Amenability and exactness for groups, group actions and operator algebras

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Abstract. These expository notes aim to introduce some weakenings of the notion of amenability for groups, and to develop their connections with the theory of operator algebras as well as with recent remarkable applications.

Introduction

The class of amenable groups was isolated in 1929 by J. von Neumann [66] in the course of study of the Banach-Tarski paradox. This notion turned out to be very important for many areas of mathematics and is still a very active subject of research. Let us mention for instance the many new examples of amenable groups arising from automata groups.

Much later, motivated by the study of Poisson boundaries of random walks, R. Zimmer introduced in the late 1970s [71] the notion of measured amenable group action. Very soon after, J. Renault [58] began the study of this notion in the settings of measured and topological groupoids. This was further developed in [3] and finally a comprehensive exposition was published in [5].

One of the striking applications of this notion of amenable action come from the C^*-algebraic approach to the Novikov Conjecture. We shall not describe here this geometric conjecture, but refer to [10, 70] for introductions to this subject. M. Gromov had suggested in 1993 [31] that finitely generated groups which are uniformly embeddable into Hilbert spaces[1] should satisfy the Novikov Conjecture. This was proved at the end of the 1990s by G. Yu [69]. In the same paper, Yu introduced a Følner type condition on finitely generated groups, weaker than amenability, he called property (A), which guaranties the existence of such uniform embeddings. N. Higson and J. Roe discovered in [39] that Yu's property (A) for a finitely generated group Γ is equivalent to the fact that Γ has an amenable action on a *compact* Hausdorff space[2]. The hope that every finitely generated group has this property was soon negated by Gromov [32, 33].

It turns out that the class of finitely generated groups with property (A), had been already investigated, in another guise, for reasons pertaining to amenability properties of operator algebras. This subject dates back to the major breakthrough, due to M. Takesaki in 1964 [61], giving the first example of two C^*-algebras whose tensor product could be completed in more than one way to give a C^*-algebra. The well-behaved C^*-algebras A, such that for any C^*-algebra B

[1] Unexplained terms will be defined in the main part of the text.

[2] For that reason, property (A) is also called boundary amenability.

there is only one possible C^*-algebra completion of the algebraic tensor product $A \odot B$, are called nuclear. They were first investigated by Lance [48] in 1973 and completely understood around the mid-1970s, mainly thanks to Choi, Effros, Kirchberg and Lance. In particular it was proved in [48] that the reduced C^*-algebra of a discrete group is nuclear if and only if the group is amenable.

In 1977, E. Kirchberg introduced the notion of exact C^*-algebra, related to nuclearity but strictly weaker : the reduced C^*-algebra of free groups with $n \geq 2$ generators are exact, whereas they are not nuclear. Kirchberg's important contributions to the subject began to be published only after 1990 [42, 43, 45]. For the reduced C^*-algebra of a discrete group Γ, the remarkable link between exactness and amenability was discovered around 2000 [34, 52, 4] : the reduced C^*-algebra of Γ is exact if and only if Γ has property (A).

After this brief and far from exhaustive history of the subject, let us describe now more precisely the contents of these lectures. We give in Section 1 some very short preliminaries on amenability and we introduce in a more detailed way some basic notions on C^*-algebras, intended to an audience not specialized in the theory of operator algebras. In Section 2 we discuss the relations between amenability and nuclearity and introduce also another important weak amenability property, the Haagerup approximation property. Section 3 is devoted to the study of exactness, boundary amenability and property (A), which are three equivalent properties of a finitely generated group. An important observation is that they can be expressed in a way which only involves the metric defined by any word length function. More generally, property (A) makes sense for any discrete metric space. In Section 4, we show that any exact metric space is uniformly embeddable into some Hilbert space. Whether the converse is true or not for metric spaces underlying finitely generated groups is an important open problem. We end these notes by a short study of the Hilbert space compression of groups, an invariant (currently the subject of active research) introduced by Guentner and Kaminker in order to tackle this problem.

In these notes we shall only only consider *countable* groups (for simplicity) and *Hausdorff* topological spaces. This will often be implicit in our statements.

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1 Preliminaries

1.1 Amenability for groups. Let Γ be a discrete group. We shall denote by $\ell^1(\Gamma)_1^+$ the space of probability measures on Γ and by $\ell^2(\Gamma)_1$ the unit sphere of $\ell^2(\Gamma)$. For $f : \Gamma \to \mathbb{C}$ and $t \in \Gamma$ the function tf is defined by $(tf)(s) = f(t^{-1}s)$.

Recall first that a complex valued function φ on Γ is said to be *positive definite*

(or of *positive type*) if for every $n \geq 1$ and every $t_1, \ldots, t_n$ in Γ, the matrix $[\varphi(t_i^{-1}t_j)]$ is positive, that is

$$\forall n \in \mathbb{N}, \forall t_1, \ldots, t_n \in \Gamma, \forall \lambda_1, \ldots, \lambda_n \in \mathbb{C}, \quad \sum_{i,j=1}^{n} \overline{\lambda_i}\lambda_j \varphi(t_i^{-1}t_j) \geq 0.$$

Given $(\sigma, \mathcal{H}, \xi)$ where ξ is a non-zero vector in the Hilbert space $\mathcal{H}$ of the unitary representation σ of Γ, the coefficient $t \mapsto \langle \xi, \sigma(t)\xi \rangle$ of σ is positive definite. The Gelfand-Naimark-Segal (GNS) construction asserts that every positive definite function is of this form.

Definition 1.1 (-Proposition). Γ is *amenable* if and only if one of the four following equivalent conditions holds :

(i) *There exists a net* (f_i) *in* $\ell^1(\Gamma)_1^+$ *such that for every* $t \in \Gamma$ *:*

$$\lim_i \|tf_i - f_i\|_1 = 0.$$

(ii) *There exists a net* (ξ_i) *in* $\ell^2(\Gamma)_1$ *such that for every* $t \in \Gamma$ *:*

$$\lim_i \|t\xi_i - \xi_i\|_2 = 0.$$

(iii) *There exists a net* (φ_i) *of positive definite functions on* Γ*, with finite support, such that for every* $t \in \Gamma$ *:*

$$\lim_i |1 - \varphi_i(t)| = 0.$$

(iv) *There exists an invariant state* M *on* $\ell^\infty(\Gamma)$*, that is such that* $M(tf) = M(f)$ *for* $t \in \Gamma$ *and* $f \in \ell^\infty(\Gamma)$*.*

Note that condition (ii) is Reiter's property saying that the (left) *regular representation* λ *of* Γ *almost has invariant vectors* or, in other terms, that *the trivial representation of* Γ *is weakly contained in its regular one.*
Reference : [30]

1.2 Amenability for actions. Let X be a locally compact space and assume that Γ acts on X from the left by homeomorphisms. We denote this action by $(t, x) \mapsto tx$ and we shall use the symbol $\Gamma \curvearrowright X$ in this situation. The amenability of the action is defined by the equivalent conditions below, which are the versions "with parameter" of the conditions (i) to (iii) in 1.1.

The space $\ell^1(\Gamma)_1^+$ of probability measures on Γ is equipped with the topology of pointwise convergence, which is the same here that the norm topology.

Definition 1.2 (-Proposition). The action $\Gamma \curvearrowright X$ is *amenable* if and only if one of the three following equivalent conditions holds :

(i) *There exists a net* (f_i) *of continuous functions* $f_i : x \mapsto f_i^x$ *from* X *into* $\ell^1(\Gamma)_1^+$, *such that for every* $t \in \Gamma$ *:*

$$\lim_i \|tf_i^x - f_i^{tx}\|_1 = 0,$$

uniformly on compact subsets of X.

(ii) *There exists a a net* (ξ_i) *of continuous functions* $\xi_i : x \mapsto \xi_i^x$ *from* X *into* $\ell^2(\Gamma)_1$, *such that for every* $t \in \Gamma$

$$\lim_i \|t\xi_i^x - \xi_i^{tx}\|_2 = 0,$$

uniformly on compact subsets of X.

(iii) *There exists a net* (h_i) *of continuous positive definite function* h_i *on* $X \times \Gamma$, *with compact support, such that* $lim_i|1 - h_i(x,t)| = 0$, *uniformly on compact subsets of* $X \times \Gamma$.

A net (f_i) as in (i) is called an *approximate invariant continuous mean* (a.i.c.m. for short).

A complex valued function h on $X \times \Gamma$ is said to be *positive definite* (or of *positive type*) if for every $x \in X$, every $n \geq 1$ and every $t_1, \ldots, t_n$ in Γ, the $n \times n$ matrix $[h(t_i^{-1}x, t_i^{-1}t_j)]$ is positive. To better understand this definition it is useful to see $\mathcal{G} = X \times \Gamma$ as a groupoid :

- the range and source are respectively given by

$$r(x,t) = x \quad \text{and} \quad s(x,t) = t^{-1}x;$$

- the inverse is given by $(x,t)^{-1} = (t^{-1}x, t^{-1})$
- the product is defined by $(x,t)(t^{-1}x, t') = (x, tt')$.

Then the above definition of positiveness reads as : for every $x \in X = \mathcal{G}^{(0)}$ (the space of units of the groupoid) and every $g_1 = (x, t_1), \ldots, g_n = (x, t_n)$ in $r^{-1}(x)$, the matrix $[h(g_i^{-1}g_j)]$ is positive. When X is reduced to a point, we recover the usual definition of a positive definite function on Γ.

Observe that a positive definite fonction h on the trivial groupoid $X \times X$ is just a positive definite kernel in the usual sense, that is, for every $y_1, \ldots y_n$ in X the matrix $[h(y_i, y_j)]$ is positive.

Remark 1.3. The equivalence between (i) and (ii) in 1.2 follows the inequalities

- $\big\||\xi_1|^2 - |\xi_2|^2\big\|_1 \leq 2\|\xi_1 - \xi_2\|_2$ for $\xi_1, \xi_2 \in \ell^2(\Gamma)_1$;
- $\big\|\sqrt{f_1} - \sqrt{f_2}\big\|_2^2 \leq \|f_1 - f_2\|_1$ for $f_1, f_2 \in \ell^1(\Gamma)^+$.

The implication (ii) ⇒ (iii) follows an approximation argument and from the fact that, given $\xi : x \mapsto \xi^x$ from X to $\ell^2(\Gamma)$, then

$$(x,t) \mapsto h(x,t) = \left\langle \xi^x, t\xi^{t^{-1}x} \right\rangle$$

is positive definite. For more details on the proof of proposition 1.2 we refer for example to [4] or [6].

Examples 1.4. (1) If Γ is an amenable group, every action $\Gamma \curvearrowright X$ is amenable. Indeed, given $(f_i)_{i\in I}$ as in definition 1.1 (i), we define $\tilde{f}_i$ to be the constant map on X whose value is f_i. Then $(\tilde{f}_i)_{i\in I}$ is an a.i.c.m.

Note that if $\Gamma \curvearrowright X$ is amenable and if there is a Γ-invariant probability measure μ on X, then Γ is an amenable group. Indeed, let us consider an a.i.c.m. $(f_i)_{i\in I}$ and set $k_i(t) = \int f_i^x(t) d\mu(x)$. Then $(k_i)_{i\in I}$ has the properties stated in definition 1.1 (i) and therefore Γ is amenable.

(2) For every group Γ, its left action on itself is amenable. Indeed, $x \in \Gamma \mapsto f^x = \delta_x$ is invariant[3] : $tf^x = f^{tx}$ for every $t, x \in \Gamma$. Such an action, having a continuous invariant system of probability measures, is called *proper*. This is equivalent to the usual properness of the map $(t,x) \mapsto (tx,x)$ from $\Gamma \times X$ into $X \times X$ (see [5, Cor. 2.1.17]).

(3) Let $\mathbb{F}_2$ be the free group with two generators a and b. The boundary $\partial\mathbb{F}_2$ is the set of all infinite reduced words $\omega = a_1 a_2 \dots a_n \dots$ in the alphabet $S = \{a, a^{-1}, b, b^{-1}\}$. It is equipped with the topology induced by the product topology on $S^{\mathbb{N}^*}$. The group $\mathbb{F}_2$ acts continuously to the left by concatenation on the Cantor discontinuum $\partial\mathbb{F}_2$. This action is amenable. Indeed, for $n \geq 1$ and $\omega = a_1 a_2 \dots a_n \dots$, define

$$\begin{aligned} f_n^\omega(t) &= \frac{1}{n} \quad \text{if} \quad t = a_1 \dots a_k, \quad k \leq \frac{1}{n}, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Then $(f_n)_{n\geq 1}$ is an a.i.c.m. This observation holds for any free group.

(4) Another convenient way to show that group actions are amenable is to use the invariance of this notion by Morita equivalence [5, Th. 2.2.17]. Let us consider for instance a locally compact group G, an amenable closed subgroup H and a discrete subgroup Γ. Then the left Γ-action on G/H is amenable.

1.3 Group C^*-algebras. We first recall some basic facts about C^*-algebras.

1.3.1 ***C^*-algebras.*** A *C^*-algebra* A is a closed involutive subalgebra of the involutive Banach algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on some (complex) Hilbert space $\mathcal{H}$.

Hence, for any element a of a C^*-algebra, we have $\|a\|^2 = \|a^*a\|$.

[3] δ_x is the Dirac function at x.

Theorem 1.5 (Gelfand-Naimark). *Let A be a Banach $*$-algebra (over $\mathbb{C}$) such that $\|a\|^2 = \|a^*a\|$ for every $a \in A$. Then there is an isometric $*$-isomorphism from A onto a closed involutive subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.*

First examples:

- $\mathcal{B}(\mathcal{H})$ is a C^*-algebra. When $\mathcal{H} = \mathbb{C}^n$, we get the algebra $M_n(\mathbb{C})$ of $n \times n$ matrices with complex entries. Every finite dimensional C^*-algebra is a finite product of full matrix algebras.
- (*Commutative case.*) Given a localement compact space X, the Banach $*$-algebra $C_0(X)$ of continuous functions from X to $\mathbb{C}$, vanishing at infinity, is a commutative C^*-algebra. A celebrated theorem of Gelfand states that every commutative C^*-algebra is of this form.

We shall see below that groups and group actions provide a wealth of examples. Before, we make some important observations on C^*-algebras.

Basic facts:

- If A is a C^*-algebra and $a = a^* \in A$, then $\left\|a^2\right\| = \|a\|^2$. It follows that $\|a\|$ is equal to the spectral radius of a. Hence the norm on a C^*-algebra only depends on algebraic properties.
- Let B be another C^*-algebra and $\pi : A \to B$ a homomorphism[4]. Then $\|\pi(a)\| \leq \|a\|$ for every $a \in A$ and $\pi(A)$ is closed in B. Moreover, if π is injective then it is isometric. It follows that on a C^*-algebra there is only one C^*-norm[5].
- A C^*-algebra A has a natural closed positive cone, namely the cone A^+ of elements of the form a^*a with $a \in A$. When A is concretely represented as a subalgebra of some $\mathcal{B}(\mathcal{H})$, this is the cone of positive operators belonging to A.

References : [49], [57], [25].

1.3.2 ***Group C^*-algebras.*** Let Γ be a discrete group and let $\mathbb{C}[\Gamma]$ be the corresponding *group algebra*, that is the $*$-algebra of formal sums $\sum_{t\in\Gamma} c_t t$ where $t \mapsto c_t$ is a finitely supported function from Γ to $\mathbb{C}$. Recall that the product in $\mathbb{C}[\Gamma]$ is the obvious extension of the product of Γ and that the involution is given by $(ct)^* = \overline{c}t^{-1}$ for $c \in \mathbb{C}$ and $t \in \Gamma$. Traditionally, two norms are defined on $\mathbb{C}[\Gamma]$, reflecting which unitary representations of Γ one wishes to consider, and this gives rise to two completions of $\mathbb{C}[\Gamma]$, the *full C^*-algebra* $C^*(\Gamma)$ and the *reduced C^*-algebra* $C^*_r(\Gamma)$, that we describe now.

[4] In the sequel, a homomorphism between C^*-algebras is a map preserving the algebraic operations and the involution.

[5] A C^*-norm on a $*$-algebra A is a norm such that $\|xy\| \leq \|x\|\|y\|$ and $\|x^*x\| = \|x\|^2$ for every $x, y \in A$.

Let σ be a unitary representation on some Hilbert space $\mathcal{H}$. We can extend σ to a $*$-homomorphism from $\mathbb{C}[\Gamma]$ into $\mathcal{B}(\mathcal{H})$ by

$$\sigma(\sum_{t\in\Gamma} c_t t) = \sum c_t \sigma(t).$$

Observe that $\|\sigma(\sum c_t t)\| \leq \sum |c_t|$.

The most important representation of Γ is the *left regular representation*, that is the representation λ in $\ell^2(\Gamma)$ by left translations : $\lambda(s)\delta_t = \delta_{st}$ (where δ_t is the Dirac function at t). One may equivalently consider the *right regular representation* ρ. Observe that λ defines an injective $*$-homomorphism from $\mathbb{C}[\Gamma]$ into $\mathcal{B}(\ell^2(\Gamma))$. Thus we may view any element $\sum c_t t$ of $\mathbb{C}[\Gamma]$ as the convolution operator on $\ell^2(\Gamma)$ by the function $t \mapsto c_t$. Then $c = \sum c_t t \in \mathbb{C}[\Gamma]$ will also be written $c = \sum c_t \lambda(t) \in C_r^*(\Gamma)$.

Definition 1.6. The *reduced C^*-algebra* $C_r^*(\Gamma)$ is the completion of $\mathbb{C}[\Gamma]$ in the norm given, for $c \in \mathbb{C}[\Gamma]$, by $\|c\|_r = \|\lambda(c)\|$. Equivalently, it is the closure of $\mathbb{C}[\Gamma]$ in $\mathcal{B}(\ell^2(\Gamma))$, when $\mathbb{C}[\Gamma]$ is identified with its image under the left regular representation.

Definition 1.7. The *full C^*-algebra* $C^*(\Gamma)$ is the completion of $\mathbb{C}[\Gamma]$ in the norm given, for $c \in \mathbb{C}[\Gamma]$, by

$$\|c\| = \sup_\sigma \|\sum_{t\in\Gamma} c_t \sigma(t)\|,$$

where σ ranges over all the unitary representations of Γ.

Obviously, there is a canonical bijective correspondence between unitary representations of Γ and non degenerate representations of $C^*(\Gamma)$ [6].

Thanks to the universal property of the full C^*-algebra with respect to unitary representations of Γ, the left regular representation induces a surjective homomorphism, still denoted by λ, from $C^*(\Gamma)$ onto $C_r^*(\Gamma)$.

Theorem 1.8 (Hulanicki). *Γ is amenable if and only if $\lambda : C^*(\Gamma) \to C_r^*(\Gamma)$ is an isomorphism.*

Proof. This boils down to showing that $\lambda : C^*(\Gamma) \to C_\lambda^*(\Gamma)$ is injective if and only if the trivial representation of Γ is weakly contained in λ. We refer to [25, §18] or [57, Th. 7.3.9] for the proof. □

1.4 Crossed products. Let $\alpha : \Gamma \curvearrowright A$ be an action of Γ on a C^*-algebra A. That is, α is a homomorphism from the group Γ into the group $\mathrm{Aut}(A)$ of automorphisms of A. We denote by $A[\Gamma]$ the $*$-algebra of formal sums $a = \sum a_t t$

[6] A representation of a C^*-algebra A in a Hilbert space $\mathcal{H}$ is a homomorphism σ from A into $\mathcal{B}(\mathcal{H})$. It is non degenerated if $\sigma(A)\mathcal{H}$ is total in $\mathcal{H}$.

where $t \mapsto a_t$ is a map from Γ into A with finite support and where the operations are given by the following rules:

$$(at)(bs) = a\alpha_t(b)\, ts, \quad (at)^* = \alpha_{t^{-1}}(a)\, t^{-1},$$

for $a, b \in A$ and $s, t \in \Gamma$.

For such a dynamical system $\alpha : \Gamma \curvearrowright A$, the notion of unitary representation is replaced by that of covariant representation.

Definition 1.9. A *covariant representation* of $\alpha : \Gamma \curvearrowright A$ is a pair (π, σ) where π and σ are respectively a representation of A and a unitary representation of Γ in the same Hilbert space $\mathcal{H}$, satisfying the covariance rule

$$\forall a \in A, \forall t \in \Gamma, \quad \sigma(t)\pi(a)\sigma(t)^* = \pi(\alpha_t(a)).$$

A covariant representation gives rise to a $*$-homomorphism $\pi \times \sigma$ from $A[\Gamma]$ into $\mathcal{B}(\mathcal{H})$ by

$$(\pi \times \sigma)(\sum a_t\, t) = \sum \pi(a_t)\sigma(t).$$

Clearly we have

$$\Big\|(\pi \times \sigma)(\sum a_t\, t)\Big\| \leq \sum_{t \in \Gamma} \|\sigma(a_t)\|.$$

Definition 1.10. The *full crossed product* $A \rtimes \Gamma$ associated with $\alpha : \Gamma \curvearrowright A$ is the C^*-algebra obtained as the completion of $A[\Gamma]$ in the norm

$$\|a\| = \sup \|(\pi \times \sigma)(a)\|$$

where (π, σ) runs over all covariant representations of $\alpha : \Gamma \curvearrowright A$.

By definition, every covariant representation (π, σ) extends to a representation of $A \rtimes \Gamma$, denoted by $\pi \times \sigma$. Conversely, it is not difficult to see that every non degenerate representation of $A \rtimes \Gamma$ comes in this way from a covariant representation. In other terms, $A \rtimes \Gamma$ is the universal C^*-algebra describing the covariant representations of $\alpha : \Gamma \curvearrowright A$.

We now describe the analogues of the regular representation, the *induced covariant representations*. Let π be a representation of A on a Hilbert space $\mathcal{H}_0$ and set $\mathcal{H} = \ell^2(\Gamma, \mathcal{H}_0) = \ell^2(\Gamma) \otimes \mathcal{H}_0$. We define a covariant representation $(\tilde{\pi}, \tilde{\lambda})$ of $\alpha : \Gamma \curvearrowright A$, acting on $\mathcal{H}$ by

$$\tilde{\pi}(a)\xi(t) = \pi\big(\alpha_{t^{-1}}(a)\big)\xi(t)$$
$$\tilde{\lambda}(s)\xi(t) = \xi(s^{-1}t),$$

for all $a \in A$, all $s, t \in \Gamma$ and all $\xi \in \ell^2(\Gamma, \mathcal{H}_0)$. The covariant representation $(\tilde{\pi}, \tilde{\lambda})$ is said to be *induced* by π.

Definition 1.11. The *reduced crossed product* $A \rtimes_r \Gamma$ is the C^*-algebra obtained as the completion of $A[\Gamma]$ in the norm

$$\|a\|_r = \sup \|(\tilde{\pi} \times \tilde{\lambda})(a)\|$$

for $a \in A[\Gamma]$, where π runs over all representations of A.

When $(\pi, \mathcal{H}_0)$ is a faithful representation of A one has $\|a\|_r = \|(\tilde{\pi} \times \tilde{\lambda})(a)\|$ for all $a \in A[\Gamma]$ and therefore $A \rtimes_r \Gamma$ is faithfully represented into $\ell^2(\Gamma) \otimes \mathcal{H}_0$.

As is subsection 1.3.2, there is a canonical surjective homomorphism from $A \rtimes \Gamma$ onto $A \rtimes_r \Gamma$. Note that when $A = \mathbb{C}$, we have $A \rtimes \Gamma = C^*(\Gamma)$ and $A \rtimes_r \Gamma = C^*_r(\Gamma)$.

In the sequel we shall only consider the case where A is a commutative C^*-algebra. Any action $\Gamma \curvearrowright X$ on a locally compact space X lifts to an action on $A = C_0(X)$ as

$$\alpha_t(a)(x) = a(t^{-1}x)$$

for all $a \in C_0(X)$ and $t \in \Gamma$.

Theorem 1.12. *If $\Gamma \curvearrowright X$ is an amenable action, the canonical surjection from $C_0(X) \rtimes \Gamma$ onto $C_0(X) \rtimes_r \Gamma$ is an isomorphism.*

For the proof, see [4, Th. 3.4] or [6, Th. 5.3].

Problem: Is the converse true ?

References : [21, Chaper VIII], [57, Chapter 7].

1.5 Tensor products of C^*-algebras. It is interesting to compare the theory of tensor products for Banach spaces and for C^*-algebras. Therefore we begin by recalling some facts concerning Banach spaces tensor products.

1.5.1 *Tensor products of Banach spaces.*

Definition 1.13. Let E and F be two Banach spaces. A *cross-norm* is a norm β on the algebraic tensor product $E \odot F$ such that $\beta(x \otimes y) = \|x\|\|y\|$ for every $x \in E$ and $y \in F$.

Let us introduce the two most important cross-norms that are considered on $E \odot F$. The *projective cross-norm* γ is defined by

$$\|z\|_\gamma = \inf \sum_{i=1}^{n} \|x_{1,i}\|\|x_{2,i}\|$$

where the infimum runs over all decompositions $z = \sum_{i=1}^n x_{1,i} \otimes x_{2,i}$. The *injective cross-norm* λ is defined by

$$\|z\|_\lambda = \sup |(f \otimes g)(z)|,$$

where f and g run over the unit balls of the duals E' and F' respectively. The Banach space completions of $E \odot F$ with respect to these two norms are denoted by $E \otimes_\gamma F$ and $E \otimes_\lambda F$.

Clearly, γ is the greatest cross-norm on $E \odot F$. Moreover, a cross-norm β is such that the dual norm on $E' \odot F'$ is a cross-norm if and only if one has $\lambda \leq \beta \leq \gamma$.

1.5.2 ***Tensor products of C^*-algebras.*** Let us now go back to C^*-algebras. Let A and B be two C^*-algebras, and denote by $A \odot B$ their algebraic tensor product. It is an $*$-algebra in an obvious way. A *C^*-norm* on $A \odot B$ is a norm of involutive algebra such that $\|x^*x\| = \|x\|^2$ for all $x \in A \odot B$. The situation is similar to the situation met before for crossed products : there are two natural ways to define C^*-norms on the $*$-algebra $A \odot B$. These norms are defined thanks to representations into Hilbert spaces.

Definition 1.14. The *minimal C^*-norm* of $x \in A \odot B$ is defined by

$$\|x\|_{\min} = \sup \|(\pi_1 \otimes \pi_2)(x)\|$$

where π_1, π_2 run over all representations of A and B respectively. The *minimal tensor product* is the completion $A \otimes_{\min} B$ of $A \odot B$ for this C^*-norm.

Takesaki has shown that when π_1 and π_2 are faithful, we have

$$\|x\|_{\min} = \|(\pi_1 \otimes \pi_2)(x)\|.$$

Therefore, if A and B are concretely represented as C^*-subalgebras of $\mathcal{B}(\mathcal{H}_1)$ and $\mathcal{B}(\mathcal{H}_2)$ respectively, then $A \otimes_{\min} B$ is (up to isomorphism) the closure of $A \odot B$, viewed as a subalgebra of $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. It is why $\|\cdot\|_{\min}$ is also called the *spatial tensor product.*

Definition 1.15. The *maximal C^*-norm* of $x \in A \odot B$ is defined by

$$\|x\|_{\max} = \sup \|\pi(x)\|$$

where π runs over all $*$-homomorphisms from $A \odot B$ into some $\mathcal{B}(\mathcal{H})$ [7]. The *maximal tensor product* is the completion $A \otimes_{\max} B$ of $A \odot B$ for this C^*-norm.

Theorem 1.16 ([62], Th.4.19). *Let A and B be two C^*-algebras. Then $\|\cdot\|_{\max}$ and $\|\cdot\|_{\min}$ are respectively the largest and the smallest C^*-norm on $A \odot B$. Moreover, they are both cross-norms.*

The following results are easily proved :

• If Γ_1, Γ_2 are two discrete groups. Then

$$C^*(\Gamma_1) \otimes_{\max} C^*(\Gamma_2) = C^*(\Gamma_1 \times \Gamma_2),$$
$$C_r^*(\Gamma_1) \otimes_{\min} C_r^*(\Gamma_2) = C_r^*(\Gamma_1 \times \Gamma_2).$$

[7] It is easily shown that this supremum is actually finite.

• If Γ acts trivially on a C^*-algebra A, then

$$A \rtimes \Gamma = A \otimes_{\max} C^*(\Gamma),$$
$$A \rtimes_r \Gamma = A \otimes_{\min} C_r^*(\Gamma).$$

Remark 1.17. Clearly, $A \otimes_{\min} B$ is a quotient of $A \otimes_{\max} B$. Whether the canonical surjection is injective is related to an amenability property for C^*-algebras, called nuclearity. It will be studied in Section 2.

Takesaki considered the case where $A = B$ is the reduced C^*-algebra $C_r^*(\mathbb{F}_2)$ of the free group $\mathbb{F}_2$ and proved in [61] that the norms $\|\cdot\|_{\max}$ and $\|\cdot\|_{\min}$ are different on the algebraic tensor product $C_r^*(\mathbb{F}_2) \odot C_r^*(\mathbb{F}_2)$.

Example 1.18. Given a C^*-algebra $A \subset \mathcal{B}(\mathcal{H})$, the $*$-algebra $M_n(A) = M_n(\mathbb{C}) \odot A$ of $n \times n$ matrices with entries in A is closed in $\mathcal{B}(\mathbb{C}^n \otimes \mathcal{H})$. It follows that there is only one C^*-norm on $M_n(A)$. In other terms we have

$$M_n(A) = M_n(\mathbb{C}) \odot A = M_n(\mathbb{C}) \otimes_{\min} A = M_n(\mathbb{C}) \otimes_{\max} A.$$

Remark 1.19. If $T_1 : A_1 \to B_1$ and $T_2 : A_2 \to B_2$ are bounded linear maps between C^*-algebras, it is not true in general that $T_1 \odot T_2 : A_1 \odot A_2 \to B_1 \odot B_2$ extends to a bounded linear map $T_1 \otimes T_2$ between the completions with respect to the minimal (or maximal) C^*-norms, such that $\|T_1 \otimes T_2\| \leq \|T_1\|\|T_2\|$[8]. For instance, given a bounded linear map $T : A \to B$ between C^*-algebras, let us denote by $T_{(n)} : M_n(A) \to M_n(B)$ the map $\mathrm{Id}_n \otimes T$, that is $T_{(n)}([a_{i,j}]) = [T(a_{i,j})]$ for $[a_{i,j}] \in M_n(A)$. It may happen (see [56]) that $\sup_n \|T_{(n)}\| = +\infty$[9]. In the next section, we shall introduce a class of morphisms between C^*-algebras, well behaved, in particular, with respect to tensor products.

Reference : [62, Chapter IV].

1.6 Completely positive maps.. Let A and B be C^*-algebras. A linear map $T : A \to B$ is said to be positive if $T(A^+) \subset B^+$. It is not true that $T_{(n)}$ is then positive for all n. A good reference for this subject is the book [56]. In particular, it is observed at the end of the first chapter that the transposition T on $M_2(\mathbb{C})$ is positive but $T_{(2)}$ is not positive.

Definition 1.20. A linear map $T : A \to B$ is said to be *completely positive* (c.p.) if $T_{(n)}$ is positive for all $n \geq 1$.

Proposition 1.21. *A linear map $T : A \to B$ is c.p. if and only if, for every $n \geq 1$, every $a_1, \ldots, a_n \in A$ and every $b_1, \ldots, b_n \in B$, we have*

$$\sum_{i,j=1}^{n} b_i^* T(a_i^* a_j) b_j \in B^+.$$

[8] This contrasts with the behaviour of the usual tensor norms λ and γ in the theory of Banach spaces tensor products.

[9] We set $\|T\|_{cb} = \sup_{n \geq 1} \|T_{(n)}\|$. If $\|T\|_{cb} < +\infty$, one says that T is *completely bounded*.

This is an easy consequence, left as an exercise, of the following lemma.

Lemma 1.22. *Given $a_1, \dots, a_n$ in a C^*-algebra A the matrix $[a_i^* a_j]$ is positive in $M_n(A)$. Moreover every element of $M_n(A)^+$ is a finite sum of matrices of this form.*

Proof. Obviously the matrix

$$[a_i^* a_j] = \begin{bmatrix} a_1^* \\ \vdots \\ a_n^* \end{bmatrix} \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix}$$

is positive. Let now $[a_{i,j}] \in M_n(A)^+$. Then there exists a matrix $[c_{i,j}] \in M_n(A)$ such that

$$[a_{i,j}] = [c_{i,j}]^* [c_{i,j}].$$

It follows that

$$[a_{i,j}] = \sum_{k=1}^{n} [c_{k,i}^* c_{k,j}].$$

□

Example 1.23. Every homomorphism from A into B is completely positive. Also, for $a \in A$, the map $x \mapsto a^* x a$ is completely positive from A into itself.

As it is shown now, completely positive maps have a very simple structure. We only consider the case where A has a unit, for simplicity. We shall write u.c.p. for unital completely positive.

Proposition 1.24 (**Stinespring theorem**, [56], Th. 4.1). *Let T be a u.c.p. map from a unital C^*-algebra A into $\mathcal{B}(\mathcal{H})$. There exist a Hilbert space $\mathcal{K}$, a representation $\pi : A \to \mathcal{B}(\mathcal{K})$ and an isometry $V : \mathcal{H} \to \mathcal{K}$ such that $T(a) = V^* \pi(a) V$ for all $a \in A$. In particular, one has $\|T\| = \|T(1)\| = \|T\|_{cb}$.*

Sketch of proof. We define on $A \odot \mathcal{H}$ the inner product

$$\langle a_1 \otimes v_1, a_2 \otimes v_2 \rangle = \langle v_1, T(a_1^* a_2) v_2 \rangle.$$

Let $\mathcal{K}$ be the Hilbert space obtained from $A \odot \mathcal{H}$ by separation and completion. We denote by $[x]$ the class of $x \in A \odot \mathcal{H}$ in $\mathcal{K}$. Then $V : v \mapsto [I_A \otimes v]$ is an isometry from $\mathcal{H}$ into $\mathcal{K}$. Let π be the representation from A into $\mathcal{K}$ defined by $\pi(a)[a_1 \otimes v] = [a a_1 \otimes v]$ for $a, a_1 \in A$ and $v \in \mathcal{H}$. Then π and V fulfill the required properties. □

Theorem 1.25 (**Arveson's extension theorem**, [56], Th. 7.5). *Let A be a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ and $T : A \to \mathcal{B}(\mathcal{K})$ a completely positive map. There exists a completely positive map $\overline{T} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ which extends T.*

Proposition 1.26 ([62], Prop. IV.4.23). *Let $T_i : A_i \to B_i$, $i = 1, 2$, be completely positive maps between C^*-algebras.*

(1) *$T_1 \odot T_2$ extends to a c.p. map $T_1 \otimes_{\min} T_2 : A_1 \otimes_{\min} A_2 \to B_1 \otimes_{\min} B_2$ and $\|T_1 \otimes_{\min} T_2\| \leq \|T_1\|\|T_2\|$.*

(2) *The same result holds for the maximal tensor norm. More generally, if T_1 and T_2 are two c.p. maps from A_1 and A_2 into a C^*-algebra B such that $T_1(A_1)$ and $T_2(A_2)$ commute, then the map from $A_1 \odot A_2$ into B sending $a_1 \otimes a_2$ to $T_1(a_1)T_2(a_2)$ extends to a c.p. map from $A_1 \otimes_{\max} A_2$ into B.*

The relations between positive definite functions on a discrete group Γ and completely positive maps on $C^*_\lambda(\Gamma)$ are described in the following lemma.

Lemma 1.27. *Let Γ be a discrete group.*

(i) *Let φ be a positive definite function on Γ. Then*

$$m_\varphi : \sum c_t \lambda(t) \mapsto \sum \varphi(t) c_t \lambda(t)$$

extends to a completely positive map ϕ from $C^*_\lambda(\Gamma)$ into itself.

(ii) *Let $\phi : C^*_\lambda(\Gamma) \to C^*_\lambda(\Gamma)$ be a completely positive map. For $t \in \Gamma$ we set $\varphi(t) = \big\langle \delta_e, \phi\big(\lambda(t)\big)\lambda(t)^*\delta_e \big\rangle$. Then φ is a positive definite function on Γ.*

Proof. (i) Let $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$ be given by the GNS construction, so that for $t \in \Gamma$ one has $\varphi(t) = \langle \xi_\varphi, \pi_\varphi(t)\xi_\varphi \rangle$. Let S from $\ell^2(\Gamma)$ into $\ell^2(\Gamma, \mathcal{H}_\varphi)$ defined by

$$(Sf)(t) = f(t)\pi_\varphi(t)^*\xi_\varphi.$$

It is a bounded linear map and its adjoint S^* satisfies $S^*(F)(t) = \langle \xi_\varphi, \pi_\varphi(t)F(t) \rangle$ for $F \in \ell^2(\Gamma, \mathcal{H}_\varphi)$. A straightforward computation shows that

$$\forall c \in \mathbb{C}[\Gamma], \quad m_\varphi(c) = S^*(c \otimes \mathrm{Id}_{\mathcal{H}_\varphi})S.$$

It follows that m_φ extends to the completely positive map $a \mapsto S^*(a \otimes \mathrm{Id}_{\mathcal{H}_\varphi})S$ from $C^*_\lambda(\Gamma)$ into itself.

(ii) Using the fact that $\lambda(s)\delta_e = \rho(s^{-1})\delta_e$ for $s \in \Gamma$, and since the right regular representation ρ commutes with the image of ϕ, we get

$$\varphi(t_i^{-1}t_j) = \big\langle \delta_e, \phi\big(\lambda(t_i)^*\lambda(t_j)\big)\rho(t_i^{-1})\rho(t_j)\delta_e \big\rangle = \big\langle \rho_{t_i}\delta_e, \phi\big(\lambda(t_i)^*\lambda(t_j)\big)\rho(t_j)\delta_e \big\rangle.$$

From the positivity of the matrix $[\phi\big(\lambda(t_i)^*\lambda(t_j)\big)]$ we deduce that φ is positive definite. □

Observe that when φ has a finite support, the associated c.p. map ϕ has a finite rank. Moreover, if Γ is amenable and if (φ_i) is a net of normalized (i.e. $\varphi_i(e) = 1$) positive definite functions, with finite support, converging pointwise to one, then the corresponding net (ϕ_i) of u.c.p. maps goes to the identity map of $C^*_r(\Gamma)$ in the topology of pointwise convergence in norm. We shall study this property more in details in the next section.

References : [56], [62].

2 Nuclearity and amenability

2.1 Nuclear C^*-algebras and amenable groups.

Definition 2.1. A completely positive contraction $T : A \to B$ is said to be *nuclear* if there exist a net (n_i) of positive integers and nets of completely positive contractions[10] $\phi_i : A \to M_{n_i}(\mathbb{C}), \tau_i : M_{n_i}(\mathbb{C}) \to B$, such that for every $a \in A$,

$$\lim_i \|\tau_i \circ \phi_i(a) - T(a)\| = 0.$$

Theorem 2.2. *Let A be a C^*-algebra. The following conditions are equivalent :*

(i) *The identity map of A is nuclear.*

(ii) *For every C^*-algebra B, there is only one C^*-norm on $A \odot B$.*

(iii) *There exists a net (ϕ_i) of finite rank completely positive maps $\phi_i : A \to A$ converging to the identity map of A in the topology of pointwise convergence in norm.*

This is the Choi-Effros theorem 3.1 in [18]. The implication (ii) $\Rightarrow$ (iii) is due to Choi-Effros [18] and Kirchberg [41] independently. For a short proof of the equivalence between (i) and (ii) we refer to [59, Prop. 1.2]. Let us just show the easy direction.

Proof of (i) $\Rightarrow$ (ii). Let $T : A \to A$ be a completely positive contraction of the form $T = \tau \circ \phi$ where $\phi : A \to M_n(\mathbb{C})$ and $\tau : M_n(\mathbb{C}) \to A$ are completely positive contractions. Then the map $T \otimes \mathrm{Id}_B : A \otimes_{\max} B \to A \otimes_{\max} B$ admits the factorization

$$A \otimes_{\max} B \xrightarrow{\theta} A \otimes_{\min} B \xrightarrow{\phi \otimes Id_B} M_n(\mathbb{C}) \otimes_{\min} B = M_n(\mathbb{C}) \otimes_{\max} B \xrightarrow{\tau \otimes Id_B} A \otimes_{\max} B,$$

where θ is the canonical homomorphism from $A \otimes_{\max} B$ onto $A \otimes_{\min} B$. In particular the kernel of $T \otimes \mathrm{Id}_B : A \otimes_{\max} B \to A \otimes_{\max} B$ contains the kernel of θ. Let now $(T_i = \tau_i \circ \phi_i)$ be a net of completely positive contractions factorizable through matrix algebras and converging to Id_A. Then $\mathrm{Id}_{A \otimes_{\max} B}$ is the norm pointwise limit of $T_i \otimes \mathrm{Id}_B : A \otimes_{\max} B \to A \otimes_{\max} B$, and therefore its kernel contains the kernel of θ. It follows that $\ker \theta = \{0\}$. □

Definition 2.3. A C^*-algebra A satisfying any of these equivalent properties is said to be *nuclear*.

The simplest examples of nuclear C^*-algebras are the commutative ones, matrix algebras and more generally the C^*-algebra of compact operators on some Hilbert space. There are also many important examples coming from group and group actions, as we shall see now.

[10] When A and B have a unit and T is unital, one may chose ϕ_i and τ_i to be unital.

Proposition 2.4 (Lance, [48]). *The reduced C^*-algebra $C^*_r(\Gamma)$ of a discrete group Γ is nuclear if and only if Γ is amenable.*

Proof. We have already shown the "if" part at the end of Section 1. To prove the converse, assume the existence of a net (ϕ_i) of finite rank u.c.p. maps from $C^*_r(\Gamma)$ into itself such that $\lim_i \|\phi_i(a) - a\| = 0$ for every $a \in C^*_r(\Gamma)$. Then the net (φ_i) of positive definite functions associated to these u.c.p. maps as in lemma 1.27 goes to one, in the topology of pointwise convergence. The main point is that we need finitely supported functions. We shall see that if $\phi : C^*_r(\Gamma) \to C^*_r(\Gamma)$ is a finite rank c.p.u. map then $\varphi : t \mapsto \big\langle \delta_e, \phi\big(\lambda(t)\big)\lambda(t)^*\delta_e \big\rangle$ is uniformly approximated by positive definite functions that are finitely supported. This will conclude the proof. There exists bounded linear forms $f_1, \ldots, f_n$ on $C^*_r(\Gamma)$ and elements $b_1, \ldots, b_n$ in $C^*_r(\Gamma)$ such that for every $a \in C^*_r(\Gamma)$,

$$\phi(a) = \sum_{i=1}^{n} f_i(a) b_i.$$

Therefore, we have

$$\varphi(t) = \sum_{i=1}^{n} f_i(\lambda(t)) \langle \delta_e, b_i \lambda(t)^* \delta_e \rangle.$$

Since each function $t \mapsto \langle \delta_e, b_i \lambda(t)^* \delta_e \rangle = \langle \delta_t, b_i^* \delta_e \rangle$ is in $\ell^2(\Gamma)$, it follows that $\varphi \in \ell^2(\Gamma)$. By Godement's theorem [25, Th.13.8.6], there exists $\xi \in \ell^2(\Gamma)$ such that $\varphi(t) = \langle \xi, \lambda(t)\xi \rangle$ for all $t \in \Gamma$. It suffices now to approximate ξ in ℓ^2-nom by finitely supported functions. □

More generally, we have :

Theorem 2.5. *Let $\Gamma \curvearrowright X$ be an action of a discrete group on a locally compact space X. The two following conditions are equivalent :*

(i) *The action is amenable.*

(ii) *The reduced cross product $C_0(X) \rtimes_r \Gamma$ is a nuclear C^* algebra.*

Proof. For (i) ⇒ (ii), one may proceed in two different ways. It is possible to construct explicitely a net of finite rank completely positive contractions from $C_0(X) \rtimes_r \Gamma$ into itself, approximating the identity map. It is also not too difficult to show directly the characterization (ii) of nuclearity in theorem 2.2 (see [6, Th. 5.8 and §8]). The converse is more easily shown with the help of Hilbert C^*-modules over $C_0(X)$. They are versions of Hilbert spaces, with parameters in X. We refer the reader to [4, Th. 3.4] or [6, Th. 5.8] for the proof. □

References : [68].

2.2 Other approximation properties.

2.2.1 *Metric and completely bounded approximation property.*

Definition 2.6. A Banach space E is said to have the *approximation property* if there is a net (T_i) of finite rank bounded operators $T_i \in \mathcal{B}(E)$ converging to Id_E uniformly on compact subsets of E. It is said to have the *bounded approximation property* if the T_i's may be chosen such that $\sup_i \|T_i\| < +\infty$[11]. When there exist such T_i's with $\sup_i \|T_i\| = 1$, the space E is said to have the *metric approximation property*.

The first example of a Banach space not having the approximation property was found by Enflo in 1972. Later on, around 1980, Szankowsky proved that the Banach space $\mathcal{B}(\ell^2)$ fails the approximation property.

Let us recall the following important result of Grothendieck, saying that E has the approximation property if and only if for every Banach space F the canonical map from $E \otimes_\gamma F$ into $E \otimes_\lambda F$ is one-to-one. Theorem 2.2 is an analogue of this result for C^*-algebras.

Obviously, a nuclear C^*-algebra has the metric approximation property. On the other hand, Haagerup proved in [37] that the reduced C^*-algebra $C^*_r(\mathbb{F}_n)$ has the metric approximation property, a very remarkable result since $C^*_r(\mathbb{F}_n)$, $n \geq 2$, is not nuclear, $\mathbb{F}_n$ not being amenable. Later, de Cannière and Haagerup proved a stronger property for $C^*_r(\mathbb{F}_2)$, the metric completely bounded approximation property.

Definition 2.7. We say that a C^*-algebra A has the *completely bounded approximation property* ($CBAP$) is there is a net of finite rank bounded maps (ϕ_i) from A to A, such that $\sup_i \|\phi_i\|_{cb} < +\infty$, converging pointwise to the identity. If there is such a net with $\sup_i \|\phi_i\|_{cb} = 1$, we say that A has the *metric completely bounded approximation property*.

2.2.2 *Haagerup approximation property, or a-T-menability.*

In his proof [37] that $C^*_r(\mathbb{F}_n)$ has the metric approximation property, Haagerup established that the word length function ℓ defined by the free generators of $\mathbb{F}_n$ is conditionally negative definite. Let us recall some basic facts on this notion.

Definition 2.8. A *conditionally negative definite kernel* on a space X is a function $k : X \times X \to \mathbb{R}$ with the following properties:

(a) $k(x, x) = 0$ for all $x \in X$;

(b) $k(x, y) = k(y, x)$ for all $x, y \in X$;

(c) for any $n \geq 1$, any elements $x_1, \ldots, x_n$ in X, and any real numbers $c_1, \ldots, c_n$ with $\sum_{i=1}^n c_i = 0$, then $\sum_{i=1}^n c_i c_i k(x_i, x_j) \leq 0$.

[11] then the convergence to Id_E may be equivalently required to be the norm pointwise convergence.

We summarize in the following theorem several useful results. For proofs we refer for example to [12, Appendix C].

Theorem 2.9. *Let $h, k : X \times X \to \mathbb{R}$ be two kernels on a space X.*

(i) *h is a positive definite (or positive type) kernel if and only if there exist a real Hilbert space $\mathcal{H}$ and a function $\xi : X \to \mathcal{H}$ such that $h(x, y) = \langle \xi_x, \xi_y \rangle$ for every $x, y \in \mathcal{H}$.*

(ii) *k is a conditionally negative definite kernel if and only if there exist a real Hilbert space $\mathcal{H}$ and a function $f : X \to \mathcal{H}$ such that $k(x, y) = \|f(x) - f(y)\|^2$ for all $x, y \in X$.*

(iii) *Let us assume that k satisfies conditions (a) and (b) of the previous definition. Then k is a conditionally negative definite kernel if and only if for every $\alpha > 0$, the kernel $\exp(-\alpha k)$ is positive definite.*

Assertion (i) above also holds when replacing real numbers by complex numbers. Assertion (iii) is known as Schöenberg theorem.

A function $\psi : \Gamma \to \mathbb{R}$ is said to be conditionally negative definite if $(s, t) \mapsto \ell(s^{-1}t)$ is a conditionally negative definite kernel.

The fact that the length function ℓ on $\mathbb{F}_n$ is conditionally negative definite is briefly explained at the beginning of Section 4. By considering the positive definite functions $\varphi_k : t \mapsto \exp(-\ell(t)/k)$, which vanishes to 0 at infinity, we see that $\mathbb{F}_n$ satisfies the following property :

Definition 2.10. We say that a discrete group Γ has the *Haagerup approximation property* or is *a-T-menable* if there exists a net (φ_i) of positive definite functions on Γ, vanishing to 0 at infinity and converging pointwise to 1.

For details on this property and examples we refer to [16].

3 Exactness and boundary amenabililiy

Nuclearity has several nice stability properties. In particular, given an exact sequence $0 \to J \to A \to A/J \to 0$ of C^*-algebras, then A is nuclear if and only if J and A/J are nuclear ([19, Cor.4]. On the other hand, a C^*-subalgebra of a nuclear C^*-algebra need no be nuclear. The first example is due to Choi who constructed in [17] an explicit embedding of the non nuclear C^*-algebra $C_r^*(\mathbb{Z}_2 * \mathbb{Z}_3)$ into a nuclear C^*-algebra.

Another simple example comes from the action of $\mathbb{F}_n$ on its boundary. Since the action $\mathbb{F}_n \curvearrowright \partial\mathbb{F}_n$ is amenable, the crossed product $C(\mathbb{F}_n) \rtimes_r \mathbb{F}_2$ is nuclear. On the other hand, its C^*-subalgebra $C_r^*(\mathbb{F}_n)$ is not nuclear for $n \geq 2$.

However, C^*-subalgebras of nuclear ones still have nice properties. They form the class of exact C^*-algebras. They will be studied in this section as well as the corresponding notion for discrete groups. In this setting this notion is also named boundary amenability (or amenability at infinity) or propery (A) for reasons which will become clear in the sequel.

3.1 Equivalent characterizations of exactness for C^*-algebras.

Definition 3.1. A C^*-algebra A is said to be *exact* (or *nuclearly embeddable*) if there exists a nuclear embedding $A \hookrightarrow D$ into some C^*-algebra D.

Using Arveson extension theorem 1.25, we see that whenever A is exact, any embbeding into some $\mathcal{B}(\mathcal{H})$ is nuclear. To compare with nuclearity, recall that A is nuclear if $\mathrm{Id}_A : A \to A$ is nuclear. The reduced C^*-algebras of most of the usual discrete groups are known to be exact (see a non exhaustive list below) while we have seen that they are nuclear only when the group is amenable. Obviously, every C^*-subalgebra of an exact C^*-algebra (and in particular of a nuclear C^*-algebra) is exact. A celebrated (and deep) result of Kirchberg-Phillips [45] says that conversely, every separable exact C^*-algebra is a C^*-subalgebra of a nuclear one.

The terminology comes from the following important result, saying that A is exact if and only if the functor $B \mapsto B \otimes_{\min} A$ preserves short exact sequences. Indeed this was the original definition of exactness by Kirchberg.

More precisely, let $0 \to I \to B \to B/I \to 0$ be a short exact sequence of C^*-algebras. It can happen that

$$0 \to I \otimes_{\min} A \to B \otimes_{\min} A \to (B/I) \otimes_{\min} A \to 0$$

is not exact in the middle. An explicit example was given in [68] by S. Wassermann. He proved that if I is the kernel of the canonical surjection $C^*(\mathbb{F}_2) \to C^*_r(\mathbb{F}_2)$, then the sequence

$$0 \to I \otimes_{\min} C^*(\mathbb{F}_2) \to C^*_r(\mathbb{F}_2) \otimes_{\min} C^*(\mathbb{F}_2) \to C^*(\mathbb{F}_2) \otimes_{\min} C^*(\mathbb{F}_2) \to 0$$

is not exact.

Theorem 3.2 (Kirchberg). *Let A be a C^*-algebra. The following conditions are equivalent:*

(i) *A is exact (or nuclearly embeddable).*

(ii) *For every short exact sequence $0 \to I \to B \to C \to 0$ of C^*-algebras, the sequence*

$$0 \to I \otimes_{\min} A \to B \otimes_{\min} A \to C \otimes_{\min} A \to 0$$

is exact.

Proof. (i) $\Rightarrow$ (ii). Denote by q the canonical homomorphism from B onto C. The only point is to show that $I \otimes_{\min} A$ is the kernel of $q \otimes \mathrm{Id}_A : B \otimes_{\min} A \to C \otimes_{\min} A$. Let $A \hookrightarrow D$ be a nuclear embedding and let (ϕ_i) be net of finite rank completely positive contractions $\phi_i : A \to D$ such that $\lim_i \|\phi_i(a) - a\| = 0$ for every $a \in A$. It is enough to prove that for $x \in \mathrm{Ker}\,(q \otimes \mathrm{Id}_A)$, we have $(\mathrm{Id}_B \otimes \phi_i)(x) \in I \otimes_{\min} D$. Indeed, considering $B \otimes_{\min} A$ as a C^*-subalgebra of $B \otimes_{\min} D$, we shall obtain

$$x \in (B \otimes_{\min} A) \cap (I \otimes_{\min} D) = I \otimes_{\min} A,$$

since $\lim_i \|x - (\mathrm{Id}_B \otimes \phi_i)(x)\| = 0$.

Hence, what is left to prove is that for any finite rank completely positive map $\phi : A \to D$ we have $(\mathrm{Id}_B \otimes T)(x) \in I \otimes_{\min} D$ whenever $x \in \mathrm{Ker}\,(q \otimes \mathrm{Id}_A)$. Since ϕ has a finite rank, there exist elements $d_1, \ldots, d_n \in D$ and bounded linear forms $f_1, \ldots, f_n$ on A such that $\phi(a) = \sum_{k=1}^n f_k(a) d_k$ for $a \in A$. For $k = 1, \ldots, n$, denote by R_k the bounded linear map from $B \otimes_{\min} A$ to B such that $R_k(b \otimes a) = f_k(a)b$ if $b \otimes a \in B \odot A$. The corresponding map from $C \otimes_{\min} A$ to C is also denoted by R_k. It is easily checked that $R_k \circ (q \otimes \mathrm{Id}_A) = q \circ R_k$ and therefore we have $R_k(\mathrm{Ker}\,(q \otimes \mathrm{Id}_A)) \subset I$. It follows that $(\mathrm{Id}_B \otimes \phi)(x) = \sum_{k=1}^n R_k(x) \otimes d_k \in I \otimes_{\min} D$ and this ends the proof.

(ii) $\Rightarrow$ (i) is a hard result due to Kirchberg (see [43, Th. 4.1] or [68, Theorem 7.3]). □

Remark 3.3. A close inspection of the above proof shows that it only uses the fact that (ϕ_i) is a net of finite rank bounded maps with

$$\lim_i \|x - \mathrm{Id}_B \otimes \phi_i(x)\| = 0$$

for every $x \in B \otimes_{\min} A$. This holds in particular when A has the completely bounded approximation property.

Note that Wassermann's example shows that $C^*(\mathbb{F}_2)$ is not exact. On the other hand we have already observed that the reduced C^*-algebra $C_r^*(\mathbb{F}_2)$ is exact.

Problem : Let Γ be a discrete group such that $C^*(\Gamma)$ is exact. Is Γ amenable ?

This problem has been solved positively for many groups (e.g. maximally almost periodic groups) but the general statement remains open [42]. We shall see now that the situation is quite different for reduced group C^*-algebras : the discrete groups Γ such $C_r^*(\Gamma)$ is exact form a huge class.

Reference : [68].

3.2 Equivalence of exactness and boundary amenability for groups.

Definition 3.4. We say that a *discrete group is exact* if its reduced C^*-algebra is exact. We say that Γ is *boundary amenable* or *amenable at infinity* if it has an amenable action on a *compact* space.

The following result links the exactness of a discrete group with boundary amenability and the amenability of its action on its Stone-Čech compactification[12].

Theorem 3.5 ([34, 52, 4, 6]). *Let Γ be a discrete group. The following conditions are equivalent:*

(i) *There exists an amenable action $\Gamma \curvearrowright X$ on a compact space X.*

[12]We endow the universal compactification $\beta\Gamma$ of the discrete space Γ with the continuous extension of the left action of Γ on itself.

(ii) *The natural left action of* Γ *on the Stone-Čech compactification* $\beta\Gamma$ *of* Γ *is amenable*[13].

(iii) *The* C^**-algebra* $C(\beta\Gamma) \rtimes_r \Gamma$ *is nuclear.*

(iv) Γ *is exact.*

Proof. (i) $\Rightarrow$ (ii). Let X be a compact space on which Γ acts amenably. We choose $x_0 \in X$. The map $s \mapsto sx_0$ from Γ to X extends to a continuous map $p : \beta\Gamma \to X$, by the universal property of the Stone-Čech compactification. Since p is Γ-equivariant, given an a.i.c.m (f_i) for $\Gamma \curvearrowright X$, the net of maps $y \mapsto f_i^{p(y)}$ defines an a.i.c.m. for $\Gamma \curvearrowright \beta\Gamma$. The converse (i) $\Rightarrow$ (ii) is obvious.

The equivalence between (ii) and (iii) follows from theorem 2.5. Now, assuming the nuclearity of $C(\beta\Gamma) \rtimes_r \Gamma$, we immediately get that $C_r^*(\Gamma)$ is exact since it is contained into $C(\beta\Gamma) \rtimes_r \Gamma$. It remains to show that (iv) $\Rightarrow$ (ii). Our proof follows the same pattern as the proof of proposition 2.4. If $C_r^*(\Gamma)$ is exact, there is a net $(\Phi_k = \tau_k \circ \phi_k)$ where $\phi_k : C_r^*(\Gamma) \to M_{n_k}(\mathbb{C})$ and $\tau_k : M_{n_k}(\mathbb{C}) \to \mathcal{B}(\ell^2(\Gamma))$ are u.c.p., such that $\lim_k \|\Phi_k(a) - a\| = 0$ for every $a \in C_r^*(\Gamma)$. For $s, t \in \Gamma$ we set

$$h_k(s,t) = \big\langle \delta_s, \Phi_k\big(\lambda(t)\big)\lambda(t)^*\delta_s \big\rangle.$$

For every t, we have $\sup_{s\in\Gamma} |h_k(s,t)| \leq 1$. It follows that h_k extends continuously to a function on $\beta\Gamma \times \Gamma$, that we still denote by h_k. Obviously, (h_k) goes to 1 uniformly on compact subsets of $\beta\Gamma \times \Gamma$. To check that h_k is positive definite on $\beta\Gamma \times \Gamma$, it is enough to show, by continuity, that for any $x \in \Gamma$, any $n \geq 1$ and any $t_1, \ldots, t_n \in \Gamma$, the matrix $[h_k(t_i^{-1}x, t_i^{-1}t_j)]$ is positive. It is a straightforward computation, using the complete positivity of Φ_k. As in the proof of proposition 2.4, the supports of the h_k's might not be compact. This technical point can be overcome by appropriate approximation (see [4, 6]). □

Examples 3.6. Boundary amenability has now been established for a long list of groups (see [55] for more details). Among them we mention

- amenable groups ;
- hyperbolic groups [1, 29], hyperbolic groups relative to a family of exact subgroups ([54, 23]) ;
- Coxeter groups [26] ;
- linear groups [36] ;
- countable subgroups of almost connected Lie groups [36] ;
- discrete subgroups of almost connected groups (use (4) in Examples 1.4).

[13] We endow the universal compactification $\beta\Gamma$ of the discrete space Γ with the continuous extension of the left action $\Gamma \curvearrowright \Gamma$. Recall that $\beta\Gamma$ is the spectrum of the C^*-algebra $\ell^\infty(\Gamma)$, so that $\ell^\infty(\Gamma) = C(\beta(\Gamma)$

This class of boundary amenable groups is stable by extension [44], amalgamated free products and HNN-extensions [27, 64]. More generally, a group acting on a countable simplicial tree is exact provided all isotropy subgroups are exact (see [53]).

The existence of finitely generated groups which are not boundary amenable has been established by Gromov [32, 33][14].

Open problems:

- It is not known whether exactness is a consequence of the Haagerup approximation property. In particular the Thompson group F has Haagerup's property [28], but it is not known whether it is exact.

- Are the following groups exact : $\mathrm{Out}(\mathbb{F}_n)$, automatic groups, three manifold groups ?

3.3 Exactness as a metric property of a discrete group. When dealing with continuous actions and continuous functions on $\beta\Gamma$ it is enough to consider their restrictions to Γ. We use now this observation to get a simpler description of $C(\beta\Gamma) \rtimes_r \Gamma$ and of the notion of exactness.

Let Γ be a discrete group. We shall view elements of $\mathcal{B}(\ell^2(\Gamma))$ as kernels on $\Gamma \times \Gamma$. The kernel $k : (s,t) \mapsto k(s,t)$ associated with $T \in \mathcal{B}(\ell^2(\Gamma))$ is defined by $k(s,t) = \langle \delta_s, T\delta_t \rangle$. For $\xi \in \ell^2(\Gamma)$ we have $T\xi(s) = \sum_t k(s,t)\xi(t)$. There is no known characterization of the set of kernels associated with bounded operators. However, we are going to consider a class of kernels which obviously define bounded operators.

For every finite subset E of Γ, we set Δ_E the strip $\big\{(s,t) \in \Gamma \times \Gamma, s^{-1}t \in E\big\}$. We say that a kernel k on $\Gamma \times \Gamma$ has *finite propagation* if there is a finite subset $E \subset \Gamma$ such that k is supported in the strip Δ_E. If moreover k is bounded, it clearly defines an element of $\mathcal{B}(\ell^2(\Gamma))$, denoted $\mathrm{Op}(k)$.

Definition 3.7. Let Γ be a discrete group. The *uniform Roe algebra* $C^*_u(|\Gamma|)$ is the norm closure in $\mathcal{B}(\ell^2(\Gamma))$ of the $*$-subalgebra formed by the operators $\mathrm{Op}(k)$, where k ranges over the bounded kernels with finite propagation.

We shall now describe a natural identification of $C(\beta\Gamma) \rtimes_r \Gamma$ with $C^*_u(|\Gamma|)$. First, let us observe that every element $f = \sum f_t t$ of $C(\beta\Gamma)[\Gamma]$ may by identified with the continuous function with compact support $\Theta(f) : (x,t) \mapsto f_t(x)$ on $\beta\Gamma \times \Gamma$. This defines an isomorphism between the $*$-algebra $C(\beta\Gamma)[\Gamma]$ and the $*$-algebra $C_c(\beta\Gamma \times \Gamma)$ of continous functions with compact support on $\beta\Gamma \times \Gamma$.[15] In the following we make no distinction between the elements of $C_c(\beta\Gamma \times \Gamma)$ and their restrictions to $\Gamma \times \Gamma$.

[14] They are not even uniformly embeddable into a Hilbert space (see Subsection 4.1).

[15] The operations of the latter algebra are derived from the groupoid structure of $\beta\Gamma \times \Gamma$, that is $F^*(x,s) = \overline{F}(s^{-1}x, s^{-1})$ and $(F * G)(x,s) = \sum_t F(x,t)G(t^{-1}x, t^{-1}s)$.

Next, we need to introduce the involution $J : (s,t) \mapsto (s^{-1}, s^{-1}t)$ of $\Gamma \times \Gamma$. It is easily checked that $F \mapsto F \circ J$ is an isomorphism between the $*$-algebra $C_c(\beta\Gamma \times \Gamma)$ and the $*$-algebra of bounded kernels with finite propagation on $\Gamma \times \Gamma$.

Proposition 3.8. *The map $\Pi : f \mapsto \mathrm{Op}(\Theta(f) \circ J)$ extends to an isomorphism between the C^*-algebras $C(\beta\Gamma) \rtimes_r \Gamma$ and $C^*_u(|\Gamma|)$.*

Proof. To define $C(\beta\Gamma) \rtimes_r \Gamma$ we use the faithful representation π of $C(\beta\Gamma)$ in $\ell^2(\Gamma)$ given by $\pi(f)\xi(t) = f(t)\xi(t)$ for $f \in C(\beta\Gamma)$ and $\xi \in \ell^2(\Gamma)$. Therefore $C(\beta\Gamma) \rtimes_r \Gamma$ is concretely represented into $\mathcal{B}(\ell^2(\Gamma \times \Gamma))$. More precisely, $f = \sum f_s s \in C(\beta\Gamma)[\Gamma]$ acts on $\ell^2(\Gamma \times \Gamma)$ by

$$(f.\xi)(x,t) = \sum_{s \in \Gamma} f_s(tx)\xi(x, s^{-1}t).$$

Let W be the unitary operator on $\ell^2(\Gamma \times \Gamma)$ defined by $W\xi(x,t) = \xi(x, t^{-1}x^{-1})$. A straightforward verification shows that $WfW^* = \mathrm{Id}_{\ell^2(\Gamma)} \otimes \mathrm{Op}(\Theta(f) \circ J)$ for every $f \in C(\beta\Gamma)[\Gamma] \subset C(\beta\Gamma) \rtimes_r \Gamma$. This shows that Π is isometric and concludes the proof by a density argument. □

We now observe that the amenability of $\Gamma \curvearrowright \beta\Gamma$ can be expressed in terms of positive definite kernels on $\Gamma \times \Gamma$. Recall that a kernel $k : \Gamma \times \Gamma \to \mathbb{C}$ is positive definite (in the usual sense) if for every n and $t_1, \ldots, t_n \in \Gamma$ the matrix $[k(t_i, t_j)]$ is positive.

Proposition 3.9. *Let Γ be a discrete group. The following conditions are equivalent:*

(i) *$\Gamma \curvearrowright \beta\Gamma$ is amenable (i.e. Γ is exact).*

(ii) *For every $\varepsilon > 0$ and every finite subset $E \subset \Gamma$, there exists a function $f : s \mapsto f_s$ from Γ to $\ell^1(\Gamma)^+_1$ and a finite subset E' of Γ such that*

  (a) *$\|f_s - f_t\|_1 \leq \varepsilon$ whenever $s^{-1}t \in E$;*

  (b) *$\mathrm{supp}(f_s) \subset sE'$ for all $s \in \Gamma$.*

(iii) *For every $\varepsilon > 0$ and every finite subset $E \subset \Gamma$, there exists a function $\xi : s \to \xi_s$ from Γ to $\ell^2(\Gamma)_1$ and a finite subset E' of Γ such that*

  (a) *$\|\xi_s - \xi_t\|_2 \leq \varepsilon$ whenever $s^{-1}t \in E$;*

  (b) *$\mathrm{supp}(\xi_s) \subset sE'$ for all $s \in \Gamma$.*

(iv) *For every $\varepsilon > 0$ and every finite subset $E \subset \Gamma$, there exists a bounded positive definite kernel k on $\Gamma \times \Gamma$ and a finite subset E' of Γ such that*

  (a) *$|1 - k(s,t)| \leq \varepsilon$ whenever $s^{-1}t \in E$;*

  (b) *$\mathrm{supp}(k) \subset \{(s,t), s^{-1}t \in E'\} = \Delta_{E'}$.*

Proof. We pass from the three equivalent conditions characterizing the amenability of $\Gamma \curvearrowright \beta\Gamma$ recalled in Definition 1.2 to the above conditions by a change of variables. For example let $F : x \mapsto F^x$ be a continuous function from $\beta\Gamma \to \ell^1(\Gamma)_1^+$ and set $f_s(u) = F^{s^{-1}}(s^{-1}u)$ for $s, u \in \Gamma$. Then $f_s \in \ell^1(\Gamma)_1^+$ and for $s, t \in \Gamma$, setting $g = s^{-1}t$, one has

$$\|f_s - f_t\|_1 = \left\|F^{gt^{-1}} - gF^{t^{-1}}\right\|_1.$$

Therefore, given $E \subset \Gamma$, we get

$$\sup_{(s,t)\in\Delta_E} \|f_s - f_t\|_1 = \sup_{g\in E, t\in\Gamma} \left\|gF^t - F^{gt}\right\|_1. \tag{1}$$

Starting from an a.i.c.m (F_i) as in Definition 1.2 (i), where by approximation we may assume that each function F_i has a finite support, independent of x, we deduce from equation (1) that property (ii) of proposition 3.9 holds. The converse is proved by reversing this construction.

The same observation holds for the equivalence between 1.2 (ii) and 3.9 (iii). As for (iv), we pass from (iii) in Definition 1.2 to (iv) above by replacing h_i by $k_i : (s,t) \mapsto h_i(s^{-1}, s^{-1}t)$. □

Remark 3.10. A kernel $k : X \times X \to \mathbb{C}$ is positive definite if and only if there exist a Hilbert space $\mathcal{H}$ and a map $\xi : X \ni x \to \xi_x \in \mathcal{H}$ such that $k(x,y) = \langle \xi_x, \xi_y \rangle$ for all $x, y \in X$ (see Theorem 2.9). It follows that in (iii) of the previous proposition, it suffices to have the existence of ξ with values in some Hilbert space $\mathcal{H}$, instead of $\ell^2(\Gamma)$, satifying (a) and condition (b') :

$$\langle \xi_s, \xi_t \rangle = 0 \text{ when } (s,t) \notin \Delta_{E'} \text{ for a finite set } E' \subset \Gamma.$$

It is also easily seen that in Proposition 3.9 we may limit the study to real valued functions and real Hilbert spaces.

Definition 3.11. A *length function* ℓ on a discrete group Γ is a function $\ell : \Gamma \to \mathbb{N}$ such that $\ell(s) = 0$ if and only if $s = e$, $\ell(st) \leq \ell(s) + \ell(t)$ and $\ell(s) = \ell(s^{-1})$ for all $s, t \in \Gamma$. We say that the length function ℓ is proper if, in addition, $\lim_{s\to\infty} \ell(s) = +\infty$.

On every discrete group Γ there is a proper length function ℓ (see [64, Lemma 2.1] for instance). When Γ is finitely generated, we may choose word length functions. This provides a left invariant metric d_ℓ on Γ by $d_\ell(s,t) = \ell(s^{-1}t)$. From Proposition 3.9 we deduce that the exactness of Γ only depends on the metric d_ℓ and not on the group structure, since the equivalent properties (ii), (iii), (iv) may obviously be expressed in terms of d_ℓ [16]. Similarly, the Roe algebra also only depends on d_ℓ and this explains why we use the notation $C_u^*(|\Gamma|)$ instead of $C_u^*(\Gamma)$.

In fact, most of the results of this section may be transposed to the case of metric spaces.

[16] This also shows that the exactness of Γ does not depend on the choice of the proper length function.

3.4 Property (A) for metric spaces. Let (X, d) be a discrete metric space. For $r > 0$ we denote by Δ_r the strip $\{(x, y) \in X \times X, d(x, y) \leq r\}$. The C^*-algebra $C^*_u(X)$ is the completion of the $*$-algebra of operators in $\mathcal{B}(\ell^2(X))$ with bounded kernels supported in some strip Δ_r. One also defines in an obvious way the analogues of conditions (ii), (iii) and (iv) of Proposition 3.9.

We spell out below the generalization of (ii) to any discrete metric space and call it exactness.

Definition 3.12. We say that a discrete metric space (X, d) is *exact* if for every $R > 0$ and $\varepsilon > 0$, there exist a map $f : X \to \ell^1(X)^+_1$ and a number $S > 0$ such that

(a) $\|f_x - f_y\|_1 \leq \varepsilon$ whenever $d(x, y) \leq R$;

(b) $\mathrm{supp}(f_x) \subset B(x, S)$[17] for every $x \in X$.

A slightly stronger notion (expressed by a Følner type condition) was introduced by Yu [69] under the name of property (A). For discrete metric spaces with bounded geometry[18], Yu's property (A) was proved by Higson and Roe [39] to be equivalent to exactness just defined above. This is why exact groups are also named groups with property (A).

Proposition 3.13. *Let (X, d) be a discrete metric space with bounded geometry. The following conditions are equivalent :*

(i) *(X, d) is exact.*

(ii) *For every $\varepsilon > 0$ and $R > 0$, there exist a function $f : X \to \ell^2(X)_1$ (or from X into the unit sphere $\mathcal{H}_1$ of some complex (or real) Hilbert space $\mathcal{H}$) and a number $S > 0$ such that*

  (a) *$\|f_x - f_y\|_1 \leq \varepsilon$ whenever $d(x, y) \leq R$;*

  (b) *$\mathrm{supp}(f_x) \subset B(x, S)$ for every $x \in X$.*

(iii) *For every $\varepsilon > 0$ and $R > 0$, there exist a Hilbert space $\mathcal{H}$, a function $\xi : X \to \mathcal{H}_1$ and $S > 0$ such that*

  (a) *$\|\xi_x - \xi_y\| \leq \varepsilon$ whenever $d(x, y) \leq R$;*

  (b) *$\langle \xi_x, \xi_y \rangle = 0$ whenever $d(x, y) \geq S$.*

(iv) *For every $\varepsilon > 0$ and $R > 0$, there exist a positive definite function h on $X \times X$ and $S > 0$ such that*

  (a) *$|1 - h(x, y)| \leq \varepsilon$ whenever $d(x, y) \leq R$;*

[17] $B(x, S)$ is the ball of centre x and radius S.

[18] This means that for every $r > 0$ there exists N such that every ball of radius r has at most N elements.This condition is fulfilled for the metric associated with any proper length function on a discrete group.

(b) $h(x,y) = 0$ *whenever* $d(x,y) \geq S$.

(v) *The* C^**-algebra* $C^*_u(X)$ *is nuclear.*

Proof. See [64] for the equivalence between (i), (ii), (iii) and (iv). The equivalence of (i) with (v) is proved in [60]. □

4 Exactness and uniform embeddability

Let $\Gamma = \mathbb{F}_r$ be the free group with $r \geq 2$ generators, A its set of free generators and ℓ the corresponding word length function. Recall that the *Cayley graph* is the graph (Γ, E) where the set of edges is defined by

$$E = \{(s, sa), s \in \Gamma, a \in A \cup A^{-1}\}.$$

If $w = a_1 a_2 \cdots a_n$, we set $w_0 = e$ (the unit of $\mathbb{F}_r$), $w_k = a_1 a_2 \cdots a_k$, and we denote by $e_k(w) = (w_{k-1}, w_k)$ the k-th edge of w, $k = 1, \ldots n$.

Let ξ be the map from Γ into $\ell^2(E)$ defined by $\xi_e = 0$ and, if w is a word of length n,

$$\xi_w = \sum_{k=1}^{n} \delta_{e_k(w)}.$$

Given two elements w, w' of Γ, an easy computation shows that

$$d_\ell(w, w') = \ell(w^{-1}w') = \|\xi_w - \xi_{w'}\|^2.$$

This means that free groups can be drawn in a Hilbert space without too much distorsion. This is a particular case of the notion uniform embedding we shall study now.

4.1 Uniform embeddability.

Definition 4.1 (Gromov, [31], §7.E). Let (X, d_X), (Y, d_Y) be two metric spaces. A map $f : X \to Y$ is said to be a *uniform embedding*[19] if there exist two non-decreasing functions ρ_1, ρ_2 from $\mathbb{R}^+$ into $\mathbb{R}$ such that

(1) for all $x, y \in X$,

$$\rho_1(d_X(x,y)) \leq d_Y(f(x), f(y)) \leq \rho_2(d_X(x,y)); \tag{2}$$

(2) $\lim_{r \to +\infty} \rho_i(r) = +\infty$, $i = 1, 2$.

Definition 4.2. We say that the metric space (X, d) is *uniformly embeddable*[20] if there exists an uniform embedding in the metric space underlying some Hilbert space.

[19] The term *coarse embedding* is also widely used.

[20] implicitely, in a Hilbert space

We shall give characterizations of uniform embeddability similar to those of exactness stated in Proposition 3.13.

Theorem 4.3 ([22]). *Let (X, d) be a metric space. The following conditions are equivalent :*

(i) *(X, d) is uniformly embeddable.*

(ii) *For every $\varepsilon > 0$ and $R > 0$, there exist a Hilbert space $\mathcal{H}$ and a function $\xi : X \to \mathcal{H}_1$ such that*

  (a) *$\|\xi_x - \xi_y\| \leq \varepsilon$ whenever $d(x, y) \leq R$;*

  (b) *$\lim_{r\to+\infty} \sup\{|\langle \xi_x, \xi_y\rangle|, d(x, y) \geq r\} = 0$.*

(iii) *For every $\varepsilon > 0$ and $R > 0$, there exist a positive definite function h on $X \times X$ such that $h(x, x) = 1$ for every $x \in X$ and*

  (a) *$|1 - h(x, y)| \leq \varepsilon$ whenever $d(x, y) \leq R$;*

  (b) *$\lim_{r\to+\infty} \sup\{|h(x, y|, d(x, y) \geq r\} = 0$.*

Proof. Obviously, we may assume that the Hilbert spaces we work with are real Hilbert spaces and that h in (iii) is a real valued function, as well. Since h may be written $h(x, y) = \langle \xi_x, \xi_y\rangle$ where ξ is a map from X into the unit sphere of a real Hilbert space $\mathcal{H}$, the equivalence between (ii) and (iii) follows from the equality $\|\xi_x - \xi_y\|^2 = 2(1 - h(x, y))$.

Let us assume that (i) holds. Let $f : X \to \mathcal{H}$ be a uniform embedding and let ρ_1, ρ_2 be two non-decreasing real valued functions on $[0, +\infty[$ such that $\lim_{r\to+\infty} \rho_i(r) = +\infty$, $i = 1, 2$, and

$$\rho_1(d_X(x, y)) \leq \|f(x) - f(y)\| \leq \rho_2(d_X(x, y))$$

for all $x, y \in X$. Given $t > 0$ we set $h_t(x, y) = \exp(-t\|f(x) - f(y)\|^2)$. This gives a positive definite kernel with $h_t(x, x) = 1$ for all $x \in X$. Morever for $d(x, y) \leq R$ we have

$$|1 - h_t(x, y)| \leq |1 - \exp(-t\rho_2^2(R))| \leq \varepsilon$$

for t small enough. Let us choose such a t. If $d(x, y) \geq r$, we have

$$h_t(x, y) \leq \exp(-t\rho_1^2(r)),$$

and since $\lim_{r\to+\infty} \rho_1(r) = +\infty$, we see that h_t satisfies condition (b) of (iii).

Finally, let us prove that (ii) implies (i). Assuming that (ii) holds, there exists a sequence $(\xi_n)_{n\geq1}$ of maps ξ_n from X into the unit sphere of a Hilbert space $\mathcal{H}_n$ and a sequence $(r_n)_{n\geq1}$ of positive real numbers such that, for $n \geq 1$,

(a) $\|\xi_n(x) - \xi_n(y)\| \leq 1/2^n$ whenever $d(x, y) \leq \sqrt{n}$;

(b) $\|\xi_n(x) - \xi_n(y)\|^2 = 2(1 - \langle \xi_n(x), \xi_n(y)\rangle \geq 1$ whenever $d(x, y) \geq r_n$.

We set $r_0 = 0$. We may assume thet the sequence (r_n) is strictly increasing with $\lim_{n\to\infty} r_n = +\infty$.

We chose a base point $x_0 \in X$ and define $f : X \to \bigoplus_{n=1}^{\infty} \mathcal{H}_n$ (hilbertian direct sum) by

$$f(x) = \bigoplus_{n=1}^{\infty} \big(\xi_n(x) - \xi_n(x_0)\big).$$

Obviously this map is well defined. Moreover for $x, y \in X$, if n is such that $\sqrt{n-1} \leq d(x,y) < \sqrt{n}$, we have

$$\begin{aligned}\|f(x) - f(y)\|^2 &= \sum_{k\leq n-1} \|\xi_k(x) - \xi_k(y)\|^2 + \sum_{k\geq n} \|\xi_k(x) - \xi_k(y)\|^2 \\ &\leq 4(n-1) + \sum_{k\geq n} \frac{1}{2^{2n}} \leq 4d(x,y)^2 + 1.\end{aligned}$$

Therefore we have

$$\forall x, y \in X, \quad \|f(x) - f(y)\| \leq 2d(x,y) + 1.$$

On the other hand, if $r_n \leq d(x,y)$, we have

$$\|f(x) - f(y)\|^2 \geq \sum_{k\leq n} \|\xi_k(x) - \xi_k(y)\|^2 \geq n.$$

We define ρ_1 on $[0, +\infty[$ by $\rho_1(t) = \sqrt{n}$ when $t \in [r_n, r_{n+1}[$. Then we get $\|f(x) - f(y)\| \geq \rho_1(d(x,y))$ for every $x, y \in X$, and therefore

$$\forall x, y \in X, \quad \rho_1(d(x,y)) \leq \|f(x) - f(y)\| \leq 2d(x,y) + 1. \tag{3}$$

□

Corollary 4.4 (Yu, [69]). *Every exact discrete metric space is uniformly embeddable.*

Proof. This follows immediately from Proposition 3.13 and Theorem 4.3 .Indeed, bounded geometry is not needed for the proof of (i) ⇒ (iii), in Proposition 3.13. □

Remark 4.5. Using sequences of expanding graphs, Gromov has obtained [32, 33] explicit examples of metric spaces with bounded geometry which are not uniformly embeddable into Hilbert spaces. P. Nowak [51] has constructed a locally finite metric space (i.e. every ball is finite) which is uniformly embeddable into a Hilbert space without being exact.

Let us come back to the case of a discrete countable group Γ.

Definition 4.6. We say that Γ is *uniformly embeddable* (in a Hilbert space) if the metric space $|\Gamma|$ associated with any of its proper length functions is uniformly embeddable.

Remark 4.7. The definition does not depend on the choice of the proper length function, because (for instance) it can be expressed as follows : Γ is uniformly embeddable if and only if for every $\varepsilon > 0$ and any finite subset $E \subset \Gamma$ there is a positive definite kernel h on Γ such that $h(s,s) = 1$ for all $s \in \Gamma$ and

(a) $|1 - h(s,t)| \leq \varepsilon$ whenever $s^{-1}t \in E$;

(b) given $\eta > 0$ there exists a finite subset $E' \subset \Gamma$ with $|h(s,t)| \leq \eta$ when $s^{-1}t \notin E'$.

Proposition 4.8. *Exact groups and groups with the Haagerup property are uniformly embedable.*

This follows immediately from the definitions and Theorem 4.3.

As already said, Gromov has shown in [32, 33] the existence of finitely generated groups that are not uniformly embeddable. They are random groups whose Cayley graphs "quasi" contain some infinite family of expanders.

The class of uniformly embeddable countable groups is closed under subgroups, products, free products with amalgams, and extensions by exact groups [**?**, 22].

Open problems:

- Is any extension of uniformly embeddable groups still uniformly embeddable ?

- Are there uniformly embeddable groups which are not exact ? The Thompson group is known to be uniformly embeddable, but it is still open whether it is exact or not.

4.2 Compression functions; compression constants. To get a better understanding of uniform embeddability, we shall look more closely to the functions ρ_1 and ρ_2 occuring in definition 4.1.

We are only interested in the asymptotic properties of these functions and we shall use the following notation : given two non-decreasing functions f and g on $\mathbb{R}^+$ we set $f \preceq g$ if there exist $c > 0$ such that $f(t) \leq cg(t) + c$ for all $t > 0$.

We have seen in the proof of theorem 4.3 that a uniform embeddable metric space always has a uniform embedding with $\rho_2(t) \preceq t$. Morever when (X, d_X) is a quasi-geodesic metric space[21], then for any uniform embedding $f : X \to Y$ in a metric space (Y, d_Y), there exists a constant $c > 0$ such that for all $x, y \in X$,

$$d_Y(f(x), f(y)) \leq c d_X(x,y) + c$$

(see [35, Prop. 2.9] for a proof of this observation due to Gromov). For instance, let us consider the case of a finitely generated group Γ equipped with a word length

[21] that is, there exist $\delta > 0$ and $\lambda \geq 1$ such that for any $x, y \in X$, there exist $x_0 = x, x_1, \ldots, x_n = y$ with $\sum_{i=0}^{n-1} d(x_i, x_{i+1}.) \leq \lambda d(x,y)$ and $d(x_i, x_{i+1}) \leq \delta$ for $i = 0, \ldots, n-1$.

function ℓ. Let $f : X = \Gamma \to (Y, d_Y)$ be a uniform embedding with ρ_2 as above. Let $s, t \in \Gamma$ be such that $\ell(s^{-1}t) = n$ and choose a geodesic path $x_0 = s, x_1, \dots, x_n = t$. We have

$$d_Y(f(s), f(t)) \leq \sum_{i=0}^{n-1} d_Y(f(x_i), f(x_{i+1})) \leq \sum_{i=0}^{n-1} \rho_2\big(\ell(x_i^{-1}x_{i+1})\big) = \rho_2(1)\ell(s^{-1}t).$$

A non-decreasing function $f : \mathbb{R}^+ \to \mathbb{R}$ is said to be *large-scale Lipschitz* if $f(t) \preceq t$. Since in definition 4.1, we may choose ρ_2 to be large-scale Lipschitz when (X, d_X) is quasi-geodesic, the main subject of interest is the behaviour of possible choices for ρ_1.

In the following, we shall always assume that the metric spaces we consider are unbounded, to avoid uninteresting considerations.

Definition 4.9 ([31, 35]). Let $f : (X, d_X) \to (Y, d_Y)$ be a large-scale Lipschitz embedding.

(a) The *compression function* ρ_f of f is defined by

$$\rho_f(r) = \inf_{d_X(x,y) \geq r} d_Y(f(x), f(y)).$$

(b) The *asymptotic compression constant* R_f of f is defined by

$$R_f = \sup\{\alpha \geq 0, t^\alpha \preceq \rho_f(t)\}.$$

Remarks 4.10. (a) Note that ρ_f is a non-decreasing function on $\mathbb{R}^+$ satisfying the first inequality of (2). Moreover if ρ_1 is another such function, then $\rho_1 \leq \rho_f$. It follows that the embedding f is uniform if and only if $\lim_{r \to +\infty} \rho_f(r) = +\infty$.

(b) Observe that $0 \leq R_f \leq 1$, and that the embedding is uniform whenever $R_f > 0$.

(c) Let $f : (X, d_X) \to (Y, d_Y)$ be a large-scale Lipschitz embedding. We say that f is a *quasi-isometry* if there exists $c > 0$ such that $c^{-1}d_X(x, y) - c \leq d_Y(f(x), f(y))$ for all $x, y \in X$ (that is ρ_1 may be chosen to be affine). In this case $R_f = 1$.

(d) The embedding of $\mathbb{F}_r$, $r \geq 2$, into a Hilbert space considered at the beginning of this section has compression constant $1/2$.

Definition 4.11. Let (X, d_X) be a metric space. The *Hilbert space compression constant* of (X, d) is

$$R(X) = \sup_f R_f$$

where f runs over all large-scale Lipschitz embeddings f into Hilbert spaces.

This number $R(X) \in [0, 1]$ is a quasi-isometry invariant of X.

Definition 4.12. Let Γ be a finitely generated group. Its *Hilbert space compression constant* $R(|\Gamma|)$ is the Hilbert space compression constant defined by any word length metric.

Example 4.13. For any $\varepsilon > 0$, a uniform embedding f of $\mathbb{F}_r$, $r \geq 2$, into a Hilbert space, such that $R_f \geq 1 - \varepsilon$, can be constructed by a modification of the embedding described at the beginning of this section (see [13] and also [35, Prop. 4.2]). It follows that $R(\mathbb{F}_r) = 1$. However, this supremum is not attained, because if it were the free group $\mathbb{F}_r$ would embed quasi-isometrically into a Hilbert space. This is not possible, due to a result of Bourgain [13] showing that the 3-regular tree does not embed quasi-isometrically into a Hilbert space.

Theorem 4.14 (Guentner-Kaminker [35]). *Let Γ be a finitely generated group equipped with the word length function defined by a symmetric set S of generators. Assume that there exists a large-scale Lipschitz uniform embedding $f : \Gamma \to \mathcal{H}$ such that $\lim_n \rho_f(n)/\sqrt{n} = +\infty$ (for instance assume that the Hilbert space compression constant $R(|\Gamma|)$ is $> 1/2$). Then Γ is exact.*

For the proof ot theorem 4.14, we shall need the following lemma.

Lemma 4.15. *Let $k : \Gamma \times \Gamma \to \mathbb{R}^+$ be a positive definite kernel such that*

$$\sup_{s\in\Gamma} \sum_{t\in\Gamma} k(s,t) < +\infty$$
$$\lim_{n\to\infty} \sup_{s\in\Gamma} \Big(\sum_{\{t, d(s,t)\geq n\}} k(s,t) \Big) = 0. \tag{4}$$

Then for every $\varepsilon > 0$, there exists a positive definite kernel h on $\Gamma \times \Gamma$ with finite propagation such that

$$\sup_{(s,t)\in\Gamma\times\Gamma} |k(s,t) - h(s,t)| \leq \varepsilon.$$

Proof. For $n \in \mathbb{N}$, we define the kernel cut-off k_n by $k_n(s,t) = k(s,t)$ if $d(s,t) < n$, and $k_n(s,t) = 0$ otherwise. Of course, for n large enough, k_n uniformly approximate k up to ε, but there is no reason why k_n should be positive definite.

We set $C_n = \sup_{s\in\Gamma} \sum_{\{t,d(s,t)\geq n\}} k(s,t)$. We leave it as an easy exercise (only using the Cauchy-Schwarz inequality) to show that for every $\xi \in \ell^2(\Gamma)$,

$$\sum_{s\in\Gamma} \Big| \sum_{\{t,d(s,t)\geq n\}} k(s,t)\xi(t) \Big|^2 \leq C_n^2 \sum_{s\in\Gamma} |\xi(s)|^2.$$

Therefore the operator $\mathrm{Op}(k - k_n)$, corresponding to the matrix $[(k - k_n)(s,t)]$ is well-defined and bounded, with $\|\mathrm{Op}(k - k_n)\| \leq C_n$. This holds in particular for $n = 0$ where $k_0 = 0$. It follows that $\mathrm{Op}(k)$ belongs to the Roe algebra $C_u^*(|\Gamma|)$ since the operators $\mathrm{Op}(k_n)$ belong to $C_u^*(|\Gamma|)$ and $\lim_{n\to\infty} \|\mathrm{Op}(k) - \mathrm{Op}(k_n)\| = 0$.

Since k is a positive definite kernel, the operator $\mathrm{Op}(k)$ is positive. Let T be its square root. Given $\eta > 0$ we choose a bounded kernel k' with finite propagation such that $\|T - \mathrm{Op}(k')\| \leq \eta$. We set $V = \mathrm{Op}(k')$ and

$$h(s,t) = \langle V\delta_s, V\delta_t \rangle.$$

Then h is a positive definite kernel with finite propagation and for $s,t\in\Gamma$,

$$\begin{aligned}|k(s,t)-h(s,t)&=|\langle(\mathrm{Op}(k)-V^*V)\delta_s,\delta_t\rangle|\\&\leq\|T^*T-V^*V\|\leq(\|T\|+\|V\|)\eta.\end{aligned}$$

The conclusion follows by choosing η small enough. □

Proof of theorem 4.14. For $p\in\mathbb{N}^*$, we set $h_p(s,t)=\exp(-(1/p)\|f(s)-f(t)\|^2)$. Then (h_p) is a sequence of positive definite bounded kernels on $\Gamma\times\Gamma$, going to 1 uniformly on the strips $\{(s,t),d(s,t)\leq R\}$ since f is a large-scale Lipschitz function. The main point is that these kernels h_p are not supported by strips. We shall use the previous lemma to show that they can be uniformly approximated by finite propagation positive definite kernels. Let us prove that condition (4) holds for h_p. For $s\in\Gamma$ and $n\geq 0$, we have

$$\begin{aligned}\sum_{\{t,d(s,t)\geq n\}}h_p(s,t)&=\sum_{m\geq n}\sum_{d(s,t)=m}h_p(s,t)\\&\leq\sum_{m\geq n}(\#S)^m\exp\big(-(1/p)\rho_f(m)^2\big).\end{aligned}$$

If we set $\rho_f(m)=\sqrt{m}g(m)$ we get

$$(\#S)^m\exp(-(1/p)\rho_f(m)^2)=\big((\#S)\exp\big((-(1/p)g(m)^2\big)^m,$$

which is the general term of a converging series, since $\lim_m g(m)=+\infty$. It follows that $\lim_{n\to\infty}\sup_{s\in\Gamma}\big(\sum_{\{t,d(s,t)\geq n\}}h_p(s,t)\big)=0$. □

Remark 4.16. Free groups [13, 35], more generally, hyperbolic groups [14], lattices in semi-simple Lie groups and co-compact lattices in all Lie groups [63] have compression constant 1. The compression constant of the wreath product $\mathbb{Z}\wr\mathbb{Z}$ was proved to be between 1/2 and 3/4 in [8], to be $\geq 2/3$ in [63, 50], and finally the exact value 2/3 was recently obtained in [9]. The compression constant of the Thompson group F is 1/2 (see [8]). Unfortunately Theorem 4.14 gives no information about the exactness of F.

Very recently, it has been proved in [7] that for every $\alpha\in[0,1]$, there exists an exact finitely generated discrete group with Hilbert space compression constant $R(|\Gamma|)=\alpha$. This remarkable result shows that the sufficient condition for exactness given in Theorem 4.14 is far from being necessary.

4.3 Equivariant case. In this section we consider a finitely generated group with a word length metric. We shall give a short survey of the equivariant analogues of exactness and uniform embeddability.

Definition 4.17. A *kernel* h on $\Gamma\times\Gamma$ is said to be *equivariant* if $h(st_1,st_2)=h(t_1,t_2)$ for all $s,t_1,t_2\in\Gamma$.

Then h is completely determined by $\varphi(s) = h(e, s)$. Note that h is a positive definite kernel if and only if φ is a positive definite function and that h has finite propagation if and only if φ has finite support.

The *equivariant analogue of exactness for* Γ *is amenability* since

- Γ *is amenable if and only if there exists a sequence* (φ_n) *of positive definite functions on* Γ*, with finite support, going to* 1 *uniformly on finite subsets (i.e. pointwise).*
- Γ *is exact if and only if there exists a sequence* (h_n) *of bounded positive definite kernels on* $\Gamma \times \Gamma$*, with finite propagation, going to* 1 *uniformly on strips* $\{(s,t) \in \Gamma \times \Gamma, s^{-1}t \in E\}$*,* E *finite.*

The equivariant analogue of *uniform embeddability for* Γ *is the Haagerup approximation property* since

- Γ *has the Haagerup approximation property if and only if there exists a sequence* (φ_n) *of positive definite functions on* Γ*, vanishing to infinity and going to* 1 *uniformly on finite subsets.*
- Γ *is uniformly embeddable if and only if there exists a sequence* (h_n) *of bounded positive definite kernels on* $\Gamma \times \Gamma$*, vanishing to zero at infinity outside strips, going to* 1 *uniformly on strips.*

Let us make this latter observation more precise.

Definition 4.18. An *equivariant uniform embedding in a Hilbert space* is a triple $(\sigma, f, \mathcal{H})$ where σ is a representation of Γ by affine isometries in the Hilbert space $\mathcal{H}$ and f is a uniform embedding such that $f(st) = \sigma(s)f(t)$ for all $s, t \in \Gamma$.

Recall that if σ is a representation of Γ by affine isometries, there exist a representation π of Γ by linear isometries and a 1-cocycle $b : \Gamma \to \mathcal{H}$ such that $\sigma(s)(\cdot) = \pi(s)(\cdot) + b(s)$ for all $s \in \Gamma$ (see [65] in these Proceedings). The cocycle property is :

$$\forall s, t \in \Gamma, \quad b(st) = b(s) + \pi(s)(b(t)).$$

Note that if we set $v = f(e)$ then $f(s) = \pi(s)v + b(s)$ for all $s \in \Gamma$. Since $\|f(s) - b(s)\| = \|v\|$, we may replace the study of the asymptotic behaviour of f by that of b. Moreover, using the cocycle property of b, we have

$$\|b(s) - b(t)\| = \left\|b(s^{-1}t\right\|, \quad \|b(s)\| \leq C\ell(s)$$

for all $s, t \in \Gamma$, where $C = \max_{t \in S} \|b(t)\|$ and S is a generating set defining the metric. Hence, the embedding b is Lipschitz. It is uniform if and only if the cocycle is proper, that is $\lim_{t \to \infty} \|b(t)\| = +\infty$. In this case, we say that the action σ on $\mathcal{H}$ is *proper*. This is equivalent to the property that for all bounded subset B of $\mathcal{H}$, the set $\{t \in \Gamma, tB \cap B \neq \emptyset\}$ is finite.

Proposition 4.19 ([2]). *A discrete group Γ has the Haagerup approximation property if and only if there exists an equivariant uniform embedding of Γ into a Hilbert space.*

This follows from the fact that conditionally negative definite functions on Γ are those of the form $t \mapsto \|b(t)\|^2$ where b is a 1-cocycle as above. For more details see [16, Th. 2.1.1].

Definition 4.20. The *equivariant Hilbert space compression constant* of Γ is

$$R_\Gamma(|\Gamma|) = \sup R_b(|\Gamma|)$$

where b runs over all cocycles associated with representations by affine isometries on Hilbert spaces.

We observe that $0 \leq R_\Gamma(|\Gamma|) \leq R(|\Gamma|) \leq 1$ and that Γ has the Haagerup property when $R_\Gamma(|\Gamma|) > 0$.

Theorem 4.21 (Guentner-Kaminker, [35]). *If the equivariant Hilbert space compression constant is $> 1/2$, then Γ is amenable.*

The compression constant $R_\Gamma(|\Gamma|)$ of the amenable group $\mathbb{Z} \wr (\mathbb{Z} \wr \mathbb{Z})$ is $\leq 1/2$ (see [8]). So far, there is no example of a finitely generated amenable group with Hilbert space compression strictly less than $1/2$.

An observation due to Gromov (see [20, Prop. 4.4]) shows that for a finitely generated amenable group, the equivariant compression constant is the same as the non-equivariant one. The equivariant compression constant of the Thompson group is $1/2$ (see [8]). It is still an open problem whether this group is amenable.

Further developments. Uniform embeddability of metric spaces (and in particular of finitely generated groups) into Hilbert spaces or more generally into Banach spaces is a very active domain of research. It has been proved in [15] that every metric space with bounded geometry is uniformly embeddable into a strictly convex reflexive Banach space B. Moreover, for a discrete group Γ, such a uniform embedding may be defined through a metrically proper affine isometric action.

On the other hand, recent results of V. Lafforgue [47] yield a family of expanders that is *not uniformly embeddable into any uniformly convex Banach space*. However, Gromov's argument allowing to pass from families of expanders to random groups do not apply in order to construct, from Lafforgues expanders, groups not uniformly embeddable into uniformly convex Banach spaces. Whether such groups exist is a crucial open problem in view of the recent result of Kasparov and Yu showing that the Novikov Conjecture holds for finitely generated discrete groups which are uniformly embeddable into some uniformly convex Banach space [46].

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