processed in real-time. However, the response to a single sample of broadband excitation can be corrupted with noise; this noise can be minimized by averaging over many samples but this tends to offset the time advantage gained by real-time spectral analysis.

Impact excitation can be imposed by hammer blows and as a first approximation, mathematically represented by a Dirac function of force. The Fourier transform of a Dirac function gives a continuous and constant frequency spectrum, the same type of spectrum associated with broadband white noise. One can then utilize a shock excitation in place of white noise excitation. Shock excitation caused by impact with specially instrumented hammers is now widely used by specialists in the identification of the dynamic characteristics of structures. As for white noise, the spectral analysis of the response from impact excitation can be carried out in real-time. However, control over the signal bandwidth is limited and too high initial excitation levels will tend to excite structural nonlinearities and to overload the measurement system.

The ability to generate frequency spectrums rapidly has not only permitted real-time analysis for measurement purposes but it has facilitated the development of the field of machinery health monitoring. A single instrument can repeatedly scan a large number of transducers and compute a spectrum analysis of the signal for each one. These spectra can then be stored digitally and compared with their predecessors so that changes in the vibration signal can be identified. Such changes are often the precursors to failure or serious performance degradation.

7
Computer Programs

The computer programs presented here can be used on a microcomputer. They are useful for both beginners and experienced workers in mechanical vibration. For the former, these programs allow experience with the dynamic behavior of multi-degree of freedom structures without the labor of lengthy hand calculation. For the latter, they offer reference solutions, means of comparing various solution techniques, and the possibility of solving real problems on a desk-top computer.

The programs are written in BASIC and can be used on a computer which has at least 16K bytes of core memory. In addition, they use some specialized matrix commands and have the ability to present plots on a graphic terminal. They are easily adaptable to another language and their size can be greatly reduced by suppressing the comments and eliminating options, such as plotting.

When the algorithms have not been presented in the preceding chapters, they are briefly explained during the presentation of each program.

The contents of the chapter are as follows:

7.1 Language and Commentary

7.1.1 Language

The language used is standard BASIC. In addition, there are some matrix commands and plotting commands which are specific to the microcomputer used to develop the programs, and they will be briefly presented in this section.

Matrix commands

Addition, subtraction, and equality of matrices are done in the same way as for ordinary variables. Also, all the elements of a matrix can be set to the

Appendix: Lagrange's Equations

Lagrange's equations are deduced from the principle of virtual work applied to the forces, including inertia forces, acting on a system. These equations are easy to use and well suited for mechanical systems.

In what follows $q_1, \ldots, q_N$ are a set of N independent generalized coordinates for a system having N degrees of freedom. The N Lagrange's equations which will give the N coupled differential equations of motion are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i \qquad i = 1, \ldots, N \tag{1}$$

with

T, kinetic energy

Q_i, generalized force corresponding to the coordinate q_i.

In most cases, forces induced by strains can be deduced from the potential strain energy U and the forces due to viscous damping from a Rayleigh's dissipation function R. Then (1) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} + \frac{\partial R}{\partial \dot{q}_i} = Q_i \qquad i = 1, \ldots, N \tag{2}$$

In (2), Q_i is the vector of external forces. This expression is commonly used in this book.

When the generalized coordinates are no longer independent it is necessary to eliminate some of the coordinates in T, U, R before applying (1) or (2). If this elimination cannot be performed, Lagrange multipliers have to be used. In this book the generalized coordinates are always independent.

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The above bibliography presents the existing state of knowledge in mechanical vibrations for engineers; advances in this knowledge are published in a wide variety of research journals. A comprehensive set of abstracts from these journals as well as surveys of the latest technology in shock and vibration can be found in *The Shock and Vibration Digest* which is available from the Shock and Vibration Information Center, Naval Research Laboratory, Washington, DC 20375, U.S.A.

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MECHANICAL VIBRATIONS FOR ENGINEERS

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This is a compact text which covers both analytical and experimental methods in modern mechanical vibration. Approximate and numerical methods are used in preference to classical methods of exact solution. This emphasis on modern methods extends to the use of the microcomputer – 12 vibration analysis programs written in BASIC are included. There are also over 100 exercises, each with an answer and most with the method of solution presented in detail. By working through these problems, any student should learn the important methods of modern vibration engineering.

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