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## Introduction to Conics and Quadrics

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# MECH 576

## Computer Graphics & Geometric Modeling

June 24, 2008

### Introduction to Conics and Quadrics

## 1 Conics and Quadrics by Other Means

Second order planar curves and spatial surfaces can be defined on five and nine points, respectively. Now some other, possibly more efficient ways, will be discussed.

## 2 Conic on Two Line Pairs and a Point

A given line pair  $p$  and  $r$  intersects another given pair  $q$  and  $s$  on four points.

$$p \cap q, q \cap r, r \cap s, s \cap p$$

A fifth point  $T$  on the conic is also supplied. This is shown in Fig. 1

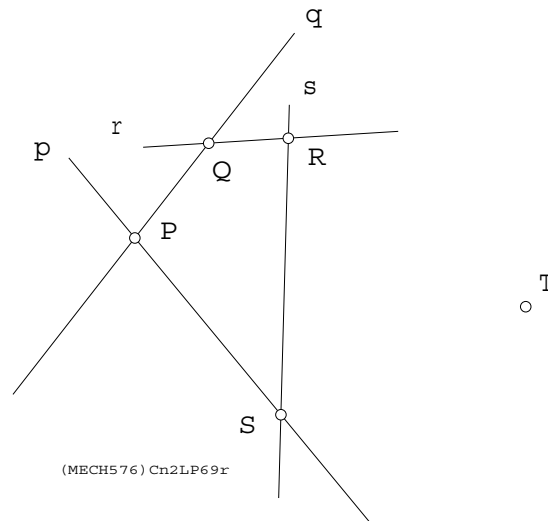


Figure 1: Conic on Line Pairs and Fifth Point

An efficient way to compute the six coefficients of the conic equation is available as the sum of the products of the first and second line pairs, individually scaled by constants  $\lambda$  and  $\mu$  so as to accommodate the fifth point, thus.

$$\lambda pr + \mu qs = 0 \tag{1}$$

The coplanar lines and the fifth point are specified by their homogeneous coordinates.

$$p\{P_0 : P_1 : P_2\}, \quad q\{Q_0 : Q_1 : Q_2\}, \quad r\{R_0 : R_1 : R_2\}, \quad s\{S_0 : S_1 : S_2\}$$

$$T\{t_0 : t_1 : t_2\}$$

Then Eq. 1 becomes

$$\lambda(P_0t_0 + P_1t_1 + P_2t_2)(R_0t_0 + R_1t_1 + R_2t_2) + \mu(Q_0t_0 + Q_1t_1 + Q_2t_2)(S_0t_0 + S_1t_1 + S_2t_2) = 0 \quad (2)$$

which defines  $\lambda$  and  $\mu$  as

$$M\lambda + L\mu = 0$$

and the conic equation can be written by expanding the following relation.

$$L(P_0x_0 + P_1x_1 + P_2x_2)(R_0x_0 + R_1x_1 + R_2x_2) - M(Q_0x_0 + Q_1x_1 + Q_2x_2)(S_0x_0 + S_1x_1 + S_2x_2) = 0 \quad (3)$$

The familiar form of the desired conic, Eq. 4, is written below.

$$\begin{aligned} & (LP_1R_1 - MQ_1S_1)x_1^2 + (LP_2R_2 - MQ_2S_2)x_2^2 \\ & + [L(P_1R_2 + P_2R_1) - M(Q_1S_2 + Q_2S_1)]x_1x_2 \\ & + [L(P_0R_1 + P_1R_0) - M(Q_0S_1 + Q_1S_0)]x_0x_1 \\ & + [L(P_0R_2 + P_2R_0) - M(Q_0S_2 + Q_2S_0)]x_0x_2 \\ & + (LP_0R_0 - MQ_0S_0)x_0^2 = 0 \end{aligned} \quad (4)$$

It is most satisfying to note that to get  $L$  and  $M$  from Eq 2 requires 22 FLOPS while the six coefficients of Eq. 4 needs only 42 more. This assumes that the four lines are given in homogeneous coordinate form. With the four points on the conic which define these line pairs or *degenerate conics*, the twelve line equation coefficients would incur  $12 \times 3 = 36$  additional FLOPS.

### 3 Quadric on Nine Points via Three Sums of Four Plane Pair Products

#### 3.1 Problem

Nine given points  $A, B, C, D, E, F, G, H, J$  define a quadric. One may obtain the ten coefficients of the homogeneous equation

$$\begin{aligned} & a_{00}x_0^2 + 2a_{01}x_0x_1 + 2a_{02}x_0x_2 + 2a_{03}x_0x_3 + a_{11}x_1^2 \\ & + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + a_{22}x_2^2 + 2a_{23}x_2x_3 + a_{33}x_3^2 = 0 \end{aligned} \quad (5)$$

where  $X\{x_0 : x_1 : x_2 : x_3\}$  represents any point on this second order surface by evaluating ten  $9 \times 9$  determinants by expanding on the top row minors of the following singular matrix whose

determinant vanishes.

$$\begin{vmatrix}
 x_0^2 & x_0x_1 & x_0x_2 & x_0x_3 & x_1^2 & x_1x_2 & x_1x_3 & x_2^2 & x_2x_3 & x_3^2 \\
 a_0^2 & a_0a_1 & a_0a_2 & a_0a_3 & a_1^2 & a_1a_2 & a_1a_3 & a_2^2 & a_2a_3 & a_3^2 \\
 b_0^2 & b_0b_1 & b_0b_2 & b_0b_3 & b_1^2 & b_1b_2 & b_1b_3 & b_2^2 & b_2b_3 & b_3^2 \\
 c_0^2 & c_0c_1 & c_0c_2 & c_0c_3 & c_1^2 & c_1c_2 & c_1c_3 & c_2^2 & c_2c_3 & c_3^2 \\
 d_0^2 & d_0d_1 & d_0d_2 & d_0d_3 & d_1^2 & d_1d_2 & d_1d_3 & d_2^2 & d_2d_3 & d_3^2 \\
 e_0^2 & e_0e_1 & e_0e_2 & e_0e_3 & e_1^2 & e_1e_2 & e_1e_3 & e_2^2 & e_2e_3 & e_3^2 \\
 f_0^2 & f_0f_1 & f_0f_2 & f_0f_3 & f_1^2 & f_1f_2 & f_1f_3 & f_2^2 & f_2f_3 & f_3^2 \\
 g_0^2 & g_0g_1 & g_0g_2 & g_0g_3 & g_1^2 & g_1g_2 & g_1g_3 & g_2^2 & g_2g_3 & g_3^2 \\
 h_0^2 & h_0h_1 & h_0h_2 & h_0h_3 & h_1^2 & h_1h_2 & h_1h_3 & h_2^2 & h_2h_3 & h_3^2 \\
 j_0^2 & j_0j_1 & j_0j_2 & j_0j_3 & j_1^2 & j_1j_2 & j_1j_3 & j_2^2 & j_2j_3 & j_3^2
 \end{vmatrix} = 0 \tag{6}$$

On the other hand a more computationally efficient way may be formulated by using an octahedron with vertices on six of the points, say,  $A, B, C, D, E, F$  and forming the sum of four binary products of four pairs of plane equations of plane pairs, none of which intersect on one of the twelve octahedral edges because each plane pair must span all six points. The four constant coefficients, each multiplying one of the products, are evaluated with three homogeneous linear equations provided by the three remaining points  $G, H, J$ . This involves evaluating only four  $3 \times 3$  determinants after the homogeneous coordinates of the eight planes are obtained with 32 such determinants.

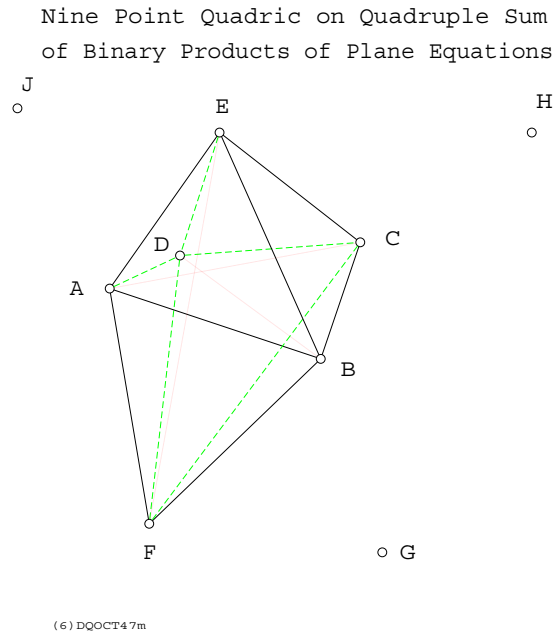


Figure 2: Quadric on Plane Pairs and Three Other Points

## 3.2 Formulation

Given six points there are ten distinct plane pairs that span all six because there are  ${}_6C_3 = 20$  distinct planes in all. Any four planes and their partners, together with the three remaining points, may be used to generate the ten coefficients of the quadric equations. Referring to Fig. 2 let us use the following associations of points into plane pairs.

$$\begin{aligned} ABC &\leftrightarrow DEF, & p &\leftrightarrow q \\ BCD &\leftrightarrow AEF, & r &\leftrightarrow s \\ CDA &\leftrightarrow BEF, & t &\leftrightarrow u \\ DAB &\leftrightarrow CEF, & v &\leftrightarrow w \end{aligned}$$

A typical point or plane is specified by its four homogeneous coordinates, thus.

$$A\{a_0 : a_1 : a_2 : a_3\}, \quad ABC \equiv p\{P_0 : P_1 : P_2 : P_3\}$$

The equation sought is

$$\alpha p'q' + \beta r's' + \gamma t'u' + \delta v'w' = 0 \quad (7)$$

where  $p'$ , for example, is the plane equation

$$P_0x_0 + P_1x_1 + P_2x_2 + P_3x_3 = 0$$

### 3.2.1 Sum of Binary Plane Products

Now Eq. 7 must be satisfied by the remaining three points  $G, H, J$ . This provides three linear equations in  $\alpha, \beta, \gamma, \delta$ . These equations can be solved homogeneously to obtain the four coefficients. The constraint equations are written below.

$$\begin{bmatrix} \mathbf{P} \cdot \mathbf{gQ} \cdot \mathbf{g} & \mathbf{R} \cdot \mathbf{gS} \cdot \mathbf{g} & \mathbf{T} \cdot \mathbf{gU} \cdot \mathbf{g} & \mathbf{V} \cdot \mathbf{gW} \cdot \mathbf{g} \\ \mathbf{P} \cdot \mathbf{hQ} \cdot \mathbf{h} & \mathbf{R} \cdot \mathbf{hS} \cdot \mathbf{h} & \mathbf{T} \cdot \mathbf{hU} \cdot \mathbf{h} & \mathbf{V} \cdot \mathbf{hW} \cdot \mathbf{h} \\ \mathbf{P} \cdot \mathbf{jQ} \cdot \mathbf{j} & \mathbf{R} \cdot \mathbf{jS} \cdot \mathbf{j} & \mathbf{T} \cdot \mathbf{jU} \cdot \mathbf{j} & \mathbf{V} \cdot \mathbf{jW} \cdot \mathbf{j} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (8)$$

Note that the boldface symbols in the matrix represent elements formed by products of two inner vector products of plane and point coordinate vectors, thus.

$$\mathbf{P} \cdot \mathbf{gQ} \cdot \mathbf{g} = \left( [P_0 \ P_1 \ P_2 \ P_3] \begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{bmatrix} \right) \left( [Q_0 \ Q_1 \ Q_2 \ Q_3] \begin{bmatrix} g_0 \\ g_1 \\ g_2 \\ g_3 \end{bmatrix} \right) \quad (9)$$

## 3.3 Solution

Consider the elements of the symmetric  $4 \times 4$  quadric coefficient matrix which expresses a general second order surface as

$$[x_0 \ x_1 \ x_2 \ x_3] \begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} \\ a_{01} & a_{11} & a_{12} & a_{13} \\ a_{02} & a_{12} & a_{22} & a_{23} \\ a_{03} & a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

These are computed, once  $\alpha, \beta, \gamma, \delta$  are known, with a rearrangement of Eq. 7.

$$\begin{aligned}
a_{00} &= P_0Q_0\alpha + R_0S_0\beta + T_0U_0\gamma + V_0W_0\delta \\
a_{01} &= [(P_0Q_1 + P_1Q_0)\alpha + (R_0S_1 + R_1S_0)\beta + (T_0U_1 + T_1U_0)\gamma + (V_0W_1 + V_1W_0)\delta]/2 \\
a_{02} &= [(P_0Q_2 + P_2Q_0)\alpha + (R_0S_2 + R_2S_0)\beta + (T_0U_2 + T_2U_0)\gamma + (V_0W_2 + V_2W_0)\delta]/2 \\
a_{03} &= [(P_0Q_3 + P_3Q_0)\alpha + (R_0S_3 + R_3S_0)\beta + (T_0U_3 + T_3U_0)\gamma + (V_0W_3 + V_3W_0)\delta]/2 \\
a_{11} &= P_1Q_1\alpha + R_1S_1\beta + T_1U_1\gamma + V_1W_1\delta \\
a_{12} &= [(P_1Q_2 + P_2Q_1)\alpha + (R_1S_2 + R_2S_1)\beta + (T_1U_2 + T_2U_1)\gamma + (V_1W_2 + V_2W_1)\delta]/2 \\
a_{13} &= [(P_1Q_3 + P_3Q_1)\alpha + (R_1S_3 + R_3S_1)\beta + (T_1U_3 + T_3U_1)\gamma + (V_1W_3 + V_3W_1)\delta]/2 \\
a_{22} &= P_2Q_2\alpha + R_2S_2\beta + T_2U_2\gamma + V_2W_2\delta \\
a_{23} &= [(P_2Q_3 + P_3Q_2)\alpha + (R_2S_3 + R_3S_2)\beta + (T_2U_3 + T_3U_2)\gamma + (V_2W_3 + V_3W_2)\delta]/2 \\
a_{33} &= P_3Q_3\alpha + R_3S_3\beta + T_3U_3\gamma + V_3W_3\delta \tag{10}
\end{aligned}$$

## 4 Hyperboloid of One Sheet

Consider the formulation of the implicit *point* equation  $f(s_1, s_2, s_3) = 0$  of a hyperboloid of one sheet which is defined as a ruled surface swept by a radial line

$$\mathcal{S}_r\{s_{01} : s_{01} : s_{01} : s_{01} : s_{01} : s_{01}\}$$

moving so as to remain in intersection on three, given by their axial coordinates, as expressed below.

$$\begin{aligned}
P_{01}s_{01} + P_{02}s_{02} + P_{03}s_{03} + P_{23}s_{23} + P_{31}s_{31} + P_{12}s_{12} &= 0 \\
Q_{01}s_{01} + Q_{02}s_{02} + Q_{03}s_{03} + Q_{23}s_{23} + Q_{31}s_{31} + Q_{12}s_{12} &= 0 \\
R_{01}s_{01} + R_{02}s_{02} + R_{03}s_{03} + R_{23}s_{23} + R_{31}s_{31} + R_{12}s_{12} &= 0 \tag{11}
\end{aligned}$$

These three lines and a typical line in the other regulus of a hyperboloid of one sheet is shown in Fig. 3. A one parameter quadric line equation can be written immediately by eliminating  $s_{23}, s_{31}, s_{12}$  from the three equations Eq. 11 and the Plücker condition. After simplification, this is

$$\begin{aligned}
&[(R_{31}Q_{12} - R_{12}Q_{31})s_{01} + (R_{12}Q_{23} - R_{23}Q_{12})s_{02} \\
&+ (R_{23}Q_{31} - R_{31}Q_{23})s_{03}](P_{01}s_{01} + P_{02}s_{02} + P_{03}s_{03}) \\
&[(P_{31}R_{12} - P_{12}R_{31})s_{01} + (P_{12}R_{23} - P_{23}R_{12})s_{02} \\
&+ (P_{23}R_{31} - P_{31}R_{23})s_{03}](Q_{01}s_{01} + Q_{02}s_{02} + Q_{03}s_{03}) \\
&[(Q_{31}P_{12} - Q_{12}P_{31})s_{01} + (Q_{12}P_{23} - Q_{23}P_{12})s_{02} \\
&+ (Q_{23}P_{31} - Q_{31}P_{23})s_{03}](R_{01}s_{01} + R_{02}s_{02} + R_{03}s_{03}) = 0 \tag{12}
\end{aligned}$$

Eq. 12 can be used to generate one regulus of a general hyperboloid of one sheet, a hyperboloid of revolution or a hyperbolic paraboloid by choosing the given lines appropriately. Eq. 12 may not seem to be bivariate but recall the Plücker coordinates are homogeneous so one may parameterize on a ratio, say  $s_{01} : s_{02}$ , and the resulting third direction number,  $s_{03}$ , will completely specify

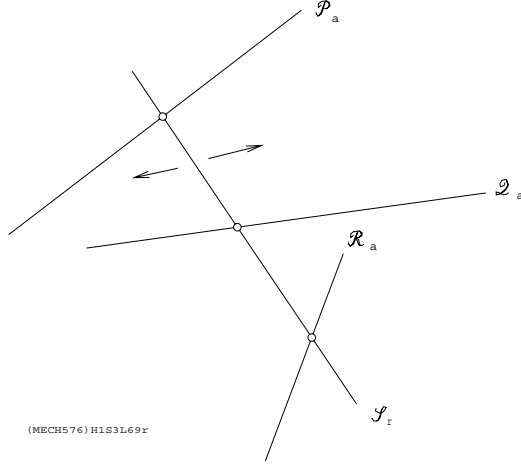


Figure 3: A Line Intersecting Three Given Ones

the direction of the ruling line selected by the ratio variable parameter. Furthermore, since one generally seeks line rulings in Euclidean space, the elimination of  $s_{23}$ ,  $s_{31}$ ,  $s_{12}$  is reasonable because  $s_{01}^2 + s_{02}^2 + s_{03}^2 \neq 0$ .

To get the point equation one uses  $S \in \mathcal{S}_r$  like in the cone of revolution formulation. The two middle equations and the Plücker condition are used to yield the following three substitution relations.

$$\begin{aligned} s_0 s_{23} &= s_2 s_{03} - s_3 s_{02}, & s_3 s_{31} &= s_3 s_{01} - s_1 s_{03} \\ s_0 s_3 s_{12} &= -(s_2 s_{03} - s_3 s_{02}) s_{01} - (s_3 s_{01} - s_1 s_{03}) s_{02} \end{aligned}$$

These are substituted into Eq. 11 which reduce, after simplification, to

$$\begin{aligned} s_0(P_{01}s_{01} + P_{02}s_{02} + P_{03}s_{03})s_{03} + P_{23}(s_2s_{03} - s_3s_{02})s_{03} \\ + P_{31}(s_3s_{01} - s_1s_{03})s_{03} + P_{12}(s_1s_{02} - s_2s_{01})s_{03} &= 0 \\ s_0(Q_{01}s_{01} + Q_{02}s_{02} + Q_{03}s_{03})s_{03} + Q_{23}(s_2s_{03} - s_3s_{02})s_{03} \\ + Q_{31}(s_3s_{01} - s_1s_{03})s_{03} + Q_{12}(s_1s_{02} - s_2s_{01})s_{03} &= 0 \\ s_0(R_{01}s_{01} + R_{02}s_{02} + R_{03}s_{03})s_{03} + R_{23}(s_2s_{03} - s_3s_{02})s_{03} \\ + R_{31}(s_3s_{01} - s_1s_{03})s_{03} + R_{12}(s_1s_{02} - s_2s_{01})s_{03} &= 0 \end{aligned}$$

which further simplifies to

$$\begin{aligned} (P_{01}s_0 - P_{12}s_2 + P_{31}s_3)s_{01} + (P_{02}s_0 - P_{23}s_3 + P_{12}s_1)s_{02} \\ + (P_{03}s_0 - P_{31}s_1 + P_{23}s_2)s_{03} = 0 \end{aligned} \tag{13}$$

$$\begin{aligned} (Q_{01}s_0 - Q_{12}s_2 + Q_{31}s_3)s_{01} + (Q_{02}s_0 - Q_{23}s_3 + Q_{12}s_1)s_{02} \\ + (Q_{03}s_0 - Q_{31}s_1 + Q_{23}s_2)s_{03} = 0 \end{aligned} \tag{14}$$

$$\begin{aligned} (R_{01}s_0 - R_{12}s_2 + R_{31}s_3)s_{01} + (R_{02}s_0 - R_{23}s_3 + R_{12}s_1)s_{02} \\ + (R_{03}s_0 - R_{31}s_1 + R_{23}s_2)s_{03} = 0 \end{aligned} \tag{15}$$

Solving Eqs. 13 and 14 above for  $s_{02}$  and  $s_{03}$

$$D = \begin{vmatrix} (P_{02}s_0 - P_{23}s_3 + P_{12}s_1) & (P_{03}s_0 - P_{31}s_1 + P_{23}s_2) \\ (Q_{02}s_0 - Q_{23}s_3 + Q_{12}s_1) & (Q_{03}s_0 - Q_{31}s_1 + Q_{23}s_2) \end{vmatrix}$$

$$N_{02} = \begin{vmatrix} -(P_{01}s_0 - P_{12}s_2 + P_{31}s_3)s_{01} & (P_{03}s_0 - P_{31}s_1 + P_{23}s_2) \\ -(Q_{01}s_0 - Q_{12}s_2 + Q_{31}s_3)s_{01} & (Q_{03}s_0 - Q_{31}s_1 + Q_{23}s_2) \end{vmatrix}$$

$$N_{03} = \begin{vmatrix} (P_{02}s_0 - P_{23}s_3 + P_{12}s_1) & -(P_{01}s_0 - P_{12}s_2 + P_{31}s_3)s_{01} \\ (Q_{02}s_0 - Q_{23}s_3 + Q_{12}s_1) & -(Q_{01}s_0 - Q_{12}s_2 + Q_{31}s_3)s_{01} \end{vmatrix}$$

and substituting the sum-of-products from the three Cramer determinants above into Eq. 15 produces a point equation from which the last remaining line coordinate  $s_{01}$  can be factored out.

$$\begin{aligned} & (R_{01}s_0 - R_{12}s_2 + R_{31}s_3) [(P_{02}s_0 - P_{23}s_3 + P_{12}s_1)(Q_{03}s_0 - Q_{31}s_1 + Q_{23}s_2) \\ & \quad - (Q_{02}s_0 - Q_{23}s_3 + Q_{12}s_1)(P_{03}s_0 - P_{31}s_1 + P_{23}s_2)] s_{01} \\ & + (R_{02}s_0 - R_{23}s_3 + R_{12}s_1) [(P_{03}s_0 - P_{31}s_1 + P_{23}s_2)(Q_{01}s_0 - Q_{12}s_2 + Q_{31}s_3) \\ & \quad - (Q_{03}s_0 - Q_{31}s_1 + Q_{23}s_2)(P_{01}s_0 - P_{12}s_2 + P_{31}s_3)] s_{01} \\ & + (R_{03}s_0 - R_{31}s_1 + R_{23}s_2) [(P_{01}s_0 - P_{12}s_2 + P_{31}s_3)(Q_{02}s_0 - Q_{23}s_3 + Q_{12}s_1) \\ & \quad - (Q_{01}s_0 - Q_{12}s_2 + Q_{31}s_3)(P_{02}s_0 - P_{23}s_3 + P_{12}s_1)] s_{01} = 0 \end{aligned}$$

The final step, collecting on coefficients of the triple point coordinate variable product

$$s_i^l s_j^m s_k^n, \quad i + j + k = l + m + n = 3$$

yields

$$\begin{aligned} & \{ [P_{01}(Q_{31}R_{12} - Q_{12}R_{31}) + Q_{01}(R_{31}P_{12} - R_{12}P_{31}) \\ & \quad + R_{01}(P_{31}Q_{12} - P_{12}Q_{31})] s_1^2 \\ & + [P_{02}(Q_{12}R_{23} - Q_{23}R_{12}) + Q_{02}(R_{12}P_{23} - R_{23}P_{12}) \\ & \quad + R_{02}(P_{12}Q_{23} - P_{23}Q_{12})] s_2^2 \\ & + [P_{03}(Q_{23}R_{31} - Q_{31}R_{23}) + Q_{03}(R_{23}P_{31} - R_{31}P_{23}) \\ & \quad + R_{03}(P_{23}Q_{31} - P_{31}Q_{23})] s_3^2 \\ & + [P_{23}(Q_{31}R_{02} - Q_{02}R_{31} + R_{12}Q_{03} - R_{03}Q_{12}) \\ & \quad + Q_{23}(R_{31}P_{02} - R_{02}P_{31} + P_{12}R_{03} - P_{03}R_{12}) \\ & + R_{23}(P_{31}Q_{02} - P_{02}Q_{31} + Q_{12}P_{03} - Q_{03}P_{12})] s_2 s_3 \\ & + [P_{31}(Q_{12}R_{03} - Q_{03}R_{12} + R_{23}Q_{01} - R_{01}Q_{23}) \\ & \quad + Q_{31}(R_{12}P_{03} - R_{03}P_{12} + P_{23}R_{01} - P_{01}R_{23}) \\ & + R_{31}(P_{12}Q_{03} - P_{03}Q_{12} + Q_{23}P_{01} - Q_{01}P_{23})] s_3 s_1 \\ & + [P_{12}(Q_{23}R_{01} - Q_{01}R_{23} + R_{31}Q_{02} - R_{02}Q_{31}) \\ & \quad + Q_{12}(R_{23}P_{01} - R_{01}P_{23} + P_{31}R_{02} - P_{02}R_{31}) \\ & + R_{12}(P_{23}Q_{01} - P_{01}Q_{23} + Q_{31}P_{02} - Q_{02}P_{31})] s_1 s_2 \} s_0 \\ & + \{ [P_{01}(Q_{31}R_{02} - Q_{02}R_{31} - R_{12}Q_{03} + R_{03}Q_{12}) \end{aligned}$$



$$\begin{aligned}
& +Q_{01}(R_{31}P_{02} - R_{02}P_{31} - P_{12}R_{03} + P_{03}R_{12}) \\
& +R_{01}(P_{31}Q_{02} - P_{02}Q_{31} - Q_{12}P_{03} + Q_{03}P_{12})] s_1 \\
& + [P_{02}(Q_{12}R_{03} - Q_{03}R_{12} - R_{23}Q_{01} + R_{01}Q_{23}) \\
& \quad +Q_{02}(R_{12}P_{03} - R_{03}P_{12} - P_{23}R_{01} + P_{01}R_{23}) \\
& +R_{02}(P_{12}Q_{03} - P_{03}Q_{12} - Q_{23}P_{01} + Q_{01}P_{23})] s_2 \\
& + [P_{03}(Q_{23}R_{01} - Q_{01}R_{23} - R_{31}Q_{02} + R_{02}Q_{31}) \\
& \quad +Q_{03}(R_{23}P_{01} - R_{01}P_{23} - P_{31}R_{02} + P_{02}R_{31}) \\
& +R_{03}(P_{23}Q_{01} - P_{01}Q_{23} - Q_{31}P_{02} + Q_{02}P_{31})] s_3 \} s_0^2 \\
& + [P_{01}(Q_{02}R_{03} - Q_{03}R_{02}) + P_{02}(Q_{03}R_{01} - Q_{01}R_{03}) \\
& \quad +P_{03}(Q_{01}R_{02} - Q_{02}R_{01})] s_0^3 = 0
\end{aligned} \tag{16}$$

Producing the ten numerical coefficients of Eq. 16 to derive the implicit equation of a desired surface requires the specification of the three given lines  $\mathcal{Q}, \mathcal{R}, \mathcal{S}$ . This can be done quite easily if one thinks “geometrically”. Imagine the following six points.

1. The first point locates the desired centre of symmetry of the desired ruled surface.
2. The second specifies another point on the axis.
3. The third is on the surface, on a principal axis of the minimal elliptical section, on the first point, perpendicular to the axis.
4. The fourth is another on a generator, on the third. Now the first line in the regulus to be ruled by moving  $\mathcal{S}$  is defined by the third and fourth points.
5. The fifth is on the surface, on the *other* principal axis of the minimal elliptical section, on the first point, perpendicular to the axis.
6. The sixth is another on a generator, on the fifth. Now the second line in the regulus to be ruled by moving  $\mathcal{S}$  is defined by the fifth and sixth points.

The third line is produced by perpendicular reflection, on the axis, of the first. A hyperbolic paraboloid is specified differently. One chooses two lines to be ruled and a third line which is in the opposite regulus, connecting the first two. Let us assume this line is

$$\mathcal{R}'_r \{r'_{01} : r'_{02} : r'_{03} : r'_{23} : r'_{31} : r'_{12}\}$$

Then the third specification line  $\mathcal{R}$  becomes a special line at infinity whose moment is the direction of the connecting line.

$$\mathcal{R}_r \{r_{01} : r_{02} : r_{03} : r_{23} : r_{31} : r_{12}\} \equiv \{0 : 0 : 0 : r'_{01} : r'_{02} : r'_{03}\}$$

In all events, do not forget that the three given lines must be specified as *axial*.

# 5 Conic on Four or Three Given Points and One or Two Given Tangent Lines

Consider the ellipse shown in Fig. 4. Although it is in standard form to simplify any computation one may wish to undertake in order to verify any concepts introduced below note that there is no loss of generality on this account. The ellipse is shown with five points on it and each supports a tangent line,  $p, q, r, s, t$ , respectively.

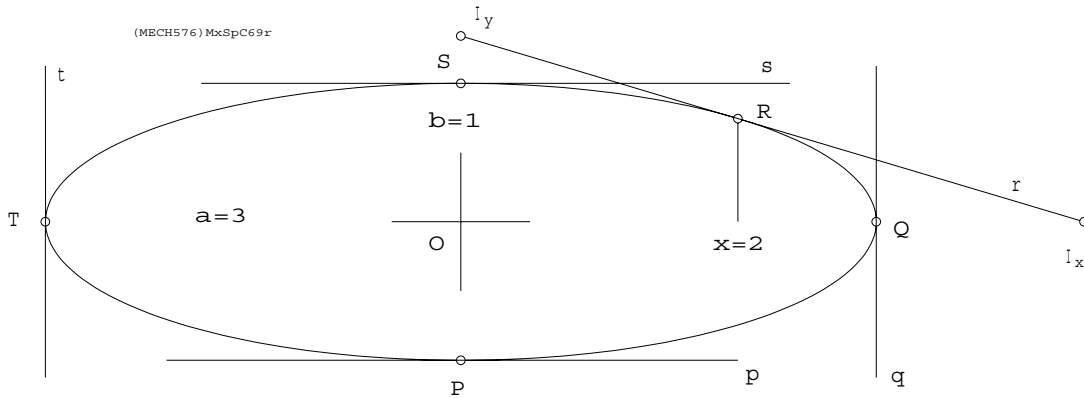


Figure 4: Conics on Five Linear Constraints

## 5.1 Conic Coefficients

The task is to evaluate the six coefficients  $a_{ij}$ .

$$[x_0 \ x_1 \ x_2] \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & a_{11} & a_{12} \\ a_{02} & a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = 0$$

With five given points  $P, Q, R, S, T$

$$\begin{bmatrix} p_0^2 & 2p_0p_1 & 2p_0p_2 & p_1^2 & 2p_1p_2 & p_2^2 \\ q_0^2 & 2q_0q_1 & 2q_0q_2 & q_1^2 & 2q_1q_2 & q_2^2 \\ r_0^2 & 2r_0r_1 & 2r_0r_2 & r_1^2 & 2r_1r_2 & r_2^2 \\ s_0^2 & 2s_0s_1 & 2s_0s_2 & s_1^2 & 2s_1s_2 & s_2^2 \\ t_0^2 & 2t_0t_1 & 2t_0t_2 & t_1^2 & 2t_1t_2 & t_2^2 \end{bmatrix} \begin{bmatrix} a_{00} \\ a_{01} \\ a_{02} \\ a_{11} \\ a_{12} \\ a_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

coefficients  $a_{ij}$  are evaluated with six  $5 \times 5$  determinants of alternating sign formed by deleting columns in sequence from the left.

## 5.2 Autopolar Tangent Lines

### 5.2.1 Four Points and a Tangent Line on One

With four given points  $P, Q, R, S$  and tangent line  $s$  on  $S$

$$\begin{bmatrix} p_0^2 & 2p_0p_1 & 2p_0p_2 & p_1^2 & 2p_1p_2 & p_2^2 & 0 & 0 \\ q_0^2 & 2q_0q_1 & 2q_0q_2 & q_1^2 & 2q_1q_2 & q_2^2 & 0 & 0 \\ r_0^2 & 2r_0r_1 & 2r_0r_2 & r_1^2 & 2r_1r_2 & r_2^2 & 0 & 0 \\ s_0^2 & 2s_0s_1 & 2s_0s_2 & s_1^2 & 2s_1s_2 & s_2^2 & 0 & 0 \\ 0 & s_0 & 0 & s_1 & s_2 & 0 & -S_1 & 0 \\ 0 & 0 & s_0 & 0 & s_1 & s_2 & -S_2 & 0 \end{bmatrix} \begin{bmatrix} a_{00} \\ a_{01} \\ a_{02} \\ a_{11} \\ a_{12} \\ a_{22} \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

coefficients  $a_{ij}$  are evaluated with the first six  $6 \times 6$  determinants of alternating sign formed by deleting columns in sequence from the left. Notice that a row containing  $S_0$  cannot be used because it is the normal direction information pertaining to line  $s$  that is required its position is already established as  $S \in s$ .

### 5.3 Three Points and Tangent Lines on Two

With three given points  $P, Q, R$  and tangent lines  $q$  and  $r$  on  $Q$  and  $R$ , respectively,

$$\begin{bmatrix} p_0^2 & 2p_0p_1 & 2p_0p_2 & p_1^2 & 2p_1p_2 & p_2^2 & 0 & 0 \\ q_0^2 & 2q_0q_1 & 2q_0q_2 & q_1^2 & 2q_1q_2 & q_2^2 & 0 & 0 \\ r_0^2 & 2r_0r_1 & 2r_0r_2 & r_1^2 & 2r_1r_2 & r_2^2 & 0 & 0 \\ 0 & q_0 & 0 & q_1 & q_2 & 0 & -Q_1 & 0 \\ 0 & 0 & q_0 & 0 & q_1 & q_2 & -Q_2 & 0 \\ 0 & r_0 & 0 & r_1 & r_2 & 0 & 0 & -R_1 \\ 0 & 0 & r_0 & 0 & r_1 & r_2 & 0 & -R_2 \end{bmatrix} \begin{bmatrix} a_{00} \\ a_{01} \\ a_{02} \\ a_{11} \\ a_{12} \\ a_{22} \\ \lambda \\ \mu \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

coefficients are evaluated with the first six  $7 \times 7$  determinants of alternating sign formed by deleting columns in sequence from the left.

#### 5.3.1 Autopolar Examples and the Polar Triangle

The following exercise will begin with finding the conic, an ellipse in this case, on five given points.

$$P(1, 1), Q(2, 4), R(5, 5), S(7, 3), T(6, 2)$$

Below is the detached coefficient form of the five linear homogeneous equations in six variables.

$$\begin{bmatrix} 1 & 2 & 2 & 1 & 2 & 1 \\ 1 & 4 & 8 & 4 & 16 & 16 \\ 1 & 10 & 10 & 25 & 50 & 25 \\ 1 & 14 & 6 & 49 & 43 & 9 \\ 1 & 12 & 4 & 36 & 24 & 4 \end{bmatrix} \begin{bmatrix} a_{00} \\ a_{01} \\ a_{02} \\ a_{11} \\ a_{12} \\ a_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

With coefficients  $a_{ij}$  the conic scalar equation is written as

$$-120 + 38x_1 + 106x_2 - 11x_1^2 + 16x_1x_2 - 29x_2^2 = 0$$

These coefficients and the equation could equally well have been produced with the following two systems. In the first, the point  $S$  is dropped and the tangent line  $t$ ,  $T \in t$  is added. In this example the line equation

$$100 - 31x_1 + 43x_2 = 0$$

and its coefficients were obtained with the conic, above, using the autopolar relation.

$$\begin{bmatrix} 1 & 2 & 2 & 1 & 2 & 1 & 0 \\ 1 & 4 & 8 & 4 & 16 & 16 & 0 \\ 1 & 10 & 10 & 25 & 50 & 25 & 0 \\ 1 & 12 & 4 & 36 & 43 & 24 & 0 \\ 0 & 1 & 0 & 6 & 2 & 0 & 31 \\ 0 & 0 & 1 & 0 & 6 & 2 & -43 \end{bmatrix} \begin{bmatrix} a_{00} \\ a_{01} \\ a_{02} \\ a_{11} \\ a_{12} \\ a_{22} \\ \lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

In the second, the point  $Q$  is dropped and the tangent line  $p$ ,  $P \in p$  is added. The line equation of  $p$  is

$$-3 + x_1 + 2x_2 = 0$$

Equations for the second system appear below.

$$\begin{bmatrix} 1 & 2 & 2 & 1 & 2 & 1 & 0 & 0 \\ 1 & 10 & 10 & 25 & 50 & 25 & 0 & 0 \\ 1 & 12 & 4 & 36 & 24 & 4 & 0 & 0 \\ 0 & 1 & 0 & 6 & 2 & 0 & 31 & 0 \\ 0 & 0 & 1 & 0 & 6 & 2 & -43 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} a_{00} \\ a_{01} \\ a_{02} \\ a_{11} \\ a_{12} \\ a_{22} \\ \lambda \\ \mu \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Essentials are illustrated in Fig. 5.

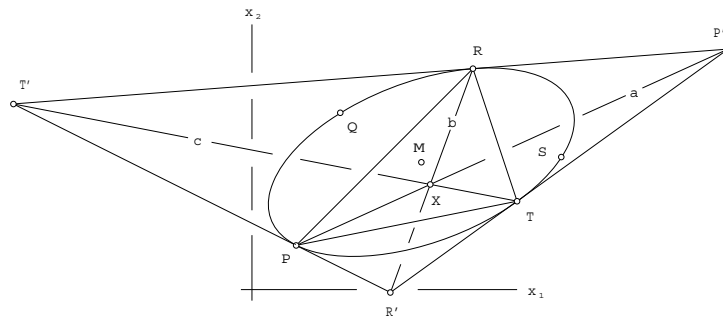


Figure 5: Conic, Tangent Lines and Polar Triangle

However there is more than meets the eye at a glance. Note the tangent line  $r$ ,  $R \in r$ . Its coefficients, too, were calculated with the conic coefficient matrix premultiplying the homogeneous column vector of point  $R\{1 : 5 : 5\}$ .

$$60 + x_1 - 13x_2 = 0$$

The triangle  $P'R'T'$  is called a polar triangle with respect to the conic.  $P'$  is polar to the line  $RT$ . Similarly, lines  $p$  and  $r$  intersect on  $T'$  and subtend line  $PR$ , polar to  $T'$ . Finally  $R'$  is polar to line  $PT$ . Joining lines  $a = PP'$ ,  $b = RR'$  and  $c = TT'$  produce a common point of intersection  $X$ . This is the fundamental property of polar triangles that circumscribe a conic with respect to the inscribed triangle on the three tangent points. Looking at this more closely, the three tangent lines  $p, r, t$  are produced by a *linear transformation*, *i.e.*, the symmetric conic coefficient matrix operating successively on points  $P, R, T$  that are on the conic.

$$[A][P], [R], [T] = [p], [r], [t]$$

$$\begin{bmatrix} -120 & 19 & 53 \\ 19 & -11 & 4 \\ 53 & 4 & -29 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 6 \\ 2 \end{bmatrix} = \lambda \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \mu \begin{bmatrix} 60 \\ 1 \\ -13 \end{bmatrix}, \nu \begin{bmatrix} 100 \\ -31 \\ 43 \end{bmatrix}$$

Then these three lines  $p, r, t$  intersect as  $r \cap p = T'$ ,  $r \cap t = P'$ ,  $p \cap t = R'$ , the vertices of the polar triangle. In terms of the given numerical data

$$\begin{bmatrix} X_0 & X_1 & X_2 \\ 60 & 1 & -13 \\ 100 & -31 & 43 \end{bmatrix} \rightarrow -1 \begin{bmatrix} 9 \\ 97 \\ 49 \end{bmatrix}, \begin{bmatrix} X_0 & X_1 & X_2 \\ -3 & 1 & 1 \\ 100 & -31 & 43 \end{bmatrix} \rightarrow 7 \begin{bmatrix} 15 \\ 47 \\ -1 \end{bmatrix}, \begin{bmatrix} X_0 & X_1 & X_2 \\ 60 & 1 & -13 \\ -3 & 1 & 2 \end{bmatrix} \rightarrow 3 \begin{bmatrix} 5 \\ -27 \\ 21 \end{bmatrix}$$

Now the line equation coefficients of  $a = P \cap P'$ ,  $b = R \cap R'$ ,  $c = T \cap T'$  are similarly produced. Note that common multipliers of the resulting vector elements have been omitted. Finally these lines are shown to be linearly dependent, *i.e.*, share the common intersection point  $X$ .

$$\begin{bmatrix} x_0 & x_1 & x_2 \\ 1 & 1 & 1 \\ 9 & 97 & 49 \end{bmatrix} \rightarrow \begin{bmatrix} 6 \\ 5 \\ -11 \end{bmatrix}, \begin{bmatrix} x_0 & x_1 & x_2 \\ 1 & 5 & 5 \\ 15 & 47 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} -60 \\ 19 \\ -7 \end{bmatrix}$$

$$\begin{bmatrix} x_0 & x_1 & x_2 \\ 1 & 6 & 2 \\ 5 & -27 & 21 \end{bmatrix} \rightarrow \begin{bmatrix} 180 \\ -11 \\ -57 \end{bmatrix}, \begin{vmatrix} 6 & 5 & -11 \\ -60 & 19 & -7 \\ 180 & -11 & -57 \end{vmatrix} = 0$$

Any pair of lines in the determinant yield  $X$ .

$$\begin{bmatrix} X_0 & X_1 & X_2 \\ 6 & 5 & -11 \\ 180 & -11 & -57 \end{bmatrix} \rightarrow \begin{bmatrix} 29 \\ 117 \\ 69 \end{bmatrix}$$

A polar triangle need not be composed of three tangents to a conic. In Fig. 6 one sees an arbitrary triangle with vertices  $A, B, C$  whose sides  $a, b, c$  provide chords of the conic, a circle in this case, that are polar lines whose tangents subtend the vertices  $A', B', C'$  of the polar triangle with sides  $a', b', c'$ . These sides are certainly not on the tangents, shown in red. Similarly none of  $A, B, C$  is on the circle circumference. Nevertheless lines  $a'' = A \cap A'$ ,  $b'' = B \cap B'$ ,  $c'' = C \cap C'$  intersect on a common point  $X$ .

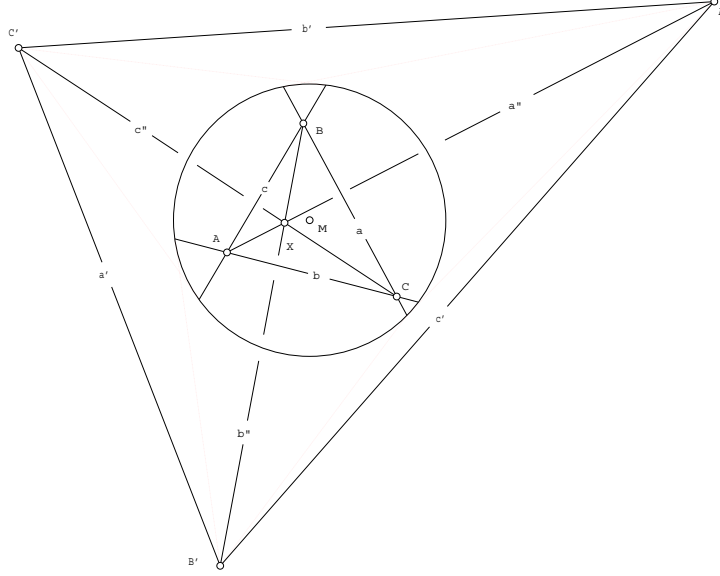


Figure 6: Polar Triangle Need Not Contain Tangents

## 5.4 Points and Discrete Tangent Lines

Given four points  $P, Q, R, S$  and a tangent line  $t$  one obtains a linear equation for each point, *e.g.*, point  $P$  produces

$$p_0^2 a_{00} + 2p_0 p_1 a_{01} + 2p_0 p_2 a_{02} + p_1^2 a_{11} + 2p_1 p_2 a_{12} + p_2^2 a_{22} = 0$$

Then making use of the property, that the adjoint of the conic point coefficient matrix plays the same rôle with respect to the dual conic defined on tangent lines, one may write

$$[T_0 \ T_1 \ T_2] \begin{bmatrix} a_{11}a_{22} - a_{12}^2 & a_{02}a_{12} - a_{01}a_{22} & a_{01}a_{12} - a_{02}a_{11} \\ a_{02}a_{12} - a_{01}a_{22} & a_{00}a_{22} - a_{02}^2 & a_{01}a_{02} - a_{00}a_{12} \\ a_{01}a_{12} - a_{02}a_{11} & a_{01}a_{02} - a_{00}a_{12} & a_{00}a_{11} - a_{01}^2 \end{bmatrix} \begin{bmatrix} T_0 \\ T_1 \\ T_2 \end{bmatrix} = 0$$

to yield a second order equation in  $a_{ij}$ .

$$\begin{aligned} T_2^2 a_{00} a_{11} - 2T_1 T_2 a_{00} a_{12} + T_1^2 a_{00} a_{22} - T_2^2 a_{01}^2 + 2T_1 T_2 a_{01} a_{02} + 2T_0 T_2 a_{01} a_{12} \\ - 2T_0 T_1 a_{01} a_{22} - T_1^2 a_{02}^2 - 2T_0 T_2 a_{02} a_{11} + 2T_0 T_1 a_{02} a_{12} + T_0^2 a_{11} a_{22} - T_0^2 a_{12}^2 = 0 \end{aligned}$$

With three given points  $P, Q, R$  there are only three linear equations and the two given tangent lines  $s, t$  give us two equations of second order in  $S_i, T_i, a_{ij}$ . Two conics satisfy the specification with one tangent line while four different conics will satisfy cases where two tangent lines are given.

## 6 Rectification of a Quadric on Three Lines

It will be shown how to find the principal axis of a quadric, a hyperboloid of one sheet specified by three given lines, by finding the centre of its absolute conic using the procedure outlined in

subsection **5.3**, above. If the lines  $\mathcal{P}, \mathcal{Q}, \mathcal{R}$  are given as radial Plücker coordinates then it is easy to see that these pierce the absolute plane on points  $P, Q, R$  and map to absolute lines  $p, q, r$ .

$$\begin{aligned}\mathcal{P}\{p_{01} : p_{02} : p_{03} : p_{23} : p_{31} : p_{12}\} &\rightarrow P\{p_{01} : p_{02} : p_{03}\}, \quad p\{p_{23} : p_{31} : p_{12}\} \\ \mathcal{Q}\{q_{01} : q_{02} : q_{03} : q_{23} : q_{31} : q_{12}\} &\rightarrow Q\{q_{01} : q_{02} : q_{03}\}, \quad q\{q_{23} : q_{31} : q_{12}\} \\ \mathcal{R}\{r_{01} : r_{02} : r_{03} : r_{23} : r_{31} : r_{12}\} &\rightarrow R\{r_{01} : r_{02} : r_{03}\}, \quad r\{r_{23} : r_{31} : r_{12}\}\end{aligned}$$

The detached coefficient form of the seven constraint equations necessary to compute the six coefficients  $a_{ij}$  of the absolute conic is written below.

$$\begin{bmatrix} p_{01}^2 & 2p_{01}p_{02} & 2p_{01}p_{03} & p_{02}^2 & 2p_{02}p_{03} & p_{03}^2 & 0 & 0 \\ q_{01}^2 & 2q_{01}q_{02} & 2q_{01}q_{03} & q_{02}^2 & 2q_{02}q_{03} & q_{03}^2 & 0 & 0 \\ r_{01}^2 & 2r_{01}r_{02} & 2r_{01}r_{03} & r_{02}^2 & 2r_{02}r_{03} & r_{03}^2 & 0 & 0 \\ q_{01} & q_{02} & q_{03} & 0 & 0 & 0 & -q_{23} & 0 \\ 0 & q_{01} & 0 & q_{02} & q_{03} & 0 & 0 & -q_{31} \\ r_{01} & r_{02} & r_{03} & 0 & 0 & 0 & -r_{23} & 0 \\ 0 & r_{01} & 0 & r_{02} & r_{03} & 0 & 0 & -r_{31} \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{22} \\ a_{23} \\ a_{33} \\ \lambda \\ \mu \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

In the case of the numerical example introduced below

$$[0 \ 0 \ q_{01} \ 0 \ 0 \ q_{02} \ q_{03} \ -q_{12} \ 0]$$

replaces the fifth row to avoid rank deficiency. The three lines chosen represent three skew sides of an origin centred rectangular “box” with bottom front, upper right horizontal and rear right vertical edges of respective lengths of 8, 4 and 2 units; the lines are  $\mathcal{P}, \mathcal{Q}, \mathcal{R}$ .

$$\mathcal{P}\{1 : 0 : 0 : 0 : -1 : 1\}, \quad \mathcal{Q}\{0 : 1 : 0 : -1 : 0 : 4\}, \quad \mathcal{R}\{0 : 0 : 1 : 2 : 4 : 0\}$$

They produce the following matrix.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -4 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \\ a_{22} \\ a_{23} \\ a_{33} \\ \lambda \\ \mu \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Solving homogeneously for  $a_{ij}$  and dividing out a common factor 4 yields the absolute conic coefficient matrix.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 0 & -4 \\ -2 & -4 & 0 \end{bmatrix}$$

Its eigenvalues, obtained by numerical solution of the characteristic equation, are

$$4.924343992 - 0.3 \times 10^{-9}i, \quad -4.139409960 + 0.2 \times 10^{-9}i, \quad -0.784934032 + 0.2 \times 10^{-9}i$$

A cubic can have one or three real roots because complex roots must be accompanied by their conjugate. Since there are no such pairs and the three imaginary residues are small, the real parts are taken as three real eigenvalues; a negative pair and one positive one.

### 6.0.1 Eigenvectors

The opportunity is taken here to recall the geometric implication of eigenvalues and eigenvectors. The latter are *homogeneous* vectors or sets of *direction numbers* that are not transformed, *i.e.*, are mapped to themselves, when multiplied by a matrix operator, in this case the coefficient matrix of an absolute conic. This may be stated as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \lambda \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \text{ or } \frac{1}{\lambda} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix}$$

The multiplier  $\lambda$  serves the same purpose here as it and  $\mu$  did in the vector of variables used in formulating the simultaneous equations to find  $a_{ij}$  the coefficients of the absolute conic coefficient matrix; to serve as multipliers that convert expressions involving proportionality between equivalent sets of ratios into equalities, *i.e.*, *equations*. Note how the transformed vector  $e\{E_1 : E_2 : E_3\}$  is “planar line-like” while  $E\{e_1 : e_2 : e_3\}$  is “point-like”. This may be illustrated by the polarity relation between absolute points that close the principal axes of a conic in the affine, not absolute, plane and the respective axes (lines) themselves as shown in Fig. 7.

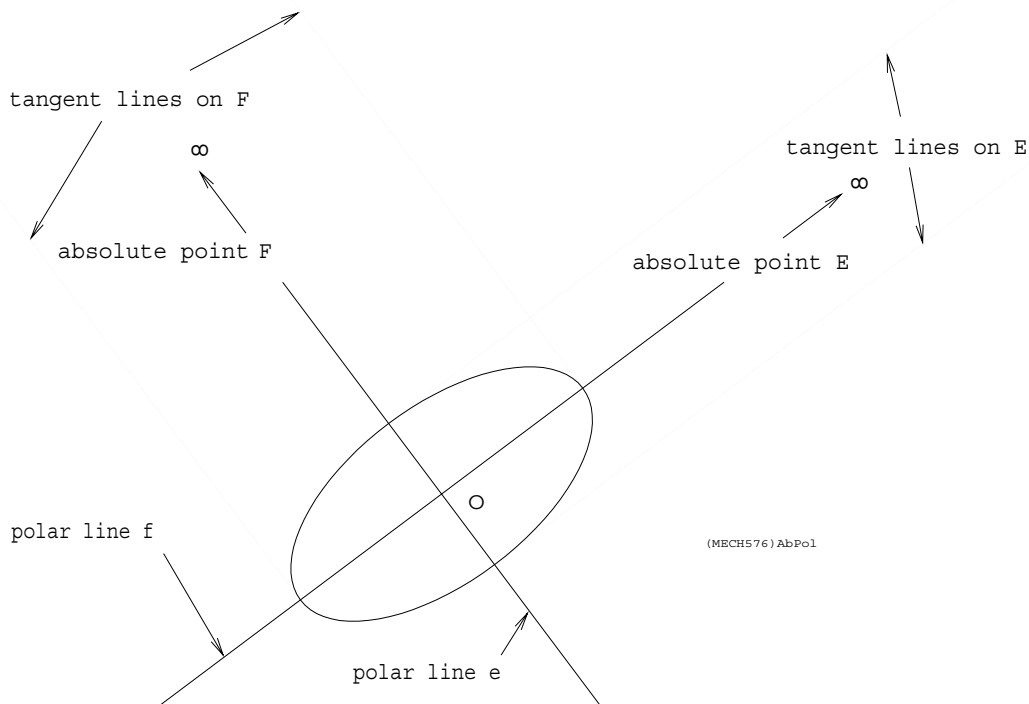


Figure 7: Eigenvectors, Absolute Points and Polar Lines

Returning to the numerical problem, the following eigenvectors are obtained.

$$\begin{bmatrix} -13.84868798 \\ -21.69737597 \\ 23.24916375 \end{bmatrix}, \begin{bmatrix} 4.278819920 \\ 14.55763984 \\ 16.13471482 \end{bmatrix}, \begin{bmatrix} -2.430131936 \\ 1.139736128 \\ -.3838785654 \end{bmatrix}$$



These line-like elements can be intersected pairwise to produce the points that close the principal axis of the quadric. The first and second eigenvectors above intersect on a absolute point  $X$  in the direction opposite to the direction of the third, labeled 3 in Fig. 8. Similarly the second and third intersect on  $Y$  in a direction opposite to the line labeled 1. First the Grassmanian determinants are presented below, then the absolute points  $X, Y, Z$ . These are on the shorter axis triad.

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ -13.84868798 & -21.69737597 & 23.24916375 \\ 4.278819920 & 14.55763984 & 16.13471482 \end{bmatrix}$$

$$\begin{bmatrix} y_1 & y_2 & y_3 \\ 4.278819920 & 14.55763984 & 16.13471482 \\ -2.430131936 & 1.139736128 & -0.3838785654 \end{bmatrix}$$

$$\begin{bmatrix} z_1 & z_2 & z_3 \\ -2.430131936 & 1.139736128 & -0.3838785654 \\ -13.84868798 & -21.69737597 & 23.24916375 \end{bmatrix}$$

$$X\{0 : -688.5339261 : 322.9236162 : -108.7650474\}$$

$$Y\{0 : -23.9776833 : -37.56693851 : 40.25371114\}$$

$$Z\{0 : 18.16875431 : 61.81474978 : 68.51133629\}$$

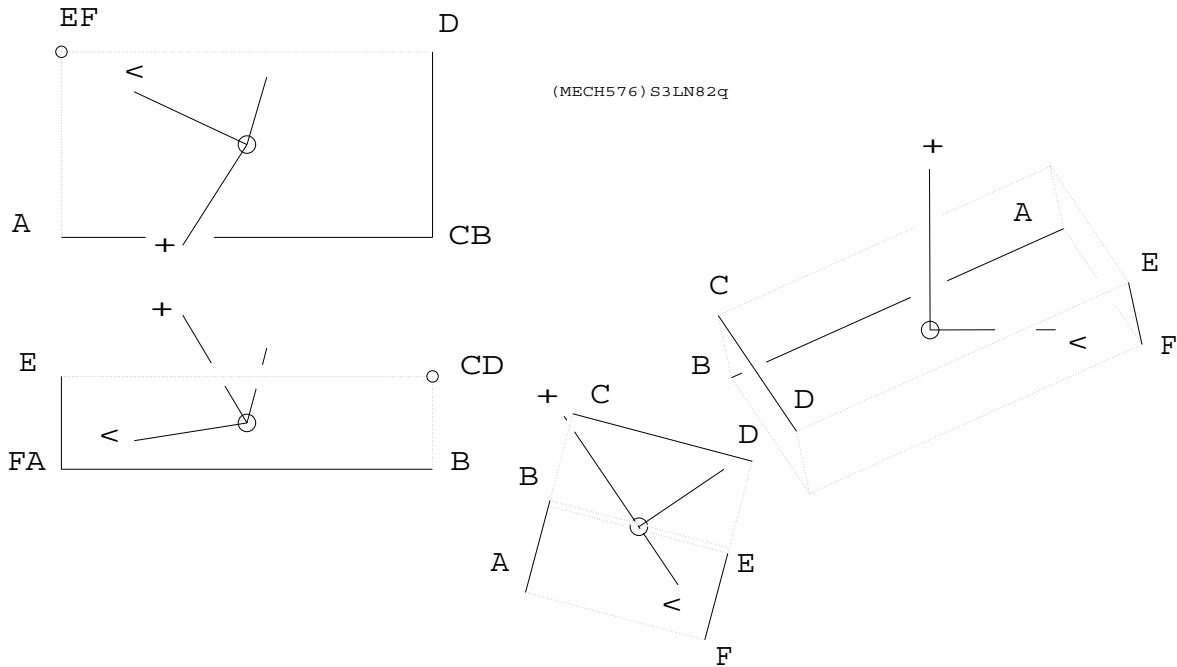


Figure 8: Three Orthogonal Lines and the Principal Axes of the Quadric They Define

One may note that the three given lines on a rectangular box are symmetrically distributed about the quadric centre but there is no discernable relation between the directions of the quadric axes

to those of the box edges or diagonals. However if the box were a cube the quadric would be a one sheet hyperboloid of revolution with its ruling circle axis on a space diagonal of the cube. Substituting the given lines into Eq. 16 produces the following equation of the quadric which is then plotted, in Fig. 9 in a box only slightly larger than that defined by the given lines. Compare the second auxiliary projection to the pictorial of the quadric. The axis marked (+) is associated with the single positive eigenvalue while the axis marked (<) is associated with the negative eigenvalue of lesser absolute magnitude.

$$16s_0^2 + 2s_1s_2 - 4s_1s_3 - 8s_2s_3 = 0$$

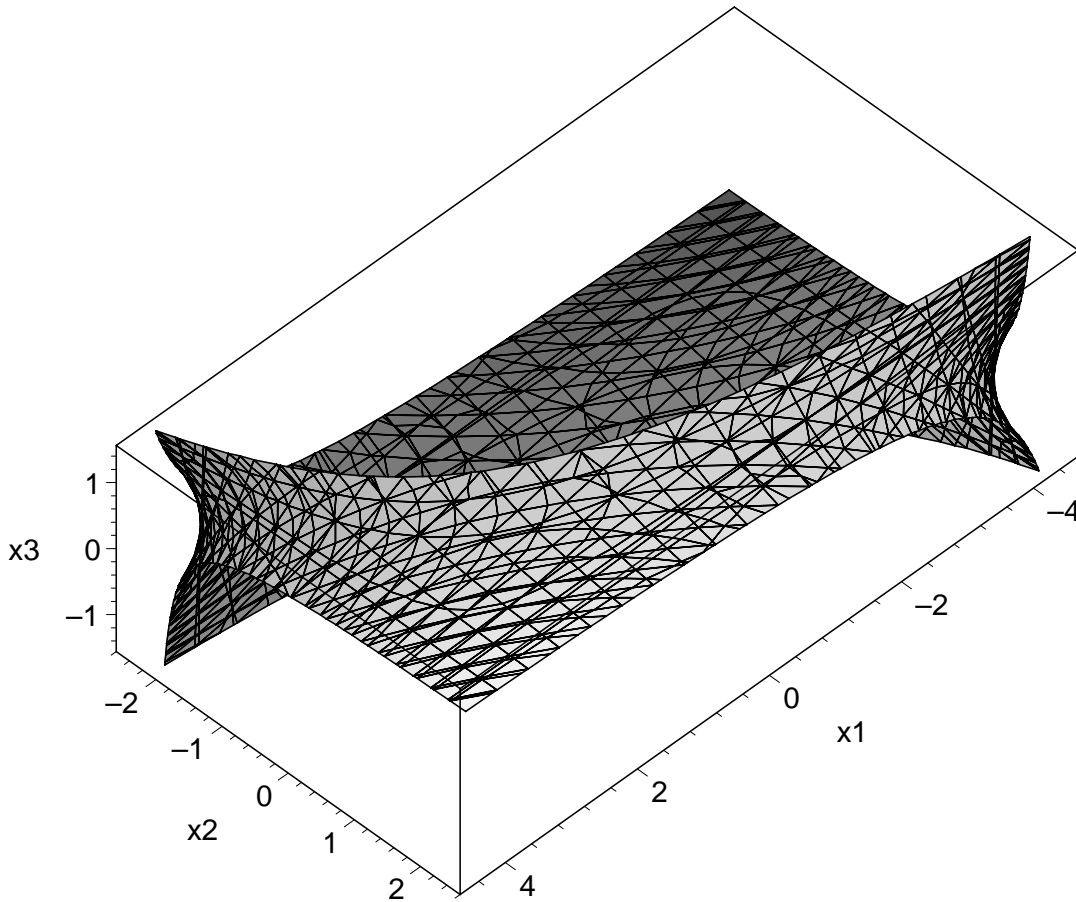


Figure 9: The Region Near the Hyperboloid Throat and the Given Lines

## 7 Rectification of a Conic

A much simpler exercise is to take an arbitrary conic, expressed by its implicit equation, say,

$$-120 + 38x_1 + 106x_2 - 11x_1^2 + 16x_1x_2 - 29x_2^2 = 0$$

and reduce it to standard form

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - 1 = 0$$

This is done via a two step process. First the conic is translated so as to be origin centred. Then the origin centred conic is rotated to orient the principal axes of the conic along the axes of the Cartesian frame. The equation of the conic is the following vector-matrix product.

$$\begin{bmatrix} 1 & x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & a_{11} & a_{12} \\ a_{02} & a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} 1 \\ x_1 \\ x_2 \end{bmatrix} = a_{00} + 2a_{01}x_1 + 2a_{02}x_2 + a_{11}x_1^2 + 2a_{12}x_1x_2 + a_{22}x_2^2 = 0$$

In the example above the matrix is

$$\begin{bmatrix} -120 & 19 & 53 \\ 19 & -11 & 8 \\ 53 & 8 & -29 \end{bmatrix}$$

Translation is effected by premultiplying this matrix by the transpose of the cofactor matrix of that matrix which will translate a point  $(x_1, x_2)$  to  $(t_1 + x_1, t_2 + x_2)$ , *i.e.*,

$$\begin{bmatrix} 1 & 0 & 0 \\ t_1 & 1 & 0 \\ t_2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ t_1 + x_1 \\ t_2 + x_2 \end{bmatrix}$$

and postmultiplying the same conic coefficient matrix by the cofactor matrix, thus.

$$\begin{bmatrix} 1 & 0 & 0 \\ -t_1 & 1 & 0 \\ -t_2 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{01} & a_{11} & a_{12} \\ a_{02} & a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} 1 & -t_1 & -t_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1793}{17} & 0 & 0 \\ 0 & -11 & 8 \\ 0 & 8 & -29 \end{bmatrix}$$

The top row off-diagonal elements of this triple product matrix provide the necessary pair of linear equations in  $t_1$  and  $t_2$  to zero these elements and their first column off diagonal partners.

$$a_{01} = 19 \rightarrow 19 + 11t_1 - 8t_2 = 0, \quad a_{02} = 53 \rightarrow 53 - 8t_1 + 29t_2 = 0, \quad t_1 = -\frac{65}{17}, \quad t_2 = -\frac{49}{17}$$

For reference, the origin centred conic equation is written below.

$$-\frac{1793}{17} + 11x_1^2 - 16x_1x_2 + 29x_2^2 = 0$$

Rotation is similarly effected by premultiplication of the coefficient matrix, now with  $a_{01} = a_{02} = 0$ , by the transpose of the cofactor of an appropriate rotation matrix and postmultiplying by the cofactor matrix itself. Note that the cofactor of a homogeneous orthogonal matrix is itself. That is why the transpose of a rotation matrix is its inverse.

Rather than using the triple matrix product approach to obtain the slope of the principal axes one may invoke eigenvalues and eigenvectors of the  $2 \times 2$  minor of  $a_{00}$ . This elegant method makes use of the conic property that the pencil of lines parallel to a principal axis are in polar correspondence

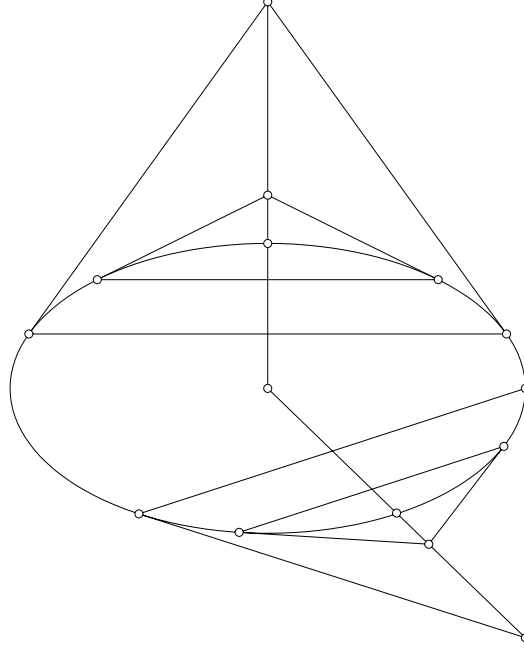


Figure 10: Normal and Skew Polar Point Pencils and Their Corresponding Parallel Polar Lines

to a linear pencil of points that lie on a line normal to the axis. This property is illustrated in Fig. 10. This is expressed by the following transformation.

$$\begin{bmatrix} -11 & 8 \\ 8 & -29 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix} = \lambda \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

Rearranging produces two linear homogeneous equations in the ratio  $e_2/e_1$  and redundant or *linearly dependent* in  $\lambda$ .

$$\begin{bmatrix} -(11 + \lambda) & 8 \\ 8 & -(29 + \lambda) \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \frac{e_2}{e_1} = \frac{11 + \lambda}{8} \quad \text{or} \quad \frac{e_2}{e_1} = \frac{8}{29 + \lambda}$$

Equating the two fractions in  $\lambda$  yields the same characteristic equation as one obtains by taking the determinant of the rearranged matrix above.

$$\lambda^2 + 40\lambda + 255 = 0 \rightarrow \lambda = -20 \pm \sqrt{145}$$

Since one is dealing with a homogeneous system the  $e_2/e_1$  ratio will provide the ratio of  $\sin \theta / \cos \theta$  necessary to construct the appropriate rotation matrix to form, together with the origin centred conic coefficient matrix, the diagonalized triple product, *e.g.*, taking the first expression for the ratio and  $+\sqrt{145}$  gives

$$\sin \theta \propto -9 + \sqrt{145}, \quad \cos \theta \propto 8$$

Normalization yields the rotation matrix that operates on a point so as to rotate it about the origin through the positive angle measured from the  $x_1$ -axis to the direction vector in the first

quadrant.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{8}{\sqrt{8^2+(\sqrt{145}-9)^2}} & -\frac{\sqrt{145}-9}{\sqrt{8^2+(\sqrt{145}-9)^2}} \\ 0 & \frac{\sqrt{145}-9}{\sqrt{8^2+(\sqrt{145}-9)^2}} & \frac{8}{\sqrt{8^2+(\sqrt{145}-9)^2}} \end{bmatrix}$$

The triple matrix product is formed by the sequence of the transpose of the above matrix, the origin centred coefficient matrix and the rotation matrix itself since it is its own cofactor. This effects the desired *backwards* rotation on the origin centred coefficient matrix in the same way that a similar triple product, using the forward translation of a point, brought the conic *back* to an origin centred position. The result is the diagonal matrix below.

$$\begin{bmatrix} -a^2b^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & a^2 \end{bmatrix} = \begin{bmatrix} \frac{1792}{17} & 0 & 0 \\ 0 & -5 \left( \frac{65\sqrt{145}-841}{9\sqrt{145}-145} \right) & 0 \\ 0 & 0 & -5 \left( \frac{7\sqrt{145}-319}{9\sqrt{145}-145} \right) \end{bmatrix} \rightarrow -1 + \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 0$$

The principal semi-major and -minor axis lengths can now be determined as

$$a \simeq 3.639414426, \quad b \simeq 1.813791998$$

The eigenvector that produced these results is shown in Fig. 11 in the first quadrant.

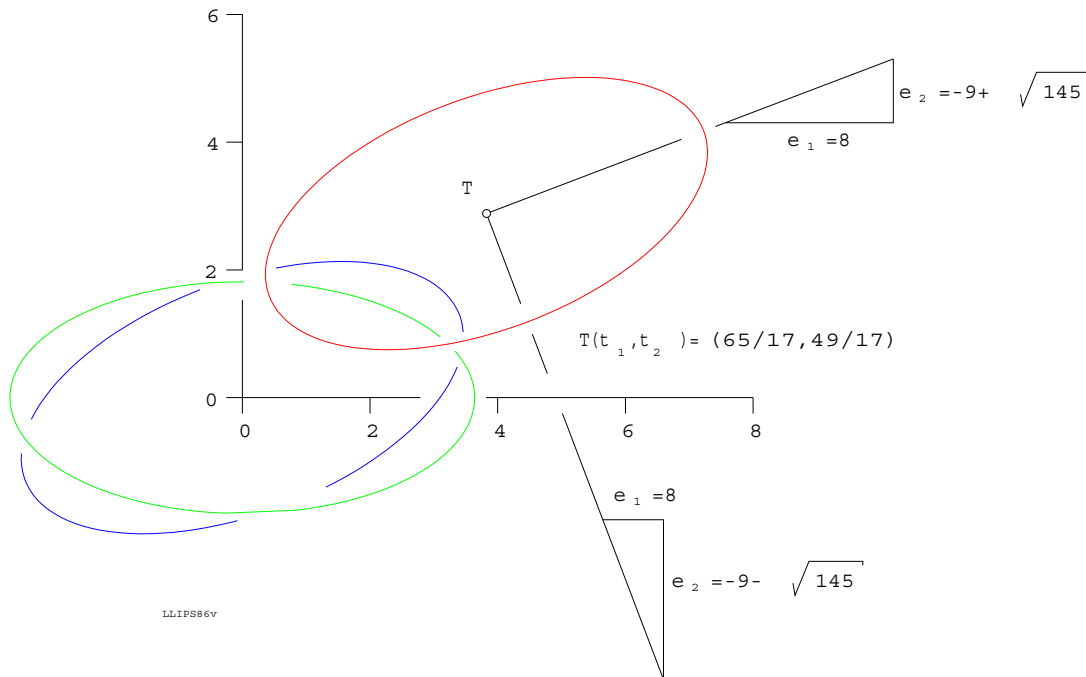


Figure 11: Three Positions of the Ellipse