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COMBINATORICS OF TRIANGULATIONS AND HILBERT SERIES

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1. LECTURE: GRÖBNER BASIS THEORY

In this lecture we describe the basics of Gröbner basis theory from the point of weight vectors and polyhedral combinatorics. All material can be found in the book by Sturmfels [11].

Let $S = k[x_1, \dots, x_n]$ be the polynomial ring in n variables over the field k . For a vector $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ we write \mathbf{x}^α for the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. A termorder \preceq on the set of monomials in S is a linear order such that $1 \preceq m$ for all monomials m and such that $m \prec m'$ implies $mn \preceq m'n$ for all monomials m, m', n .

Let $f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha \mathbf{x}^\alpha$ be a polynomial in S ; that is $c_\alpha \in k$ and for almost all α we have $c_\alpha = 0$. For a given term order \preceq the leading term $\text{lt}_\preceq(f)$ is $c_\alpha \mathbf{x}^\alpha$ for the largest monomial \mathbf{x}^α for which $c_\alpha \neq 0$. For an ideal I in S the initial ideal $\text{in}_\preceq(I)$ is the ideal generated by the leading terms of the polynomials in I .

Clearly, for $n \geq 2$ there are infinitely many different term orders. But for a fixed ideal I the number of possible initial ideals is finite.

Theorem 1.1. *Let I be an ideal in S then there are only finitely many initial ideals of I .*

This result has an interesting consequence. It shows that for a given ideal I there is a finite set of polynomials that is a Gröbner basis for I for any term order \preceq . Before we proceed to that consequence, we review some basic definitions.

A set of polynomials $G \subseteq I$ is a Gröbner basis for the ideal I with respect to a given term order \preceq if $\text{in}_\preceq(I) = \langle \text{lt}_\preceq(f) \mid f \in G \rangle$. We call a Gröbner basis G reduced if $\text{lt}_\preceq(f)$ does not divide any monomial occurring in any $g \in G \setminus \{f\}$. Usually, in addition one assumes that in a reduced Gröbner basis the leading term of any polynomial has coefficient 1. The conditions for reduced Gröbner bases imply:

Lemma 1.2. *For any termorder \preceq and any ideal I there is a unique finite Gröbner basis of I .*

The preceding Lemma together with Theorem 1.1 then imply:

Corollary 1.3. *For any ideal I there is a finite set G of polynomials that is a Gröbner basis of I for all term order \preceq .*

For a given ideal I a set of polynomials G satisfying Corollary 1.3 is called an universal Gröbner basis.

Corollary 1.3 is proved by taken G as the union of reduced Gröbner basis of I , one for each of the finitely many initial ideals.

Motivated by this fact it becomes an interesting task to determine all initial ideals of a given ideal. It has turned out that for that purpose it is most suitable to consider a geometric structure.

For an $\omega \in \mathbb{R}^n$ we consider the following partial order on the set of monomials in S . We set $\mathbf{x}^\alpha < \mathbf{x}^\beta$ if and only if $\alpha \cdot \omega < \beta \cdot \omega$. Clearly, \leq depends on ω and in general is not a term order. It usually fails to be a linear order and if ω contains negative entries it may fail $1 \preceq \mathbf{x}^\alpha$ for some α . Now for a polynomial $f = \sum_{\alpha \in \mathbb{N}^n} c_\alpha \mathbf{x}^\alpha$ the leading term $\text{lt}_\omega(f)$ with respect to \leq is the sum $\sum_{\alpha \in J} c_\alpha \mathbf{x}^\alpha$, where J is the set of α for which $c_\alpha \neq 0$ and there is no $\mathbf{x}^\alpha < \mathbf{x}^\beta$ such that $c_\beta \neq 0$. Analogously, the initial ideal $\text{in}_\omega(I)$ is defined as the ideal generated by $\text{lt}_\omega(f)$ for $f \in I$.

Even though in general ω fails the conditions for a term order, for a given ideal any term order in some sense can be represented by some ω .

Theorem 1.4. *For any ideal I and any term order \preceq there is an $\omega \in \mathbb{R}^n$ such that $\text{in}_\prec(I) = \text{in}_\omega(I)$.*

We call a vector $\omega \in \mathbb{R}^n$ a representable vector if there is a termorder \prec such that $\text{in}_\omega(I) = \text{in}_\prec(I)$.

In a next step one collects for a given ideal I the ω which lead to the same $\text{in}_\omega(I)$. For $\omega \in \mathbb{R}^n$ we write $C[\omega]$ for the set of all $\omega'' \in \mathbb{R}^n$ such that $\text{in}_\omega(I) = \text{in}_{\omega''}(I)$.

Proposition 1.5. *For a given ideal I and $\omega \in \mathbb{R}^n$ the set $C[\omega] \subseteq \mathbb{R}^n$ is a relatively open polyhedral cone.*

For basic definition from convex geometry we refer to the book by Barvinok [3]. We call the union of the closures $\overline{C[\omega]}$ for representable $\omega \in \mathbb{R}^n$ the Gröbner region of I .

Proposition 1.6. *If I is a homogeneous ideal then $\text{GR}(I) = \mathbb{R}^n$.*

Clearly, by definition for two representable $\omega, \omega' \in \mathbb{R}^n$ we have either $C[\omega] = C[\omega']$ or $C[\omega] \cap C[\omega'] = \emptyset$. Moreover, the face structure of the closure of $C[\omega]$ is nice.

Proposition 1.7. *Let $\omega \in \mathbb{R}^n$. If F is a face of the closure $\overline{C[\omega]}$ of $C[\omega]$ then there is an ω' such that $\overline{C[\omega']} = F$.*

Now we are in position to state the final result on the geometric structure of the set of initial ideal of a given ideal I in the case I is homogeneous.

Theorem 1.8. *Let I be a homogeneous ideal. Then the set of cones $C[\omega]$ for $\omega \in \mathbb{R}^n$ is a finite complete polyhedral fan.*

2. LECTURE: GRÖBNER BASIS, HILBERT SERIES AND EHRHART RINGS

In this section we seek applications of Gröbner basis theory in Combinatorics by means of Hilbert series. For that we review another part of standard Gröbner basis theory (see e.g. [1]) and outline applications to combinatorial questions.

Of a given ideal I and term order \preceq a monomial m is called standard if m does not lie in $\text{in}_{\preceq}(I)$.

Proposition 2.1. *The standard monomials form a k -basis of S/I .*

The preceding proposition becomes interesting when considering the Hilbert series of S/I . Recall that for a homogeneous ideal I the quotient S/I is a standard graded k -algebra A ; that is $A \cong \bigoplus_{i \geq 0} A_i$ as k -vectorspaces and $A_1 = k$, A is generated by A_1 , $A_i A_j \subseteq A_{i+j}$ and $\dim_k A_1 < \infty$.

Theorem 2.2 (Hilbert-Serre). *Let $A = \bigoplus_{i \geq 0} A_i$ be a standard graded k -algebra. Then*

$$\sum_{i \geq 0} \dim_k A_i t^i = \frac{h_0 + \cdots + h_s t^s}{(1-t)^d}$$

for d the Krull dimension of A and $1 = h_0, h_1, \dots, h_s \neq 0$ some integers.

For a standard graded k -algebra A the series $\sum_{i \geq 0} \dim_k A_i$ is called Hilbert series of A and denote by $\text{Hilb}(A, t)$.

Now Proposition 2.1 implies:

Lemma 2.3. *Let I be a homogeneous ideal and \preceq a term order. Then*

$$\text{Hilb}(S/I, t) = \text{Hilb}(S/\text{in}_{\preceq}(I), t).$$

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For applications in Combinatorics it is often useful to find for a given finite sequence $1 = h_0, \dots, h_s$ of integers a standard graded k -algebra A for which

$$\sum_{i \geq 0} \dim_k A_i t^i = \frac{h_0 + \cdots + h_s t^s}{(1-t)^d}.$$

Indeed this happens surprisingly often. For example if $h_i = \binom{n}{i}$ then $s = n$. For $I = \langle x_1^2, \dots, x_n^2 \rangle$ we have

$$\text{Hilb}(S/I, t) = \sum_{i=0}^n \binom{n}{i} t^i = (1+t)^n.$$

Let us now come to a more interesting example. For a permutation $\pi : [n] \rightarrow [n]$ we call $1 \leq i \leq n$ a descent if $\pi(i) > \pi(i+1)$. Let $\text{des}(\pi)$ be the number of descents of π .

Now if S_n denotes as usual the set of all permutations of $[n]$ then $A_n(t) := \sum_{\pi \in S_n} t^{\text{des}(\pi)}$ is the Eulerian polynomial. More generally, we are interested in the following situation:

Let P be a natural partial order on $[n]$; that is \leq_P is a partial order such that if $i \leq_P j$ then $i \leq j$ in the natural order. We call a permutation $\pi \in S_n$ a linear extension of P if for $i \leq_P j$ we have $\pi^{-1}(i) \leq \pi^{-1}(j)$. For a partial order P we write $\mathcal{L}(P)$ for the set of $\pi \in S_n$ that are linear extensions of P .

The polynomial

$$W_P(t) := \sum_{\pi \in \mathcal{L}(P)} t^{\text{des}(\pi)}$$

is called the W -polynomial of P . If P is the n -element antichain then all $\pi \in S_n$ are linear extensions of P and hence $\mathcal{L}(P) = S_n$. Then $W_P(t) = A_n(t)$ is the Eulerian polynomial. The polynomial $W_P(t)$ is the main object in the so called poset conjecture by Neggers.

Conjecture 2.4 (Neggers). *For any naturally labelled poset P the polynomial $W_P(t)$ has only real roots.*

The conjecture was recently shown to fail by Petter Brändén: It was motivated by the fact that the Eulerian polynomials are known to have only real roots. Nevertheless, it is still open whether unimodality of the coefficient sequence of the polynomials $W_P(t)$ holds. Recall that a sequence of numbers h_0, \dots, h_s is unimodal if there is an index j such that $h_0 \leq \dots \leq h_j \geq h_{j+1} \geq \dots \geq h_s$. Since it is easily seen that the coefficients of $W_P(t)$ are all strictly positive up to the degree by well known facts from analysis the poset conjecture implies unimodality of the coefficient sequence of the $W_P(t)$. So our goal now is to prove unimodality in certain cases using an algebraic construction and Gröbner basis theory.

In order to construct an algebra H_P for which $\text{Hilb}(H_P, t) = W_P(t)/(1-t)^d$ we need to introduce more concepts from Combinatorics.

If P is a partial order on $[n]$ then a lower order ideal O is a subset $O \subseteq P$ such that if $i \in O$ and $j \leq_P i$ then $j \in O$. By the well known basic theorem of distributive lattices we know that if P is a partial order on $[n]$ then the set of its order ideals ordered by inclusion is a distributive lattice L_P and conversely for any distributive lattice L there is a poset P such that $L = L_P$.

Consider $S_P = k[x_O \mid O \text{ order ideal in } P]$, the polynomial rings whose variables are indexed by order ideals of P . Let I_P be the ideal generated by $x_O x_{O'} - x_{O \cap O'} x_{O \cup O'}$. Note that if O and O' are order ideals in P then $O \cap O'$ and $O \cup O'$ are again order ideals in P . For a poset P on $[n]$ the ring $H_P := S_P/I_P$ is called the Hibi ring of P . Clearly, I_P is homogeneous and therefore H_P is a standard graded k -algebra.

Theorem 2.5 (Hibi). *Let P be a naturally labelled partial order on $[n]$. Then*

$$\text{Hilb}(H_P, t) = \frac{W_P(t)}{(1-t)^{\ell(P)}},$$

where $\ell(P)$ denotes that number of order ideals in a maximal chain of order ideals of P .

Already the initial proof of Theorem 2.5 uses Lemma 2.3. Here one shows that there is a term order \preceq for which $\text{in}_{\preceq}(I_P)$ is generated by all $x_O x_{O'}$ such that O and O' are incomparable lower order ideals. From that one deduced that the Hilbert series of H_P equals the Hilbert series of the Stanley-Reisner ring of the order complex of the distributive lattice associated to P . For this ring the desired equality is well known and can for example deduced via a shelling argument.

But the preceding proof using a reduction to order complexes of distributive lattices does not provide much insight in the enumerative structure of the polynomials $W_P(t)$. Since H_P is Cohen-Macaulay it tells us that the coefficient sequence of $W_P(t)$ is always an M-sequence but this does not imply unimodality. Recall that a sequence h_0, \dots, h_s of strictly positive natural numbers is called an M-sequence if there is a 0-dimensional standard graded k -algebra $B = B_0 \oplus \dots \oplus B_s$ such that $\dim_k B_i = h_i$.

This failure comes as there is no further structural insight into the Hilbert series of Stanley-Reisner rings of order complexes of distributive lattices than that the coefficient sequence of the numerator polynomial is an M-sequences. For more details on Stanley-Reisner rings and M-sequences we refer the reader to the book by Bruns & Herzog [4]. Let us consider a more restricted class of distributive lattices.

Proposition 2.6 (Hibi). *Let P be a naturally labelled partial order. Then the ring H_P is Gorenstein if and only if P is graded.*

Recall that a graded partial order is a partial order in which all maximal unrefinable chains have the same cardinality.

We will use deep and substantial information on the Hilbert series of boundary complexes of simplicial polytopes.

Theorem 2.7 (g-Theorem by Stanley and Billera & Lee). *Let h_0, \dots, h_d be a sequence of strictly positive natural numbers. Then there is a d -dimensional simplicial polytope P such that the Stanley-Reisner ring $k[\partial P]$ of the boundary complex ∂P of P has Hilbert series*

$$\text{Hilb}(k[\partial P], t) = \frac{h_0 + \dots + h_s}{(1-t)^d}$$

if and only if

- (i) $h_0 = 1$.
- (ii) $h_i = h_{d-i}$.
- (iii) For $g_0 = 1$ and $g_i = h_i - h_{i-1}$, $1 \leq i \leq \lfloor \frac{d}{2} \rfloor$ the sequence $g_0, \dots, g_{\lfloor \frac{d}{2} \rfloor}$ is an M-sequence.

Note that condition (i) - (iii) of the g-Theorem 2.7 imply that h_0, \dots, h_d is unimodal. Before we can state the application of the g-Theorem 2.7 to the unimodality consequence of the poset conjecture we need the following simple fact.

Lemma 2.8. *Let $\Delta = 2^V * \Delta'$ be a simplicial complex. Then*

$$\text{Hilb}(k[\Delta], t) = \text{Hilb}(k[\Delta'], t) \cdot \frac{1}{(1-t)^{\#V}}.$$

Now key result connecting Gröbner basis theory, g-Theorem and poset conjecture is the following.

Theorem 2.9 (Reiner & Welker [7]). *Let P be a graded naturally labelled poset. Then there is a term order \preceq such that $\text{in}_{\preceq}(I_P)$ is the Stanley-Reisner ideal of the simplicial complex $\partial P * 2^V$ for a simplicial polytope P and some set V .*

The desired consequence now is:

Corollary 2.10. *Let P be a graded naturally labelled poset. Then the coefficient sequence of $W_P(t)$ satisfies conditions (i) - (iii) of the g-Theorem. In particular, the coefficient series is unimodal.*

It turns out the the method of proof leading to Corollary 2.10 is surprisingly general. It was applied by Athanasiadis [2] to verify a long standing conjecture by Stanley on the

unimodality of the so called h^* -vector of the Birkhoff-Polytope. Finally Bruns & Römer [5] found the so far most general setting.

For a description of the result by Bruns & Römer we need the concept of an Ehrhart ring. Let $Q \subseteq \mathbb{R}^n$ be an integer polytope; that is a polytope such that all its vertices lie in \mathbb{Z}^n . Then consider the embedding of the polytope into \mathbb{R}^{n+1} by sending $\underline{x} \in \mathbb{R}^n$ to $(\underline{x}, \mathbf{1}) \in \mathbb{R}^{n+1}$. Clearly, the vertices of the embedded polytope are again integral. In order to avoid technical difficulties we assume that all the vertices of Q have actually non-negative integer coordinates. Consider, the algebra $k[Q] \subseteq k[x_1, \dots, x_{n+1}]$ that is generated by $\underline{x}^{(\alpha, \mathbf{1})}$ for $\alpha \in Q \cap \mathbb{Z}^n$. The algebra $k[Q]$ is called the Ehrhardt ring of Q .

Now let us return to the poset conjecture and the W -polynomial. Consider the following polytope. For any order ideal O in the partial order P let e_O be the 0/1-vector in \mathbb{R}^n whose i th entry is 0 if $i \notin O$ and 1 if $i \in O$. Then let $\mathcal{O}(P)$ be the convex hull of the e_O . The polytope $\mathcal{O}(P)$ is known as the order polytope of P and its vertices are the e_O . It is also known that it does not contain any integer point in its interior.

Lemma 2.11. *Let P be a naturally labelled partial order. Then $H_P \cong k[\mathcal{O}(P)]$.*

In order to be able to use Gröbner basis theory, we need to define the toric ideal I_Q corresponding to Q . For that let S_Q be the polynomial ring in the variables y_α where α runs through all integers points in Q . Then I_Q is the kernel of the map from S_Q to $k[Q]$ sending y_α to $\underline{x}^{(\alpha, \mathbf{1})}$.

Theorem 2.12 (Bruns & Römer [5]). *Let Q be an integer polytope that has a regular and unimodular triangulation and such that $k[Q]$ is Gorenstein. Then there is a term order \preceq for which $\text{in}_\prec(I_Q)$ is the Stanley-Reisner ideal of $\partial R * 2^V$ for the boundary complex ∂R of a simplicial polytope R and some set V .*

This indeed generalizes Theorem 2.9 since all assumptions are satisfied for $Q = \mathcal{O}(P)$. As a consequence one obtains:

Corollary 2.13. *Let Q be an integer polytope that has a regular and unimodular triangulation and such that $k[Q]$ is Gorenstein. Let*

$$\text{Hilb}(k[Q], t) = \frac{h_0 + \dots + h_s}{(1-t)^d}.$$

Then the sequence h_0, \dots, h_s satisfies (i) - (iii) of the g -Theorem 2.7. In particular, h_0, \dots, h_s is unimodal.

3. LECTURE: GRÖBNER BASIS FOR SOME GORENSTEIN DOMAINS

The assumption that all $k[Q]$ in Theorem 2.12 are Gorenstein domains is crucial. Indeed Hibi and Stanley conjecture that the coefficient sequence of the numerator polynomial of the Hilbert series of a Gorenstein domain is unimodal. Moreover, it is known that unimodality does not hold for Gorenstein algebras that are not domains. Therefore, it seems worthwhile to search for other Gorenstein domains for which a similar Gröbner approach works.

Here we outline two classical example.

- (i) **Determinantal Rings** Let $S_D = k[x_{ij} \mid 1 \leq i, j \leq n]$ be the polynomial ring in n^2 variables. For a number $r \geq 1$ let $I_{n,r}$ be the ideal generated by all $r \times r$ -minors of the matrix $(x_{ij})_{1 \leq i, j \leq n}$. It is well known that $S_D/I_{n,r}$ is a Gorenstein domain.
- (ii) **Pfaffian Rings** Let $S_P = k[x_{ij} \mid 1 \leq i < j \leq n]$. For a fixed r such that $2r \leq n$ let $J_{n,r}$ be the ideal generated by the degree r Pfaffians of the skew-symmetric matrix $(a_{ij})_{1 \leq i, j \leq n}$, where $a_{ii} = 0$, $a_{ij} = x_{ij}$ if $i < j$ and $a_{ij} = -x_{ij}$ if $i > j$. Recall that a degree r Pfaffian is obtained by taking the $2r \times 2r$ -minor corresponding to $2r$ column indices and the same row indices and then taking the square root of that minor. Again it is classical that $S_P/J_{n,r}$ is a Gorenstein domain.

Now one can prove the following:

Theorem 3.1 ([6]). *For each $2 \leq 2r \leq n$ there is a term order \preceq such that $\text{in}_{\preceq}(J_{n,r}) = I_{\Delta_{n,r}} * 2^V$ where $\Delta_{n,r}$ is the triangulation of a PL-sphere and V some set.*

Theorem 3.2 ([10]). *For each $1 \leq r \leq n$ there is a term order \preceq such that $\text{in}_{\preceq}(J_{n,r}) = I_{\Delta_{n,r}^B} * 2^V$ where $\Delta_{n,r}^B$ is the triangulation of homology sphere over the field with 2 elements and V some set.*

Neither of the preceding results allows the application of the g-Theorem 2.7 since neither $\Delta_{n,r}$ nor $\Delta_{n,r}^B$ have been shown to be boundary complexes of simplicial polytopes. But in either case the Stanley-Reisner ring of the complexes is Gorenstein* (over the field with 2 elements in the case of $\Delta_{n,r}^B$). So the consequence on the coefficient sequence of the numerator polynomial of the Hilbert series depends so far on the validity of the g-conjecture.

Conjecture 3.3 (g-Conjecture). *Let h_0, \dots, h_d be a sequence of strictly positive natural numbers. Then there is a Gorenstein* simplicial complex Δ such that the Stanley-Reisner ring $k[\Delta]$ has Hilbert series*

$$\text{Hilb}(k[\partial P], t) = \frac{h_0 + \dots + h_s}{(1-t)^d}$$

if and only if

- (i) $h_0 = 1$.
- (ii) $h_i = h_{d-i}$.
- (iii) For $g_0 = 1$ and $g_i = h_i - h_{i-1}$, $1 \leq i \leq \lfloor \frac{d}{2} \rfloor$ the sequence $g_0, \dots, g_{\lfloor \frac{d}{2} \rfloor}$ is an M-sequence.

Note that only the forward direction of the g-Conjecture is open, the backward direction already follows from the g-Theorem 2.7.

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