



HAL
open science

COMPUTING IN COMMUTATIVE ALGEBRA

Gerhard Pfister

► **To cite this version:**

Gerhard Pfister. COMPUTING IN COMMUTATIVE ALGEBRA. 3rd cycle. Lahore (Pakistan), 2009, pp.18. cel-00374623

HAL Id: cel-00374623

<https://cel.hal.science/cel-00374623>

Submitted on 9 Apr 2009

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

COMPUTING IN COMMUTATIVE ALGEBRA

GERHARD PFISTER

1. STANDARD BASES AND SINGULAR

SINGULAR is available, free of charge, as a binary programme for most common hardware and software platforms. Release versions of SINGULAR can be downloaded through ftp from our FTP site

`ftp://www.mathematik.uni-kl.de/pub/Math/Singular/`,

or, using your favourite WWW browser, from

`http://www.singular.uni-kl.de/download.html`

The basis of SINGULAR is multivariate polynomial factorization and standard bases computations.

We explain first of all the notion of a Gröbner basis (with respect to any ordering) as the basis for computations in localizations of factorings of polynomial rings. The presentation of a polynomial as a linear combination of monomials is unique only up to an order of the summands, due to the commutativity of the addition. We can make this order unique by choosing a total ordering on the set of monomials. For further applications it is necessary, however, that the ordering is compatible with the semigroup structure on Mon_n .

We give here only the important definitions, theorems and examples. Proofs can be found in [7]. The SINGULAR examples can be found on the CD in [7].

Definition 1.1. A monomial ordering or semigroup ordering is a total (or linear) ordering $>$ on the set of monomials $\text{Mon}_n = \{x^\alpha \mid \alpha \in \mathbf{N}^n\}$ in n variables satisfying

$$x^\alpha > x^\beta : \implies : x^\gamma x^\alpha > x^\gamma x^\beta$$

for all $\alpha, \beta, \gamma \in \mathbf{N}^n$. We say also $>$ is a monomial ordering on $A[x_1, \dots, x_n]$, A any ring, meaning that $>$ is a monomial ordering on Mon_n .

Definition 1.2. Let $>$ be a fixed monomial ordering. Write $f \in K[x]$, $f \neq 0$, in a unique way as a sum of non-zero terms

$$f = a_\alpha x^\alpha + a_\beta x^\beta + \dots + a_\gamma x^\gamma, \quad x^\alpha > x^\beta > \dots > x^\gamma,$$

and $a_\alpha, a_\beta, \dots, a_\gamma \in K$. We define:

- (1) $LM(f) := \text{leadmonom}(f) := x^\alpha$, the leading monomial of f ,
- (2) $LE(f) := \text{leadexp}(f) := \alpha$, the leading exponent of f ,
- (3) $LT(f) := \text{lead}(f) := a_\alpha x^\alpha$, the leading term or head of f ,
- (4) $LC(f) := \text{leadcoef}(f) := a_\alpha$, the leading coefficient of f

- (5) $\text{tail}(f) := f - \text{lead}(f) = a_\beta x^\beta + \dots + a_\gamma x^\gamma$, the tail.
(6) $\text{ecart}(f) := \deg(f) - \deg(\text{LM}(f))$.

SINGULAR Example 1.

```
ring A = 0, (x,y,z), lp;
poly f = y4z3+2x2y2z2+3x5+4z4+5y2;
f; //display f in a lex-ordered way
//-> 3x5+2x2y2z2+y4z3+5y2+4z4
leadmonom(f); //leading monomial
//-> x5
leadexp(f); //leading exponent
//-> 5,0,0
lead(f); //leading term
//-> 3x5
leadcoef(f); //leading coefficient
//-> 3
f - lead(f); //tail
//-> 2x2y2z2+y4z3+5y2+4z4
```

Definition 1.3. Let $>$ be a monomial ordering on $\{x^\alpha \mid \alpha \in \mathbf{N}^n\}$.

- (1) $>$ is called a global ordering if $x^\alpha > 1$ for all $\alpha \neq (0, \dots, 0)$,
- (2) $>$ is called a local ordering if $x^\alpha < 1$ for all $\alpha \neq (0, \dots, 0)$,
- (3) $>$ is called a mixed ordering if it is neither global nor local.

Lemma 1.4. Let $>$ be a monomial ordering, then the following conditions are equivalent:

- (1) $>$ is a well-ordering.
- (2) $x_i > 1$ for $i = 1, \dots, n$.
- (3) $x^\alpha > 1$ for all $\alpha \neq (0, \dots, 0)$, that is, $>$ is global.

In the following examples we fix an enumeration x_1, \dots, x_n of the variables, any other enumeration leads to a different ordering.

%beginenumerate GLOBAL ORDERINGS

(i) *Lexicographical ordering $>_{lp}$* (also denoted by *lex*):

$$x^\alpha >_{lp} x^\beta : \iff \exists 1 \leq i \leq n : \alpha_1 = \beta_1, \dots, \alpha_{i-1} = \beta_{i-1}, \alpha_i > \beta_i.$$

(ii) *Degree reverse lexicographical ordering $>_{dp}$* (denoted by *degrevlex*):

$$x^\alpha >_{dp} x^\beta : \iff : \deg x^\alpha > \deg x^\beta$$

$$\text{or} : (\deg x^\alpha = \deg x^\beta \text{ and } \exists 1 \leq i \leq n : \alpha_n = \beta_n, \dots, \alpha_{i+1} = \beta_{i+1}, \alpha_i < \beta_i),$$

where $\deg x^\alpha = \alpha_1 + \dots + \alpha_n$.

%le i

LOCAL ORDERINGS

(i) *Negative lexicographical ordering* $>_{ls}$:

$$x^\alpha >_{ls} x^\beta : \iff \exists 1 \leq i \leq n, \alpha_1 = \beta_1, \dots, \alpha_{i-1} = \beta_{i-1}, \alpha_i < \beta_i.$$

(ii) *Negative degree reverse lexicographical ordering*:

$$\begin{aligned} x^\alpha >_{ds} x^\beta &: \iff: \deg x^\alpha < \deg x^\beta, \text{ where } \deg x^\alpha = \alpha_1 + \dots + \alpha_n, \\ &\text{or : } (\deg x^\alpha = \deg x^\beta \text{ and } \exists 1 \leq i \leq n : \\ &\quad \alpha_n = \beta_n, \dots, \alpha_{i+1} = \beta_{i+1}, \alpha_i < \beta_i). \end{aligned}$$

Let $>$ be a monomial ordering on the set of monomials $\text{Mon}(x_1, \dots, x_n) = \{x^\alpha \mid \alpha \in \mathbf{N}^n\}$, and $K[x] = K[x_1, \dots, x_n]$ the polynomial ring in n variables over a field K . Then the leading monomial function LM has the following properties for polynomials $f, g \in K[x] \setminus \{0\}$:

- (1) $\text{LM}(gf) = \text{LM}(g)\text{LM}(f)$.
- (2) $\text{LM}(g+f) \leq \max\{\text{LM}(g), \text{LM}(f)\}$ with equality if and only if the leading terms of f and g do not cancel.

In particular, it follows that

$$S_{>} := \{u \in K[x] \setminus \{0\} \mid \text{LM}(u) = 1\}$$

is a multiplicatively closed set.

Definition 1.5. For any monomial ordering $>$ on $\text{Mon}(x_1, \dots, x_n)$, we define

$$K[x]_{>} := S_{>}^{-1}K[x] = \left\{ \frac{f}{u} \mid f, u \in K[x], \text{LM}(u) = 1 \right\},$$

the localization of $K[x]$ with respect to $S_{>}$ and call $K[x]_{>}$ the ring associated to $K[x]$ and $>$.

Note that $S_{>} = K^*$ if and only if $>$ is global and $S_{>} = K[x] \setminus \langle x_1, \dots, x_n \rangle$ if and only if $>$ is local.

Definition 1.6. Let $>$ be any monomial ordering:

- (1) For $f \in K[x]_{>}$ choose $u \in K[x]$ such that $\text{LT}(u) = 1$ and $uf \in K[x]$. We define

$$\begin{aligned} \text{LM}(f) &:= \text{LM}(uf), \\ \text{LC}(f) &:= \text{LC}(uf), \\ \text{LT}(f) &:= \text{LT}(uf), \\ \text{LE}(f) &:= \text{LE}(uf), \end{aligned}$$

and $\text{tail}(f) = f - \text{LT}(f)$.

- (2) For any subset $G \subset K[x]_{>}$ define the ideal

$$L_{>}(G) := L(G) := \langle \text{LM}(g) \mid g \in G \setminus \{0\} \rangle_{K[x]}.$$

$L(G) \subset K[x]$ is called the leading ideal of G .

Definition 1.7. Let $I \subset R = K[x]_{>}$ be an ideal.

(1) A finite set $G \subset R$ is called a standard basis of I if

$$G \subset I, \text{ and } L(I) = L(G).$$

That is, G is a standard basis, if the leading monomials of the elements of G generate the leading ideal of I , or, in other words, if for any $f \in I \setminus \{0\}$ there exists a $g \in G$ satisfying $LM(g) \mid LM(f)$.

(2) If $>$ is global, a standard basis is also called a Gröbner basis.

(3) If we just say that G is a standard basis, we mean that G is a standard basis of the ideal $\langle G \rangle_R$ generated by G .

Standard bases can be characterized using the notion of the normal form. We need the following definitions:

Definition 1.8. Let $f, g \in R \setminus \{0\}$ with $LM(f) = x^\alpha$ and $LM(g) = x^\beta$, respectively. Set

$$\gamma := \text{lcm}(\alpha, \beta) := (\max(\alpha_1, \beta_1), \dots, \max(\alpha_n, \beta_n))$$

and let $\text{lcm}(x^\alpha, x^\beta) := x^\gamma$ be the least common multiple of x^α and x^β . We define the s -polynomial (spoly, for short) of f and g to be

$$\text{spoly}(f, g) := x^{\gamma-\alpha} f - \frac{LC(f)}{LC(g)} \cdot x^{\gamma-\beta} g.$$

If $LM(g)$ divides $LM(f)$, say $LM(g) = x^\beta$, $LM(f) = x^\alpha$, then the s -polynomial is particularly simple,

$$\text{spoly}(f, g) = f - \frac{LC(f)}{LC(g)} \cdot x^{\alpha-\beta} g,$$

and $LM(\text{spoly}(f, g)) < LM(f)$.

Definition 1.9. Let \mathcal{G} denote the set of all finite lists $G \subset R = K[x]_{>}$.

$$NF : R \times \mathcal{G} \rightarrow R, (f, G) \mapsto NF(f \mid G),$$

is called a normal form on R if, for all $G \in \mathcal{G}$,

$$(0) \quad NF(0 \mid G) = 0,$$

and, for all $f \in R$ and $G \in \mathcal{G}$,

$$(1) \quad NF(f \mid G) \neq 0 \implies LM(NF(f \mid G)) \notin L(G).$$

(2) If $G = \{g_1, \dots, g_s\}$, then f has a standard representation with respect to $NF(- \mid G)$, that is, there exists a unit $u \in R^*$ such that

$$uf - NF(f \mid G) = \sum_{i=1}^s a_i g_i, \quad a_i \in R, \quad s \geq 0,$$

satisfying $LM(\sum_{i=1}^s a_i g_i) \geq LM(a_i g_i)$ for all i such that $a_i g_i \neq 0$.

The existence of a normal form is given by the following algorithm:

Algorithm 1.10. $NF(f \mid G)$

Let $>$ be any monomial ordering.

Input: $f \in K[x]$, G a finite list in $K[x]$

Output: $h \in K[x]$ a polynomial normal form of f with respect to G .

- $h := f$;
- $T := G$;
- *while*($h \neq 0$ and $T_h := \{g \in T \mid LM(g) \mid LM(h)\} \neq \emptyset$)
 - choose* $g \in T_h$ with *ecart*(g) minimal;
 - if* (*ecart*(g) > *ecart*(h))
 - $T := T \cup \{h\}$;
 - $h := \text{spoly}(h, g)$;
- *return* h ;

Theorem 1.11. *Let $I \subset R$ be an ideal and $G = \{g_1, \dots, g_s\} \subset I$. Then the following are equivalent:*

- (1) G is a standard basis of I .
- (2) $NF(f \mid G) = 0$ if and only if $f \in I$.

We will explain now how to use standard bases to solve problems in algebra.

Ideal membership

Problem: Given $f, f_1, \dots, f_k \in K[x]$, and let $I = \langle f_1, \dots, f_k \rangle_R$. We wish to decide whether $f \in I$, or not.

Solution: We choose any monomial ordering $>$ such that $K[x]_> = R$ and compute a standard basis $G = \{g_1, \dots, g_s\}$ of I with respect to $>$. $f \in I$ if and only if $NF(f \mid G) = 0$.

SINGULAR Example 2.

```
ring A = 0, (x,y), dp;
ideal I = x10+x9y2, y8-x2y7;
ideal J = std(I);
poly f = x2y7+y14;
reduce(f, J, 1);          //3rd parameter 1 avoids tail reduction
//-> -xy12+x2y7          //f is not in I
      f = xy13+y12;
reduce(f, J, 1);
//-> 0                   //f is in I
```

Intersection with Subrings (Elimination of variables)

Problem: Given $f_1, \dots, f_k \in K[x] = K[x_1, \dots, x_n]$, $I = \langle f_1, \dots, f_k \rangle_{K[x]}$, we should like to find generators of the ideal

$$I' = I \cap K[x_{s+1}, \dots, x_n], \quad s < n.$$

Elements of the ideal I' are said to be obtained from f_1, \dots, f_k by *eliminating* x_1, \dots, x_s . The following lemma is the basis for solving the elimination problem.

Lemma 1.12. Let $>$ be an elimination ordering for x_1, \dots, x_s on the set of monomials $\text{Mon}(x_1, \dots, x_n)$, and let $I \subset K[x_1, \dots, x_n]_{>}$ be an ideal. If $S = \{g_1, \dots, g_k\}$ is a standard basis of I , then

$$S' := \{g \in S \mid LM(g) \in K[x_{s+1}, \dots, x_n]\}$$

is a standard basis of $I' := I \cap K[x_{s+1}, \dots, x_n]_{>'}$. In particular, S' generates the ideal I' .

SINGULAR Example 3.

```
ring A =0, (t,x,y,z), dp;
ideal I=t2+x2+y2+z2, t2+2x2-xy-z2, t+y3-z3;

eliminate(I,t);
//-> _[1]=x2-xy-y2-2z2      _[2]=y6-2y3z3+z6+2x2-xy-z2
```

Alternatively choose a product ordering:

```
ring A1=0, (t,x,y,z), (dp(1), dp(3));
ideal I=imap(A,I);
ideal J=std(I);
J;
//-> J[1]=x2-xy-y2-2z2      J[2]=y6-2y3z3+z6+2x2-xy-z2
//-> J[3]=t+y3-z3
```

Radical Membership

Problem: Let $f_1, \dots, f_k \in K[x]_{>}$, $>$ a monomial ordering on $\text{Mon}(x_1, \dots, x_n)$ and $I = \langle f_1, \dots, f_k \rangle_{K[x]_{>}}$. Given some $f \in K[x]_{>}$ we want to decide whether $f \in \sqrt{I}$. The following lemma, which is sometimes called *Rabinowich's trick*, is the basis for solving this problem.¹

Lemma 1.13. Let A be a ring, $I \subset A$ an ideal and $f \in A$. Then

$$f \in \sqrt{I} : \iff : 1 \in \tilde{I} := \langle I, 1 - tf \rangle_{A[t]}$$

where t is an additional new variable.

SINGULAR Example 4.

```
ring A =0, (x,y,z), dp;
ideal I=x5,xy3,y7,z3+xyz;
poly f =x+y+z;

ring B =0, (t,x,y,z), dp; //need t for radical test
ideal I=imap(A,I);
poly f =imap(A,f);
I=I,1-t*f;
std(I);
//-> _[1]=1                //f is in the radical
```

¹We can even compute the full radical \sqrt{I} , but this is a much harder computation.

```
LIB"primdec.lib"; //just to see, we compute the radical
setring A;
radical(I);
//-> _[1]=z _[2]=y _[3]=x
```

Intersection of Ideals

Problem: Given $f_1, \dots, f_k, h_1, \dots, h_r \in K[x]$ and $>$ a monomial ordering. Let $I_1 = \langle f_1, \dots, f_k \rangle K[x]_{>}$ and $I_2 = \langle h_1, \dots, h_r \rangle K[x]_{>}$. We wish to find generators for $I_1 \cap I_2$.

Consider the ideal $J := \langle t f_1, \dots, t f_k, (1-t)h_1, \dots, (1-t)h_r \rangle (K[x]_{>})[t]$.

Lemma 1.14. *With the above notations, $I_1 \cap I_2 = J \cap K[x]_{>}$.*

SINGULAR Example 5.

```
ring A=0, (x,y,z), dp;
ideal I1=x,y;
ideal I2=y^2,z;
intersect(I1,I2); //the built-in SINGULAR command
//-> _[1]=y^2 _[2]=yz _[3]=xz
```

```
ring B=0, (t,x,y,z), dp; //the way described above
ideal I1=imap(A,I1);
ideal I2=imap(A,I2);
ideal J=t*I1+(1-t)*I2;
eliminate(J,t);
//-> _[1]=yz _[2]=xz _[3]=y^2
```

Quotient of Ideals

Problem: Let I_1 and $I_2 \subset K[x]_{>}$. We want to compute

$$I_1 : I_2 = \{g \in K[x]_{>} \mid gI_2 \subset I_1\}.$$

Since, obviously, $I_1 : \langle h_1, \dots, h_r \rangle = \bigcap_{i=1}^r (I_1 : \langle h_i \rangle)$, we can compute $I_1 : \langle h_i \rangle$ for each i . The next lemma shows a way to compute $I_1 : \langle h_i \rangle$.

Lemma 1.15. *Let $I \subset K[x]_{>}$ be an ideal, and let $h \in K[x]_{>}$, $h \neq 0$. Moreover, let $I \cap \langle h \rangle = \langle g_1 \cdot h, \dots, g_s \cdot h \rangle$. Then $I : \langle h \rangle = \langle g_1, \dots, g_s \rangle K[x]_{>}$.*

SINGULAR Example 6.

```
ring A=0, (x,y,z), dp;
ideal I1=x,y;
ideal I2=y^2,z;
quotient(I1,I2); //the built-in SINGULAR command
//-> _[1]=y _[2]=x
```


Kernel of a Ring Map

Let $\varphi : R_1 := (K[x]_{>_1})/I \rightarrow (K[y]_{>_2})/J =: R_2$ be a ring map defined by polynomials $\varphi(x_i) = f_i \in K[y] = K[y_1, \dots, y_m]$ for $i = 1, \dots, n$ (and assume that the monomial orderings satisfy $1 >_2 \text{LM}(f_i)$ if $1 >_1 x_i$).

Define $J_0 := J \cap K[y]$, and $I_0 := I \cap K[x]$. Then φ is induced by

$$\tilde{\varphi} : K[x]/I_0 \rightarrow K[y]/J_0, \quad x_i \mapsto f_i,$$

and we have a commutative diagram

$$\begin{array}{ccc} K[x]/I_0 & \xrightarrow{\tilde{\varphi}} & K[y]/J_0 \\ \downarrow & & \downarrow \\ R_1 & \xrightarrow{\varphi} & R_2. \end{array}$$

Problem: Let I, J and φ be as above. Compute generators for $\text{Ker}(\varphi)$.

Solution: Assume that $J_0 = \langle g_1, \dots, g_s \rangle_{K[y]}$ and $I_0 = \langle h_1, \dots, h_t \rangle_{K[x]}$.

Set $H := \langle h_1, \dots, h_t, g_1, \dots, g_s, x_1 - f_1, \dots, x_n - f_n \rangle \subset K[x, y]$, and compute $H' := H \cap K[x]$ by eliminating y_1, \dots, y_m from H . Then H' generates $\text{Ker}(\varphi)$ by the following lemma.

Lemma 1.16. *With the above notations, $\text{Ker}(\varphi) = \text{Ker}(\tilde{\varphi})R_1$ and*

$$\text{Ker}(\tilde{\varphi}) = (I_0 + \langle g_1, \dots, g_s, x_1 - f_1, \dots, x_n - f_n \rangle_{K[x,y]} \cap K[x]) \text{ mod } I_0.$$

In particular, if $>_1$ is global, then $\text{Ker}(\varphi) = \text{Ker}(\tilde{\varphi})$.

SINGULAR Example 7.

```
ring A=0, (x,y,z), dp;
ring B=0, (a,b), dp;
map phi=A, a2, ab, b2;
ideal zero; //compute the preimage of 0
setring A;
preimage(B, phi, zero); //the built-in SINGULAR command
//-> _[1]=y2-xz

ring C=0, (x,y,z,a,b), dp; //the method described above
ideal H=x-a2, y-ab, z-b2;
eliminate(H, ab);
//-> _[1]=y2-xz
```

2. LECTURE: POLYNOMIAL SOLVING AND PRIMARY DECOMPOSITION

Solvability of Polynomial Equations

Problem: Given $f_1, \dots, f_k \in K[x_1, \dots, x_n]$, we want to assure whether the system of polynomial equations

$$f_1(x) = \dots = f_k(x) = 0$$

has a solution in \bar{K}^n , where \bar{K} is the algebraic closure of K .

Let $I = \langle f_1, \dots, f_k \rangle_{K[x]}$, then the question is whether the algebraic set $V(I) \subset \bar{K}^n$ is empty or not.

Solution: By Hilbert's Nullstellensatz, $V(I) = \emptyset$ if and only if $1 \in I$. We compute a Gröbner basis G of I with respect to any global ordering on $\text{Mon}(x_1, \dots, x_n)$ and normalize it (that is, divide every $g \in G$ by $\text{LC}(g)$). Since $1 \in I$ if and only if $1 \in L(I)$, we have $V(I) = \emptyset$ if and only if 1 is an element of a normalized Gröbner basis of I . Of course, we can avoid normalizing, which is expensive in rings with parameters. Since $1 \in I$ if and only if G contains a non-zero constant polynomial, we have only to look for an element of degree 0 in G .

SINGULAR Example 8.

```
ring A=0,(x,y,z),lp;
ideal I=x2+y+z-1,
      x+y2+z-1,
      x+y+z2-1;
ideal J=groebner(I); //the lexicographical Groebner basis
J;
//-> J[1]=z6-4z4+4z3-z2      J[2]=2yz2+z4-z2
//-> J[3]=y2-y-z2+z         J[4]=x+y+z2-1
```

We use the multivariate solver based on triangular sets.

```
LIB"solve.lib";
list s1=solve(I,6);
//-> // name of new current ring: AC
s1;
//-> [1]:          [2]:          [3]:          [4]:          [5]:
//->   [1]:          [1]:          [1]:          [1]:          [1]:
//->     0.414214      0          -2.414214      1          0
//->   [2]:          [2]:          [2]:          [2]:          [2]:
//->     0.414214      0          -2.414214      0          1
//->   [3]:          [3]:          [3]:          [3]:          [3]:
//->     0.414214      1          -2.414214      0          0
```

If we want to compute the zeros with multiplicities then we use 1 as a third parameter for the command:

```

setring A;
list s2=solve(I,6,1);
s2;
//-> [1]: [2]:
//-> [1]: [1]:
//-> [1]: [1]:
//-> -2.414214 0
//-> [2]: [2]:
//-> -2.414214 1
//-> [3]: [3]:
//-> -2.414214 0
//-> [2]: [2]:
//-> [1]: [1]:
//-> 0.414214 1
//-> [2]: [2]:
//-> 0.414214 0
//-> [3]: [3]:
//-> 0.414214 0
//-> [2]: [3]:
//-> 1 [1]:
//-> 0
//-> [2]:
//-> 0
//-> [3]:
//-> 1
//-> [2]:
//-> 2

```

The output has to be interpreted as follows: there are two zeros of multiplicity 1 and three zeros $((0, 1, 0), (1, 0, 0), (0, 0, 1))$ of multiplicity 2.

Definition 2.1.

- (1) A maximal ideal $M \subset K[x_1, \dots, x_n]$ is called in general position with respect to the lexicographical ordering with $x_1 > \dots > x_n$, if there exist $g_1, \dots, g_n \in K[x_n]$ with $M = \langle x_1 + g_1(x_n), \dots, x_{n-1} + g_{n-1}(x_n), g_n(x_n) \rangle$.
- (2) A zero-dimensional ideal $I \subset K[x_1, \dots, x_n]$ is called in general position with respect to the lexicographical ordering with $x_1 > \dots > x_n$, if all associated primes P_1, \dots, P_k are in general position and if $P_i \cap K[x_n] \neq P_j \cap K[x_n]$ for $i \neq j$.

Proposition 2.2. Let K be a field of characteristic 0, and let $I \subset K[x]$, $x = (x_1, \dots, x_n)$, be a zero-dimensional ideal. Then there exists a non-empty, Zariski open subset $U \subset K^{n-1}$

such that for all $\underline{a} = (a_1, \dots, a_{n-1}) \in U$, the coordinate change $\varphi_{\underline{a}} : K[x] \rightarrow K[x]$ defined by $\varphi_{\underline{a}}(x_i) = x_i$ if $i < n$, and

$$\varphi_{\underline{a}}(x_n) = x_n + \sum_{i=1}^{n-1} a_i x_i$$

has the property that $\varphi_{\underline{a}}(I)$ is in general position with respect to the lexicographical ordering defined by $x_1 > \dots > x_n$.

Proposition 2.3. Let $I \subset K[x_1, \dots, x_n]$ be a zero-dimensional ideal. Let $\langle g \rangle = I \cap K[x_n]$, $g = g_1^{v_1} \dots g_s^{v_s}$, g_i monic and prime and $g_i \neq g_j$ for $i \neq j$. Then

$$(1) I = \bigcap_{i=1}^s \langle I, g_i^{v_i} \rangle.$$

If I is in general position with respect to the lexicographical ordering with $x_1 > \dots > x_n$, then

$$(2) \langle I, g_i^{v_i} \rangle \text{ is a primary ideal for all } i.$$

SINGULAR Example 9 (zero-dim primary decomposition).

We give an example for a zero-dimensional primary decomposition.

```
option(redSB);
ring R=0, (x,y), lp;
ideal I=(y^2-1)^2, x^2-(y+1)^3;
```

The ideal I is not in general position with respect to lp , since the minimal associated prime $\langle x^2 - 8, y - 1 \rangle$ is not.

```
map phi=R,x,x+y; //we choose a generic coordinate change
map psi=R,x,-x+y; //and the inverse map
I=std(phi(I));
I;
//-> I[1]=y^7-y^6-19y^5-13y^4+99y^3+221y^2+175y+49
//-> I[2]=112xy+112x-27y^6+64y^5+431y^4-264y^3-2277y^2-2520y-847
//-> I[3]=56x^2+65y^6-159y^5-1014y^4+662y^3+5505y^2+6153y+2100
factorize(I[1]);
//-> [1]:
//-> _[1]=1
//-> _[2]=y^2-2y-7
//-> _[3]=y+1
//-> [2]:
//-> 1,2,3

ideal Q1=std(I,(y^2-2y-7)^2); //the candidates for the
//primary ideals
ideal Q2=std(I,(y+1)^3); //in general position
Q1; Q2;

//-> Q1[1]=y^4-4y^3-10y^2+28y+49 Q2[1]=y^3+3y^2+3y+1
```

```

//-> Q1[2]=56x+y3-9y2+63y-7      Q2[2]=2xy+2x+y2+2y+1
                                   Q2[3]=x2

factorize(Q1[1]); //primary and general position test
                //for Q1

//-> [1]:
//->   _[1]=1
//->   _[2]=y2-2y-7
//-> [2]:
//->   1,2

factorize(Q2[1]); //primary and general position test
                //for Q2

//-> [1]:
//->   _[1]=1
//->   _[2]=y+1
//-> [2]:
//->   1,3

```

Both ideals are primary and in general position.

```

Q1=std(psi(Q1)); //the inverse coordinate change
Q2=std(psi(Q2)); //the result
Q1; Q2;

//-> Q1[1]=y2-2y+1      Q2[1]=y2+2y+1
//-> Q1[2]=x2-12y+4     Q2[2]=x2

```

We obtain that I is the intersection of the primary ideals Q_1 and Q_2 with associated prime ideals $\langle y-1, x^2-8 \rangle$ and $\langle y+1, x \rangle$.

The following proposition reduces the higher dimensional case to the zero-dimensional case:

Proposition 2.4. *Let $I \subset K[x]$ be an ideal and $u \subset x = \{x_1, \dots, x_n\}$ be a maximal independent set of variables² with respect to I .*

- (1) $IK(u)[x \setminus u] \subset K(u)[x \setminus u]$ is a zero-dimensional ideal.
- (2) Let $S = \{g_1, \dots, g_s\} \subset I \subset K[x]$ be a Gröbner basis of $IK(u)[x \setminus u]$, and let $h := \text{lcm}(LC(g_1), \dots, LC(g_s)) \in K[u]$, then

$$IK(u)[x \setminus u] \cap K[x] = I : \langle h^\infty \rangle,$$

and this ideal is equidimensional of dimension $\dim(I)$.

²It is maximal such that $I \cap K[u] = \langle 0 \rangle$.

- (3) Let $IK(u)[x \setminus u] = Q_1 \cap \cdots \cap Q_s$ be an irredundant primary decomposition, then also $IK(u)[x \setminus u] \cap K[x] = (Q_1 \cap K[x]) \cap \cdots \cap (Q_s \cap K[x])$ is an irredundant primary decomposition.

Finally we explain how to compute the radical.

Proposition 2.5. Let $I \subset K[x_1, \dots, x_n]$ be a zero-dimensional ideal and $I \cap K[x_i] = \langle f_i \rangle$ for $i = 1, \dots, n$. Moreover, let g_i be the squarefree part of f_i , then $\sqrt{I} = I + \langle g_1, \dots, g_n \rangle$.

The higher dimensional case can be reduced similarly to the primary decomposition to the zero-dimensional case.

3. LECTURE: INVARIANTS

The computation of the Hilbert function will be discussed and explained. Let K be a field.

Definition 3.1. Let $A = \bigoplus_{v \geq 0} A_v$ be a Noetherian graded K -algebra, and let $M = \bigoplus_{v \in \mathbb{Z}} M_v$ be a finitely generated graded A -module. The Hilbert function $H_M : \mathbb{Z} \rightarrow \mathbb{Z}$ of M is defined by

$$H_M(n) := \dim_K(M_n),$$

and the Hilbert–Poincaré series HP_M of M is defined by

$$HP_M(t) := \sum_{v \in \mathbb{Z}} H_M(v) \cdot t^v \in \mathbb{Z}[[t]][t^{-1}].$$

Theorem 3.2. Let $A = \bigoplus_{v \geq 0} A_v$ be a graded K -algebra, and assume that A is generated, as K -algebra, by $x_1, \dots, x_r \in A_1$. Then, for any finitely generated (positively) graded A -module $M = \bigoplus_{v \geq 0} M_v$,

$$HP_M(t) = \frac{Q(t)}{(1-t)^r} \text{ for some } Q(t) \in \mathbb{Z}[t].$$

Note that SINGULAR has a command which computes the numerator $Q(t)$ for the Hilbert–Poincaré series:

SINGULAR Example 10.

```
ring A=0, (t,x,y,z), dp;
ideal I=x5y2,x3,y3,xy4,xy7;
intvec v = hilb(std(I),1);
v;
//-> 1,0,0,-2,0,0,1,0
```

We obtain $Q(t) = t^6 - 2t^3 + 1$.

The latter output has to be interpreted as follows: if $v = (v_0, \dots, v_d, 0)$ then $Q(t) = \sum_{i=0}^d v_i t^i$.

Theorem 3.3. *Let $>$ be any monomial ordering on $K[x] := K[x_1, \dots, x_r]$, and let $I \subset K[x]$ be a homogeneous ideal. Then*

$$HP_{K[x]/I}(t) = HP_{K[x]/L(I)}(t),$$

where $L(I)$ is the leading ideal of I with respect to $>$.

Examples how to compute the Hilbert polynomial, the Hilbert–Samuel function, the degree respectively and the multiplicity and the dimension of an ideal can be found in [7]. As above all computations are reduced to compute the corresponding invariants for the leading ideal.

4. LECTURE: HOMOLOGICAL ALGEBRA

Here we will show different approaches how to test Cohen–Macaulayness using SINGULAR. More details about the underlying theory can be found in [7].

SINGULAR Example 11 (first test for Cohen–Macaulayness).

Let (A, \mathfrak{m}) be a local ring, $\mathfrak{m} = \langle x_1, \dots, x_n \rangle$. Let M be an A -module given by a presentation $A^\ell \rightarrow A^s \rightarrow M \rightarrow 0$. To check whether M is Cohen–Macaulay we use that the equality

$$\begin{aligned} \dim(A/\text{Ann}(M)) &= \dim(M) = \text{depth}(M) \\ &= n - \sup\{i \mid H_i(x_1, \dots, x_n, M) \neq 0\}. \end{aligned}$$

is necessary and sufficient for M to be Cohen–Macaulay. The following procedure computes $\text{depth}(\mathfrak{m}, M)$, where $\mathfrak{m} = \langle x_1, \dots, x_n \rangle \subset A = K[x_1, \dots, x_n]_{>}$ and M is a finitely generated A -module with $\mathfrak{m}M \neq M$.

The following procedures use the procedures Koszul Homology from `homolog.lib` and `Ann` from `primdec.lib` to compute the Koszul Homology $H_i(x_1, \dots, x_n, M)$ and the annihilator $\text{Ann}(M)$. They have to be loaded first.

```
LIB "homolog.lib";
proc depth(module M)
{
  ideal m=maxideal(1);
  int n=size(m);
  int i;
  while(i<n)
  {
    i++;
    if(size(KoszulHomology(m,M,i))==0){return(n-i+1);}
  }
  return(0);
}
```

Now the test for Cohen–Macaulayness is easy.

```
LIB "primdec.lib";
proc CohenMacaulayTest(module M)
{
  return(depth(M)==dim(std(Ann(M))));
}
```

The procedure returns 1 if M is Cohen–Macaulay and 0 if not.

As an application, we check that a complete intersection is Cohen–Macaulay and that $K[x, y, z]_{\langle x, y, z \rangle} / \langle xz, yz, z^2 \rangle$ is not Cohen–Macaulay.

```
ring R=0, (x, y, z), ds;
ideal I=xz, yz, z2;
module M=I*freemodule(1);
CohenMacaulayTest(M);
//-> 0
```

```
I=x2+y2, z7;
M=I*freemodule(1);
CohenMacaulayTest(M);
//-> 1
```

SINGULAR Example 12 (second test for Cohen–Macaulayness).

Let $A = K[x_1, \dots, x_n]_{\langle x_1, \dots, x_n \rangle} / I$. Using Noether normalization, we may assume that $A \supset K[x_{s+1}, \dots, x_n]_{\langle x_{s+1}, \dots, x_n \rangle} =: B$ is finite. We choose a monomial basis $m_1, \dots, m_r \in K[x_1, \dots, x_s]$ of $A|_{x_{s+1}=\dots=x_n=0}$.

Then m_1, \dots, m_r is a minimal system of generators of A as B -module. A is Cohen–Macaulay if and only if A is a free B -module, that is, there are no B -relations between m_1, \dots, m_r , in other words, $\text{syz}_A(m_1, \dots, m_r) \cap B^r = \langle 0 \rangle$. This test can be implemented in SINGULAR as follows:

```
proc isCohenMacaulay(ideal I)
{
  def A    = basering;
  list L   = noetherNormal(I);
  map phi  = A, L[1];
  I        = phi(I);
  int s    = nvars(basering)-size(L[2]);
  execute("ring B=( "+charstr(A)+" ), x(1..s), ds;");
  ideal m  = maxideal(1);
  map psi  = A, m;
  ideal J  = std(psi(I));
  ideal K  = kbase(J);
  setring A;
  execute("
    ring C=( "+charstr(A)+" ), (" +varstr(A)+" ), (dp(s), ds);");
```



```

ideal I = imap(A,I);
qring D = std(I);
ideal K = fetch(B,K);
module N = std(syz(K));
intvec v = leadexp(N[size(N)]);
int i=1;
while((i<s)&&(v[i]==0)){i++;}
setring A;
if(!v[i]){return(0);}
return(1);
}

```

As the above procedure uses `noetherNormal` from `algebra.lib`, we first have to load this library.

```

LIB"algebra.lib";
ring r=0,(x,y,z),ds;
ideal I=xz,yz;
isCohenMacaulay(I);
//-> 0

```

```

I=x2-y3;
isCohenMacaulay(I);
//-> 1

```

SINGULAR Example 13 (3rd test for Cohen–Macaulayness).

We use the Auslander–Buchsbaum formula to compute the depth of M and then check if $\text{depth}(M) = \dim(M) = \dim(A/\text{Ann}(M))$.

We assume that $A = K[x_1, \dots, x_n]_{\langle x_1, \dots, x_n \rangle} / I$ and compute a minimal free resolution. Then $\text{depth}(A) = n - \text{pd}_{K[x_1, \dots, x_n]_{\langle x_1, \dots, x_n \rangle}}(A)$. If M is a finitely generated A -module of finite projective dimension, then we compute a minimal free resolution of M and obtain $\text{depth}(M) = \text{depth}(A) - \text{pd}_A(M)$.

```

proc projdim(module M)
{
  list l=mres(M,0);          //compute the resolution
  int i;
  while(i<size(l))
  {
    i++;
    if(size(l[i])==0){return(i-1);}
  }
}

```

Now it is easy to give another test for Cohen–Macaulayness.

```

proc isCohenMacaulay1(ideal I)

```

```

{
  int de=nvars(basing)-projdim(I*freemodule(1));
  int di=dim(std(I));
  return(de==di);
}

```

```

ring R=0,(x,y,z),ds;
ideal I=xz,yz;
isCohenMacaulay1(I);
//-> 0

```

```

I=x2-y3;
isCohenMacaulay1(I);
//-> 1

```

```

I=xz,yz,xy;
isCohenMacaulay1(I);
//-> 1
kill R;

```

The following procedure checks whether the depth of M is equal to d . It uses the procedure `Ann` from `primdec.lib`.

```

proc CohenMacaulayTest1(module M, int d)
{
  return((d-projdim(M))==dim(std(Ann(M))));
}

```

```

LIB"primdec.lib";
ring R=0,(x,y,z),ds;
ideal I=xz,yz;
module M=I*freemodule(1);
CohenMacaulayTest1(M,3);
//-> 0

```

```

I=x2+y2,z7;
M=I*freemodule(1);
CohenMacaulayTest1(M,3);
//-> 1

```

REFERENCES

- [1] Cox, D.; Little, J.; O'Shea, D.: Ideals, Varieties and Algorithms. Springer (1992).
- [2] Decker, W.; Lossen, Chr.: Computing in Algebraic Geometry; A quick start using SINGULAR. Springer, (2006).
- [3] Decker, W.; Greuel, G.-M.; Pfister, P.: Primary Decomposition: Algorithms and Comparisons. In: Algorithmic Algebra and Number Theory, Springer, 187–220 (1998).

- [4] Decker, W.; Greuel, G.-M.; de Jong, T.; Pfister, G.: The Normalization: a new Algorithm, Implementation and Comparisons. In: Proceedings EUROCONFERENCE Computational Methods for Representations of Groups and Algebras (1.4. – 5.4.1997), Birkhäuser, 177–185 (1999).
- [5] Dickenstein, A.; Emiris, I.Z.: Solving Polynomial Equations; Foundations, Algorithms, and Applications. Algorithms and Computations in Mathematics, Vol. 41, Springer, (2005).
- [6] Eisenbud, D.; Grayson, D.; Stillman, M., Sturmfels, B.: Computations in Algebraic Geometry with Macaulay2. Springer, (2001).
- [7] Greuel, G.-M.; Pfister G.: A Singular Introduction to Commutative Algebra. Springer 2008.
- [8] Greuel, G.-M.; Pfister, G.: SINGULAR and Applications, Jahresbericht der DMV 108 (4), 167-196, (2006).
- [9] Kreuzer, M.; Robbiano, L.: Computational Commutative Algebra 1. Springer (2000).
- [10] Vasconcelos, W.V.: Computational Methods in Commutative Algebra and Algebraic Geometry. Springer (1998).

Computer Algebra Systems

- [11] ASIR (Noro, M.; Shimoyama, T.; Takeshima, T.): <http://www.asir.org/>.
- [12] CoCoA (Robbiano, L.): A System for Computation in Algebraic Geometry and Commutative Algebra. Available from cocoa.dima.unige.it/cocoa
- [13] Macaulay 2 (Grayson, D.; Stillman, M.): A Computer Software System Designed to Support Research in Commutative Algebra and Algebraic Geometry. Available from <http://math.uiuc.edu/Macaulay2>.
- [14] SINGULAR (Greuel, G.-M.; Pfister, G.; Schönemann, H.): A Computer Algebra System for Polynomial Computations. Centre for Computer Algebra, University of Kaiserslautern, free software under the GNU General Public Licence (1990-2007). <http://www.singular.uni-kl.de>.