CGAL - the Computational Geometry Algorithms Library

Pierre Alliez, Andreas Fabri, Efi Fogel

Computational Geometry Algorithms Library

Pierre Alliez	Andreas Fabri	Efi Fogel
INRIA	GeometryFactory	Tel-Aviv University

Abstract

The CGAL Open Source Project provides *easy access to* efficient and reliable geometric algorithms in the form of a C++ library, offering geometric data structures and algorithms, which are efficient, robust, easy to use, and easy to integrate in existing software. The usage of de facto standard libraries increases productivity, as it allows software developers to focus on the application layer. This course is targeted at software developers with geometric needs, and course graduates will be able to select and use the appropriate algorithms and data structures provided by CGAL in their current or upcoming projects.

Key Facts

CGAL 3.3, released in June 2007: 90 software components, 600,000 lines of code, 3,500 user and reference manual pages, Cross platform support, Annual release with 12,000 downloads, 1,000 subscribers on the user mailing list, 40 subscribers on the developer mailing list. CGAL is used in many application areas by companies as Total (Oil&gas), British Telecom (Telecom), Cadence (VLSI), Leica Geosystems (GIS), Dassault Systèmes (CAD), The Moving Picture Company (Visual effects).

CGAL Project

The project is steered by an Editorial Board, it has a well defined development process, and the infrastructure for distributed development. The following research institutes and companies are actively involved or made contributions to the library: INRIA-Sophia-Antipolis, Max-Planck Institute for Computer Science, Tel-Aviv University, GeometryFactory, ETH Zurich, FU Berlin, University of Groningen, University of Utrecht, Stanford University, Athens University, and the Foundation of Research and Technology – Hellas. For more information on the project see www.cgal.org For an overview on what is in CGAL:
http://www.cgal.org/Manual/3.3/doc_html/cgal_manual/packages.html

Organization of the Course

This course starts with an overview followed by three in-depth sessions covering central data structures: The overview session presents the CGAL project and the design principles of CGAL. CGAL has adopted the *exact computing* paradigm, which yields robust and at the same time fast algorithms. CGAL further has adopted the *generic programming* paradigm, which makes CGAL particularly easy to customize and to integrate. Finally, we show how CGAL fits naturally with the STL, and the Boost graph library.

The session on *polyhedral surfaces* presents the underlying halfedge data structure and how it can be customized to user needs. We further present algorithms for polyhedral surfaces like parameterization, mesh subdivision and simplification, Boolean operations, and intersection detection.

The session on *arrangements* presents the arrangement API and several data structures built on top of it. These are 3D Minkowski sums, which can be used for collision detection, and 3D lower envelopes, which can be used for visibility map computations.

The last session covers the 2D and 3D *triangulation* API as well as the surface and volume mesh generators, which are based on Delaunay refinement.

The source code of the examples will be made available at http://www.cgal.org/Courses/SIGGRAPH2008.

Presenters

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Biographies

Andreas Fabri, PhD, GeometryFactory As member of the initial development team of the CGAL project, Andreas Fabri is one of the architects of the CGAL software. For several years he chaired the CGAL Editorial Board. In 2003 Andreas founded the GeometryFactory as spin-off of the CGAL project, offering licenses, service and support to commercial users, who cannot comply with the Open Source license of CGAL. Andreas received his PhD in computer science in 2004 from Ecole de Mines de Paris while working on geometric algorithms for parallel machines at INRIA.

Pierre Alliez obtained his PhD from Ecole nationale supérieure des Télécommunications, did his postdoc at Caltech, and is researcher at INRIA since 2001. His main research interests are on topics commonly referred to as Geometry Processing: geometry compression, surface approximation, mesh parameterization, surface remeshing and mesh generation. He is this year co-chair of the EUROGRAPHICS Symposium on Geometry Processing. In 2005 Pierre Alliez received the Eurographics Young Researcher Award.

Efi Fogel, MSc, Tel-Aviv University Efi Fogel is a co-founder of LucidLogix Ltd., a startup company that intends to deliver high performance 3D graphics systems. Efi Fogel is completing his Ph.D. studies at Tel-Aviv University. 3D Graphics and Computational Geometry are his main areas of interests. He is a member of the CGAL Editorial Board, and he is deeply involved with the design and implementation of the arrangement package of CGAL and its derivatives. Efi Fogel received his M.Sc. from Stanford University in 1989. He worked for Silicon Graphics Inc. (SGI) between 1989-1997 at the Advanced Graphics Devision, where he contributed to the specification of OpenGl among the other. After that Efi worked for Immersia Ltd, and he served as the CTO of Enbaya Ltd.

Course Syllabus

Overview	Andreas	30′
Polyhedron	Pierre	40′
Arrangements	Efi	40′
Break		20′
Triangulations & Meshes	Andreas, Pierre	80′
Wrap-up, Q&A	All	15′

Bibliography

- **Bibliographic entries for individual chapters of CGAL manuals**
- **CGAL User and Reference Manual Bibliography**

CGAL Contributors

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Computational Geometry Algorithms Library

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Efi Fogel Tel Aviv University

SIGGRAPH2008

Course Outline

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Polyhedron	Pierre	40′
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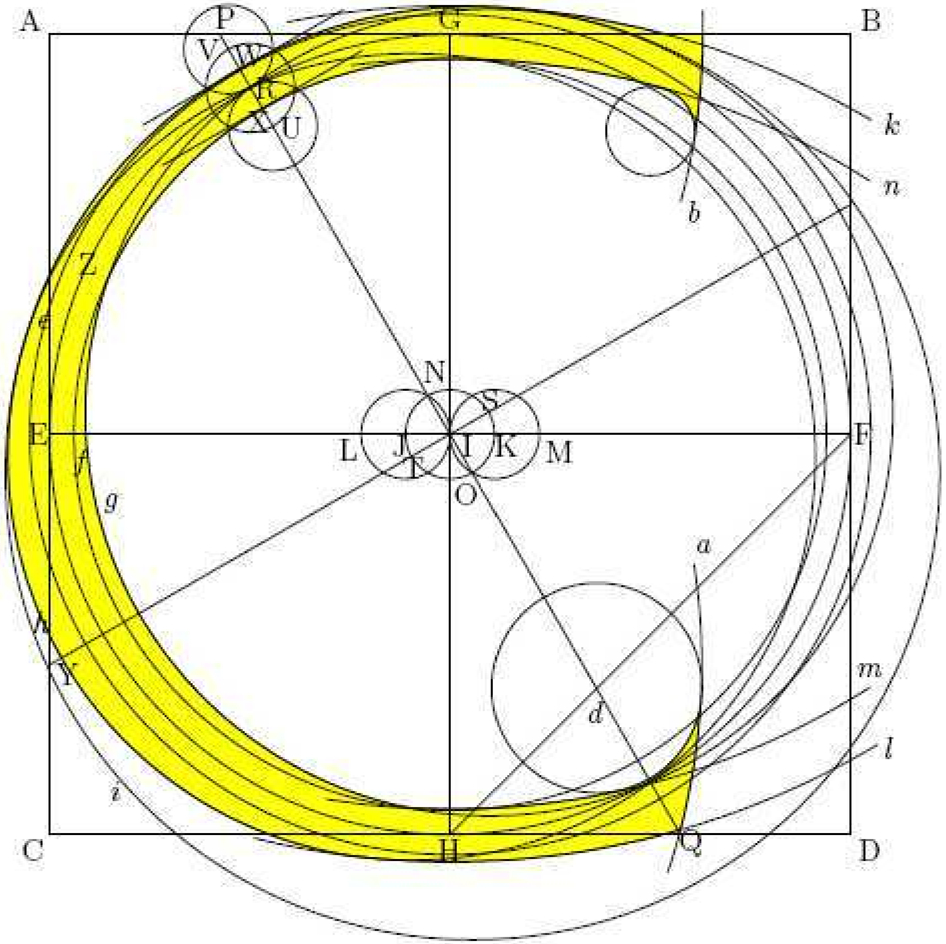
Break		15′
2D Triangulations & Meshes	Andreas	40′
3D Triangulations & Meshes	Pierre	40′
Wrap-up, Q&A		15′



Overview

Andreas Fabri

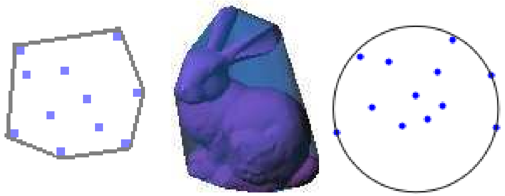
GeometryFactory

Mission Statement

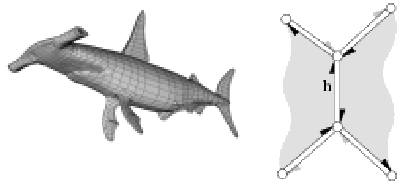
"Make the large body of geometric algorithms developed in the field of computational geometry available for industrial applications"

CGAL Project Proposal, 1996

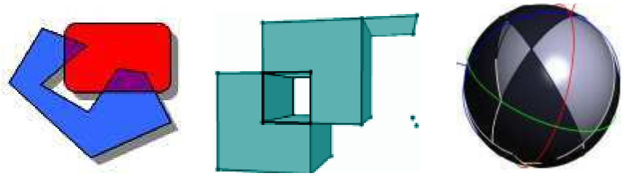
Algorithms and Datastructures



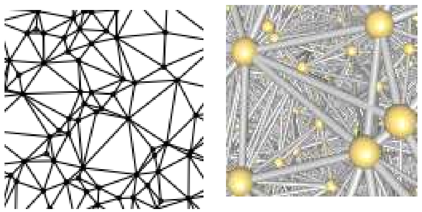
Bounding Volumes



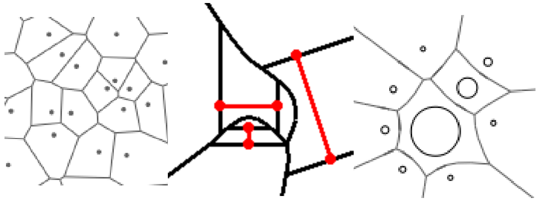
Polyhedral Surface



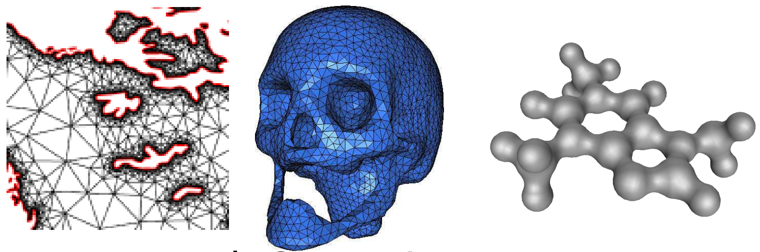
BooleanOperations



Triangulations



Voronoi Diagrams



Mesh Generation

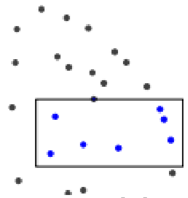
Subdivision

Simplification

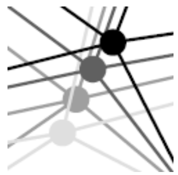
Parametrisation

Streamlines

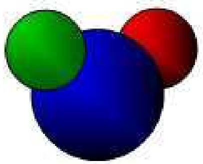
Ridge Detection



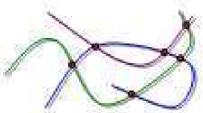
Neighbor Search



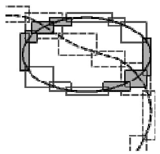
Kinetic Datastructures



Lower Envelope

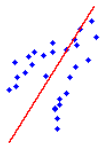


Arrangement

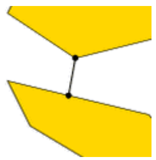


Intersection Detection

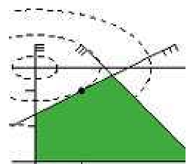
Minkowski Sum



PCA



Polytope distance

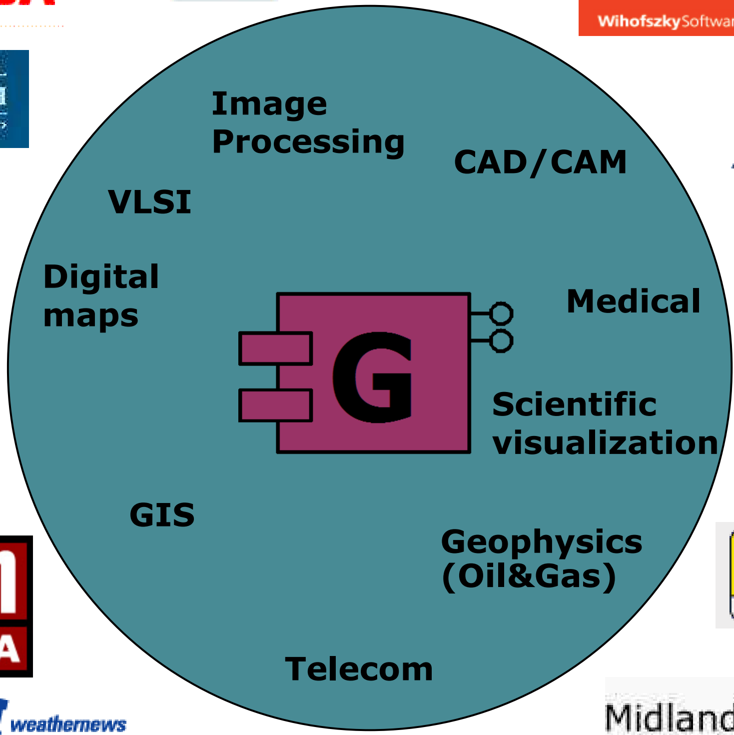


QP Sover

CGAL in Numbers

500,000 lines of C++ code
10,000 downloads/year (+ Linux distributions)
3,500 manual pages
3,000 subscribers to cgal-announce
1,000 subscribers to cgal-discuss
120 packages
60 commercial users
20 active developers
12 months release cycle
2 licenses: Open Source and commercial

Some Commercial Users



Why They Use CGAL

“ I recommended to the senior management that we start a policy of buying-in as much functionality as possible to reduce the quantity of code that our development team would have to maintain.

This means that we can concentrate on the application layer and concentrate on our own problem domain.”

Senior Development Engineer
& Structural Geologist

Midland Valley Exploration

Why They Use CGAL

" My research group JYAMITI at the Ohio State University uses CGAL because it provides an efficient and robust code for Delaunay triangulations and other primitive geometric predicates. Delaunay triangulation is the building block for many of the shape related computations that we do. [...]

Without the robust and efficient codes of CGAL, these codes could not have been developed. "

Tamal Dey
Professor, Ohio State University

CGAL Open Source Project

Project = « Planned Undertaking »

- Institutional members make a long term commitment: Inria, MPI, Tel-Aviv U, Utrecht U, Groningen U, ETHZ, GeometryFactory, FU Berlin, Forth, U Athens
- Editorial Board
  - Steers and animates the project
  - Reviews submissions
- Development Infrastructure
  - Gforge: svn, tracker, nightly testsuite,...
  - 120p developer manual and mailing list
  - Two 1-week developer meetings per year

Contributions

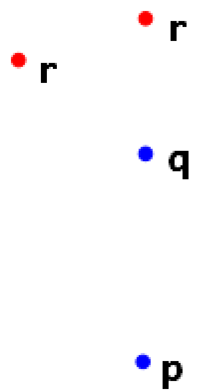
- Submission of specifications of new contributions
- Review and decision by the Editorial Board

- Value for contributor
  - Integration in the CGAL community
  - Gain visibility in a mature project
  - Publication value for accepted contributions

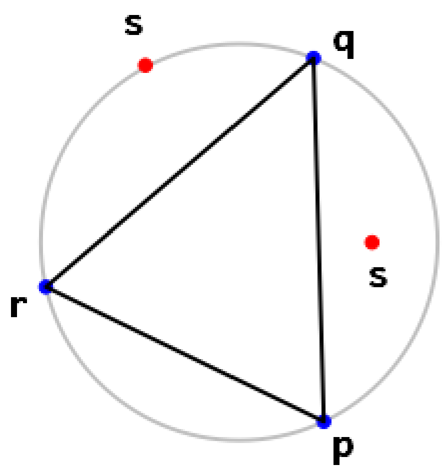
Exact Geometric Computing

Predicates and Constructions

Predicates

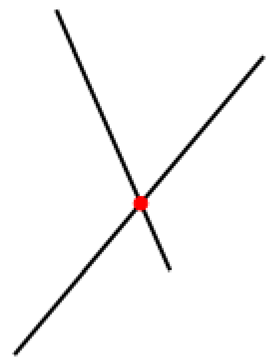


orientation

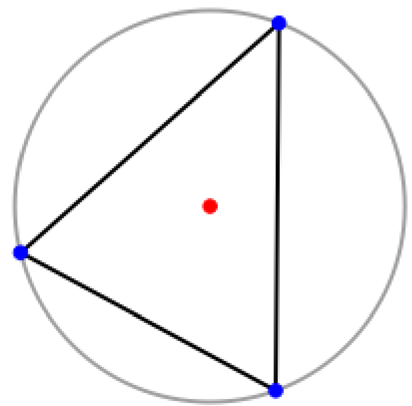


in_circle

Constructions



intersection



circumcenter

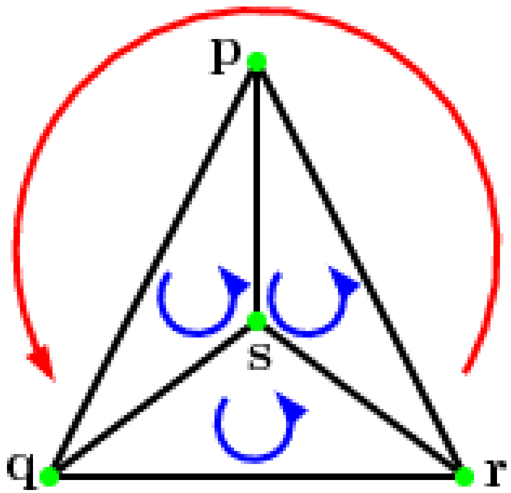
Robustness Issues

- Naive use of floating-point arithmetic causes geometric algorithms to:
  - Produce [slightly] wrong output
  - Crash after invariant violation
  - Infinite loop

- There is a gap between
  - Geometry in theory
  - Geometry with floating-point arithmetic

Geometry in Theory

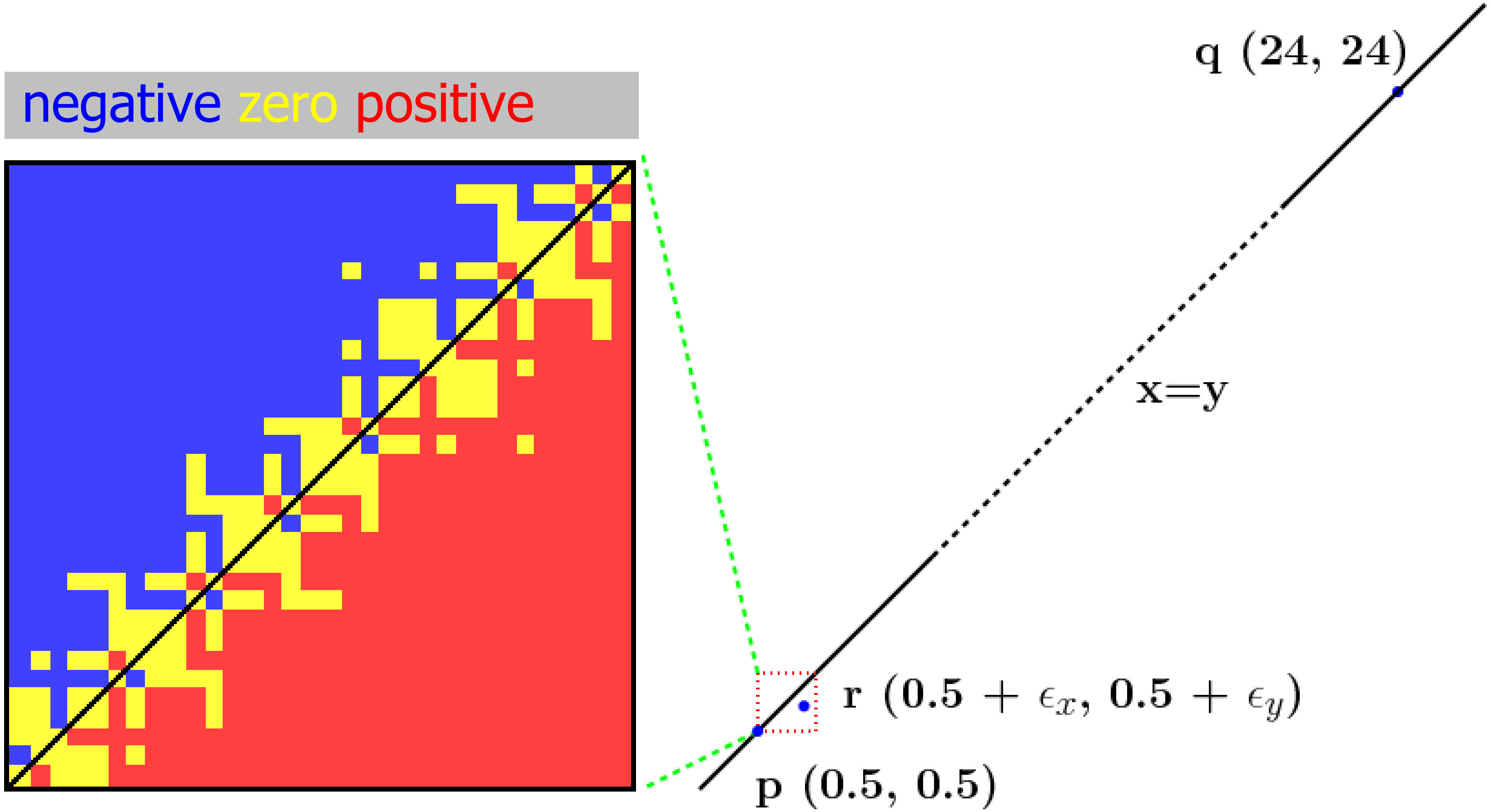
ccw(s,q,r) & ccw(p,s,r) & ccw(p,q,s) $\Rightarrow$ ccw(p,q,r)



Correctness proofs of algorithms rely on such theorems

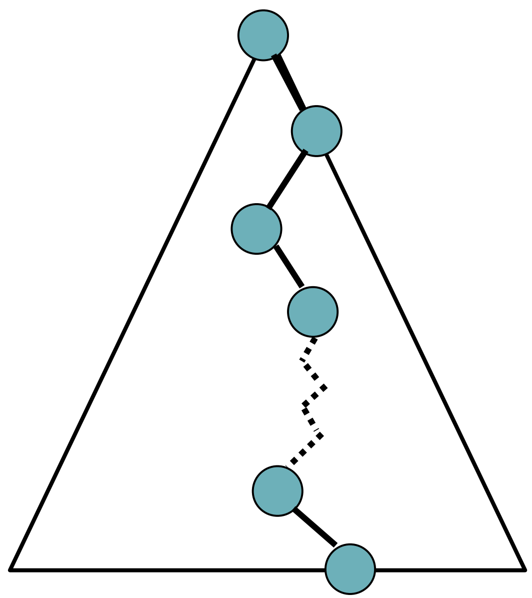
The Trouble with Double

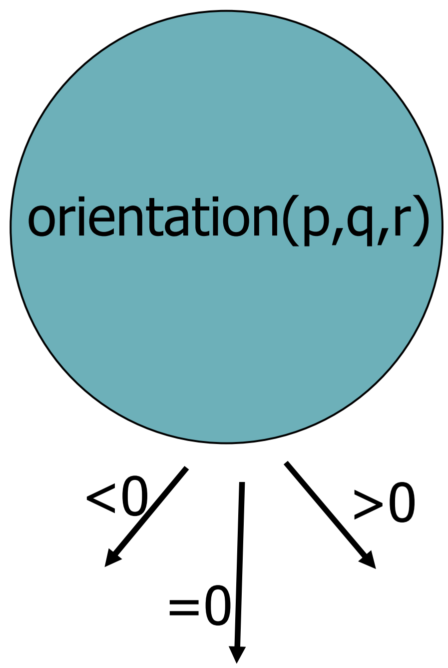
$$\text{orientation}(p,q,r) = \text{sign}((p_x-r_x)(q_y-r_y)-(p_y-r_y)(q_x-r_x))$$



Exact Geometric Computing [Yap]

Make sure that the control flow in the implementation corresponds to the control flow with exact real arithmetic





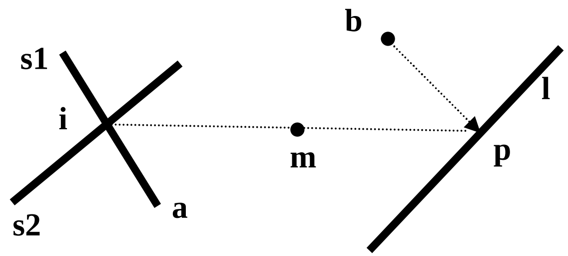
Filtered Predicates

- Generic functor adaptor Filtered_predicate<>
  - Try the predicate instantiated with intervals
  - In case of uncertainty, evaluate the predicate with multiple precision arithmetic
- Refinements:
  - Static error analysis
  - Progressively increase precision

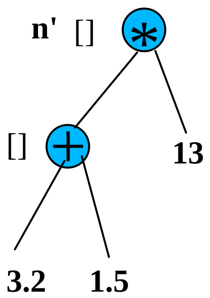
Filtered Constructions

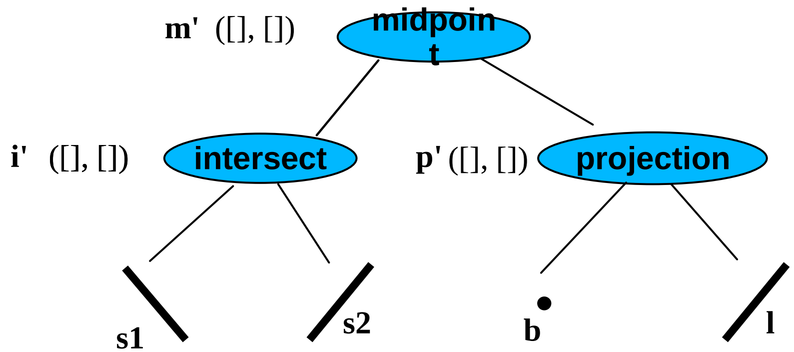
Lazy number = interval and arithmetic expression tree

Lazy object = approximated object and geometric operation tree



(3.2 + 1.5) * 13





Test that may trigger an exact re-evaluation:

if (n' < m')

if (collinear(a',m',b'))

The User Perspective

- Convenience Kernels
  - **Exact_predicates_inexact_constructions_kernel**
  - **Exact_predicates_exact_constructions_kernel**
  - **Exact_predicates_exact_constructions_kernel_with_sqrt**
- Number Types
  - **double, float**
  - **CGAL::Gmpq (rational), Core (algebraic)**
  - **CGAL::Lazy_exact_nt<ExactNT>**
- Kernels
  - **CGAL::Cartesian<NT>**
  - **CGAL::Filtered_kernel<Kernel>**
  - CGAL::Lazy_kernel<NT>

CGAL manual

Merits and Limitations

- Ultimate robustness inside the black box
- The time penalty is reasonable, e.g. 10% for 3D Delauny triangulation of 1M random points
- Limitations of Exact Geometric Computing
  - Topology preserving rounding is non-trivial
  - Construction depth must be reasonable
  - Cannot handle trigonometric functions

Generic Programming

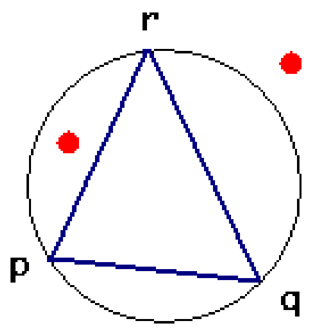
STL Genericity

```
template <class Key, class Less>
class set {
  Less less;

  insert(Key k)
  {
     if (less(k, treenode.key))
       insertLeft(k);
     else
       insertRight(k);
  }
};
```

STL manual

CGAL Genericity

```
template < class Geometry >
class Delaunay_triangulation_2 {
    Geometry::Orientation orientation;
    Geometry::In_circle in_circle;

    void insert(Geometry::Point t) {
        ...
        if(in_circle(p,q,r,t)) {...}
        ...
        if(orientation(p,q,r){...}
    }
};
```



CGAL Genericity

```
template < class Geometry, class TDS >
class Delaunay_triangulation_2 {


};



template < class Vertex, class Face >
class Triangulation_data_structure_2 {


};
```

Iterators

```
template <class Geometry>
class Delaunay_triangulation_2 {

  typedef .. Vertex_iterator;
  typedef .. Face_iterator;

  Vertex_iterator vertices_begin();
  Vertex_iterator vertices_end();

  template <class OutputIterator>
  incident_faces(Vertex_handle v, OutputIterator it);
};
std::list<Face_handle> faces;
dt.incident_faces(v, std::back_inserter(faces));
```

Iterators

```
template <class Geometry>
class Delaunay_triangulation_2 {

  template <class T>
  void insert(T begin, T end);// typeof(*begin)==Point
};
```

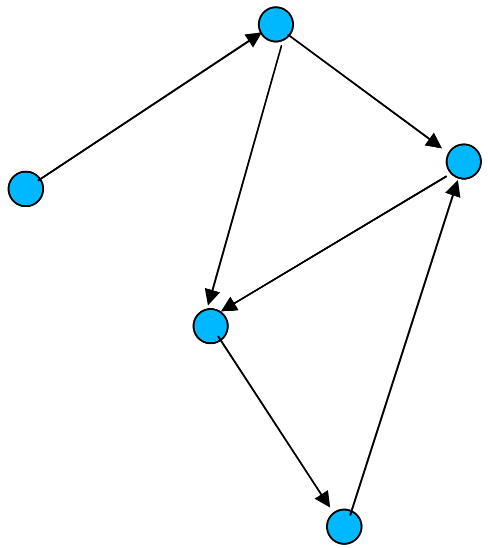
```
list<Kernel::Point_2> points;

Delaunay_triangulation<Kernel> dt;

dt.insert(points.begin(), points.end());
```

Boost Graph Library (BGL)

- Rich collection of graph algorithms:
  shortest paths, minimum spanning tree, flow, etc.
- Design that
  - decouples data structure from algorithm
  - links them through a thin glue layer
- BGL and CGAL
  - Provide glue layer for CGAL
  - Extension to embedded graphs
    inducing the notion of faces

BGL manual

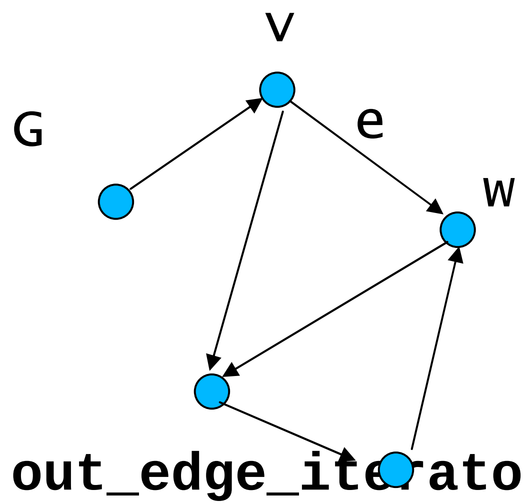
BGL Glue Layer: Traits Class



```
template <typename Graph >
struct boost::graph_traits {
    typedef ... vertex_descriptor;
    typedef ... edge_descriptor;
    typedef ... vertex_iterator;
    typedef ... out_edge_iterator;
};
```

BGL Glue Layer: Free Functions

```
vertex_descriptor v, w;
edge_descriptor e;

v = source(e,G);
w = target(e,G);
```




```
std::pair<out_edge_iterator, out_edge_iterator> ipair;

ipair = out_edges(v,G);
```

BGL Glue Layer for CGAL

CGAL provides partial specializations:

```
template <typename T>
graph_traits<Polyhedron<T>>;
```

```
template <typename T>
Polyhedron<T>::Vertex
source(Polyhedron<T>::Edge);
```

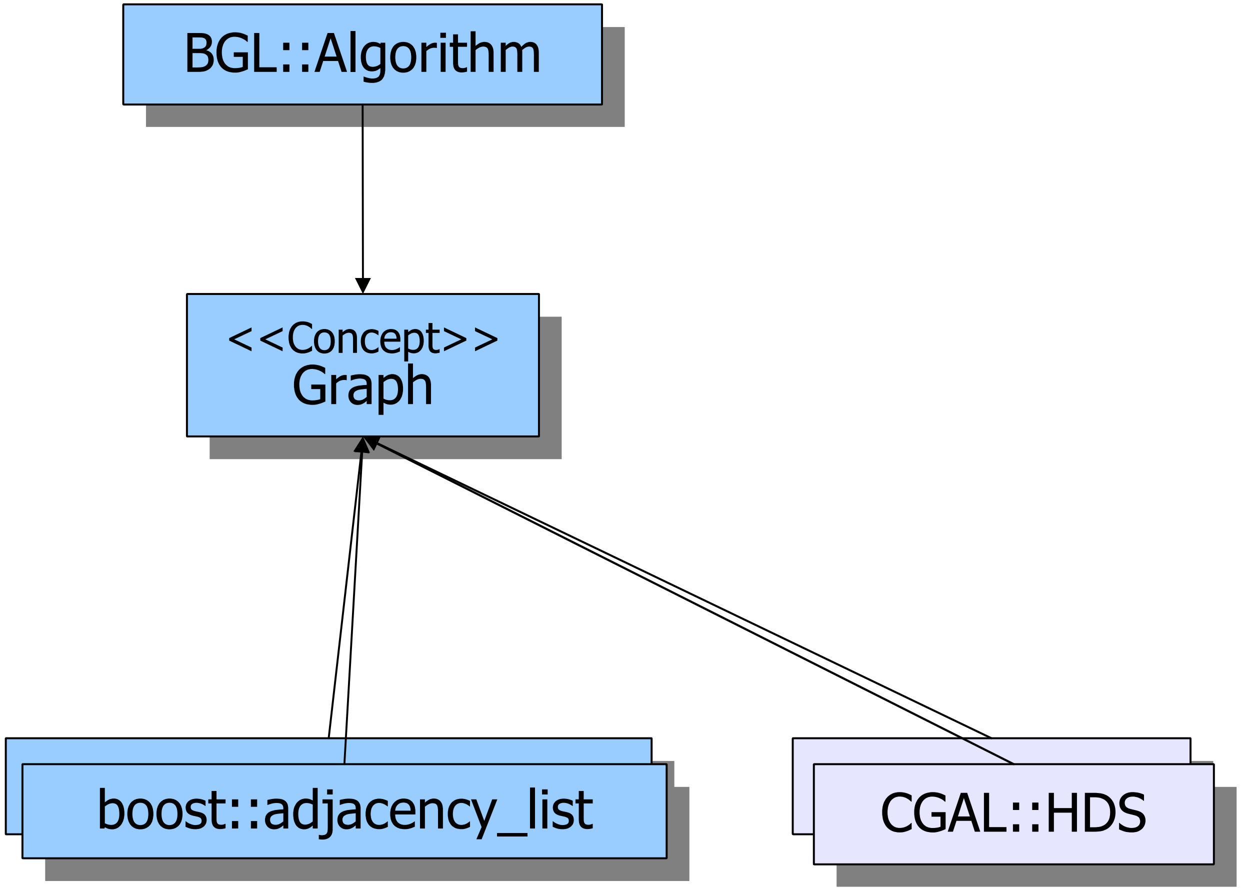
Users can run:

```
boost::kruskal_mst(P);
```

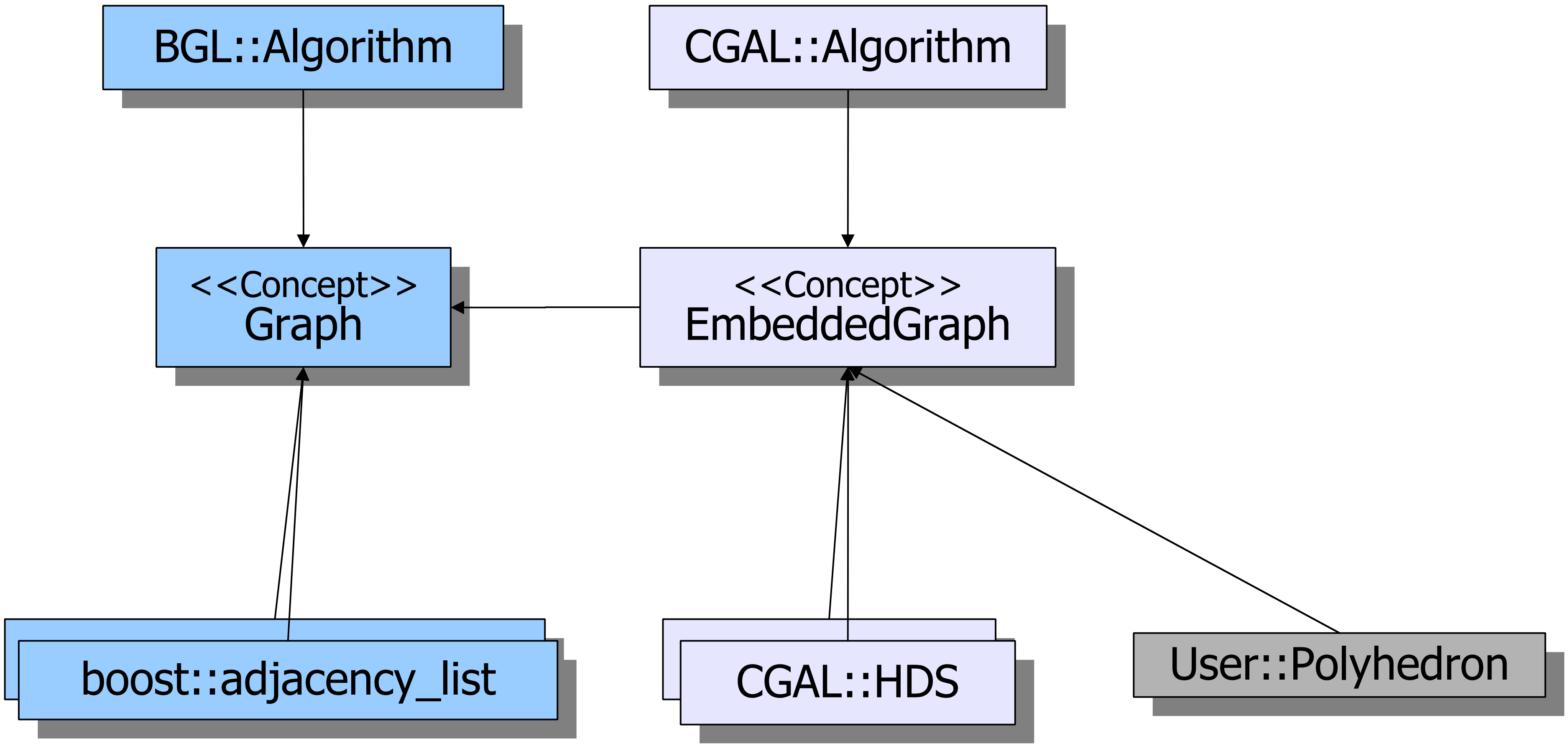
CGAL manual

Courtesy: P.Schroeder, Caltech

From A BGL Glue Layer for CGAL



To BGL Style CGAL Algorithms

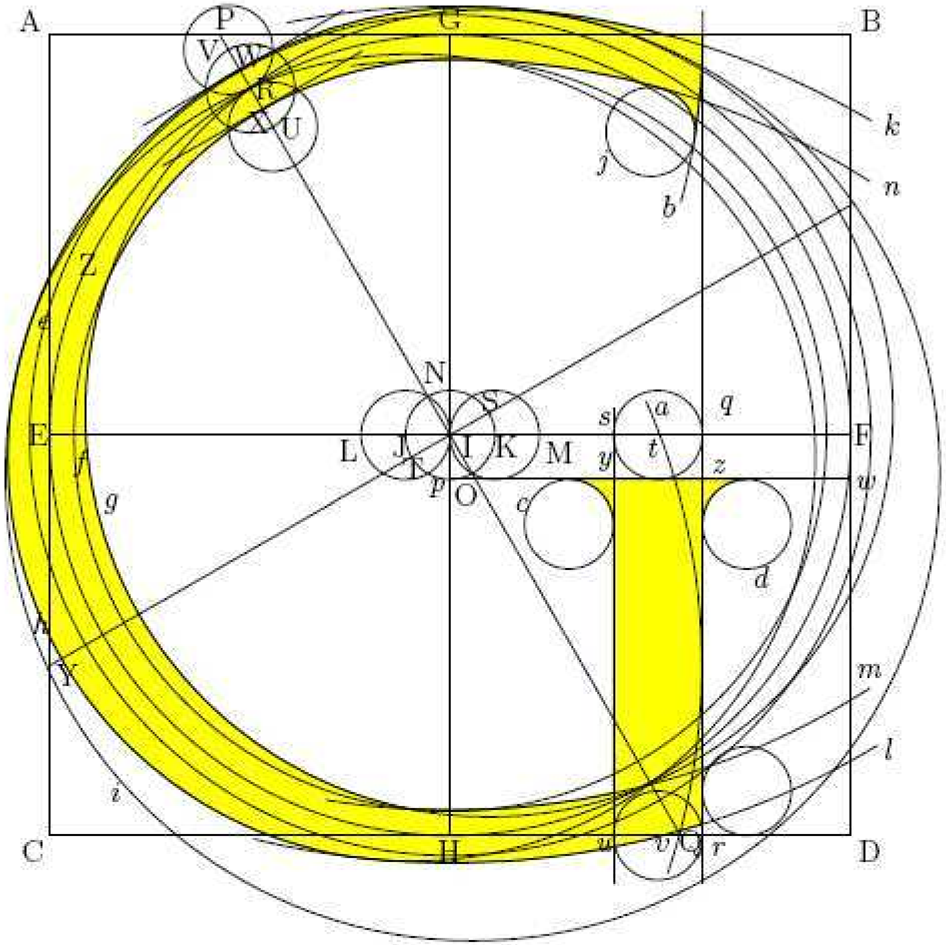


Summary: Overview

- Huge collection with uniform APIs
- Modular and not monolithic
- Open Source and commercial licenses

- Clear focus on geometry
- Interfaces with de facto standards/leaders:
  STL, Boost, GMP, Qt, blas

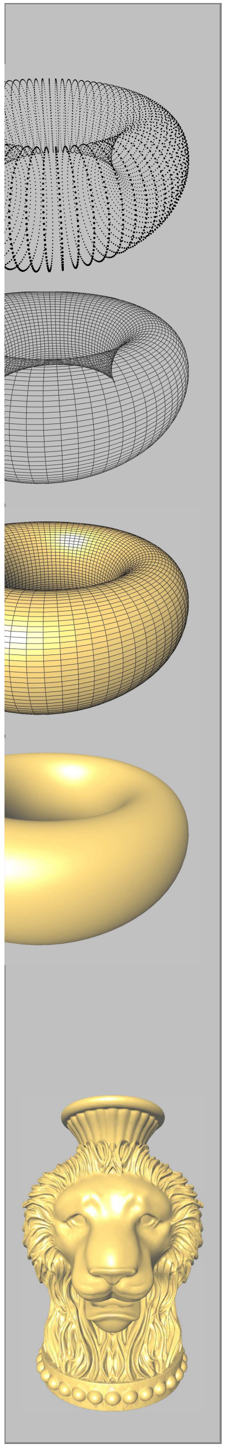
- Robust and fast through exact geometric computing
- Easy to integrate through generic programming



Polyhedral Surfaces

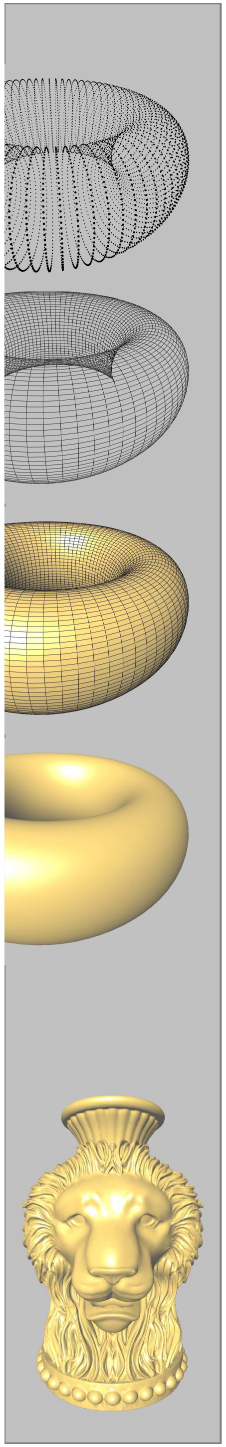
Pierre Alliez

INRIA



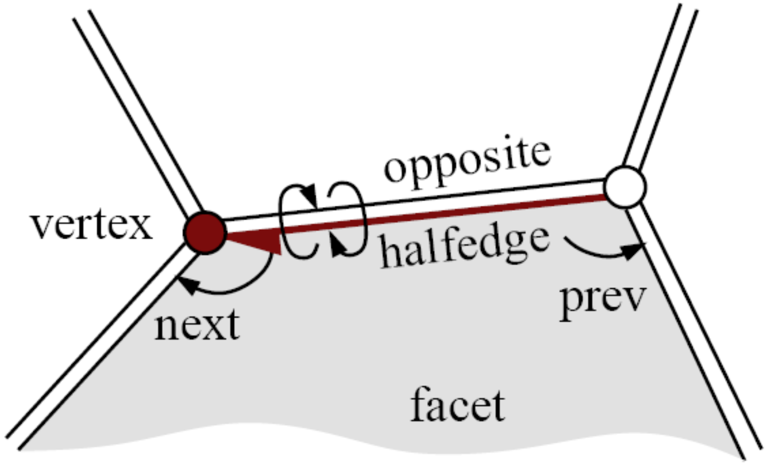
Outline

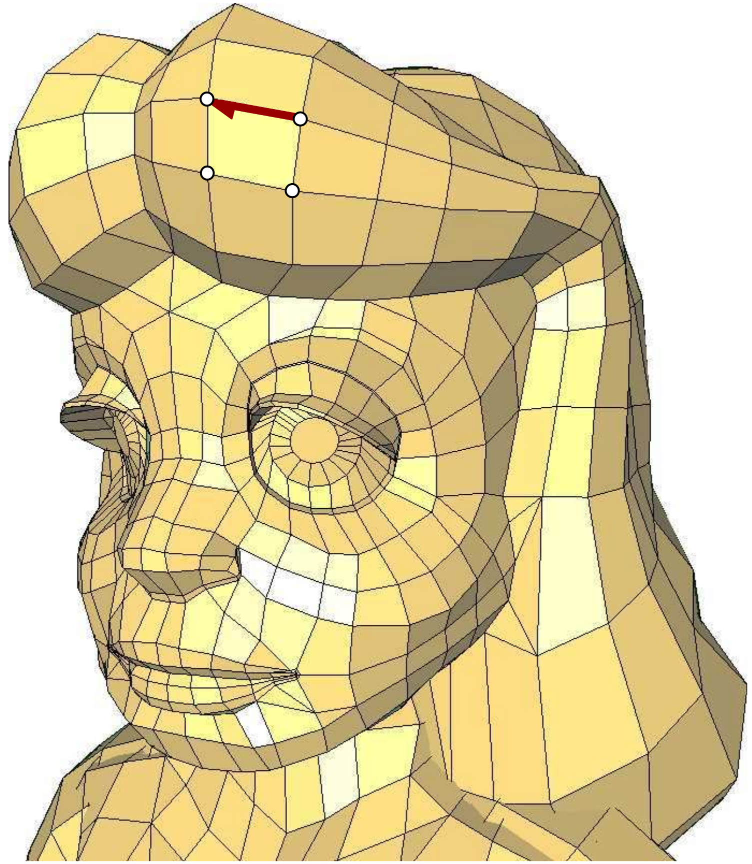
- Halfedge Data Structure and Polyhedron
- Euler Operators
- Customization
- Algorithms for Geometric Modelling and Geometry Processing



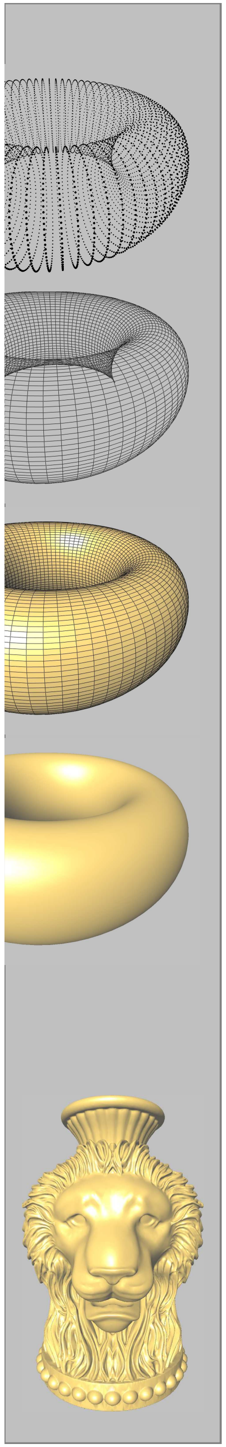
Halfedge Data Structure

Represented by vertices, edges, facets and an **incidence relation** on them, restricted to orientable 2-manifolds with boundary.



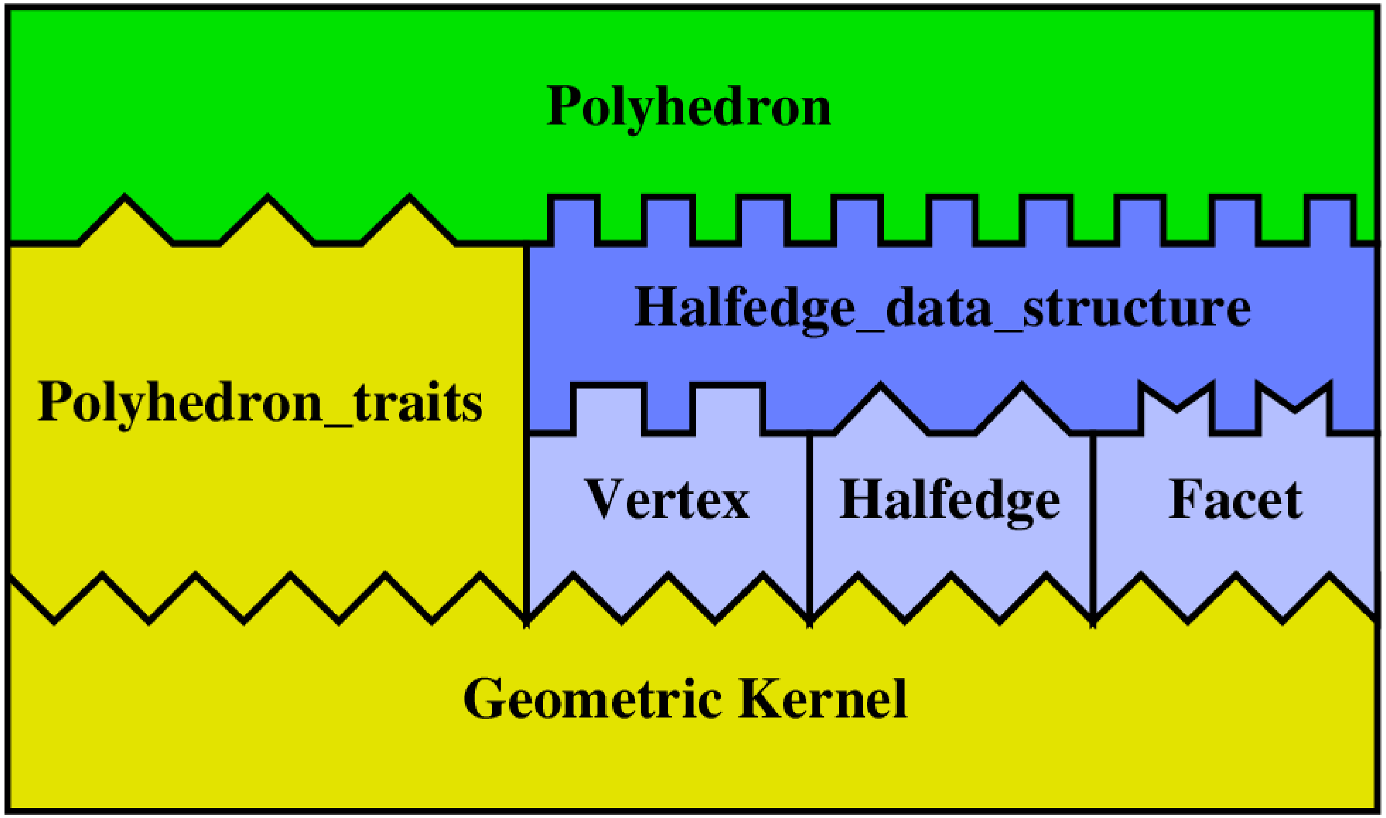


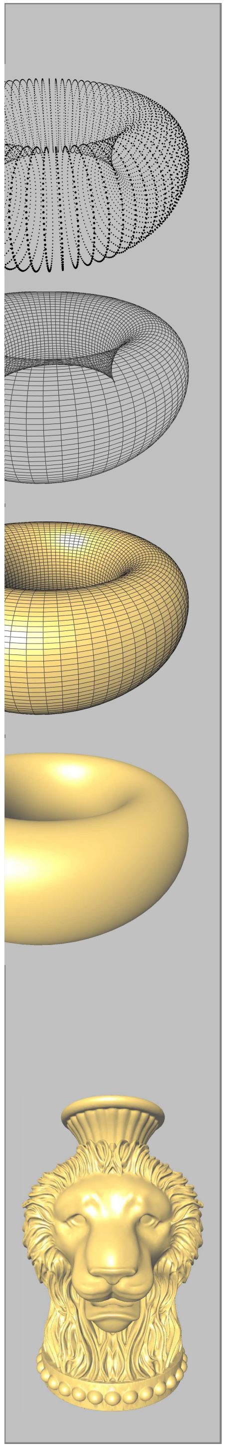
CGAL manual



Polyhedron

Building blocks assembled with C++ templates



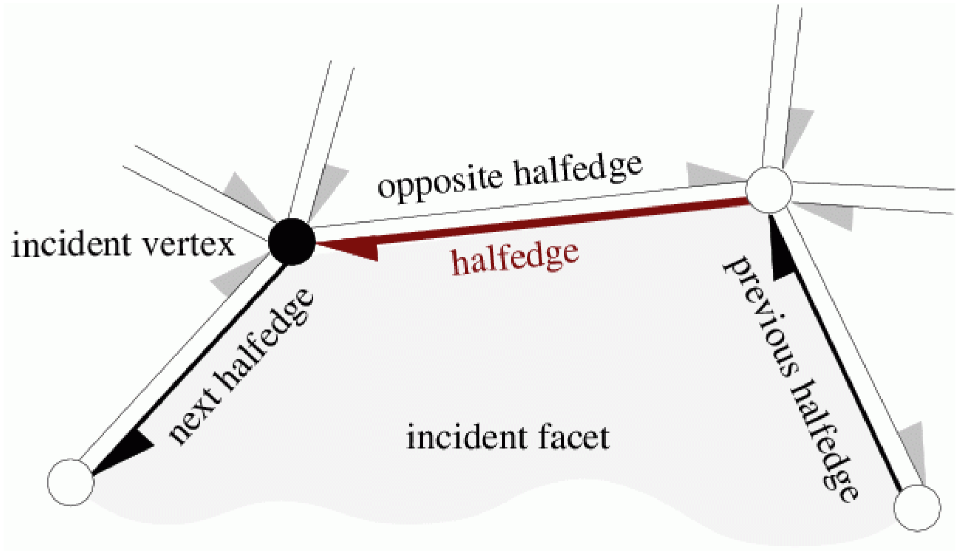


Default Polyhedron

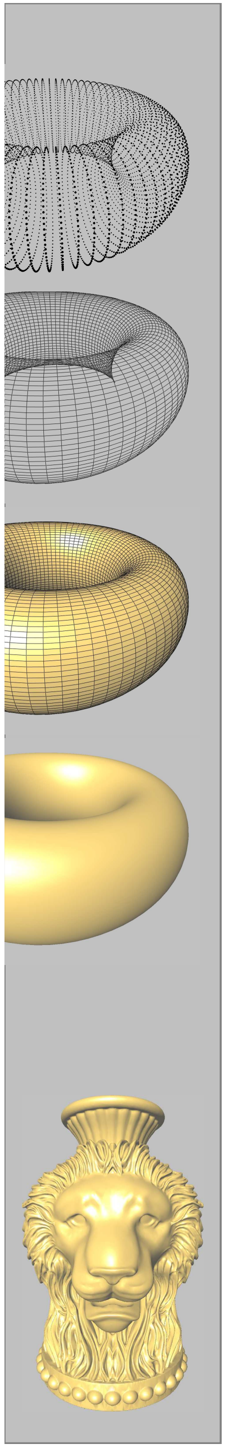
Vertex	
Halfedge_handle	halfedge()
Point&	point()
......	...

Halfedge	
Halfedge_handle	opposite()
Halfedge_handle	next()
Halfedge_handle	prev()
Vertex_handle	vertex()
Facet_handle	facet()
......	...

Facet	
Halfedge_handle	halfedge()
Plane&	plane()
Normal&	normal()
Color&	color()
......	...

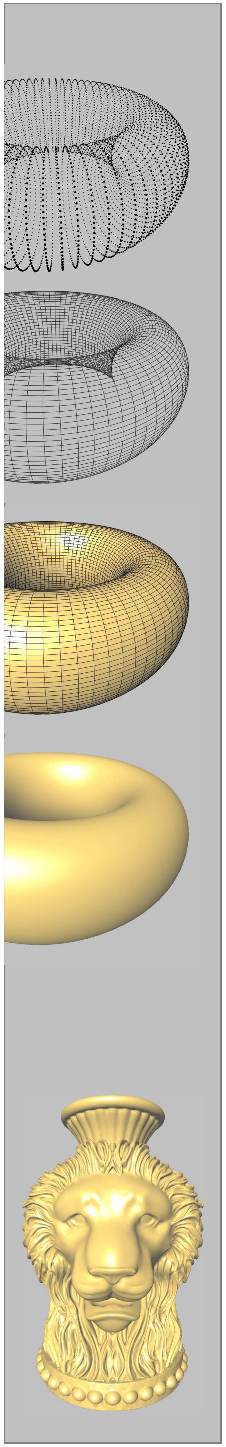


CGAL manual



Example

```
#include <CGAL/Cartesian.h>
#include <CGAL/Polyhedron_3.h>

typedef CGAL::Simple_cartesian<double> Kernel;
typedef CGAL::Polyhedron_3<Kernel>     Polyhedron;
typedef Polyhedron::Vertex_iterator    Vertex_iterator;

int main()
{
   Polyhedron p;
   // ... read from file or build

    Vertex_iterator v;
    for(v = p.vertices_begin();
        v != p.vertices_end();
        ++v)
    std::cout << v->point() << std::endl;
}
```

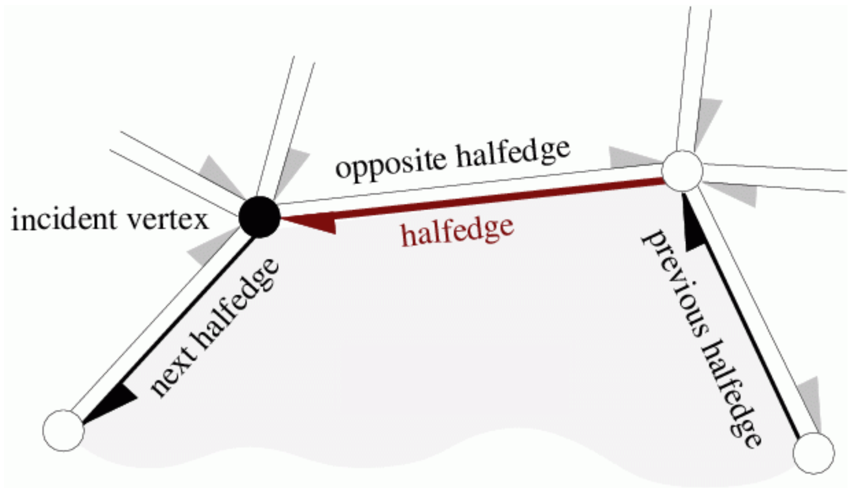


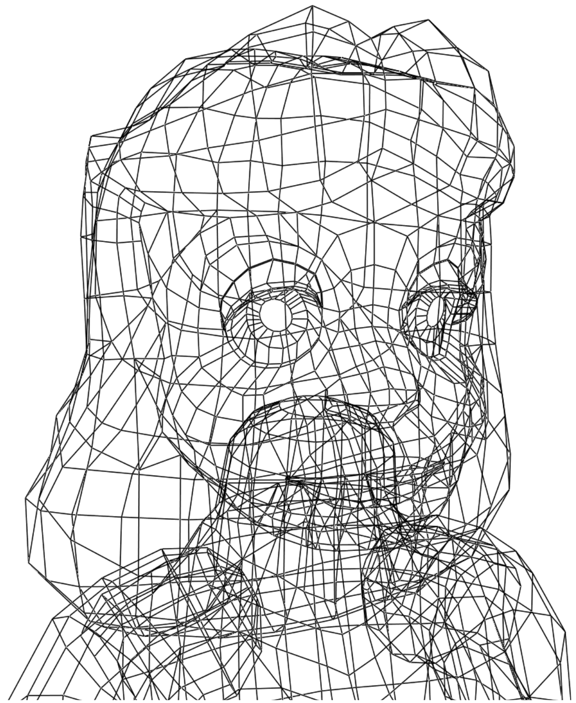
Flexible Data Structure

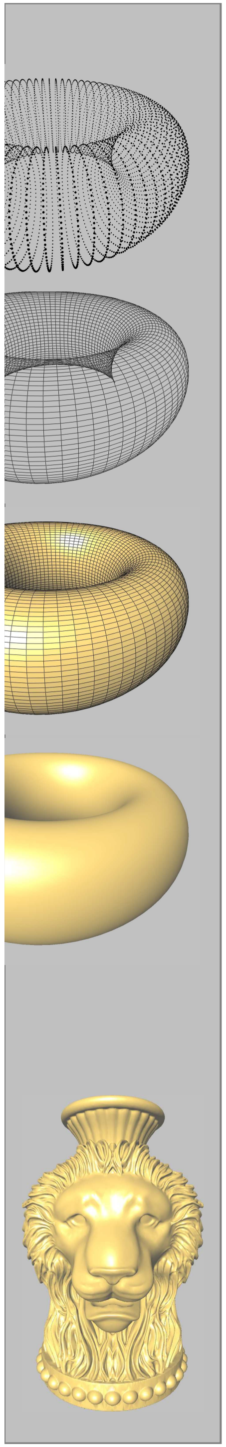
Vertex	
Halfedge_handle	halfedge()
Point&	point()
……	…

Halfedge	
Halfedge_handle	opposite()
Halfedge_handle	next()
Halfedge_handle	prev()
Vertex_handle	vertex()
Facet_handle	facet()
……	…

Facet	
Halfedge_handle	halfedge()
Plane&	plane()
Normal&	normal()
Color&	color()
……	…

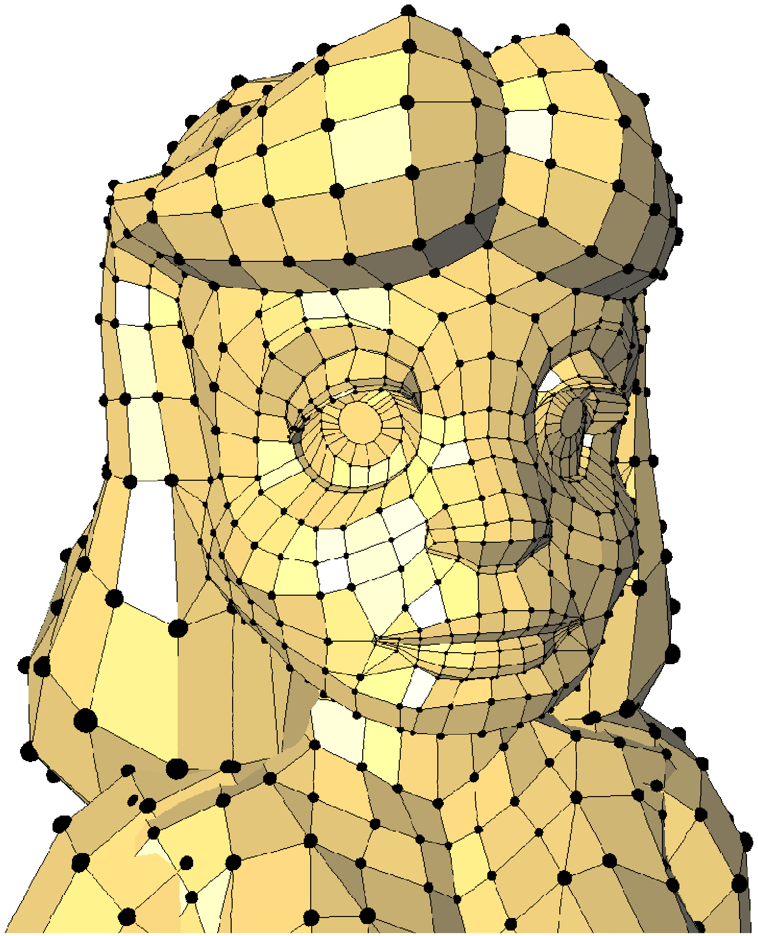


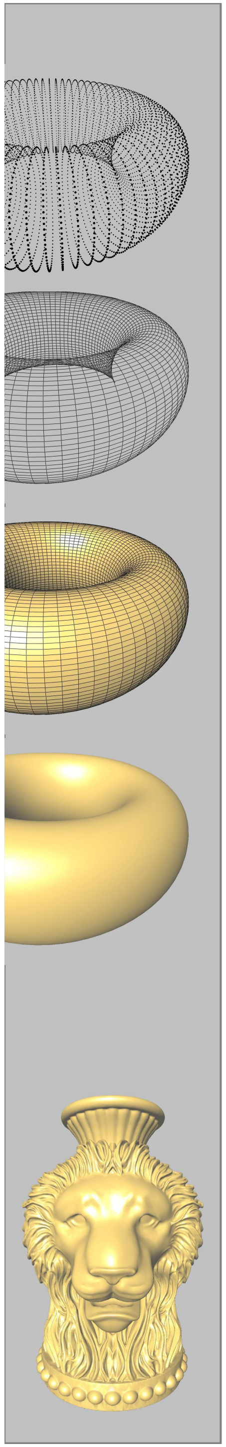




Iterate over all Vertices

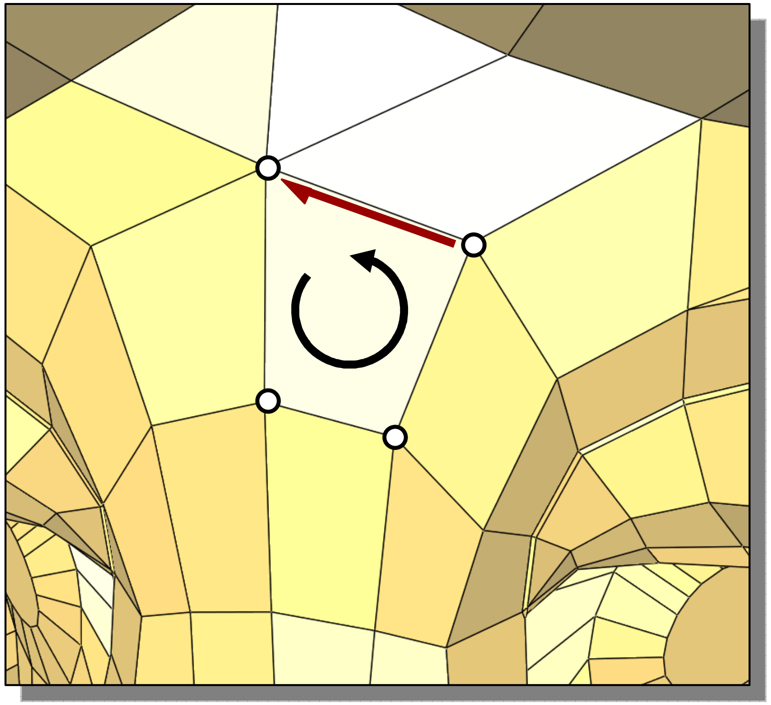
```
Vertex_iterator v;
for( v = polyhedron.vertices_begin();
     v != polyhedron.vertices_end();
     ++v )
{
  // do something with v
}
```

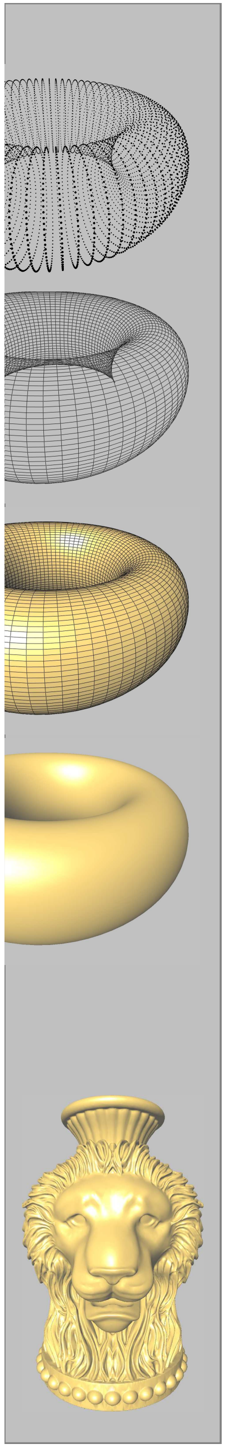




Circulate around Facet

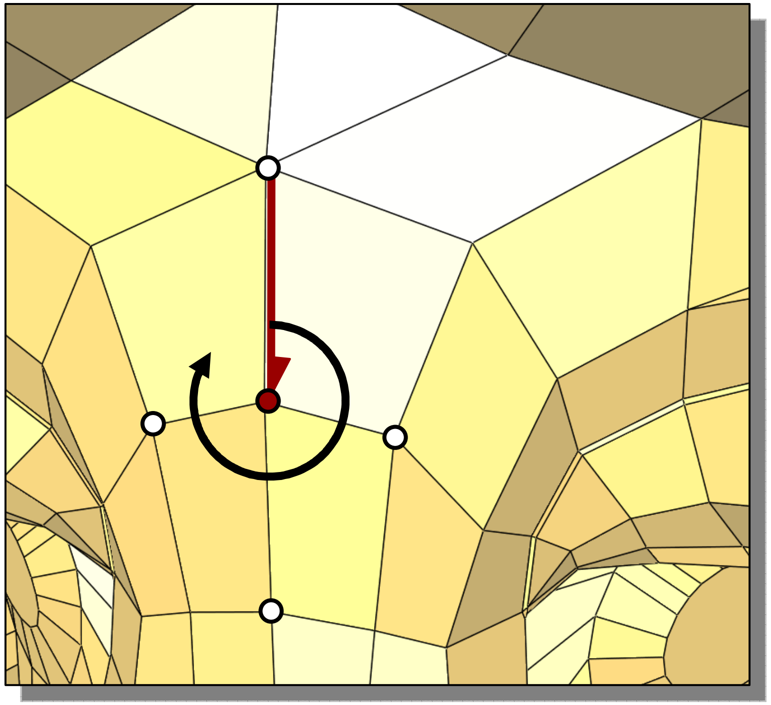
```
Halfedge_around_facet_circulator he,end;
he = end = f->facet_begin();
CGAL_For_all(he,end)
{
  // do something with he
}
```

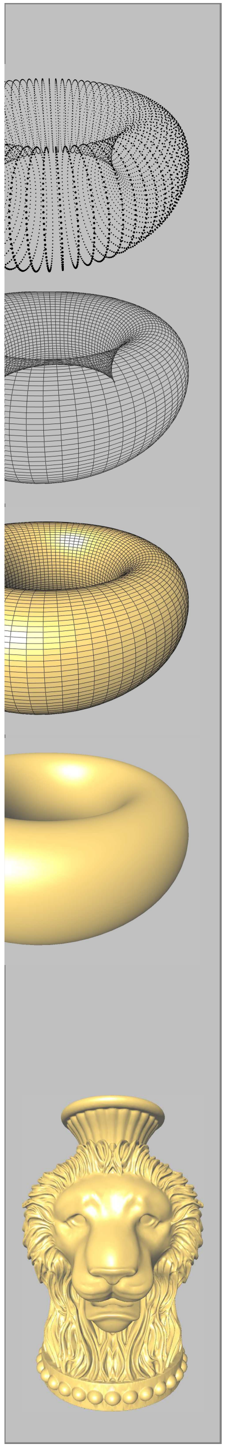




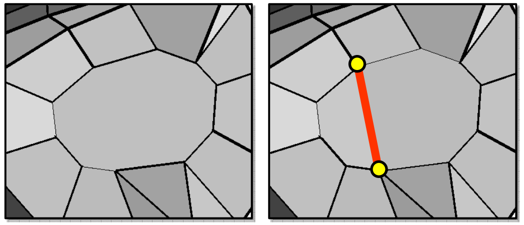
Circulate around Vertex

```
Halfedge_around_vertex_circulator he,end;
he = end = v->vertex_begin();
CGAL_For_all(he,end)
{
   // do something with he
}
```

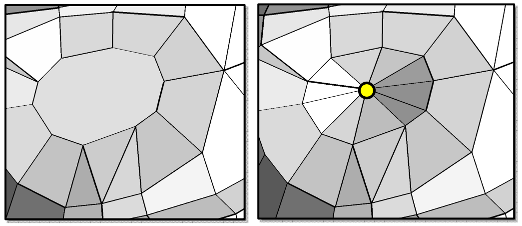




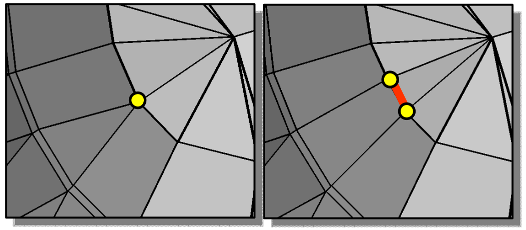
Euler Operators



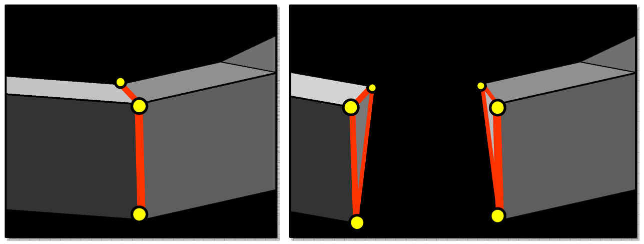
split_facet
join_facet



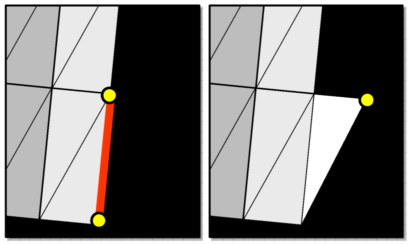
create_center_vertex
erase_center_vertex



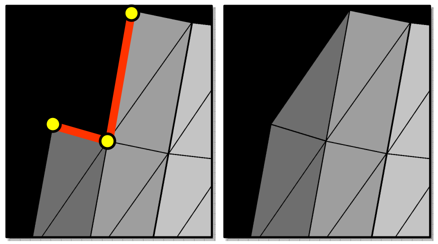
split_vertex
join_vertex
(aka edge collapse)



split_loop
join_loop

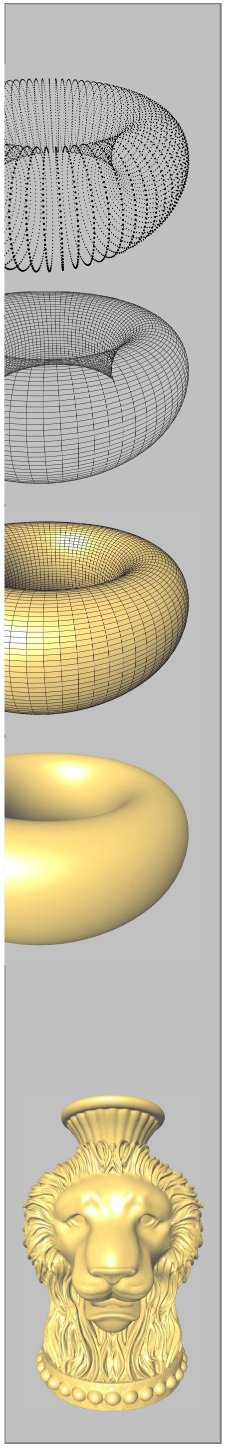


add_vertex_and_facet
_to_border
erase_facet



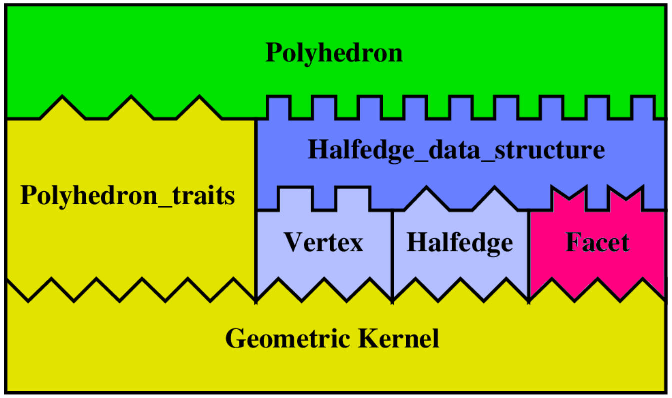
add_facet_to_border
erase_facet

CGAL manual

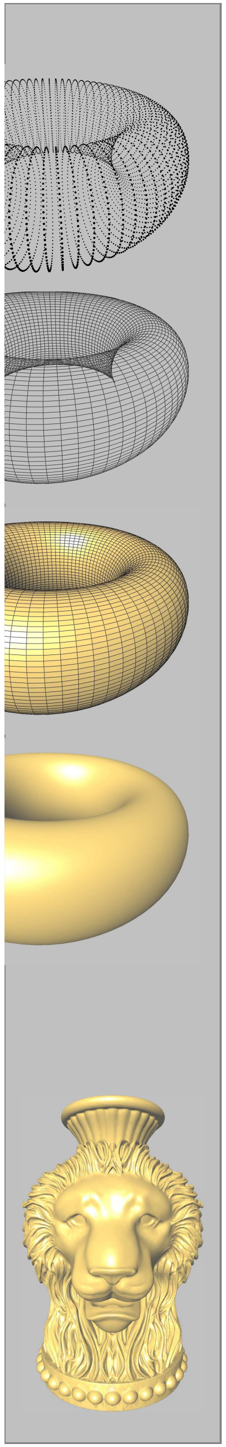


Customization

```
template <class Refs>
struct MyFace : public CGAL::HalfedgeDS_face_base<Refs> {
    CGAL::Color color;
};

struct MyItems : public CGAL::Polyhedron_items_3 {
    template <class Refs, class Traits>
    struct Face_wrapper {
        typedef MyFace<Refs> Face;
    };
};

typedef CGAL::Simple_cartesian<double>       Kernel;
typedef CGAL::Polyhedron_3<Kernel,MyItems>   MyPolyhedron;
```

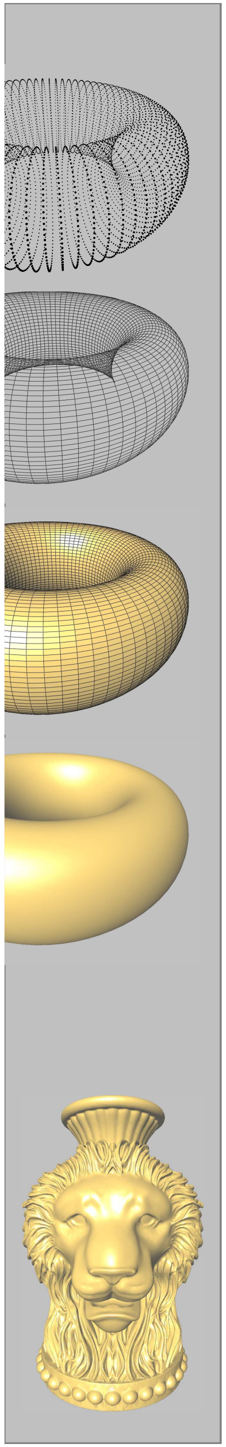


CGAL manual

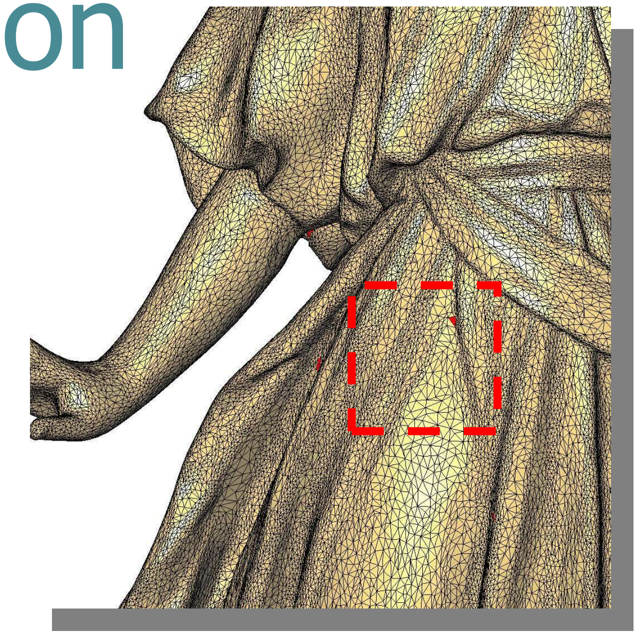


Algorithms

- Intersection detection
- Convex hull
- Boolean operations
- Kernel
- Parameterization
- Subdivision
- Principal component analysis
- Estimation of curvatures
- Extraction of ridges
- Simplification

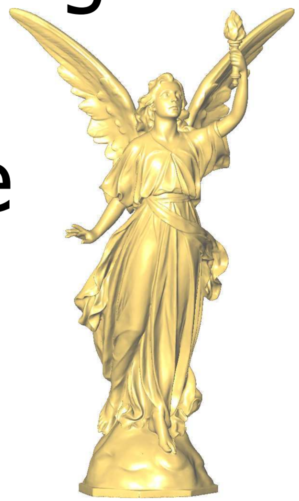


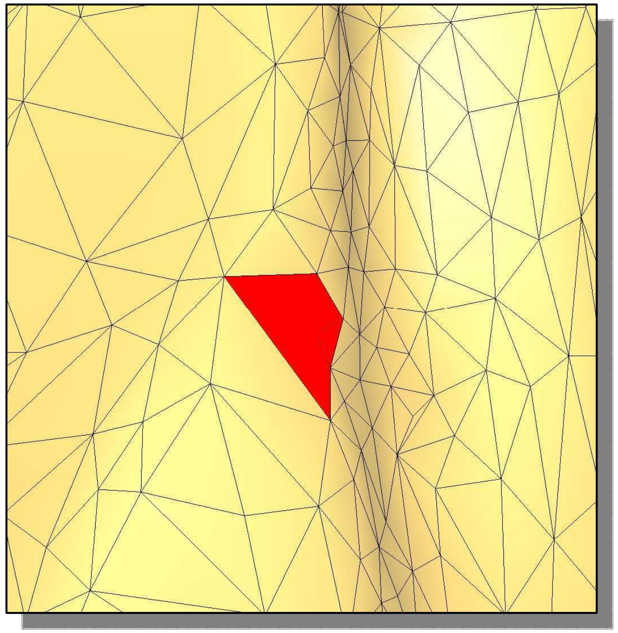
Intersection Detection



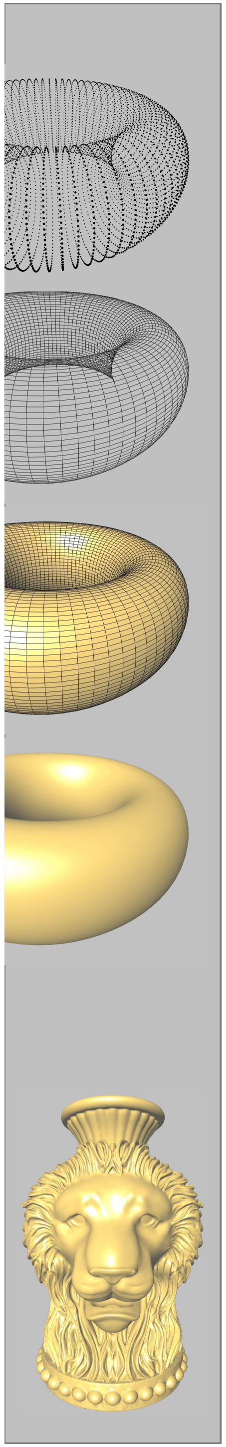
Efficient algorithm for finding all intersecting pairs for large numbers of axis-aligned bounding boxes.

Generic programming: Boxes can contain objects of any type





CGAL manual

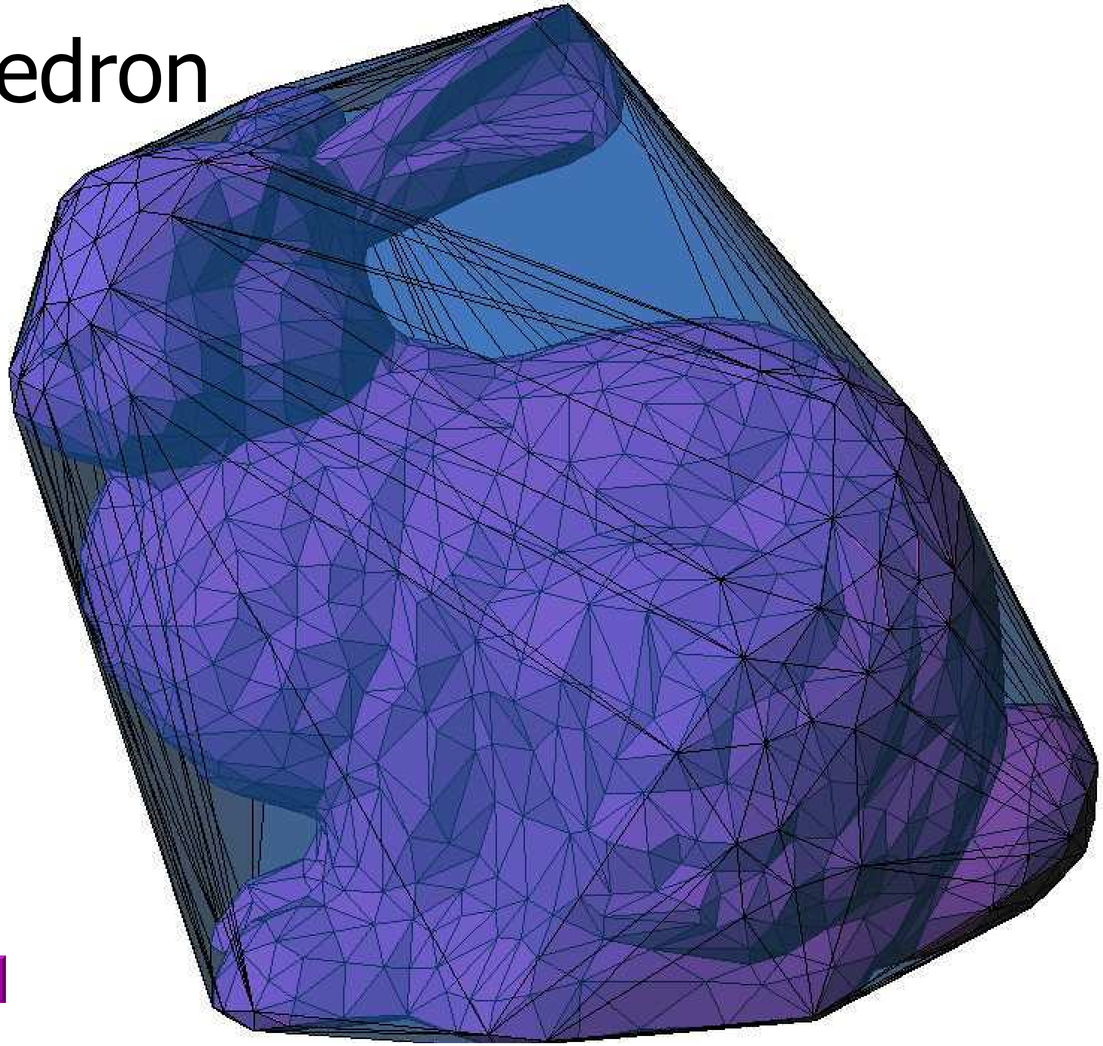


Example: Intersecting 3D Triangles

```
#include <CGAL/Exact_predicates_inexact_constructions_kernel.h>
#include <CGAL/intersections.h>
#include <CGAL/box_intersection_d.h>

typedef CGAL::Exact_predicates_inexact_constructions_kernel Kernel;
typedef Kernel::Triangle_3 Triangle_3;
typedef std::list<Triangle_3>::iterator Iterator;
typedef CGAL::Box_intersection_d::Box_with_handle_d<double,3,Iterator>
Box;


void callback(const Box& a, const Box& b)
{
  Triangle_3 ta = *a.handle();
  Triangle_3 tb = *b.handle();
  if(CGAL::do_intersect(ta,tb)) {
    // do something
  }
}


std::list<Triangle_3> triangles(...);

std::list<Box> boxes;
Iterator it;
for(it = triangles.begin(); it != triangles.end(); ++it)
  boxes.push_back(Box((*it).bbox(),it));


CGAL::box_self_intersection_d(boxes.begin(), boxes.end(), callback);
```

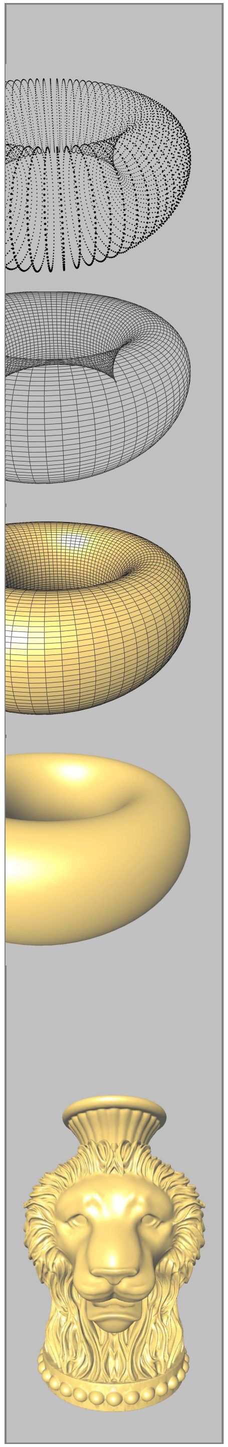
Convex Hull

- From point set
- Outputs a polyhedron



Example

```
#include <CGAL/convex_hull_3.h>
#include <CGAL/ Exact_predicates_inexact_constructions_kernel.h>
typedef CGAL::Exact_predicates_inexact_constructions_kernel Kernel;
typedef CGAL::Convex_hull_traits_3<Kernel> Traits;
typedef Traits::Polyhedron_3 Polyhedron;
typedef Kernel::Point_3 Point;

int main()
{
  std::list<Point> points;
  // fill container...
  Polyhedron polyhedron;
  CGAL::convex_hull_3(points.begin(), points.end(), polyhedron);
  return 0;
}
```



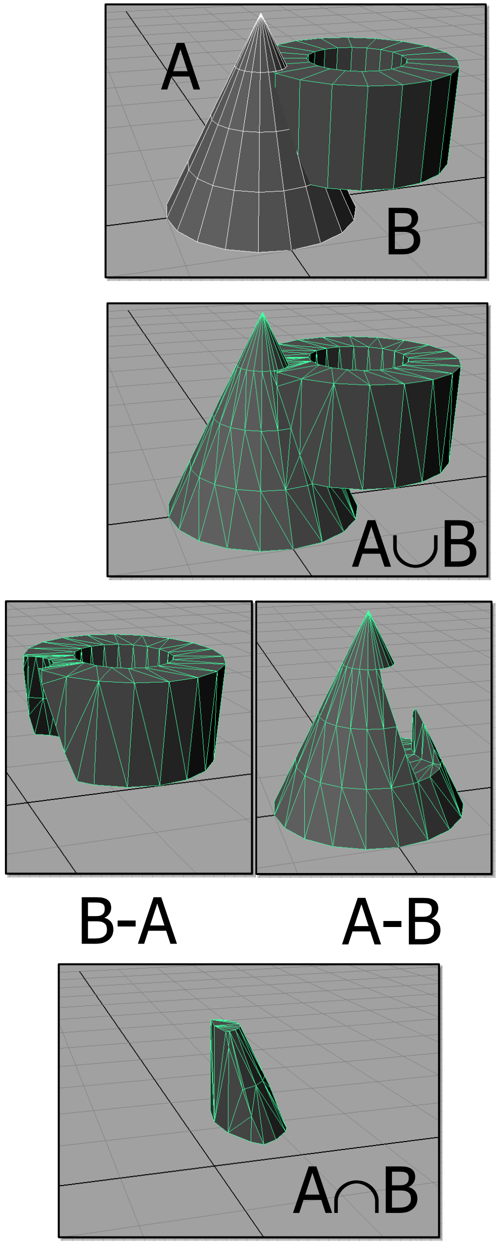
Boolean Operations

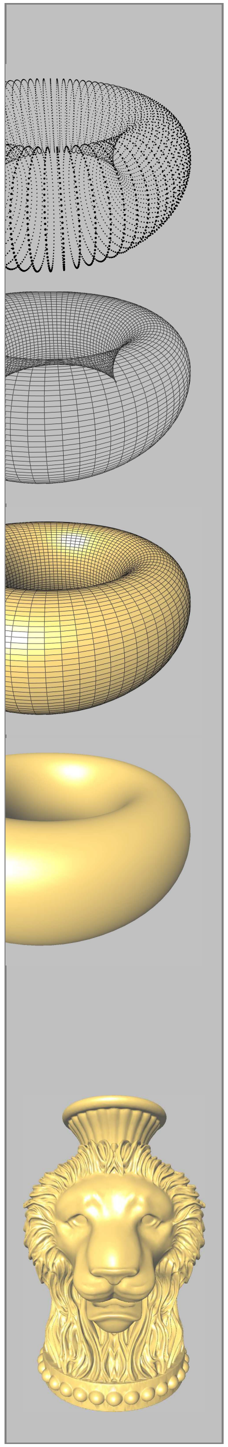
Operations:

- Union
- Difference
- Intersection
- Complement

Problem: not closed, i.e., result of a Boolean operation is not necessarily a Polyhedron

CGAL manual

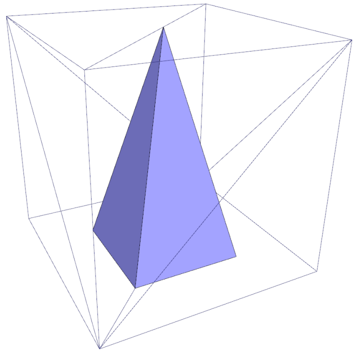




Nef Polyhedron

The **3D Nef polyhedron** is a B-rep data structure which is

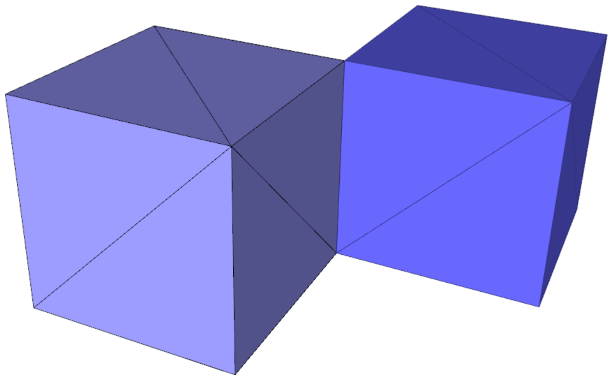
- closed under Boolean operations
- without enforcing regularization



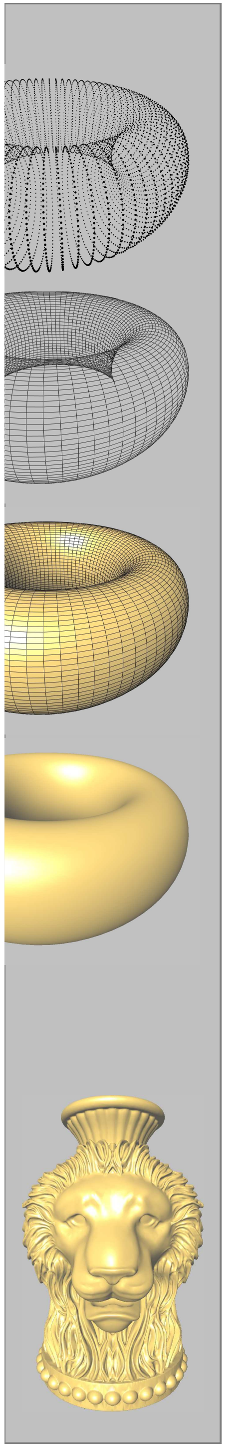
Operations:

- Union
- Intersection
- Difference
- Complement
- Interior
- Exterior
- Boundary
- Closure
- Regularization

can evaluate a CSG-tree with halfspaces as primitives and convert it to B-rep

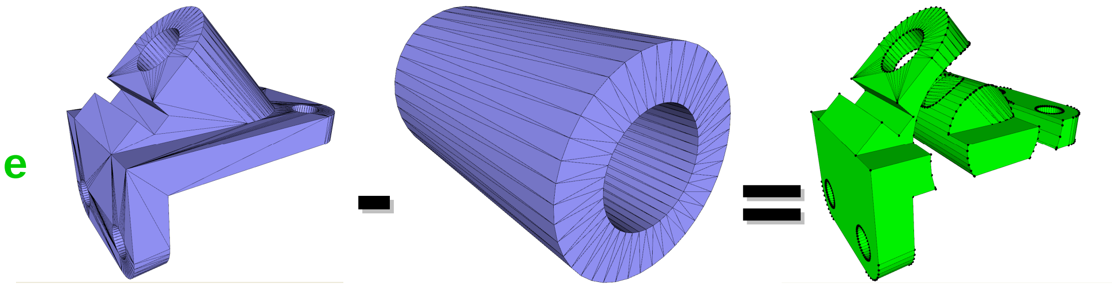


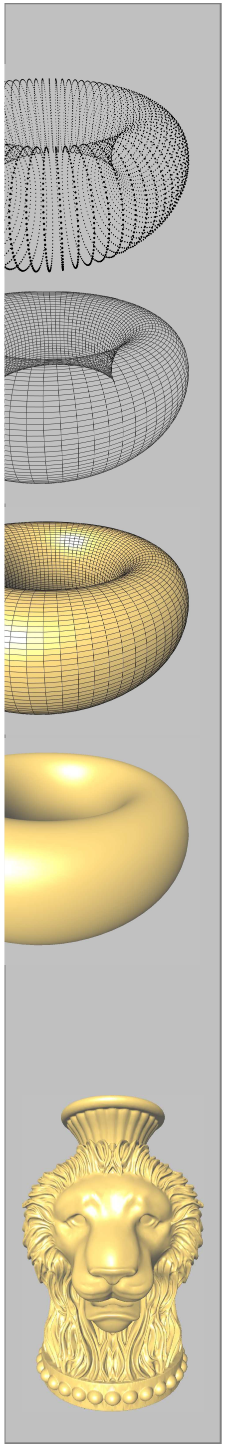
CGAL manual



Example

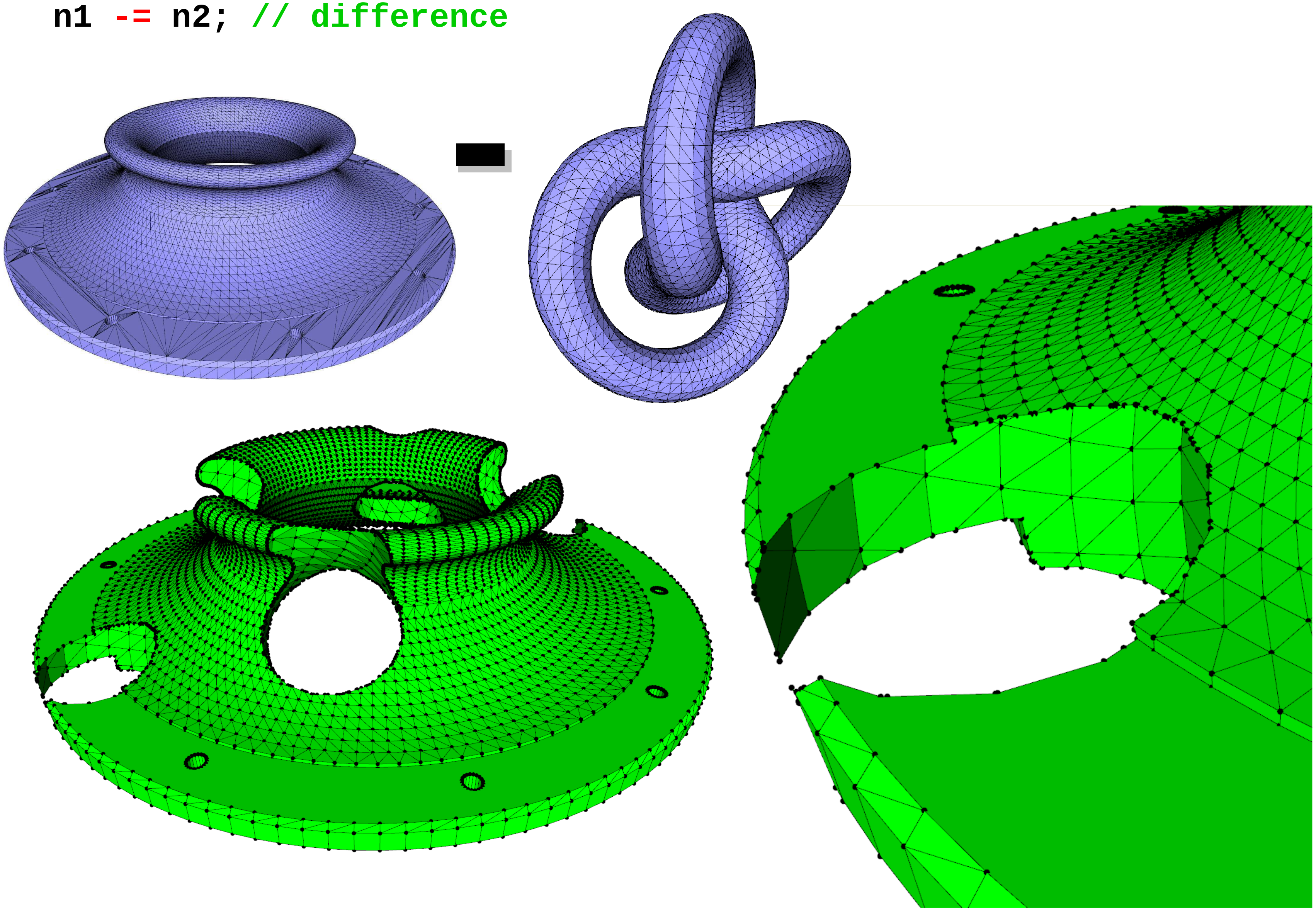
```
#include <CGAL/Exact_predicates_exact_constructions_kernel.h>
#include <CGAL/Polyhedron_3.h>
#include <CGAL/Nef_polyhedron_3.h>

typedef CGAL::Exact_predicates_exact_constructions_kernel Kernel;
typedef CGAL::Polyhedron_3<Kernel> Polyhedron;
typedef CGAL::Nef_polyhedron_3<Kernel> Nef_polyhedron;

Polyhedron p1;
Polyhedron p2;

Nef_polyhedron n1(p1);
Nef_polyhedron n2(p2);

n1 -= n2; // difference

Polyhedron p3;
if(n1.is_simple())
  n1.convert_to_Polyhedron(p3);
else
 // analyze/process n1 and do something...
```





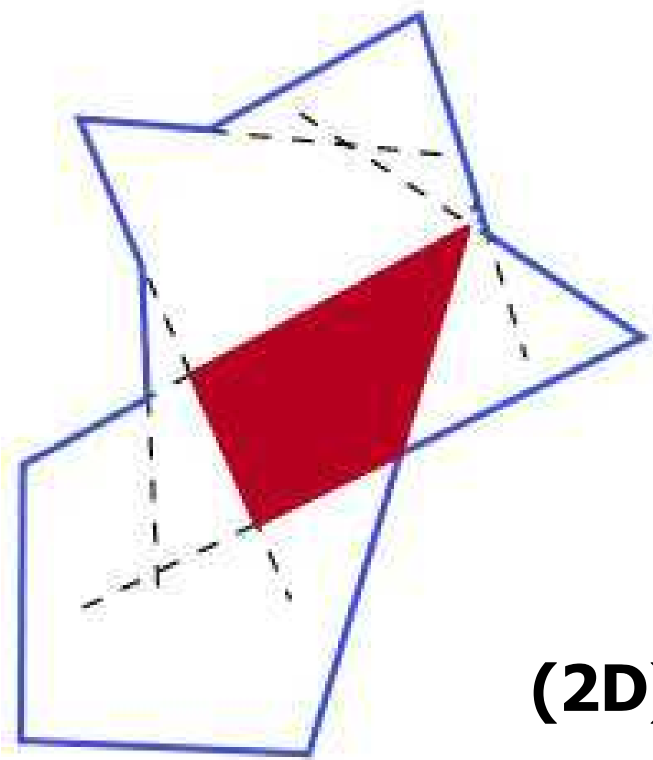
Example

```
n1 -= n2; // difference
```



Kernel of a Polyhedron

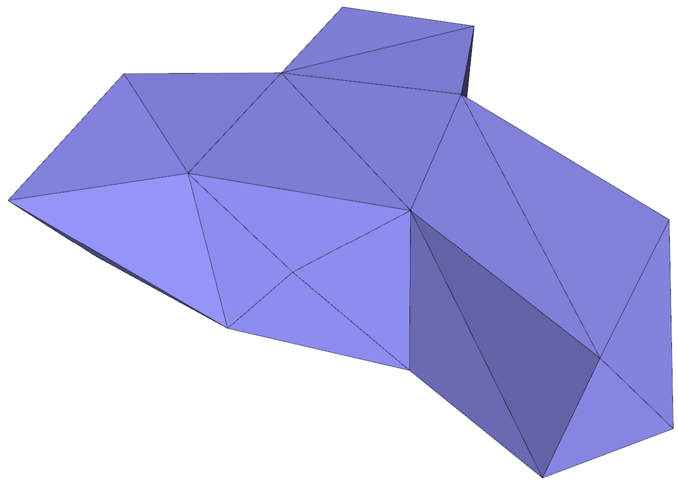
- Intersection of all its interior half-spaces.



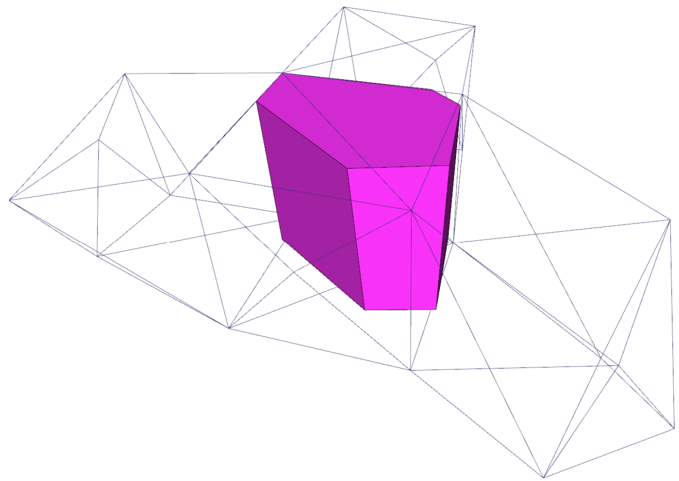
(2D)

Kernel of a Polyhedron

- Intersection of all its interior half-spaces
- Uses linear programming (CGAL::QP_solver)

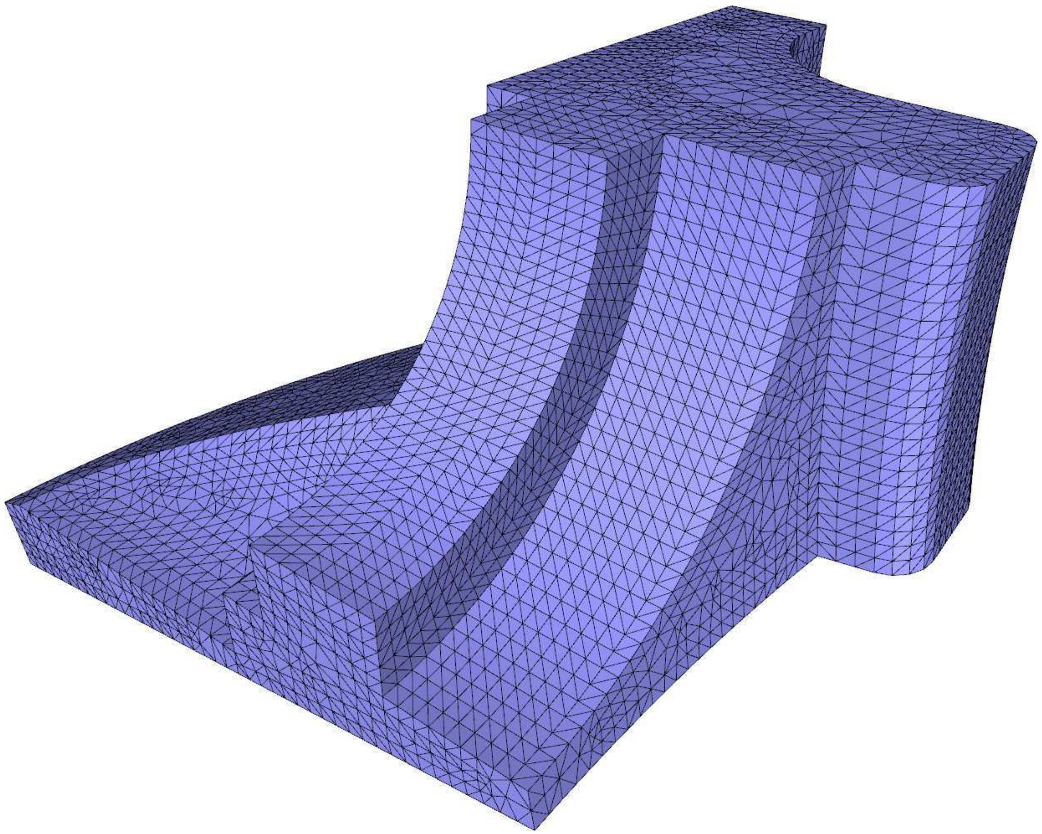


input polyhedron



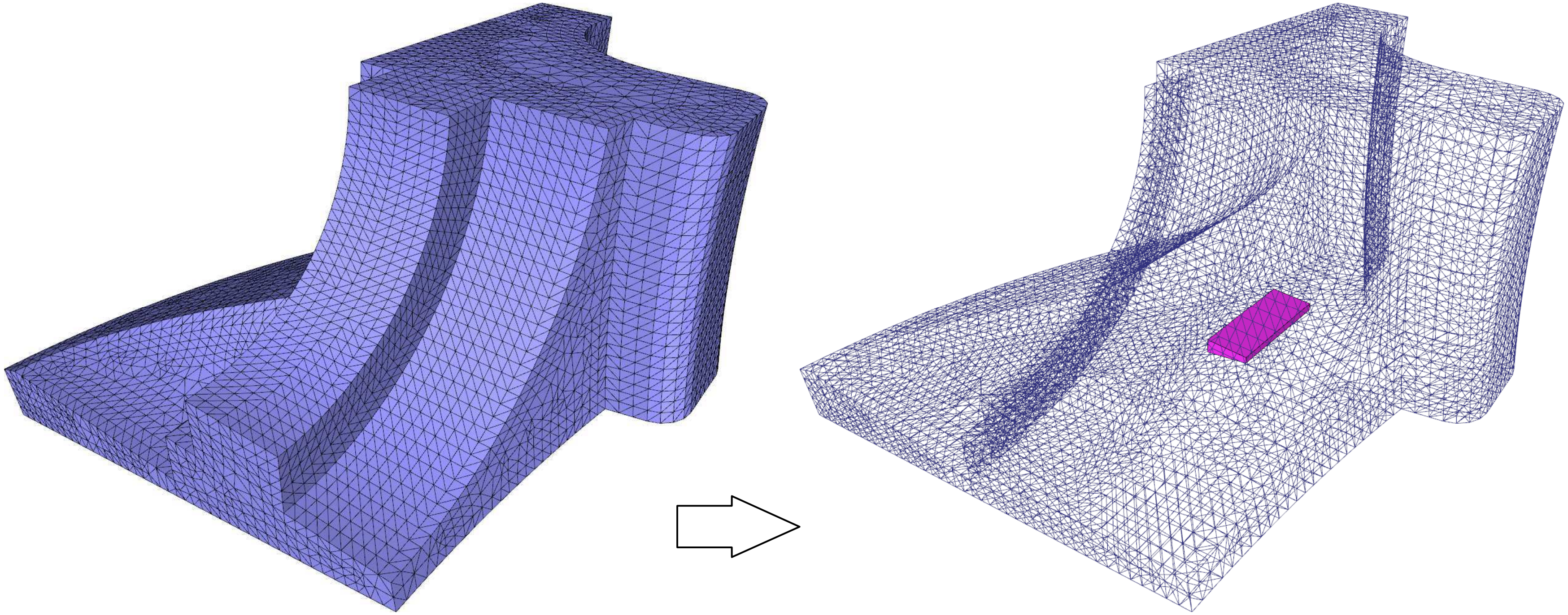
kernel

Kernel w/ Linear Programming

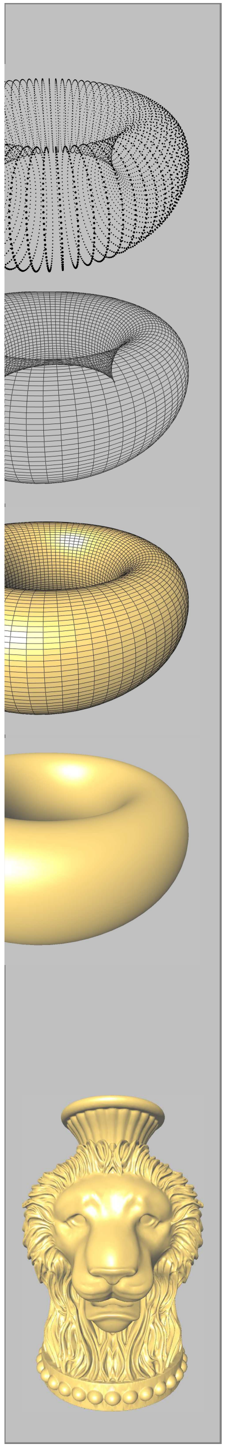


Does it have
a kernel?

Kernel w/ Linear Programming



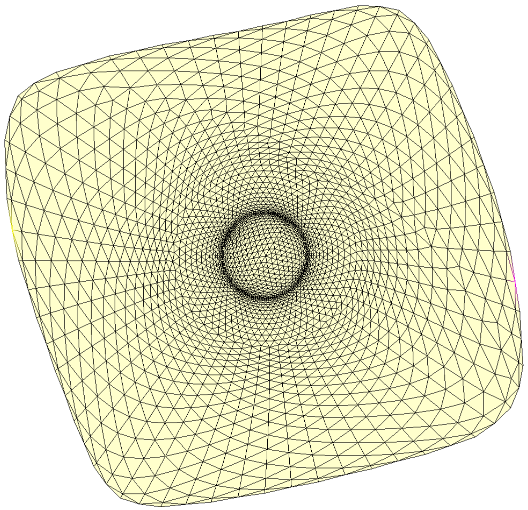
Demo

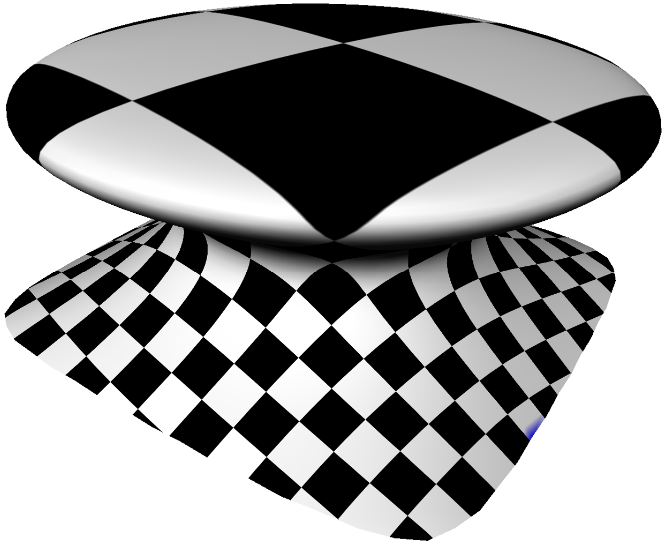


Parameterization

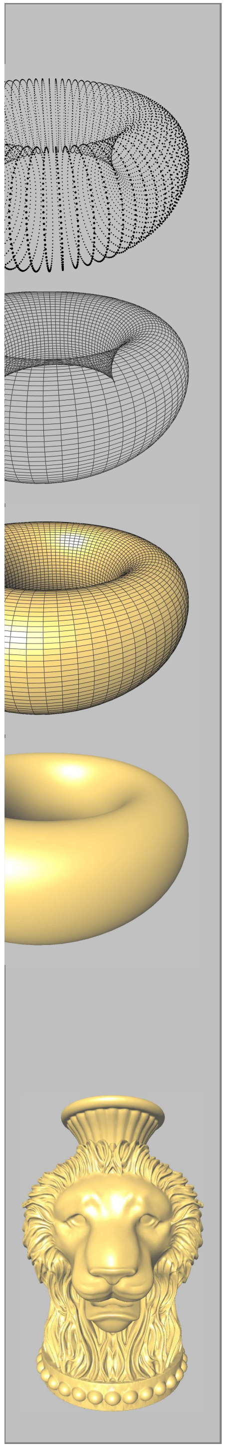
Planar

- Conformal [Eck et al., Levy et al., Desbrun et al.]
- Mean value coordinates [Floater]
- …



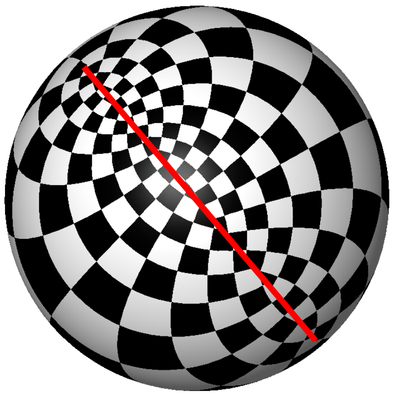


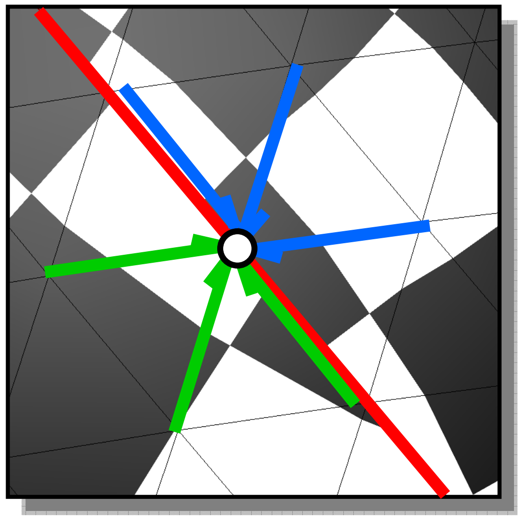
CGAL manual

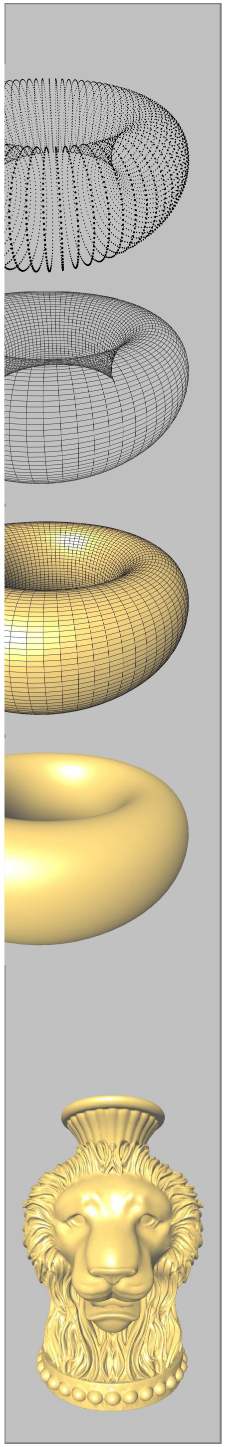


Example

```
#include <CGAL/Cartesian.h>
#include <CGAL/Polyhedron_3.h>
#include <CGAL/Parameterization_polyhedron_adaptor_3.h>
#include <CGAL/parameterize.h>

Polyhedron mesh;
Mesh_adaptor_polyhedron mesh_adaptor(&mesh);
CGAL::parameterize(&mesh_adaptor);
Point_2 uv = mesh_adaptor.info(he)->uv();
```

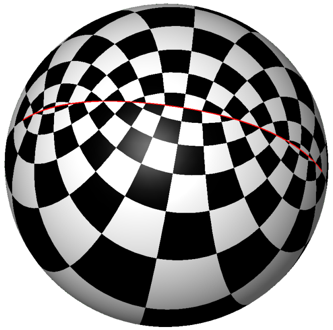


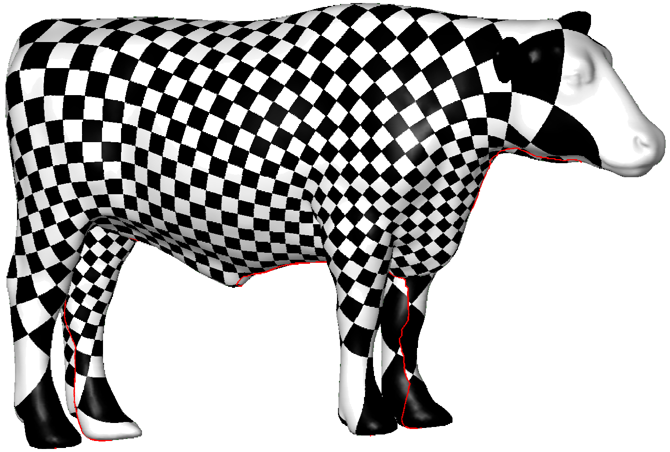


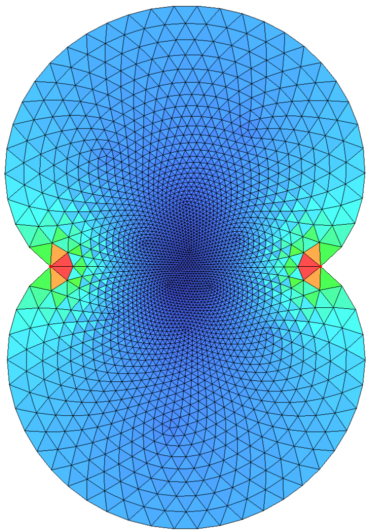


Parameterization

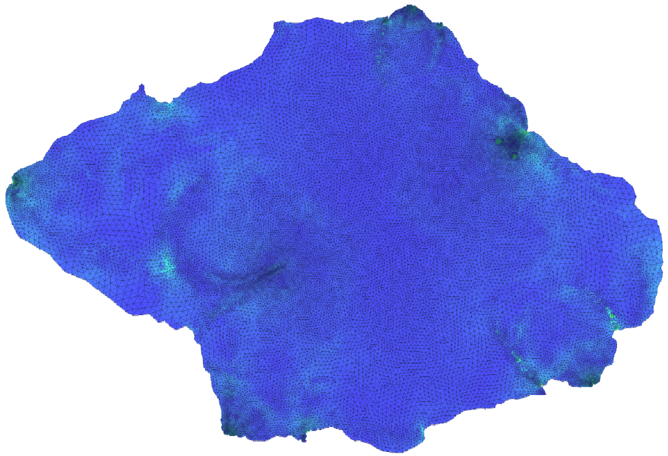
User-provided cut graph for closed or high genus surfaces.

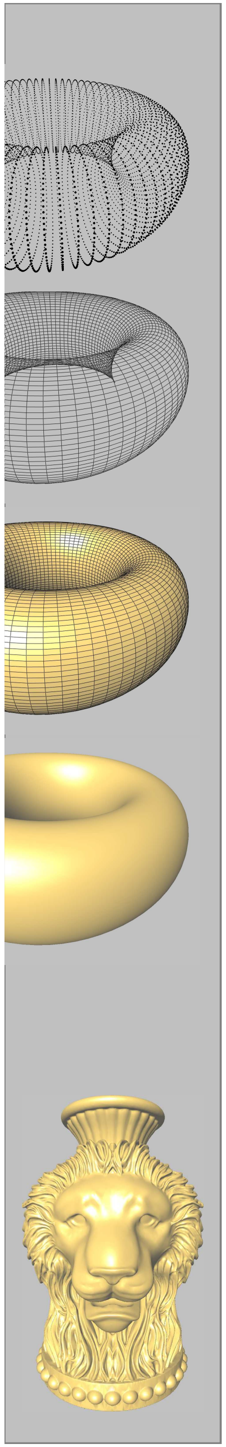




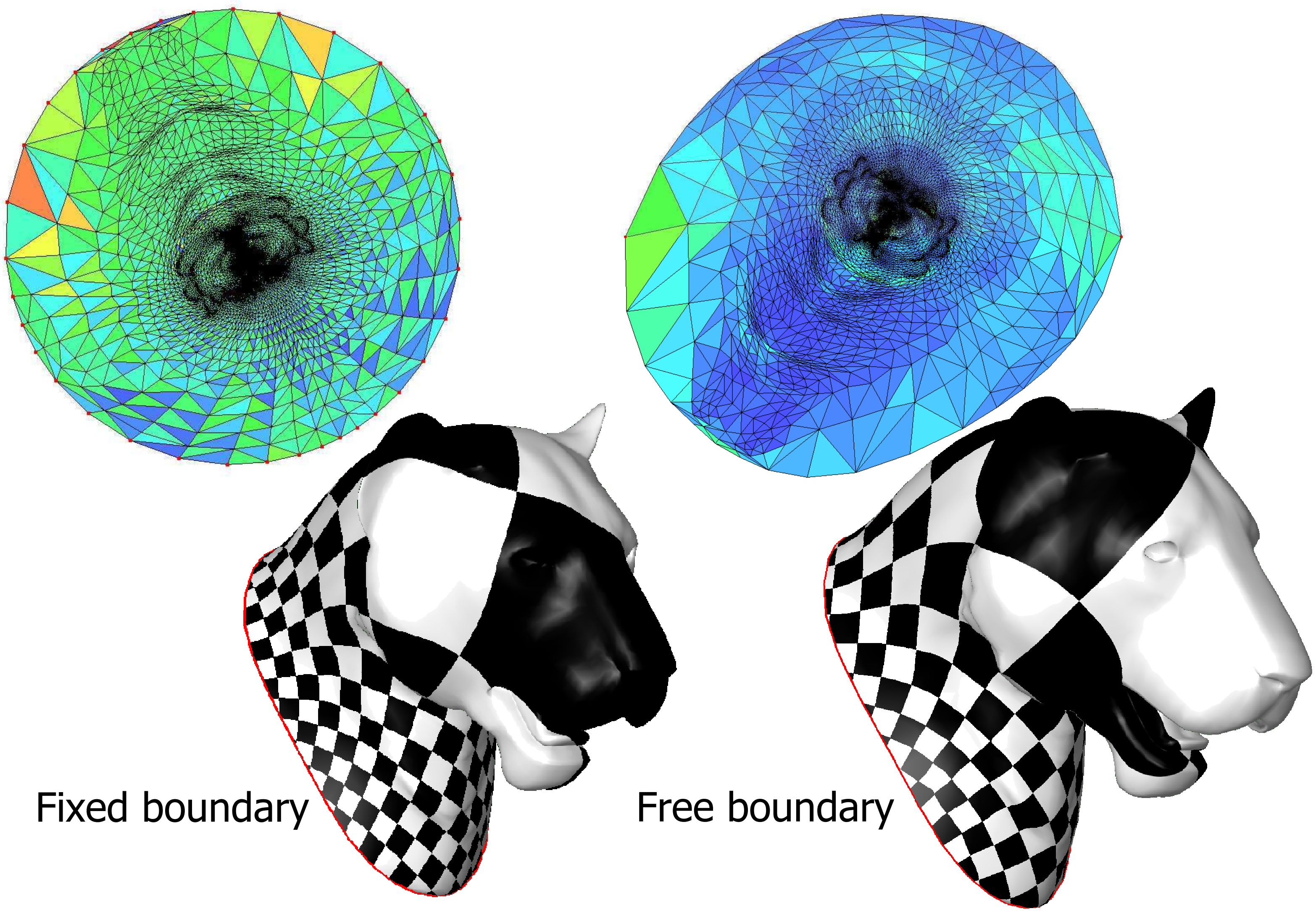


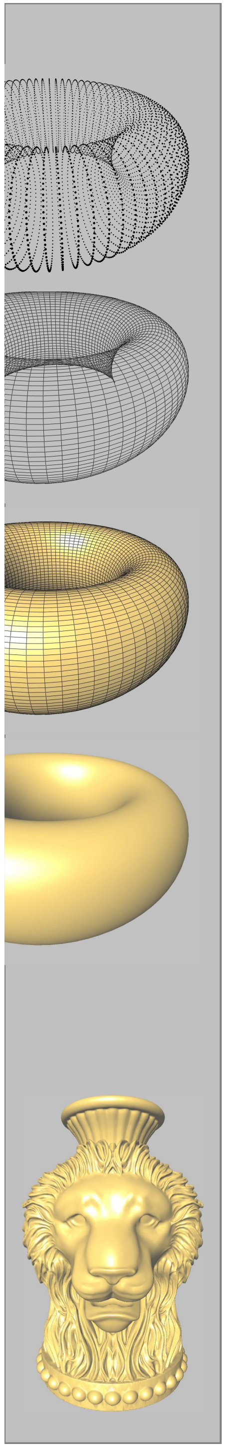
(conformal distortion)





Parameterization

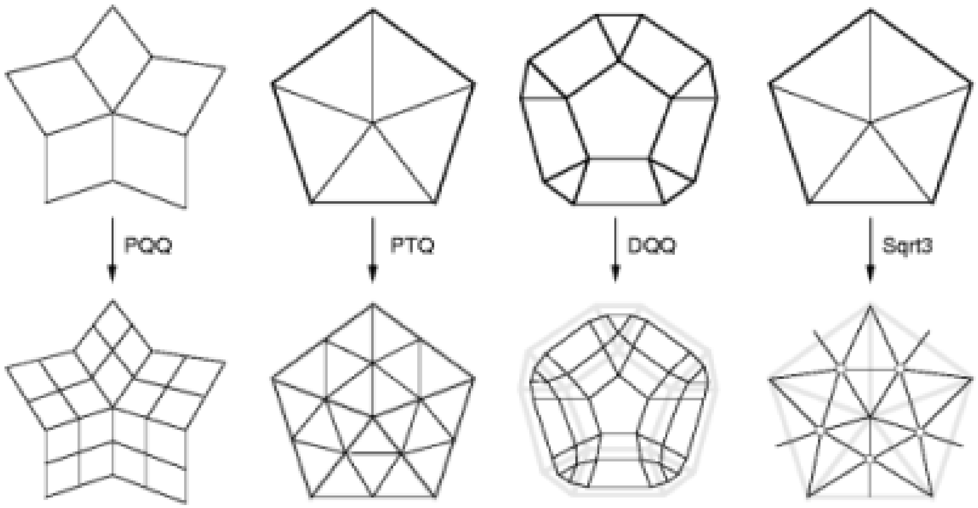




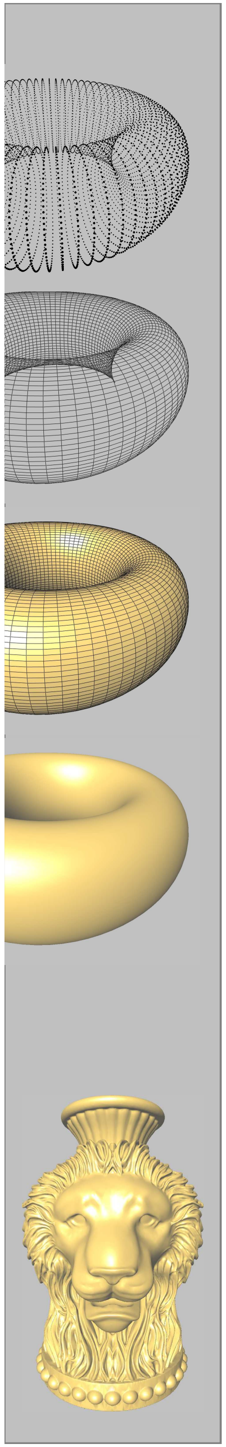
Subdivision

Designed to work on CGAL polyhedron

- Catmull-Clark
- Loop
- Doo-Sabin
- Sqrt3
- …

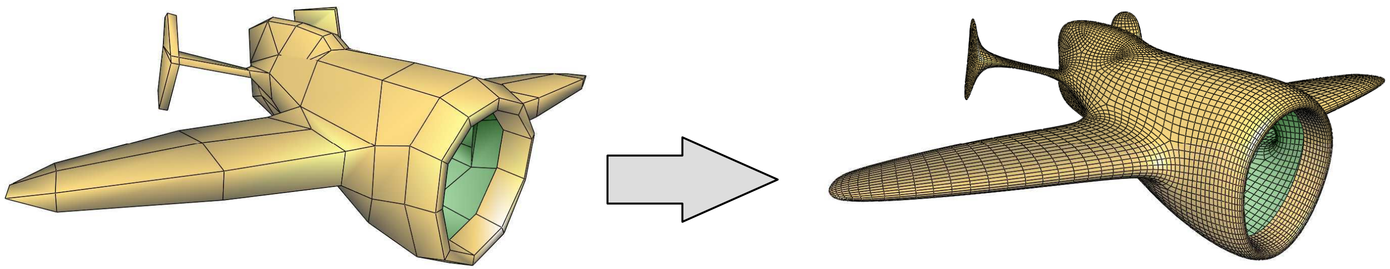


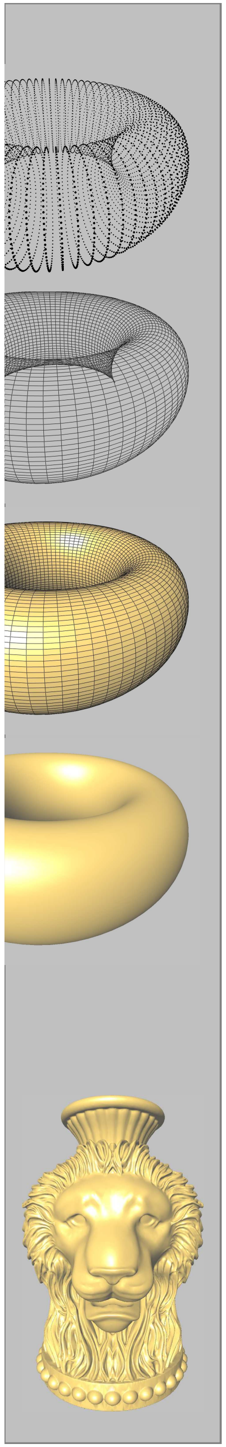
CGAL manual



Example

```
#include <CGAL/Cartesian.h>
#include <CGAL/Polyhedron_3.h>
#include <CGAL/Subdivision_method_3.h>

typedef CGAL::Cartesian<double>        Kernel;
typedef CGAL::Polyhedron_3<Kernel>     Polyhedron;

using CGAL::Subdivision_method_3;

Polyhedron polyhedron;
int subdivision_depth = 3;
CatmullClark_subdivision(polyhedron, subdivision_depth);
```





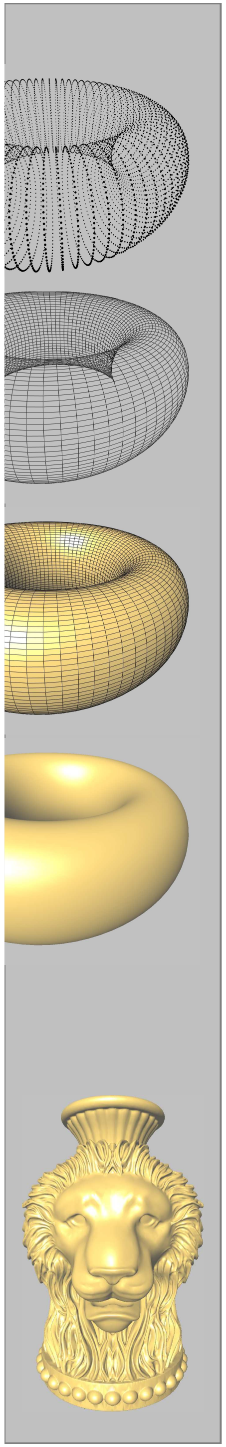
Principal Component Analysis

Linear least squares fitting on sets of 3D kernel objects:

- points
- triangles

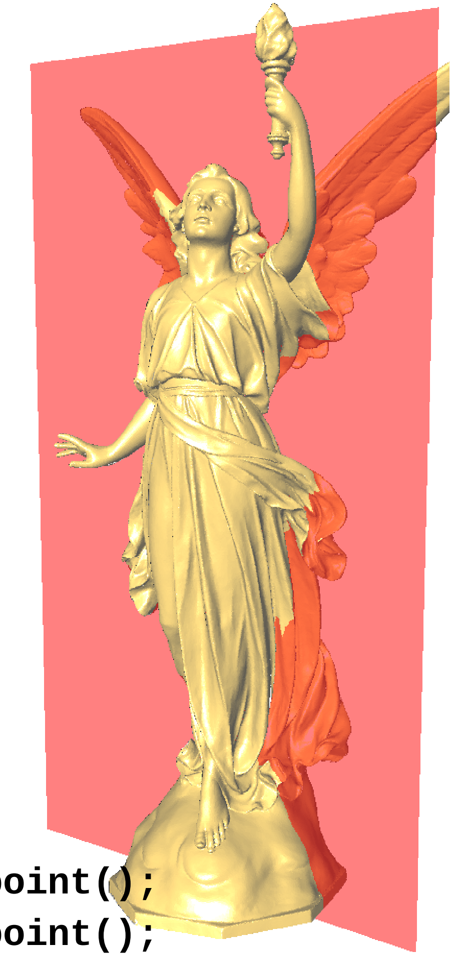
for triangle meshes

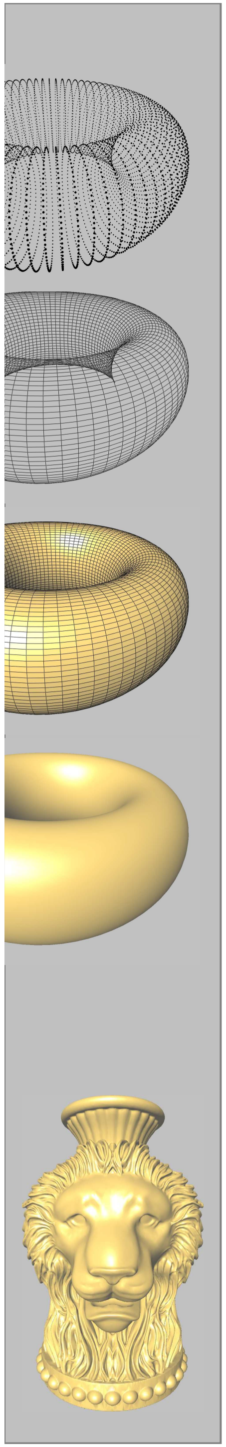
CGAL manual



Example

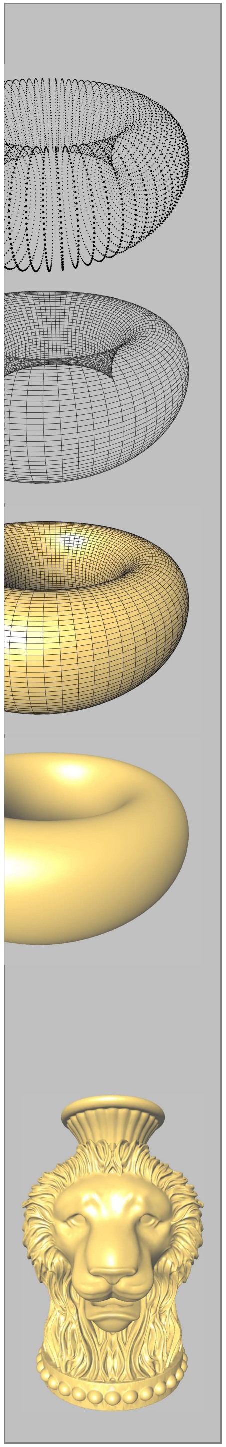
```
#include <CGAL/linear_least_squares_fitting_3.h>
// more #includes and typedefs

Polyhedron mesh;

std::list<Triangle_3> triangles;
Polyhedron::Facet_iterator f;
for(f = mesh.facets_begin();
    f != mesh.facets_end();
    ++f) {
    const Point& a = f->halfedge()->vertex()->point();
    const Point& b = f->halfedge()->next()->vertex()->point();
    const Point& c = f->halfedge()->prev()->vertex()->point();
    triangles.push_back(Triangle_3(a,b,c));
}
Plane_3 plane;
CGAL::linear_least_squares_fitting_3( triangles.begin(),
                                      triangles.end(),
                                      plane,
                                      CGAL::PCA_dimension_2_tag()  );
```



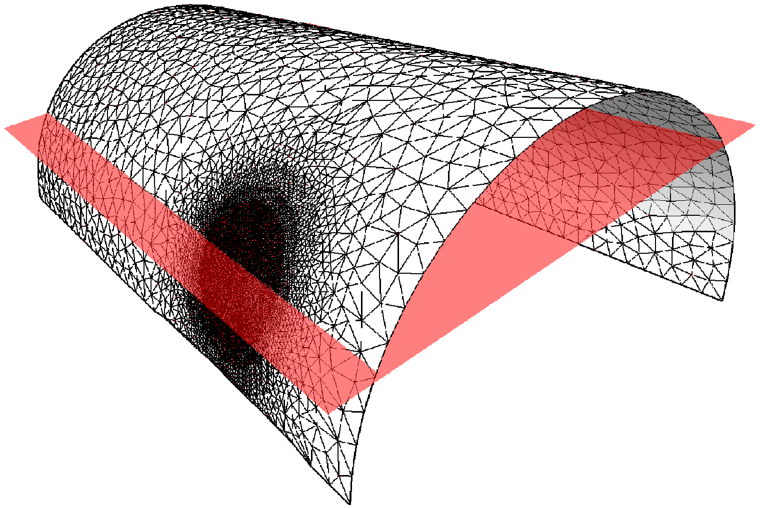


Same for Boost freaks ☺

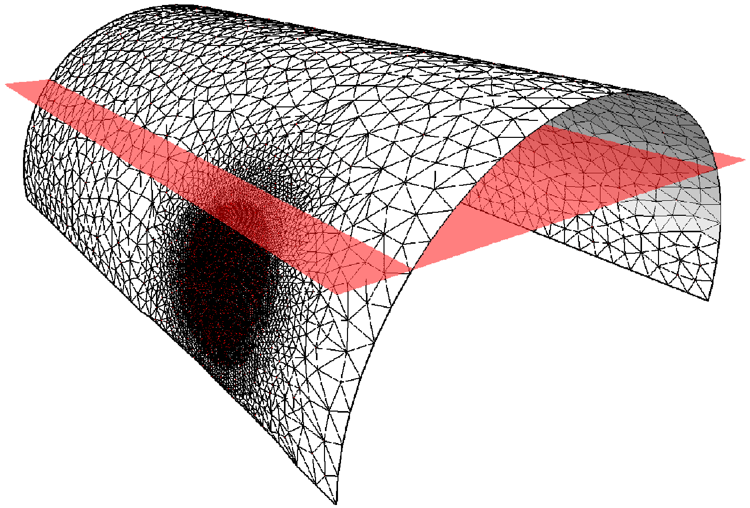
```
#include <CGAL/linear_least_squares_fitting_3.h>
#include <boost/iterator/transform_iterator.hpp>

Polyhedron mesh;

class ToTriangle {
  Triangle_3 operator(Polyhedron::FacetIterator f) {
   return Triangle_3( f->halfedge()->vertex()->point();
                      f->halfedge()->next()->vertex()->point();
                      f->halfedge()->prev()->vertex()->point() );
  }
};

Plane_3 plane;

CGAL::linear_least_squares_fitting_3(

  boost::make_transform_iterator(mesh.facets_begin(), ToTriangle()),

  boost::make_transform_iterator(mesh.facets_end(),   ToTriangle()),

  plane, CGAL::PCA_dimension_2_tag());
```



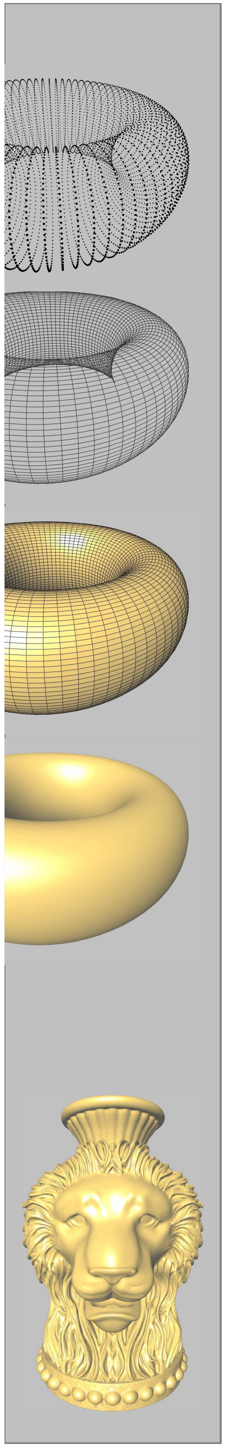
Fitting Points vs Triangles



fit points



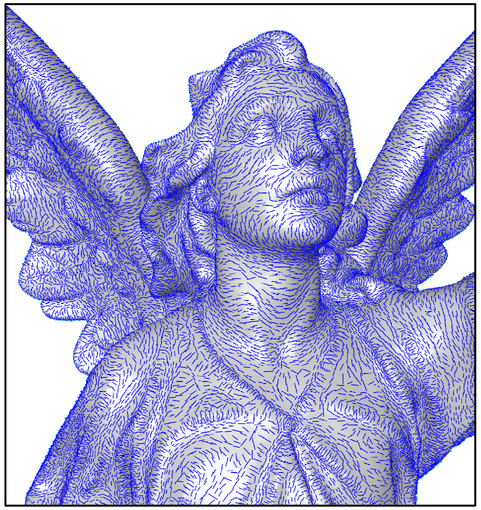
fit triangles



Estimation of Curvatures

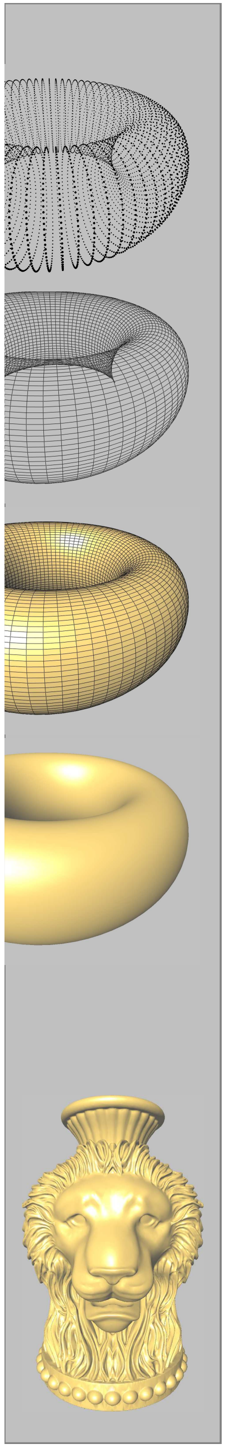
- Estimates general differential properties (Monge form) on point sets.
- Through polynomial (d-jet) fitting

CGAL manual



min curvature directions

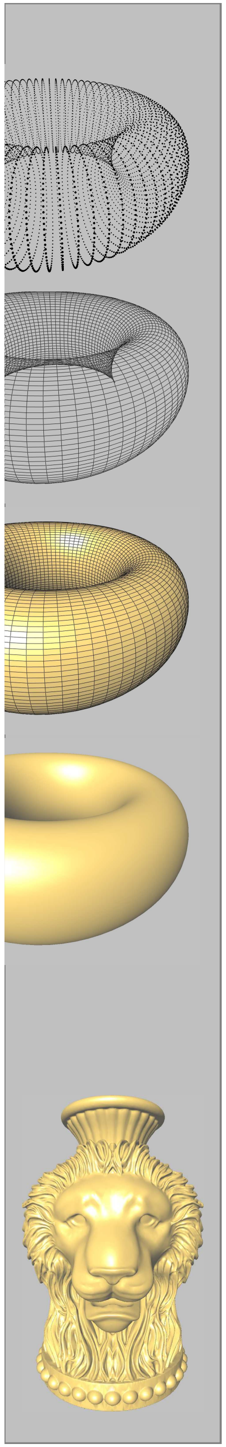
max curvature directions



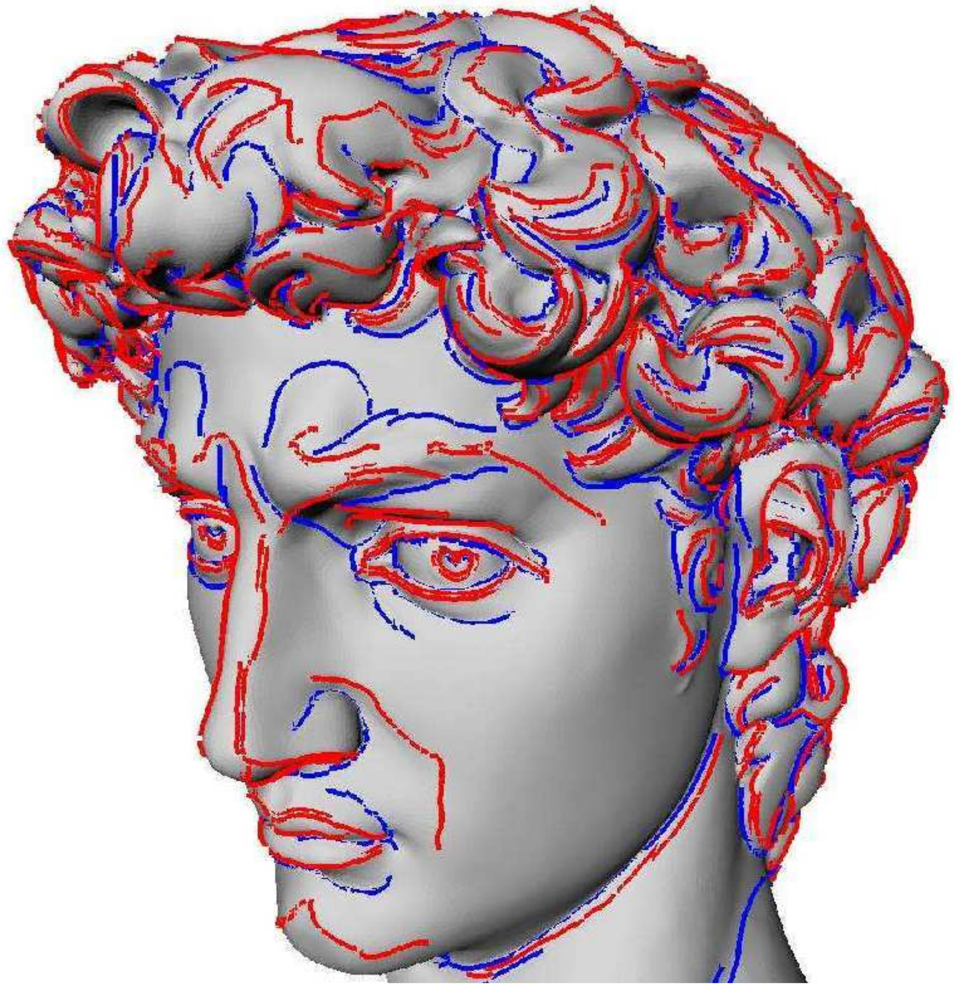
Example

```
#include <CGAL/Monge_via_jet_fitting.h>
typedef CGAL::Cartesian<double> Kernel;
typedef CGAL::Monge_via_jet_fitting<Kernel> Monge_fit;
typedef Monge_fit::Monge_form Monge_form;

Monge_fit monge_fit;
Monge_form monge_form =
   monge_fit(  points.begin(),
               points.end(),
               dim_fitting, dim_monge);

Vector_3 kmin = monge_form.minimal_principal_direction();
Vector_3 kmax = monge_form.maximal_principal_direction();
Vector_3 normal = monge_form.normal_direction();
```

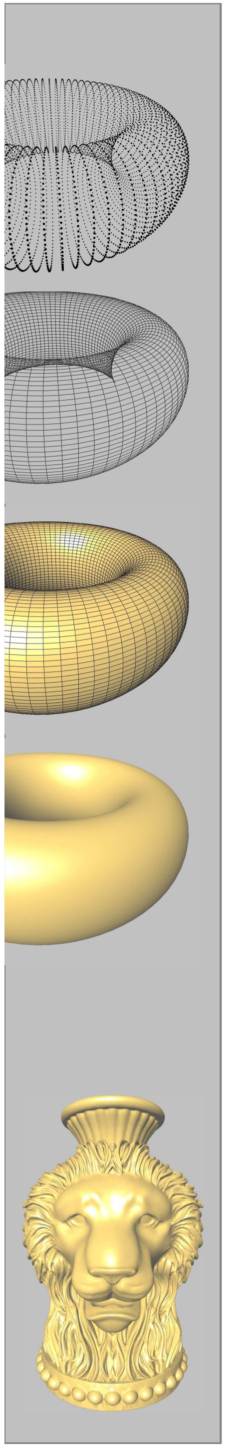
Extraction of Ridges



Ridge: curve along which one of the principal curvatures has an extremum along its curvature line.



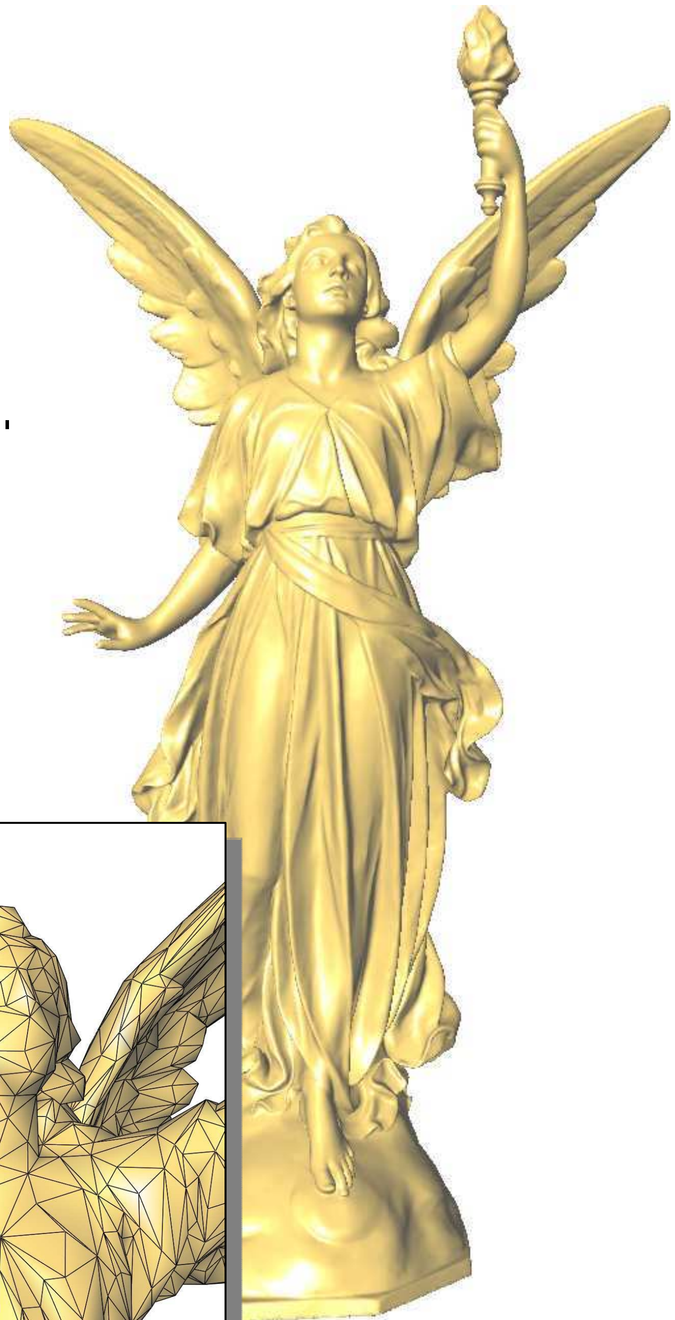
CGAL manual

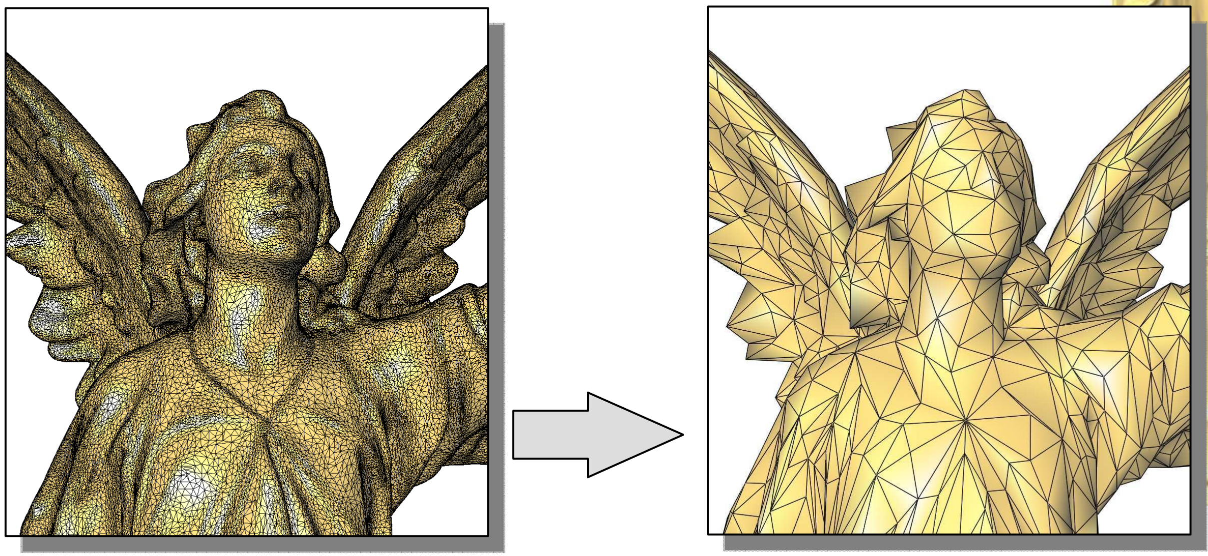


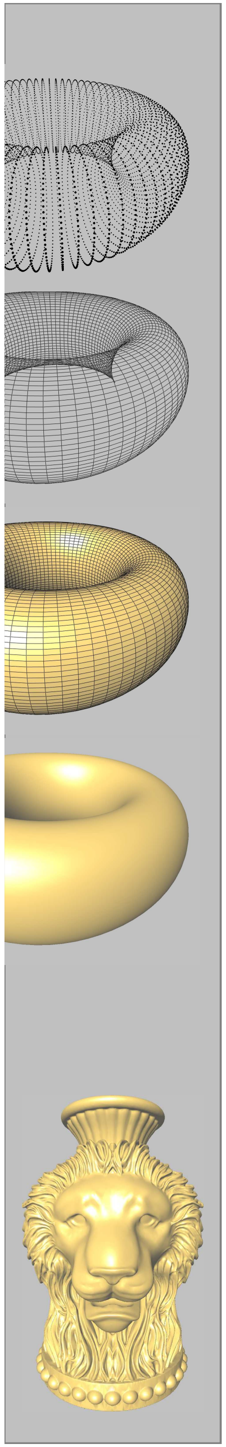
Simplification

Implementation of **[Lindstrom-Turk]** volume-preserving method.

BGL-style allows simplification of any model of EmbeddedGraph



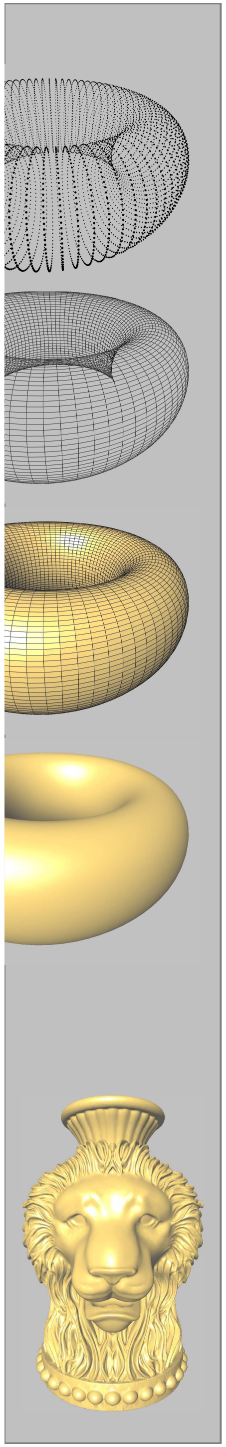




Simplification: Example

```
#include <CGAL/Simple_cartesian.h>
#include <CGAL/Polyhedron_3.h>
#include <CGAL/Surface_mesh_simplification/HalfedgeGraph_Polyhedron_3.h>
#include <CGAL/Surface_mesh_simplification/edge_collapse.h>
#include
    <CGAL/Surface_mesh_simplification/Policies/Edge_collapse/Count_stop_predicate.h
    >

typedef CGAL::Simple_cartesian<double> Kernel;
typedef CGAL::Polyhedron_3<Kernel> Mesh;
namespace SMS = CGAL::Surface_mesh_simplification ;

Mesh mesh;
SMS::Count_stop_predicate< Mesh > stop(1000); // target # edges
SMS::edge_collapse(mesh, stop,
            CGAL::vertex_index_map(boost::get(CGAL::vertex_external_index, mesh))
            .edge_index_map(boost::get(CGAL::edge_external_index, mesh)));
```

CGAL manual   boost::property_map   bgl_named_params



Summary and Outlook

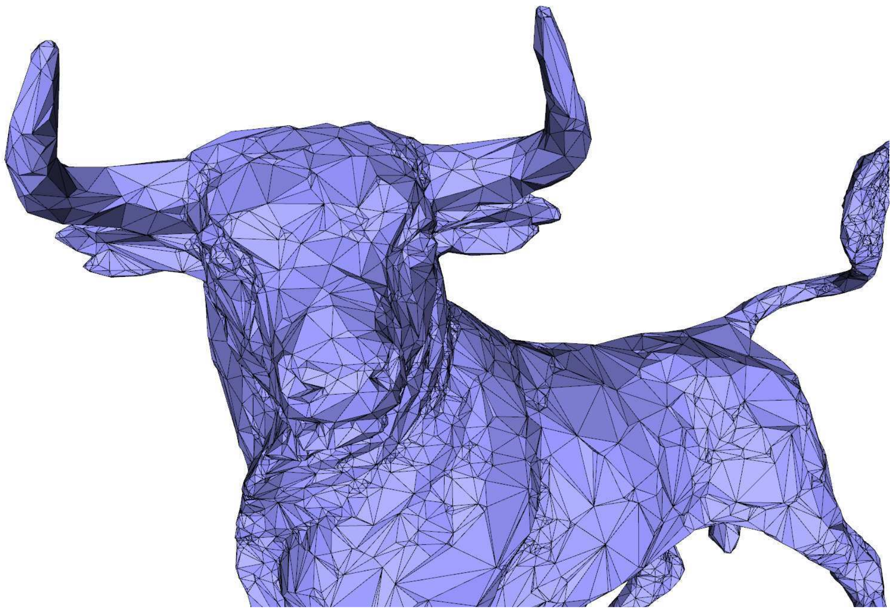
- The halfedge data structure and the polyhedron are highly flexible
- CGAL provides algorithms for geometric modeling and geometry processing
- Polyhedral surface as output of surface mesh generation algorithms (Part IV)

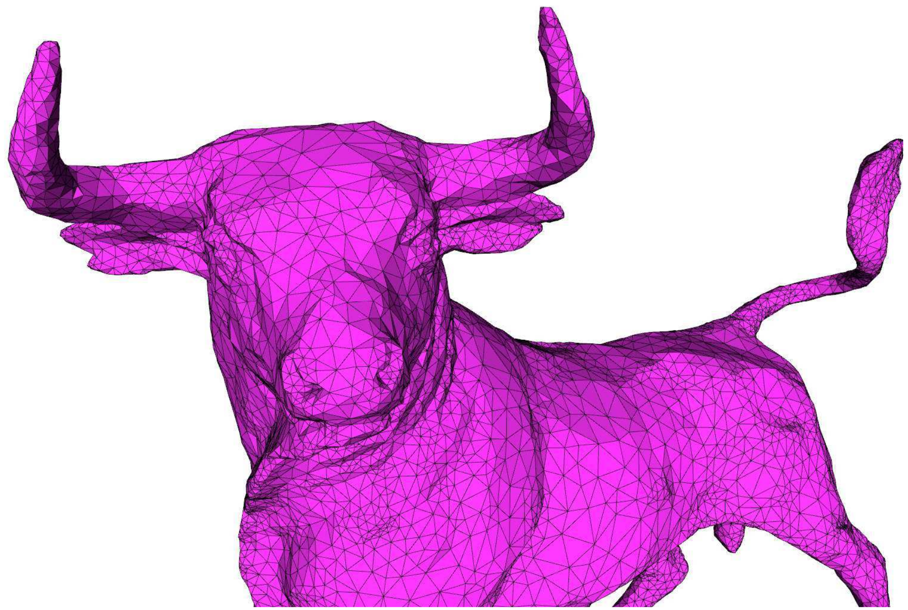
Under Development

- BGL-ization of existing CGAL algorithms

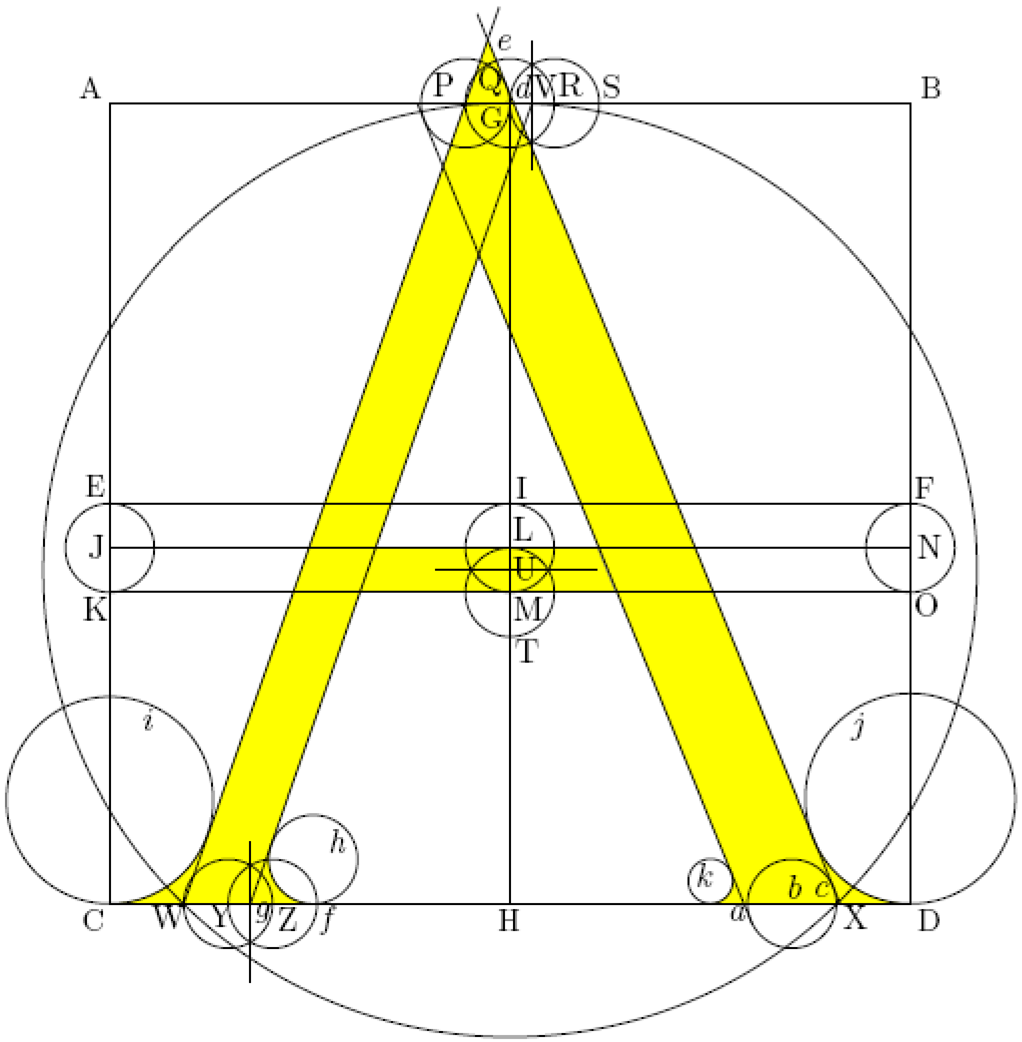
Under Development

- BGL-ization of existing CGAL algorithms
- Remeshing





Questions?



Arrangements

Efi Fogel

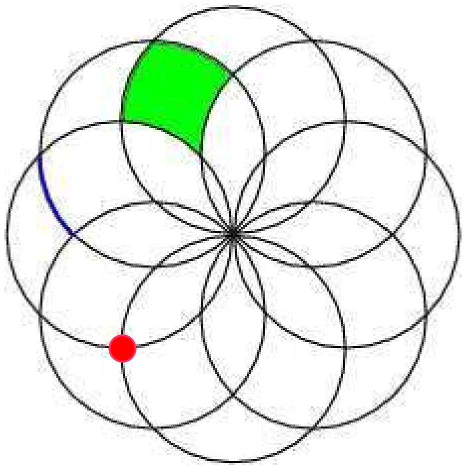
Tel Aviv University

Outline

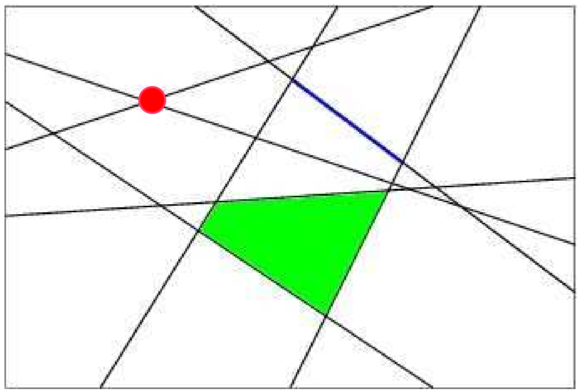
- Arrangements
- Algorithms based on Arrangements
  - Boolean Set Operations
  - Minkowski Sums and Polygon Offset
  - Envelopes
- Arrangements on Surfaces

Arrangement Definition

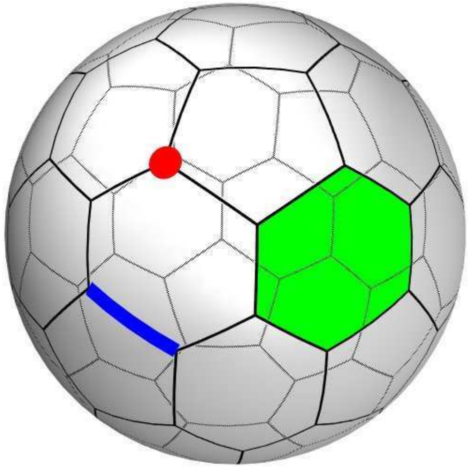
Given a collection of curves on a surface, the **arrangement** is the partition of the surface into **vertices**, **edges** and **faces** induced by the curves



An arrangement of circles in the plane



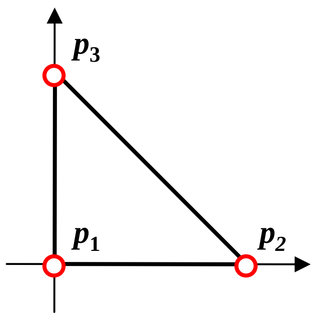
An arrangement of lines in the plane



An arrangement of geodesic arcs on the sphere

Code: A Simple Arrangement

```
#include <CGAL/Exact_predicates_exact_constructions_kernel.h>
#include <CGAL/Arr_segment_traits_2.h>
#include <CGAL/Arrangement_2.h>

typedef CGAL::Exact_predicates_exact_constructions_kernel Kernel;
typedef CGAL::Arr_segment_traits_2<Kernel>                Traits;
typedef Traits::Point_2                                   Point;
typedef Traits::X_monotone_curve_2                        Segment;
typedef CGAL::Arrangement_2<Traits>                       Arrangement;

int main() {
  Point p1(0, 0), p2(1, 0), p3(0, 1);
  Segment cv[3] = { Segment(p1,p2), Segment(p2,p3), Segment(p3,p1) };
  Arrangement arr;
  CGAL::insert(arr, cv, cv+3);
  return (arr.is_valid()) ? 0 : -1;
}
```



CGAL::Arrangement_2

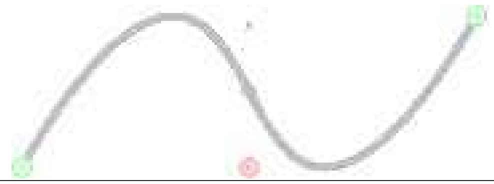
- Constructs, maintains, modifies, traverses, queries, and presents subdivisions of the plane
- Robust and exact
  - All inputs are handled correctly (including degenerate)
  - Exact number types are used to achieve exact results
- Efficient
- Generic
  - Easy to interface, extend, and adapt
  - Notification mechanism for change propagation
- Modular
  - Geometric and topological aspects are separated

CGAL manual

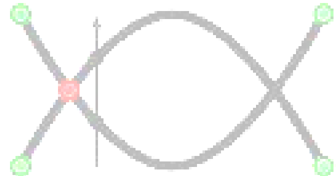
Geometric Traits

- Define the family of curves
- Aggregate geometric types and operations over the types

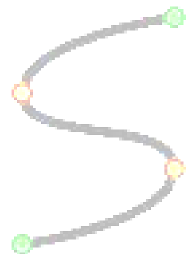
- Compare two points
- Determine the relative position of a point and an *x*-monotone curve



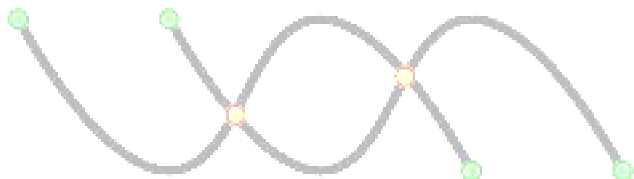
- Determine the relative position of two *x*-monotone curves to the left (right) of a point



- Subdivide a curve into x-monotone curves



- Find all intersections of two x-monotone curves



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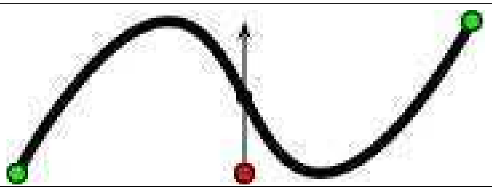

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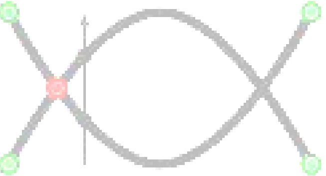

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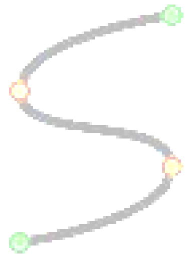
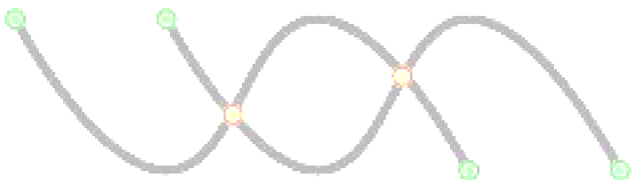


- Find all intersections of two x-monotone curves



Geometric Traits

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- Compare two points
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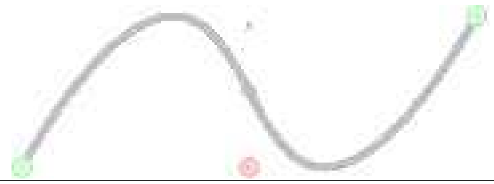

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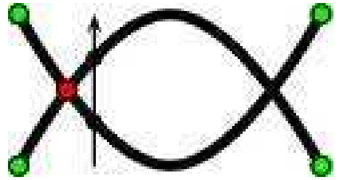
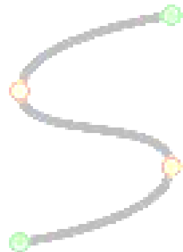
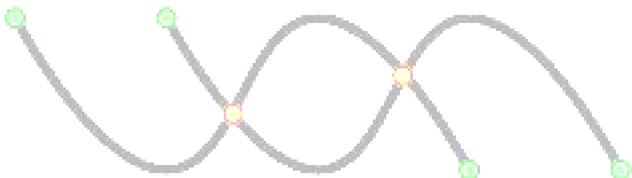


- Subdivide a curve into x-monotone curves

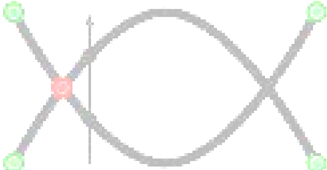
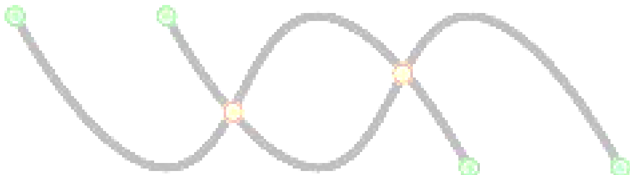


- Find all intersections of two x-monotone curves



Geometric Traits

- Define the family of curves
- Aggregate geometric types and operations over the types

• Compare two points
• Determine the relative position of a point and an *x*-monotone curve 
• Determine the relative position of two *x*-monotone curves to the left (right) of a point 
• Subdivide a curve into x-monotone curves 
• Find all intersections of two x-monotone curves 

Geometric Traits

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- Compare two points
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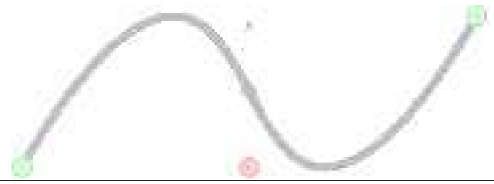

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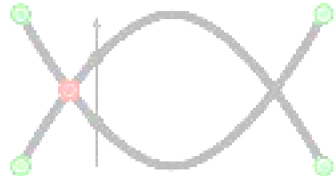

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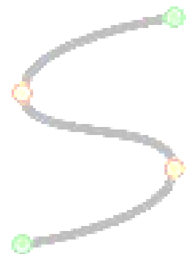
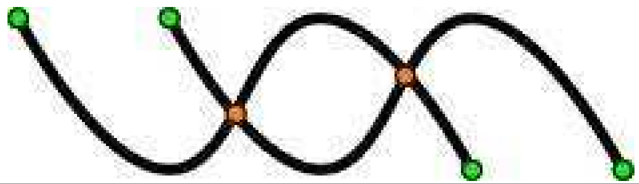


- Find all intersections of two x-monotone curves



Arrangement Traits Classes

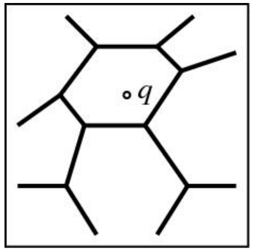
Curve Family	Degree	Surface	Boundness	Arithmetic	Attribute	
linear segments	1	plane	bounded	rational	caching noncaching	
linear segments, rays, lines	1	plane	unbounded	rational		
piecewise linear curves	∞	plane	bounded	rational	caching noncaching	
circular arcs, linear segments	≤ 2	plane	bounded	rational	CK	
algebraic curves	≤ 2	plane	Bounded unbounded	algebraic	CORE CKvA_2	
quadric projections	≤ 2	plane	unbounded	algebraic		
algebraic curves	≤ 3	plane	unbounded	algebraic		
algebraic curves	≤ n	plane	unbounded	algebraic		
planar Bézier curves	≤ n	plane	unbounded	algebraic		
univariate polynomials	≤ n	plane	unbounded	algebraic	RS	
rational function arcs	≤ n	plane	unbounded	algebraic		
geodesic arcs on sphere	≤ 2	sphere	bounded	rational		
quadric intersection arcs	≤ 2	quadric	unbounded	algebraic		
dupin cyclide intersection. arcs	≤ 2	dupin cyclides	bounded	algebraic		

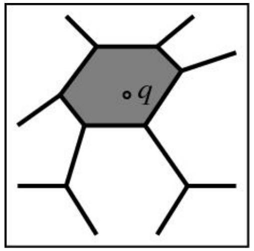
Enrich Vertices, Edges, Faces

```
template <typename Traits, typename Dcel = Arr_default_dcel<Traits_> >
class Arrangement_2 {
  . . .
};

enum Color {BLUE, RED, WHITE};

typedef CGAL::Arr_segment_traits_2<Kernel>                         Traits;
typedef Traits::Point_2                                            Point;
typedef Traits::X_monotone_curve_2                                 Segment;
typedef CGAL::Arr_extended_dcel<Traits, Color, Color, Color> Dcel;
typedef CGAL::Arrangement_2<Traits, Dcel>                          Arrangement;
```

Point Location

Given a subdivision *A* of the space into cells and *a* query point *q*, find the cell of *A* containing *q*





- Fast query processing
- Reasonably fast preprocessing
- Small space data structure

	Naive	Walk	RIC	Landmarks
Preprocessing time	none	none	O(n log n)	O(k log k)
Memory space	none	none	O(n)	O(k)
Query time	bad	reasonable	good	good
Applicability	all	limited	limited	limited

Walk — Walk along a line

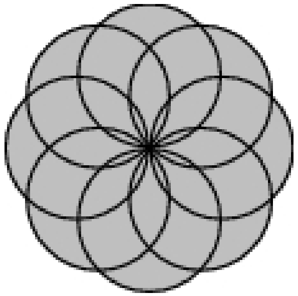
RIC — Random Incremental Construction based on trapezoidal decomposition

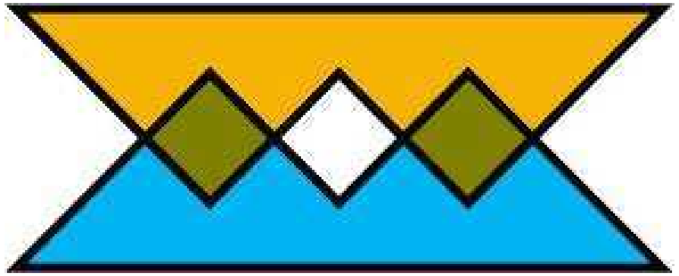
k — number of landmarks

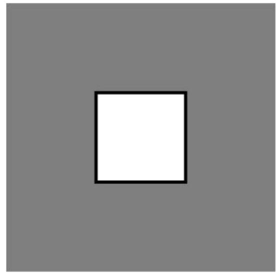
Code: Point Location

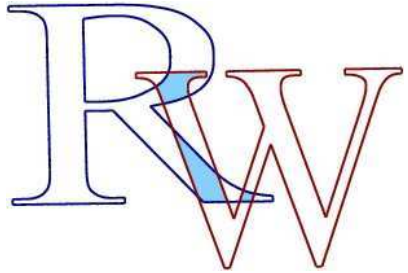
```
typedef CGAL::Arr_naive_point_location<Arrangement_2>     Naive_pl;
typedef CGAL::Arr_landmarks_point_location<Arrangement_2> Landmarks_pl;

int main() {
  Arrangement arr;
  construct_arr(arr);
  Point p(1, 4);

  Naive_pl naive_pl(arr);   // Associate arrangement to naïve point location
  CGAL::Object obj1 = naive_pl.locate(p);

  Landmarks landmarks_pl;
  landmarks_pl.attach(arr); // Attach landmarks point location to arrangement
  CGAL::Object obj2 = landmarks_pl.locate(p);

  return (obj1 == obj2) ? 0 : -1;
}
```

Boolean Set Operations









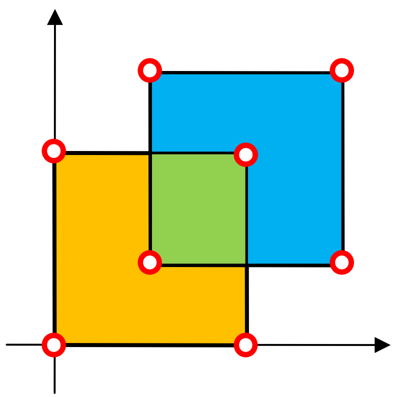
Union Intersection Complement

For two point sets P and Q and a point r :

Complement	$R = \overline{P}$	
Union	$R = P \cup Q$	
Intersection	$R = P \cap Q$	$= \overline{\overline{P} \cup \overline{Q}}$
Difference	$R = P \setminus Q$	$= P \cap \overline{Q}$
Symmetric Difference	$R = (P \setminus Q) \cup (Q \setminus P)$	
Intersection predicate	$P \cap Q \neq \varnothing$	Overlapping cell(s) are not explicitly computed
Containment predicate	$r \in P$	
Interior, Boundary, Closure		
Regularization	R = closure(interior(P))	

Code: Simple BOPs

```
int main() {
  Polygon p, q;
  p.push_back(Point(0, 0)); p.push_back(Point(2, 0));
  p.push_back(Point(1, 1)); p.push_back(Point(0, 2));
  q.push_back(Point(1, 1)); q.push_back(Point(3, 1));
  q.push_back(Point(2, 2)); q.push_back(Point(1, 3));

  Polygon_with_holes comp_p, comp_q;
  CGAL::complement(p, comp_p);
  CGAL::complement(q, comp_q);

  Polygon_with_holes a;
  CGAL::join(comp_p, comp_q, a);

  std::list<Polygon_with_holes> l1, l2;
  CGAL::complement(a, std::back_inserter(l1));
  CGAL::intersection(p, q, l2);

  return std::compare(l1, l2);
}
```



CGAL::Boolean_set_operation_2

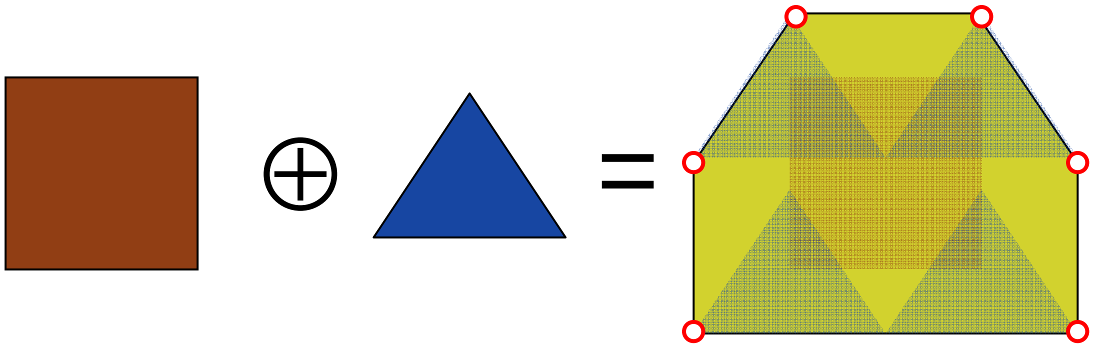
- Supports
  - regularized Boolean set-operations
  - intersection predicates
  - point containment predicates
- Operands and results are regularized point sets bounded by *x*-monotone curves referred to as general polygons
  - General polygons may have holes
- Extremely efficient aggregated operations
- Based on the **`Arrangement_2`** and **`Polygon_2`** packages

Online demo CGAL manual

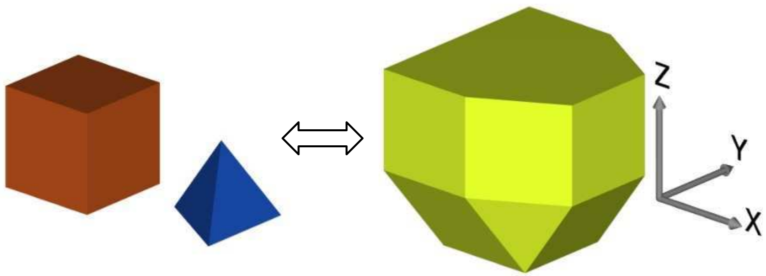
Minkowski Sum in R^d

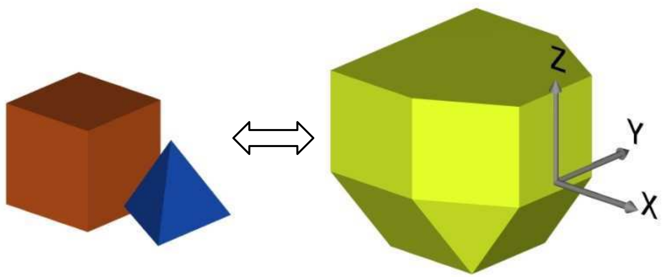
P and Q are 2 polytopes in R^d

$P \oplus Q = \{p + q \mid p \in P, q \in Q\}$ Minkowski sum



$P \cap Q \neq \emptyset \Leftrightarrow \text{Origin} \in P \oplus (-Q)$ collision





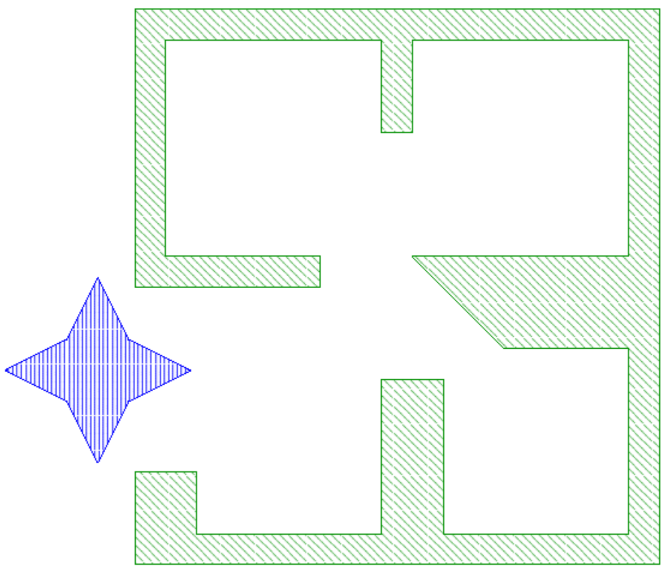
CGAL::Minkowski_sum_2

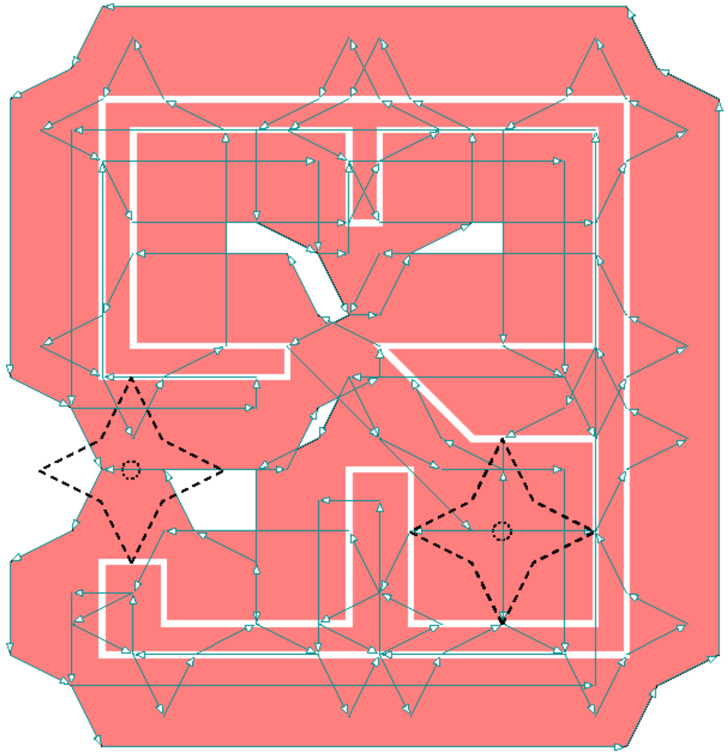
- Based on the **`Arrangement_2`**, **`Polygon_2`**, and **`Partition_2`** packages
- Supports Minkowski sums of two simple polygons
  - Implemented using either decomposition or convolution
  - Exact
- Interoperable with **`Boolean_set_operations_2`**, e.g., Compute the union of offset polygons
- Supports Minkowski sums of a simple polygon and a disc (polygon offsetting)
  - Offers either an exact computation or a conservative approximation scheme
  - Disk radius can be negative (inner offset)

CGAL manual

Motion Planning

- The input robot and the obstacle are non-convex
- Exploits the convolution method
- The output sum contains four holes, isolated points, and antennas



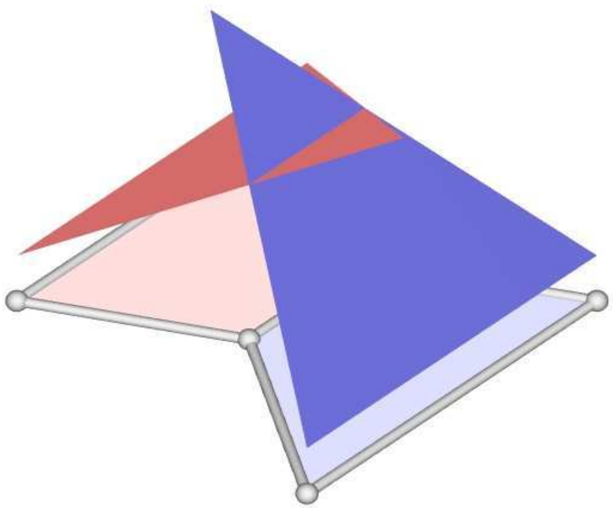


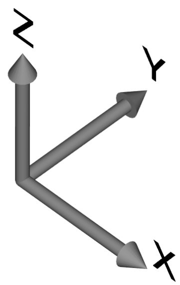
Envelopes in R^3

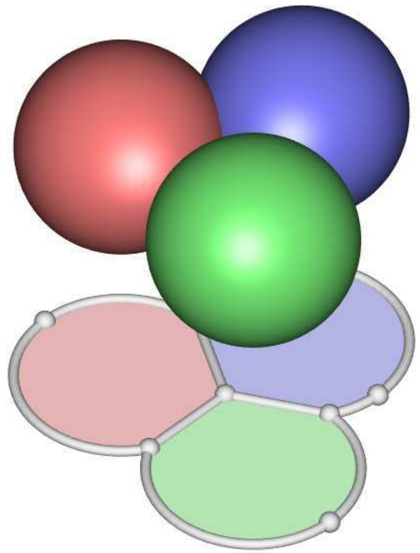
The lower envelope of a set of *xy*-monotone surfaces $S = \{S_1, S_2, \ldots, S_n\}$, is the point-wise minimum of all surfaces

The minimization diagram of S is an arrangement

- The identity of the surfaces that induce the lower envelope over a specific cell (vertex, edge, face) of the arrangement is the same







Code: Lower Envelope

```
#include <CGAL/Exact_predicates_exact_constructions_kernel.h>
#include <CGAL/Env_triangle_traits_3.h>
#include <CGAL/Env_surface_data_traits_3.h>
#include <CGAL/envelope_3.h>

typedef CGAL::Exact_predicates_exact_constructions_kernel Kernel;
typedef CGAL::Env_triangle_traits_3<Kernel>               Traits;
typedef Traits::Point_3                                   Point;
typedef Traits::Surface_3                                 Tri;
enum Color {RED, BLUE};
typedef CGAL::Env_surface_data_traits_3<Traits, Color>    Data_traits;
typedef Data_traits::Surface_3                            Dtri;
typedef CGAL::Envelope_diagram_2<Data_traits>             Arrangement;

int main() {
  Point p1(0,0,1), p2(0,6,1), p3(5,3,5), p4(6,0,1), p5(6,6,1), p6(1,3,5);
  Tri tris[] = {Dtri(Tri(p1,p2,p3), RED), Dtri(Tri(p4,p5,p6), BLUE)};
  Arrangement arr;
  CGAL::lower_envelope_3(tris, tris+2, arr);
  return (arr.is_valid()) ? 0 : -1;
}
```

CGAL::Envelope_3

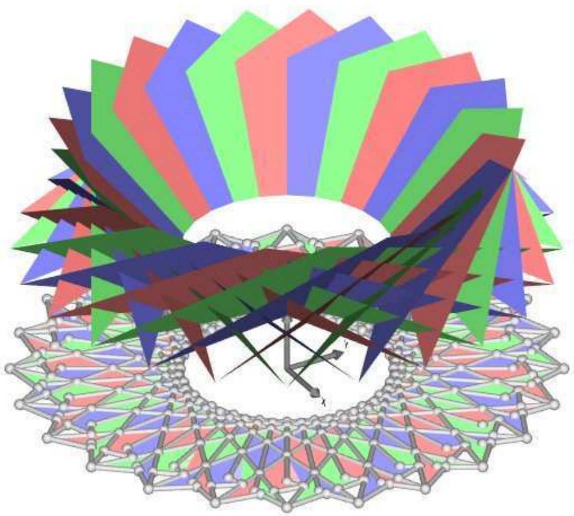
- Constructs lower and upper envelopes of surfaces

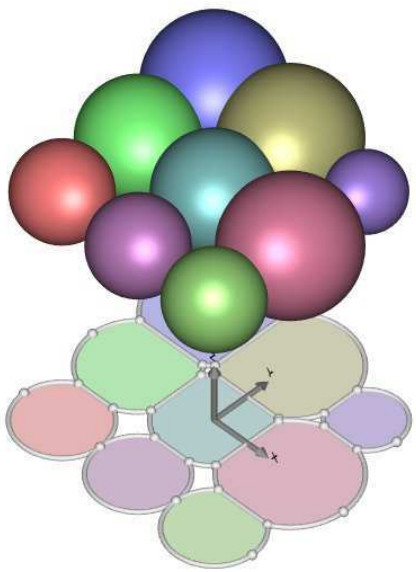
Surface Family	Class Name	
triangles	Env_triangle_traits_3	
spheres	Env_sphere_traits_3	
planes and half planes	Env_plane_traits_3	
quadrics	Env_quadric_traits_3	mpii

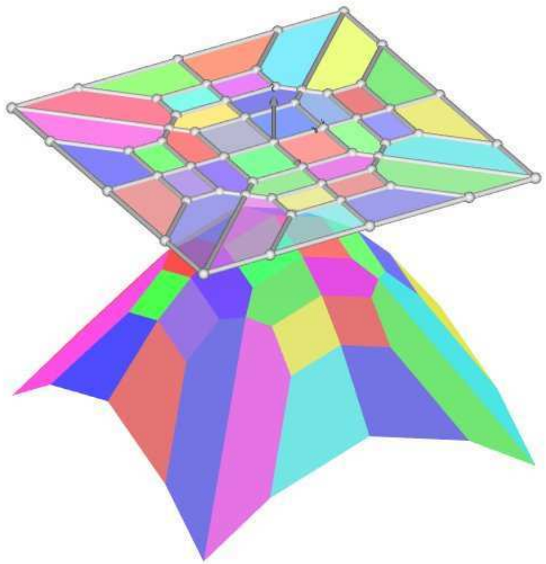
- Based on the **`Arrangement_2`** package
- Exploits
  - Overlay computation (using the sweep line framework)
  - Isolated points
  - Zone traversal

CGAL manual

Lower Envelopes



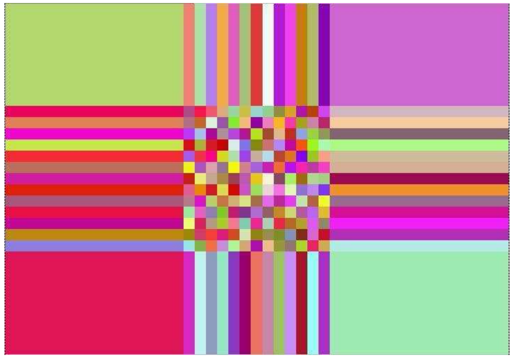


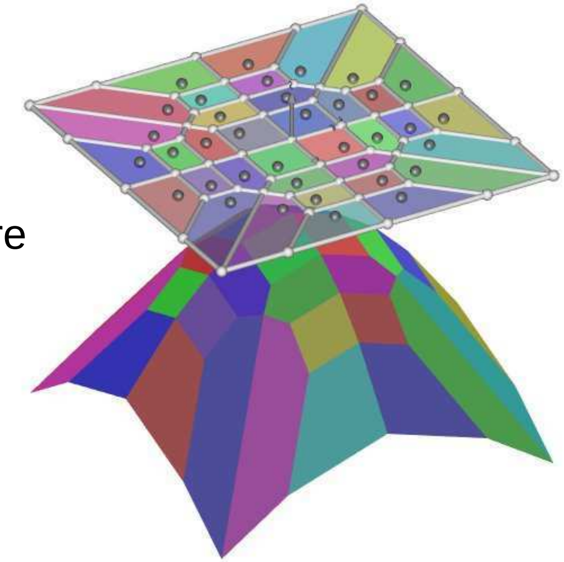


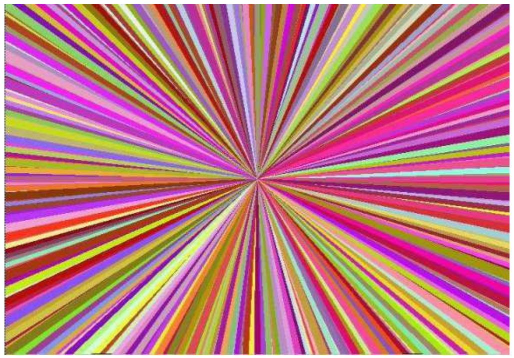
Voronoi Diagrams via Envelopes

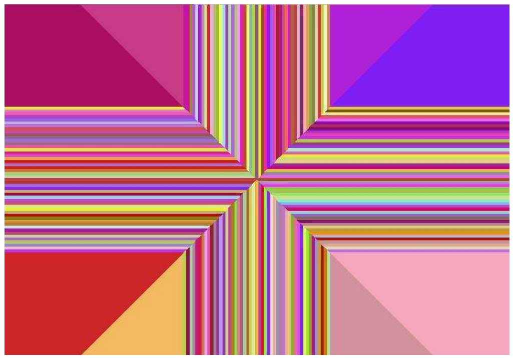
- Computed as lower envelopes of planes
- Represented as planar arrangements of unbounded curves


points along a line segment


points on a grid inside a square




points on a circle

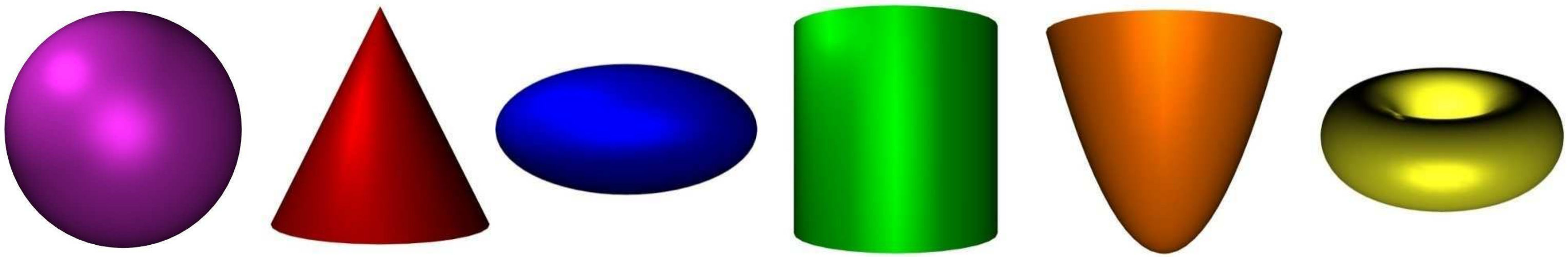

points on a square boundary

Arrangements on Surfaces in R^3

A parametric surface *S* of two parameters is a surface defined by parametric equations involving two parameters *u* and *v*:

$$f_S(u, v) = (x(u, v), y(u, v), z(u, v))$$

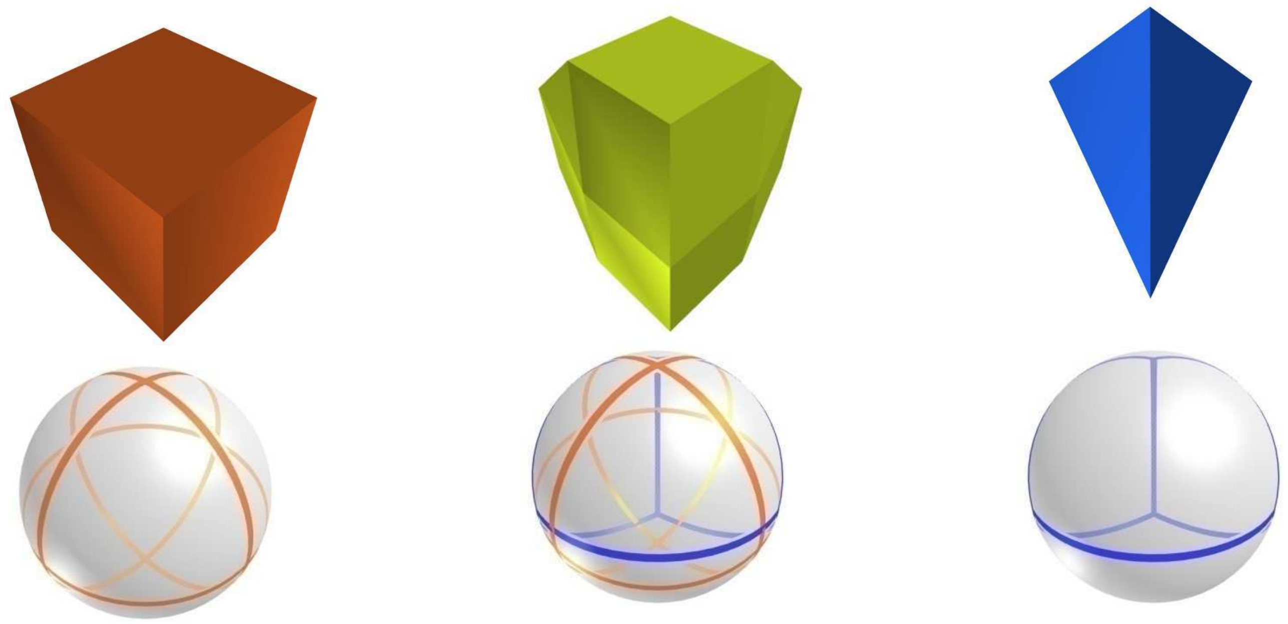
Thus, $f_s : P \rightarrow R^3$ and $S = f_s(P)$, where P is a continuous and simply connected two-dimensional parameter space



We deal with orientable parametric surfaces

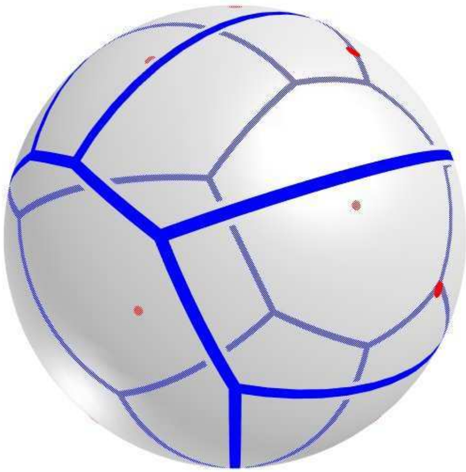
Minkowski-Sums of Polytopes

- The Gaussian map of a polytope is the decomposition of S^2 into maximal connected regions so that the extremal point is the same for all directions within one region
- The overlay of the Gaussian maps of two polytopes *P* and *Q* is the Gaussian map of the Minkowski sum of *P* and *Q*

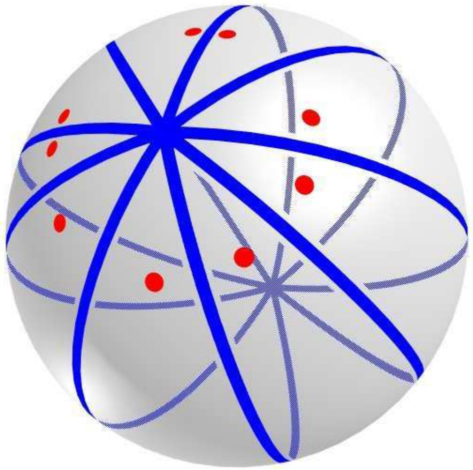


Voronoi Diagrams on the Sphere

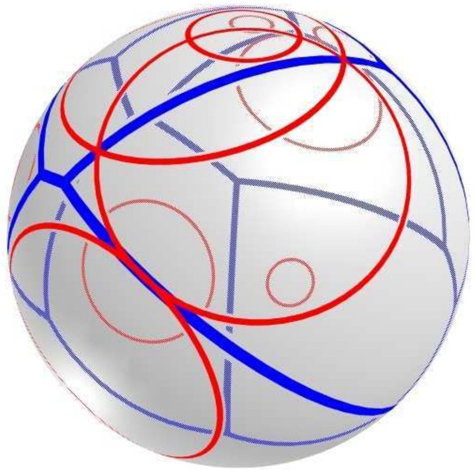
- All algorithms supported by the **`Arrangement_2`** package can also be used on the sphere
- We compute lower envelopes defined over the sphere
- We can compute Voronoi diagrams on the sphere, the edges of which are geodesic arcs



Voronoi diagram on the sphere



Degenerate Voronoi diagram on the sphere



Power (Laguerre Voronoi) diagram on the sphere

Arrangements on the Sphere

- The overlay of
  - An arrangement on the sphere induced by
    - the continents and some of the islands on earth
    - 5 cities
      - New Orleans
      - Los Angeles
      - San Antonio
      - San Diego
      - Boston
  - Voronoi diagram of the cities

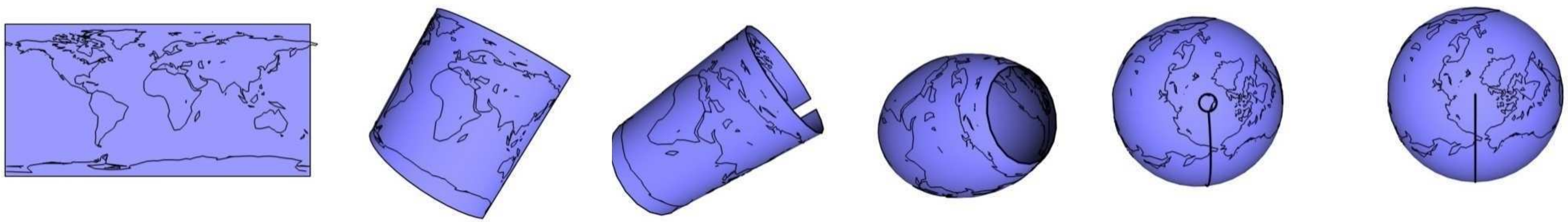
The sphere is oriented such that Cambridge is at the center

- Diagrams, envelopes, etc. are represented as arrangements
  - Can be passed as input to consecutive operations

Online demo

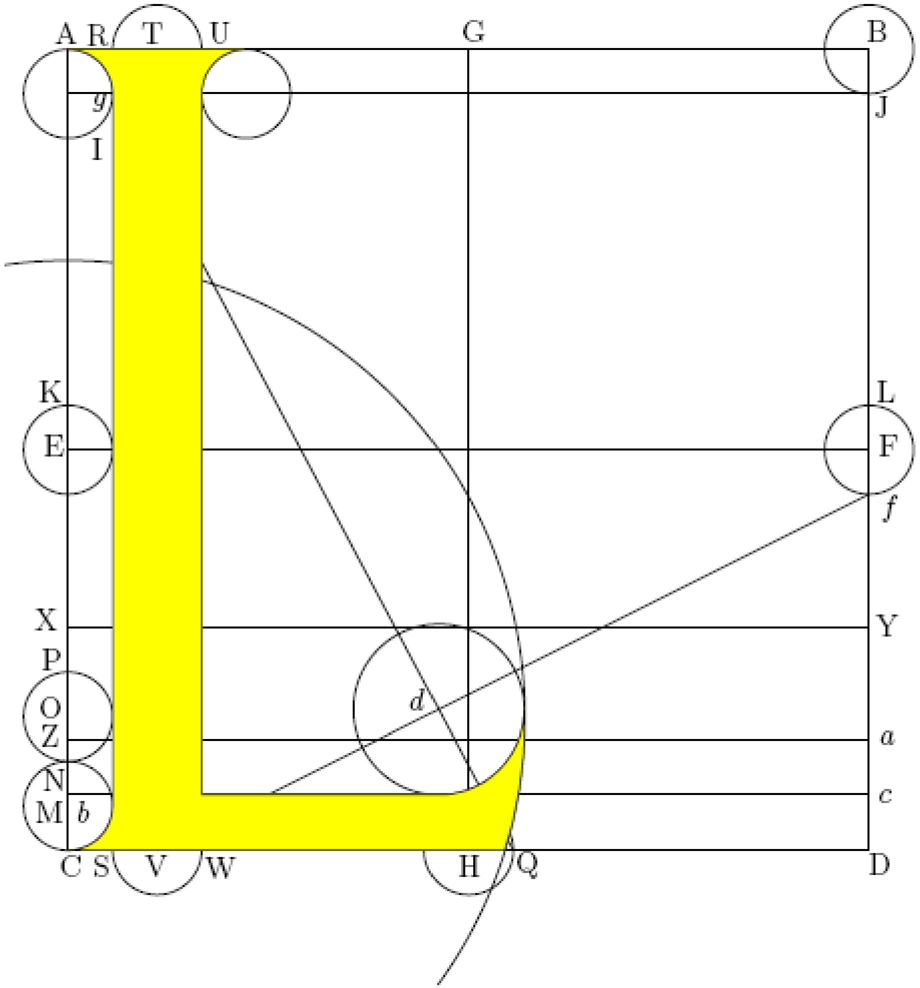
Video

- [Arrangements of Geodesic Arcs on the Sphere]
- Was presented at the 24th ACM Symposium on Computational Geometry, College Park, Maryland, July 2008



Summary

- Arrangements are versatile tools
- Arrangements are used as foundation for higher level geometric data structures
- Arrangements are not bound to the plane



Triangulations and Mesh Generation

Andreas Fabri
GeometryFactory

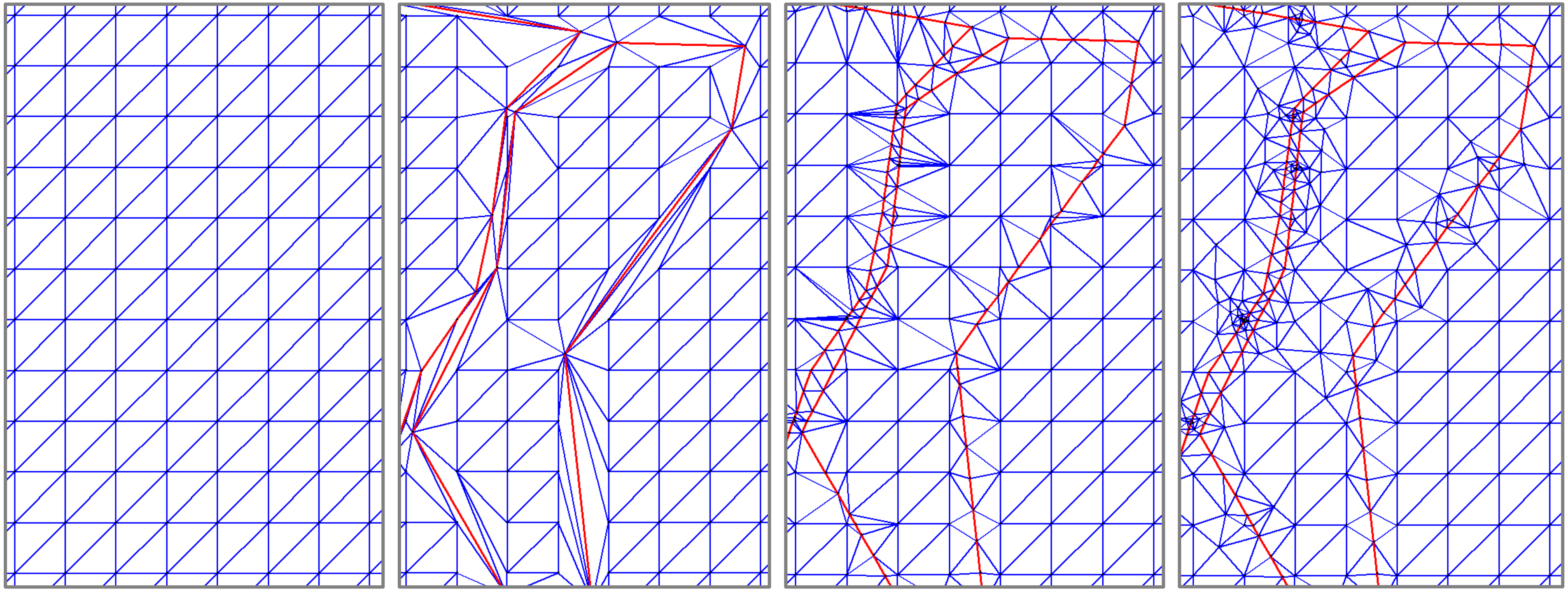
Pierre Alliez
INRIA

Outline

- 2D
  - From triangulations to quality meshes
  - Related components
- 3D
  - Triangulations
  - Mesh generation
    - Key concepts
    - Surfaces
    - Volumes
    - Next release and work in progress
- Summary

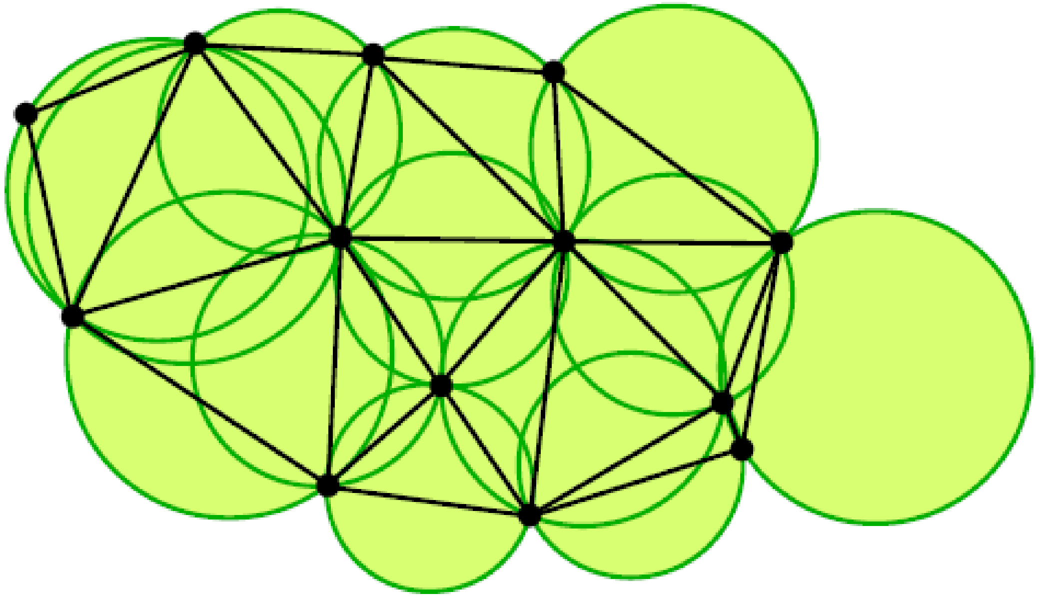
2D

From Triangulations to Quality Meshes



Delaunay Triangulation

- A triangulation is a Delaunay triangulation, if the circumscribing circle of any facet of the triangulation contains no vertex in its interior

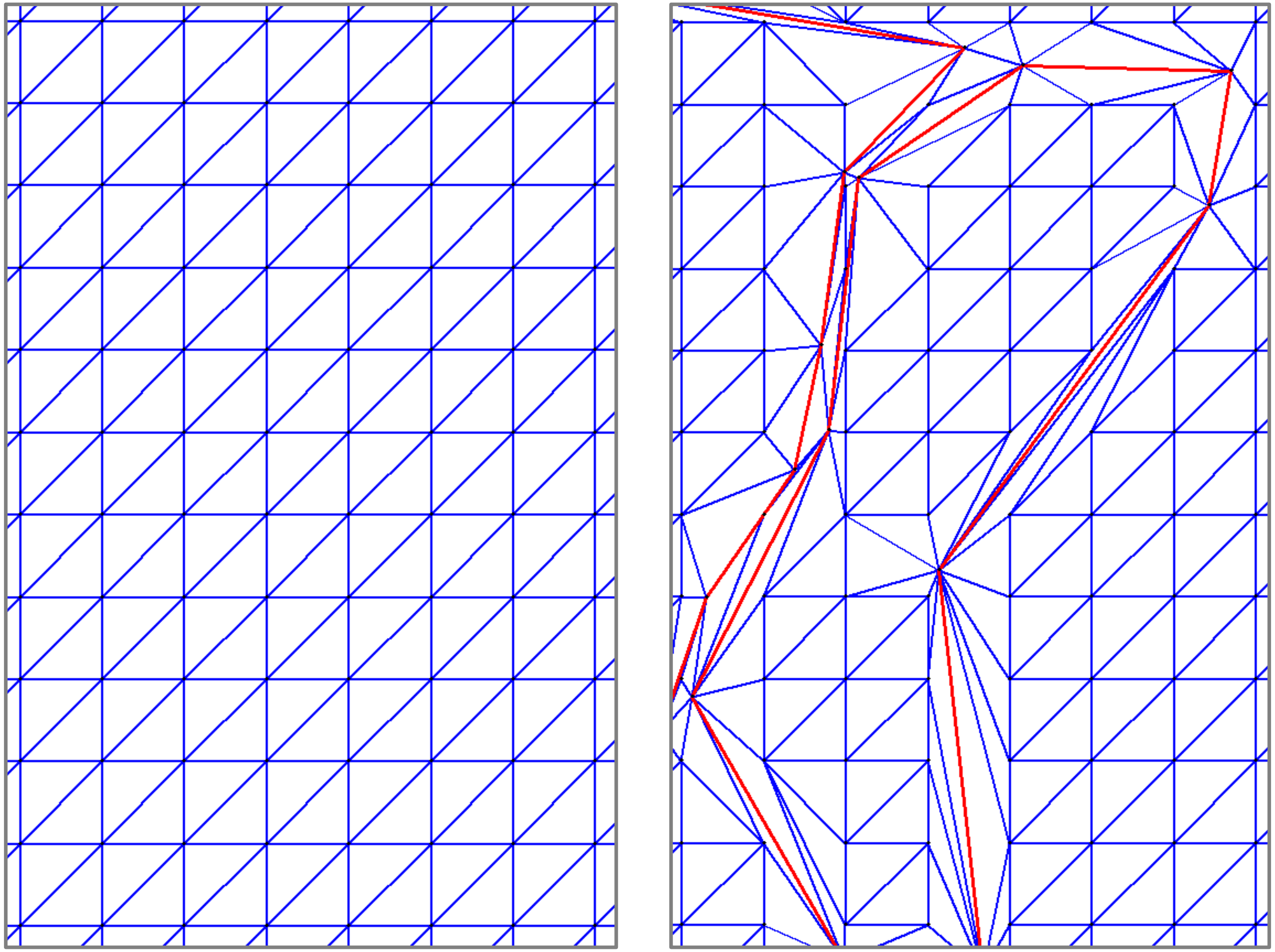


CGAL manual

Code Example

```
#include <CGAL/Exact_predicates_inexact_constructions_kernel.h>
#include <CGAL/Delaunay_triangulation_2.h>

typedef CGAL::Exact_predicates_inexact_constructions_kernel Kernel;
typedef Kernel::Point_2 Point;

typedef CGAL::Delaunay_triangulation_2<Kernel> Delaunay;
typedef Delaunay::Vertex_handle Vertex_handle;

int main()
{
  Delaunay dt;

  dt.insert( std::istream_iterator<Point>(std::cin),
             std::istream_iterator<Point>() );

  Vertex_handle v = dt.nearest_vertex(Point(0.0,0.0));

  std::cout << “Nearest vertex to origin: “ << v->point() << std::endl;

  return 0;
}
```

Adding Constraints

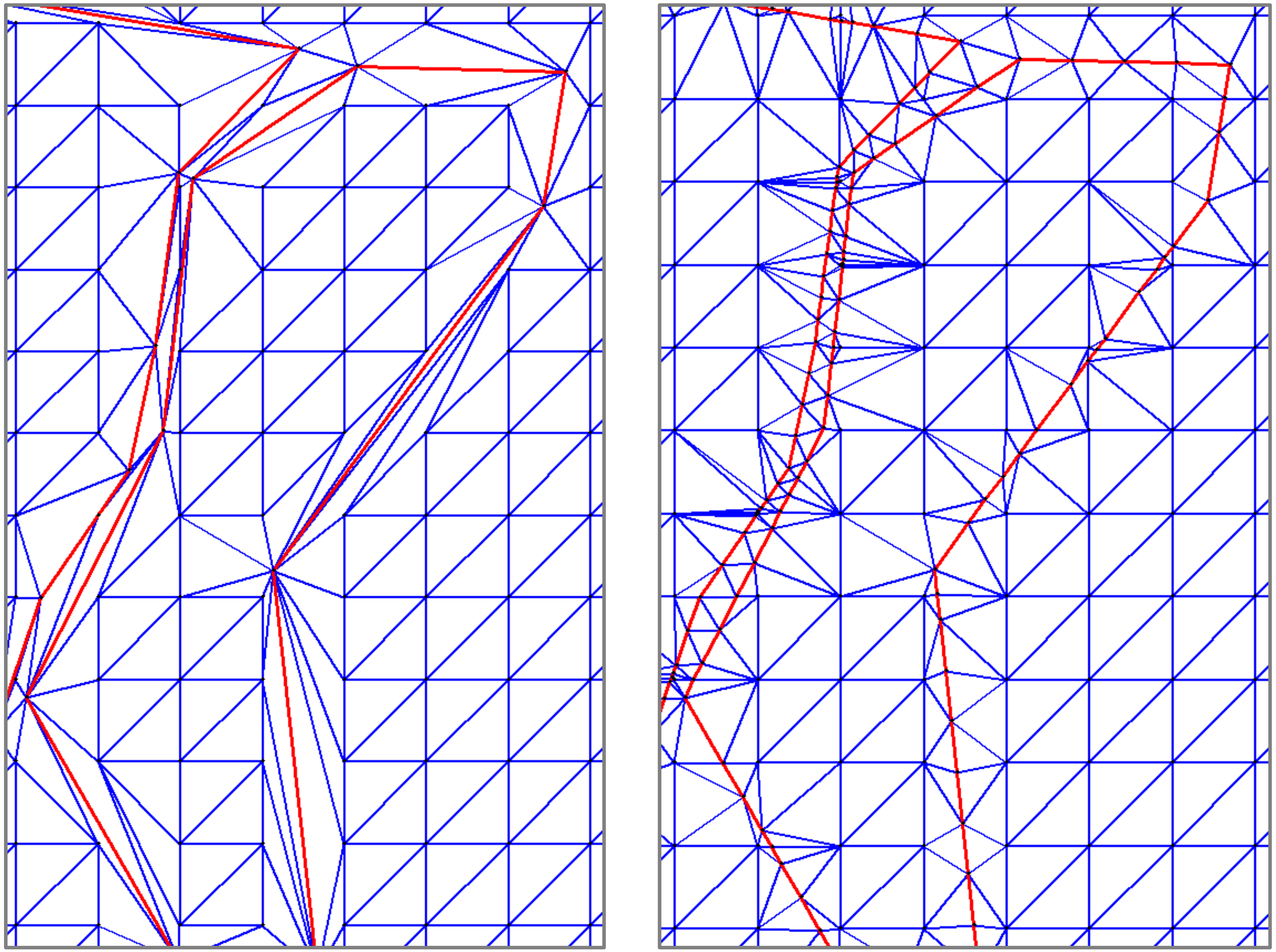


CGAL manual

Code Example

```
#include <CGAL/Exact_predicates_inexact_constructions_kernel.h>
#include <CGAL/Constrained_Delaunay_triangulation_2.h>

typedef CGAL::Exact_predicates_inexact_constructions_kernel Kernel;
typedef Kernel::Point_2 Point;

typedef CGAL::Constrained_Delaunay_triangulation_2<Kernel> CDT;
typedef CDT::Vertex_handle Vertex_handle;

int main()
{
  CDT cdt;

  // from points
  cdt.insert_constraint(Point(0.0,0.0), Point(1.0,0.0));

  // from vertices
  Vertex_handle v1 = cdt.insert(Point(2.0,3.0));
  Vertex_handle v2 = cdt.insert(Point(4.0,5.0));
  cdt.insert_constraint(v1,v2);

  return 0;
}
```

Conforming Delaunay

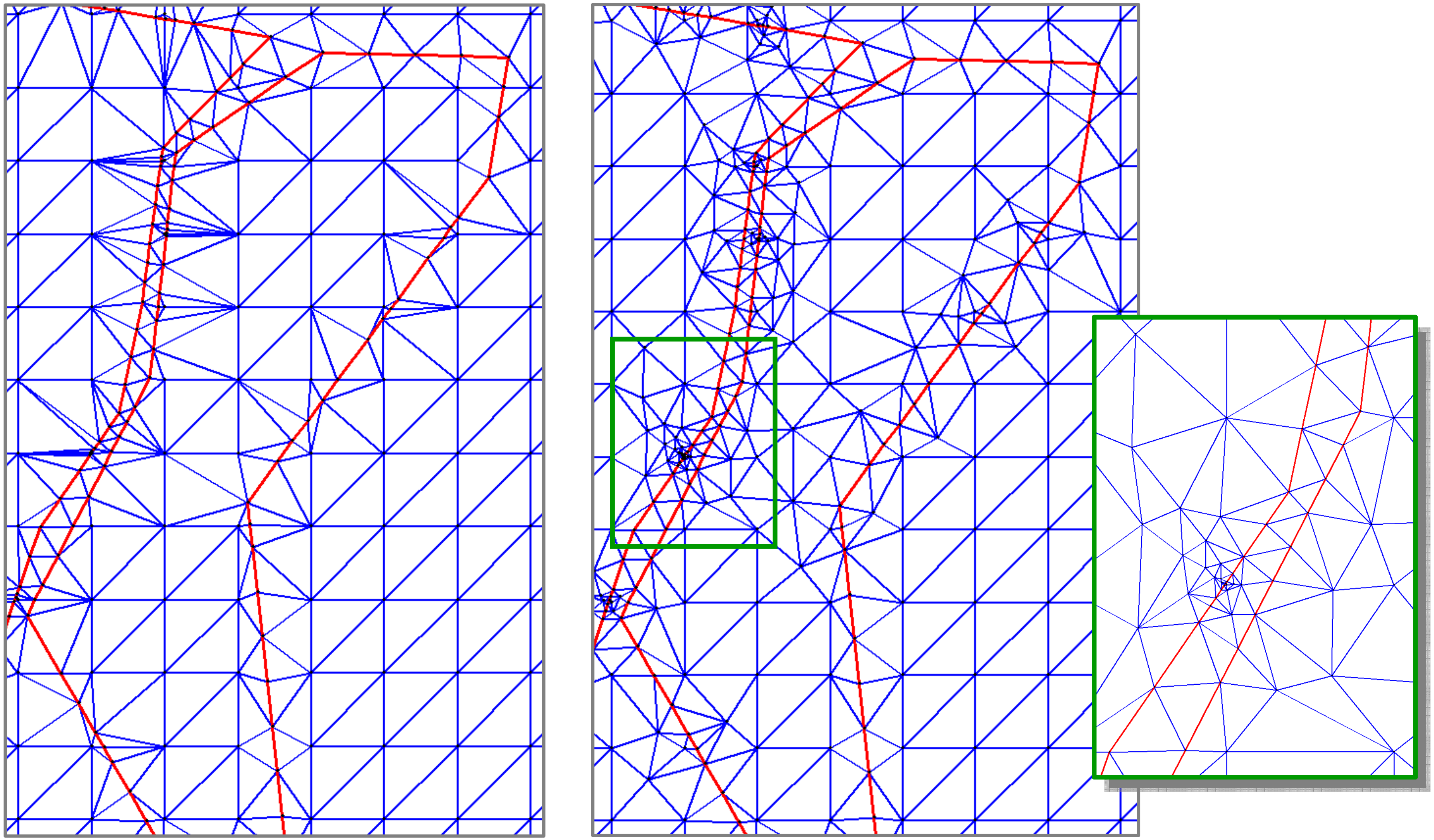


CGAL manual

Code Example

```
#include <CGAL/Triangulation_conformer_2.h>

// constrained Delaunay triangulation
CDT cdt;
… // insert points & constraints

CGAL::make_conforming_Delaunay_2(cdt);
```

Delaunay Meshing



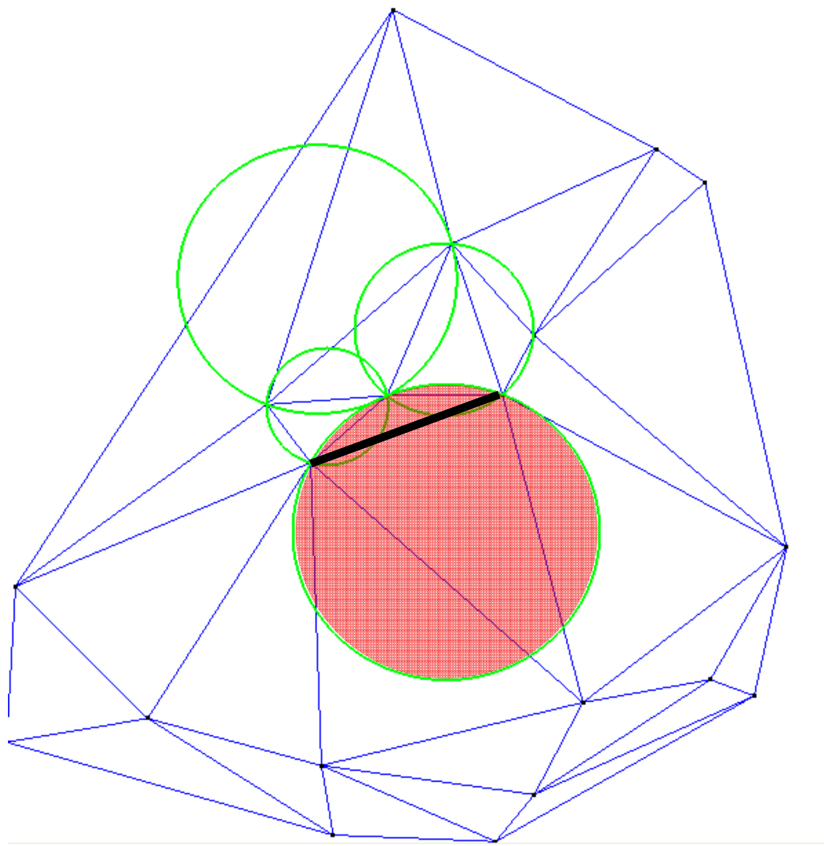
Code Example

```
#include <CGAL/Exact_predicates_inexact_constructions_kernel.h>
#include <CGAL/Constrained_Delaunay_triangulation_2.h>
#include <CGAL/Delaunay_mesher_2.h>
typedef CGAL::Exact_predicates_inexact_constructions_kernel Kernel;
typedef CGAL::Constrained_Delaunay_triangulation_2<Kernel> CDT;

int main()
{
  CDT cdt;
  … // insert points and constraints
  CGAL::refine_Delaunay_mesh_2(cdt);
  return 0;
}
```

Background

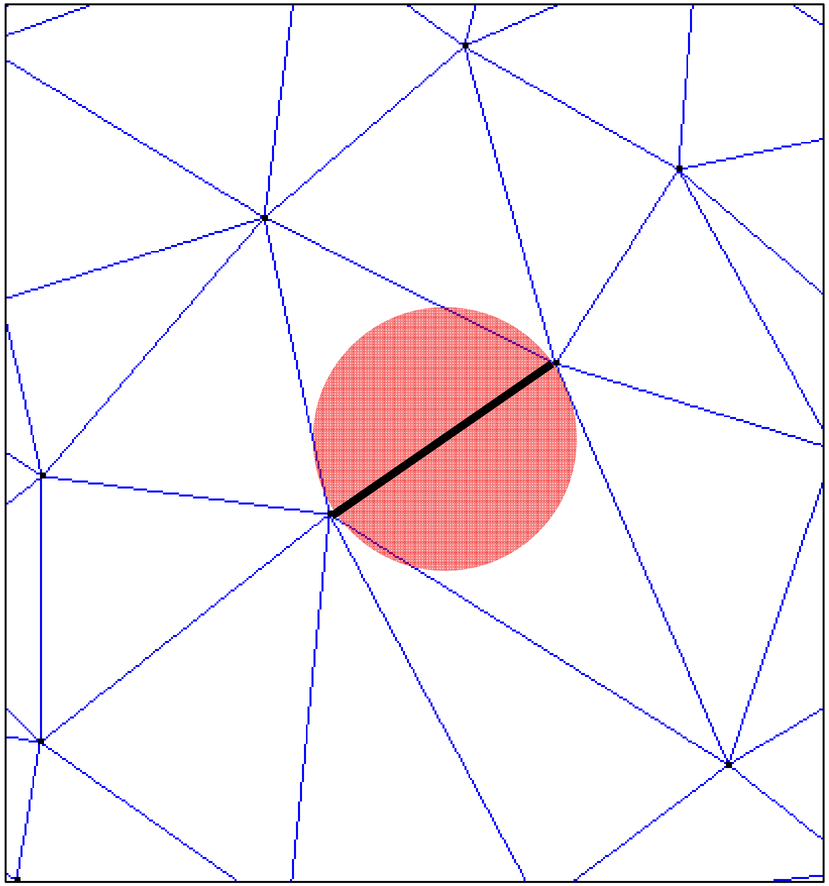
Delaunay Edge

An edge is said to be a Delaunay edge, if it is inscribed in an empty circle



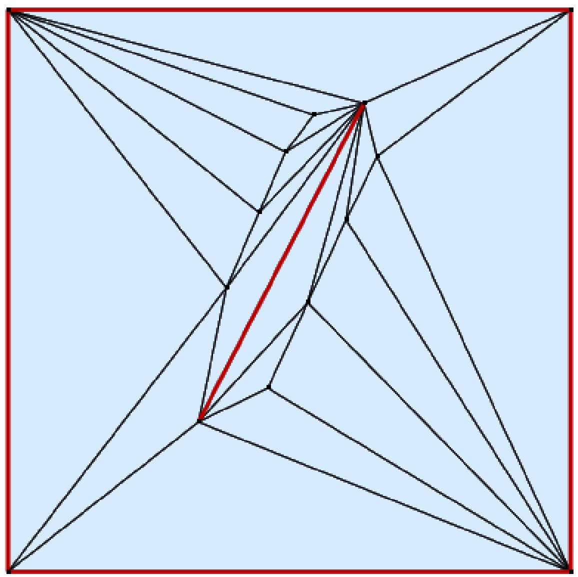
Gabriel Edge

An edge is said to be a Gabriel edge, if its diametral circle is empty

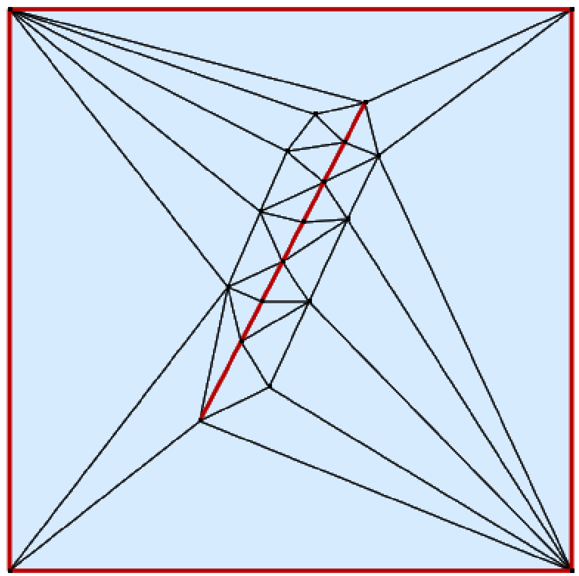


Conforming Delaunay Triangulation

A constrained Delaunay triangulation is a conforming Delaunay triangulation, if every constrained edge is a Delaunay edge



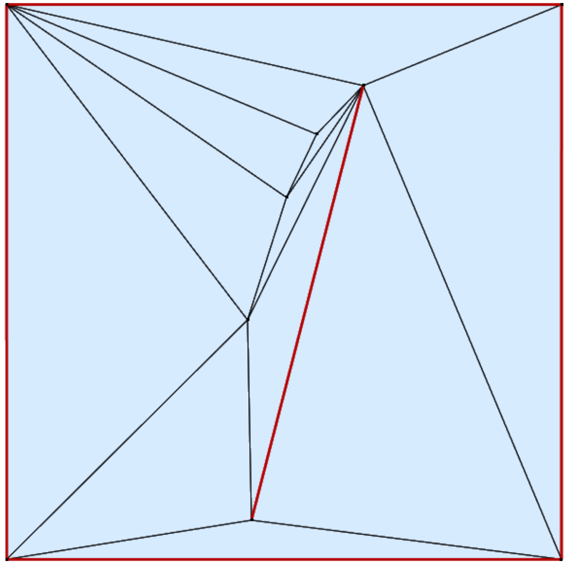
non conforming



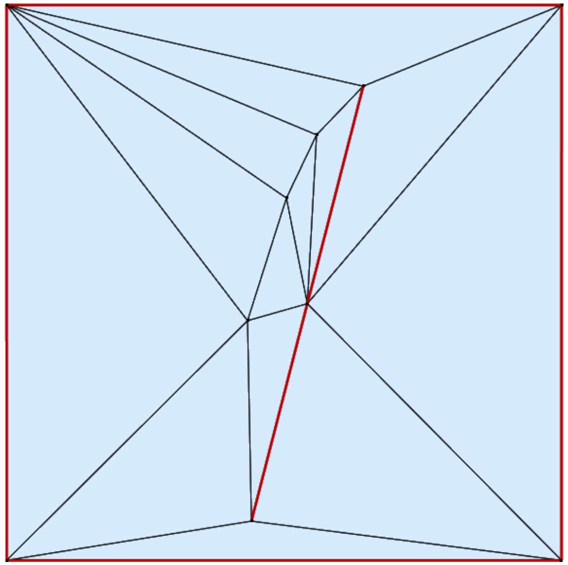
conforming

Conforming Gabriel Triangulation

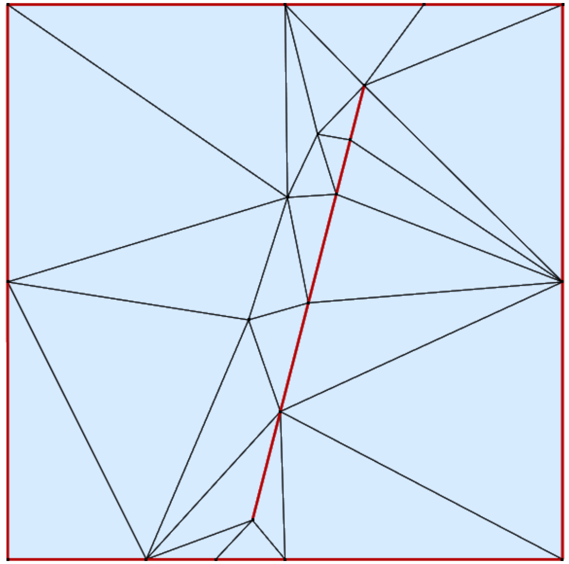
A constrained Delaunay triangulation is a conforming Gabriel triangulation, if every constrained edge is a Gabriel edge



non conforming



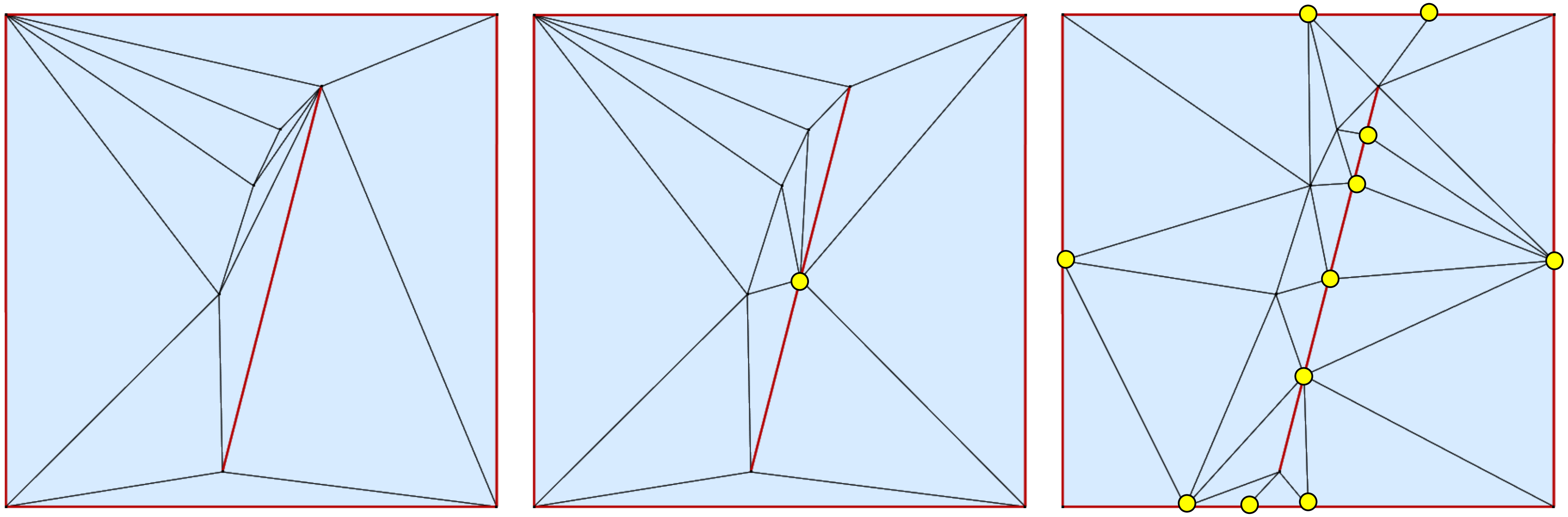
conforming



Gabriel

Steiner Vertices

Any constrained Delaunay triangulation can be refined into a conforming Delaunay or Gabriel triangulation by adding Steiner vertices.

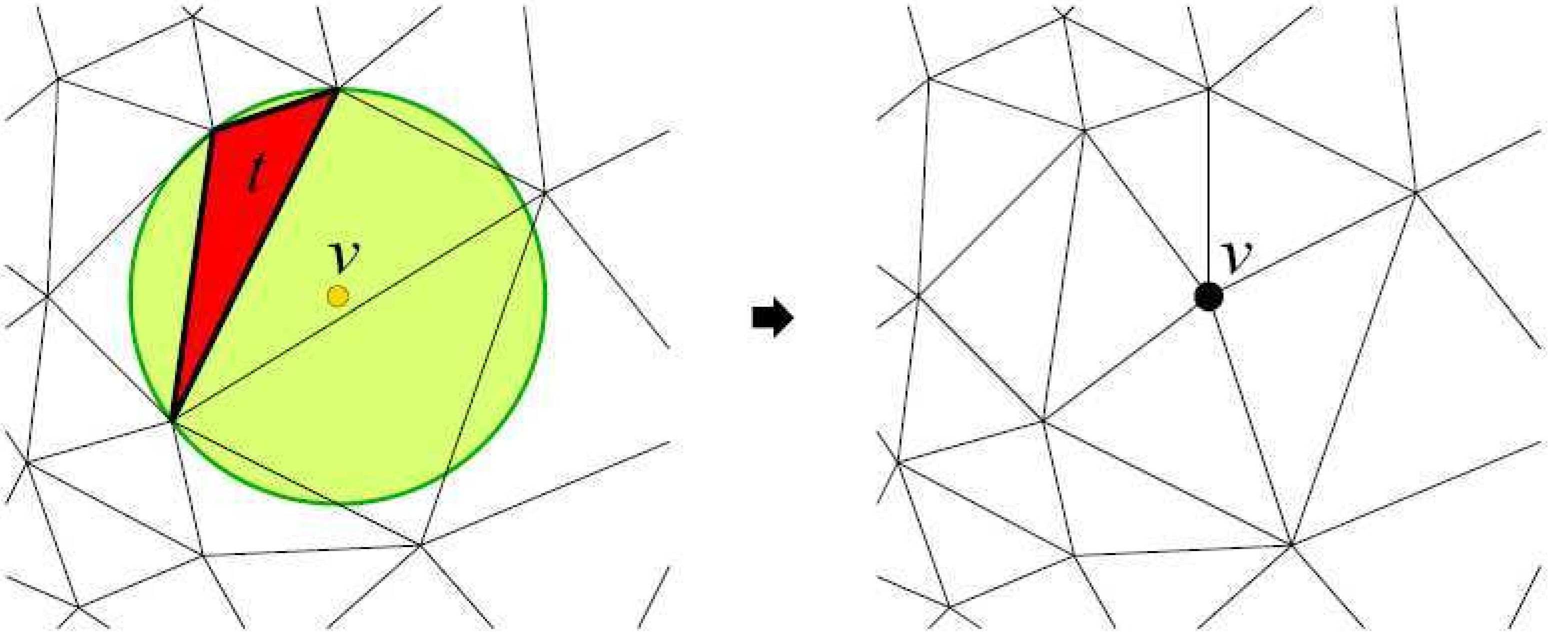


non conforming conforming Gabriel

Delaunay Refinement

Rule #1: break bad elements by inserting circumcenters (Voronoi vertices)

- “bad” in terms of size or shape (too big or skinny)

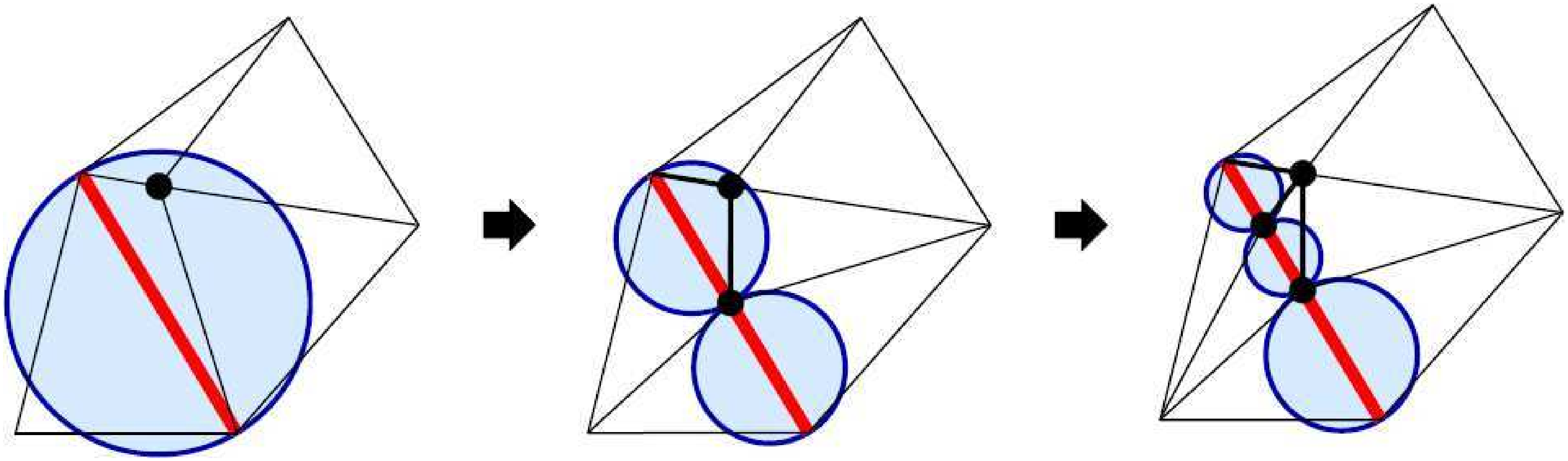


Picture taken from [Shewchuk]

Delaunay Refinement

Rule #2: Midpoint vertex insertion

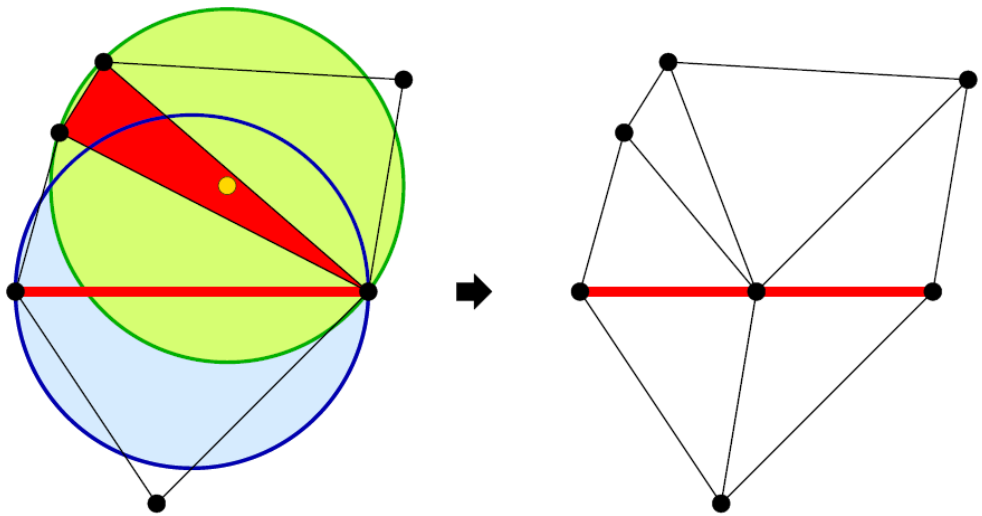
A constrained segment is said to be encroached, if there is a vertex inside its diametral circle



Picture taken from [Shewchuk]

Delaunay Refinement

Encroached subsegments have priority over skinny triangles

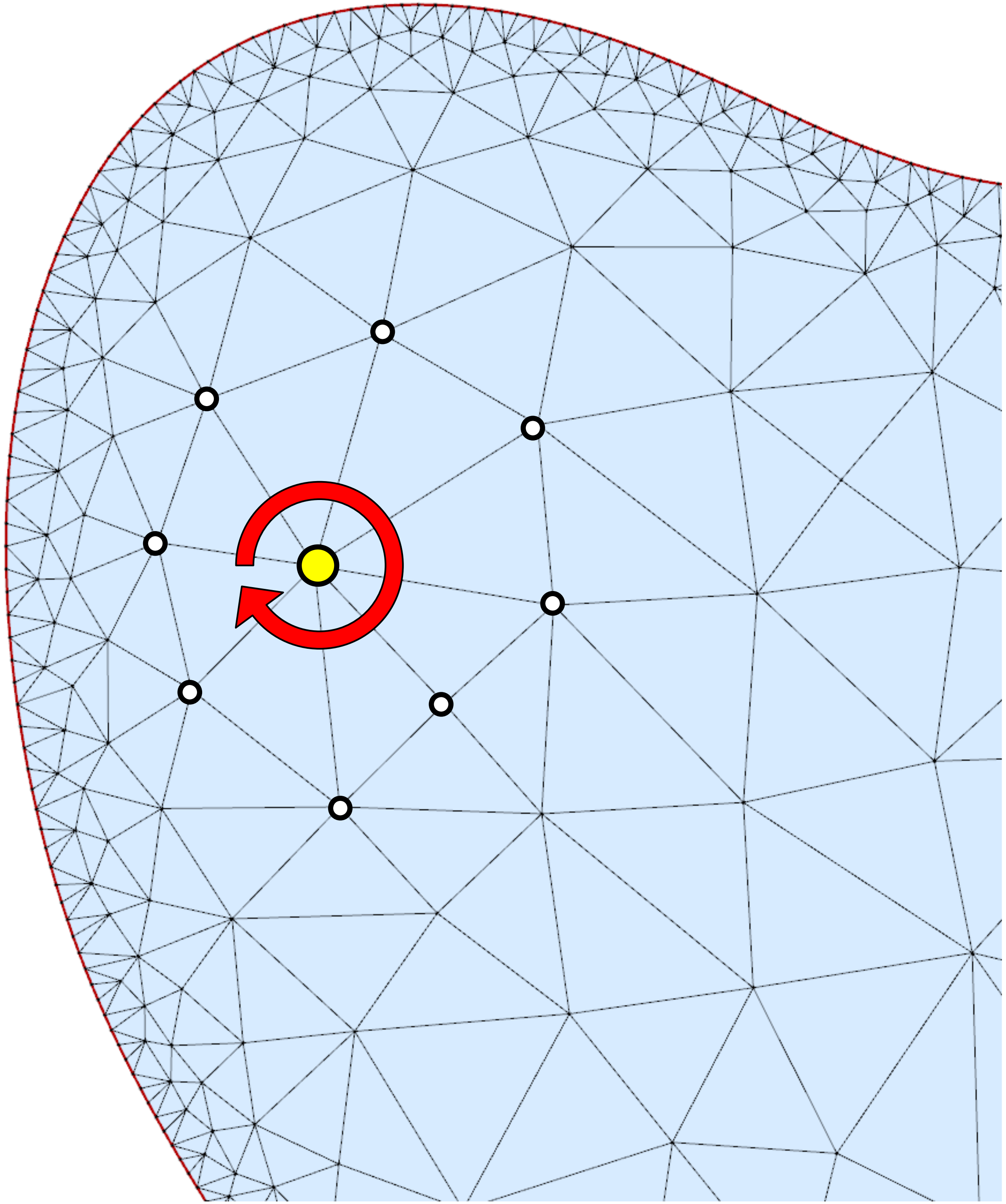


Picture taken from [Shewchuk]

API

Rich API

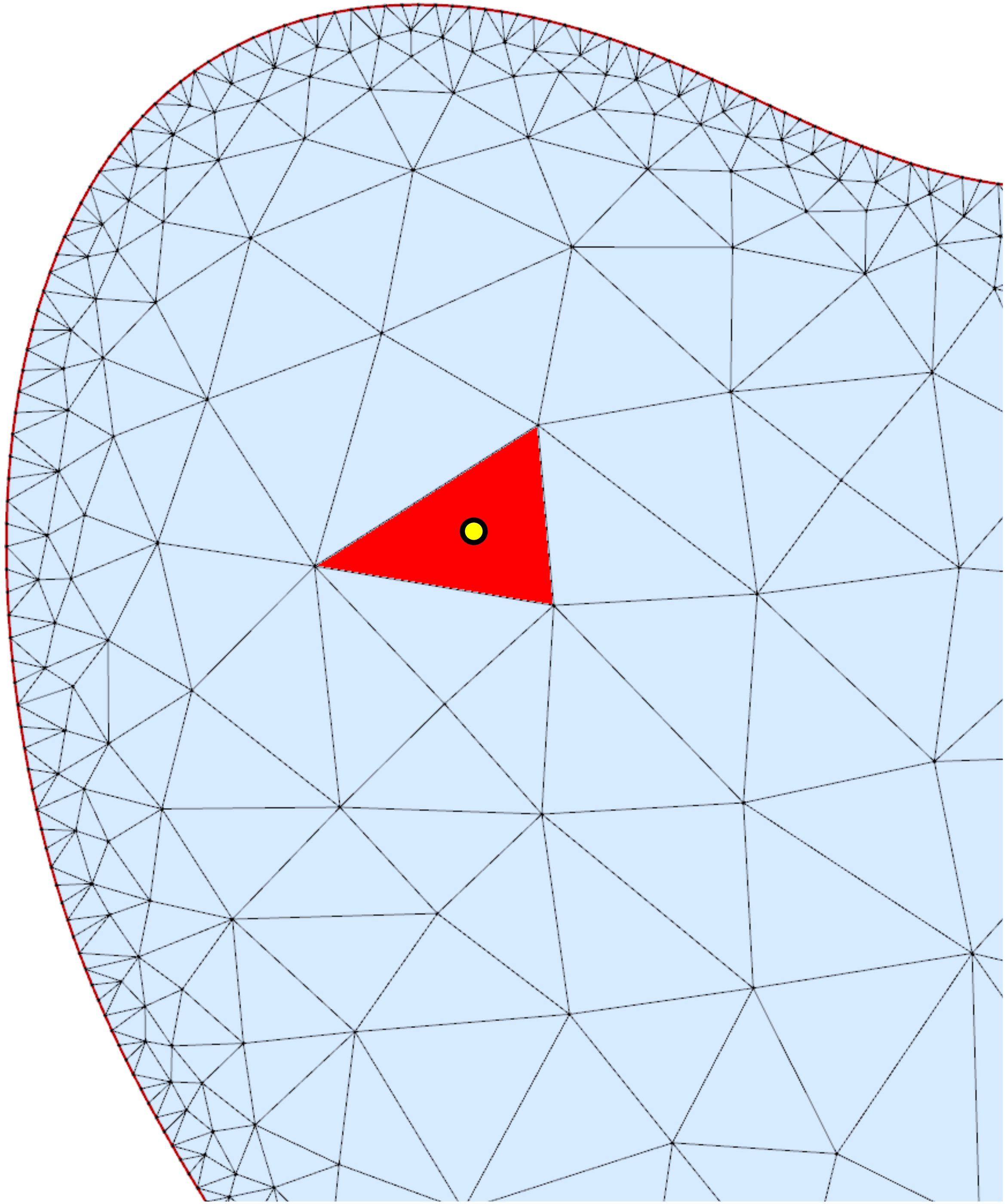
- Traversal



Online manual

Rich API

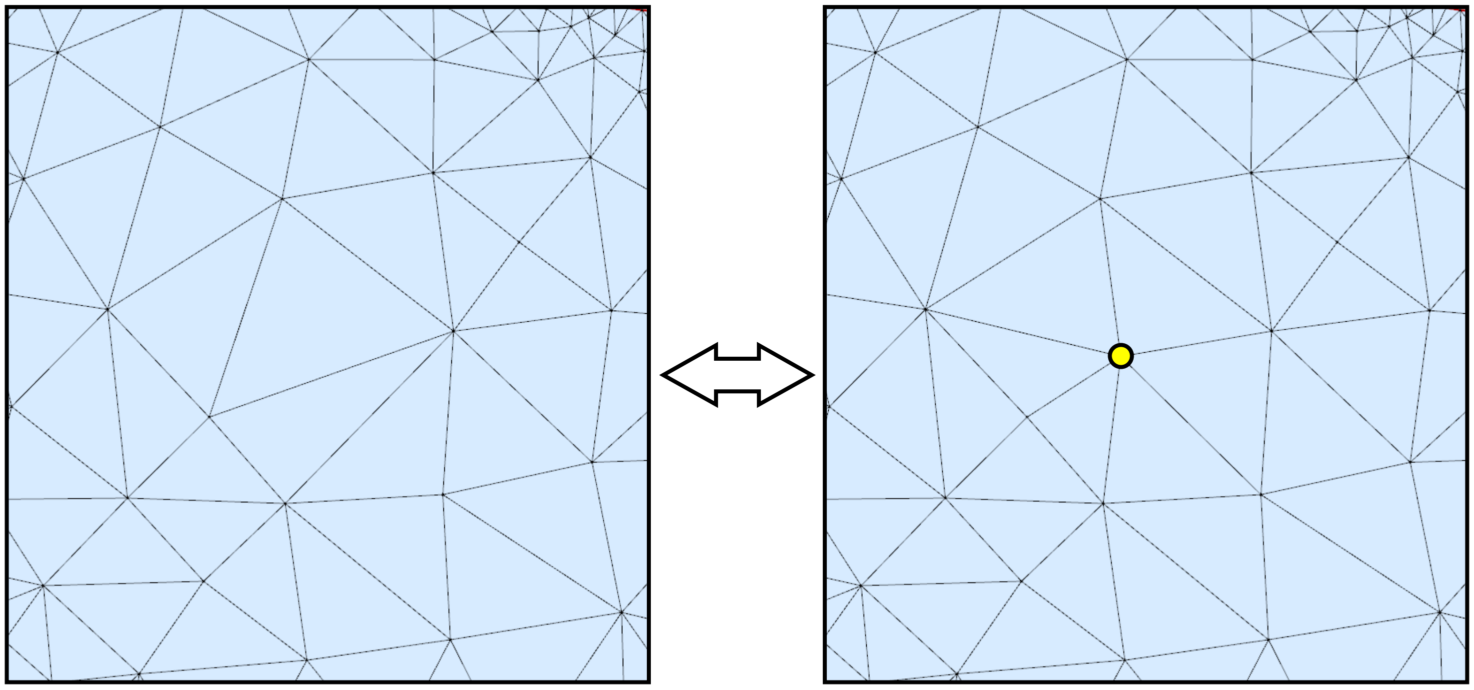
- Traversal
- Localization



Online manual

Rich API

- Traversal
- Localization
- Dynamic: insertion & removal



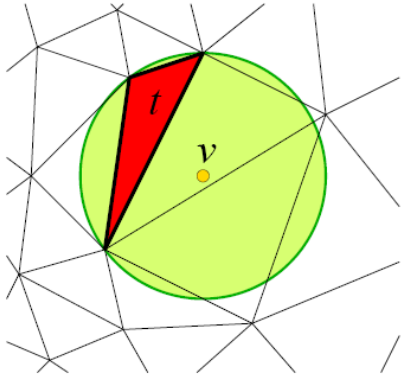
Online manual

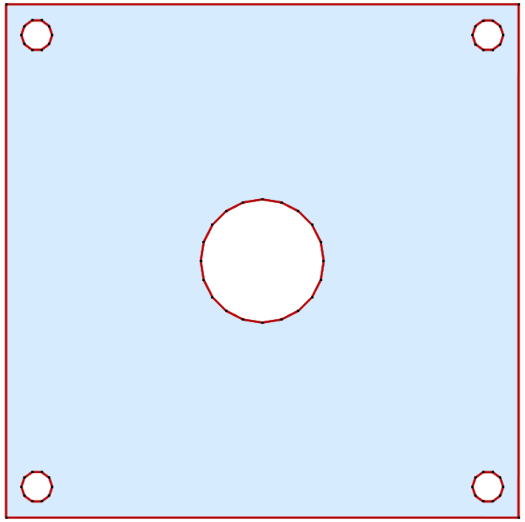
Rich API

- Traversal
- Localization
- Dynamic: insertion & removal
- Parameters for mesh generation

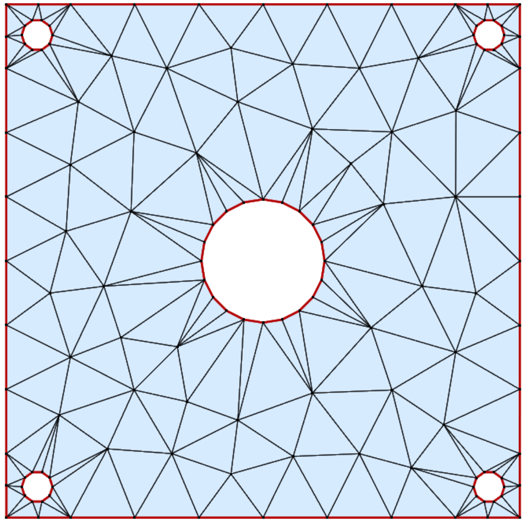
Parameters for Mesh Generation

- **Shape**
  - Lower bound on triangle angles

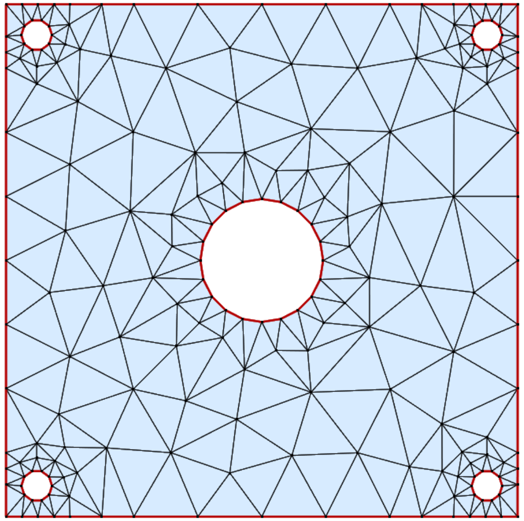




Input PLSG



5 deg

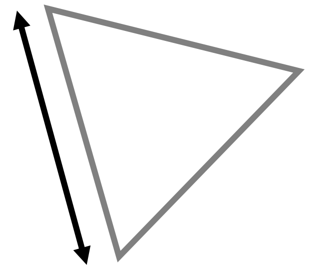


20.7 deg

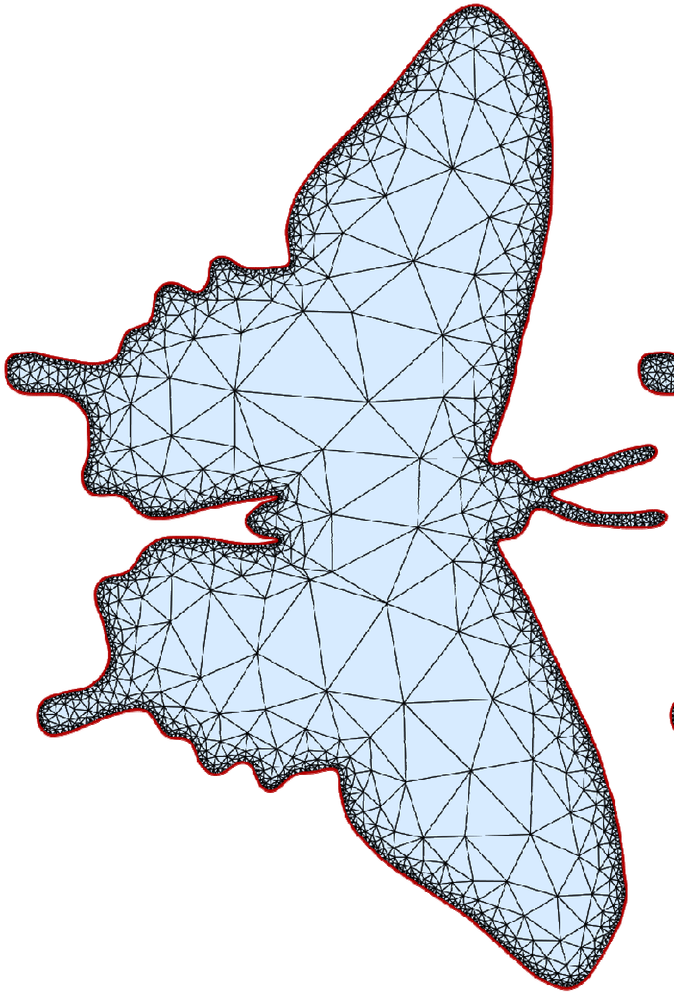
Online manual

Parameters for Mesh Generation

- Shape
  - Lower bound on triangle angles
- **Size**
  - No constraint
  - Uniform sizing
  - Sizing function

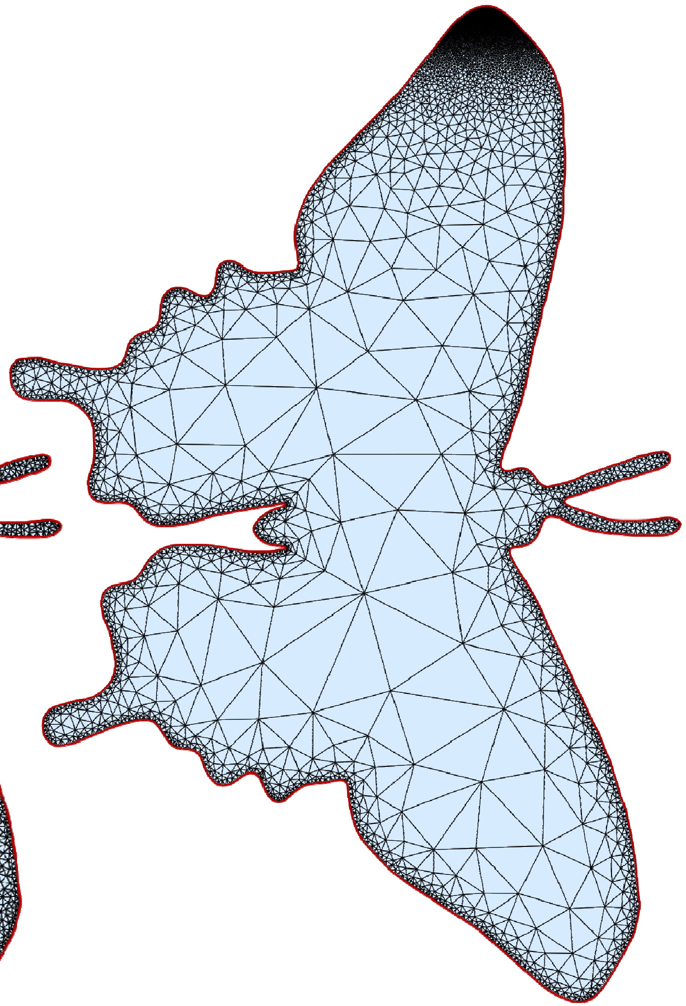


Sizing Parameter



No constraint

Uniform



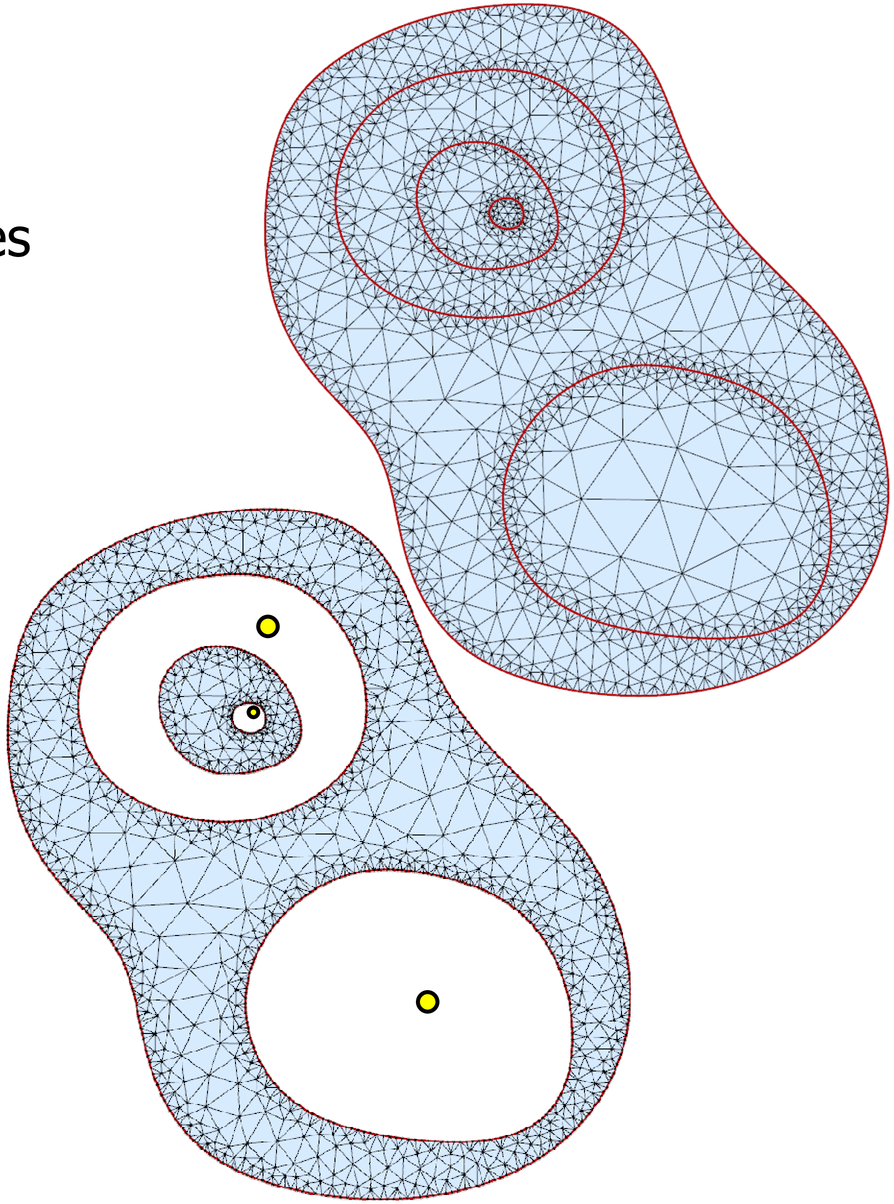
Sizing function

Example Code

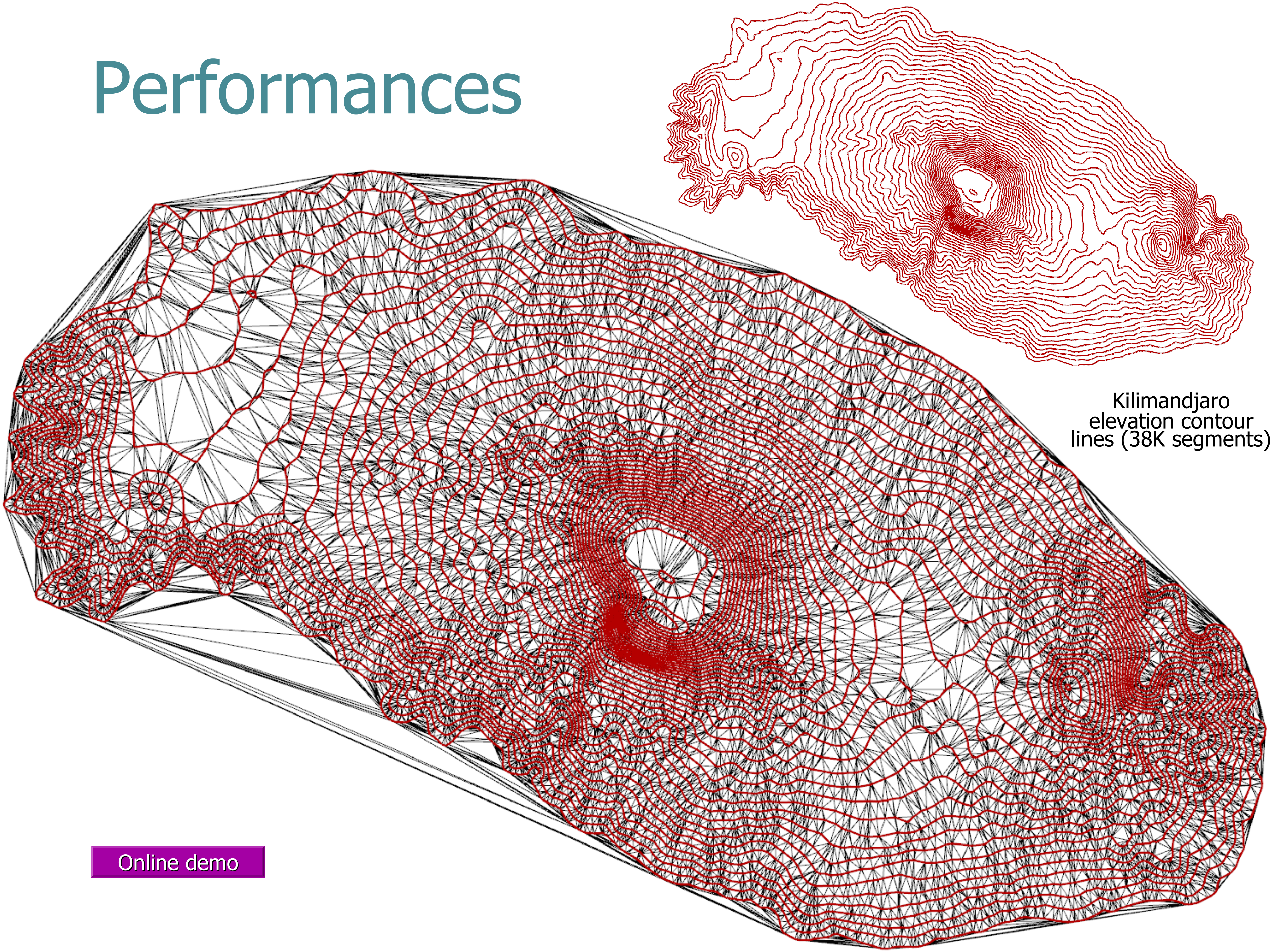
```
#include <CGAL/Exact_predicates_inexact_constructions_kernel.h>
#include <CGAL/Constrained_Delaunay_triangulation_2.h>
#include <CGAL/Delaunay_mesher_2.h>
#include <CGAL/Delaunay_mesh_size_criteria_2.h>
typedef CGAL::Exact_predicates_inexact_constructions_kernel Kernel;
typedef CGAL::Constrained_Delaunay_triangulation_2<Kernel> CDT;
typedef CGAL::Delaunay_mesh_size_criteria_2<CDT> Criteria;
typedef CGAL::Delaunay_mesher_2<CDT, Criteria> Meshing_engine;
int main()
{
  CDT cdt;
  Meshing_engine engine(cdt);
  engine.refine_mesh();
  engine.set_criteria(Criteria(0.125, 0.5)); // min 20.6 deg
                                             // 0.5 for sizing
  engine.refine_mesh(); // refine once more, etc.
  return 0;
}
```

Parameters for Mesh Generation

- Shape
  - Lower bound on triangle angles
- Size
  - No constraint
  - Uniform sizing
  - Sizing function
- **Seeds**
  - Exclude/include components



Performances

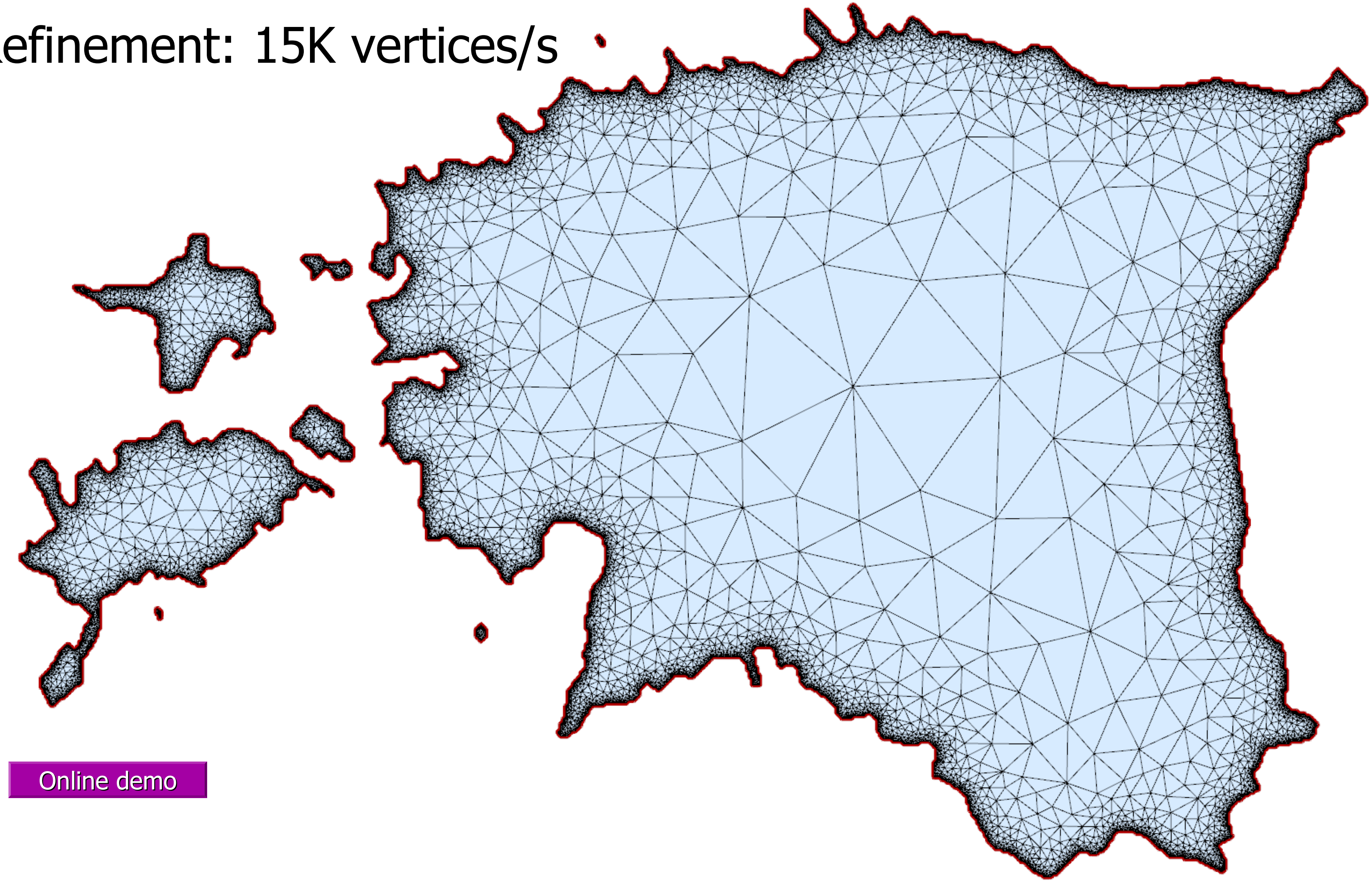


Kilimandjaro
elevation contour
lines (38K segments)

Online demo

Performances

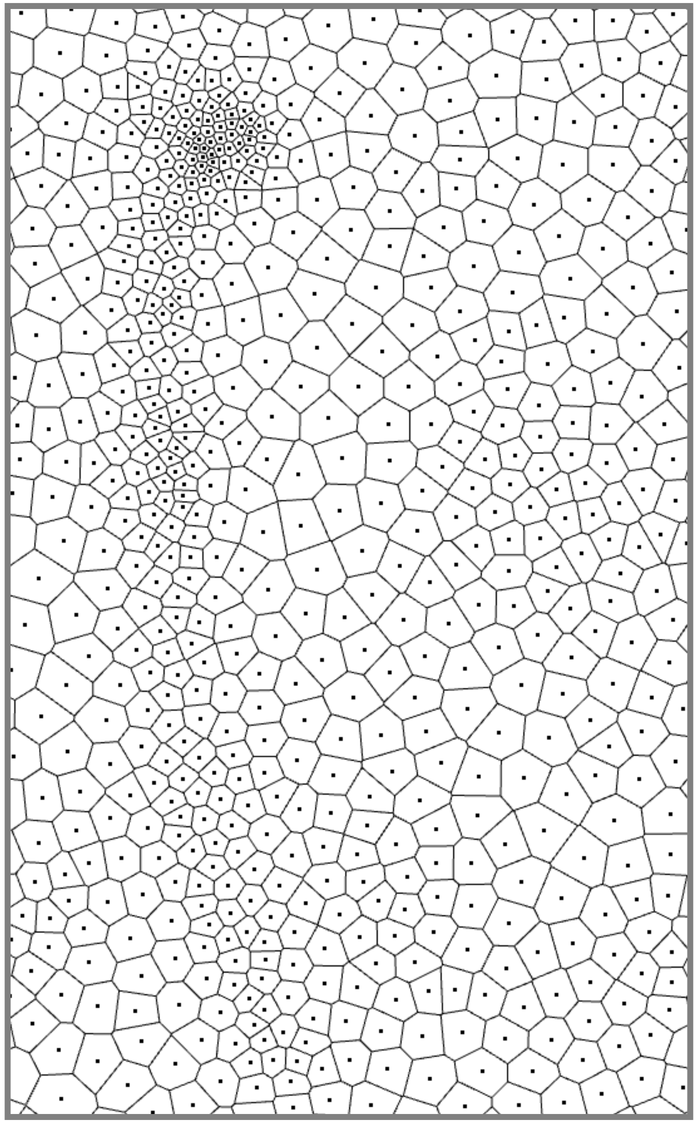
Refinement: 15K vertices/s



Online demo

Related Components

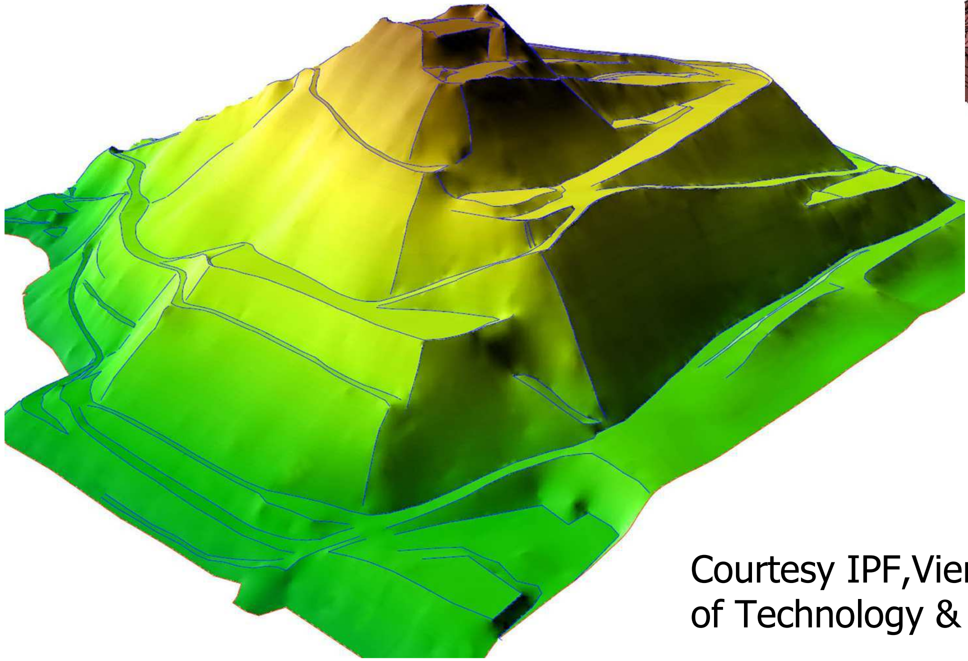
- Voronoi diagram

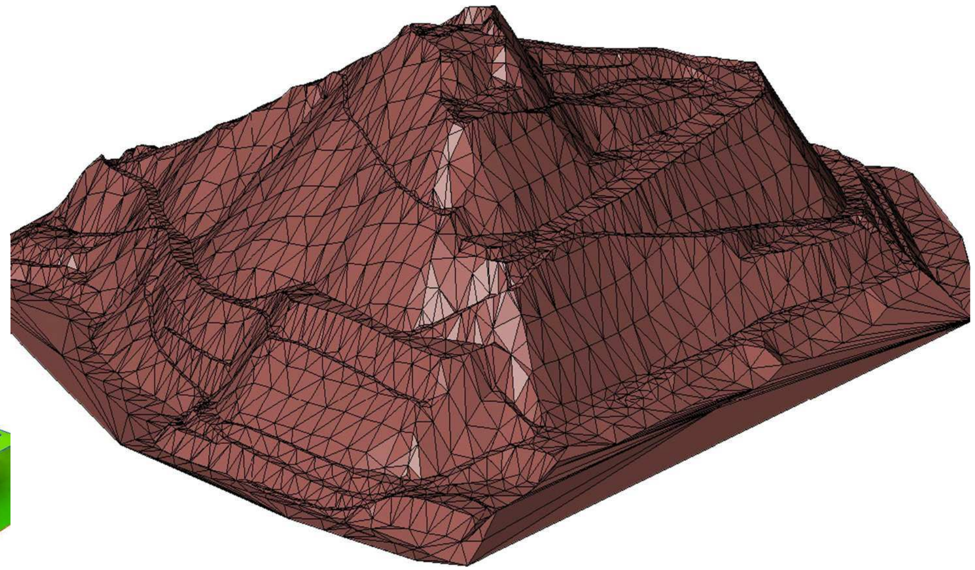


Online manual

Related Components

- Voronoi diagram
- Elevation (through traits class)

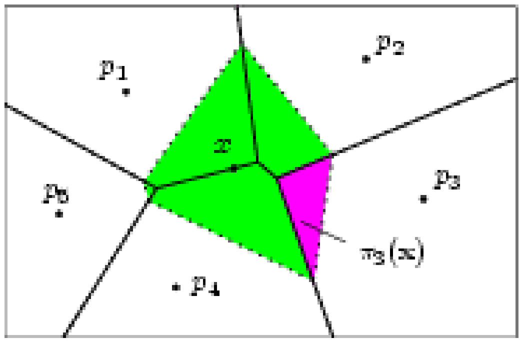




Courtesy IPF,Vienna University
of Technology & Inpho GmbH

Related Components

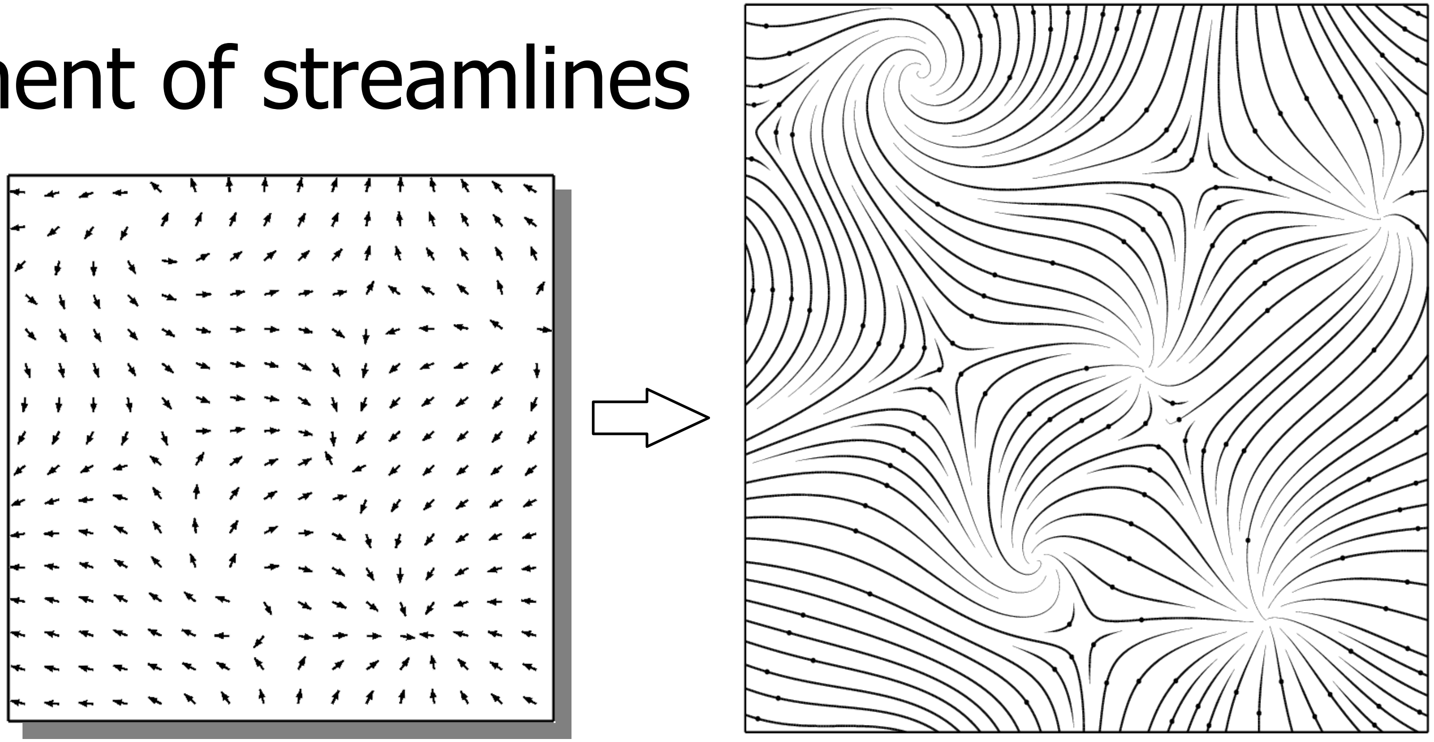
- Voronoi diagram
- Elevation
- Interpolation (natural neighbors)



Online manual

Related Components

- Voronoi diagram
- Elevation
- Interpolation
- Placement of streamlines

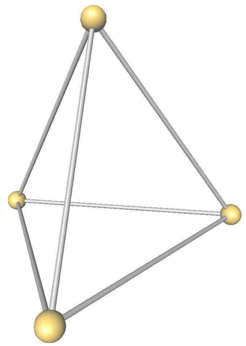


Online manual

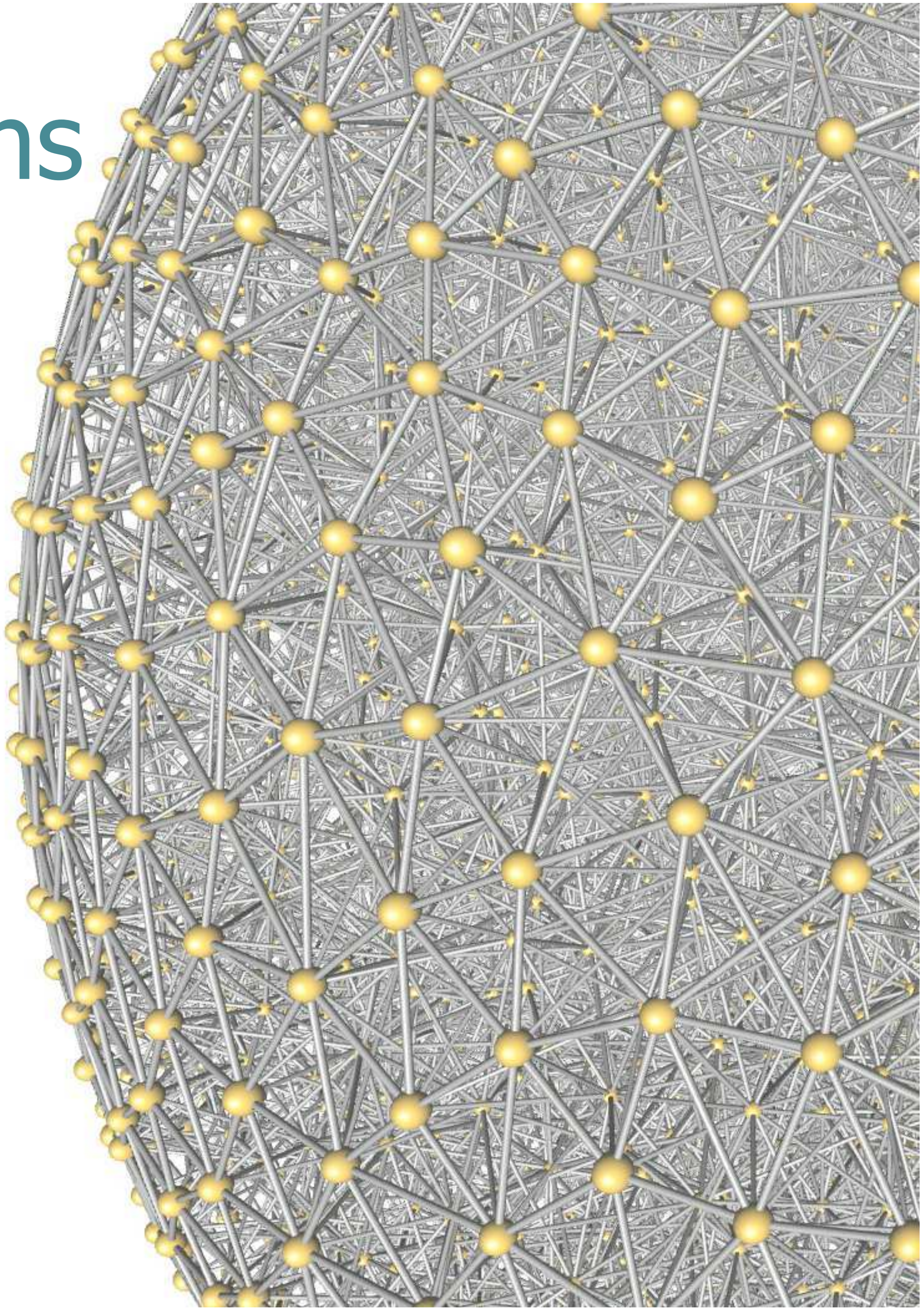
3D

3D Triangulations

- Delaunay
- Regular
- Rich API
- Fully dynamic
- 1M points in 16s



Tetrahedron

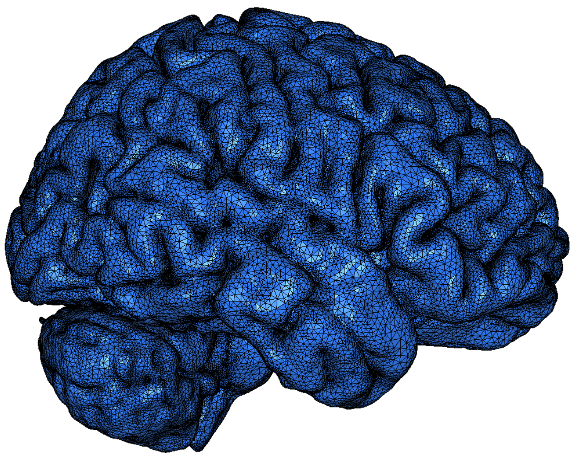
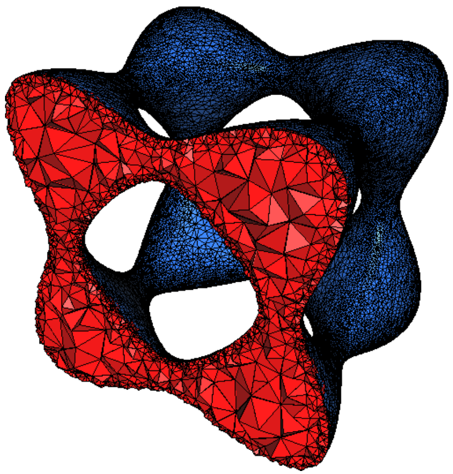
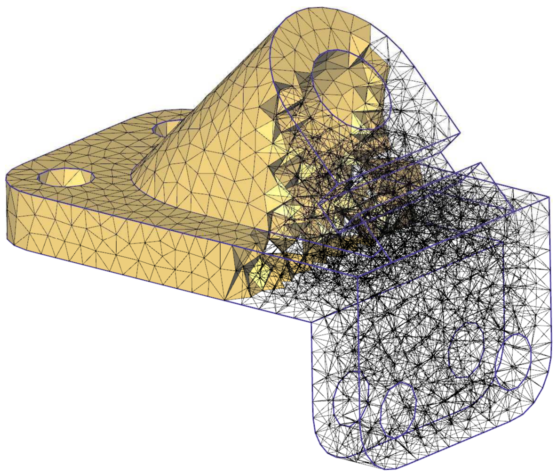


Online manual

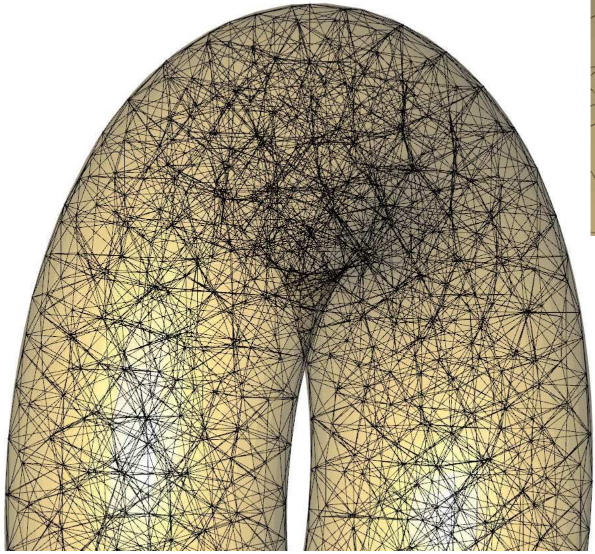
Mesh Generation

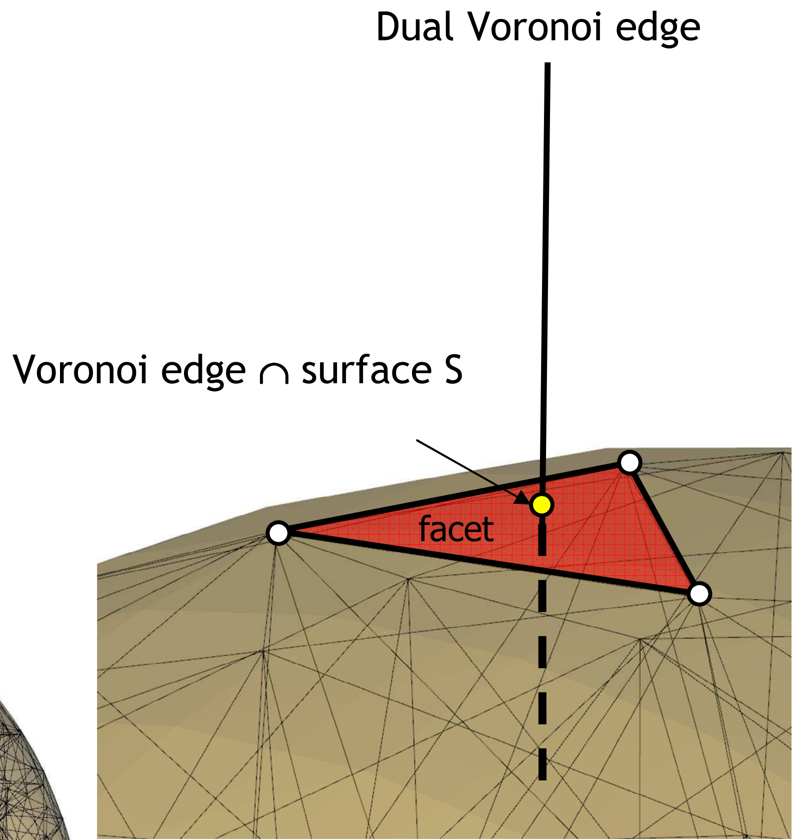
Key concepts:

- Delaunay filtering
- Delaunay refinement

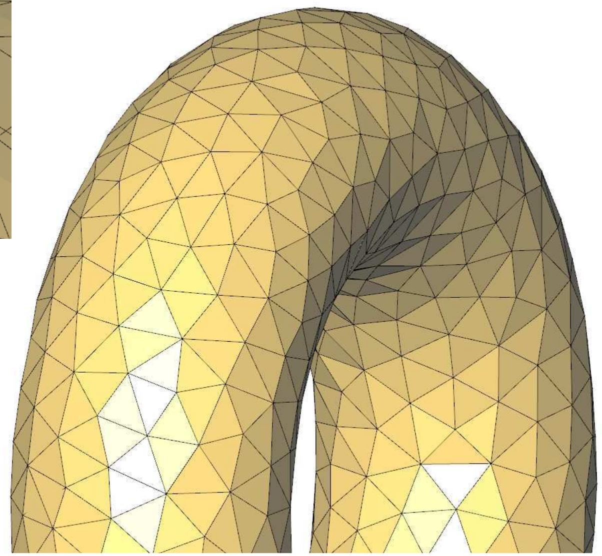




Delaunay Filtering





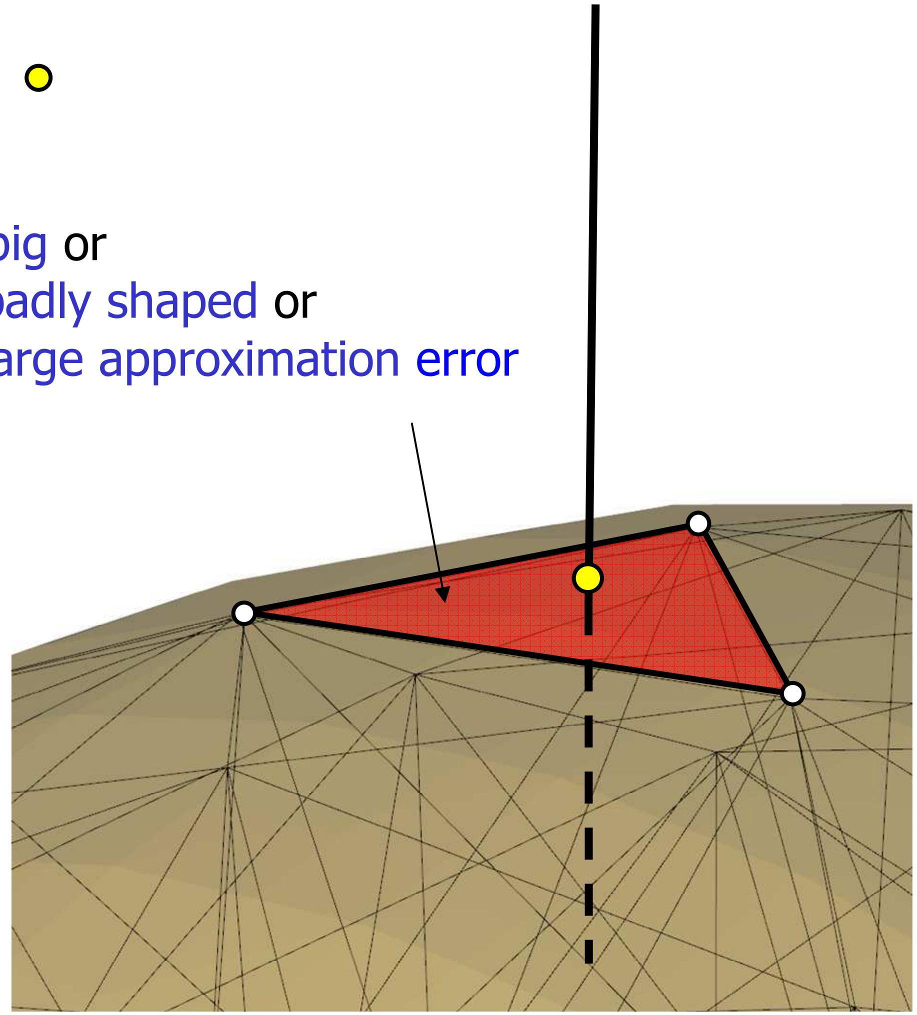
Delaunay triangulation restricted to surface S



Delaunay Refinement

Steiner point

Bad facet = big or
badly shaped or
large approximation error



Surface Mesh Generation Algorithm

repeat

{

pick bad facet **f**

insert furthest (dual(**f**) $\cap$ **S**) in Delaunay triangulation

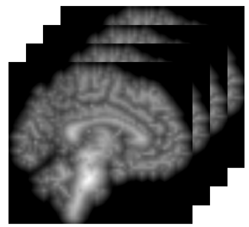
update Delaunay triangulation restricted to **S**

}

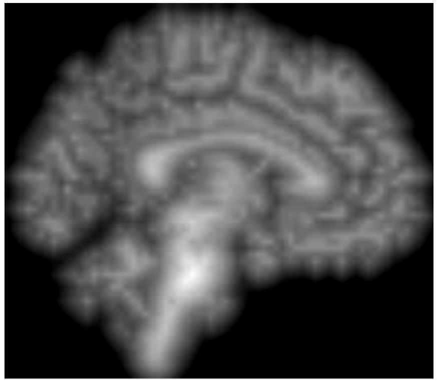
until all facets are good

Isosurface from 3D Grey Level Image

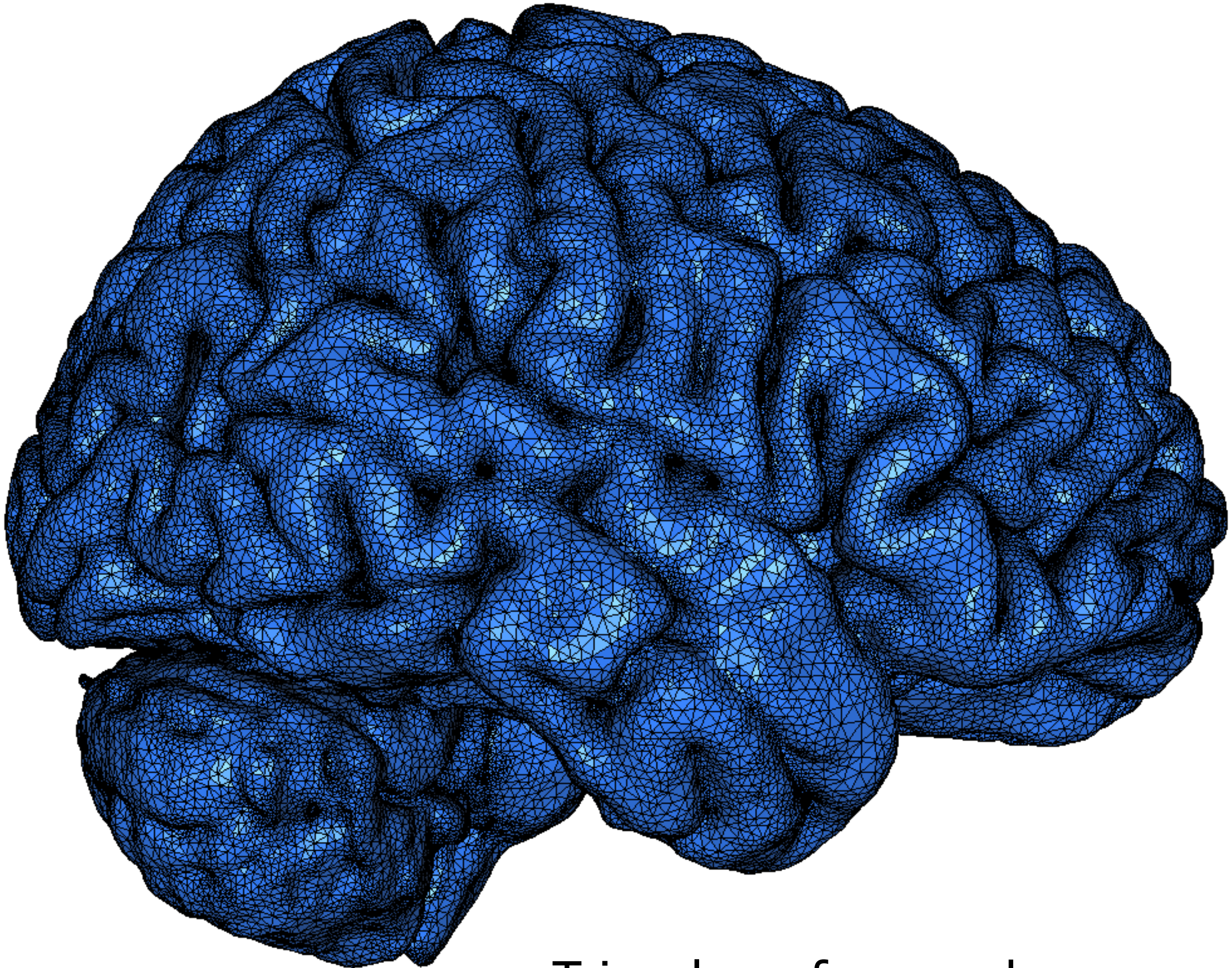
```
#include <CGAL/Surface_mesh_default_triangulation_3.h>
#include <CGAL/Complex_2_in_triangulation_3.h>
#include <CGAL/make_surface_mesh.h>
#include <CGAL/Gray_level_image_3.h>
#include <CGAL/Implicit_surface_3.h>
typedef CGAL::Surface_mesh_default_triangulation_3 Tr;
typedef CGAL::Complex_2_in_triangulation_3<Tr> C2t3;
typedef CGAL::Implicit_surface_3<Kernel, Gray_level_image> Surface_3;

Tr tr;                  // 3D-Delaunay triangulation
C2t3 c2t3 (tr);         // 2D-complex in 3D-Delaunay triangulation
Gray_level_image image("data/brain",128);
Surface_3 surface(image, bounding_sphere, 1e-2);

// Criteria: min triangle angles, size, approximation error,
CGAL::Surface_mesh_default_criteria_3<Tr> criteria(30.,5.,5.);
CGAL::make_surface_mesh(c2t3, surface, criteria, CGAL::Manifold_tag());
```



Output Mesh



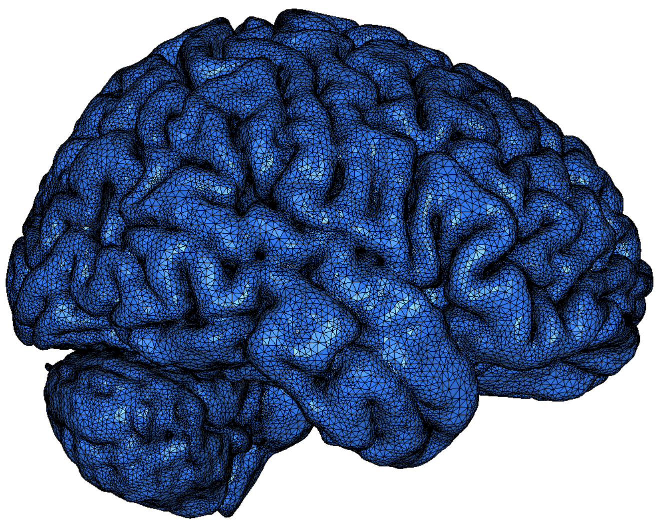
input



Triangle surface mesh
approximating S

Output Mesh Properties

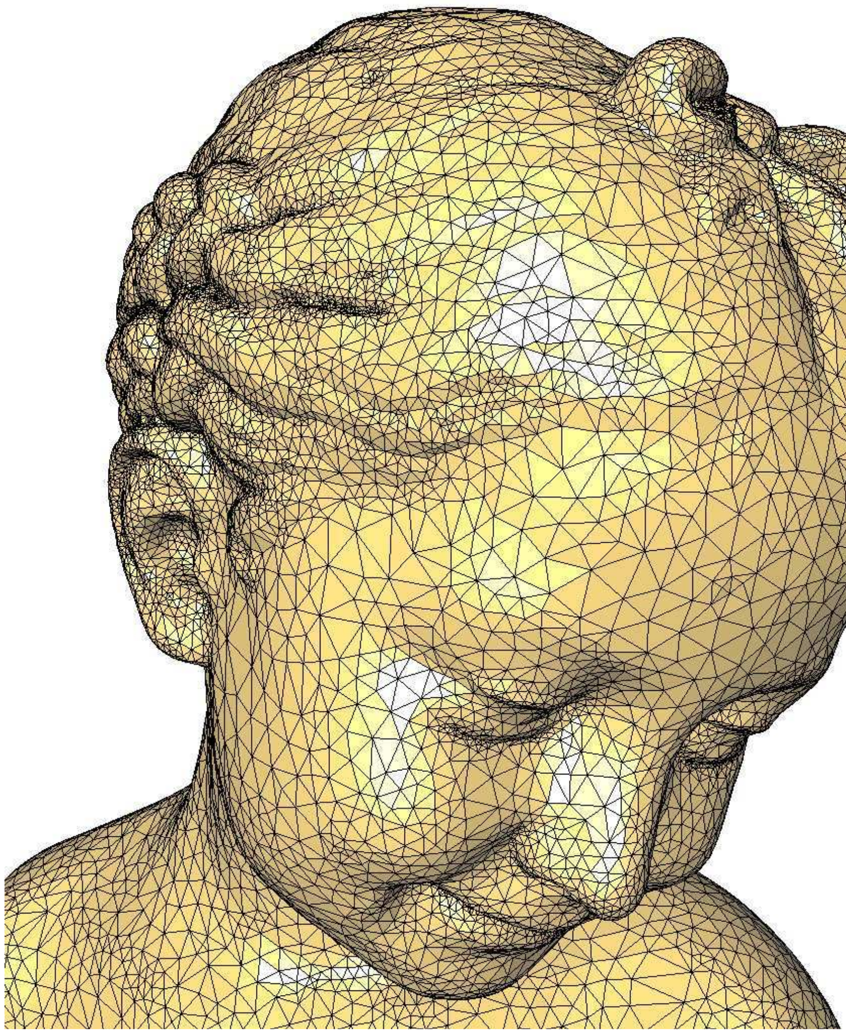
- Well shaped triangles
  - Lower bound on triangle angles
- Homeomorphic to input surface
- Manifold
  - not only combinatorially, i.e., no self-intersection
- Faithful Approximation of input surface
  - Hausdorff distance
  - Normals



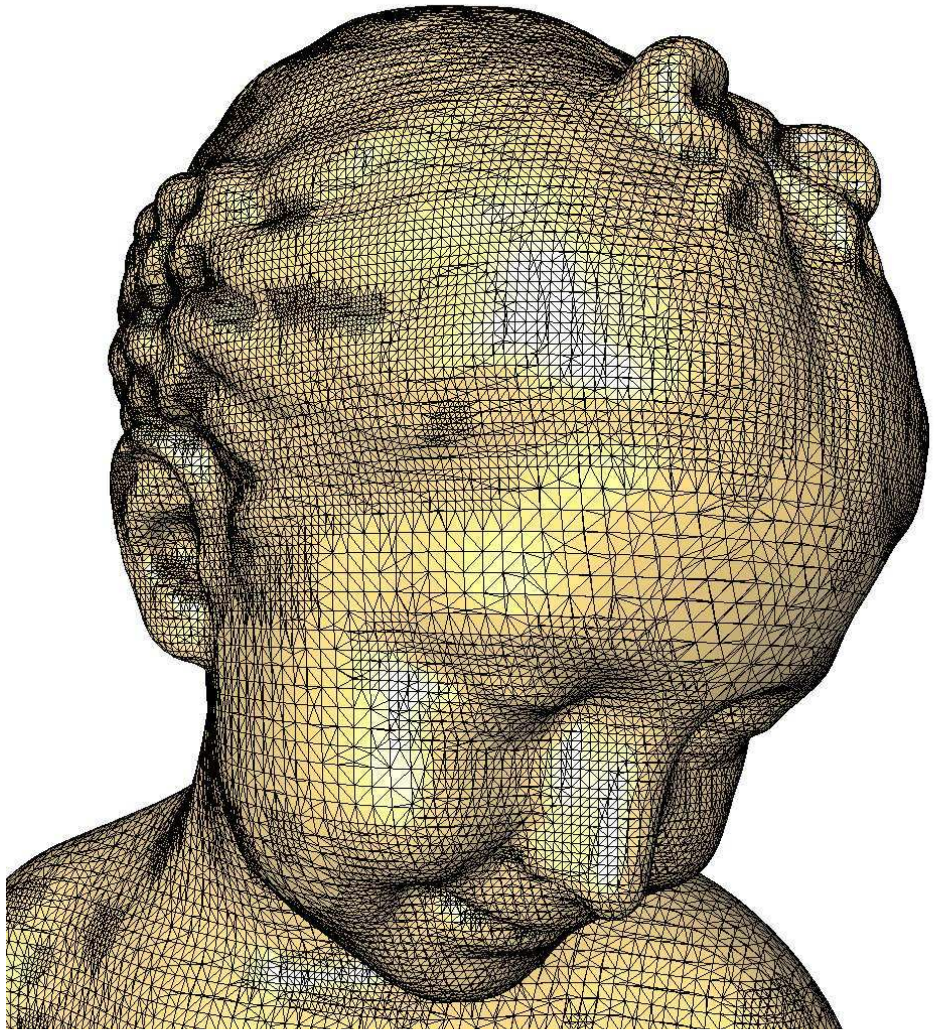
Online manual

Delaunay Refinement vs Marching Cubes

Delaunay refinement



Marching cubes in octree



Surface Remeshing

Input is a polyhedral surface

(requires efficient data structures for intersection computations)

Parameters

- Shape of triangles
  - lower bound on triangle angles
- Size

Parameters

- Shape of triangles
  - lower bound on triangle angles
- Size
  - No constraint

Parameters

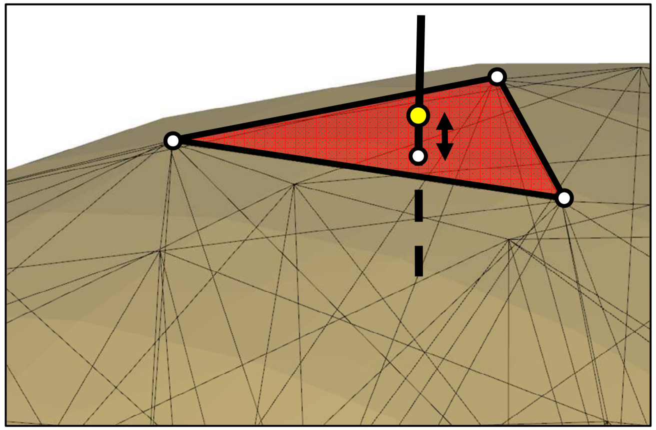
- Shape of triangles
  - lower bound on triangle angles
- Size
  - No constraint
  - Uniform sizing

Parameters

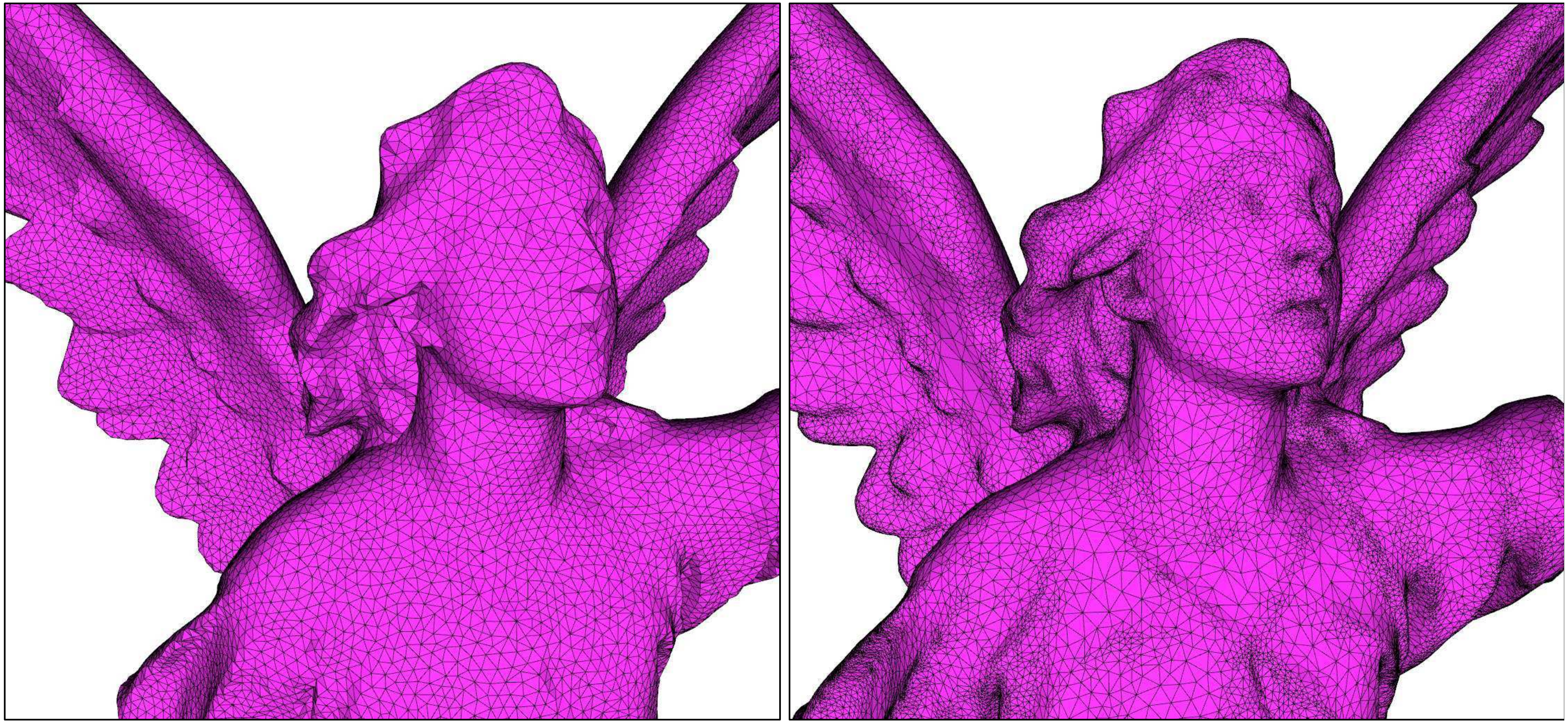
- Shape of triangles
  - lower bound on triangle angles
- Size
  - No constraint
  - Uniform sizing
  - Sizing function

Parameters

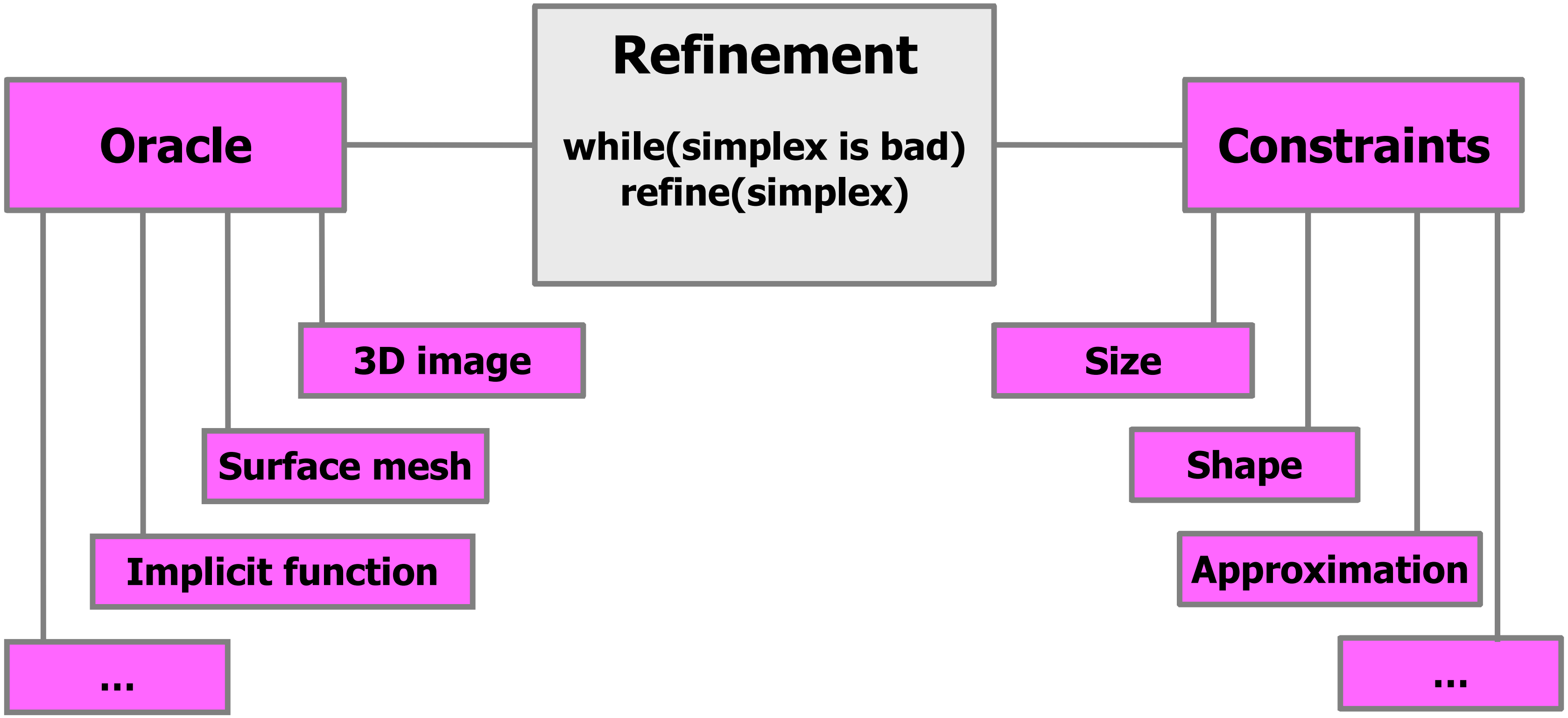
- Shape of triangles
  - lower bound on triangle angles
- Size
  - No constraint
  - Uniform sizing
  - Sizing function
- Approximation error



Uniform vs Adapted



Mesh Generation Framework

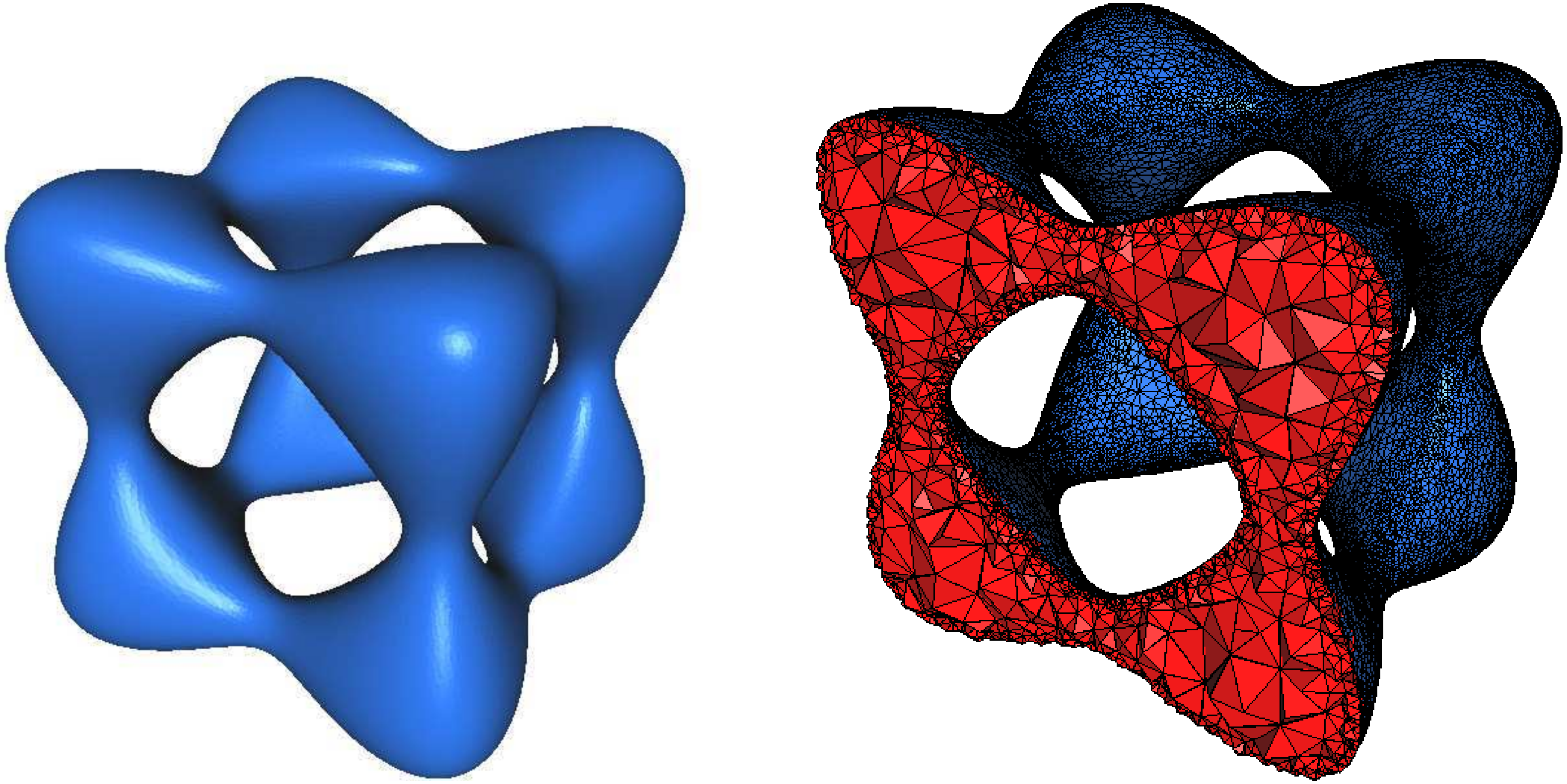


A Versatile Framework

- 3D grey level images
- 3D multi-domain images
- Implicit function: $f(x, y, z) = \text{constant}$
- Surface mesh (remeshing)
- Point set (surface reconstruction)
- Anything which provides intersections

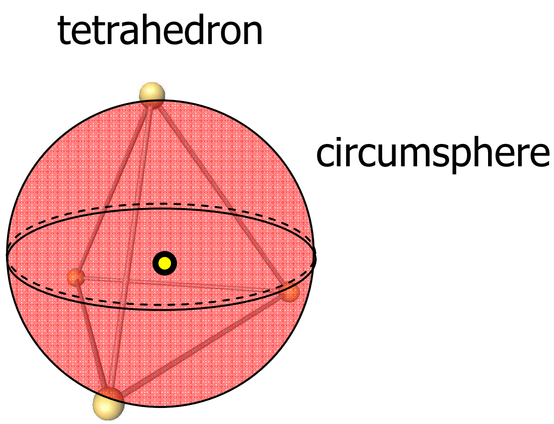
Next Release

Volume Mesh Generation

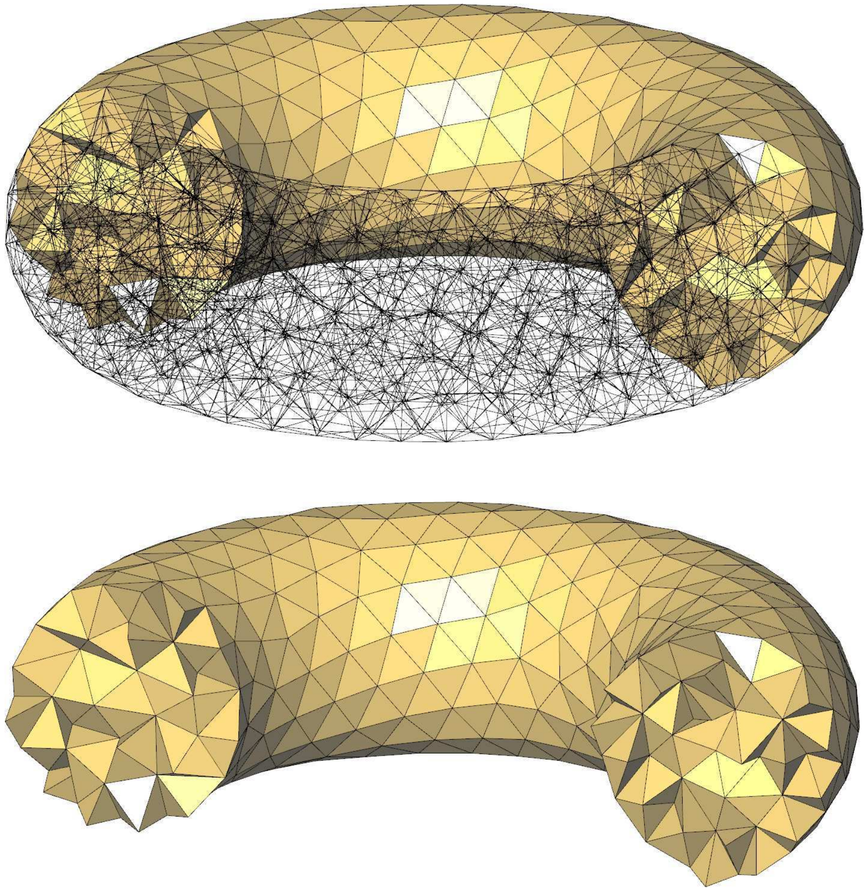


More Delaunay Filtering

Delaunay triangulation restricted to domain Ω



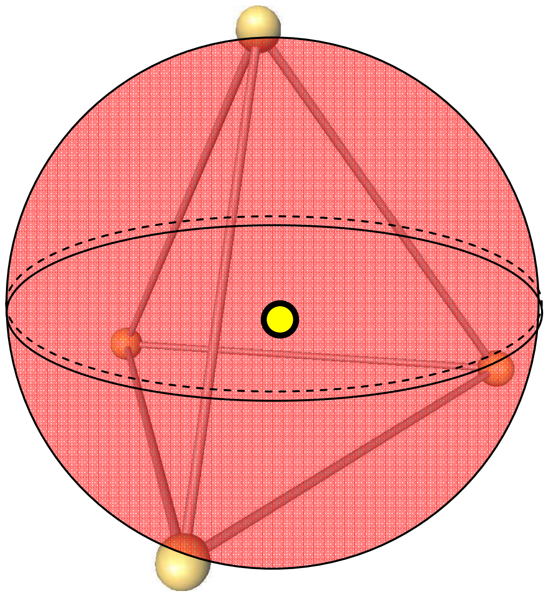
Dual Voronoi vertex
inside domain Ω
(“oracle”)



Delaunay Refinement

Steiner point

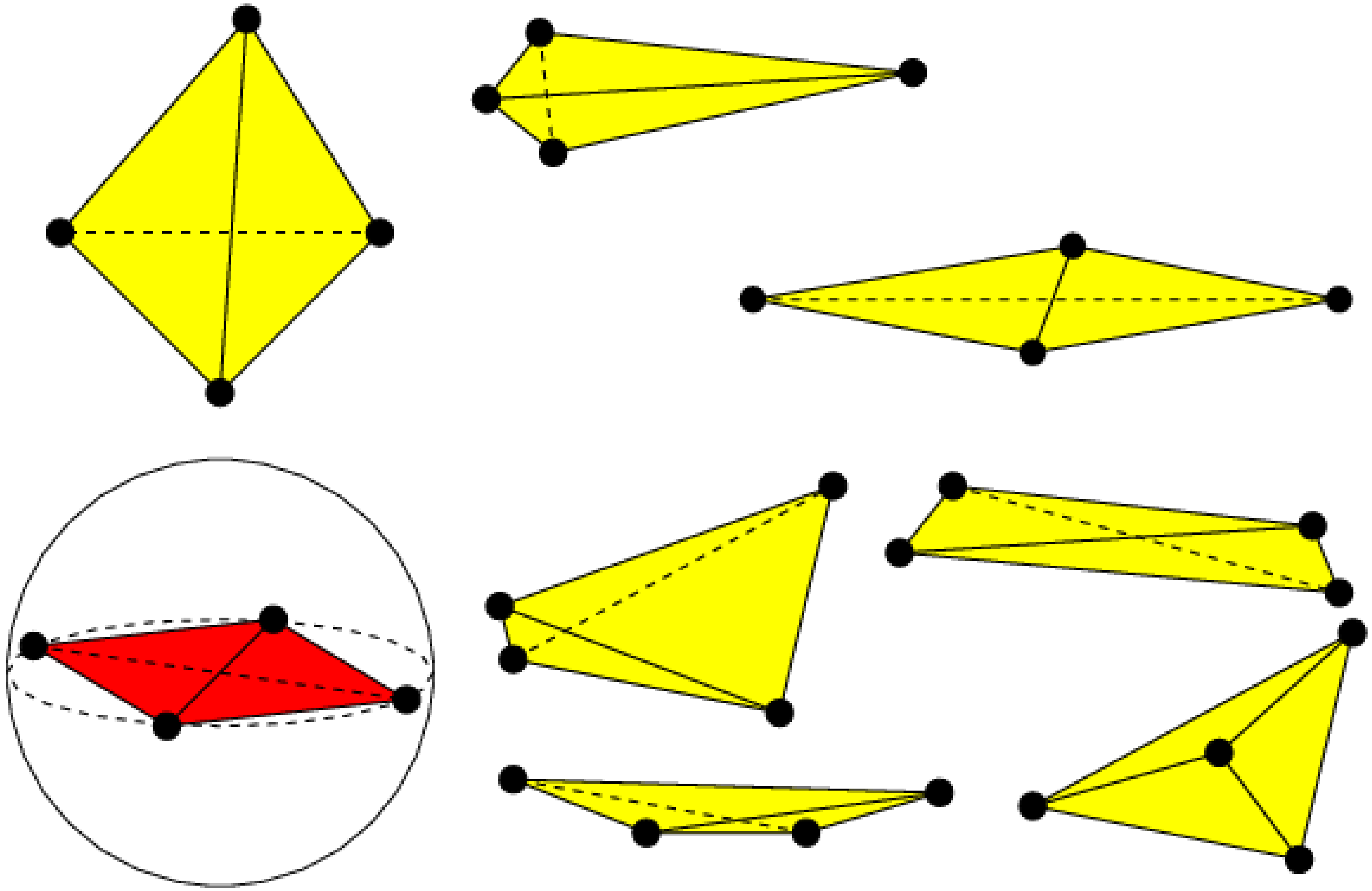
Bad tetrahedron = big or badly shaped



Volume Mesh Generation Algorithm

```
repeat
{
    pick bad simplex
    if(Steiner point encroaches a facet)
      refine facet
    else
      refine simplex
    update Delaunay triangulation restricted to domain
}
until all simplices are good
Exude slivers
```

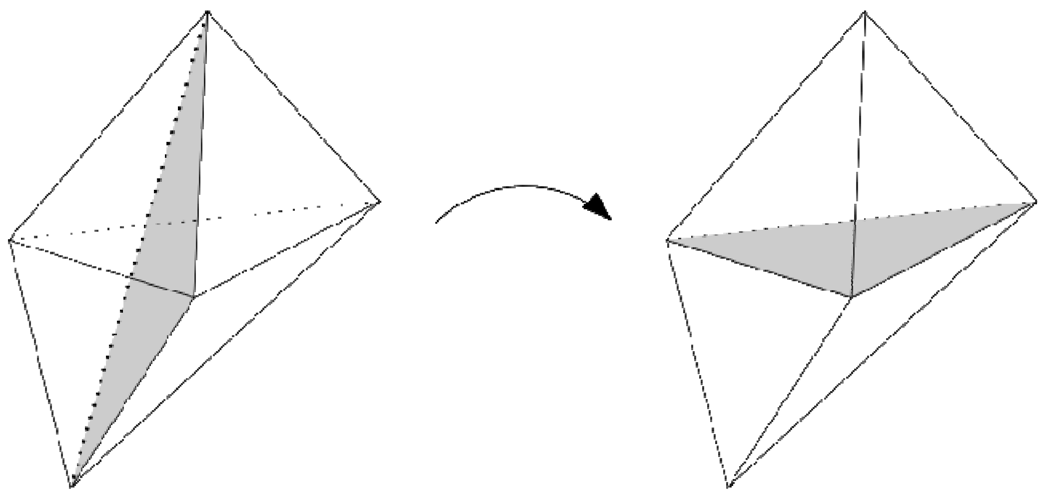
Tetrahedron Zoo



sliver

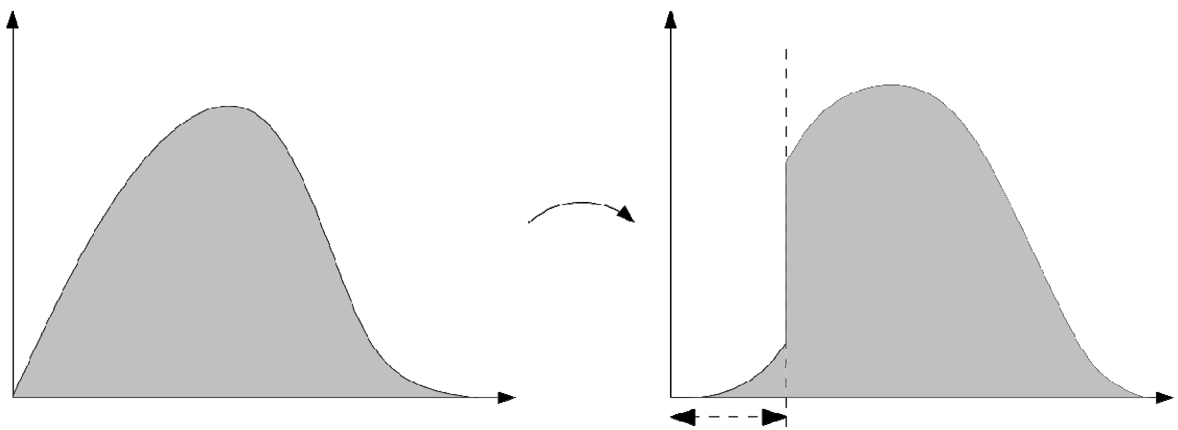
Sliver Exudation [Edelsbrunner-Guoy]

- Delaunay triangulation turned into a regular triangulation with null weights.
- Small increase of weights triggers edge-facets flips to remove slivers.



Sliver Exudation Process

- Try improving all tetrahedra with an aspect ratio lower than a given bound
- Never flips a boundary facet

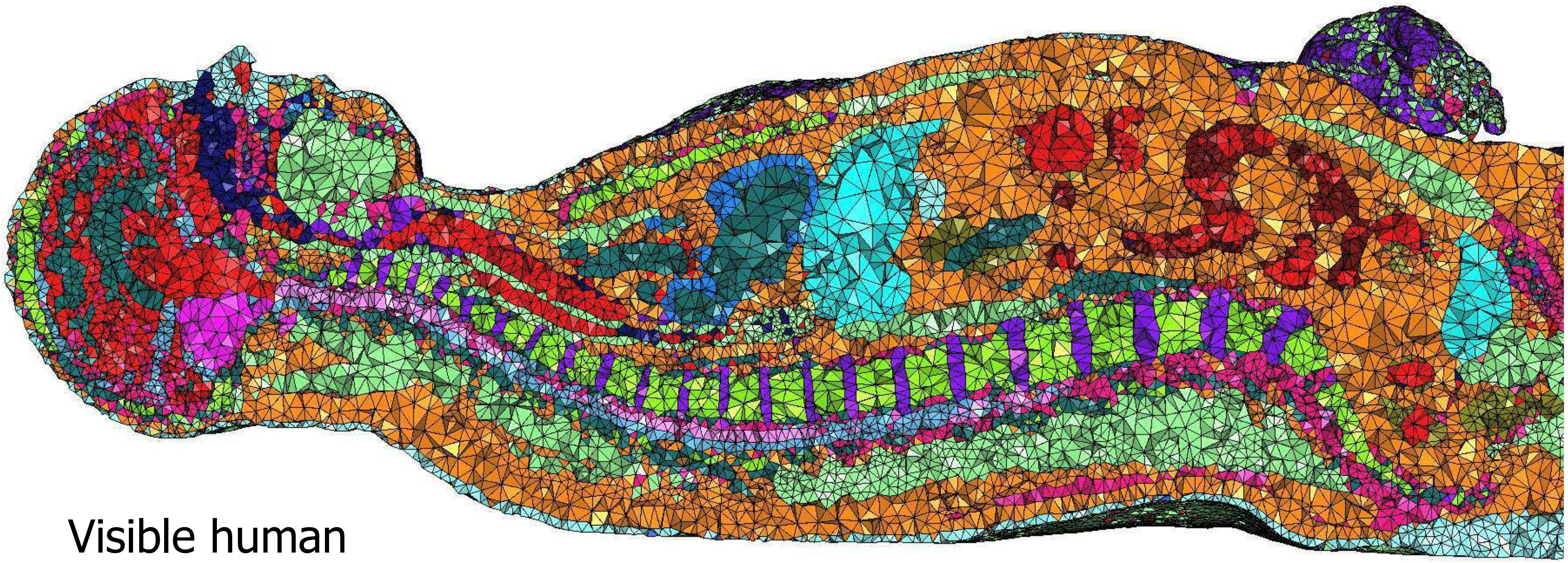


distribution of aspect ratios

3D Mesh from Multi-Domain Images

```
Tr tr;  // 3D Delaunay triangulation
C2t3 c2t3(tr); // 2D complex in 3D-Delaunay triangulation

Image_3 image(“segmented_visible_human");
Mesh_traits mesh_traits(image);
Facets_criteria facets_criteria(5, 1); // facet sizing
                                       // approximation error
Tets_criteria tets_criteria(5);        // tet sizing

// 0.5 = radius-radius ratio upper bound for sliver exudation
CGAL::make_mesh_3_for_multivolumes(c2t3, mesh_traits,
                                   facets_criteria,
                                   tets_criteria, 0.5);
```

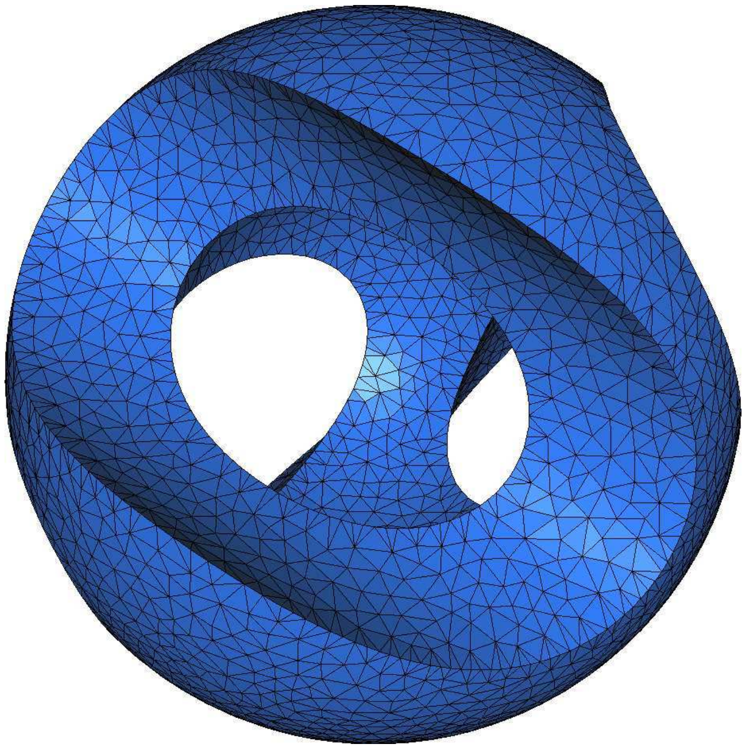
Output Volume Mesh

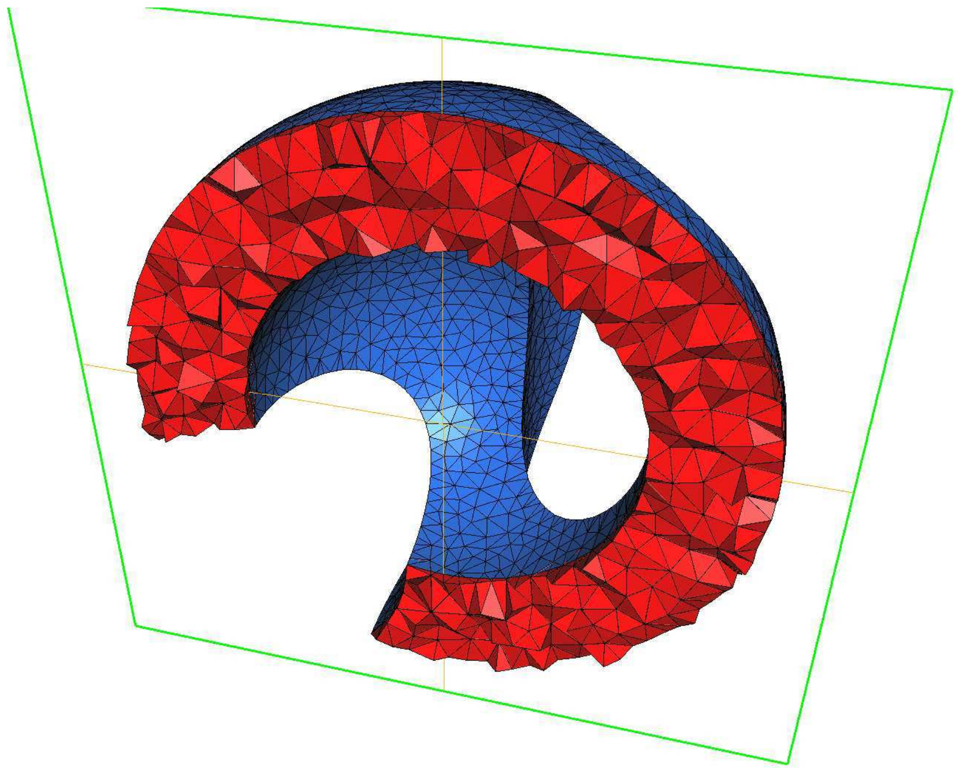


Visible human

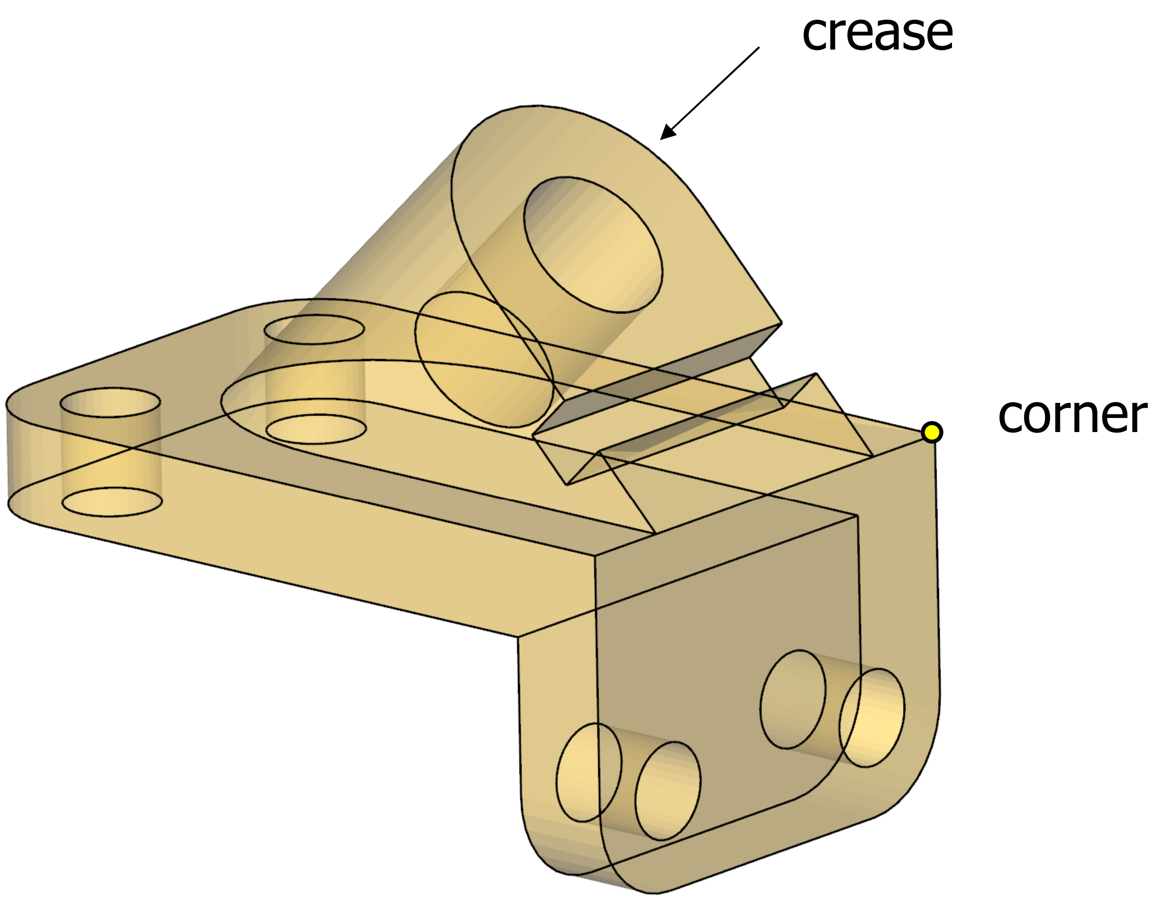
Work in Progress

Piecewise Smooth Surfaces



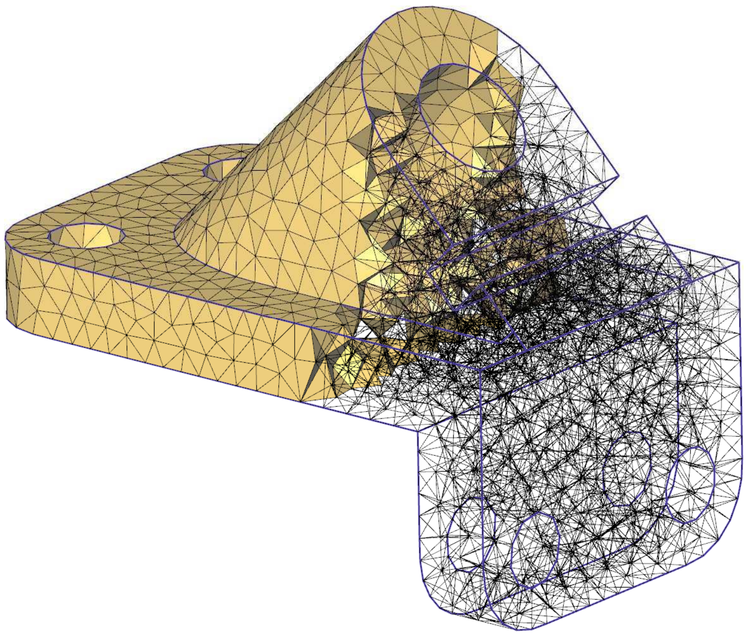


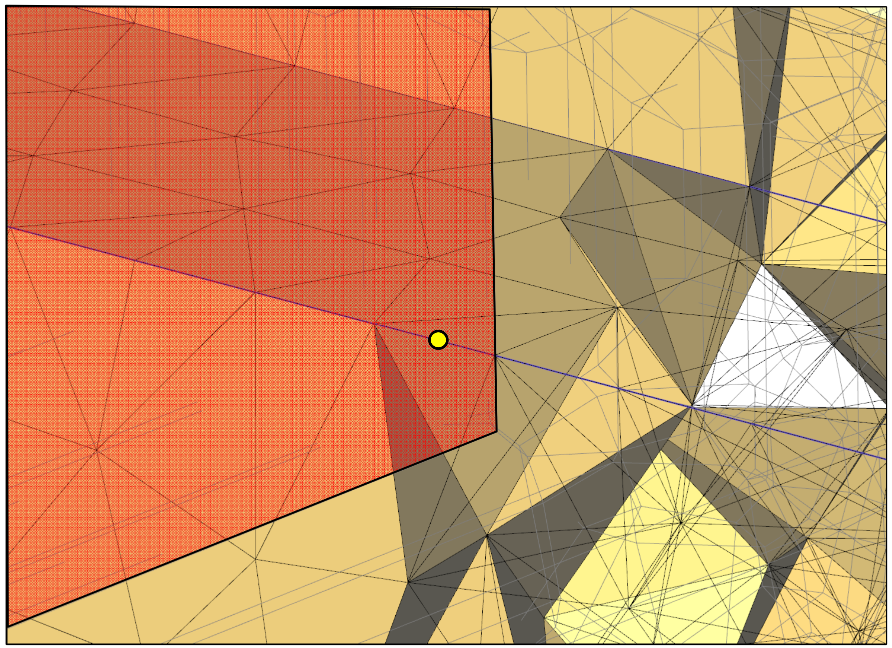
Input: Piecewise smooth complex



More Delaunay Filtering

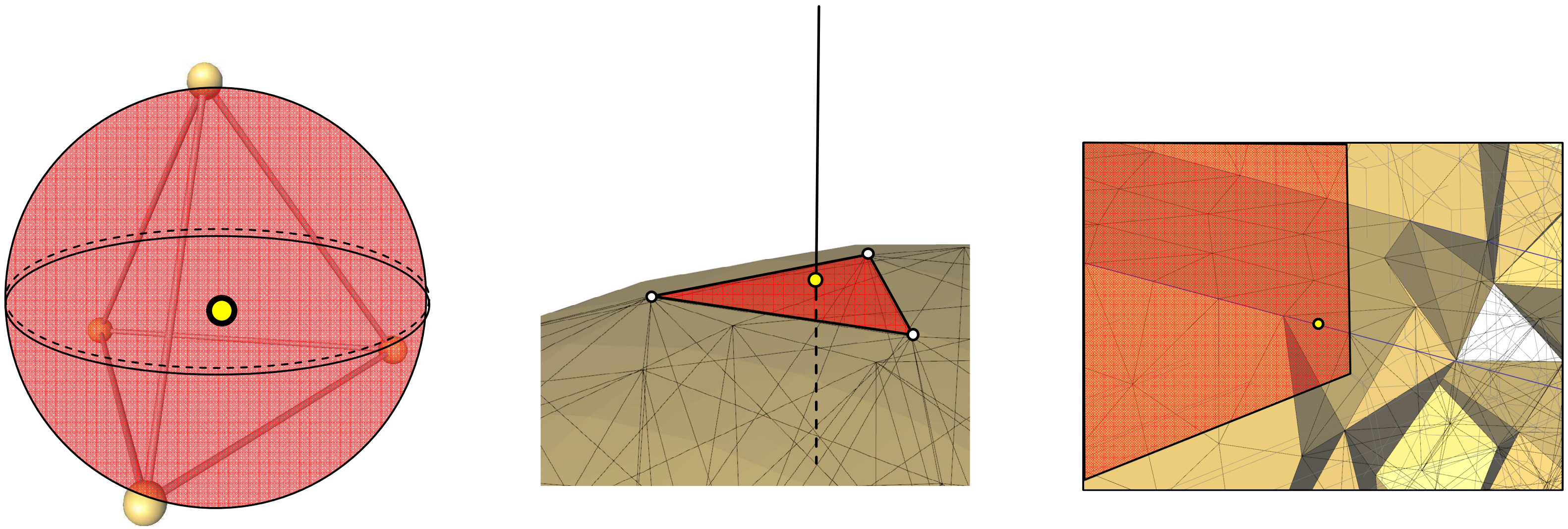
primitive	dual of	test	against
Voronoi vertex	tetrahedron	inside	domain
Voronoi edge	facet	intersect	domain boundary
Voronoi face	edge	intersect	crease





Delaunay Refinement

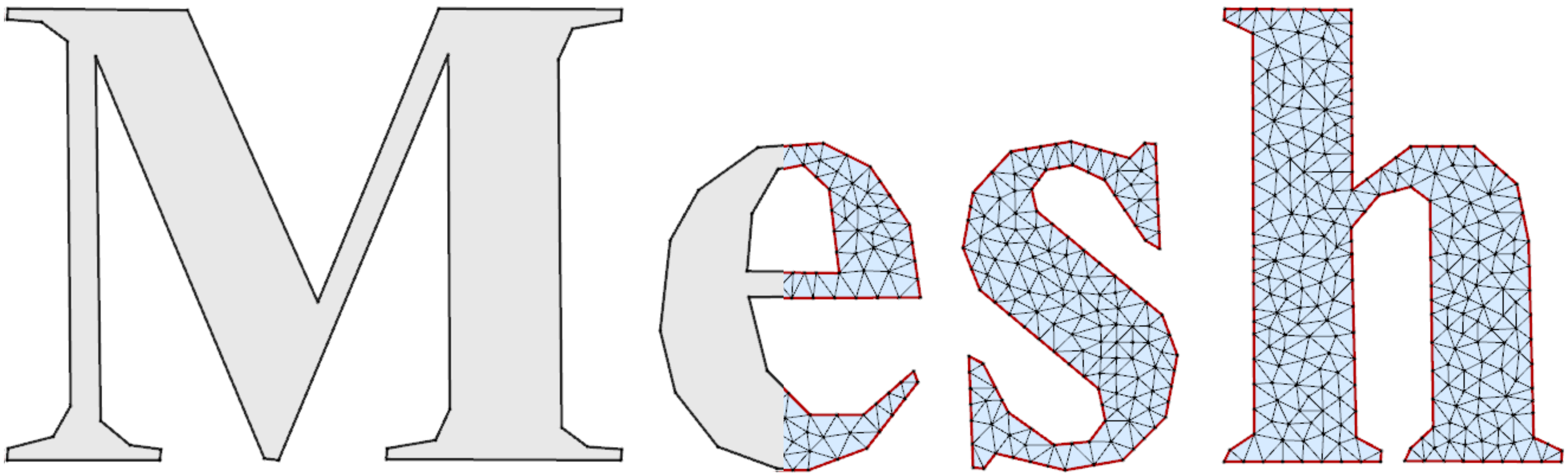
- Steiner points



Summary

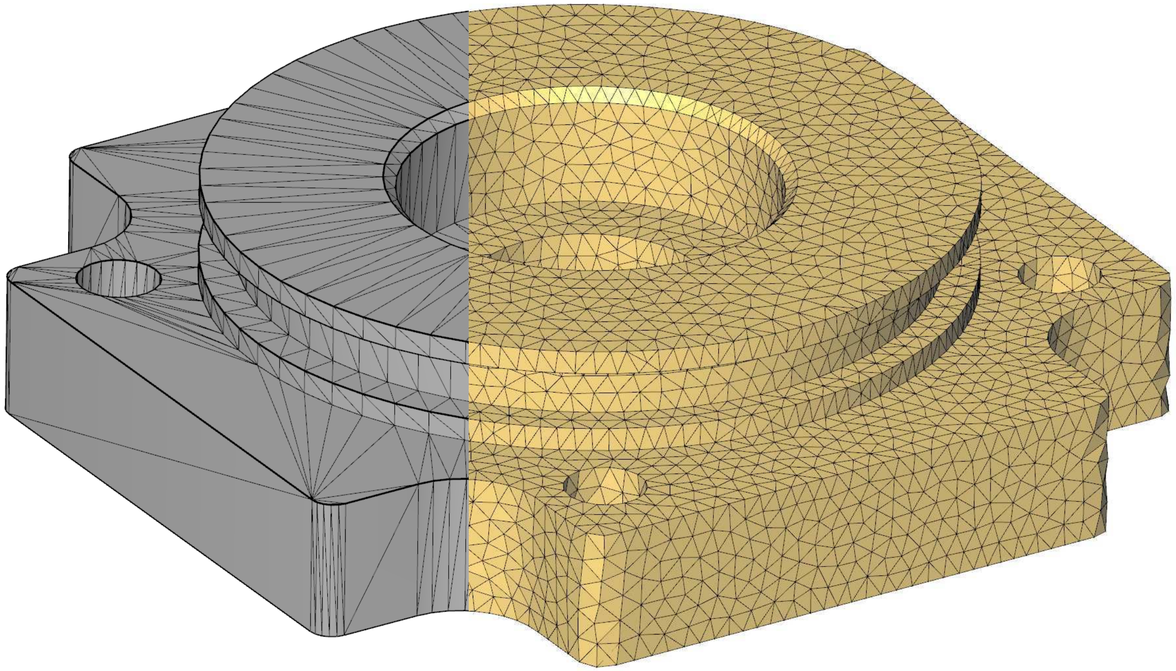
Summary

- From triangulation to quality meshes
- Mesh generation:
  - 2D: Preserves constraints exactly.



Summary

- From triangulation to quality meshes
- Mesh generation:
  - 2D: Preserves constraints exactly.
  - 3D: Interpolates boundary and sharp creases.

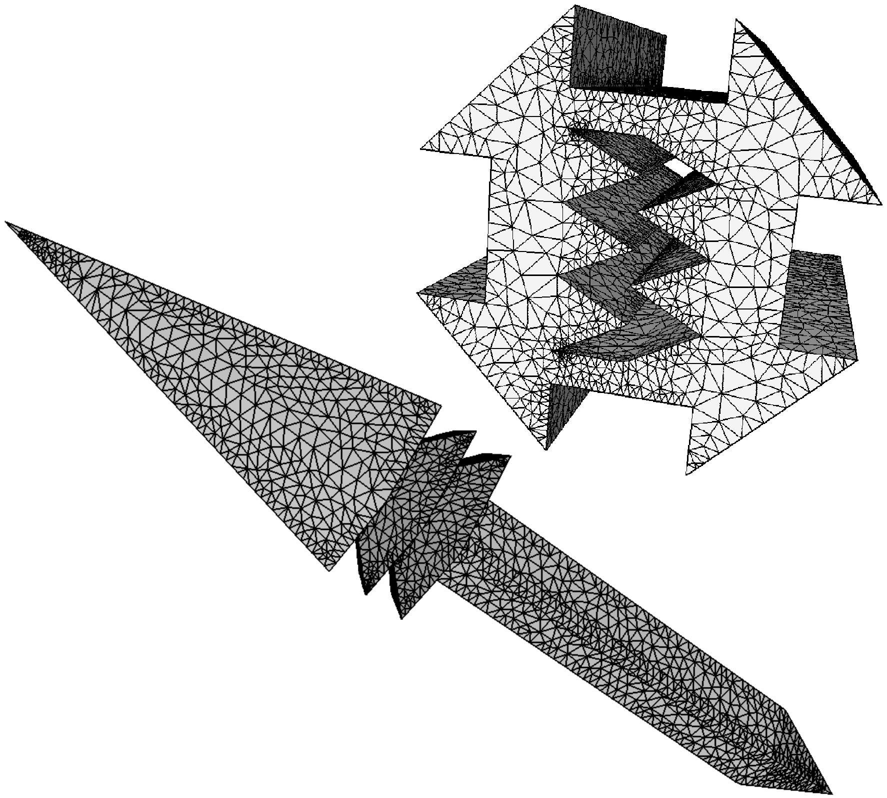


Summary

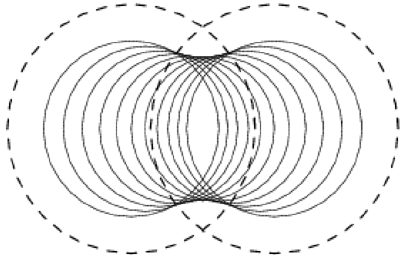
- From triangulation to quality meshes
- Mesh generation:
  - 2D: Preserves constraints exactly.
  - 3D:
    - Interpolates boundary and sharp creases.
    - Versatile through oracle-based design

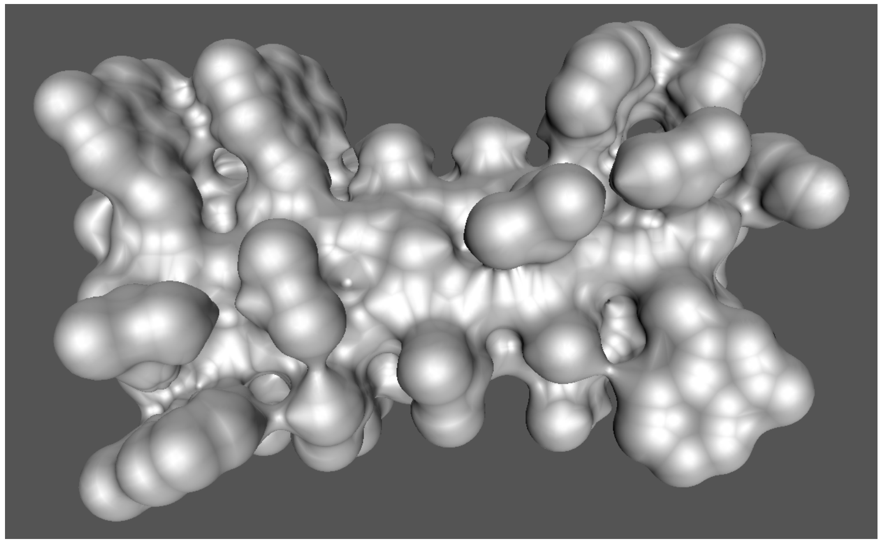
See Also

DelPSC software
(based on CGAL)
[Dey-Levine]



Skin surfaces





Online manual