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# Notes on Deformation Theory

*Nitin Nitsure*  
*Guanajuato 2006*

## Abstract

These expository notes give an introduction to the elements of deformation theory, which is meant for graduate students interested in the theory of vector bundles and their moduli.

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## 1 Introduction

There are two basic examples, which motivate the subject of deformation theory. In each example, we have a natural notion of a family of deformations of a given type of geometric structure. This has a functorial formulation, which we now explain.

Let  $\mathbf{Schemes}_*$  be the **category of pointed schemes** over a chosen base field  $k$  (which may be assumed to be algebraically closed for simplicity), whose objects are defined to be pairs  $(S, s)$  where  $S$  is a scheme over  $k$  and  $s : \text{Spec } k \rightarrow S$  is a  $k$ -valued point, called the **base point**. Morphisms in this category are morphisms

of  $k$ -schemes which preserve the chosen base point. When considering deformations of various objects or structures, there naturally arise contravariant functors  $\varphi : \mathbf{Schemes}_* \rightarrow \mathbf{Sets}_*$  where  $\mathbf{Sets}_*$  is the category of pointed sets. We now give the two archetypal examples of such functors  $\varphi$ .

**Deformations of a variety  $X$ .** If  $X$  is a complete variety over  $k$ , and  $(S, s)$  is a pointed scheme, a deformation of  $X$  parametrised by  $(S, s)$  is a pair  $(\mathfrak{X}, i)$  where  $\mathfrak{X} \rightarrow S$  is a flat proper morphism of schemes and  $i : X \rightarrow \mathfrak{X}_s$  is an isomorphism of  $X$  with the fiber of  $\mathfrak{X}$  over  $s \in S$ . We say that two deformations  $(\mathfrak{X}, i)$  and  $(\mathfrak{Y}, j)$  parametrised by  $(S, s)$  are equivalent if there exists an isomorphism  $\alpha : \mathfrak{X} \rightarrow \mathfrak{Y}$  over  $S$  which takes  $i$  to  $j$ . We call the projection  $X \times S \rightarrow S$ , together with the identity isomorphism of  $X$  with its fiber over  $s$ , as a trivial deformation. The set  $\varphi(S, s)$  of all equivalence classes of deformations over  $(S, s)$  becomes a pointed set with base point the class of the trivial deformation. Given any morphism  $(T, t) \rightarrow (S, s)$  of pointed scheme, the pull-back of a deformation is a deformation, and as pull-backs preserve equivalences, this defines a contravariant functor  $\mathbf{Def}_X : \mathbf{Schemes}_* \rightarrow \mathbf{Sets}_*$ .

**Deformations of a vector bundle on  $X$ .** Let  $X$  be a variety over  $k$ , and let  $E$  be a vector bundle on  $X$ . We fix  $X$  and will vary  $E$ . Given any pointed scheme  $(S, s)$ , we consider all pairs  $(\mathcal{E}, i)$  where  $\mathcal{E}$  is a vector bundle on  $X \times S$ , and  $i : E \rightarrow \mathcal{E}_s$  is an isomorphism of  $E$  with the restriction  $\mathcal{E}_s$  of  $\mathcal{E}$  to  $X_s$  (which is naturally identified with  $X$ ). We say that two deformations  $(\mathcal{E}, i)$  and  $(\mathcal{F}, j)$  parametrised by  $(S, s)$  are equivalent if there exists an isomorphism  $\alpha : \mathcal{E} \rightarrow \mathcal{F}$  which takes  $i$  to  $j$ . We call the pullback of  $E$  to  $X \times S$ , together with the identity isomorphism of  $E$  with its restriction to  $X_s$ , as a trivial deformation. The set  $\varphi(S, s)$  of all equivalence classes of deformations over  $(S, s)$  becomes a pointed set with base point the class of the trivial deformation. Given any morphism  $(T, t) \rightarrow (S, s)$  of pointed scheme, the pull-back of a deformation is a deformation, and again this defines a contravariant functor  $\mathcal{D}_E : \mathbf{Schemes}_* \rightarrow \mathbf{Sets}_*$ .

**Relation with moduli problems.** Thus, so far one may say that we are looking at moduli problems of certain structures, with a chosen base point on the moduli. If a fine moduli space  $M$  exists, and if a point  $m_0$  of it corresponds to the starting structure (variety  $X$  or bundle  $E$  in the above examples), then  $\varphi(S, s_0)$  is just the set of all morphisms  $f : S \rightarrow M$  with  $f(s_0) = m_0$ . However, we will not assume that a fine moduli exists, and indeed it will not exist in the majority of examples where deformation theory can still give us interesting and important insights. But for that, we have to put a certain condition on the parameter space  $S$ , as follows.

**Local deformations.** We now introduce a condition on our parameter scheme  $(S, s_0)$  of deformations, which amounts to focussing attention on ‘infinitesimal’ deformations of the starting structure. We will assume that  $S$  is of the form  $\mathrm{Spec} A$  where  $A$  is a finite local  $k$ -algebra with residue field  $k$  (equivalently,  $A$  is an Artin local  $k$ -algebra with residue field  $k$ ). The unique  $k$ -valued point of  $\mathrm{Spec} A$  will be the base point  $s_0$ , and so there is no need to specify the base point. This means we will look at covariant functors  $D$  from the category  $\mathbf{Art}_k$  of Artin local  $k$ -algebras with residue field  $k$  to the category  $\mathbf{Sets}_*$  of pointed sets.

**Local structure of moduli.** If a fine moduli space  $M$  exists, then studying all possible deformations parametrised by objects of  $\mathbf{Art}_k$  is enough to recover the completion of the local ring of  $M$  at the point  $m_0$ . In this way, studying deformation theory sheds light on the local structure of moduli. In particular, we get to know what is the dimension of  $M$  at  $m_0$ , and whether  $M$  is non-singular at  $m_0$  via deformation theory done over  $\mathbf{Art}_k$ .

**The plan of these lecture notes.** These notes give an introduction to the elementary aspects of deformation theory, focussing on the deformation of vector bundles. The approach is algebraic, based on functors of Artin rings. In section 2 we begin with some basic definitions, and then focus on first order deformations, giving important basic examples. Section 3 gives the proofs of the theorems of Grothendieck and Schlessinger on pro-representability of a deformation functor and existence of versal families of deformations. This is applied to some important basic examples. In section 4, the obstruction space for prolongation of a deformation is calculated for some examples.

All the above material is standard, with no originality on my part except in minor points of arguments.

**Literature.** There is a vast amount of literature on deformation theory. What follows is a short (and very incomplete) list of some reading material, to start with. For quick look at the theory, a beginner can see the chapter 6 by Fantechi and Göttsche, followed by chapter 8 by Illusie of the multi-author book ‘Fundamental Algebraic Geometry: Grothendieck’s FGA Explained’. An quick introduction, focusing on applications to vector bundles, is given in the book of Huybrechts and Lehn ‘The geometry of moduli spaces of sheaves’. A very readable elementary introduction in lecture-note format is given by the notes of Ravi Vakil (MIT lecture course, available on the web). For a more complete treatment, one can see the recent book by Sernesi titled ‘Deformation of Schemes’.

There are also other approaches to deformation theory. A good account of the classical results of Kodaira-Spencer, with which the modern subject of deformation theory started, is in Kodaira’s book ‘Complex Manifolds and Deformation of Complex Structures’. A more advanced algebraic approach, via the cotangent complex, is due to Illusie, as expounded in his book ‘Complexe Cotangent et Déformations’ parts I and II. Yet another modern approach, based on differential graded lie algebras, can be read in the lecture notes of Kontsevich which are widely circulated (available on web).

## 2 First order deformations and tangent spaces to functors

For simplicity, we will work over a fixed base field  $k$  which we assume to be algebraically closed. All schemes and morphisms between them will be assumed to be over the base  $k$ , unless otherwise indicated. We denote by  $\mathbf{Rings}_k$  the category of all commutative  $k$ -algebras with unity, by  $\mathbf{Schemes}$  the category of all schemes over  $k$ , by  $\mathbf{Schemes}_*$  the category of all pointed schemes over  $k$ ,  $\mathbf{Sets}$  the category of all sets, and by  $\mathbf{Sets}_*$  the category of all pointed sets.

### 2.1 Functor of points

To any scheme  $X$ , we associate a covariant functor  $h_X$  from  $\mathbf{Rings}_k$  to  $\mathbf{Sets}$  called **the functor of points** of  $X$ . By definition, given any  $k$ -algebra  $R$ ,  $h_X(R)$  is the set of all morphisms of  $k$ -schemes from  $\text{Spec } R$  to  $X$ . The set  $h_X(R)$  is called the **set of  $R$ -valued points** of  $X$ .

**Example** If  $X$  is a variety over  $k$  (or more generally, a scheme of locally finite type over  $k$ ), then a  $k$ -valued point of  $X$  is the same as a closed point  $x \in X$ . (Recall that we have assumed  $k$  to be algebraically closed.)

Any scheme  $X$  can be recovered from its functor of points  $h_X$ . The set of all morphisms  $X \rightarrow Y$  between two schemes is naturally bijective with the set of all natural transformations  $h_X \rightarrow h_Y$ . Note that these statements are stronger than just the purely categorical Yoneda lemma, as we have confined ourselves to points with values in affine schemes.

We say that a functor  $\mathfrak{X} : \mathbf{Rings}_k \rightarrow \mathbf{Sets}$  is **representable** if  $\mathfrak{X}$  is naturally isomorphic to the functor of points  $h_X$  of some scheme  $X$  over  $k$ . If  $X$  is a scheme over  $k$  and  $\alpha : h_X \rightarrow \mathfrak{X}$  is a natural isomorphism, then we say that the pair  $(X, \alpha)$  **represents** the functor  $\mathfrak{X}$ . The scheme  $X$  is called a **representing scheme** or **moduli scheme** for  $\mathfrak{X}$ , and the natural isomorphism  $\alpha$  is called a **universal family** or a **Poincaré family** over  $X$ . The pair  $(X, \alpha)$  is unique up to a unique isomorphism.

A scheme  $X$  can be recovered from its functor of points  $h_X$ , therefore in principle all possible data concerning  $X$  can be read off from  $h_X$ . In order to see how to recover the tangent space  $T_x X$  at a  $k$ -valued point  $x \in X$ , we need some elementary facts involving linear algebra and Artin local rings.

### 2.2 Linear algebraic preliminaries

**Lemma 1** *Let  $\mathbf{Vect}_k$  be the category of all vector spaces over  $k$ , with  $k$ -linear maps as morphisms, and let  $\mathbf{FinVect}_k$  be its full subcategory consisting of all finite di-*

dimensional vector spaces. Let

$$f : \mathbf{FinVect}_k \rightarrow \mathbf{Sets}$$

be a functor which satisfies the following:

(1) For the zero vector space  $0$ , the set  $f(0)$  is a singleton set.

(2) The natural map  $\beta_{V,W} : f(V \times W) \rightarrow f(V) \times f(W)$  induced by applying  $f$  to the projections  $V \times W \rightarrow V$  and  $V \times W \rightarrow W$  is bijective.

Then for each  $V$  in  $\mathbf{FinVect}_k$ , there exists a unique structure of a  $k$  vector space on the set  $f(V)$  which gives a lift of  $f$  to a  $k$ -linear functor

$$F : \mathbf{FinVect}_k \rightarrow \mathbf{Vect}_k.$$

Let  $T = F(k)$ . Then there exists an isomorphism

$$\Psi_{F,V} : F(V) \rightarrow T \otimes_k V$$

which is functorial in both  $V$  and  $F$ . If  $f$  and  $g$  are two functors from  $\mathbf{FinVect}_k$  to  $\mathbf{Sets}$  which satisfy the conditions (1) and (2), and if  $\alpha : f \rightarrow g$  is a morphism of functors, then for each  $V$  in  $\mathbf{FinVect}_k$ , the map  $\alpha_V : f(V) \rightarrow g(V)$  is linear with respect to the vector space structure on  $f(V)$  and  $g(V)$ , consequently  $\alpha$  induces a natural transformation between the lifts of the functors  $f$  and  $g$  to  $\mathbf{Vect}_k$ .

**Proof** A functor  $\phi : \mathbf{FinVect}_k \rightarrow \mathbf{Vect}_k$  is called  $k$ -linear if the induced map  $\text{Hom}(U, V) \rightarrow \text{Hom}(\phi(U), \phi(V))$  is  $k$ -linear for any two  $U, V$  in  $\mathbf{FinVect}_k$ .

The requirement of  $k$ -linearity of the functor  $F$  forces us to define the addition map  $f(V) \times f(V) \rightarrow f(V)$  to be the composite map

$$f(V) \times f(V) \xrightarrow{\beta_{V,V}^{-1}} f(V \times V) \xrightarrow{f(+)} f(V)$$

where  $\beta_{V,V}^{-1}$  is the inverse of the natural isomorphism given by the assumption on  $f$ , and  $f(+)$  is obtained by applying  $f$  to the addition map  $+: V \times V \rightarrow V$ . Also, for any  $\lambda \in k$ , the requirement of  $k$ -linearity of the functor  $F$  forces us to define the scalar multiplication map  $\lambda_{f(V)} : f(V) \rightarrow f(V)$  to be the map  $f(\lambda_V)$ , as we must have  $\lambda_{f(V)} = \lambda 1_{f(V)} = \lambda f(1_V) = f(\lambda 1_V) = f(\lambda_V)$ . It can be verified directly that these operations indeed give a vector space structure on  $f(V)$ .

The rest is a simple exercise. □

**Lemma 2** Let  $T$  be a finite-dimensional vector space. Then the  $k$ -linear functor  $F : \mathbf{FinVect}_k \rightarrow \mathbf{FinVect}_k$  defined by  $V \mapsto T \otimes_k V$  is representable. Let  $\mathbf{1}_T \in F(T^*) = T \otimes_k T^* = \text{End}_k(T)$  be the identity map on  $T$ . Then the pair  $(T^*, \mathbf{1}_T)$  represents  $F$ . □

## 2.3 Artin local algebras

Let  $k$  be a field. Let  $\mathbf{Art}_k$  be the category of all artin local  $k$ -algebras, with residue field  $k$ . The morphisms in this category are all  $k$ -algebra homomorphisms, and it can be seen that these are automatically local (take the maximal ideal into the maximal ideal). Any such  $k$ -algebra is finite over  $k$ .

Note that  $k$  is both an initial and a final object of  $\mathbf{Art}_k$ . In particular, any functor  $F : \mathbf{Art}_k \rightarrow \mathbf{Sets}$  has a natural lift to the category  $\mathbf{Sets}_*$  of pointed sets.

If  $f : B \rightarrow A$  and  $g : C \rightarrow A$  are homomorphisms in  $\mathbf{Art}_k$ , the fibred product

$$B \times_A C = \{(b, c) \mid f(b) = g(c) \in A\}$$

with component-wise operations is again an object in  $\mathbf{Art}_k$  (Exercise). Also, for homomorphisms  $A \rightarrow B$  and  $A \rightarrow C$  in  $\mathbf{Art}_k$ , the tensor product  $B \otimes_A C$  is again an object in  $\mathbf{Art}_k$  (Exercise). Thus,  $\mathbf{Art}_k$  admits both fibred products (pullbacks)  $B \times_A C$  and tensor products (pushouts)  $B \otimes_A C$ .

As  $k$  is the final object in  $\mathbf{Art}_k$ , the fibred product  $A \times_k B$  serves as the direct product in the category  $\mathbf{Art}_k$ , and as  $k$  is the initial object in  $\mathbf{Art}_k$ , the tensor product  $B \otimes_A C$  serves as the coproduct in the category  $\mathbf{Art}_k$ .

The monics in  $\mathbf{Art}_k$  are clearly the same as the injections and the epics in  $\mathbf{Art}_k$  are the same as the surjections as can be seen by applying the Nakayama lemma (Exercise).

An important full subcategory of  $\mathbf{Art}_k$  consists all objects  $A$  in  $\mathbf{Art}_k$  whose maximal ideal  $\mathfrak{m}_A$  satisfies  $\mathfrak{m}_A^2 = 0$ . This subcategory is equivalent to the category  $\mathbf{FinVect}_k$  of all finite dimensional  $k$ -vector spaces as follows. For a  $k$ -vector space  $V$ , let  $k\langle V \rangle = k \oplus V$  with ring multiplication defined by putting  $(a, v)(b, w) = (ab, aw + bv)$ , and obvious  $k$ -algebra structure. Note that  $k\langle V \rangle$  is artinian if and only if  $V$  is finite dimensional. It can be seen that  $V \mapsto k\langle V \rangle$  defines a fully faithful functor  $\mathbf{FinVect}_k \rightarrow \mathbf{Art}_k$ , and any  $A$  in  $\mathbf{Art}_k$  with  $\mathfrak{m}_A^2 = 0$  is naturally isomorphic to  $k\langle \mathfrak{m}_A \rangle$ . The functor  $V \mapsto k\langle V \rangle$  takes the zero vector space (which is both an initial and final object of  $\mathbf{FinVect}_k$ ) to the algebra  $k$  (which is both an initial and final object of  $\mathbf{Art}_k$ ). If  $V \rightarrow U$  and  $W \rightarrow U$  are morphisms in  $\mathbf{FinVect}_k$ , then it can be seen that the natural map

$$k\langle V \times_U W \rangle \rightarrow k\langle V \rangle \times_{k\langle U \rangle} k\langle W \rangle$$

(which is induced by the projections from  $V \times_U W$  to  $V$  and  $W$ ) is an isomorphism. Therefore the functor  $\mathbf{FinVect}_k \rightarrow \mathbf{Art}_k$  preserves all finite limits, in particular, it preserves equalisers.

**Caution** The functor  $\mathbf{FinVect}_k \rightarrow \mathbf{Art}_k : V \mapsto k\langle V \rangle$  does not preserve co-products.

## 2.4 Tangent space of a functor

Let  $\varphi : \mathbf{Art}_k \rightarrow \mathbf{Sets}$  be any functor such that

- (1)  $\varphi(k)$  is a singleton set,
- (2) For any objects  $A, B$  in  $\mathbf{Art}_k$ , the induced map  $\varphi(A \times_k B) \rightarrow \varphi(A) \times \varphi(B)$  is a bijection.

Then the composite functor  $\mathbf{FinVect}_k \rightarrow \mathbf{Art}_k \rightarrow \mathbf{Sets}$  sending  $V \mapsto \varphi(k\langle V \rangle)$  satisfies hypothesis of Lemma 1. Let  $T(\varphi)$  denote the vector space

$$T(\varphi) = \varphi(k\langle k^1 \rangle) = \varphi(k[\epsilon]/(\epsilon^2)),$$

so that the composite functor  $\mathbf{FinVect}_k \rightarrow \mathbf{Art}_k \rightarrow \mathbf{Sets}$  is isomorphic to the functor which maps  $V \mapsto T(\varphi) \otimes_k V$ . We call  $T(\varphi)$  the **tangent vector space to the functor**  $\varphi$ . We denote it simply by  $T$  if  $\varphi$  is understood.

**Example** Let  $R$  be a local  $k$ -algebra with residue field  $k$ . Then the functor  $\varphi = \text{Hom}_{k\text{-alg}}(R, -) : \mathbf{Art}_k \rightarrow \mathbf{Sets}$  satisfies the above conditions. We determine the corresponding  $T$ . Note that a  $k$ -homomorphism  $R \rightarrow \varphi(k\langle V \rangle)$  is determined by the induced linear map  $\mathfrak{m}_R \rightarrow V$ , which must map  $\mathfrak{m}_R^2$  to 0. Conversely, any linear map  $\mathfrak{m}_R \rightarrow V$  which map  $\mathfrak{m}_R^2$  to 0, prolongs to a unique  $k$ -algebra homomorphism  $R \rightarrow k\langle V \rangle$ . This defines a natural isomorphism of the composite functor  $\mathbf{FinVect}_k \rightarrow \mathbf{Art}_k \rightarrow \mathbf{Sets}$  with the functor  $V \mapsto \text{Hom}_{\text{Vect}_k}(\mathfrak{m}_R/\mathfrak{m}_R^2, V) = (\mathfrak{m}_R/\mathfrak{m}_R^2)^* \otimes_k V$ , where  $(\mathfrak{m}_R/\mathfrak{m}_R^2)^*$  denotes the dual vector space of  $\mathfrak{m}_R/\mathfrak{m}_R^2$ . Hence we get

$$T = (\mathfrak{m}_R/\mathfrak{m}_R^2)^*.$$

### Application to the tangent space of a scheme

Let  $X$  be a scheme over  $k$ , and  $x \in X$  a  $k$ -valued point (such a point is necessarily closed in  $X$ , and all closed points of  $X$  are of this form if  $X$  is of locally finite type over the algebraically closed field  $k$ ). Let  $h_{X,x} : \mathbf{Art}_k \rightarrow \mathbf{Sets}_*$  be the functor defined by putting  $h_{X,x}(A)$  to be the pointed set consisting of all morphisms  $\text{Spec } A \rightarrow X$  such that the composite morphism

$$\text{Spec } k \rightarrow \text{Spec } A \rightarrow X$$

is the  $k$ -valued point  $x$ . The distinguished element of the pointed set  $h_{X,x}(A)$  is defined to be the composite morphism

$$\text{Spec } A \rightarrow \text{Spec } k \xrightarrow{x} X.$$

**Proposition 3** *The functor  $h_{X,x} : \mathbf{Art}_k \rightarrow \mathbf{Sets}_*$  preserves the (initial and) final object, equalisers, and direct products, and so it preserves all finite inverse limits including fibered products.*



**Proof** Let  $\mathcal{O}_{X,x}$  be the local ring of  $X$  at  $x$ . If  $A$  is in  $\mathbf{Art}_k$ , then a morphism  $\mathrm{Spec} A \rightarrow X$  has image  $x$  if and only if it factors through the inclusion  $\mathrm{Spec} \mathcal{O}_{X,x} \rightarrow X$ , and such a factorization (when it exists) is unique. Thus,  $h_{X,x}$  is naturally isomorphic to the functor  $\mathrm{Hom}_{k\text{-alg}}(\mathcal{O}_{X,x}, -)$ , and so the result follows from the general fact that in any category a functor of the form  $\mathrm{Hom}(X, -)$  preserves finite inverse limits.  $\square$

Let  $X$  be a scheme over  $k$ , and  $x \in X$  a  $k$ -valued point. Let  $\mathfrak{m}_x \subset \mathcal{O}_{X,x}$  be the maximal ideal in its local ring. The above discussion shows that the functor  $h_{X,x} : \mathbf{Art}_k \rightarrow \mathbf{Sets}_*$  has as its tangent space the vector space  $(\mathfrak{m}_x/\mathfrak{m}_x^2)^*$ , which is just the usual tangent space to  $X$  at  $x$ , defined as the vector space of  $k$ -valued derivations on the  $k$ -algebra  $\mathcal{O}_{X,x}$ . This shows the definition of the tangent space to a functor generalizes the usual definition of tangent space to a scheme.

## 2.5 Examples: tangent spaces to various functors

### 1. Tangent space to Grassmannian.

This is the most basic and well-known example, and we sketch it in brief. If  $W$  is a finite dimensional vector space over  $k$  and  $0 \leq r \leq \dim W$  an integer, the Grassmannian  $X = \mathrm{Grass}(W, r)$  of  $r$ -dimensional quotients of  $V$  is a scheme which represents the functor  $h_X$  defined as follows. For any scheme  $S$ , the set  $h_X(S)$  consists of all equivalence classes of pairs  $(\mathcal{E}, q)$  where  $\mathcal{E}$  is a locally free  $\mathcal{O}_S$ -module of constant rank  $r$ , and  $q : V \otimes_k \mathcal{O}_S \rightarrow \mathcal{E}$  is a surjective  $\mathcal{O}_S$ -linear homomorphism. Two such pairs  $(E, q)$  and  $(E', q')$  are defined to be equivalent if there exists an  $\mathcal{O}_S$ -linear isomorphism  $g : \mathcal{E} \rightarrow \mathcal{E}'$  with  $q' = g \circ q$ .

Let  $E$  be a  $k$ -vector space of dimension  $r$  and let  $p : W \rightarrow E$  be a  $k$ -linear surjection. Then  $x = (E, p)$  is a  $k$ -valued point of  $X = \mathrm{Grass}(V, r)$ . We now describe the tangent space  $T_x X$ .

As any vector bundle on  $\mathrm{Spec} k\langle V \rangle$  is trivial, any element of  $h_{X,x}(k\langle V \rangle)$  can be represented by a pair  $(E \otimes_k k\langle V \rangle, q)$  such that  $q|_{\mathrm{Spec} k} = g_0 \circ p$  for some  $g_0 \in GL_E(k)$ . Note that

$$\mathrm{Hom}_{k\langle V \rangle}(W \otimes_k k\langle V \rangle, E \otimes_k k\langle V \rangle) = \mathrm{Hom}_k(W, E) \otimes_k k\langle V \rangle = \mathrm{Hom}_k(W, E) \oplus \mathrm{Hom}_k(W, E) \otimes_k V.$$

In terms of the above decomposition, let  $q = q_0 + q_1$  (the ‘Taylor series’ of  $q$ ), where  $q_0 = q|_{\mathrm{Spec} k} \in \mathrm{Hom}_k(W, E)$  and  $q_1 \in \mathrm{Hom}_k(W, E) \otimes_k V$ . Every possible  $q_1$  can occur in the above decomposition for any given value of  $q_0$ . Let  $F \subset W$  be the kernel of  $p$ . Then restricting  $q_1$  to  $F$  gives an element

$$q_1|_F \in \mathrm{Hom}_k(F, E) \otimes_k V.$$

Two elements  $q, q' \in \mathrm{Hom}_k(W, E) \otimes_k k\langle V \rangle$  are equivalent if and only if there exists  $g \in GL_E(k\langle V \rangle)$  such that  $q' = g \circ q$ . Let  $g = g_0 + g_1$  be the Taylor series of  $g$ ,

where  $g_0 \in GL_E(k)$  and  $g_1 \in \text{End}(E) \otimes V$ . As  $V \cdot V = 0$ , a simple argument using elementary linear algebra shows that any two elements  $q$  and  $q'$  are equivalent if and only if  $(q_1)|_F = (q'_1)|_F$  for the corresponding elements  $q_1, q'_1$ . This shows that

$$h_{X,x}(k\langle V \rangle) = \text{Hom}_k(F, E) \otimes V.$$

From this we conclude that

$$T_x X = \text{Hom}_k(F, E).$$

## 2. Tangent space to $\text{Pic}_{X/k}$ .

Let  $X$  be a projective variety over  $k$  (or more generally a projective scheme over  $k$ ), where  $k$  is algebraically closed. In particular, if such an  $X$  is non-empty then it has a  $k$ -rational point. Any projective module on an Artin local ring is free. Therefore, restricted to  $\mathbf{Art}_k$ , the functor  $\text{Pic}_{X/k}$  is described as

$$\text{Pic}_{X/k}(A) = \text{Pic}(X_A) = H^1(X_A, \mathcal{O}_{X_A}^\times)$$

where  $X_A = X \otimes_k A$ , and  $\mathcal{O}_{X_A}^\times \subset \mathcal{O}_{X_A}$  is the sheaf of invertible elements. (Note that a global description of the functor  $\text{Pic}_{X/k}$  is more complicated.) For any line bundle  $L$  on  $X$ , we have a functor  $\mathcal{D}_L$  (deformations of  $L$ , defined in the introduction) for which  $\mathcal{D}_L(A)$  is the subset of  $\text{Pic}(X_A)$  consisting of isomorphism classes of all line bundles  $\mathcal{L}$  on  $X_A$  such that  $\mathcal{L}|_X \cong L$ . It will follow from a more general result below for deformations of a vector bundle or of a coherent sheaf, that

$$T(\mathcal{D}_L) = \text{Ext}^1(L, L) = H^1(X, \mathcal{O}_X).$$

## 3. Tangent space to deformation functor of coherent sheaves.

Let  $X$  be a proper scheme over a field  $k$ , and let  $E$  be a coherent sheaf of  $\mathcal{O}_X$ -modules. The deformation functor  $\mathcal{D}_E$  of  $E$  is the covariant functor  $\text{Art}_k \rightarrow \text{Sets}$  defined as follows. For any  $A$  in  $\text{Art}_k$ , we take  $\mathcal{D}_E(A)$  to be the set of all equivalence classes of pairs  $(F, \theta)$  where  $F$  is a coherent sheaf on  $X_A = X \otimes_k A$  which is flat over  $A$ , and  $\theta : i^*F \rightarrow E$  is an isomorphism where  $i : X \hookrightarrow X_A$  is the closed embedding induced by  $A \rightarrow k$ , with  $(F, \theta)$  and  $(F', \theta')$  to be regarded as equivalent when there exists some isomorphism  $\eta : F \rightarrow F'$  such that  $\theta' \circ i^*(\eta) = \theta$ . It can be seen that  $\mathcal{D}_E(A)$  is indeed a set. Given any morphism  $f : \text{Spec } B \rightarrow \text{Spec } A$  in  $(\text{Art}_k)^{op}$  and an equivalence class  $(F, \theta)$  in  $\mathcal{D}_E(A)$ , we define  $f^*(F, \theta)$  in  $\mathcal{D}_E(B)$  to be obtained by pull-back under the morphism  $f : X_B \rightarrow X_A$ . This operation preserves equivalences, and thus it gives us a functor  $\mathcal{D}_E : \text{Art}_k \rightarrow \text{Sets}$ .

**Theorem 4** *Let  $X$  be a proper scheme over a field  $k$ . Let  $E$  be a coherent sheaf on  $X$ . Then the deformation functor  $\mathcal{D}_E : \text{Art}_k \rightarrow \text{Sets}$  of  $E$  as defined above satisfies*

$$\mathcal{D}_E(k\langle V \rangle \times_k k\langle W \rangle) = \mathcal{D}_E(k\langle V \times W \rangle)$$

*and its tangent space is  $\text{Ext}^1(E, E)$ .*

We first prove this result for the special case where  $E$  is a vector bundle (that is,  $E$  is locally free), though this also follows from the general case which is proved later.

**Special case of vector bundles:** Let  $(F, \theta) \in \mathcal{D}_E(k[\epsilon]/(\epsilon^2))$ . Then  $F$  is a vector bundle on  $X[\epsilon] = X \otimes_k k[\epsilon]/(\epsilon^2)$ . Any open subscheme  $V$  of  $X[\epsilon]$  is of the form  $U[\epsilon]$ , where  $U$  is an open subscheme of  $X$ . Let  $V_i = U_i[\epsilon]$  be an affine open cover of  $X[\epsilon]$  and let  $f_{i,\alpha}$  be a free basis for  $F|_{V_i}$ . The transition functions for  $F$  have the form  $g_{i,j} + \epsilon h_{i,j}$ . The  $g_{i,j}$  will be the transition functions for  $E$  for the basis  $e_{i,\alpha} = \theta(f_{i,\alpha}|_{U_i})$ . Note that  $(h_{i,j})$  defines a 1-cocycle for  $\underline{End}(E)$  with respect to the trivialization  $(U_i, e_{i,\alpha})$ , which gives us an element of  $H^1(X, \underline{End}(E))$ . Converse is similar. This shows that  $\mathcal{D}_E(k[\epsilon]/(\epsilon^2))$  has a bijection with  $H^1(X, \underline{End}(E))$ . We leave it to the reader to see that an obvious generalisation of the above argument in fact gives a functorial bijection from  $\mathcal{D}_E(k\langle V \rangle)$  to  $H^1(X, \underline{End}(E)) \otimes_k V$  on the category of finite dimensional vector spaces. Hence  $T_{\mathcal{D}_E} = H^1(X, \underline{End}(E))$ . As by assumption  $X$  is proper over  $k$ , the vector space  $H^1(X, \underline{End}(E))$  is finite dimensional.

This completes the proof of the Theorem 4 in the special case of vector bundles.

**General case of coherent sheaves:** Next, we give a proof that for a general  $E$ , the tangent space is  $Ext^1(E, E)$ . This proof is very different in spirit, and in particular it gives another proof in the vector bundle case. For any finite dimensional vector space  $V$  over  $k$ , we define a map

$$f_V : V \otimes_k Ext^1(E, E) = Ext^1(V \otimes_k E, E) \rightarrow \mathcal{D}_E(k\langle V \rangle)$$

as follows, where  $k\langle V \rangle$  is the Artin local  $k$ -algebra generated by  $V$  with  $V^2 = 0$ . An element of  $Ext^1(V \otimes_k E, E)$  is represented by a short exact sequence  $S$  of  $\mathcal{O}_X$ -modules

$$S = (0 \rightarrow V \otimes_k E \xrightarrow{i} F \xrightarrow{j} E \rightarrow 0)$$

We give  $F$  the structure of an  $\mathcal{O}_{X\langle V \rangle}$ -module (where  $X\langle V \rangle = X \otimes_k k\langle V \rangle$ ) by defining the scalar-multiplication map  $V \otimes_k F \rightarrow F$  as the composite

$$V \otimes_k F \xrightarrow{(\text{id}_V, j)} V \otimes_k E \xrightarrow{i} F$$

We denote the resulting  $\mathcal{O}_{X\langle V \rangle}$ -module by  $\mathcal{F}_S$ . Note that the induced homomorphism

$$V \otimes_k \frac{\mathcal{F}_S}{V\mathcal{F}_S} \rightarrow V\mathcal{F}_S$$

is an isomorphism, as it is just the identity map on  $V \otimes_k E$ . Hence by Lemma 25 below, it follows that  $\mathcal{F}_S$  is flat over  $k\langle V \rangle$ . Hence we indeed get an element of  $\mathcal{D}_E(k\langle V \rangle)$ , which completes the definition of the map  $f_V : V \otimes_k Ext^1(E, E) \rightarrow \mathcal{D}_E(k\langle V \rangle)$ .

Next, we check its linearity. The fact that  $f_V$  preserves addition follows from the definition of addition on  $\mathcal{D}_E(k\langle V \rangle)$  together with the exercise below:

**Exercise 5** Let  $M$  and  $N$  be objects of an abelian category  $\mathcal{C}$  which has enough injectives. Let  $\alpha_N : N \oplus N \rightarrow N$  be the addition morphism. Then composite map

$$Ext^1(M, N) \oplus Ext^1(M, N) = Ext^1(M, N \oplus N) \xrightarrow{\alpha_N} Ext^1(M, N)$$

is the addition map on the abelian group  $Ext^1(M, N)$ .  $\square$

Next, we give an inverse  $g_V : \mathcal{D}_E(k\langle V \rangle) \rightarrow V \otimes_k Ext^1(E, E)$  to  $f_V$  as follows. Given any  $(\mathcal{F}, \theta) \in \mathcal{D}_E(k\langle V \rangle)$ , let

$$F = \pi_*(\mathcal{F})$$

where  $\pi : X\langle V \rangle \rightarrow X$  is the projection induced by the ring homomorphism  $k \hookrightarrow k\langle V \rangle$ . Let

$$j : F \rightarrow E$$

be the  $\mathcal{O}_X$ -linear map which is obtained from the  $\mathcal{O}_{X\langle V \rangle}$ -linear map  $\mathcal{F} \rightarrow \mathcal{F}|_X \xrightarrow{\theta} E$  by forgetting scalar multiplication by  $V$ . By flatness of  $\mathcal{F}$  over  $k\langle V \rangle$ , the sequence  $0 \rightarrow V \otimes_{k\langle V \rangle} \mathcal{F} \rightarrow \mathcal{F} \rightarrow \mathcal{F}|_X \rightarrow 0$  obtained by applying  $- \otimes_{k\langle V \rangle} \mathcal{F}$  to  $0 \rightarrow V \rightarrow k\langle V \rangle \rightarrow k \rightarrow 0$  is again exact. As  $V \otimes_{k\langle V \rangle} \mathcal{F} = V \otimes_k (\mathcal{F}/V\mathcal{F})$ , by composing with  $\theta$  (and its inverse) this gives an exact sequence

$$S = (0 \rightarrow V \otimes_k E \xrightarrow{i} \mathcal{F} \xrightarrow{j} E \rightarrow 0)$$

We define  $g_V : \mathcal{D}_E(k\langle V \rangle) \rightarrow V \otimes_k Ext^1(E, E)$  by putting  $g_V(\mathcal{F}, \theta) = S$ .

It can be seen that  $f_V$  is functorial in  $V$  and  $g_V$  is the inverse of  $f_V$ . Hence, we have given a natural isomorphism  $f$  from the functor  $V \mapsto V \otimes_k Ext^1(E, E)$  to the functor  $V \mapsto \mathcal{D}_E(k\langle V \rangle)$  on the category of finite dimensional vector spaces  $V$ . Even though we have only checked this as an isomorphism of set-valued functor, it is automatically a  $k$ -linear isomorphism by Lemma 1. This completes the proof of the Theorem 28 in the general case of coherent sheaves.  $\square$

#### 4. Tangent space to deformation functor of Higgs bundles flat connections, logarithmic connections.

We refer the reader to the papers [BR], [N1] and [N2] where the tangent space is calculated to be a certain hypercohomology.

#### 5. Tangent space to Hilbert and Quot functors.

Let  $X$  be a proper scheme over  $k$ . Let  $E_o$  be a coherent  $\mathcal{O}_X$ -module over  $X$ , and let  $q_o : E_o \rightarrow F_o$  be a coherent quotient  $\mathcal{O}_X$ -module. For any object  $A$  of  $Art_k$ , let  $E_A$  denote the pullback of  $E_o$  to  $X_A = X \otimes_k A$ . Let  $i : X \hookrightarrow X_A$  be the special fiber of  $X_A$ . Consider pairs  $(q : E_A \rightarrow F, \theta : i^*F \rightarrow F_o)$  where  $q$  is an  $\mathcal{O}_{X_A}$ -linear surjection on a coherent  $\mathcal{O}_{X_A}$ -module  $F$  **which is flat over  $A$** , and  $\theta$  is an isomorphism such

that the following square commutes.

$$\begin{array}{ccc} i^*E_A & = & E_o \\ i^*q \downarrow & & \downarrow q_o \\ i^*F & \xrightarrow{\theta} & F_o \end{array}$$

We say that two such pairs  $(q : E_A \rightarrow F, \theta : i^*F \rightarrow F_o)$  and  $(q' : E_A \rightarrow F', \theta' : i^*F' \rightarrow F_o)$  are equivalent if there exists an isomorphism  $f : F \rightarrow F'$  with  $f \circ q = q'$  and  $\theta' \circ (i^*f) = \theta$ . For any object  $A$  of  $\mathbf{Art}_k$ , let  $Q(A)$  be the set of all equivalence classes of such pairs (it can be seen that  $Q(A)$  is indeed a set). For any morphism  $A \rightarrow B$  in  $\mathbf{Art}_k$ , we get by pull-back (applying  $- \otimes_A B$ ) a set map  $Q(A) \rightarrow Q(B)$ , so we have a functor  $Q : \mathbf{Art}_k \rightarrow \mathbf{Sets}$ .

The following result is due to Grothendieck.

**Theorem 6** *Let  $k$  be any field,  $X$  proper over  $k$ , and  $E_o \rightarrow F_o$  a surjective morphism of coherent  $\mathcal{O}_X$ -modules. Let  $Q : \mathbf{Art}_k \rightarrow \mathbf{Sets}$  be the functor defined above on the category  $\mathbf{Art}_k$  of artin local  $k$ -algebras with residue field  $k$ . This functor preserves fibered products in  $\mathbf{Art}_k$ , and the tangent vector space to this functor is the  $k$ -vector space  $\mathrm{Hom}_X(G_o, F_o)$  where  $G_o = \ker(q_o)$ .*

This result is proven later in these notes.

## 8. Tangent space to deformation functor of smooth proper varieties.

The following is proved later in the notes.

**Theorem 7** *Let  $k$  be a field, and let  $X$  be a smooth proper variety over  $k$ . Then the deformation functor  $\mathbf{Def}_X$  of  $X$  satisfies  $\mathcal{D}_E(k\langle V \rangle \times_k k\langle W \rangle) = \mathcal{D}_E(k\langle V \times W \rangle)$ , and the tangent space to the deformation functor  $\mathbf{Def}_X$  is the  $k$ -vector space  $H^1(X, T_X)$  where  $T_X = (\Omega_{X/k}^1)^*$  is the tangent bundle to  $X$ .*

# 3 Existence theorems for universal and miniversal families

## 3.1 Universal, versal and miniversal families (hulls)

Pro-families and the limit Yoneda lemma

Let  $F : \mathbf{Art}_k \rightarrow \mathbf{Sets}$  be a covariant functor. This functor can be naturally prolonged to the larger category  $\widehat{\mathbf{Art}}_k$  (which contains  $\mathbf{Art}_k$  as a full subcategory) as follows. For any complete local noetherian  $k$ -algebra  $R$  with residue field  $k$ , let  $\widehat{F}(R)$  be the set defined by

$$\widehat{F}(R) = \varprojlim F(R/\mathfrak{m}^n)$$

Given a  $k$ -homomorphism  $\theta : R \rightarrow S$  of complete local noetherian  $k$ -algebra with residue field  $k$ , let  $\widehat{F}(\theta) : \widehat{F}(R) \rightarrow \widehat{F}(S)$  be the set map induced in the obvious way. Then  $\widehat{F} : \widehat{Art}_k \rightarrow Sets$  is a covariant functor, which restricts to  $F$  on the subcategory  $Art_k \subset \widehat{Art}_k$ .

The prolongation  $\widehat{F}$  is natural in the following sense: if  $F$  and  $G$  are functors from  $Art_k$  to  $Sets$  and  $f : F \rightarrow G$  is a morphism of functors, then  $f$  prolongs to a morphism  $\widehat{f} : \widehat{F} \rightarrow \widehat{G}$ .

**Remark 8 (How formal schemes and sheaves arise from  $\widehat{F}$ ):** Let  $LocAlg_k$  be the category of local algebras  $R$  over  $k$  with residue field  $k$ . Sometimes, there may already be a functor  $F : LocAlg_k \rightarrow Sets$  already given to us, for example, for a finite type  $k$ -scheme  $X$  and a coherent sheaf  $E$  on  $X$ , we can define  $F(R)$  to be the set of equivalence classes of **flat deformations**  $(\mathcal{E}, \eta)$  of  $E$ , where  $\mathcal{E}$  is a coherent sheaf on  $X \otimes R$  that is flat over  $R$ , and  $\eta : \mathcal{E}|_X \rightarrow E$  is an isomorphism. The functor  $\widehat{F} : \widehat{Art}_k \rightarrow Sets$  defined by  $\widehat{F}(R) = \lim_{\leftarrow} F(R/\mathfrak{m}^n)$  on  $\widehat{Art}_k$  then does not in general coincide with  $F : Art_k \rightarrow Sets$ . In the above example, elements of  $\widehat{F}(R)$  are pairs  $((E_n), (\eta_n))$  where  $(E_n)$  will be a formal sheaf on a certain formal scheme  $\mathfrak{X}$ , and  $(\eta_n)$  will be an isomorphism  $(E_n)|_X \rightarrow E$ .

A **pro-family** for a covariant functor  $F : Art_k \rightarrow Sets$  is a pair  $(R, r)$  where  $R$  is a complete local noetherian  $k$ -algebra with residue field  $k$ , and  $r \in \widehat{F}(R)$  where by definition

$$\widehat{F}(R) = \lim_{\leftarrow} F(R/\mathfrak{m}^n)$$

where  $\mathfrak{m} \subset R$  is the maximal ideal. By the following lemma, this is same as a morphism of functors

$$r : h_R \rightarrow F$$

**Lemma 9 ('Limit Yoneda Lemma')**

Let  $F : Art_k \rightarrow Sets$  be a covariant functor, and let  $\widehat{F} : \widehat{Art}_k \rightarrow Sets$  be its prolongation as constructed above, where  $\widehat{F}(R) = \lim_{\leftarrow} F(R/\mathfrak{m}^n)$  for any complete local noetherian  $k$ -algebra  $R$  with residue field  $k$ . Let  $\alpha_R : Hom(h_R, F) \rightarrow \widehat{F}(R)$  (where  $h_R = Hom_{k\text{-alg}}(R, -)$ ) be the map of sets defined as follows. Given  $f : h_R \rightarrow F$ , for any  $n \geq 1$  we get a map  $f(R/\mathfrak{m}^n) : Hom_{k\text{-alg}}(R, R/\mathfrak{m}^n) \rightarrow F(R/\mathfrak{m}^n)$ , under which the quotient map  $q_n \in Hom_{k\text{-alg}}(R, R/\mathfrak{m}^n)$  maps to  $f(R/\mathfrak{m}^n)(q_n)$ , which defines an inverse system as  $n$  varies, so gives an element  $(f(R/\mathfrak{m}^n)(q_n))_{n \in \mathbb{N}} \in \widehat{F}(R)$ .

Then the above map  $\alpha_R : Hom(h_R, F) \rightarrow \widehat{F}(R)$  is a natural bijection, functorial in both  $R$  and  $F$ . □

Definition of versal, miniversal, universal families

For a quick review of basic notions about smoothness and formal smoothness, the reader can consult, for example, the first chapter of Milne's 'Etale Cohomology'.

Let  $F : \mathbf{Art}_k \rightarrow \mathbf{Sets}$  and  $G : \mathbf{Art}_k \rightarrow \mathbf{Sets}$  be functors. Recall that a morphism of functors  $\phi : F \rightarrow G$  is called **formally smooth** if given any surjection  $q : B \rightarrow A$  in  $\mathbf{Art}_k$  and any elements  $\alpha \in F(A)$  and  $\beta \in G(B)$  such that

$$\phi_A(\alpha) = G(q)(\beta) \in F(A),$$

there exists an element  $\gamma \in F(B)$  such that

$$\phi_B(\gamma) = \beta \in G(B) \text{ and } F(q)(\gamma) = \alpha \in F(A)$$

In other words, the following diagram of functors commutes, where the diagonal arrow  $h_B \rightarrow F$  is defined by  $\gamma$ .

$$\begin{array}{ccc} h_A & \xrightarrow{\alpha} & F \\ q \downarrow & \nearrow & \downarrow \phi \\ h_B & \xrightarrow{\beta} & G \end{array}$$

The morphism  $\phi : F \rightarrow G$  is called **formally étale** if it is formally smooth, and moreover the element  $\gamma \in F(B)$  is unique.

**Caution** If the functors  $F$  and  $G$  are of the form  $h_R$  and  $h_S$  for rings  $R$  and  $S$ , then  $\phi$  is formally étale if and only if it is formally smooth and the tangent map  $T_R \rightarrow T_S$  is an isomorphism. However, if  $F$  and  $G$  are not both of the above form, then a functor can  $\phi$  be formally smooth, and moreover the map  $T_F \rightarrow T_G$  can be an isomorphism, yet  $\phi$  need not be formally étale. It is because of this subtle difference that a miniversal family can fail to be universal, as we will see in examples later.

A **versal family** for a covariant functor  $F : \mathbf{Art}_k \rightarrow \mathbf{Sets}$  is a pro-family  $(R, r)$  (where  $R$  is a complete local noetherian  $k$ -algebra with residue field  $k$ , and  $r \in \widehat{F}(R)$ ) such that the morphism of functors  $r : h_R \rightarrow F$  is formally smooth.

**Remark 10** If  $(R, r)$  is a versal family, then for any  $A$  in  $\mathbf{Art}_k$ , the induced set map  $r(A) : h_R(A) \rightarrow F(A)$  is surjective. For, given any  $v \in F(A)$ , we can regard it as a morphism  $v : h_A \rightarrow F$ . Now consider the following commutative square.

$$\begin{array}{ccc} h_k & \longrightarrow & h_R \\ \downarrow & & \downarrow \\ h_A & \xrightarrow{v} & F \end{array}$$

By formal smoothness of  $h_R \rightarrow F$ , there exists a morphism  $u : h_A \rightarrow h_R$  which makes the above diagram commute. But such a morphism is just an element of  $h_R(A)$  which maps to  $v \in F(A)$ , which proves that  $r(A) : h_R(A) \rightarrow F(A)$  is surjective.

For any covariant functor  $F : \mathbf{Art}_k \rightarrow \mathbf{Sets}$ , the pointed set

$$T_F = F(k[\epsilon]/(\epsilon^2))$$

is called the **tangent set** to  $F$ , or the set of **first order deformations** under  $F$ .

A **minimal versal** (‘**miniversal**’) **family** (also called as a **hull**) for a covariant functor  $F : Art_k \rightarrow Sets$  is a versal family for which the set map

$$dr : T_R = h_R(k[\epsilon]/(\epsilon^2)) \rightarrow F(k[\epsilon]/(\epsilon^2)) = T_F$$

is a bijection.

A **universal family** for a covariant functor  $F : Art_k \rightarrow Sets$  is a pro-family  $(R, r)$  such that  $r : h_R \rightarrow F$  is a natural bijection. If a universal family exists, it is clearly unique up to a unique isomorphism. A covariant functor  $F : Art_k \rightarrow Sets$  is called **pro-representable** if a universal family exists. (The reason for the prefix ‘pro-’ is that  $R$  need not be in the subcategory  $Art_k$  of  $\widehat{Art_k}$ .)

**Remark 11** A pro-family  $(R, r)$  is universal if and only if the morphism of functors  $r : h_R \rightarrow F$  is formally étale.

**Example 12** **A miniversal family that is not universal.** Let  $F : Art_k \rightarrow Sets$  be the functor

$$A \mapsto \mathfrak{m}_A/\mathfrak{m}_A^2$$

It can be verified that  $F$  satisfies the Schlessinger conditions **(H1)**, **(H2)**, **(H3)** so admits a hull. It can be seen that it does not satisfy Schlessinger conditions **(H4)** by taking  $A = k[x]/(x^2)$  and  $B = k[x]/(x^3)$  with quotient map  $B \rightarrow A : x \mapsto x$ . Then we have  $F(B \times_A B) = k^2$ , while  $F(B) \times_{F(A)} F(B) = k^1$  with map given by first projection  $k^2 \rightarrow k^1$ , which is not injective, violating **(H4)**.

In fact, a hull  $(R, r)$  for  $F$  is given by  $R = k[[t]]$  with  $r$  given by  $dt \in \mathfrak{m}_R/\mathfrak{m}_R^2 = \widehat{F}(R)$ .

Note that the hull is not unique up to unique isomorphism, as it admits non-trivial automorphisms  $f : R \rightarrow R$  defined by  $f(t) = t + t^2g(t)$  for arbitrary  $g(t) \in k[[t]]$  (so that  $(df/dt)_0 = 1$ , which means  $f$  preserves  $dt$ ). This again shows that  $F$  is not pro-representable, for whenever a functor  $G$  is pro-representable, any hull is universal, so is unique up to unique isomorphism.

Also note that the functor pro-represented by  $R = k[[t]]$  is given by  $h_R(A) = \mathfrak{m}_A$ .

**Exercise 13** If  $F : Art_k \rightarrow Sets$  admits a hull and moreover if  $T_F = 0$  then  $F(A) = F(k)$  for all  $A$  in  $Art_k$ .

### 3.2 Grothendieck’s theorem on pro-representability

**Theorem 14** Let  $F : Art_k \rightarrow Sets$  be a functor such that  $F(k)$  is a singleton set. Then  $F$  is pro-representable if and only if the following two conditions **(lim)** and



**(fin)** are satisfied.

**(lim)**  $F$  preserves fibered products: for any pair of homomorphisms  $B \rightarrow A$  and  $C \rightarrow A$  in  $\mathbf{Art}_k$ , the induced map  $F(B \times_A C) \rightarrow F(B) \times_{F(A)} F(C)$  is bijective.<sup>1</sup>

(As a consequence of **(lim)**, note that the set  $T_F = F(k[\epsilon]/(\epsilon^2))$  acquires a natural  $k$ -vector space structure.)

**(fin)** The  $k$ -vector space  $T_F$  is finite dimensional.

**Proof** Consider the category  $Fam$  whose objects are all families  $(A, \alpha)$  for the functor  $F$ , consisting of an Artin local  $k$ -algebra  $A$  with residue field  $k$  together with an element  $\alpha \in F(A)$ . A morphism  $(B, \beta) \rightarrow (A, \alpha)$  in  $Fam$  is a  $k$ -algebra homomorphism  $f : B \rightarrow A$  such that  $f_*(\beta) = \alpha$ . Consider the induced natural map  $h_f : h_A \rightarrow h_B$ , and the resulting direct system  $(h_A, h_f)$  in  $Fun(\mathbf{Art}_k, \mathbf{Sets})$  indexed by the category  $Fam$ . The morphisms  $\alpha : h_A \rightarrow F$  induce a morphism of functors

$$\Phi : \operatorname{colim}_{Fam} h_A \rightarrow F$$

where the colimit (that is, direct limit) is taken over the category  $Fam$ . (The set-theoretic difficulties involved in this limit – and later such limits – can be easily bypassed by replacing  $Fam$  by a suitable small category.)

Note that the category  $Fam$  has a final element, namely  $(k, *)$ . Moreover, as  $Art_k$  admits fibered products, and these are preserved by  $F$ , so the category  $Fam$  has fibered products. In particular, the category  $Fam$  is cofiltered.

We now show that  $\Phi$  is an isomorphism, that is, for each  $C$  in  $Art_k$  the map  $\Phi_C : \operatorname{colim} h_A(C) \rightarrow F(C)$  is bijective. If  $\gamma \in F(C)$ , then the element  $id_C$  of  $h_C$  indexed by  $(C, \gamma)$  is an element of  $\operatorname{colim} h_A(C)$  which maps to  $\gamma \in F(C)$ , showing  $\Phi_C$  is surjective. As  $Fam$  is cofiltered, each element  $x$  of  $\operatorname{colim} h_A(C)$  is represented by a homomorphism  $u : A \rightarrow C$  for some  $(A, \alpha)$  in  $Fam$ , and any two  $x, y \in \operatorname{colim} h_A(C)$  are represented by homomorphisms  $u, v : A \rightarrow C$  where  $(A, \alpha)$  is common. Let  $E \subset A$  be the equalizer of  $u$  and  $v$ , that is,  $E = \{a \in A \mid u(a) = v(a) \in C\}$ . As  $F$  preserves fibered products, it also preserves equalizers, hence  $F(E)$  is the equalizer of  $F(u), F(v) : F(A) \rightarrow F(C)$ . Note that  $\Phi_C(x) = F(u)\alpha$ , and  $\Phi_C(y) = F(v)\alpha$ , so if  $\Phi_C(x) = \Phi_C(y)$  then  $\alpha$  comes from an element  $\gamma \in F(E)$  under the inclusion  $E \hookrightarrow A$ . This defines an object  $(E, \gamma)$  of  $Fam$ , with a morphism  $(A, \alpha) \rightarrow (E, \gamma)$  defined by the inclusion  $E \hookrightarrow A$ . Then  $x$  and  $y$  are represented by the composite homomorphisms  $E \hookrightarrow A \xrightarrow{u} C$  and  $E \hookrightarrow A \xrightarrow{v} C$ . As these are equal, we have  $x = y$ , showing  $\Phi_C$  is injective. Thus we have proved that  $\Phi$  is an isomorphism.

Given any  $(A, \alpha)$  in  $Fam$ , as  $A$  is a finite dimensional vector space over  $k$ , the intersection  $A' \subset A$  of the images of all  $f : (B, \beta) \rightarrow (A, \alpha)$  is a finite intersection, and as  $Fam$  has fibered products,  $A'$  equals the image of some  $(D, \delta) \rightarrow (A, \alpha)$ . Let

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<sup>1</sup>As  $\mathbf{Art}_k$  has a final object and admits fibered products, this is equivalent to the statement that  $F$  preserves all finite inverse limits, hence the name **(lim)**.

$\alpha'$  be the restriction of  $\delta$  to  $A'$ . Hence, replacing each  $(A, \alpha)$  by the corresponding  $(A', \alpha')$ , we get a full subcategory  $Fam'$  of  $Fam$  in which every homomorphism is surjective at the level of the underlying rings, and which is cofinal in  $Fam$  (since given any  $(A, \alpha)$  in  $Fam$  we have the corresponding  $(A', \alpha')$  in  $Fam$  together with a morphism  $(A', \alpha') \rightarrow (A, \alpha)$  in  $Fam$  induced by the inclusion  $A' \hookrightarrow A$ ). Hence we get an isomorphism

$$\text{colim}_{Fam'} h_A \rightarrow \text{colim}_{Fam} h_A.$$

Composing with  $\Phi$ , we get an isomorphism

$$\Phi' : \text{colim}_{Fam'} h_A \rightarrow F.$$

Let  $Fam''$  be the full subcategory of  $Fam'$  which consists of objects  $(A, \alpha)$  for which the induced map

$$\alpha(k[\epsilon]/(\epsilon^2)) : T_A \rightarrow T_F$$

is an isomorphism, where  $T_A = h_A(k[\epsilon]/(\epsilon^2))$  is the tangent space to  $A$  and  $T_F = F(k[\epsilon]/(\epsilon^2))$  is the tangent space to  $F$ . Note that when  $B \rightarrow A$  is a surjective homomorphism in  $\mathbf{Art}_k$ , the induced tangent map  $T_A \rightarrow T_B$  is injective. As the  $k$ -vector space  $F(k[\epsilon]/(\epsilon^2))$  is the direct limit

$$\text{colim}_{Fam'} h_A(k[\epsilon]/(\epsilon^2)) = \text{colim}_{Fam'} T_A$$

as this direct system consists of injective  $k$ -linear maps, and as  $F(k[\epsilon]/(\epsilon^2))$  is finite dimensional by **(fn)**, it follows that  $Fam''$  is cofinal in  $Fam'$ . Therefore to prove the theorem, we just have to show that  $\text{colim}_{Fam''} h_A$  is isomorphic to the functor  $h_R$  for some noetherian complete local  $k$ -algebra  $R$  with residue field  $k$ .

For each integer  $i \geq 1$ , let  $Fam(i)$  be the full subcategory of  $Fam''$  formed by the families  $(A, \alpha)$  where  $\mathfrak{m}_A^{i+1} = 0$ . This category is co-filtered, for if  $(A, \alpha)$  and  $(B, \beta)$  are families in  $Fam(i)$ , and  $(C, \gamma)$  is a family in  $Fam''$  with morphisms  $f : (C, \gamma) \rightarrow (A, \alpha)$  and  $g : (C, \gamma) \rightarrow (B, \beta)$ , then  $(C/\mathfrak{m}_C^{i+1}, \gamma/\mathfrak{m}_C^{i+1})$  is a family in  $Fam(i)$  with morphisms  $f/\mathfrak{m}_C^{i+1}$  and  $g/\mathfrak{m}_C^{i+1}$  to the two families. Note that if  $\dim_k(T_F) = n$ , then for any  $(A, \alpha)$  in  $Fam(i)$  we must have

$$\dim_k(A) \leq \dim_k(k[[x_1, \dots, x_n]]/(x_1, \dots, x_n)^{i+1})$$

as  $A$  must be a quotient of  $k[[x_1, \dots, x_n]]$ . An object  $X$  in a category  $\mathcal{C}$  is called a co-final object if given any other object  $Y$ , there exists a morphism  $X \rightarrow Y$ . As each homomorphism in  $Fam(i)$  is by assumption surjective, and as  $Fam(i)$  is co-filtered, the above bound on dimension shows that  $Fam(i)$  has a co-final element  $(R_i, \alpha_i)$ , which we can choose to be any family with  $\dim_k(R_i)$  the maximum possible.

Note that we have a homomorphism  $f_{i+1} : (R_{i+1}, \alpha_{i+1}) \rightarrow (R_i, \alpha_i)$  in  $Fam''$ , which is surjective. Recall that the induced map  $T_{R_i} \rightarrow T_{R_{i+1}}$  is an isomorphism. Consider the following inverse system in  $\mathbf{Art}_k$ .

$$R_1 \xleftarrow{f_2} R_2 \xleftarrow{f_3} R_3 \dots$$

As the  $f_i$  are surjective maps which are tangent-level isomorphisms, the inverse limit ring

$$R = \lim (R_i, f_i)$$

is a complete noetherian local  $k$ -algebra with residue field  $k$ . As the collection  $(R_i, \alpha_i)$  is cofinal in  $Fam''$ , we get

$$h_R = \operatorname{colim}_i h_{R_i} = \operatorname{colim}_{Fam''} h_A$$

and thereby the theorem is proved.  $\square$

### 3.3 Schlessinger's theorem on hull and pro-representability

**15** Let  $G$  be a group, and  $p : E \rightarrow B$  a map of sets, and let there be given an action  $E \times G \rightarrow E$  over  $B$  (means  $p(x \cdot g) = p(x)$  for all  $x \in E$  and  $g \in G$ ). We say that this data defines a **relative principal  $G$ -set over  $B$**  if the resulting map

$$E \times G \rightarrow E \times_B E : (x, g) \mapsto (x, x \cdot g)$$

is bijective. In particular, this means that the non-empty fibers of  $p$  (if any) have a bijection with  $G$  which is well-defined up to left translations on  $G$ .

**Example 16** Let  $\emptyset$  be the empty set. Then for any set  $B$  and any group  $G$ , the unique map  $p : \emptyset \rightarrow B$  defines a relative principal  $G$ -set over  $B$ .

#### Theorem 17 (Schlessinger)

**Existence of a hull :** *Let  $F : \mathit{Art}_k \rightarrow \mathit{Sets}$  be a covariant functor such that  $F(k)$  is a singleton set. Then  $F$  admits a hull if and only if the following three conditions **(H1)**, **(H2)**, **(H3)** are satisfied.*

**(H1)** *Given any three objects  $A, B$ , and  $C$  of  $\mathit{Art}_k$ , with morphisms  $B \rightarrow A$  and  $C \rightarrow A$  such that  $C \rightarrow A$  is surjective with kernel a principal ideal  $I$  which satisfies  $\mathfrak{m}_C I = 0$ , consider the diagram*

$$\begin{array}{ccc} B \times_A C & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & A \end{array}$$

*Then the induced map  $F(B \times_A C) \rightarrow F(B) \times_{F(A)} F(C)$  is surjective.*

**(H2)** *Let  $B$  be any object in  $\mathit{Art}_k$ . Consider the diagram*

$$\begin{array}{ccc} B \times_k k[\epsilon]/(\epsilon^2) & \longrightarrow & k[\epsilon]/(\epsilon^2) \\ \downarrow & & \downarrow \\ B & \longrightarrow & k \end{array}$$

Then the induced map  $F(B \times_k k[\epsilon]/(\epsilon^2)) \rightarrow F(B) \times_{F(k)} F(k[\epsilon]/(\epsilon^2)) = F(B) \times F(k[\epsilon]/(\epsilon^2))$  is bijective.

(As a consequence, the tangent set  $T_F = F(k[\epsilon]/(\epsilon^2))$  gets the structure of a  $k$ -vector space, with the base point of  $T_F$  as the zero vector, such that given any family  $(R, r)$ , the map  $T_R \rightarrow T_F$  becomes linear.)

**(H3)** With the above  $k$ -linear structure, the  $k$ -vector space  $T_F$  is finite dimensional.

**Pro-representability :** A covariant functor  $F : \text{Art}_k \rightarrow \text{Sets}$ , for which  $F(k)$  is a singleton set, is pro-representable if and only if it satisfies conditions **(H1)**, **(H2)**, **(H3)** (as above) and **(H4)**:

**(H4)** If  $B \rightarrow A$  is a surjection in  $\text{Art}_k$  with kernel  $I$  such that  $\mathfrak{m}_B I = 0$  where  $\mathfrak{m}_B \subset B$  is the maximal ideal of  $B$ , then the following map of sets is bijective.

$$F(B \times_A B) \rightarrow F(B) \times_{F(A)} F(B)$$

**Proof Equivalent versions: (H1)  $\Leftrightarrow$  (H1') and (H2)  $\Leftrightarrow$  (H2')**

**(H1')** Given any three objects  $A$ ,  $B$ , and  $C$  of  $\text{Art}_k$ , with morphisms  $B \rightarrow A$  and  $C \rightarrow A$  such that  $C \rightarrow A$  is surjective, consider the diagram

$$\begin{array}{ccc} B \times_A C & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & A \end{array}$$

Then the induced map  $F(B \times_A C) \rightarrow F(B) \times_{F(A)} F(C)$  is surjective.

Clearly, **(H1')**  $\Rightarrow$  **(H1)**. We now show implication **(H1)**  $\Rightarrow$  **(H1')**. If  $\dim_k(C) = \dim_k(A)$  as  $k$ -vector space, then the surjection  $C \rightarrow A$  is an isomorphism, and we are done. Otherwise, we can reduce to the case where  $\dim_k(C) = \dim_k(A) + 1$  (the case of a small extension) as follows. The surjective homomorphism  $p : C \rightarrow A$  can be factored in  $\text{Art}_k$  as the composite of a finite sequence of surjections

$$C = C_n \rightarrow C_{n-1} \rightarrow \dots \rightarrow C_1 \rightarrow C_0 = A$$

where  $N \geq 1$  is an integer such that  $\mathfrak{m}_C^n = 0$ , and  $C_j = C/\mathfrak{m}^j I$  where  $I$  is the kernel of  $C \rightarrow A$ . We can construct an element of  $F(B \times_A C)$  above a given element of  $F(B) \times_{F(A)} F(C)$  by step-by-step constructing elements of  $F(B \times_A C_1)$ ,  $F(B \times_A C_2)$ , etc. Therefore without loss of generality we can assume that  $\mathfrak{m}_C I = 0$ . This means  $I$  is just a finite dimensional  $k$ -vector space. Next, we can filter  $I$  by subspaces  $I_1 \subset I_2 \subset \dots \subset I_d = I$  where  $d = \dim_k(I)$  and  $\dim(I_j) = j$ . The  $I_j$  are automatically ideals in  $C$ . The surjection  $C \rightarrow A$  factors as the composite

$$C \rightarrow C/I_1 \rightarrow \dots \rightarrow C/I_d = A$$

So again can construct an element of  $F(B \times_A C)$  above a given element of  $F(B) \times_{F(A)} F(C)$  by step-by-step constructing elements of  $F(B \times_A C/I_1)$ ,  $F(B \times_A C/I_2)$ , etc. This completes the proof that **(H1)**  $\Rightarrow$  **(H1')**.

**(H2')** The set  $F(k)$  is a singleton set. Moreover the following holds. Let  $B$  be any object in  $Art_k$ , and let  $C = k\langle V \rangle$  where  $V$  is a finite dimensional  $k$ -vector space. Consider the diagram

$$\begin{array}{ccc} B \times_k C & \longrightarrow & C \\ \downarrow & & \downarrow \\ B & \longrightarrow & k \end{array}$$

Then the induced map  $F(B \times_k C) \rightarrow F(B) \times_{F(k)} F(C) = F(B) \times F(C)$  is bijective.

Clearly, **(H2')**  $\Rightarrow$  **(H2)** by taking  $V = k^1$ . To show the converse, we choose a basis  $(v_1, \dots, v_n)$  for  $V$ , which gives an isomorphism

$$\frac{k[\epsilon_1, \dots, \epsilon_n]}{(\epsilon_1, \dots, \epsilon_n)^2} \rightarrow k\langle V \rangle : \epsilon_i \mapsto v_i$$

Then by repeated application of **(H2)**, it follows that **(H2)**  $\Rightarrow$  **(H2')**.

**Versal implies (H1) :** Let  $(R, r)$  be a versal family for  $F$ , where  $R$  is a noetherian complete local  $k$ -algebra with residue field  $k$ , and  $r \in \widehat{F}(R) = Hom(h_R, F)$  is such that  $r : h_R \rightarrow F$  is formally smooth. We wish to show that  $F(B \times_A C) \rightarrow F(B) \times_{F(A)} F(C)$  is surjective when  $C \rightarrow A$  is surjective. For this, let  $(b, c) \in F(B) \times_{F(A)} F(C)$ , with both  $b$  and  $c$  mapping to the same element  $a \in F(A)$ . We will construct an element  $d \in F(B \times_A C)$  which lies above  $(b, c)$ , by constructing a suitable element

$$\delta \in h_R(B \times_A C) = h_R(B) \times_{h_R(A)} h_R(C)$$

and then defining  $d$  as the image of  $\delta$  under  $h_R \rightarrow F$ .

By Remark 10, the induced map  $r(B) : h_R(B) \rightarrow F(B)$  is surjective for any  $B$  in  $Art_k$ . Therefore given any element  $(b, c) \in F(B) \times_{F(A)} F(C)$ , we can choose  $\beta \in h_R(B)$  which maps to  $b \in F(B)$ . Let  $\beta \mapsto \alpha \in h_R(A)$  under the map induced by the homomorphism  $B \rightarrow A$ . In particular,  $\alpha \mapsto a \in F(A)$  under  $h_R \rightarrow F$ .

Now consider the commutative square

$$\begin{array}{ccc} h_A & \xrightarrow{\alpha} & h_R \\ \downarrow & & \downarrow \\ h_C & \xrightarrow{c} & F \end{array}$$

By surjectivity of  $C \rightarrow A$  and formal smoothness of  $h_R \rightarrow F$ , there exists a morphism  $\gamma : h_C \rightarrow h_R$  which makes the above diagramme commute. We regard  $\gamma$  as an element of  $h_R(C)$ . So we get an element  $\delta = (\beta, \gamma) \in h_R(B) \times_{h_R(A)} h_R(C)$ . It can be seen that this element is what we were looking for. This completes the proof that versal implies **(H1)**.

**Miniversal implies (H2) :** We wish to show that

$$F(B \times_k k[\epsilon]/(\epsilon^2)) \rightarrow F(B) \times T_F$$

is bijective, where  $T_F = F(k[\epsilon]/(\epsilon^2))$ . As surjectivity is already proved above, we just have to check injectivity. For this, let  $e_1, e_2 \in F(B \times_k k[\epsilon]/(\epsilon^2))$  such that both map to the same element  $(b, u) \in F(B) \times T_F$ . As  $r : h_R \rightarrow F$  induces a surjection  $h_R(B) \rightarrow F(B)$ , there exists an element  $\beta \in h_R(B)$  (that is, a morphism  $\beta : \text{Spec } B \rightarrow \text{Spec } R$  over  $\text{Spec } k$ ) such that  $\beta \mapsto b$ . Consider the following commutative square where  $C$  denotes  $k[\epsilon]/(\epsilon^2)$ .

$$\begin{array}{ccc} h_B & \xrightarrow{\beta} & h_R \\ \downarrow & & \downarrow \\ h_{B \times_k C} & \xrightarrow{e_i} & F \end{array}$$

By surjectivity of  $B \times_k C \rightarrow B$  and formal smoothness of  $h_R \rightarrow F$ , there exists  $f_i : h_{B \times_k C} \rightarrow h_R$  (that is, a morphism  $f_i : \text{Spec } B \times_k C \rightarrow \text{Spec } R$  over  $\text{Spec } k$ ) which makes the above diagram commute. We can regard  $f_i$  to be an element of  $h_R(B \times_k C) = h_R(B) \times h_R(C)$ . As such, by commutativity of the diagram we must have  $f_i = (\beta, w_i)$  for  $w_i \in h_R(C)$ . As both  $w_1, w_2$  must map to  $u$  under  $h_R(C) \rightarrow F(C)$ , and as by assumption,  $h_R(C) \rightarrow F(C)$  is bijective, we must have  $w_1 = w_2$ . Therefore  $e_1 = e_2$ , proving **(H2)**.

**Linear structure on  $T_F$  given by (H2) :** We have a functor  $FinVect_k \rightarrow Art_k$  which sends  $V \mapsto k\langle V \rangle = k \oplus V$  with obvious  $k$ -algebra structure. Given functor  $F : Art_k \rightarrow Sets$ , by composition we get  $f : FinVect_k \rightarrow Sets$ , under which  $V \mapsto f(V) = F(k\langle V \rangle)$ . The condition **(H2)** means that we can apply Lemma 1 to this functor  $f$ , which gives us a structure of a vector space on the set  $T_F$ . The zero vector is the distinguished point of the set  $T_F$ , as the zero vector space in  $FinVect_k$  maps to the  $k$ -algebra  $k\langle 0 \rangle = k$ . The linearity of the map  $T_R \rightarrow T_F$  for any family  $(R, r)$  is clear.

**Miniversal implies (H3) :** As **(H2)** holds,  $T_F$  acquires a natural structure of a  $k$ -vector space as described above, such that  $T_R \rightarrow T_F$  becomes a linear map for any family  $(R, r)$ . If moreover  $(R, r)$  is miniversal, the map  $T_R \rightarrow T_F$  is bijective by definition of miniversality. Therefore,  $T_R \rightarrow T_F$  is a linear isomorphism for any miniversal family  $(R, r)$ , hence as  $T_R$  is finite dimensional, so is  $T_F$ .

This completes the proof that existence of a hull implies that the conditions **(H1)**, **(H2)**, **(H3)** are satisfied.

**Pro-representability implies (H1), (H2), (H3), (H4) :** Obvious.

**Existence of hull together with (H4) implies pro-representability :** We will show that any hull  $(R, r)$  is in fact a universal family. Let  $B \rightarrow A$  be a surjection in  $Art_k$  with kernel  $I$  such that  $\mathfrak{m}_B I = 0$  where  $\mathfrak{m}_B \subset B$  is the maximal ideal of  $B$ . Then we have an isomorphism of  $k$ -algebras

$$B \times_A B \rightarrow B \times_k k\langle I \rangle : (x, y) \mapsto (x, (\bar{x}, x - y))$$

where  $k\langle I \rangle = k \oplus I$  with  $I^2 = 0$  and  $\bar{x} \in k$  denotes the image of  $x \in B$  under  $B \rightarrow B/\mathfrak{m}_B = k$ . (The fact that the above map preserves ring multiplication follows

from  $\mathfrak{m}_B I = 0$ ). As we have shown that existence of hull implies **(H2)**, the above isomorphism gives a bijection

$$F(B \times_A B) \xrightarrow{\sim} F(B) \times F(k\langle I \rangle)$$

Now, repeated application of **(H2)** gives for any finite dimensional  $k$ -vector space  $V$  a bijection

$$F(k\langle V \rangle) = T_F \otimes V$$

so the above bijection becomes

$$F(B \times_A B) \xrightarrow{\sim} F(B) \times (T_F \otimes I)$$

If  $F(p_1) : F(B \times_A B) \rightarrow F(B)$  is induced by the first projection  $p_1 : B \times_A B \rightarrow B$  and if  $F(B) \times (T_F \otimes I) \rightarrow F(B)$  is the first projection, then the following diagram commutes.

$$\begin{array}{ccc} F(B \times_A B) & \rightarrow & F(B) \times (T_F \otimes I) \\ F(p_1) \downarrow & & \downarrow \\ F(B) & = & F(B) \end{array}$$

The map  $F(B \times_A B) \rightarrow F(B) \times_{F(A)} F(B)$  therefore becomes a map

$$\alpha : F(B) \times (T_F \otimes I) \rightarrow F(B) \times_{F(A)} F(B)$$

which commutes with the first projections on  $F(B)$ .

It can be verified that the second projection on  $F(B)$  in the above map in fact defines an action of the group  $T_F \otimes I$  on the set  $F(B)$ .

By **(H1)** the map  $\alpha$  is surjective, which shows that the group  $T_F \otimes I$  acts transitively on each fibre of  $F(B) \rightarrow F(A)$ .

If **(H4)** holds, then the following map of sets is bijective.

$$F(B \times_A B) \rightarrow F(B) \times_{F(A)} F(B)$$

Therefore, the map

$$\alpha : F(B) \times (T_F \otimes I) \rightarrow F(B) \times_{F(A)} F(B)$$

is a bijection, which means that each fibre of  $F(B) \rightarrow F(A)$  is a principal set (possibly empty) under the group  $T_F \otimes I$ .

Now we assume that there exists a miniversal family  $(R, r)$  for  $F$ . We will show that  $(R, r)$  is universal. For this, we must show that the map  $r(B) : h_R(B) \rightarrow F(B)$  is a bijection for each object  $B$  of  $Art_k$ . This is clear for  $B = k$ . So now we proceed by induction on the smallest positive integer  $n(B)$  for which  $\mathfrak{m}_B^{n(B)} = 0$  (for  $B = k$  we have  $n = 1$ ). For a given  $B$ , suppose  $n(B) \geq 2$ . Let  $I = \mathfrak{m}_B^{n(B)-1}$  so that  $\mathfrak{m}_B I = 0$ . Let  $A = B/I$ , so that  $n(A) = n(B) - 1$ , which by induction gives a bijection  $r(A) : h_R(A) \rightarrow F(A)$ . Consider the commutative square

$$\begin{array}{ccc} h_R(B) & \rightarrow & F(B) \\ \downarrow & & \downarrow \\ h_R(A) & = & F(A) \end{array}$$

Note that  $h_R(B) \rightarrow h_R(A)$  is a relative principal  $T_R \otimes I$ -set over  $h_R(A)$  (see definition 15), and the map  $r(B) : h_R(B) \rightarrow F(B)$  is  $T_F \otimes I$ -equivariant, where we identify  $T_R$  with  $T_F$  via  $r : h_R \rightarrow F$ . It follows that  $r(B) : h_R(B) \rightarrow F(B)$  is injective. As  $r(B) : h_R(B) \rightarrow F(B)$  is already known to be surjective by versality, this shows that  $r(B)$  is bijective, thus  $(R, r)$  pro-represents  $F$ .

**Construction of a universal first order family assuming (H2) and (H3) :**

By **(H2)**, the set  $T_F$  has a natural structure of a  $k$ -vector space, and by **(H3)** it is finite dimensional. Let  $T_F^*$  be its dual vector space, and let  $A = k\langle T_F^* \rangle \in \text{Art}_k$ . Note that  $T_A = T_F$ . The identity endomorphism  $\theta \in \text{End}(T_F) = T_F \otimes T_F^* = F(A)$  defines a family  $(A, \theta)$ , which can be seen to have the following properties.

(i) The map  $\theta : h_A \rightarrow F$  induces the identity isomorphism  $T_F \rightarrow T_F$ .

(ii) Let  $(R, r)$  be any family for  $F$  parametrised by  $R \in \widehat{\text{Art}}_k$ . Let  $R_1 = R/\mathfrak{m}_R^2$  and let  $r_1 = r|_{R_1}$ . Then there exists a unique  $k$ -homomorphism  $A \rightarrow R_1$  such that  $r_1 \in F(R_1)$  is the image of  $\theta \in F(A)$ .

**(H1), (H2), (H3) imply the existence of a hull :** The proof will go in two stages. First, we will construct a family  $(R, r)$ , which will be our candidate for a hull. Next, we prove that the family  $(R, r)$  is indeed a hull.

**Construction of a family  $(R, r)$  :** Let  $S$  be the completion of the local ring at the origin of the affine space  $\text{Spec Sym}_k(T_F^*)$ . If  $x_1, \dots, x_d$  is a linear basis for  $T_F^*$ , then  $S = k[[x_1, \dots, x_d]]$ . Let  $\mathfrak{n} = (x_1, \dots, x_d) \subset S$  denote the maximal ideal of  $S$ . We will construct a versal family  $(R, r)$  where  $R = S/J$  for some ideal  $J$ . The ideal  $J$  will be constructed as the intersection of a decreasing chain of ideals

$$\mathfrak{n}^2 = J_2 \supset J_3 \supset J_4 \supset \dots \supset \bigcap_{q=2}^{\infty} J_q = J$$

such that at each stage we will have

$$J_q \supset J_{q+1} \supset \mathfrak{n}J_q$$

Consequently, we will have  $J_q \supset \mathfrak{n}^q$  which in particular means  $R/J_q \in \text{Art}_k$ , and  $J_q/J$  is a fundamental system of open neighbourhoods in  $R = S/J$  for the  $\mathfrak{m}$ -adic topology on  $R$ , where  $\mathfrak{m} = \mathfrak{n}/J$  is the maximal ideal of  $R$ . Note that  $R$  is complete for the  $\mathfrak{m}$ -adic topology.

In fact, if  $S$  is any noetherian local ring with maximal ideal  $\mathfrak{n}$ , then any ideal  $J \subset S$  is automatically closed in the  $\mathfrak{n}$ -adic topology as  $\bigcap_{i \geq 1} (J + \mathfrak{n}^i) = J$  (which follows from Krull's theorem that  $\bigcap_{i \geq 1} \mathfrak{n}^i = 0$ ). If  $S$  is complete, then the quotient  $R$  is again a complete local ring which means complete for  $\mathfrak{m}$ -adic topology where  $\mathfrak{m} = J/\mathfrak{n}$  is the maximal ideal of  $R$ .

Starting with  $q = 2$ , we will define for each  $q$  an ideal  $J_q$  and a family  $(R_q, r_q)$  parametrised by  $R_q = S/J_q$ , such that  $r_{q+1}|_{R_q} = r_q$ . We take  $J_2 = \mathfrak{n}^2$ . On  $R_2 = S/\mathfrak{n}^2 = k\langle T_F^* \rangle$  we take  $q_2$  to be the 'universal first order family'  $\theta$  constructed earlier. Having already constructed  $(R_q, r_q)$ , we next take  $J_{q+1}$  to be the unique smallest ideal in the set  $\Psi$  of all ideals  $I \subset S$  which satisfy the following two conditions:



(1) We have inclusions  $J_q \supset I \supset \mathfrak{n}J_q$ .

(2) There exists a family  $\alpha$  (need not be unique) parametrised by  $R/I$  which prolongs  $r_q$ , that is,  $\alpha|_{R_q} = r_q$ .

Note that  $\Psi$  is non-empty as  $J_q \in \Psi$ . Also, as  $S/\mathfrak{n}J_q$  is artinian (being a quotient of  $S/\mathfrak{n}^{q+1}$ ), the set  $\Psi$  has at least one minimal element.

We will now show that  $\Psi$  has a unique minimal element, by showing that if  $I_1, I_2 \in \Psi$  then  $I_0 = I_1 \cap I_2 \in \Psi$ .

Consider the vector space  $J_q/\mathfrak{n}J_q$  and its subspaces  $I_1/\mathfrak{n}J_q$ ,  $I_2/\mathfrak{n}J_q$ , and  $I_0/\mathfrak{n}J_q$ . Let

$$u_1, \dots, u_a, v_1, \dots, v_b, w_1, \dots, w_c, z_1, \dots, z_d \in J_q$$

be elements such that

(i)  $u_1, \dots, u_a \pmod{\mathfrak{n}J_q}$  is a linear basis of  $I_0/\mathfrak{n}J_q$ ,

(ii)  $u_1, \dots, u_a, v_1, \dots, v_b \pmod{\mathfrak{n}J_q}$  is a linear basis of  $I_1/\mathfrak{n}J_q$ ,

(iii)  $u_1, \dots, u_a, w_1, \dots, w_c \pmod{\mathfrak{n}J_q}$  is a linear basis of  $I_2/\mathfrak{n}J_q$ , and

(iv)  $u_1, \dots, u_a, v_1, \dots, v_b, w_1, \dots, w_c, z_1, \dots, z_d \pmod{\mathfrak{n}J_q}$  is a linear basis of  $J_q/\mathfrak{n}J_q$ .

Let  $I_3 = (u_1, \dots, u_a, w_1, \dots, w_c, z_1, \dots, z_d) + \mathfrak{n}J_q$ . Then we have  $I_2 \subset I_3$ ,  $I_1 \cap I_3 = I_0$  and  $I_1 + I_3 = J_q$ . Note that we have

$$\frac{S}{I_1} \times_{\left(\frac{S}{I_1+I_3}\right)} \frac{S}{I_3} = \frac{S}{I_1 \cap I_3}$$

As  $I_1 + I_3 = J_q$  and  $I_1 \cap I_3 = I_0$ , this reads

$$\frac{S}{I_1} \times_{\left(\frac{S}{\mathfrak{n}J_q}\right)} \frac{S}{I_3} = \frac{S}{I_0}$$

As **(H1)** is satisfied, this gives surjection

$$F\left(\frac{S}{I_0}\right) = F\left(\frac{S}{I_1} \times_{\left(\frac{S}{\mathfrak{n}J_q}\right)} \frac{S}{I_3}\right) \rightarrow F\left(\frac{S}{I_1}\right) \times_{F\left(\frac{S}{\mathfrak{n}J_q}\right)} F\left(\frac{S}{I_3}\right)$$

Let  $\alpha_1 \in F(S/I_1)$  and  $\alpha_2 \in F(S/I_2)$  be any prolongation of  $r_q \in F(S/J_q)$ , which exist as  $I_1, I_2 \in \Psi$ . Let  $\alpha_3 = \alpha_2|_{S/I_3}$ . This defines an element

$$(\alpha_1, \alpha_3) \in F\left(\frac{S}{I_1}\right) \times_{F\left(\frac{S}{\mathfrak{n}J_q}\right)} F\left(\frac{S}{I_3}\right)$$

Therefore by **(H1)** there exists  $\alpha_0 \in F(S/I_0)$  which prolongs both  $\alpha_1$  and  $\alpha_3$  (it might not prolong  $\alpha_2$ ). This means  $\alpha_0$  prolongs  $r_q$ , so  $I_0 \in \Psi$  as was to be shown.

Therefore  $\Psi$  has a unique minimal element  $J_{q+1}$ , and we choose  $r_{q+1} \in F(S/J_{q+1})$  to be an arbitrary prolongation of  $r_q$  (not claimed to be unique).

Now let  $J$  be the intersection of all the  $J_n$ , and let  $R = S/J$ . We want to define an element  $r \in \widehat{F}(R)$  which restricts to  $r_q$  on  $S/J_q$  for each  $q$ . This makes sense and is indeed possible, as we will show using the following lemma.

**Lemma 18** *Let  $R$  be a complete noetherian local ring with maximal ideal  $\mathfrak{m}$ . Let  $I_1 \supset I_2 \supset \dots$  be a decreasing sequence of ideals such that (i) the intersection  $\bigcap_{n \geq 1} I_n$  is 0, and (ii) for each  $n \geq 1$ , we have  $I_n \supset \mathfrak{m}^n$ . Then the natural map  $f : R \rightarrow \varprojlim R/I_n$  is an isomorphism. Moreover, for any  $n \geq 1$  there exists an  $q \geq n$  such that  $\mathfrak{m}^n \supset I_q$ .*

**Proof** Recall that an inverse system  $(E_n)$  indexed by natural numbers is said to satisfy the **Mittag-Leffler condition** if for each  $n$  the decreasing filtration

$$E_n \supset \text{im}(E_{n+1}) \supset \text{im}(E_{n+2}) \supset \text{im}(E_{n+3}) \supset \dots$$

stabilises in finitely many steps. Whenever  $0 \rightarrow (E_n) \rightarrow (F_n) \rightarrow (G_n) \rightarrow 0$  is a short exact sequence of inverse systems such that  $(E_n)$  satisfies the Mittag-Leffler condition, then the resulting limit sequence

$$0 \rightarrow \varprojlim E_n \rightarrow \varprojlim F_n \rightarrow \varprojlim G_n \rightarrow 0$$

is again short exact.

As  $I_n \supset \mathfrak{m}^n$  by assumption, we get the following short exact sequence of inverse systems:

$$0 \rightarrow (I_n/\mathfrak{m}^n) \rightarrow (R/\mathfrak{m}^n) \rightarrow (R/I_n) \rightarrow 0$$

The inverse system  $(I_n/\mathfrak{m}^n)$  satisfies the Mittag-Leffler condition, as it consists of finite dimensional  $k$ -vector spaces and  $k$ -linear maps. This gives a short exact sequence

$$0 \rightarrow \varprojlim I_n/\mathfrak{m}^n \rightarrow R \xrightarrow{f} \varprojlim R/I_n \rightarrow 0$$

where we have put  $R = \varprojlim R/\mathfrak{m}^n$  by assumption of completeness of  $R$ . In other words,  $f$  is surjective.

As  $\bigcap_{n \geq 1} I_n = 0$ , it follows directly from its definition that  $f : R \rightarrow \varprojlim R/I_n$  is injective. Therefore  $f$  is an isomorphism.

As  $f$  is injective, it follows that  $\varprojlim I_n/\mathfrak{m}^n = 0$ , which means that the decreasing filtration  $I_n/\mathfrak{m}^n \supset \text{im}(I_{n+1}/\mathfrak{m}^{n+1}) \supset \text{im}(I_{n+2}/\mathfrak{m}^{n+2}) \supset \dots$  stabilises to zero. As we have already argued (the Mittag-Leffler condition), the decreasing filtration must stabilise in finitely many steps. Therefore there is some  $q \geq n$  for which the map  $I_q/\mathfrak{m}^q \rightarrow I_n/\mathfrak{m}^n$  is zero. This means for each  $n$  there exists an  $q \geq n$  such that  $\mathfrak{m}^n \supset I_q$  as desired.

This completes the proof of the Lemma 18.  $\square$

**Construction of the family  $(R, r)$  (continued)** : We will apply Lemma 18 to the following. Let  $R = S/J$ , which is a complete noetherian local ring with maximal ideal  $\mathfrak{m} = \mathfrak{n}/J$ , and let  $I_q = J_q/J$  for  $q \geq 2$ . (It does not matter, but can take  $J_1 = \mathfrak{n}$  and  $I_1 = \mathfrak{m}$  for the sake of notation). By construction, we have

$J_q \supset J_{q+1} \supset \mathfrak{n}J_q$ , which means  $I_q \supset I_{q+1} \supset \mathfrak{m}I_q$ . In particular, this means  $I_q \supset \mathfrak{m}^q$ . As  $J = \bigcap J_q$ , we get  $\bigcap I_q = 0$ . Therefore by Lemma 18, for each  $n \geq 1$  there exists a  $q \geq n$  with

$$I_n \supset \mathfrak{m}^n \supset I_q$$

and in particular the natural map  $R \rightarrow \lim_{\leftarrow} R/I_n$  is an isomorphism.

Note that  $S/J_q = R/I_q$ . Recall that we have already chosen an inverse system of elements  $r_q \in F(R/I_q)$ . For each  $n$  choose the smallest  $q_n \geq n$  such that  $\mathfrak{m}^{q_n} \supset I_{q_n}$ . We have a natural surjection  $R/I_{q_n} \rightarrow R/\mathfrak{m}^{q_n}$ . Let

$$\theta_n = r_{q_n}|_{R/\mathfrak{m}^{q_n}}$$

Then from its definition it follows that under  $R/\mathfrak{m}^{q_{n+1}} \rightarrow R/\mathfrak{m}^{q_n}$ , we have

$$\theta_n = \theta_{q_{n+1}}|_{R/\mathfrak{m}^{q_n}}$$

Therefore  $(\theta_n)$  defines an element

$$r = (\theta_n) \in \lim_{\leftarrow} F(R/\mathfrak{m}^n) = \widehat{F}(R)$$

**Verification that  $(R, r)$  is a hull for  $F$  :** By its construction, the map  $T_R \rightarrow T_F$  is an isomorphism. So all that remains is to show that  $h_R \rightarrow F$  is formally smooth. This means given any surjection  $p : B \rightarrow A$  in  $\text{Art}_k$  and a commutative square

$$\begin{array}{ccc} h_A & \xrightarrow{u^*} & h_R \\ p^* \downarrow & & \downarrow r \\ h_B & \xrightarrow{b} & F \end{array}$$

there exists a diagonal morphism  $v^* : h_B \rightarrow h_R$  which makes the above square commute. (Here we have used the following notation:  $u^* : h_A \rightarrow h_R$  corresponds to a homomorphism  $u : R \rightarrow A$ ,  $p^* : h_A \rightarrow h_B$  corresponds to  $p : B \rightarrow A$ ,  $b : h_B \rightarrow F$  corresponds to  $b \in F(B)$  by Yoneda,  $r : h_R \rightarrow F$  corresponds to  $r \in \widehat{F}(R)$  by ‘limit Yoneda’, and what we are seeking is a homomorphism  $v : R \rightarrow B$  such that the above diagram commutes.

**Reduction to a small extension :** If  $\dim_k(B) = \dim_k(A)$  as  $k$ -vector space, then the surjection  $B \rightarrow A$  is an isomorphism, and we are done. Otherwise, we can reduce to the case where  $\dim_k(B) = \dim_k(A) + 1$  (the case of a small extension) as follows. The surjective homomorphism  $p : B \rightarrow A$  can be factored in  $\text{Art}_k$  as the composite of a finite sequence of surjections

$$B = B_n \rightarrow B_{n-1} \rightarrow \dots \rightarrow B_1 \rightarrow B_0 = A$$

where  $N \geq 1$  is an integer such that  $\mathfrak{m}_B^N = 0$ , and  $B_j = B/\mathfrak{m}^j I$  where  $I$  is the kernel of  $B \rightarrow A$ . We can construct the desired homomorphism  $v : R \rightarrow B$  by step-by-step constructing  $v_1 : R \rightarrow B_1$ ,  $v_2 : R \rightarrow B_2$ , etc. Therefore without loss of generality we

can assume that  $\mathfrak{m}_B I = 0$ . This means  $I$  is just a finite dimensional  $k$ -vector space. Next, we can filter  $I$  by subspaces  $I_1 \subset I_2 \subset \dots \subset I_d = I$  where  $d = \dim_k(I)$  and  $\dim(I_j) = j$ . The  $I_j$  are automatically ideals in  $B$ . The surjection  $B \rightarrow A$  factors as the composite

$$B \rightarrow B/I_1 \rightarrow \dots \rightarrow B/I_d = A$$

Therefore, without loss of generality we can assume that the following:

**19** The surjection  $p : B \rightarrow A$  in  $\text{Art}_k$  satisfies  $\mathfrak{m}_B I = 0$  and  $\dim_k(I) = 1$  where  $I = \ker(p)$ . Equivalently,  $\dim_k(B) = \dim_k(A) + 1$ .

**It is enough to find some  $w : R \rightarrow B$  which lifts  $u$  :** Suppose there exists a homomorphism  $w : R \rightarrow B$  such that

$$u = p \circ w : R \rightarrow B \rightarrow A$$

Using such a  $w$ , we will construct a homomorphism  $v : R \rightarrow B$  as needed in the proof of formal smoothness of  $h_R \rightarrow F$ , which satisfies both

$$u = p \circ v : R \rightarrow B \rightarrow A \text{ and } r \circ v^* = b : h_B \rightarrow F(B) \rightarrow F(A)$$

(In short, using a diagonal  $w^*$  which makes only the upper triangle commute, we will construct a new diagonal  $v^*$  which makes both triangles commute, giving the desired commutative diagram.)

Consider the following commutative square:

$$\begin{array}{ccc} h_R(B) & \xrightarrow{r(B)} & F(B) \\ h_R(p) \downarrow & & \downarrow F(p) \\ h_R(A) & \xrightarrow{r(A)} & F(A) \end{array}$$

As the kernel  $I$  of  $p : B \rightarrow A$  satisfies  $\mathfrak{m}_B I = 0$ , and as the functor  $h_R$  satisfies **(H1)**, there is a natural transitive action of the additive group  $G = T_R \otimes I$  on each fibre of the set map  $h_R(B) \rightarrow h_R(A)$ . (In fact, as  $h_R$  also satisfies **(H4)**,  $h_R(B) \rightarrow h_R(A)$  is a principal  $T_R \otimes I$ -set, but we do not need this here.) As by hypothesis the functor  $F$  satisfies **(H1)**, there is a natural transitive action of the additive group  $G = T_F \otimes I$  on each fibre of the set map  $F(B) \rightarrow F(A)$ . Under the isomorphism  $T_R \rightarrow T_F$ , the top map  $r(B) : h_R(B) \rightarrow F(B)$  in the above square is  $G$ -equivariant. As  $u = p \circ w$ , the elements  $r(B)w$  and  $b$  both lie in the same fibre of  $F(B) \rightarrow F(A)$ , over  $r(A)u \in F(A)$ . Therefore, there exists some  $\alpha \in G$  (not necessarily unique) such that

$$b = r(B)w + \alpha$$

Let  $v = w + \alpha \in h_R(B)$ . By  $G$ -equivariance of  $r(B)$ , we get

$$r(B)v = r(B)(w + \alpha) = r(B)w + \alpha = b$$

Also, as the action of  $G$  preserves the fibers of  $h_R(B) \rightarrow h_R(A)$ , we have

$$p \circ v = p \circ (w + \alpha) = p \circ w = u$$

Therefore  $v$  has the desired property.

**Remark 20** Let  $B \rightarrow A$  be a surjection in  $\text{Art}_k$  such that  $\dim_k(B) = \dim_k(A) + 1$  (equivalently, the kernel  $I$  of the surjection satisfies  $\mathfrak{m}_B I = 0$  and  $\dim_k(I) = 1$ ). Suppose that  $B \rightarrow A$  does not admit a section  $A \rightarrow B$ . Then for any  $k$ -algebra homomorphism  $S \rightarrow B$ , the composite  $S \rightarrow B \rightarrow A$  is surjective (if and) only if  $S \rightarrow B$  is surjective.

**Existence of  $w : R \rightarrow B$  with  $p \circ w = u$  :** The homomorphism  $u : R \rightarrow A$  must factor via  $R_q = R/\mathfrak{m}^q$  for some  $q \geq 1$ , giving a homomorphism  $u_q : R_q \rightarrow A$ .

As before, let  $S = k[[x_1, \dots, x_d]]$  be the complete local ring at the origin of the affine space  $\text{Spec Sym}_k(T_F^*)$ , with  $R = S/J$ . We are given a diagram

$$\begin{array}{ccccc} \text{Spec } A & \xrightarrow{u_q^*} & \text{Spec } R_q & \rightarrow & \text{Spec } R \hookrightarrow & \text{Spec } S \\ \downarrow & & & & & \downarrow \\ \text{Spec } B & & & \rightarrow & & \text{Spec } k \end{array}$$

The morphism  $\text{Spec } S \rightarrow \text{Spec } k$  is formally smooth, therefore, there exists a diagonal homomorphism  $f^* : \text{Spec } B \rightarrow \text{Spec } S$  which makes the above diagram commute. Equivalently, there exists a  $k$ -algebra homomorphism  $f : S \rightarrow B$  such that  $p \circ f = u \circ \pi : S \rightarrow A$  where  $\pi : S \rightarrow R = S/J$  is the quotient map. Therefore, we get a commutative square

$$\begin{array}{ccc} S & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ R_q & \xrightarrow{u_q} & A \end{array}$$

where the vertical maps are the quotient maps  $\pi_q : S \rightarrow S/J_q = R_q$ , and  $p : B \rightarrow A$ . This defines a  $k$ -homomorphism

$$\varphi = (\pi_q, f) : S \rightarrow R_q \times_A B$$

The composite  $S \rightarrow R_q \times_A B \rightarrow R_q$  is  $\pi_q$  which is surjective. As by assumption  $\dim_k(B) = \dim_k(A) + 1$ , it follows that

$$\dim_k(R_q \times_A B) = \dim_k(R_q) + 1$$

Therefore by Remark 20, at least one of the following holds:

- (1) The projection  $R_q \times_A B \rightarrow R_q$  admits a section  $(\text{id}, s) : R_q \rightarrow R_q \times_A B$ , in other words, there exists some  $s : R_q \rightarrow B$  such that  $p \circ s = u_q : R_q \rightarrow A$ .
- (2) The homomorphism  $\varphi : S \rightarrow R_q \times_A B$  is surjective.

If (1) holds, then we immediately get a lift

$$v : R \rightarrow R_q \xrightarrow{s} B$$

of  $u : R \rightarrow A$ , completing the proof.

If (2) holds, then we claim that  $\varphi : S \rightarrow R_q \times_A B$  factors through  $S \rightarrow S/J_{q+1} = R_{q+1}$ , thereby giving a homomorphism  $s' : R_{q+1} \rightarrow B$  such that  $p \circ s' = u_{q+1} : R_{q+1} \rightarrow A$ . This immediately gives a lift

$$v : R \rightarrow R_{q+1} \xrightarrow{s'} B$$

of  $u : R \rightarrow A$ , again completing the proof.

Therefore, all that remains is to show that if  $\varphi : S \rightarrow R_q \times_A B$  is surjective, then it must factor through  $S \rightarrow S/J_{q+1} = R_{q+1}$ .

Let  $K = \ker(\varphi) \subset S$ , so that  $R_q \times_A B$  gets identified with  $S/K$  by surjectivity of  $\varphi$ . We have the families  $r_q \in F(R_q)$ ,  $a \in F(A)$  and  $b \in F(B)$  such that both  $r_q$  and  $b$  map to  $a$  under  $R_q \rightarrow A$  and  $B \rightarrow A$ . By **(H1)** the map

$$F(R_q \times_A B) \rightarrow F(R_q) \times_{F(A)} F(B)$$

is surjective, so there exists a family  $\mu \in F(R_q \times_A B) = F(S/K)$  which restricts to  $r_q \in F(R_q)$ . This means the ideal  $K$  is in the set of ideals  $\Psi$  defined earlier while constructing the nested sequence  $J_2 \supset J_3 \supset \dots$  of ideals in  $S$ . By minimality of  $J_{q+1}$ , we have  $K \supset J_{q+1}$ . Therefore  $\varphi : S \rightarrow R_q \times_A B = S/K$  factors through  $S \rightarrow S/J_{q+1} = R_{q+1}$  as desired.

This completes the proof of Schlessinger's theorem. □

## 3.4 Application to examples

### Preliminaries: Some lemmas on flatness

**Remark 21 (Nilpotent Nakayama)** Let  $A$  be a ring and  $J \subset A$  a nilpotent ideal (there exists some  $n \geq 1$  such that  $J^n = 0$ ). If  $M$  is any  $A$ -module with  $M = JM$  then  $M = 0$ . For, we have the chain of equalities  $M = JM = J^2M = \dots = J^nM = 0$ . This simple remark is generalised by the following lemma.

**Lemma 22** (Schlessinger Lemma 3.3) *Let  $A$  be a ring and  $J \subset A$  a nilpotent ideal (there exists some  $n \geq 1$  such that  $J^n = 0$ ). Let  $u : M \rightarrow N$  be a homomorphism of  $A$ -modules where  $N$  is flat over  $A$ . If  $\bar{u} : M/JM \rightarrow N/JN$  is an isomorphism, then  $u$  is an isomorphism.*

**Proof** If  $C$  is the cokernel of  $u$ , then it follows by surjectivity of  $\bar{u}$  and right-exactness of  $- \otimes_A (A/J)$  that  $C/JC = 0$ . Therefore  $C = JC$ . By iteration,  $C = JC = J^2C = \dots = J^nC = 0 \cdot C = 0$ . So we have a short exact sequence  $0 \rightarrow K \rightarrow M \rightarrow N \rightarrow 0$ , where  $K = \ker(u)$ . By flatness of  $N$ , we get the short exact sequence  $0 \rightarrow K/JK \rightarrow M/JM \rightarrow N/JN \rightarrow 0$ . As  $\bar{u}$  is injective,  $K/JK = 0$ . Therefore as above,  $K = 0$ . □

**Corollary 23** *Flat modules over an artin local ring are free.*

**Proof** Let  $M$  be flat over an artin local ring  $A$ . Let  $(v_i)_{i \in I}$  be a  $k$ -linear basis of  $M \otimes_A k$ , where  $k$  denotes the residue field of  $A$ . (The indexing set  $I$  could be infinite.) Let  $N = \bigoplus_I A$  be the direct sum of  $I$  copies of  $A$ , which is a free  $A$ -module, with standard basis denoted by  $(e_i)_{i \in I}$ . Let  $u : N \rightarrow M$  be the surjective homomorphism defined by  $e_i \mapsto v_i$ . Then  $\bar{u} : M/\mathfrak{m}M \rightarrow N/\mathfrak{m}N$  is an isomorphism, where  $\mathfrak{m} \subset A$  is the maximal ideal. As  $A$  is artinian,  $\mathfrak{m}$  is nilpotent, so the desired conclusion follows from the above lemma.  $\square$

**Lemma 24** *Let  $A$  be an artin local ring, and  $M$  an  $A$ -module (not-necessarily finitely generated). Then  $M$  is flat if and only if  $\text{Tor}_1^A(A/\mathfrak{m}, M) = 0$ .*

**Proof** If  $M$  is flat, then  $\text{Tor}_1^A(N, M) = 0$  for each  $A$ -module  $N$ , in particular, for  $N = A/\mathfrak{m}$ . For the converse, choose a basis  $(x_i)_{i \in I}$  for the vector space  $M/\mathfrak{m}M$  over the residue field  $A/\mathfrak{m}$ . Let  $(e_i)_{i \in I}$  be the standard basis for the direct sum  $F = A^{\oplus I}$ . Consider the  $A$ -linear map  $\varphi : F \rightarrow M : e_i \mapsto x_i$ . Then going modulo  $\mathfrak{m}$ , we have an isomorphism  $\bar{\varphi} : F/\mathfrak{m}F \rightarrow M/\mathfrak{m}M$ , which shows that

$$\varphi(F) + \mathfrak{m}M = M$$

This means  $\mathfrak{m}(M/\varphi(F)) = M/\varphi(F)$ , so by the Remark 21, we get  $\varphi(F) = M$ , so  $\varphi$  is surjective. Let  $N = \ker(\varphi)$  so that we have a short exact sequence  $0 \rightarrow N \rightarrow F \rightarrow M \rightarrow 0$  by surjectivity of  $\varphi$ . Applying  $(A/\mathfrak{m}) \otimes_A -$  to this we get the exact sequence

$$0 \rightarrow \text{Tor}_1^A(A/\mathfrak{m}, M) \rightarrow N/\mathfrak{m}N \rightarrow F/\mathfrak{m}F \rightarrow M/\mathfrak{m}M \rightarrow 0$$

As  $\text{Tor}_1^A(A/\mathfrak{m}, M) = 0$  and as  $F/\mathfrak{m}F \rightarrow M/\mathfrak{m}M$  is an isomorphism, we get  $N/\mathfrak{m}N = 0$ . Therefore again by Remark 21, we get  $N = 0$ , which shows  $\varphi : F \rightarrow M$  is an isomorphism.  $\square$

**Lemma 25** *Let  $k$  be a field and  $V$  a finite dimensional  $k$ -vector space. Let  $M$  a module over  $k\langle V \rangle$ , not necessarily finitely generated. Then  $M$  is flat over  $k\langle V \rangle$  if and only if the map*

$$V \otimes_k \frac{M}{VM} \rightarrow VM$$

*induced by scalar multiplication is an isomorphism.*

**Proof** Note that we have a natural isomorphism

$$V \otimes_{k\langle V \rangle} M \xrightarrow{\sim} V \otimes_k (M/VM) : v \otimes_{k\langle V \rangle} x \mapsto v \otimes_k \bar{x}$$

Consider the following short exact sequence of  $k\langle V \rangle$ -modules:

$$0 \rightarrow V \rightarrow k\langle V \rangle \rightarrow k \rightarrow 0$$

On applying the functor  $- \otimes_{k\langle V \rangle} M$ , this gives the following exact sequence:

$$0 \rightarrow \text{Tor}_1^{k\langle V \rangle}(k, M) \rightarrow V \otimes_k (M/VM) \rightarrow M \rightarrow M/VM \rightarrow 0$$

By Lemma 24,  $M$  is flat if and only if  $\text{Tor}_1^{k\langle V \rangle}(k, M) = 0$ . Therefore, the lemma follows.  $\square$

The following lemma is an example of **non-flat descent**: even though  $\text{Spec}(A') \rightarrow \text{Spec}(B)$  and  $\text{Spec}(A'') \rightarrow \text{Spec}(B)$  is not necessarily a flat cover of  $\text{Spec}(B)$ , we get a flat module  $N$  on  $B$  from flat modules  $M'$  and  $M''$  on  $A'$  and  $A''$ .

**Lemma 26** (Schlessinger Lemma 3.4) *Let  $A' \rightarrow A$  and  $A'' \rightarrow A$  be ring homomorphisms, such that  $A'' \rightarrow A$  is surjective with its kernel a nilpotent ideal  $J \subset A''$ . Let  $B = A' \times_A A''$ , with  $B \rightarrow A'$  and  $B \rightarrow A''$  the projections. Let  $M, M'$  and  $M''$  be modules over  $A, A', A''$ , together with  $A'$ -linear homomorphism  $u' : M' \rightarrow M$  and  $A''$ -linear homomorphism  $u'' : M'' \rightarrow M$  which give isomorphisms  $M' \otimes_{A'} A \rightarrow M$  and  $M'' \otimes_{A''} A \rightarrow M$ . Let  $N$  be the  $B$ -module*

$$N = M' \times_M M'' = \{(x', x'') \in M' \times M'' \mid u'(x') = u''(x'') \in M\}$$

where scalar multiplication by elements  $(a', a'') \in B$  is defined by  $(a', a'') \cdot (x', x'') = (a'x', a''x'')$ . If  $M'$  and  $M''$  are flat modules over  $A'$  and  $A''$  respectively, then  $N$  is flat over  $B$ . Moreover, the projection maps  $N \rightarrow M'$  and  $N \rightarrow M''$  induce isomorphisms  $N \otimes_B A' \xrightarrow{\sim} M'$  and  $N \otimes_B A'' \xrightarrow{\sim} M''$ .

**Proof (Only in the case where  $M'$  is a free  $A'$ -module) :** Note that if  $A'$  is artin local, then we are automatically in this case by Corollary 23. This is therefore the only case which we need in these notes.

Let  $(x'_i)_{i \in I}$  be a free basis for  $M'$  over  $A'$ . As  $M' \otimes_{A'} A \rightarrow M$  is an isomorphism, this gives a free basis  $u'(x'_i)$  of  $M$  over  $A$ .

As  $A'' \rightarrow A$  is surjective, any element  $\sum y'_j \otimes a''_j$  of  $M'' \otimes_{A''} A$  equals  $x'' \otimes 1$  for some (not necessarily uniquely determined) element  $x'' \in M''$ . Therefore the assumption of surjectivity of  $M'' \otimes_{A''} A \rightarrow M$  tells us that  $u'' : M'' \rightarrow M$  must be surjective.

Therefore, we can choose elements  $x''_i \in M''$  such that  $u''(x''_i) = u'(x'_i)$ . Let  $N = \bigoplus_I A''$  be the free  $A''$ -module on the set  $I$ , with standard basis denoted by  $(e_i)_{i \in I}$ , and let  $u : N \rightarrow M''$  be defined by  $e_i \mapsto x''_i$ . Then  $\bar{u} : N/JN \rightarrow M''/JM'' = M$  is an isomorphism. Therefore by Lemma 22,  $u$  is an isomorphism, which shows  $M''$  is free with basis  $(x''_i)_{i \in I}$ . It follows that  $N$  is free over  $B$ , with basis  $(x'_i, x''_i)_{i \in I}$ . It is now immediate that the projections  $N \rightarrow M'$  and  $N \rightarrow M''$  induce isomorphisms  $N \otimes_B A' \xrightarrow{\sim} M'$  and  $N \otimes_B A'' \xrightarrow{\sim} M''$ .  $\square$



**Corollary 27** (Schlessinger Corollary 3.6) *With hypothesis and notation as in the above lemma, let  $L$  be a  $B$  module, and  $q' : L \rightarrow M'$  and  $q'' : L \rightarrow M''$   $B$ -linear homomorphisms, such that the following diagram commutes:*

$$\begin{array}{ccc} L & \xrightarrow{q''} & M'' \\ q' \downarrow & & \downarrow u'' \\ M' & \xrightarrow{u'} & M \end{array}$$

*Suppose that  $q'$  induces an isomorphism  $L \otimes_B A' \rightarrow M'$ . Then the map  $(q', q'') : L \rightarrow N = M' \times_M M''$  is an isomorphism of  $B$ -modules.*

**Proof** The kernel of the projection  $B \rightarrow A'$  is the ideal

$$I = 0 \times J \subset A' \times_A A'' = B$$

The ideal  $I$  is nilpotent as by assumption  $J$  is nilpotent. The desired result follows by applying Lemma 22 to the  $B$ -homomorphism  $u = (q', q'') : L \rightarrow N$ , which becomes the given isomorphism  $L \otimes_B A' \rightarrow M'$  on going modulo the nilpotent ideal  $I \subset B$ .  $\square$

### Hull for coherent sheaves

Let  $X$  be a proper scheme over a field  $k$ , and let  $E$  be a coherent sheaf of  $\mathcal{O}_X$ -modules. The deformation functor  $\mathcal{D}_E$  of  $E$  is the covariant functor  $Art_k \rightarrow Sets$  defined as follows. For any  $A$  in  $Art_k$ , we take  $\mathcal{D}_E(A)$  to be the set of all equivalence classes of pairs  $(F, \theta)$  where  $F$  is a coherent sheaf on  $X_A = X \otimes_k A$  which is flat over  $A$ , and  $\theta : i^*F \rightarrow E$  is an isomorphism where  $i : X \hookrightarrow X_A$  is the closed embedding induced by  $A \rightarrow k$ , with  $(F, \theta)$  and  $(F', \theta')$  to be regarded as equivalent when there exists some isomorphism  $\eta : F \rightarrow F'$  such that  $\theta' \circ i^*(\eta) = \theta$ . It can be seen that  $\mathcal{D}_E(A)$  is indeed a set. Given any morphism  $f : \text{Spec } B \rightarrow \text{Spec } A$  in  $(Art_k)^{op}$  and an equivalence class  $(F, \theta)$  in  $\mathcal{D}_E(A)$ , we define  $f^*(F, \theta)$  in  $\mathcal{D}_E(B)$  to be obtained by pull-back under the morphism  $f : X_B \rightarrow X_A$ . This operation preserves equivalences, and thus it gives us a functor  $\mathcal{D}_E : Art_k \rightarrow Sets$ .

**Theorem 28** *Let  $X$  be a proper scheme over a field  $k$ . Let  $E$  be a coherent sheaf on  $X$ . Then the deformation functor  $\mathcal{D}_E : Art_k \rightarrow Sets$  of  $E$  as defined above admits a hull, with tangent space  $Ext^1(E, E)$ .*

**Proof** We will show that the conditions **(H1)**, **(H2)**, **(H3)** in the Schlessinger Theorem 17 are satisfied by our functor  $\mathcal{D}_E$ .

**Verification of (H1):** An element of  $\mathcal{D}_E(A') \times_{\mathcal{D}_E(A)} \mathcal{D}_E(A'')$  is an ordered tuple  $(F', \theta', F'', \theta'')$  where  $(F', \theta') \in \mathcal{D}_E(A')$  and  $(F'', \theta'') \in \mathcal{D}_E(A'')$ , such that there exists an isomorphism  $\eta : F'|_A \rightarrow F''|_A$  which makes the following diagram commutes:

$$\begin{array}{ccc} F'|_X & \xrightarrow{i^*(\eta)} & F''|_X \\ \theta' \downarrow & & \downarrow \theta'' \\ E & = & E \end{array}$$

**Caution:** We do not have a particular choice of  $\eta$  given to us. We will now arbitrarily choose one such  $\eta$  and fix it for the rest of the proof.

Let  $F = F''|_A$ , let  $u'' : F'' \rightarrow F$  be the quotient and let  $u' : F' \rightarrow F$  be induced by  $\eta$ . Let  $B = A' \times_A A''$ , and let  $G$  be the sheaf of  $\mathcal{O}_{X_B}$ -modules defined by

$$G = F' \times_{u', F, u''} F''$$

This is clearly coherent, as the construction can be done on each affine open and glued. By Lemma 26 applied stalk-wise, the sheaf  $G$  is flat over  $B$ . By Lemma 27 applied stalk-wise, this is up to isomorphism the only coherent sheaf on  $X_B$ , flat over  $B$ , which comes with homomorphisms  $p' : G \rightarrow F'$  and  $p'' : G \rightarrow F''$  which make the following square commute:

$$\begin{array}{ccc} G & \xrightarrow{p''} & F'' \\ p' \downarrow & & \downarrow u'' \\ F' & \xrightarrow{u'} & F \end{array}$$

This shows that  $\mathcal{D}_E(B) \rightarrow \mathcal{D}_E(A') \times_{\mathcal{D}_E(A)} \mathcal{D}_E(A'')$  is surjective, as desired. Thus, Schlessinger condition **(H1)** is satisfied.

**Caution:** If we choose another  $\eta$ , we might get a different  $G$ , and so the map  $\mathcal{D}_E(B) \rightarrow \mathcal{D}_E(A') \times_{\mathcal{D}_E(A)} \mathcal{D}_E(A'')$  may not be injective.

**Verification of (H2):** If we take  $A$  to be  $k$  in the above verification of the condition **(H1)**, then  $\eta$  would be unique, and so we will get a bijection  $\mathcal{D}_E(A' \times_k A'') \rightarrow \mathcal{D}_E(A') \times_{\mathcal{D}_E(k)} \mathcal{D}_E(A'')$ . In particular, this implies that **(H2)** is satisfied.

**Verification of (H3):** We have already seen that the finite dimensional vector space  $\text{Ext}^1(E, E)$  is the tangent space to  $\mathcal{D}_E$ , and hence **(H3)** holds. This completes the proof of the Theorem 28 in the general case of coherent sheaves.  $\square$

### Prorepresentability for a ‘simple’ sheaf

**Theorem 29** *Let  $X$  be a proper scheme over a field  $k$ , and let  $F$  be a coherent sheaf on  $X$ . Assume that there exists an exact sequence  $E_1 \rightarrow E_0 \rightarrow F \rightarrow 0$  of  $\mathcal{O}_X$ -modules, where  $E_1$  and  $E_0$  are locally free (note that this condition is automatically satisfied when  $F$  itself is locally free, or when  $X$  is projective over  $k$ ). If the ring homomorphism  $k \rightarrow \text{End}(F)$  (under which  $k$  acts on  $F$  by scalar multiplication) is an isomorphism, then the deformation functor  $\mathcal{D}_F$  is pro-representable.*

**Proof** Let  $A$  be artin local, and let  $I$  be a proper ideal. Let  $(\mathcal{F}, \theta) \in \mathcal{D}_F(A)$ , and let  $(\mathcal{F}', \theta')$  denote its restriction to  $A/I$ . By Lemma 44, the natural ring homomorphisms  $A \rightarrow \text{End}_{X \otimes A/I}(\mathcal{F})$  and  $A/I \rightarrow \text{End}_{X \otimes A/I}(\mathcal{F}')$ , under which  $A$  and  $A/I$  act respectively on  $\mathcal{F}$  and  $\mathcal{F}'$  by scalar multiplication, are isomorphisms. In particular,

we get induced group isomorphisms  $A^\times \rightarrow \text{Aut}(\mathcal{F})$  and  $(A/I)^\times \rightarrow \text{Aut}(\mathcal{F}')$ . The subgroups  $1 + \mathfrak{m}_A \subset A^\times$  and  $1 + \mathfrak{m}_{A/I} \subset (A/I)^\times$  therefore map isomorphically onto  $\text{Aut}(\mathcal{F}, \theta)$  and  $\text{Aut}(\mathcal{F}', \theta')$  respectively. As the homomorphism  $1 + \mathfrak{m}_A \rightarrow 1 + \mathfrak{m}_{A/I}$  is surjective, the restriction map  $\text{Aut}(\mathcal{F}, \theta) \rightarrow \text{Aut}(\mathcal{F}', \theta')$  is again surjective. From this, it follows that the Schlessinger condition **(H4)** is satisfied, and so the functor  $\mathcal{D}_F$  is pro-representable by Theorem 17.  $\square$

#### Hull for deformations of a proper smooth variety

Given a scheme  $C$  of finite type over a field  $k$ , let the deformation functor  $\mathbf{Def}_C : \text{Art}_k \rightarrow \text{Sets}$  be defined as follows. For any  $A \in \text{Art}_k$ , consider pairs  $(p : X \rightarrow \text{Spec } A, i : C \rightarrow X_0)$  where  $p$  is a flat morphism, and  $i$  is an isomorphism over  $k$  of the given scheme  $C$  with the special fibre of  $p$ . Denoting again by  $i$  the composite  $C \rightarrow X_0 \hookrightarrow X$ , this means the following square is cartesian.

$$\begin{array}{ccc} C & \xrightarrow{i} & X \\ \downarrow & \square & \downarrow p \\ \text{Spec } k & \rightarrow & \text{Spec } A \end{array}$$

We say that two such pairs  $(p, i)$  and  $(p', i')$  are equivalent if there exists an  $A$ -isomorphism between  $X$  and  $X'$  which takes  $i$  to  $i'$ . We take  $\mathbf{Def}_C(A)$  to be the set of all equivalence classes of pairs  $(p, i)$ . It can be seen that this is indeed a set, and moreover it is clear that a morphism  $A \rightarrow B$  in  $\text{Art}_k$  gives by pull-back a well-defined set map  $\mathbf{Def}_C(A) \rightarrow \mathbf{Def}_C(B)$  which indeed gives a functor  $\mathbf{Def}_C : \text{Art}_k \rightarrow \text{Sets}$ .

Note that an **automorphism** of a pair  $(p : X \rightarrow \text{Spec } A, i : C \rightarrow X_0)$  will mean an isomorphism  $f : X \rightarrow X$  over  $A$ , such that  $f$  restricts to identity on the special fibre  $X_0$ .

**Lemma 30** *Let  $A$  be a noetherian ring, and  $I \subset A$  an ideal such that  $I^n = 0$  for some  $n \geq 1$ . Then the following properties hold for any morphism of schemes  $X \rightarrow \text{Spec } A$ .*

- (i) *If  $X \otimes_A A/I$  is affine, then  $X$  is affine.*
- (ii) *If  $X \otimes_A A/I$  is of finite type over  $A/I$ , then  $X$  is of finite type over  $A$ .*
- (iii) *If  $X \otimes_A A/I$  is separated, then  $X \rightarrow \text{Spec } A$  is separated.*
- (iv) *If  $X \otimes_A A/I$  is proper, then  $X \rightarrow \text{Spec } A$  is proper.*

**Proof** This is left as an exercise.  $\square$

**Remark 31** Let  $B$  be a ring, and  $I \subset B$  an ideal with  $I^2 = 0$ . Let  $q : B \rightarrow A = B/I$  be the quotient homomorphism. Let  $A\langle I \rangle$  be the  $A$ -module  $A \oplus I$ , with ring structure defined by  $(a, x) \cdot (a', x') = (aa', ax' + a'x)$ . Suppose there exists a

ring homomorphism  $s : A \rightarrow B$  which is a section of  $q : B \rightarrow A$ , that is,  $q \circ s = \text{id}_A$ . Then the additive map

$$\phi_s : A\langle I \rangle \rightarrow B : (a, x) \mapsto s(a) + x$$

is a ring isomorphism. Next, suppose that  $s_1, s_2 : A \rightarrow B$  are two ring homomorphism sections of  $q : B \rightarrow A$ . Then we get ring isomorphisms  $\phi_{s_1} : A\langle I \rangle \rightarrow B$  and  $\phi_{s_2} : A\langle I \rangle \rightarrow B$  as above, so we get a ring automorphism

$$\phi_{s_2}^{-1} \circ \phi_{s_1} : A\langle I \rangle \rightarrow B \rightarrow A\langle I \rangle : (a, x) \mapsto s_1(a) + x \mapsto (a, x + (s_1 - s_2)a)$$

In particular,  $(0, x) \mapsto (0, x)$ , and so note that the above map is identity when restricted to  $I \subset A\langle I \rangle$ .

This leads us to consider the map

$$v = s_1 - s_2 : A \rightarrow I$$

As  $\phi_{s_2}^{-1} \circ \phi_{s_1}$  is multiplicative, we get

$$v(aa') = av(a') + a'v(a)$$

This means  $v$  is a derivation on the ring  $A$ , taking values in the  $A$ -module  $I$ . Conversely, given any ring homomorphism  $s : A \rightarrow B$  which is a section of  $q : B \rightarrow A$ , and any derivation  $v : A \rightarrow I$ , we get another ring homomorphism  $s + v : A \rightarrow B$  which is again section of  $q : B \rightarrow A$ . Therefore, the set of sections of  $B \rightarrow A$  is a principal set under the additive group  $\text{Hom}_A(\Omega_A, I)$  of all derivations on  $A$  taking values in  $I$ .

**Definition 32** Let a scheme  $C$  be separated and of finite over  $k$ , and let  $\mathcal{F}$  be a coherent sheaf on  $C$ . An **extension of  $C$  by  $\mathcal{F}$**  will mean a triple  $(\mathcal{O}_X, p, u)$  consisting of sheaf of  $k$ -algebras  $\mathcal{O}_X$  on the underlying topological space  $C^{\text{top}}$  of the scheme  $C$ , such that  $X = (C^{\text{top}}, \mathcal{O}_X)$  is a scheme over  $k$ , together with a surjective morphism  $p : \mathcal{O}_X \rightarrow \mathcal{O}_C$  of sheaves of  $k$ -algebras such that  $\ker(p)^2 = 0$ , and an  $\mathcal{O}_C$ -linear isomorphism  $u$  of the resulting  $\mathcal{O}_C$ -module  $\ker(p)$  with  $\mathcal{F}$ . We will say that two extensions  $(\mathcal{O}_X, p, u)$  and  $(\mathcal{O}_Y, q, v)$  of  $C$  by  $\mathcal{F}$  are equivalent if there exists an isomorphism of  $k$ -algebras  $f : \mathcal{O}_X \rightarrow \mathcal{O}_Y$  such that the following diagram commutes.

$$\begin{array}{ccccccc} 0 & \rightarrow & \mathcal{F} & \xrightarrow{u} & \mathcal{O}_X & \xrightarrow{p} & \mathcal{O}_C & \rightarrow 0 \\ & & \parallel & & \downarrow f & & \parallel & \\ 0 & \rightarrow & \mathcal{F} & \xrightarrow{v} & \mathcal{O}_Y & \xrightarrow{q} & \mathcal{O}_C & \rightarrow 0 \end{array}$$

We say that an extension  $(\mathcal{O}_X, p, u)$  of  $C$  by  $\mathcal{F}$  is **locally split** if  $C$  has an open cover  $U_i$  such that each  $p|_{U_i}$  admits a section ( $k$ -algebra homomorphism)  $s_i : \mathcal{O}_C|_{U_i} \rightarrow \mathcal{O}_X|_{U_i}$  with  $p|_{U_i} \circ s_i = \text{id}_{\mathcal{O}_C|_{U_i}}$ . Note that this condition is preserved under equivalence. Moreover, note that a local splitting  $s_i$  defines an isomorphism of sheaves of rings of  $k$ -algebras

$$f_i : \mathcal{O}_C\langle \mathcal{F} \rangle|_{U_i} \rightarrow \mathcal{O}_X|_{U_i} : (a, x) \mapsto s_i(a) + x$$

where  $\mathcal{O}_C\langle\mathcal{F}\rangle = \mathcal{O}_C \oplus \mathcal{F}$  with ring structure given by  $\mathcal{F}^2 = 0$ , and where we have identified  $\ker(p)$  with  $\mathcal{F}$  using  $u$ . Note further that  $p \circ f_i$  is the projection  $\mathcal{O}_C\langle\mathcal{F}\rangle|_{U_i} \rightarrow \mathcal{O}_C|_{U_i}$  on the first factor, and the inclusion  $\mathcal{F}|_{U_i} \hookrightarrow \mathcal{O}_C\langle\mathcal{F}\rangle|_{U_i}$  is taken to  $u|_{U_i}$ .

**Lemma 33** *The set of all equivalence classes of locally-split extensions of  $C$  by  $\mathcal{F}$  (which is indeed a set) has a canonical bijection (which is described in the proof) with the set  $H^1(C, \underline{Hom}(\Omega_{C/k}^1, \mathcal{F}))$ .*

**Proof** Let  $\eta \in H^1(C, \underline{Hom}(\Omega_{C/k}^1, \mathcal{F}))$ . Let  $U_i$  be an affine open cover of  $C$ , with respect to which  $\eta$  is described by a 1-cocycle  $(\eta_{i,j})$  where  $\eta_{i,j} \in \Gamma(U_i \cap U_j, \underline{Hom}(\Omega_{C/k}^1, \mathcal{F}))$ . Over  $U_i$ , consider the sheaf of algebras  $R_i = \mathcal{O}_C\langle\mathcal{F}\rangle|_{U_i}$  where  $\mathcal{O}_C\langle\mathcal{F}\rangle = \mathcal{O}_C \oplus \mathcal{F}$  with ring structure given by  $\mathcal{F}^2 = 0$ . This comes with a projection  $p_i : R_i \rightarrow \mathcal{O}_C|_{U_i} : (a, x) \mapsto a$  which has a section  $s_i : \mathcal{O}_C|_{U_i} \rightarrow R_i : a \mapsto (a, 0)$ . Consider the homomorphism

$$g_{i,j} : R_j|_{U_i \cap U_j} \rightarrow R_i|_{U_i \cap U_j} : (a, x) \mapsto (a, x + \eta_{i,j}(da))$$

Clearly,  $g_{i,j}$  is a  $k$ -algebra automorphism, which commutes with the projections  $p_i$  and  $p_j$  to  $\mathcal{O}_C|_{U_i \cap U_j}$ , and which restricts to identity on  $\mathcal{F}$ . Moreover, the cocycle condition is satisfied by  $(g_{i,j})$ . Therefore, we can glue together the  $R_i$  using  $(g_{i,j})$ , to get a locally split extension of  $C$  by  $\mathcal{F}$ . It can be seen that the equivalence class of this extension is independent of the choice of the 1-cocycle  $(\eta_{i,j})$  for  $\eta$ . This defines the map from  $H^1(C, \underline{Hom}(\Omega_{C/k}^1, \mathcal{F}))$  to the set of all equivalence classes of locally-split extensions of  $C$  by  $\mathcal{F}$ . The verification that this is indeed a bijection now follows from the arguments made in Remark 31.  $\square$

**Example 34** Let  $C = \text{Spec } k$ , and let  $\mathcal{F} = k^1$ . Then  $\text{Spec } k[\epsilon]/(\epsilon^2)$  together with the natural projection  $k[\epsilon]/(\epsilon^2) \rightarrow k$  is an extension of  $C$  by  $\mathcal{F}$ . It is split, and admits the self-equivalences  $\epsilon \mapsto \lambda\epsilon$  where  $\lambda \in k^\times$  (and no other). Any extension of  $C$  by  $\mathcal{F}$  is equivalent to the above. The set  $H^1(C, \underline{Hom}(\Omega_{C/k}^1, \mathcal{F}))$  is a singleton.

**Theorem 35** (Schlessinger Proposition 3.10) *Let  $k$  be a field, and let  $C$  be a scheme of finite type over  $k$ . Suppose that*

- (a)  $C$  is proper over  $k$ , or
- (b)  $C$  is affine with isolated singularities.

*Then the deformation functor  $\mathbf{Def}_C$  admits a hull.*

*If  $C$  is a local complete intersection, then the tangent space to the functor  $\mathbf{Def}_C$  is the  $k$ -vector space  $\text{Ext}_C^1(\Omega_{C/k}^1, \mathcal{O}_C)$ .*

*Assume that (a) or (b) holds. Then moreover  $\mathbf{Def}_C$  is pro-representable, if and only if the following condition holds:*

(c) For each small extension  $A' \rightarrow A$  and each  $(p' : X' \rightarrow \text{Spec } A', i' : C \hookrightarrow X')$  representing an element of  $\mathbf{Def}_C(A)$ , every automorphism of the restriction  $(p, i) = (p', i')|_{\text{Spec } A}$  is the restriction to  $\text{Spec } A$  of some automorphism of  $(p', i')$ .

**Proof Verification of (H1) :** Given any three objects  $A, A'$ , and  $A''$  of  $\text{Art}_k$ , with morphisms  $A' \rightarrow A$  and  $A'' \rightarrow A$  such that  $A'' \rightarrow A$  is surjective, we want to show that the induced map

$$\mathbf{Def}_C(A' \times_A A'') \rightarrow \mathbf{Def}_C(A') \times_{\mathbf{Def}_C(A)} \mathbf{Def}_C(A'')$$

is surjective. Consider an element

$$((X', i'), (X'', i'')) \in \mathbf{Def}_C(A') \times_{\mathbf{Def}_C(A)} \mathbf{Def}_C(A'')$$

where  $X' \rightarrow \text{Spec } A'$  and  $X'' \rightarrow \text{Spec } A''$  are flat morphisms, and  $i' : C \rightarrow X'_0$  and  $i'' : C \rightarrow X''_0$  are  $k$ -isomorphisms, such that there exists an isomorphism

$$f : (X', i')|_{\text{Spec } A} \xrightarrow{\sim} (Y, i) = (X'', i'')|_{\text{Spec } A}$$

of pairs over  $A$ .

Note that by Lemma 30, the schemes  $Y, X'$ , and  $X''$  are respectively of finite type over  $A, A'$ , and  $A''$ . Also, topologically, all these are just  $C$ . On the same underlying topological space  $C$ , we have the various different structure sheaves  $\mathcal{O}_C, \mathcal{O}_Y, \mathcal{O}_{X'}$  and  $\mathcal{O}_{X''}$  (where we have made identifications using  $i, i', i''$ ), and homomorphisms of sheaves of rings  $\mathcal{O}_{X'} \rightarrow \mathcal{O}_Y$  and  $\mathcal{O}_{X''} \rightarrow \mathcal{O}_Y$ . The homomorphism  $\mathcal{O}_{X''} \rightarrow \mathcal{O}_Y$  is surjective as  $A'' \rightarrow A$  is surjective. Let  $\mathcal{O}_Z$  be the fibred product  $\mathcal{O}_{X'} \times_{\mathcal{O}_Y} \mathcal{O}_{X''}$ . This means the following diagram is cartesian, where the maps from  $\mathcal{O}_Z$  are the projections.

$$\begin{array}{ccc} \mathcal{O}_Z & \longrightarrow & \mathcal{O}_{X''} \\ \downarrow & & \downarrow \\ \mathcal{O}_{X'} & \longrightarrow & \mathcal{O}_Y \end{array}$$

If  $U = \text{Spec } R$  is an affine open set in  $C$ , then it follows from Lemma 30 that the open subschemes supported on the open set  $U$  in  $Y, X'$  and  $X''$  are all affine. Note in general that if  $Z$  is any topological space, and  $E' \rightarrow E$  and  $E'' \rightarrow E$  are morphisms of sheaves on  $Z$  then the fibred product  $F = E' \times_E E''$  exists, and for any open set  $U \subset Z$  we have  $F(U) = E'(U) \times_{E(U)} E''(U)$ , and at the level of stalks we have  $F_z = E'_z \times_{E_z} E''_z$  at any  $z \in Z$ . Therefore, in the given situation we have

$$\Gamma(U, \mathcal{O}_Z) = \Gamma(U, \mathcal{O}_{X'}) \times_{\Gamma(U, \mathcal{O}_Y)} \Gamma(U, \mathcal{O}_{X''})$$

It can be seen that the ringed space  $(U, \mathcal{O}_Z|_U)$  is an affine scheme. Hence the sheaf of rings  $\mathcal{O}_Z$  defines yet another structure of a scheme on the topological space  $C$ , which we denote by  $Z$ , symbolically,

$$Z = (C, \mathcal{O}_Z)$$

The sheaf homomorphisms (projections)  $\mathcal{O}_Z \rightarrow \mathcal{O}_{X'}$  and  $\mathcal{O}_Z \rightarrow \mathcal{O}_{X''}$  correspond to morphisms of schemes  $X' \hookrightarrow Z$  and  $X'' \hookrightarrow Z$ , which makes the following a push-out (co-cartesian diagramme) in the category of schemes:

$$\begin{array}{ccc} Y & \longrightarrow & X'' \\ \downarrow & & \downarrow \\ X' & \longrightarrow & Z \end{array}$$

Note that we have a natural morphism  $Z \rightarrow \text{Spec } B$ . It follows from Lemma 26 that  $\mathcal{O}_Z$  is flat over  $\mathcal{O}_{\text{Spec } B}$ , which means the morphism  $Z \rightarrow \text{Spec } B$  is flat. By Lemma 30, it therefore follows that we get a pair  $(Z \rightarrow \text{Spec } B, i_Z)$ , where  $i_Z : C \rightarrow Z$  is the composite  $C \rightarrow Y \rightarrow X' \rightarrow Z$  (same as the composite  $C \rightarrow Y \rightarrow X'' \rightarrow Z$ ) which shows that the functor  $\mathbf{Def}_C$  satisfies **(H1)**.

**Verification of (H2)** : Let  $A \in \text{Art}_k$ . We want to show that the map

$$\mathbf{Def}_C(A \times_k k[\epsilon]/(\epsilon^2)) \rightarrow \mathbf{Def}_C(A) \times \mathbf{Def}_C(k[\epsilon]/(\epsilon^2))$$

is bijective. Let  $(X', i')$  define an element  $\xi \in \mathbf{Def}_C(A)$ , and let  $(X'', i'')$  define an element  $\eta \in \mathbf{Def}_C(k[\epsilon]/(\epsilon^2))$ . By verifying **(H1)**, we have already seen that there exists a pair  $(Z, i_Z)$  which defines an element of  $\mathbf{Def}_C(A \times_k k[\epsilon]/(\epsilon^2))$  which maps to  $(\xi, \eta) \in \mathbf{Def}_C(A) \times \mathbf{Def}_C(k[\epsilon]/(\epsilon^2))$ . Suppose there was another element of  $\mathbf{Def}_C(A \times_k k[\epsilon]/(\epsilon^2))$  over  $(\xi, \eta)$ , say represented by a pair  $(W, i_W)$ . We can identify the underlying topological space of  $W$  with that of  $C$ , using  $i_W : C \rightarrow W$ . Therefore we get the following commutative diagram of sheaves on  $C$

$$\begin{array}{ccc} \mathcal{O}_W & \longrightarrow & \mathcal{O}_{X''} \\ \downarrow & & \downarrow \\ \mathcal{O}_{X'} & \longrightarrow & \mathcal{O}_C \end{array}$$

By assumption,  $\mathcal{O}_W \rightarrow \mathcal{O}_{X'}$  is given by going modulo  $\epsilon$ . Hence by Corollary 27, the induced map  $\mathcal{O}_W \rightarrow \mathcal{O}_{X'} \times_{\mathcal{O}_C} \mathcal{O}_{X''}$  is an isomorphism, which proves **(H2)**.

**Verification of (H3)** : I will only consider the case where  $C$  is smooth over  $k$ . In this case, the tangent space will turn out to be  $H^1(C, T_C)$  where  $T_C = (\Omega_{C/k}^1)^\vee$  is the tangent sheaf. Let  $(X, i)$  represent an element of  $\mathbf{Def}_C(k[\epsilon]/(\epsilon^2))$ . Let  $\mathcal{O}_X$  be the structure sheaf of  $X$ . The morphism  $i : C \hookrightarrow X$  gives a surjection  $i^* : \mathcal{O}_X \rightarrow \mathcal{O}_C$ . As by assumption  $\mathcal{O}_X$  is flat over  $\text{Spec}(k[\epsilon]/(\epsilon^2))$ , by Lemma 25, we have the following short exact sequence of  $\mathcal{O}_X$ -modules.

$$0 \rightarrow \mathcal{O}_C \xrightarrow{\epsilon} \mathcal{O}_X \xrightarrow{i^*} \mathcal{O}_C \rightarrow 0$$

Therefore,  $(\mathcal{O}_X, i^*, \epsilon)$  is an extension of  $C$  by  $\mathcal{O}_C$  in the sense of the Definition 32. Given any affine open  $U$  in  $C$ , the scheme  $(U, \mathcal{O}_X|_U)$  is again affine as already remarked. Consider the following commutative diagram, where the top row is the open inclusion, and the first column is the closed embedding.

$$\begin{array}{ccc} (U, \mathcal{O}_C|_U) & \longrightarrow & C \\ \downarrow & & \downarrow \\ (U, \mathcal{O}_X|_U) & \longrightarrow & \text{Spec } k \end{array}$$

By formal smoothness of  $C \rightarrow \text{Spec } k$ , there exists a diagonal morphism of schemes  $f_U : (U, \mathcal{O}_X|_U) \rightarrow C$  which makes the above diagram commute. Such an  $f_U$  is the same as a homomorphism of sheaves of  $k$ -algebras  $f_U^* : \mathcal{O}_C|_U \rightarrow \mathcal{O}_X|_U$  such that the composite

$$\mathcal{O}_C|_U \xrightarrow{f_U^*} \mathcal{O}_X|_U \xrightarrow{i^*} \mathcal{O}_C|_U$$

is identity. Therefore, the extension  $(\mathcal{O}_X, i^*, \epsilon)$  of  $C$  by  $\mathcal{O}_C$  is locally split in the sense of Definition 32. Therefore, by Lemma 33, this defines an element of the set  $H^1(C, \underline{\text{Hom}}(\Omega_{C/k}^1, \mathcal{O}_C))$ . As  $\underline{\text{Hom}}(\Omega_{C/k}^1, \mathcal{O}_C) = T_C$  the tangent sheaf, this gives an element of  $H^1(C, T_C)$ . The rest is a simple exercise.

**Prolongability of automorphisms is equivalent to (H4) :** This is now clear. The process is similar to getting new bundles on  $X' \cup X''$  by gluing given bundles  $E'$  on  $X'$  and  $E''$  on  $X''$  along  $X' \cap X''$ . If there exists an isomorphism of the restrictions to  $X' \cap X''$ , then this can be done. If any automorphism  $g$  of the restriction  $E'|_{X' \cap X''}$  can be expressed as a product  $g'g''$  where  $g'$  prolongs to  $X'$  and  $g''$  prolongs to  $X''$  then the bundle on  $X' \cup X''$  is unique (up to isomorphism). When  $X'$  and  $X''$  are two copies of the same space  $U$ , and  $X' \cap X''$  is the same subspace  $V$  inside both, with identification given by identity on  $V$ , then the above condition is equivalent to the condition that  $g$  should prolong from  $V$  to  $U$ . This completes our exposition of the proof of the theorem, in the case where  $C$  is proper and smooth over  $k$ .  $\square$

## 4 Formal smoothness

### Formal smoothness for deformations of a vector bundle

**Lemma 36** *Let  $\mathfrak{Y}$  be a noetherian scheme, and let  $\mathcal{I} \subset \mathcal{O}_{\mathfrak{Y}}$  be a coherent ideal sheaf with  $\mathcal{I}^2 = 0$ . Let  $\mathfrak{Z} \subset \mathfrak{Y}$  be the closed subscheme defined by  $\mathcal{I}$ . (Note that as  $\mathcal{I}^2 = 0$ ,  $\mathcal{I}$  becomes naturally an  $\mathcal{O}_{\mathfrak{Z}}$ -module.) Let  $\mathcal{F}$  be a vector bundle on  $\mathfrak{Z}$ , such that*

$$H^2(\mathfrak{Z}, \mathcal{I} \otimes_{\mathcal{O}_{\mathfrak{Z}}} \underline{\text{End}}(\mathcal{F})) = 0$$

*Then there exists a vector bundle on  $\mathfrak{Y}$  whose restriction to  $\mathfrak{Z}$  is isomorphic to  $\mathcal{F}$ .*

**Proof** (Taken from the lecture notes [I] of Illusie) Note that the underlying topological space of  $\mathfrak{Z}$  is the same as that of  $\mathfrak{Y}$ . We associate a category  $\mathcal{C}_V$  to each open subscheme  $V \subset \mathfrak{Z}$  as follows. When  $V$  is non-empty, the objects of  $\mathcal{C}_V$  are pairs  $(\mathcal{E}, \theta)$  where  $\mathcal{E}$  is a vector bundle on the open subscheme  $U$  of  $\mathfrak{Y}$  defined by the set  $V$ , and  $\theta$  is an  $\mathcal{O}_V$ -linear isomorphism  $\mathcal{E}|_V \rightarrow \mathcal{F}|_V$ . In other words,  $(\mathcal{E}, \theta)$  is a prolongation of  $\mathcal{F}|_V$  to  $U$ . The morphisms in  $\mathcal{C}_V$  from  $(\mathcal{E}, \theta)$  to  $(\mathcal{E}', \theta')$  are  $\mathcal{O}_U$ -linear isomorphisms  $\eta : \mathcal{E} \rightarrow \mathcal{E}'$  which take  $\theta$  to  $\theta'$ . The category  $\mathcal{C}_V$  is clearly a groupoid. When  $V$  is empty, we define  $\mathcal{C}_V$  to be the trivial groupoid, which has a single object and a single morphism.



When  $V_1 \subset V_2$ , we have an obvious restriction functor from  $\mathcal{C}_{V_2}$  to  $\mathcal{C}_{V_1}$ . This gives a presheaf of groupoids on  $\mathfrak{Z}$ , which we denote by  $\mathcal{C}$ . This presheaf is clearly a sheaf in the Zariski topology. Hence we have a sheaf of groupoids  $\mathcal{C}$  on the scheme  $\mathfrak{Z}$  (in the small Zariski site of  $\mathfrak{Z}$ ).

We claim that the groupoid  $\mathcal{C}$  is a **gerb**, as it is both **locally non-empty** and **locally connected**. To see  $\mathcal{C}$  is locally non-empty, note that any point  $z \in \mathfrak{Z}$  has an open neighbourhood  $V$  on which  $\mathcal{F}$  is trivial, so  $\mathcal{F}$  has a trivial prolongation to the corresponding open subscheme  $U$  of  $\mathfrak{Y}$  showing  $\mathcal{C}_V$  is non-empty. To see  $\mathcal{C}$  is locally connected, note that given any two objects  $(\mathcal{E}, \theta)$  and  $(\mathcal{E}', \theta')$  of  $\mathcal{C}_V$ , there is an open cover  $V_i$  of  $V$  for which both  $\mathcal{E}|_{U_i}$  and  $\mathcal{E}'|_{U_i}$  are trivial where  $U_i$  are the corresponding open subschemes of  $\mathfrak{Y}$ , and so  $(\mathcal{E}, \theta)$  and  $(\mathcal{E}', \theta')$  become isomorphic on passing to the cover  $(V_i)$ .

Given any non-empty open subscheme  $V \subset \mathfrak{Z}$  and an object  $(\mathcal{E}, \theta)$  of  $\mathcal{C}_V$ , note that

$$\text{Aut}_{\mathcal{C}_V}((\mathcal{E}, \theta)) = H^0(V, \mathcal{I} \otimes_{\mathcal{O}_3} \underline{\text{End}}(\mathcal{F}))$$

Thus the gerb  $\mathcal{C}$  has as its **band** ('lien') the sheaf  $\mathcal{I} \otimes_{\mathcal{O}_3} \underline{\text{End}}(\mathcal{F})$ . Hence the obstruction to the existence of a global element (means an object of  $\mathcal{C}_3$ ) lies in the cohomology  $H^2(\mathfrak{Z}, \mathcal{I} \otimes_{\mathcal{O}_3} \underline{\text{End}}(\mathcal{F}))$ . In particular when  $H^2(\mathfrak{Z}, \mathcal{I} \otimes_{\mathcal{O}_3} \underline{\text{End}}(\mathcal{F})) = 0$ , we get the desired result, proving the lemma.  $\square$

**Theorem 37** *Let  $X$  be a proper scheme over a field  $k$ . Let  $E$  be a vector bundle on  $X$ , with deformation functor  $\mathcal{D}_E : \text{Art}_k \rightarrow \text{Sets}$ . Suppose that  $H^2(X, \underline{\text{End}}(E)) = 0$ . Then the functor  $\mathcal{D}_E$  is smooth, that is, for any  $A$  in  $\text{Art}_k$  and a proper ideal  $I \subset A$ , the restriction map  $\mathcal{D}_E(A) \rightarrow \mathcal{D}_E(A/I)$  is surjective.*

**Proof** Let  $\mathfrak{m}^n I = 0$  where  $n \geq 1$ . If  $n \geq 2$ , then by factoring  $A \rightarrow A/I$  as a composite

$$A = A/\mathfrak{m}^n I \rightarrow A/\mathfrak{m}^{n-1} I \rightarrow \dots \rightarrow A/\mathfrak{m} I \rightarrow A/I,$$

we can assume without loss of generality that  $\mathfrak{m} I = 0$ , where  $\mathfrak{m}$  denotes the maximal ideal of  $A$ . Let  $(\mathcal{F}, \theta) \in \mathcal{D}_E(A/I)$ . We put  $\mathfrak{Y} = X_A$  and  $\mathcal{I} = \mathcal{O}_X \otimes_k I \subset \mathcal{O}_X \otimes_k A = \mathcal{O}_{\mathfrak{Y}}$ . Then  $\mathcal{I}$  is a coherent ideal sheaf with  $\mathcal{I}^2 = 0$ . The subscheme  $\mathfrak{Z}$  defined by  $\mathcal{I}$  is  $X_{A/I} \subset X_A$ . In particular,  $\mathcal{F}$  is a vector bundle on  $\mathfrak{Z}$ . Therefore we have

$$\begin{aligned} & H^2(\mathfrak{Z}, \mathcal{I} \otimes_{\mathcal{O}_3} \underline{\text{End}}(\mathcal{F})) \\ &= H^2(X_{A/I}, I \otimes_{A/I} \underline{\text{End}}(\mathcal{F})) \\ &= H^2(X_{A/I}, I \otimes_k \underline{\text{End}}(E)) \text{ as } \mathfrak{m} I = 0 \text{ and as } \underline{\text{End}}(\mathcal{F})|_X = \underline{\text{End}}(E). \\ &= H^2(X, I \otimes_k \underline{\text{End}}(E)) \\ &= I \otimes_k H^2(X, \underline{\text{End}}(E)) \text{ by the projection formula for } X \rightarrow \text{Spec}(k). \\ &= 0, \text{ as by assumption } H^2(X, \underline{\text{End}}(E)) = 0. \end{aligned}$$

Hence by Lemma 36,  $\mathcal{F}$  prolongs to  $\mathfrak{Y} = X_A$  as desired.  $\square$

**Remark 38** (**Converse of Theorem 37 is false!**) The deformation functor  $\mathcal{D}_E$  can be smooth without vanishing of  $H^2(X, \underline{\text{End}}(E))$ . For example, let  $X$  be an irreducible projective variety over an algebraically closed field  $k$  of characteristic zero. Then  $\text{Pic}_{X/k}$  is a group scheme locally of finite type over  $k$ . As  $k$  has characteristic zero,  $\text{Pic}_{X/k}$  is smooth over  $k$ . From this it can be seen that for any line bundle  $L$  on  $X$ , the deformation functor  $\mathcal{D}_L$  is smooth. However, we may have  $H^2(X, \mathcal{O}_X) \neq 0$ , for example, take  $X$  to be an abelian surface.

### Formal smoothness for deformations of a coherent sheaf

**Theorem 39** *Let  $X$  be a proper scheme over a field  $k$ . Let  $F$  be a coherent sheaf on  $X$ , with deformation functor  $\mathcal{D}_E : \text{Art}_k \rightarrow \text{Sets}$ . Suppose that  $\text{Ext}^2(F, F) = 0$ . Then the functor  $\mathcal{D}_F$  is smooth, that is, for any  $A$  in  $\text{Art}_k$  and a proper ideal  $I \subset A$ , the restriction map  $\mathcal{D}_F(A) \rightarrow \mathcal{D}_F(A/I)$  is surjective.*

**Proof** (In the projective case) If  $X$  is projective over  $k$ , let  $\mathcal{O}_X(1)$  be very ample. Then by evaluating global sections we get a surjection

$$q : H^0(X, F(n)) \otimes_k \mathcal{O}_X(-n) \rightarrow F$$

whenever  $n$  is large enough.

Let  $E$  be the vector bundle  $E = H^0(X, F(n)) \otimes_k \mathcal{O}_X(-n)$ . Let  $G$  be the kernel of the above surjection  $q : E \rightarrow F$ . We get a long exact sequence

$$\text{Hom}(G, F) \rightarrow \text{Ext}^1(F, F) \rightarrow \text{Ext}^1(E, F) \rightarrow \text{Ext}^1(G, F) \rightarrow \text{Ext}^2(F, F)$$

Note that  $\text{Ext}^1(E, F) = H^1(X, E^\vee \otimes_{\mathcal{O}_X} F) = H^0(X, F(n)) \otimes_k H^1(X, F(n)) = 0$  for a large enough  $n$ . Therefore, we get the vanishing of  $\text{Ext}^1(G, F)$  and surjectivity of  $\text{Hom}(G, F) \rightarrow \text{Ext}^1(F, F)$ . The Quot functor  $Q$  which keeps  $E$  fixed and deforms the quotient  $q$  is pro-representable, as proved in a later chapter. The vanishing of  $\text{Ext}^1(G, F)$  implies that the Quot functor  $Q$  is formally smooth. The tangent to  $Q$  is  $\text{Hom}(G, F)$  and the tangent to  $\mathcal{D}_F$  is  $\text{Ext}^1(F, F)$ . The above map  $\text{Hom}(G, F) \rightarrow \text{Ext}^1(F, F)$  is the tangent map of the forgetful morphism  $Q \rightarrow \mathcal{D}_F$ . Its surjectivity, together with formal smoothness of  $Q$  give us formal smoothness of  $\mathcal{D}_F$ , as follows.

Let  $(R, r : h_R \rightarrow \mathcal{D}_F)$  be a hull for  $\mathcal{D}_F$ . Let  $(R', r' : h_{R'} \rightarrow Q)$  pro-represent  $Q$ . By formal smoothness of  $r : h_R \rightarrow \mathcal{D}_F$ , the composite morphism  $h_{R'} \rightarrow Q \rightarrow \mathcal{D}_F$  admits a lift  $f : h_{R'} \rightarrow h_R$ , that is, we have a morphism  $f : \text{Spec } R' \rightarrow \text{Spec } R$ . The map  $dr : T_R \rightarrow T_{\mathcal{D}_F} = \text{Ext}^1(F, F)$  is an isomorphism by definition of a hull, while  $dr' : T_{R'} \rightarrow T_Q = \text{Hom}(G, F)$  is an isomorphism as  $(R', r')$  is a pro-representing family (in particular, a hull) for  $Q$ . Hence the surjectivity of  $\text{Hom}(G, F) \rightarrow \text{Ext}^1(F, F)$  shows that the morphism  $f : \text{Spec } R' \rightarrow \text{Spec } R$  is tangent-level surjective.

Note that by smoothness,  $R'$  is a formal power series ring over  $k$  in finitely many variables. On the other hand,  $R$  is the quotient of one such. Let  $R' = k[[x'_1, \dots, x'_m]]$  and let  $R = S/J$  where  $S = k[[x_1, \dots, x_n]]$  and  $J \subset S$  is a proper ideal, such that  $n =$

$\dim_k \text{Ext}^1(F, F)$ , and therefore the quotient map  $q : S \rightarrow R$  induces an isomorphism  $T_R \xrightarrow{\sim} T_S$ . Let  $f^\# : R \rightarrow R'$  denote the induced  $k$ -algebra homomorphism. Then we get a composite homomorphism

$$g^\# : k[[x_1, \dots, x_n]] \xrightarrow{q} k[[x_1, \dots, x_n]]/J = R \xrightarrow{f^\#} R' = k[[x'_1, \dots, x'_m]]$$

The tangent level map  $T_{R'} \rightarrow T_S$  is a surjection, and therefore by implicit function theorem for formal power series rings over a field, we can choose new variables  $y_1, \dots, y_m$  for  $R'$  such that  $y_i = g^\#(x_i)$  for  $1 \leq i \leq n$ . In particular,  $g^\# : S \rightarrow R'$  is injective, and so  $f^\#$  is also injective, which means  $J = 0$ , which shows that  $R = k[[x_1, \dots, x_n]]$ . Hence  $h_R$  is a formally smooth functor. As  $r : h_R \rightarrow \mathcal{D}_F$  is formally smooth (being a versal family), it follows that  $\mathcal{D}_F$  is formally smooth (see the following exercise).  $\square$

**Exercise 40** Let  $\varphi : \text{Art}_k \rightarrow \text{Sets}$  have a versal family  $(R, r : h_R \rightarrow \varphi)$ , such that  $h_R$  is formally smooth. Then  $\varphi$  is formally smooth. Conversely, if  $\varphi$  is formally smooth, then each versal family is formally smooth.

### Formal smoothness for deformations of a complete smooth variety

**Theorem 41** Let  $X$  be a complete smooth variety over a field  $k$ . Suppose that  $H^2(X, T_X) = 0$ . Then the functor  $\mathbf{Def}_X$  is smooth, that is, for any  $A$  in  $\text{Art}_k$  and a proper ideal  $I \subset A$ , the restriction map  $\mathbf{Def}_X(A) \rightarrow \mathbf{Def}_X(A/I)$  is surjective.

The proof is very similar to that of Theorem 37. See the lectures of Illusie [I] for details.

## 5 Appendix on base-change

In this appendix, I begin with a small simplification in the proofs of the Base-change Theorem as in [EGA] or [H].

**Lemma 42** Let  $A$  be a noetherian local ring with residue field  $A/\mathfrak{m} = k$ , and let

$$C = (E' \xrightarrow{f} E \xrightarrow{g} E'')$$

be a complex of finite  $A$ -modules ( $g \circ f = 0$ ) such that  $E$  and  $E''$  are free. We denote  $\ker(g)/\text{im}(f)$  by  $H(C)$ . Let

$$C \otimes_A k = (\overline{E'} \xrightarrow{\overline{f}} \overline{E} \xrightarrow{\overline{g}} \overline{E''})$$

be the complex obtained by tensoring  $C$  with  $k$ . Suppose that the induced map

$$H(C) \otimes_A k \rightarrow H(C \otimes_A k)$$

is surjective. Then  $\ker(g)$  is a direct summand of  $E$ , and  $\operatorname{im}(g)$  is a direct summand of  $E''$ . Consequently, for any  $A$ -module  $M$  the induced map

$$H(C) \otimes_A M \rightarrow H(C \otimes_A M)$$

is an isomorphism.

In particular, if  $H(C \otimes_A k) = 0$  then  $H(C) = 0$ .

**Proof** Consider the following commutative diagram with exact rows.

$$\begin{array}{ccccccc} \operatorname{im}(f) \otimes_A k & \rightarrow & \ker(g) \otimes_A k & \rightarrow & H(C) \otimes_A k & \rightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 \rightarrow & \operatorname{im}(\bar{f}) & \rightarrow & \ker(\bar{g}) & \rightarrow & H(C \otimes_A k) & \rightarrow 0 \end{array}$$

The first vertical map  $\operatorname{im}(f) \otimes_A k \rightarrow \operatorname{im}(\bar{f})$  is clearly surjective, and the third vertical map  $H(C) \otimes_A k \rightarrow H(C \otimes_A k)$  is surjective by hypothesis. It follows by the snake lemma that the middle vertical map  $\ker(g) \otimes_A k \rightarrow \ker(\bar{g})$  is surjective. Therefore there exist elements  $u_1, \dots, u_p \in \ker(g)$  such that the elements  $u_i \otimes 1 \in \bar{E}$  form a  $k$ -linear basis for  $\ker(\bar{g})$ .

In particular, the elements  $u_i \otimes 1 \in \bar{E}$  are linearly independent over  $k$ . Consequently, there exist elements  $w_1, \dots, w_r \in E$  such that  $u_1, \dots, u_p, w_1, \dots, w_r$  is a free basis for  $E$  over  $A$ .

Note that as  $u_i \otimes 1, w_j \otimes 1$  is a basis of  $\bar{E}$  and  $u_i \otimes 1$  is a basis of  $\ker(\bar{g})$ , the images of  $w_j \otimes 1$  under  $\bar{g}$  are linearly independent in  $\bar{E}''$ . As  $E''$  is finite free as an  $A$ -module, this means the sequence of elements  $g(w_i) \in E''$  can be prolonged to a basis  $g(w_i), v_k \in E''$ . As  $E$  is spanned by  $u_i, w_j$  and as  $u_i \in \ker(g)$ , it follows that  $\operatorname{im}(g)$  is spanned by the  $g(w_i)$ , and so it follows that  $\operatorname{im}(g)$  is a direct summand of  $E''$ . We claim that as a submodule of  $E$ ,  $\ker(g)$  is the span of the elements  $u_i$ . To see this, let  $x = \sum a_i u_i + \sum b_j w_j \in \ker(g)$ . Then  $0 = g(x) = \sum b_j g(w_j)$ , and hence each  $b_j$  is zero by the linear independence of  $g(w_j)$  over  $A$ . This completes the proof of the lemma.  $\square$

Applying the above to the Grothendieck semi-continuity complex, we get the following:

**Theorem 43** *Let  $S = \operatorname{Spec}(A)$  where  $A$  is a noetherian local ring. Let  $\pi : \mathfrak{X} \rightarrow S$  be a proper morphism and  $\mathcal{F}$  a coherent  $\mathcal{O}_{\mathfrak{X}}$ -module which is flat over  $S$ . Let  $s \in S$  be the closed point with residue field denoted by  $k$ . Let  $\mathfrak{X}_s$  be the fiber over  $s$  and let  $\mathcal{F}_s = \mathcal{F}|_{\mathfrak{X}_s}$  denote the restriction of  $\mathcal{F}$  to  $\mathfrak{X}_s$ . Let  $i$  an integer, such that the natural map*

$$H^i(\mathfrak{X}, \mathcal{F}) \otimes_A k \rightarrow H^i(\mathfrak{X}_s, \mathcal{F}_s)$$

is surjective. Then for any  $A$ -module  $M$ , the induced map

$$H^i(\mathfrak{X}, \mathcal{F}) \otimes_A M \rightarrow H^i(\mathfrak{X}, \mathcal{F} \otimes_{\mathcal{O}_{\mathfrak{X}}} \pi^* M)$$

is an isomorphism. In particular if  $H^i(\mathfrak{X}_s, \mathcal{F}_s) = 0$  then  $H^i(\mathfrak{X}, \mathcal{F}) = 0$ .

**Remark** In the absence of our elementary Lemma 42, both [EGA] and [H] give rather complicated proofs of Theorem 43, involving inverse limits over modules of finite length (which in [H] is done by invoking the theorem on formal functions).

The following lemma is used in the deformation theory for a coherent sheaf  $E$  which is ‘simple’, to prove the theorem that the deformation functor  $\mathcal{D}_E$  of such a sheaf is pro-representable.

**Lemma 44** *Let  $A$  be a noetherian local ring, let  $S = \text{Spec } A$ , and let  $\pi : \mathfrak{X} \rightarrow S$  be a proper morphism. Let  $X$  denote the schematic fiber of  $\pi$  over the closed point  $\text{Spec } k$ , where  $k$  is the residue field of  $A$ . Let  $\mathcal{E}$  be a coherent sheaf on  $\mathfrak{X}$  such that  $\mathcal{E}$  is flat over  $S$ . Assume that there exists an exact sequence  $\mathcal{F}_1 \rightarrow \mathcal{F}_0 \rightarrow \mathcal{E} \rightarrow 0$  of  $\mathcal{O}_{\mathfrak{X}}$ -modules, where  $\mathcal{F}_1$  and  $\mathcal{F}_0$  are locally free (note that this condition is automatically satisfied when  $\mathcal{E}$  itself is locally free, or when  $\pi : \mathfrak{X} \rightarrow S$  is a projective morphism). Let  $E = \mathcal{E}|_X$  be the restriction of  $\mathcal{E}$  to  $X$ . If the ring homomorphism  $k \rightarrow \text{End}_X(E)$  (under which  $k$  acts on  $E$  by scalar multiplication) is an isomorphism, then for any morphism  $f : T \rightarrow S$ , the natural ring homomorphism*

$$H^0(T, \mathcal{O}_T) \rightarrow \text{End}_{\mathfrak{X}_T}((\text{id} \times f)^* \mathcal{E})$$

(under which  $H^0(T, \mathcal{O}_T)$  acts on  $(\text{id} \times f)^* \mathcal{E}$  by scalar multiplication) is an isomorphism.

**Proof** Consider the contravariant functor  $\text{End}(\mathcal{E})$  from  $S$ -schemes to sets, which associates to any  $S$ -scheme  $f : T \rightarrow S$  the set

$$\text{End}(\mathcal{E})(T) = \text{End}_{\mathfrak{X}_T}((\text{id} \times f)^* \mathcal{E})$$

Then by a fundamental theorem of Grothendieck (EGA III 7.7.8, 7.7.9), there exists a coherent sheaf  $\mathcal{Q}$  on  $S$  and a functorial  $H^0(T, \mathcal{O}_T)$ -module isomorphism

$$\alpha_T : \text{End}_{\mathfrak{X}_T}((\text{id} \times f)^* \mathcal{E}) \rightarrow \text{Hom}_T(f^* \mathcal{Q}, \mathcal{O}_T)$$

Note that as  $S = \text{Spec } A$ , the coherent sheaf  $\mathcal{Q}$  corresponds to the finite  $A$ -module  $Q = H^0(S, \mathcal{Q})$ . Consider the isomorphism  $\alpha_S : \text{End}_{\mathfrak{X}}(\mathcal{E}) \rightarrow \text{Hom}_S(\mathcal{Q}, \mathcal{O}_S) = \text{Hom}_A(Q, A)$ . Let  $\theta : Q \rightarrow A$  be the image of  $1_{\mathcal{E}}$  under  $\alpha_S$ . By functoriality, the restriction  $\theta_k : Q \otimes_A k \rightarrow k$  of  $\theta$  to  $\text{Spec } k$  is the image of  $1_E$  under the isomorphism  $\alpha_k : \text{End}_X(E) \rightarrow \text{Hom}_k(Q \otimes_A k, k) = \text{Hom}_A(Q, A)$ .

As by assumption  $k \rightarrow \text{End}_X(E)$  is an isomorphism, by composing with  $\alpha_k$  we get an isomorphism  $k \mapsto \text{Hom}_k(Q \otimes_A k, k)$  under which  $1 \mapsto \theta_k$ . Hence  $\text{Hom}_k(Q \otimes_A k, k)$

is 1-dimensional as a  $k$ -vector space with basis  $\theta_k$ . Therefore  $\theta_k$  is surjective, and so by Nakayama it follows that  $\theta : Q \rightarrow A$  is surjective. Hence we have a splitting  $Q = A \oplus N$  where  $N = \ker(\theta)$ , under which the map  $\theta : Q \rightarrow A$  becomes the projection  $p_1 : A \oplus N \rightarrow A$  on the first factor. But as  $\theta_k$  is an isomorphism, it again follows by Nakayama that  $N = 0$ . This shows that  $\theta : Q \rightarrow A$  is an isomorphism.

Identifying  $\mathcal{Q}$  with  $\mathcal{O}_S$  under  $\theta$ , for any  $f : T \rightarrow S$  we have  $\text{Hom}_T(f^*\mathcal{Q}, \mathcal{O}_T) = H^0(T, \mathcal{O}_T)$ , and so we get a functorial  $H^0(T, \mathcal{O}_T)$ -module isomorphism  $\alpha_T : \text{End}_{\mathfrak{X}_T}((\text{id} \times f)^*\mathcal{E}) \rightarrow H^0(T, \mathcal{O}_T)$  which maps  $1 \mapsto 1$ . The composite map

$$H^0(T, \mathcal{O}_T) \rightarrow \text{End}_{\mathfrak{X}_T}((\text{id} \times f)^*\mathcal{E}) \rightarrow H^0(T, \mathcal{O}_T)$$

is identity, so it follows that  $H^0(T, \mathcal{O}_T) \rightarrow \text{End}_{\mathfrak{X}_T}((\text{id} \times f)^*\mathcal{E})$  is an isomorphism.  $\square$

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