Optimal Control of PDE
Theory and Numerical Analysis

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Introduction.

In a control problem we find the following basic elements.

(1) A *control* u that we can handle according to our interests, which can be chosen among a family of feasible controls $\mathbb{K}$.
(2) The *state of the system* y to be controlled, which depends on the control. Some limitations can be imposed on the state, in mathematical terms $y \in \mathbb{C}$, which means that not every possible state of the system is satisfactory.
(3) A *state equation* that establishes the dependence between the control and the state. In the next sections this state equation will be a partial differential equation, y being the solution of the equation and u a function arising in the equation so that any change in the control u produces a change in the solution y. However the origin of control theory was connected with the control of systems governed by ordinary differential equations and there is a huge activity in this field; see, for instance, the classical books Pontriaguine et al. [**40**] or Lee and Markus [**36**].
(4) A *function* to be minimized, called the objective function or the cost function, depending on the control and the state (y, u).

The objective is to determine an admissible control, called optimal control, that provides a satisfactory state for us and that minimizes the value of functional J. The basic questions to study are the existence of solution and its computation. However to obtain the solution we must use some numerical methods, arising some delicate mathematical questions in this numerical analysis. The first step to solve numerically the problem requires the discretization of the control problem, which is made usually by finite elements. A natural question is how good the approximation is, of course we would like to have some error estimates of these approximations. In order to derive the error estimates it is essential to have some regularity of the optimal control, some order of differentiability is necessary, at least some derivatives in a weak sense. The regularity of the optimal control can be deduced from the first order optimality conditions. Another key tool in the proof of the error estimates is the use of the second order optimality conditions. Therefore our analysis requires to derive the first and second order conditions for optimality.

Once we have a discrete control problem we have to use some numerical algorithm of optimization to solve this problem. When the problem is not convex, the optimization algorithms typically provides local minima, the question now is if these local minima are significant for the original control problem.

The following steps must be followed when we study an optimal control problem:

(1) Existence of a solution.
(2) First and second order optimality conditions.
(3) Numerical approximation.
(4) Numerical resolution of the discrete control problem.

We will not discuss the numerical algorithms of optimization, we will only consider the first three points for a model problem. In this model problem the state equation will be a semilinear elliptic partial differential equation. Through the nonlinearity introduces some complications in the study, we have preferred to consider them to show the role played by the second order optimality conditions. Indeed, if the equation is linear and the cost functional is the typical quadratic functional, then the use of the second order optimality conditions is hidden.

There are no many books devoted to all the questions we are going to study here. Firstly let me mention the book by Profesor J.L. Lions [**38**], which is an obliged reference in the study of the theory of optimal control problems of partial differential equations. In this text, that has left an indelible track, the reader will be able to find some of the methods used in the resolution of the two first questions above indicated. More recent books are X. Li and J. Yong [**37**], H.O. Fattorini [**34**] and F. Tröltzsch [**46**].

CHAPTER 1

Existence of a Solution

1.1. Setting of the Control Problem

Let Ω be an open and bounded subset of $\mathbb{R}^n$ ($n = 2$ o 3), Γ being its boundary that we will assume to be regular; $C^{1,1}$ is enough for us in all this course. In Ω we will consider the linear operator A defined by

$$Ay = -\sum_{i,j=1}^{n} \partial_{x_j}\left(a_{ij}(x)\partial_{x_i}y(x)\right) + a_0(x)y(x),$$

where $a_{ij} \in C^{0,1}(\bar{\Omega})$ and $a_0 \in L^\infty(\Omega)$ satisfy:

$$\begin{cases} \exists m > 0 \text{ such that } \displaystyle\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq m|\xi|^2 \quad \forall \xi \in \mathbb{R}^n \text{ and } \forall x \in \Omega, \\ \\ a_0(x) \geq 0 \quad \text{a.e. } x \in \Omega. \end{cases}$$

Now let $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ be a non decreasing monotone function of class C^2, with $\phi(0) = 0$. For any $u \in L^2(\Omega)$, the Dirichlet problem

$$\begin{cases} Ay + \phi(y) = u & \text{in } \Omega \\ y = 0 & \text{on } \Gamma \end{cases} \tag{1.1}$$

has a unique solution $y_u \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

The control problem associated to this system is formulated as follows

$$\text{(P)} \begin{cases} \text{Minimize} \quad J(u) = \displaystyle\int_\Omega L(x, y_u(x), u(x))dx \\ \\ u \in \mathbb{K} = \{u \in L^\infty(\Omega) : \alpha \leq u(x) \leq \beta \quad \text{a.e. } x \in \Omega\}, \end{cases}$$

where $-\infty < \alpha < \beta < +\infty$ and L fulfills the following assumptions:
(H1) $L : \Omega \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a Carathéodory function and for all $x \in \Omega$, $L(x, \cdot, \cdot)$ is of class C^2 in $\mathbb{R}^2$. Moreover for every $M > 0$ and all

$x, x_1, x_2 \in \Omega$ and $y, y_1, y_2, u, u_1, u_2 \in [-M, +M]$, the following properties hold

$$|L(x,y,u)| \leq L_{M,1}(x), \qquad |\frac{\partial L}{\partial y}(x,y,u)| \leq L_{M,p}(x)$$

$$|\frac{\partial L}{\partial u}(x_1,y,u) - \frac{\partial L}{\partial u}(x_2,y,u)| \leq C_M|x_1 - x_2|$$

$$|L''_{(y,u)}(x,y,u)|_{\mathbb{R}^{2\times 2}} \leq C_M$$

$$|L''_{(y,u)}(x,y_1,u_1) - L''_{(y,u)}(x,y_2,u_2)|_{\mathbb{R}^{2\times 2}} \leq C_M(|y_1 - y_2| + |u_1 - u_2|),$$

where $L_{M,1} \in L^1(\Omega)$, $L_{M,p} \in L^p(\Omega)$, $p > n$, $C_M > 0$, $L''_{(y,u)}$ is the Hessian matrix of L with respect to (y,u), and $|\cdot|_{\mathbb{R}^{2\times 2}}$ is any matricial norm.

To prove our second order optimality conditions and the error estimates we will need the following additional assumption
(H2) There exists $\Lambda > 0$ such that

$$\frac{\partial^2 L}{\partial u^2}(x,y,u) \geq \Lambda \quad \forall\ (x,y,u) \in \Omega \times \mathbb{R}^2.$$

REMARK 1.1. A typical functional in control theory is

$$J(u) = \int_\Omega \left\{|y_u(x) - y_d(x)|^2 + Nu^2(x)\right\} dx, \tag{1.2}$$

where $y_d \in L^2(\Omega)$ denotes the ideal state of the system and $N \geq 0$. The term $\int_\Omega Nu^2(x)dx$ can be considered as the cost term and it is said that the control is expensive if N is big, however the control is cheap if N is small or zero. From a mathematical point of view the presence of the term $\int_\Omega Nu^2(x)dx$, with $N > 0$, has a regularizing effect on the optimal control. Hypothesis **(H1)** is fulfilled, in particular the condition $L_{M,p} \in L^p(\Omega)$, if $y_d \in L^p(\Omega)$. This condition plays an important role in the study of the regularity of the optimal control. Hypothesis **(H2)** holds if $N > 0$.

REMARK 1.2. Other choices for the set of feasible controls are possible, in particular the case $\mathbb{K} = L^2(\Omega)$ is frequent.The important question is that $\mathbb{K}$ must be closed and convex. Moreover if $\mathbb{K}$ is not bounded, then some coercivity assumption on the functional J is required to assure the existence of a solution.

REMARK 1.3. In practice $\phi(0) = 0$ is not a true restriction because it is enough to change ϕ by $\phi - \phi(0)$ and u by $u - \phi(0)$ to transform the problem under the required assumptions. Non linear terms of the form $f(x,y(x))$, with f of class C^2 with respect to the second variable and monotone non decreasing with respect to the same variable, can be considered as an alternative to the term $\phi(y(x))$. We lose some generality in order to avoid technicalities and to get a simplified and more clear presentation of our methods to study the control problem.

The existence of a solution y_u in $H_0^1(\Omega)\cap L^\infty(\Omega)$ can be proved as follows: firstly we truncate ϕ to get a bounded function ϕ_k, for instance in the way

$$\phi_k(t) = \begin{cases} \phi(t) & \text{if } |\phi(t)| \le k, \\ +k & \text{if } \phi(t) > +k, \\ -k & \text{if } \phi(t) < -k. \end{cases}$$

Then the operator $(A+\phi_k) : H_0^1(\Omega) \longrightarrow H^{-1}(\Omega)$ is monotone, continuous and coercive, therefore there exists a unique element $y_k \in H_0^1(\Omega)$ satisfying $Ay_k + \phi_k(y_k) = u$ in Ω. By using the usual methods it is easy to prove that $\{y_k\}_{k=1}^\infty$ is uniformly bounded in $L^\infty(\Omega)$ (see, for instance, Stampacchia [**45**]), consequently for k large enough $\phi_k(y_k) = \phi(y_k)$ and then $y_k = y_u \in H_0^1(\Omega)\cap L^\infty(\Omega)$ is the solution of problem (1.1). On the other hand the inclusion $Ay_u \in L^\infty(\Omega)$ implies the $W^{2,p}(\Omega)$-regularity of y_u for every $p < +\infty$; see Grisvard [**35**]. Finally, remembering that $\mathbb{K}$ is bounded in $L^\infty(\Omega)$, we deduce the next result

THEOREM 1.4. *For any control $u \in \mathbb{K}$ there exists a unique solution y_u of (1.1) in $W^{2,p}(\Omega)\cap H_0^1(\Omega)$, for all $p < \infty$. Moreover there exists a constant $C_p > 0$ such that*

$$\|y_u\|_{W^{2,p}(\Omega)} \le C_p \quad \forall u \in \mathbb{K}. \tag{1.3}$$

It is important to remark that the previous theorem implies the Lipschitz regularity of y_u. Indeed it is enough to remind that $W^{2,p}(\Omega) \subset C^{0,1}(\bar{\Omega})$ for any $p > n$.

1.2. Existence of a Solution

The goal of this section is to study the existence of a solution for problem (P), which is done in the following theorem.

THEOREM 1.5. *Let us assume that L is a Carathéodory function satisfying the following assumptions*

A1) *For every $(x,y) \in \Omega\times\mathbb{R}$, $L(x,y,\cdot) : \mathbb{R} \longrightarrow \mathbb{R}$ is a convex function.*

A2) *For any $M > 0$ there exists a function $\psi_M \in L^1(\Omega)$ such that*

$$|L(x,y,u)| \le \psi_M(x) \quad \text{a.e. } x\in\Omega, \quad \forall |y| \le M, \quad \forall |u| \le M.$$

Then problem (P) *has at least one solution.*

PROOF. Let $\{u_k\} \subset \mathbb{K}$ be a minimizing sequence of (P), this means that $J(u_k) \to \inf(\mathrm{P})$. Let us take a subsequence, again denoted in the same way, converging weakly* in $L^\infty(\Omega)$ to an element $\bar{u} \in \mathbb{K}$. Let us prove that $J(\bar{u}) = \inf(\mathrm{P})$. For this we will use Mazur's Theorem (see, for instance, Ekeland and Temam [**33**]): given $1 < p < +\infty$ arbitrary, there exists a sequence of convex combinations $\{v_k\}_{k\in\mathbb{N}}$,

$$v_k = \sum_{l=k}^{n_k} \lambda_l u_l, \quad \text{with} \quad \sum_{l=k}^{n_k} \lambda_l = 1 \quad \text{and} \quad \lambda_l \geq 0,$$

such that $v_k \to \bar{u}$ strongly in $L^p(\Omega)$. Then, using the convexity of L with respect to the third variable, the dominated convergence theorem and the assumption A1), it follows

$$J(\bar{u}) = \lim_{k\to\infty} \int_\Omega L(x, y_{\bar{u}}(x), v_k(x))dx \leq$$

$$\limsup_{k\to\infty} \sum_{l=k}^{n_k} \lambda_l \int_\Omega L(x, y_{\bar{u}}(x), u_l(x))dx \leq \limsup_{k\to\infty} \sum_{l=k}^{n_k} \lambda_l J(u_l) +$$

$$\limsup_{k\to\infty} \int_\Omega \sum_{l=k}^{n_k} \lambda_l \left| L(x, y_{u_l}(x), u_l(x)) - L(x, y_{\bar{u}}(x), u_l(x)) \right| dx =$$

$$\inf\,(\mathrm{P}) + \limsup_{k\to\infty} \int_\Omega \sum_{l=k}^{n_k} \lambda_l \left| L(x, y_{u_l}(x), u_l(x)) - L(x, y_{\bar{u}}(x), u_l(x)) \right| dx,$$

where we have used the convergence $J(u_k) \to \inf(\mathrm{P})$. To prove that the last term converges to zero it is enough to remark that for any given point x, the function $L(x,\cdot,\cdot)$ is uniformly continuous on bounded subsets of $\mathbb{R}^2$, the sequences $\{y_{u_l}(x)\}$ and $\{u_l(x)\}$ are uniformly bounded and $y_{u_l}(x) \to y_{\bar{u}}(x)$ when $l \to \infty$, therefore

$$\lim_{k\to\infty} \sum_{l=k}^{n_k} \lambda_l \left| L(x, y_{u_l}(x), u_l(x)) - L(x, y_{\bar{u}}(x), u_l(x)) \right| = 0 \quad \text{a.e. } x \in \Omega.$$

Using again the dominated convergence theorem, assumption A2) and the previous convergence, we get

$$\limsup_{k\to\infty} \int_\Omega \sum_{l=k}^{n_k} \lambda_l \left| L(x, y_{u_l}(x), u_l(x)) - L(x, y_{\bar{u}}(x), u_l(x)) \right| dx = 0,$$

which concludes the proof. □

REMARK 1.6. It is possible to formulate other similar problems to (P) by taking $\mathbb{K}$ as a closed and convex subset of $L^p(\Omega)$, with $1 < p < +\infty$. The existence of a solution can be proved as above by assuming that $\mathbb{K}$ is bounded in $L^p(\Omega)$ or J is coercive on $\mathbb{K}$. The coercivity holds if the following conditions is fulfilled: $\exists \psi \in L^1(\Omega)$ and $C > 0$ such that

$$L(x, y, u) \geq C|u|^p + \psi(x) \quad \forall (x, y, u) \in \Omega \times \mathbb{R}^2.$$

This coercivity assumption implies the boundedness in $L^p(\Omega)$ of any minimizing sequence, the rest of the proof being as in Theorem 1.5.

1.3. Some Other Control Problems

In the rest of the chapter we are going to present some control problems that can be studied by using the previous methods. First let us start with a very well known problem, which is a particular case of (P).

1.3.1. The Linear Quadratic Control Problem. Let us assume that ϕ is linear and $L(x, y, u) = (1/2)\{(y - y_d(x))^2 + Nu^2\}$, with $y_d \in L^2(\Omega)$ fixed, therefore

$$J(u) = \frac{1}{2}\int_\Omega (y_u(x) - y_d(x))^2 dx + \frac{N}{2}\int_\Omega u^2(x) dx.$$

Now (P) is a convex control problem. In fact the objective functional $J : L^2(\Omega) \to \mathbb{R}$ is well defined, continuous and strictly convex. Under these conditions, if $\mathbb{K}$ is a convex and closed subset of $L^2(\Omega)$, we can prove the existence and uniqueness of an optimal control under one of the two following assumptions:

(1) $\mathbb{K}$ is a bounded subset of $L^2(\Omega)$.
(2) $N > 0$.

For the proof it is enough to take a minimizing sequence as in Theorem 1.5, and remark that the previous assumptions imply the boundedness of the sequence. Then it is possible to take a subsequence $\{u_k\}_{k=1}^\infty \subset \mathbb{K}$ converging weakly in $L^2(\Omega)$ to $\bar{u} \in \mathbb{K}$. Finally the convexity and continuity of J implies the weak lower semicontinuity of J, then

$$J(\bar{u}) \leq \liminf_{k\to\infty} J(u_k) = \inf \text{(P)}.$$

The uniqueness of the solution is an immediate consequence of the strict convexity of J.

1.3.2. A Boundary Control Problem. Let us consider the following Neumann problem

$$\begin{cases} Ay + \phi(y) = f & \text{in } \Omega \\ \partial_{\nu_A} y = u & \text{on } \Gamma, \end{cases}$$

where $f \in L^\rho(\Omega)$, $\rho > n/2$, $u \in L^t(\Gamma)$, $t > n-1$ and

$$\partial_{\nu_A} y = \sum_{i,j=1}^{n} a_{ij}(x) \partial_{x_i} y(x) \nu_j(x),$$

$\nu(x)$ being the unit outward normal vector to Γ at the point x.

The choice $\rho > n/2$ and $t > n-1$ allows us to deduce a theorem of existence and uniqueness analogous to Theorem 1.4, assuming that $a_0 \not\equiv 0$.

Let us consider the control problem

$$\text{(P)} \begin{cases} \text{Minimize } J(u) \\ u \in \mathbb{K}, \end{cases}$$

where $\mathbb{K}$ is a closed, convex and non empty subset of $L^t(\Gamma)$. The functional $J : L^t(\Gamma) \longrightarrow \mathbb{R}$ is defined by

$$J(u) = \frac{1}{2} \int_\Omega L(x, y_u(x))dx + \frac{N}{r} \int_\Gamma |u(x)|^r d\sigma(x),$$

$L : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ being a Carathodory function such that there exists $\psi_0 \in L^1(\Omega)$ and for any $M > 0$ a function $\psi_M \in L^1(\Omega)$ satisfying

$$\psi_0(x) \leq L(x, y) \leq \psi_M(x) \quad \text{a.e. } x \in \Omega, \ \forall |y| \leq M.$$

Let us assume that $1 < r < +\infty$, $N \geq 0$ and that one of the following assumptions is fulfilled:

(1) $\mathbb{K}$ is bounded in $L^t(\Gamma)$ and $r \leq t$.
(2) $N > 0$ and $r \geq t$.

Remark that in this situation the control variable is acting on the boundary Γ of the domain, for this reason it is called a boundary control and (P) is said a boundary control problem. In problem (P) defined in §1.1, u was a distributed control in Ω.

1.3.3. Control of Evolution Equations. Let us consider the following evolution state equation

$$\begin{cases} \dfrac{\partial y}{\partial t}(x,t) + Ay(x,t) = f \text{ in } \ \Omega_T = \Omega \times (0,T), \\ \\ \partial_{\nu_A} y(x,t) + b(x,t,y(x,t)) = u(x,t) \text{ on } \ \Sigma_T = \Gamma \times (0,T), \\ \\ y(x,0) = y_0(x) \text{ in } \ \Omega, \end{cases}$$

where $y_0 \in C(\bar{\Omega})$ and $u \in L^\infty(\Sigma_T)$. If $f \in L^r([0,T], L^p(\Omega))$, with r and p sufficiently large, b is monotone non decreasing and bounded on bounded sets, then the above problem has a unique solution in $C(\bar{\Omega}_T) \cap L^2([0,T], H^1(\Omega))$; see Di Benedetto [**3**]. Thus we can formulate a control problem similar to the previous ones, taking as an objective function

$$J(u) = \int_{\Omega_T} L(x,t,y_u(x,t))dxdt + \int_{\Sigma_T} l(x,t,y_u(x,t),u(x,t))d\sigma(x)dt.$$

To prove the existence of a solution of the control problem is necessary to make some assumptions on the functional J. $L : \Omega_T \times \mathbb{R} \longrightarrow \mathbb{R}$ and $l : \Sigma_T \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ are Carathédory functions, l is convex with respect to the third variable and for every $M > 0$ there exist two functions $\alpha_M \in L^1(\Omega_T)$ and $\beta_M \in L^1(\Sigma_T)$ such that

$$|L(x,t,y)| \leq \alpha_M(x,t) \ \text{ a.e. } (x,t) \in \Omega_T, \ \forall |y| \leq M$$

and

$$|l(x,t,y,u)| \leq \beta_M(x,t) \ \text{ a.e. } (x,t) \in \Sigma_T, \ \forall |y| \leq M, \ \forall |u| \leq M.$$

Let us remark that the hypotheses on the domination of the functions L and l by α_M and β_M are not too restrictive. The convexity of l with respect to the control is the key point to prove the existence of an optimal control. In the lack of convexity, it is necessary to use some compactness argumentation to prove the existence of a solution. The compactness of the set of feasible controls has been used to get the existence of a solution in control problems in the coefficients of the partial differential operator. These type of problems appear in structural optimization problems and in the identification of the coefficients of the operator; see Casas [**11**] and [**12**].

If there is neither convexity nor compactness, we cannot assure, in general, the existence of a solution. Let us see an example.

$$\begin{cases} -\Delta y = u \ \text{ in } \ \Omega, \\ y = 0 \ \text{ on } \ \Gamma. \end{cases}$$

$$(P) \begin{cases} \text{Minimize } \ J(u) = \displaystyle\int_\Omega [y_u(x)^2 + (u^2(x) - 1)^2]dx \\ -1 \leq u(x) \leq +1, \ \ x \in \Omega. \end{cases}$$

Let us take a sequence of controls $\{u_k\}_{k=1}^\infty$ such that $|u_k(x)| = 1$ for every $x \in \Omega$ and verifying that $u_k \rightharpoonup 0$ weakly* in $L^\infty(\Omega)$. The existence of such a solution can be obtained by remarking that the unit closed ball of $L^\infty(\Omega)$ is the weak* closure of the unit sphere $\{u \in L^\infty(\Omega) : \|u\|_{L^\infty(\Omega)} = 1\}$; see Brezis [**10**]. The reader can also make a direct construction of such a sequence (include Ω in a n-cube to simplify the proof). Then, taking into account that $y_{u_k} \to 0$ uniformly in Ω, we have

$$0 \leq \inf_{-1 \leq u(x) \leq +1} J(u) \leq \lim_{k\to\infty} J(u_k) = \lim_{k\to\infty} \int_\Omega y_{u_k}(x)^2 dx = 0.$$

But it is obvious that $J(u) > 0$ for any feasible control, which proves the non existence of an optimal control.

To deal with control problems in the absence of convexity and compactness, (P) is sometimes included in a more general problem ($\bar{\text{P}}$), in such a way that inf(P)= inf($\bar{\text{P}}$), ($\bar{\text{P}}$) having a solution. This leads to the relaxation theory; see Ekeland and Temam [**33**], Warga [**47**], Young [**48**], Roubček [**42**], Pedregal [**39**].

In the last years a lot of research activity has been focused on the control problems associated to the equations of the fluid mechanics; see, for instance, Sritharan [**44**] for a first reading about these problems.

CHAPTER 2

Optimality Conditions

In this chapter we are going to study the first and second order conditions for optimality. The first order conditions are necessary conditions for local optimality, except in the case of convex problems, where they become also sufficient conditions for global optimality. In absence of convexity the sufficiency requires the establishment of optimality conditions of second order. We will prove sufficient and necessary conditions of second order. The sufficient conditions play a very important role in the numerical analysis of the problems. The necessary conditions of second order are the reference that indicate if the sufficient conditions enunciated are reasonable in the sense that its fulfillment is not a too restrictive demand.

2.1. First Order Optimality Conditions

The key tool to get the first order optimality conditions is provided by the next lemma.

Lemma 2.1. *Let U be a Banach space, $\mathbb{K} \subset U$ a convex subset and $J : U \longrightarrow \mathbb{R}$ a function. Let us assume that $\bar{u}$ is a local solution of the optimization problem*

$$(\mathrm{P}) \begin{cases} \inf J(u) \\ u \in \mathbb{K} \end{cases}$$

and that J has directional derivatives at $\bar{u}$. Then

$$J'(\bar{u}) \cdot (u - \bar{u}) \geq 0 \quad \forall u \in \mathbb{K}. \tag{2.1}$$

Reciprocally, if J is a convex function and $\bar{u}$ is an element of $\mathbb{K}$ satisfying (2.1), then $\bar{u}$ is a global minimum of (P).

Proof. The inequality (2.1) is easy to get

$$J'(\bar{u}) \cdot (u - \bar{u}) = \lim_{\lambda \searrow 0} \frac{J(\bar{u} + \lambda(u - \bar{u})) - J(\bar{u})}{\lambda} \geq 0.$$

The last inequality follows from the local optimality of $\bar{u}$ and the fact that $\bar{u} + \lambda(u - \bar{u}) \in \mathbb{K}$ for every $u \in \mathbb{K}$ and every $\lambda \in [0, 1]$ due to the convexity of $\mathbb{K}$.

Reciprocally if $\bar{u} \in \mathbb{K}$ fulfills (2.1) and J is convex, then for every $u \in \mathbb{K}$

$$0 \leq J'(\bar{u}) \cdot (u - \bar{u}) = \lim_{\lambda \searrow 0} \frac{J(\bar{u} + \lambda(u - \bar{u})) - J(\bar{u})}{\lambda} \leq J(u) - J(\bar{u}),$$

therefore $\bar{u}$ is a global solution of (P). □

In order to apply this lemma to the study of problem (P) we need to analyze the differentiability of the functionals involved in the control problem.

Proposition 2.2. *The mapping $G : L^\infty(\Omega) \longrightarrow W^{2,p}(\Omega)$ defined by $G(u) = y_u$ is of class C^2. Furthermore if $u, v \in L^\infty(\Omega)$ and $z = DG(u) \cdot v$, then z is the unique solution in $W^{2,p}(\Omega)$ of Dirichlet problem*

$$\left\{ \begin{array}{l} Az + \phi'(y_u(x))z = v \quad \text{in} \quad \Omega, \\ z = 0 \quad \text{on} \quad \Gamma. \end{array} \right. \tag{2.2}$$

Finally, for every $v_1, v_2 \in L^\infty(\Omega)$, $z_{v_1 v_2} = G''(u)v_1 v_2$ is the solution of

$$\left\{ \begin{array}{l} Az_{v_1 v_2} + \phi'(y_u(x))z_{v_1 v_2} + \phi''(y_u(x))z_{v_1} z_{v_2} = 0 \quad \text{in} \quad \Omega, \\ z_{v_1 v_2} = 0 \quad \text{on} \quad \Gamma, \end{array} \right. \tag{2.3}$$

where $z_{v_i} = G'(u)v_i$, $i = 1, 2$.

Proof. To prove the differentiability of G we will apply the implicit function theorem. Let us consider the Banach space

$$V(\Omega) = \{y \in H_0^1(\Omega) \cap W^{2,p}(\Omega) : Ay \in L^\infty(\Omega)\},$$

endowed with the norm

$$\|y\|_{V(\Omega)} = \|y\|_{W^{2,p}(\Omega)} + \|Ay\|_\infty.$$

Now let us take the function

$$F : V(\Omega) \times L^\infty(\Omega) \longrightarrow L^\infty(\Omega)$$

defined by

$$F(y, u) = Ay + \phi(y) - u.$$

It is obvious that F is of class C^2, $y_u \in V(\Omega)$ for every $u \in L^\infty(\Omega)$, $F(y_u, u) = 0$ and

$$\frac{\partial F}{\partial y}(y, u) \cdot z = Az + \phi'(y)z$$

is an isomorphism from $V(\Omega)$ into $L^\infty(\Omega)$. By applying the implicit function theorem we deduce that G is of class C^2 and $DG(u) \cdot z$ is given by (2.2). Finally (2.3) follows by differentiating twice with respect to u in the equation

$$AG(u) + \phi(G(u)) = u.$$

□

As a consequence of this result we get the following proposition.

PROPOSITION 2.3. *The function* $J : L^\infty(\Omega) \to \mathbb{R}$ *is of class* C^2. *Moreover, for every* $u, v, v_1, v_2 \in L^\infty(\Omega)$

$$J'(u)v = \int_\Omega \left(\frac{\partial L}{\partial u}(x, y_u, u) + \varphi_u \right) v\, dx \tag{2.4}$$

and

$$J''(u)v_1v_2 = \int_\Omega \left[\frac{\partial^2 L}{\partial y^2}(x, y_u, u) z_{v_1} z_{v_2} + \frac{\partial^2 L}{\partial y \partial u}(x, y_u, u)(z_{v_1} v_2 + z_{v_2} v_1) + \right.$$

$$\left. \frac{\partial^2 L}{\partial u^2}(x, y_u, u) v_1 v_2 - \varphi_u \phi''(y_u) z_{v_1} z_{v_2} \right] dx \tag{2.5}$$

where $\varphi_u \in W^{2,p}(\Omega)$ *is the unique solution of problem*

$$\left\{ \begin{array}{l} A^*\varphi + \phi'(y_u)\varphi = \dfrac{\partial L}{\partial y}(x, y_u, u) \quad \text{in } \Omega \\ \varphi = 0 \quad \text{on } \Gamma, \end{array} \right. \tag{2.6}$$

A^* *being the adjoint operator of* A *and* $z_{v_i} = G'(u)v_i$, $i = 1, 2$.

PROOF. From hypothesis **(H1)**, Proposition 2.2 and the chain rule it comes

$$J'(u) \cdot v = \int_\Omega \left[\frac{\partial L}{\partial y}(x, y_u(x), u(x)) z(x) + \frac{\partial L}{\partial u}(x, y_u(x), u(x)) v(x) \right] dx,$$

where $z = G'(u)v$. Using (2.6) in this expression we get

$$\begin{aligned} J'(u) \cdot v &= \int_\Omega \left\{ [A^*\varphi_u + \phi'(y_u)\varphi_u] z + \frac{\partial L}{\partial u}(x, y_u(x), u(x)) v(x) \right\} dx \\ &= \int_\Omega \left\{ [Az + \phi'(y_u) z] \varphi_u + \frac{\partial L}{\partial u}(x, y_u(x), u(x)) v(x) \right\} dx \\ &= \int_\Omega \left\{ \varphi_u(x) + \frac{\partial L}{\partial u}(x, y_u(x), u(x)) \right\} v(x)\, dx, \end{aligned}$$

which proves (2.4). Finally (2.5) follows again by application of the chain rule and Proposition 2.2. □

Combining Lemma 2.1 with the previous proposition we get the first order optimality conditions.

THEOREM 2.4. *Let $\bar{u}$ be a local minimum of* (P). *Then there exist $\bar{y}, \bar{\varphi} \in H_0^1(\Omega) \cap W^{2,p}(\Omega)$ such that the following relationships hold*

$$\left\{ \begin{array}{l} A\bar{y} + \phi(\bar{y}) = \bar{u} \quad \text{in} \quad \Omega, \\ \bar{y} = 0 \quad \text{on} \quad \Gamma, \end{array} \right. \tag{2.7}$$

$$\left\{ \begin{array}{l} A^*\bar{\varphi} + \phi'(\bar{y})\bar{\varphi} = \dfrac{\partial L}{\partial y}(x, \bar{y}, \bar{u}) \quad \text{in} \quad \Omega, \\ \bar{\varphi} = 0 \quad \text{on} \quad \Gamma, \end{array} \right. \tag{2.8}$$

$$\int_\Omega \left\{ \bar{\varphi}(x) + \frac{\partial L}{\partial u}(x, \bar{y}(x), \bar{u}(x)) \right\} (u(x) - \bar{u}(x))dx \geq 0 \; \forall u \in \mathbb{K}. \tag{2.9}$$

From this theorem we can deduce some regularity results of the local minima.

THEOREM 2.5. *Let us assume that $\bar{u}$ is a local minimum of* (P) *and that hypotheses **(H1)** and **(H2)** are fulfilled. Then for any $x \in \bar{\Omega}$, the equation*

$$\bar{\varphi}(x) + \frac{\partial L}{\partial u}(x, \bar{y}(x), t) = 0 \tag{2.10}$$

has a unique solution $\bar{t} = \bar{s}(x)$, where $\bar{y}$ is the state associated to $\bar{u}$ and $\bar{\varphi}$ is the adjoint state defined by (2.8). *The mapping $\bar{s} : \bar{\Omega} \longrightarrow \mathbb{R}$ is Lipschitz. Moreover $\bar{u}$ and $\bar{s}$ are related by the formula*

$$\bar{u}(x) = Proj_{[\alpha,\beta]}(\bar{s}(x)) = \max(\alpha, \min(\beta, \bar{s}(x))), \tag{2.11}$$

and $\bar{u}$ is Lipschitz too.

PROOF. The existence and uniqueness of solution of equation (2.10) is an immediate consequence of the hypothesis **(H2)**, therefore $\bar{s}$ is well defined. Let us see that $\bar{s}$ is bounded. Indeed, making a Taylor development of the first order in the relation

$$\bar{\varphi}(x) + \frac{\partial L}{\partial u}(x, \bar{y}(x), \bar{s}(x)) = 0$$

we get that

$$\frac{\partial^2 L}{\partial u^2}(x, \bar{y}(x), \theta(x)\bar{s}(x))\bar{s}(x) = -\bar{\varphi}(x) - \frac{\partial L}{\partial u}(x, \bar{y}(x), 0),$$

which along with **(H2)** lead to

$$\Lambda|\bar{s}(x)| \leq |\bar{\varphi}(x)| + \left| \frac{\partial L}{\partial u}(x, \bar{y}(x), 0) \right| \leq C \; \forall x \in \Omega.$$

Now let us prove that $\bar{s}$ is Lipschitz. For it we use **(H2)**, the properties of L enounced in **(H1)**, the fact that $\bar{y}$ and $\bar{\varphi}$ are Lipschitz functions (due to the inclusion $W^{2,p}(\Omega) \subset C^{0,1}(\bar{\Omega})$) and the equation above satisfied by $\bar{s}(x)$. Let $x_1, x_2 \in \bar{\Omega}$

$$\Lambda|\bar{s}(x_2) - \bar{s}(x_1)| \leq |\frac{\partial L}{\partial u}(x_2, \bar{y}(x_2), \bar{s}(x_2)) - \frac{\partial L}{\partial u}(x_2, \bar{y}(x_2), \bar{s}(x_1))| =$$

$$|\bar{\varphi}(x_1) - \bar{\varphi}(x_2) + \frac{\partial L}{\partial u}(x_1, \bar{y}(x_1), \bar{s}(x_1)) - \frac{\partial L}{\partial u}(x_2, \bar{y}(x_2), \bar{s}(x_1))| \leq$$

$$|\bar{\varphi}(x_1) - \bar{\varphi}(x_2)| + C_M\left(|\bar{y}(x_1) - \bar{y}(x_2)| + |x_2 - x_1|\right) \leq C|x_2 - x_1|.$$

Finally, from (2.9) and the fact that $(\partial L/\partial u)$ is an increasing function of the third variable we have

$$\begin{aligned}
\alpha < \bar{u}(x) < \beta \quad &\Rightarrow \bar{\varphi}(x) + \frac{\partial L}{\partial u}(x, \bar{y}(x), \bar{u}(x)) = 0 \;\Rightarrow\; \bar{u}(x) = \bar{s}(x), \\
\bar{u}(x) = \beta \quad &\Rightarrow \bar{\varphi}(x) + \frac{\partial L}{\partial u}(x, \bar{y}(x), \bar{u}(x)) \leq 0 \;\Rightarrow\; \bar{u}(x) \leq \bar{s}(x), \\
\bar{u}(x) = \alpha \quad &\Rightarrow \bar{\varphi}(x) + \frac{\partial L}{\partial u}(x, \bar{y}(x), \bar{u}(x)) \geq 0 \;\Rightarrow\; \bar{u}(x) \geq \bar{s}(x),
\end{aligned}$$

which implies (2.11). □

REMARK 2.6. If the assumption **(H2)** does not hold, then the optimal controls can be discontinuous. The most obvious case is the one where L is independent of u, in this case (2.9) is reduced to

$$\int_\Omega \bar{\varphi}(x)(u(x) - \bar{u}(x))\,dx \geq 0 \quad \forall u \in \mathbb{K},$$

which leads to

$$\bar{u}(x) = \begin{cases} \alpha & \text{if } \bar{\varphi}(x) > 0 \\ \beta & \text{if } \bar{\varphi}(x) < 0 \end{cases} \quad \text{a.e. } x \in \Omega.$$

If $\bar{\varphi}$ vanishes in a set of points of zero measure, then $\bar{u}$ jumps from α to β. Such a control $\bar{u}$ is called a bang-bang control. The controls of this nature are of great interest in the applications due to the easiness to automate the control process. All the results presented previously are valid without the assumption **(H2)**, except Theorem 2.5.

REMARK 2.7. A very frequent case is given by the function $L(x, y, u) = [(y - y_d(x))^2 + Nu^2]/2$, where $N > 0$ and $y_d \in L^2(\Omega)$ is a fixed element.

In this case, (2.9) leads to

$$\bar{u}(x) = Proy_{\mathbb{K}}\left(-\frac{1}{N}\bar{\varphi}\right)(x) = \begin{cases} \alpha & \text{if } -\dfrac{1}{N}\bar{\varphi}(x) < \alpha, \\ \beta & \text{if } -\dfrac{1}{N}\bar{\varphi}(x) > \beta, \\ -\dfrac{1}{N}\bar{\varphi}(x) & \text{if } \alpha \leq -\dfrac{1}{N}\bar{\varphi}(x) \leq \beta. \end{cases}$$

In this case $\bar{s} = -\bar{\varphi}/N$.

If furthermore we assume that $\mathbb{K} = L^2(\Omega)$, then (2.9) implies that $\bar{u} = -(1/N)\bar{\varphi}$. Thus $\bar{u}$ has the same regularity than $\bar{\varphi}$. Therefore $\bar{u}$ will be the more regular as much as greater be the regularity of y_d, Γ, ϕ and the coefficients of operator A. In particular we can get $C^\infty(\bar{\Omega})$-regularity if all the data of the problem are of class C^∞.

Remark 2.8. If we consider the boundary control problem formulated in §1.3.2, then the corresponding optimality system is

$$\begin{cases} A\bar{y} + \phi(\bar{y}) = f & \text{in } \Omega, \\ \partial_{\nu_A}\bar{y} = \bar{u} & \text{on } \Gamma, \end{cases}$$

$$\begin{cases} A^\star\bar{\varphi} + \phi'(\bar{y})\bar{\varphi} = \bar{y} - y_d & \text{in } \Omega, \\ \partial_{\nu_{A^\star}}\bar{\varphi} = 0 & \text{on } \Gamma, \end{cases}$$

$$\int_\Gamma \left(\bar{\varphi}(x) + N|\bar{u}(x)|^{r-2}\bar{u}(x)\right)(v(x) - \bar{u}(x))d\sigma(x) \geq 0 \quad \forall v \in \mathbb{K}.$$

Thus if $N > 0$ and $\mathbb{K} = L^r(\Gamma)$, with $r > n - 1$, we get from the last inequality

$$\bar{u}(x) = \frac{-1}{N^{1/(r-1)}}|\bar{\varphi}(x)|^{(2-r)/(r-1)}\bar{\varphi}(x),$$

which allows a regularity study of $\bar{u}$ in terms of the function $\bar{\varphi}$. If $\mathbb{K}$ is the set of controls of $L^\infty(\Gamma)$ bounded by α and β, then

$$\bar{u}(x) = \text{Proy}_{[\alpha,\beta]}\left(\frac{-1}{N^{1/(r-1)}}|\bar{\varphi}(x)|^{(2-r)/(r-1)}\bar{\varphi}(x)\right).$$

REMARK 2.9. Let us see the expression of the the optimality system corresponding to the problem formulated in §1.3.3:

$$\begin{cases} \dfrac{\partial \bar{y}}{\partial t} + A\bar{y} = f \text{ in } \Omega_T, \\ \\ \partial_{\nu_A}\bar{y} + b(x,t,\bar{y}) = \bar{u} \text{ on } \Sigma_T, \\ \\ \bar{y}(x,0) = y_0(x) \text{ in } \Omega, \end{cases}$$

$$\begin{cases} -\dfrac{\partial \bar{\varphi}}{\partial t} + A^\star\bar{\varphi} = \dfrac{\partial L}{\partial y}(x,t,\bar{y}) \text{ in } \Omega_T, \\ \\ \partial_{\nu_{A^*}}\bar{\varphi} + \dfrac{\partial b}{\partial y}(x,t,\bar{y})\bar{\varphi} = \dfrac{\partial l}{\partial y}(x,t,\bar{y},\bar{u}) \text{ on } \Sigma_T, \\ \\ \bar{\varphi}(x,T) = 0 \text{ in } \Omega, \end{cases}$$

$$\int_{\Sigma_T}\left\{\bar{\varphi} + \frac{\partial l}{\partial u}(x,t,\bar{y},\bar{u})\right\}(u-\bar{u})d\sigma(x)dt \geq 0 \quad \forall u \in \mathbb{K}.$$

Of course the convenient differentiability hypotheses on the functions b, L and l should be done to obtain the previous system, but this is a question that will not analyze here. Simply we intend to show how the adjoint state equation in problems of optimal control of parabolic equations is formulated.

REMARK 2.10. In the case of control problems with state constraints it is more difficult to derive the optimality conditions, mainly in the case of pointwise state constraints, for instance $|y(x)| \leq 1$ for every $x \in \Omega$. In fact this is an infinity number of constraints, one constraint for every point of Ω. The reader is referred to Bonnans and Casas [**4**], [**5**], [**8**].

It is possible to give an optimality system without making any derivative with respect to the control of the functions involved in the problem. These conditions are known as Pontryagin Maximum Principle. This result, first stated for control problems of ordinary differential equations (see [**40**]), has been later extended to problems governed by partial differential equations; see Bonnans and Casas [**7**], [**8**], Casas [**13**], [**15**], Casas, Raymond and Zidani [**28**], [**29**], Casas and Yong [**31**], Fattorini [**34**], Li and Yong [**37**]. This principle provides some optimality conditions more powerful than those obtained by the general optimization methods. In particular, it is possible to deduce the optimality conditions in the absence of convexity of the set of controls $\mathbb{K}$ and differentiability properties with respect to the control of the functionals involved in the control problem. As far as I know in [**29**] the reader can find the most general result on the Pontryagin principle for control problems governed by partial differential equations.

Difficulties also appear to obtain the optimality conditions in the case of state equations with more complicated linearities than those presented in these notes. This is the case for the quasilinear equations; see Casas y Fernández [**19**], [**20**]. Sometimes the non linearity causes the system to have multiple solutions for some controls while for other there is no solution; see Bonnans and Casas [**6**] and Casas, Kavian and Puel [**21**]. A situation of this nature, especially interesting by the applications, is the one that arises in the control of the Navier-Stokes equations; see Abergel and Casas [**1**], Casas [**14**], [**16**] and Casas, Mateos and Raymond [**25**].

2.2. Second Order Optimality Conditions

Let $\bar u$ be a local minimum of (P), $\bar y$ and $\bar\varphi$ being the associated state and adjoint state respectively. In order to simplify the notation we will consider the function

$$\bar d(x) = \frac{\partial L}{\partial u}(x, \bar y(x), \bar u(x)) + \bar\varphi(x).$$

From (2.9) it follows

$$\bar d(x) = \begin{cases} 0 & \text{a.e. } x \in \Omega \text{ if } \alpha < \bar u(x) < \beta, \\ \geq 0 & \text{a.e. } x \in \Omega \text{ if } \bar u(x) = \alpha, \\ \leq 0 & \text{a.e. } x \in \Omega \text{ if } \bar u(x) = \beta. \end{cases} \tag{2.12}$$

The following cone of critical directions is essential in the formulation of the second order optimality conditions.

$$C_{\bar u} = \{v \in L^2(\Omega) \text{ satisfying (2.13) and } v(x) = 0 \text{ if } \bar d(x) \neq 0\},$$

$$v(x) = \begin{cases} \geq 0 & \text{a.e. } x \in \Omega \text{ if } \bar u(x) = \alpha, \\ \leq 0 & \text{a.e. } x \in \Omega \text{ if } \bar u(x) = \beta. \end{cases} \tag{2.13}$$

Now we can to formulate the necessary and sufficient conditions for optimality.

THEOREM 2.11. *Under the hypotheses **(H1)** and **(H2)**, if $\bar u$ is a local minimum of (P), then*

$$J''(\bar u)v^2 \geq 0 \quad \forall v \in C_{\bar u}. \tag{2.14}$$

Reciprocally, if $\bar u \in \mathbb{K}$ fulfills the first order optimality conditions (2.7)–(2.9) *and the condition*

$$J''(\bar u)v^2 > 0 \quad \forall v \in C_{\bar u} \setminus \{0\}, \tag{2.15}$$

then there exist $\delta > 0$ and $\varepsilon > 0$ such that

$$J(u) \geq J(\bar{u}) + \frac{\delta}{2}\|u - \bar{u}\|^2_{L^2(\Omega)} \quad \forall u \in \mathbb{K} \cap \bar{B}_\varepsilon(\bar{u}), \tag{2.16}$$

where $\bar{B}_\varepsilon(\bar{u})$ is the unit closed ball in $L^\infty(\Omega)$ with center at $\bar{u}$ and radius ε.

Proof. *i)*- Let us assume that $\bar{u}$ is a local minimum of (P) and prove (2.14). Firstly let us take $v \in C_{\bar{u}} \cap L^\infty(\Omega)$. For every $0 < \rho < \beta - \alpha$ we define

$$v_\rho(x) = \begin{cases} 0 & \text{if } \alpha < \bar{u}(x) < \alpha + \rho \;\text{ or }\; \beta - \rho < \bar{u}(x) < \beta, \\ v(x) & \text{otherwise.} \end{cases}$$

Then we still have that $v_\rho \in C_{\bar{u}} \cap L^\infty(\Omega)$. Moreover $v_\rho \to v$ when $\rho \to 0$ in $L^p(\Omega)$ for every $p < +\infty$ and $\bar{u} + \lambda v_\rho \in \mathbb{K}$ for every $\lambda \in (0, \rho/\|v\|_{L^\infty(\Omega)}]$. By using the optimality of $\bar{u}$ it comes

$$0 \leq \frac{J(\bar{u} + \lambda v_\rho) - J(\bar{u})}{\lambda} = J'(\bar{u})v_\rho + \frac{\lambda}{2}J''(\bar{u} + \theta_\lambda \lambda v_\rho)v_\rho^2,$$

with $0 < \theta_\lambda < 1$. From this inequality and the following identity

$$J'(\bar{u})v_\rho = \int_\Omega (\frac{\partial L}{\partial u}(x, \bar{y}(x), \bar{u}(x)) + \bar{\varphi}(x))v_\rho(x)\,dx = \int_\Omega \bar{d}(x)v_\rho(x)\,dx = 0,$$

we deduce by passing to the limit when $\lambda \to 0$

$$0 \leq J''(\bar{u} + \theta_\lambda \lambda v_\rho)v_\rho^2 \to J''(\bar{u})v_\rho^2.$$

Now from the expression of the second derivative J'' given by (2.5), we can pass to the limit in the previous expression when $\rho \to 0$ and get that $J''(\bar{u})v^2 \geq 0$.

To conclude this part we have to prove the same inequality for any $v \in C_{\bar{u}}$, not necessarily bounded. Let us take v in $C_{\bar{u}}$ and consider

$$v_k(x) = Proy_{[-k,+k]}(v(x)) = \min\{\max\{-k, v(x)\}, +k\}.$$

Then $v_k \to v$ en $L^2(\Omega)$ and $v_k \in C_{\bar{u}} \cap L^\infty(\Omega)$, which implies that $J''(\bar{u})v_k^2 \geq 0$. Passing again to the the limit when $k \to +\infty$, we deduce (2.14).

ii)- Now let us assume that (2.15) holds and prove (2.16). We argue by contradiction and assume that for any $k \in \mathbb{N}$ we can find an element $u_k \in \mathbb{K}$ such that

$$\|\bar{u} - u_k\|_{L^\infty(\Omega)} < \frac{1}{k} \;\text{ y }\; J(u_k) < J(\bar{u}) + \frac{1}{k}\|u_k - \bar{u}\|^2_{L^2(\Omega)}. \tag{2.17}$$

Let us define

$$\delta_k = \|u_k - \bar{u}\|_{L^2(\Omega)} \;\text{ y }\; v_k = \frac{1}{\delta_k}(u_k - \bar{u}).$$

By taking a subsequence if necessary, we can suppose that $v_k \rightharpoonup v$ weakly in $L^2(\Omega)$. By using the equation (2.2) it is easy to check that

$$\lim_{k\to\infty} z_k = \lim_{k\to\infty} G'(\bar{u})v_k = G'(\bar{u})v = z \quad \text{strongly in} \quad H_0^1(\Omega) \cap L^\infty(\Omega).$$

On the other hand, from the properties of L established in the hypothesis **(H1)** we get for all $0 < \theta_k < 1$

$$|[J''(\bar{u} + \theta_k \delta_k v_k) - J''(\bar{u})]v_k^2| \leq$$

$$C\|\theta_k \delta_k v_k\|_{L^\infty(\Omega)} \|v_k\|^2_{L^2(\Omega)} \leq C\|u_k - \bar{u}\|_{L^\infty(\Omega)} \to 0. \tag{2.18}$$

From (2.17) it comes

$$\frac{1}{k}\|u_k - \bar{u}\|^2_{L^2(\Omega)} > J(u_k) - J(\bar{u}) = J(\bar{u} + \delta_k v_k) - J(\bar{u}) = \delta_k J'(\bar{u})v_k +$$

$$\frac{\delta_k^2}{2} J''(\bar{u} + \theta_k \delta_k v_k)v_k^2 \geq \frac{\delta_k^2}{2} J''(\bar{u} + \theta_k \delta_k v_k)v_k^2. \tag{2.19}$$

The last inequality follows from (2.9) along with the fact that $u_k \in \mathbb{K}$ and therefore

$$\delta_k J'(\bar{u})v_k = J'(\bar{u})(u_k - \bar{u}) = \int_\Omega \bar{d}(x)(u_k(x) - \bar{u}(x))\, dx \geq 0.$$

From the strong convergence $z_k \to z$ in $L^\infty(\Omega)$, the weak convergence $v_k \rightharpoonup v$ in $L^2(\Omega)$, the expression of J'' given by (2.5), the hypothesis **(H2)**, (2.18) and the inequality (2.19) we deduce

$$J''(\bar{u})v^2 \leq \liminf_{k\to\infty} J''(\bar{u})v_k^2 \leq \limsup_{k\to\infty} J''(\bar{u})v_k^2 \leq$$

$$\limsup_{k\to\infty} |[J''(\bar{u} + \theta_k \delta_k v_k) - J''(\bar{u})]v_k^2| +$$

$$\limsup_{k\to\infty} J''(\bar{u} + \theta_k \delta_k v_k)v_k^2 \leq \limsup_{k\to\infty} \frac{2}{k} = 0. \tag{2.20}$$

Now let us prove that $J''(\bar{u})v^2 \geq 0$ to conclude that $J''(\bar{u})v^2 = 0$. For it we are going to use the sufficient second order condition (2.15). First we have to prove that $v \in C_{\bar{u}}$. Let us remark that every v_k satisfies the sign condition (2.13). Since the set of functions of $L^2(\Omega)$ verifying (2.13) is convex and closed, then it is weakly closed, which implies that v belongs to this set and consequently it also satisfies (2.13). Let us see that $\bar{d}(x)v(x) = 0$. From (2.12) and (2.13) we get that $\bar{d}(x)v(x) \geq 0$. Using the mean value theorem and (2.17) we get

$$J(u_k) - J(\bar{u}) = J'(\bar{u} + \theta_k(u_k - \bar{u}))(u_k - \bar{u}) < \frac{\delta_k^2}{k},$$

which leads to

$$\int_\Omega |\bar{d}(x)v(x)|\,dx = \int_\Omega \bar{d}(x)v(x)\,dx = J'(\bar{u})v =$$

$$\lim_{k\to 0} J'(\bar{u}+\theta_k(u_k-\bar{u}))v_k = \lim_{k\to 0}\frac{1}{\delta_k}J'(\bar{u}+\theta_k(u_k-\bar{u}))(u_k-\bar{u}) = 0.$$

Thus we have that $v \in C_{\bar{u}}$. Therefore (2.15) and (2.20) is only possible if $v = 0$. Combining this with (2.20) it comes

$$\lim_{k\to\infty} J''(\bar{u})v_k^2 = 0.$$

Once again using **(H2)** and the expression (2.5), we deduce from the above identity and the weak and strong convergences of $\{v_k\}$ and $\{z_k\}$ respectively that

$$0 < \Lambda \le \liminf_{k\to\infty}\int_\Omega \frac{\partial^2 L}{\partial u^2}(x,\bar{y},\bar{u})v_k^2\,dx \le \limsup_{k\to\infty}\int_\Omega \frac{\partial^2 L}{\partial u^2}(x,\bar{y},\bar{u})v_k^2\,dx =$$

$$\lim_{k\to\infty} J''(\bar{u})v_k^2 - \lim_{k\to\infty}\int_\Omega\left[\frac{\partial^2 L}{\partial y^2}(x,\bar{y},\bar{u})z_k^2 + \frac{\partial^2 L}{\partial y\partial u}(x,\bar{y},\bar{u})v_k z_k \right.$$
$$\left. -\bar{\varphi}\phi''(\bar{y})z_k^2\right]\,dx = 0,$$

which provides the desired contradiction. □

We will finish this chapter by proving a very important result to deduce the error estimates of the approximations of problem (P).

THEOREM 2.12. *Under the hypotheses* ***(H1)*** *and* ***(H2)****, if* $\bar{u} \in \mathbb{K}$ *satisfies* (2.7)-(2.9)*, then the following statements are equivalent*

$$J''(\bar{u})v^2 > 0 \quad \forall v \in C_{\bar{u}} \setminus \{0\} \tag{2.21}$$

and

$$\exists\delta > 0 \text{ y } \exists\tau > 0 \ / \ J''(\bar{u})v^2 \ge \delta\|v\|^2_{L^2(\Omega)} \ \forall v \in C^\tau_{\bar{u}}, \tag{2.22}$$

where

$$C^\tau_{\bar{u}} = \{v \in L^2(\Omega) \text{ satisfying } (2.13) \text{ and } v(x) = 0 \text{ if } |\bar{d}(x)| > \tau\}.$$

PROOF. Since $C_{\bar{u}} \subset C^\tau_{\bar{u}}$ for all $\tau > 0$, it is obvious that (2.22) implies (2.21). Let us prove the reciprocal implication. We proceed by contradiction and assume that for any $\tau > 0$ there exists $v_\tau \in C^\tau_{\bar{u}}$ such that $J''(\bar{u})v_\tau^2 < \tau\|v_\tau\|^2_{L^2(\Omega)}$. Dividing v_τ by its norm we can assume that

$$\|v_\tau\|_{L^2(\Omega)} = 1, \quad J''(\bar{u})v_\tau^2 < \tau \text{ and } v_\tau \rightharpoonup v \text{ en } L^2(\Omega). \tag{2.23}$$

Let us prove that $v \in C_{\bar{u}}$. Arguing as in the proof of the previous theorem we get that v satisfies the sign condition (2.13). On the other hand

$$\int_\Omega |\bar{d}(x)v(x)|\,dx = \int_\Omega \bar{d}(x)v(x)\,dx =$$
$$\lim_{\tau\to 0}\int_\Omega \bar{d}(x)v_\tau(x)\,dx = \lim_{\tau\to 0}\int_{|\bar{d}(x)|\le\tau} d(x)v_\tau(x)\,dx \le$$
$$\lim_{\tau\to 0}\tau\int_\Omega |v_\tau(x)|\,dx \le \lim_{\tau\to 0}\tau\sqrt{m(\Omega)}\|v_\tau\|_{L^2(\Omega)} = 0,$$

which proves that $v(x) = 0$ if $\bar{d}(x) \neq 0$. Thus we have that $v \in C_{\bar{u}}$. Then (2.21) implies that either $v = 0$ or $J''(\bar{u})v^2 > 0$. But (2.23) leads to

$$J''(\bar{u})v^2 \le \liminf_{\tau\to 0} J''(\bar{u})v_\tau^2 \le \limsup_{\tau\to 0} J''(\bar{u})v_\tau^2 \le 0.$$

Thus we conclude that $v = 0$. Once again arguing as in the proof of the previous theorem we deduce that

$$0 < \Lambda \le \lim_{\tau\to 0}\int_\Omega \frac{\partial^2 L}{\partial u^2}(x,\bar{y},\bar{u})v_\tau^2\,dx = \lim_{\tau\to 0} J''(\bar{u})v_\tau^2 -$$
$$\lim_{\tau\to 0}\int_\Omega \left[\frac{\partial^2 L}{\partial y^2}(x,\bar{y},\bar{u})z_\tau^2 + \frac{\partial^2 L}{\partial y\partial u}(x,\bar{y},\bar{u})v_\tau z_\tau - \bar{\varphi}\phi''(\bar{y})z_\tau^2\right]dx = 0,$$

Which leads to the desired contradiction. $\square$

Remark 2.13. The fact that the control u appears linearly in the state equation and the hypothesis **(H2)** have been crucial to deduce the second order optimality conditions proved in this chapter. As a consequence of both assumptions, the functional $J''(\bar{u})$ is a Legendre quadratic form in $L^2(\Omega)$, which simplifies the proof, allowing us to follow the method of proof used in finite dimensional optimization; see Bonnans and Zidani [**9**]. In the absence of one of these assumptions, the condition (2.15) is not enough to assure the optimality; see Casas and Tröltzsch [**30**] and Casas and Mateos [**22**].

CHAPTER 3

Numerical Approximation

In order to simplify the presentation we will assume that Ω is convex.

3.1. Finite Element Approximation of (P)

Now we consider a based finite element approximation of (P). Associated with a parameter h we consider a family of triangulations $\{\mathcal{T}_h\}_{h>0}$ of $\bar{\Omega}$. To every element $T \in \mathcal{T}_h$ we assign two parameters $\rho(T)$ and $\sigma(T)$, where $\rho(T)$ denotes the diameter of T and $\sigma(T)$ is the diameter of the biggest ball contained in T. The size of the grid is given by $h = \max_{T\in\mathcal{T}_h} \rho(T)$. The following usual regularity assumptions on the triangulation are assumed.

(i) - There exist two positive constants ρ and σ such that

$$\frac{\rho(T)}{\sigma(T)} \leq \sigma, \qquad \frac{h}{\rho(T)} \leq \rho$$

for every $T \in \mathcal{T}_h$ and all $h > 0$.

(ii) - Let us set $\overline{\Omega}_h = \cup_{T\in\mathcal{T}_h} T$, Ω_h and Γ_h its interior and boundary respectively. We assume that the vertices of $\mathcal{T}_h$ placed on the boundary Γ_h are points of Γ. From [**41**, inequality (5.2.19)] we know

$$\text{measure}(\Omega \setminus \Omega_h) \leq Ch^2. \tag{3.1}$$

Associated to these triangulations we define the spaces

$$U_h = \{u \in L^\infty(\Omega_h) \mid u_{|T} \text{ is constant on each } T \in \mathcal{T}_h\},$$

$$Y_h = \{y_h \in C(\bar{\Omega}) \mid y_{h|T} \in \mathcal{P}_1, \text{ for every } T \in \mathcal{T}_h, \text{ and } y_h = 0 \text{ in } \bar{\Omega} \setminus \Omega_h\},$$

where $\mathcal{P}_1$ is the space formed by the polynomials of degree less than or equal to one. For every $u \in L^2(\Omega)$, we denote by $y_h(u)$ the unique element of Y_h satisfying

$$a(y_h(u), q_h) + \int_\Omega \phi(y_h(u))q_h\,dx = \int_\Omega uq_h\,dx \quad \forall q_h \in Y_h, \tag{3.2}$$

where $a : Y_h \times Y_h \longrightarrow \mathbb{R}$ is the bilinear form defined by

$$a(y_h, q_h) = \int_\Omega (\sum_{i,j=1}^{n} a_{ij}(x)\partial_{x_i} y_h(x)\partial_{x_j} q_h(x) + a_0(x)y_h(x)q_h(x))\, dx.$$

In other words, $y_h(u)$ is the discrete state associated with u. Let us remark that $y_h = z_h = 0$ on $\bar{\Omega} \setminus \Omega_h$, therefore the above integrals can be replaced by the integrals in Ω_h. Therefore the values of u in $\Omega \setminus \Omega_h$ do not contribute to the computation of $y_h(u)$, consequently we can define $y_h(u_h)$ for any $u_h \in U_h$. In particular for any extension of u_h to Ω, the discrete state $y_h(u)$ is the same.

The finite dimensional approximation of the optimal control problem (P) is defined in the following way

$$(\mathrm{P}_h) \begin{cases} \min J_h(u_h) = \int_{\Omega_h} L(x, y_h(u_h)(x), u_h(x))\, dx, \\ \text{such that } \ (y_h(u_h), u_h) \in Y_h \times U_h, \\ \alpha \le u_h(x) \le \beta \quad \text{a.e.} \quad x \in \Omega_h. \end{cases}$$

Let us start the study of problem (P_h) by analyzing the differentiability of the functions involved in the control problem. We just enounce the differentiability results analogous to the ones of §2.1.

Proposition 3.1. *For every $u \in L^\infty(\Omega)$, problem* (3.2) *has a unique solution $y_h(u) \in Y_h$, the mapping $G_h : L^\infty(\Omega) \longrightarrow Y_h$, defined by $G_h(u) = y_h(u)$, is of class C^2 and for all $v, u \in L^\infty(\Omega)$, $z_h(v) = G_h'(u)v$ is the solution of*

$$a(z_h(v), q_h) + \int_\Omega \phi'(y_h(u))z_h(v)q_h\, dx = \int_\Omega vq_h\, dx \quad \forall q_h \in Y_h \tag{3.3}$$

Finally, for every $v_1, v_2 \in L^\infty(\Omega)$, $z_h(v_1, v_2) = G''(u)v_1v_2 \in Y_h$ is the solution of the variational equation:

$$a(z_h, q_h) + \int_\Omega \phi'(y_h(u))z_hq_h\, dx + \int_\Omega \phi''(y_h(u))z_{h1}z_{h2}q_h dx = 0 \tag{3.4}$$

$\forall q_h \in Y_h$, where $z_{hi} = G_h'(u)v_i$, $i = 1, 2$.

Proposition 3.2. *Functional $J_h : L^\infty(\Omega) \to \mathbb{R}$ is of class C^2. Moreover for all $u, v, v_1, v_2 \in L^\infty(\Omega)$*

$$J_h'(u)v = \int_{\Omega_h} \left(\frac{\partial L}{\partial u}(x, y_h(u), u) + \varphi_h(u) \right) v\, dx \tag{3.5}$$

and

$$J_h''(u)v_1v_2 = \int_{\Omega_h} \left[\frac{\partial^2 L}{\partial y^2}(x, y_h(u), u)z_h(v_1)z_h(v_2) + \right.$$

$$\frac{\partial^2 L}{\partial y \partial u}(x, y_h(u), u)[z_h(v_1)v_2 + z_h(v_2)v_1] +$$

$$\frac{\partial^2 L}{\partial u^2}(x, y_h(u), u)v_1 v_2 - \varphi_h(u)\phi''(y_h(u))z_{h1}z_{h2}\Big] \, dx \tag{3.6}$$

where $y_h(u) = G_h(u)$ *and* $\varphi_h(u) \in Y_h$ *is the unique solution of the problem*

$$\begin{aligned} a(q_h, \varphi_h(u)) + \int_\Omega \phi'(y_h(u))\varphi_h(u) q_h \, dx = \\ \int_\Omega \frac{\partial L}{\partial y}(x, y_h(u), u) q_h \, dx \quad \forall q_h \in Y_h, \end{aligned} \tag{3.7}$$

with $z_{hi} = G_h'(u)v_i$, $i = 1, 2$.

We conclude this section by studying the existence of a solution of problem (P_h) and establishing the first order optimality conditions. The second order conditions are analogous to those proved for problem (P) and they can be obtained by the classical methods of finite dimensional optimization.

In the sequel we will denote

$$\mathbb{K}_h = \{u_h \in U_h : \alpha \leq u_h|_T \leq \beta \quad \forall T \in \mathcal{T}_h\}.$$

THEOREM 3.3. *For every* $h > 0$ *problem* (P_h) *has at least one solution. If* $\bar{u}_h$ *is a local minimum of* (P_h), *then there exist* $\bar{y}_h$, $\bar{\varphi}_h \in Y_h$ *such that*

$$a(\bar{y}_h, q_h) + \int_\Omega \phi(\bar{y}_h) q_h(x) \, dx = \int_\Omega \bar{u}_h(x) q_h(x) \, dx \quad \forall q_h \in Y_h, \tag{3.8}$$

$$a(q_h, \bar{\varphi}_h) + \int_\Omega \phi'(\bar{y}_h)\bar{\varphi}_h q_h \, dx = \int_\Omega \frac{\partial L}{\partial y}(x, \bar{y}_h, \bar{u}_h) q_h \, dx \ \forall q_h \in Y_h, \tag{3.9}$$

$$\int_{\Omega_h} \left\{ \bar{\varphi}_h + \frac{\partial L}{\partial u}(x, \bar{y}_h, \bar{u}_h) \right\} (u_h - \bar{u}_h) dx \geq 0 \ \forall u_h \in \mathbb{K}_h. \tag{3.10}$$

PROOF. The existence of a solution is an immediate consequence of the compactness of $\mathbb{K}_h$ in U_h and the continuity of J_h in $\mathbb{K}_h$. The optimality system (3.8)-(3.10) follows from Lemma 2.1 and Proposition 3.2. □

From this theorem we can deduce a representation of the local minima of (P_h) analogous to that obtained in Theorem 2.5.

THEOREM 3.4. *Under the hypotheses* ***(H1)*** *and* ***(H2)****, if* $\bar{u}_h$ *is a local minimum of* (P_h)*, and* $\bar{y}_h$ *and* $\bar{\varphi}_h$ *are the state and adjoint state associated to* $\bar{u}_h$*, then for every* $T \in \mathcal{T}_h$ *the equation*

$$\int_T [\bar{\varphi}_h(x) + \frac{\partial L}{\partial u}(x, \bar{y}_h(x), t)]\, dx = 0, \tag{3.11}$$

has a unique solution $\bar{t} = \bar{s}_T$*. The mapping* $\bar{s}_h \in U_h$*, defined by* $\bar{s}_{h|T} = \bar{s}_T$*, is related with* $\bar{u}_h$ *by the formula*

$$\bar{u}_h(x) = Proj_{[\alpha,\beta]}(\bar{s}_h(x)) = \max(\alpha, \min(\beta, \bar{s}_h(x))). \tag{3.12}$$

PROOF. The existence of a unique solution of (3.11) is a consequence of hypothesis **(H2)**. Let us denote by $\bar{u}_T$ the restriction of $\bar{u}_h$ to T. From the definition of U_h and (3.10) we deduce that

$$\int_T \left\{\bar{\varphi}_h + \frac{\partial L}{\partial u}(x, \bar{y}_h, \bar{u}_T)\right\} dx (t - \bar{u}_T) \geq 0 \quad \forall t \in [\alpha, \beta] \text{ and } \forall T \in \mathcal{T}_h.$$

From here we get

$$\begin{aligned}
\alpha < \bar{u}_T < \beta \quad &\Rightarrow \int_T \left\{\bar{\varphi}_h + \frac{\partial L}{\partial u}(x, \bar{y}_h, \bar{u}_T)\right\} dx = 0 \;\Rightarrow\; \bar{u}_T = \bar{s}_T, \\
\bar{u}_T = \beta \quad &\Rightarrow \int_T \left\{\bar{\varphi}_h + \frac{\partial L}{\partial u}(x, \bar{y}_h, \bar{u}_T)\right\} dx \leq 0 \;\Rightarrow\; \bar{u}_T \leq \bar{s}_T, \\
\bar{u}_T = \alpha \quad &\Rightarrow \int_T \left\{\bar{\varphi}_h + \frac{\partial L}{\partial u}(x, \bar{y}_h, \bar{u}_T)\right\} dx \geq 0 \;\Rightarrow\; \bar{u}_T \geq \bar{s}_T,
\end{aligned}$$

which implies (3.12). $\square$

3.2. Convergence of the Approximations

In this section we will prove that the solutions of the discrete problems (P_h) converge strongly in $L^\infty(\Omega_h)$ to solutions of problem (P). We will also prove that the strict local minima of problem (P) can be approximated by local minima of problems (P_h). In order to prove these convergence results we will use two lemmas whose proofs can be found in [**2**] and [**23**].

LEMMA 3.5. *Let* $(v, v_h) \in L^\infty(\Omega) \times U_h$ *satisfy* $\|v\|_{L^\infty(\Omega)} \leq M$ *and* $\|v_h\|_{L^\infty(\Omega_h)} \leq M$*. Let us assume that* y_v *and* $y_h(v_h)$ *are the solutions of* (1.1) *and* (3.2) *corresponding to* v *and* v_h *respectively. Moreover let* φ_v

and $\varphi_h(v_h)$ *be the solutions of* (2.6) *and* (3.7) *corresponding to* v *and* v_h *respectively. Then the following estimates hold*

$$\|y_v - y_h(v_h)\|_{H^1(\Omega)} + \|\varphi_v - \varphi_h(v_h)\|_{H^1(\Omega)} \leq C(h + \|v - v_h\|_{L^2(\Omega_h)}), \tag{3.13}$$

$$\|y_v - y_h(v_h)\|_{L^2(\Omega)} + \|\varphi_v - \varphi_h(v_h)\|_{L^2(\Omega)} \leq C(h^2 + \|v - v_h\|_{L^2(\Omega_h)}), \tag{3.14}$$

$$\|y_v - y_h(v_h)\|_{L^\infty(\Omega)} + \|\varphi_v - \varphi_h(v_h)\|_{L^\infty(\Omega)} \leq C(h + \|v - v_h\|_{L^2(\Omega_h)}), \tag{3.15}$$

where $C \equiv C(\Omega, n, M)$ *is a positive constant independent of* h.

Estimate (3.15) was not proved in [**2**], but it follows from [**2**] and the uniform error estimates for the discretization of linear elliptic equations; see for instance Schatz [**43**, Estimate (0.5)] and the references therein.

Lemma 3.6. *Let* $\{u_h\}_{h>0}$ *be a sequence, with* $u_h \in \mathbb{K}_h$ *and* $u_h \rightharpoonup u$ *weakly in* $L^1(\Omega)$*, then* $y_h(u_h) \to y_u$ *and* $\varphi_h(u_h) \to \varphi_u$ *in* $H_0^1(\Omega) \cap C(\bar{\Omega})$ *strongly. Moreover* $J(u) \leq \liminf_{h\to 0} J_h(u_h)$.

Let us remark that u_h is only defined in Ω_h, then we need to precise what $u_h \rightharpoonup u$ weakly in $L^1(\Omega)$ means. It means that

$$\int_{\Omega_h} \psi u_h \, dx \to \int_\Omega \psi u \, dx \quad \forall \psi \in L^\infty(\Omega).$$

Since the measure of $\Omega \setminus \Omega_h$ tends to zero when $h \to 0$, then the above property is equivalent to

$$\int_{\Omega} \psi \tilde{u}_h \, dx \to \int_\Omega \psi u \, dx \quad \forall \psi \in L^\infty(\Omega)$$

for any uniformly bounded extension $\tilde{u}_h$ of u_h to Ω. Analogously we can define the weak* convergence in $L^\infty(\Omega)$.

Theorem 3.7. *Let us assume that* ***(H1)*** *and* ***(H2)*** *hold. For every* $h > 0$ *let* $\bar{u}_h$ *be a solution of* (P_h)*. Then there exist subsequences of* $\{\bar{u}_h\}_{h>0}$ *converging in the weak* topology of* $L^\infty(\Omega)$*, that will be denoted in the same form. If* $\bar{u}_h \rightharpoonup \bar{u}$ *in the mentioned topology, then* $\bar{u}$ *is a solution of (P) and the following identities hold*

$$\lim_{h\to 0} J_h(\bar{u}_h) = J(\bar{u}) = \inf(P) \quad \textit{and} \quad \lim_{h\to 0} \|\bar{u} - \bar{u}_h\|_{L^\infty(\Omega_h)} = 0. \tag{3.16}$$

Proof. The existence of subsequences converging in the weak* topology of $L^\infty(\Omega)$ is a consequence of the boundedness of $\{\bar{u}_h\}_{h>0}$, $\alpha \leq \bar{u}_h(x) \leq \beta$ for every $h > 0$ and $x \in \Omega_h$. Let $\bar{u}$ be a limit point of one of these converging subsequences and prove that $\bar{u}$ is a solution of (P). Let $\tilde{u}$ be a solution of (P). From Theorem 2.5 we deduce that $\tilde{u}$ is

Lipschitz in $\bar{\Omega}$. Let us consider the operator $\Pi_h : L^1(\Omega) \longrightarrow U_h$ defined by

$$\Pi_h u|_T = \frac{1}{m(T)} \int_T u(x)\,dx \quad \forall T \in \mathcal{T}_h.$$

Let $u_h = \Pi_h \tilde{u} \in U_h$, it is easy to prove that

$$\|\tilde{u} - u_h\|_{L^\infty(\Omega)} \leq \Lambda_{\tilde{u}} h,$$

where $\Lambda_{\tilde{u}}$ is the Lipschitz constant of $\tilde{u}$. By applying Lemmas 3.5 and 3.6 we get

$$J(\bar{u}) \leq \liminf_{h\to 0} J_h(\bar{u}_h) \leq \limsup_{h\to 0} J_h(\bar{u}_h) \leq$$

$$\leq \limsup_{h\to 0} J_h(u_h) = J(\tilde{u}) = \inf\,(\mathrm{P}) \leq J(\bar{u}),$$

which proves that $\bar{u}$ is a solution of (P) and

$$\lim_{h\to 0} J_h(\bar{u}_h) = J(\bar{u}) = \inf(P).$$

Let us prove now the uniform convergence $\bar{u}_h \to \bar{u}$. From (2.11) and (3.12) follows

$$\|\bar{u} - \bar{u}_h\|_{L^\infty(\Omega_h)} \leq \|\bar{s} - \bar{s}_h\|_{L^\infty(\Omega_h)},$$

therefore it is enough to prove the uniform convergence of $\{\bar{s}_h\}_{h>0}$ to $\bar{s}$. On the other hand, from (3.11) we have that

$$\int_T [\bar{\varphi}_h(x) + \frac{\partial L}{\partial u}(x, \bar{y}_h(x), \bar{s}_{h|T})]\,dx = 0.$$

From this equality and the continuity of the integrand with respect to x it follows the existence of a point $\xi_T \in T$ such that

$$\bar{\varphi}_h(\xi_T) + \frac{\partial L}{\partial u}(\xi_T, \bar{y}_h(\xi_T), \bar{s}_h(\xi_T)) = 0. \tag{3.17}$$

Given $x \in \Omega_h$, let $T \in \mathcal{T}_h$ be such that $x \in T$. Since $\bar{s}_h$ is constant in each element T

$$|\bar{s}(x) - \bar{s}_h(x)| \leq |\bar{s}(x) - \bar{s}(\xi_T)| + |\bar{s}(\xi_T) - \bar{s}_h(\xi_T)| \leq$$

$$\Lambda_{\bar{s}}|x - \xi_T| + |\bar{s}(\xi_T) - \bar{s}_h(\xi_T)| \leq \Lambda_{\bar{s}} h + |\bar{s}(\xi_T) - \bar{s}_h(\xi_T)|,$$

where $\Lambda_{\bar{s}}$ is the Lipschitz constant of $\bar{s}$. Thus it remains to prove the convergence $\bar{s}_h(\xi_T) \to \bar{s}(\xi_T)$ for every T. For it we will use again the strict positivity of the second derivative of L with respect to u (Hypothesis **(H2)**) along with (3.17) and the fact that $\bar{s}(x)$ is the solution of the equation (2.10) to get

$$\Lambda|\bar{s}(\xi_T) - \bar{s}_h(\xi_T)| \leq \left|\frac{\partial L}{\partial u}(\xi_T, \bar{y}_h(\xi_T), \bar{s}(\xi_T)) - \frac{\partial L}{\partial u}(\xi_T, \bar{y}_h(\xi_T), \bar{s}_h(\xi_T))\right| \leq$$

$$\left|\frac{\partial L}{\partial u}(\xi_T, \bar{y}_h(\xi_T), \bar{s}(\xi_T)) - \frac{\partial L}{\partial u}(\xi_T, \bar{y}(\xi_T), \bar{s}(\xi_T))\right| +$$
$$\left|\frac{\partial L}{\partial u}(\xi_T, \bar{y}(\xi_T), \bar{s}(\xi_T)) - \frac{\partial L}{\partial u}(\xi_T, \bar{y}_h(\xi_T), \bar{s}_h(\xi_T))\right| =$$
$$\left|\frac{\partial L}{\partial u}(\xi_T, \bar{y}_h(\xi_T), \bar{s}(\xi_T)) - \frac{\partial L}{\partial u}(\xi_T, \bar{y}(\xi_T), \bar{s}(\xi_T))\right| + |\bar{\varphi}(\xi_T) - \bar{\varphi}_h(\xi_T)| \to 0$$

thanks to the uniform convergence $\bar{y}_h \to \bar{y}$ and $\bar{\varphi}_h \to \bar{\varphi}$ (Lemma 3.6). □

In a certain way, next result is the reciprocal one to the previous theorem. The question we formulate now is wether a local minimum u of (P) can be approximated by a local minimum u_h of (P$_h$). The answer is positive if the local minimum u is strict. In the sequel, $B_\rho(u)$ will denote the open ball of $L^\infty(\Omega)$ with center at u and radius ρ. $\bar{B}_\rho(u)$ will denote the corresponding closed ball.

THEOREM 3.8. *Let us assume that **(H1)** and **(H2)** hold. Let $\bar{u}$ be a strict local minimum of (P). Then there exist $\varepsilon > 0$ and $h_0 > 0$ such that (P$_h$) has a local minimum $\bar{u}_h \in B_\varepsilon(\bar{u})$ for every $h < h_0$. Moreover the convergences* (3.16) *hold.*

PROOF. Let $\varepsilon > 0$ be such that $\bar{u}$ is the unique solution of problem

$$(\mathrm{P}_\varepsilon)\begin{cases} \min J(u) \\ u \in \mathbb{K} \cap \bar{B}_\varepsilon(\bar{u}). \end{cases}$$

Let us consider the problems

$$(\mathrm{P}_{h\varepsilon})\begin{cases} \min J_h(u_h) \\ u_h \in \mathbb{K}_h \cap \bar{B}_\varepsilon(\bar{u}). \end{cases}$$

Let $\Pi_h : L^1(\Omega) \longrightarrow U_h$ be the operator introduced in the proof of the previous theorem. It is obvious that $\Pi_h\bar{u} \in \mathbb{K}_h \cap \bar{B}_\varepsilon(\bar{u})$ for every h small enough. Therefore $\mathbb{K}_h \cap \bar{B}_\varepsilon(\bar{u})$ is non empty and consequently (P$_{h\varepsilon}$) has at least one solution $\bar{u}_h$. Now we can argue as in the proof of Theorem 3.7 to conclude that $\|\bar{u}_h - \bar{u}\|_{L^\infty(\Omega_h)} \to 0$, therefore $\bar{u}_h$ is local solution of (P$_h$) in the open ball $B_\varepsilon(\bar{u})$ as desired. □

3.3. Error Estimates

In this section we will assume that **(H1)** and **(H2)** hold and $\bar{u}$ will denote a local minimum of (P) satisfying the sufficient second order condition for optimality (2.15) or equivalently (2.22). $\{\bar{u}_h\}_{h>0}$ will denote a sequence of local minima of problems (P$_h$) such that $\|\bar{u} - \bar{u}_h\|_{L^\infty(\Omega_h)} \to 0$; remind Theorems 3.7 and 3.8. The goal of this section is to estimate the error $\bar{u} - \bar{u}_h$ in the norms of $L^2(\Omega_h)$ and $L^\infty(\Omega_h)$ respectively. For it we are going to prove three previous lemmas.

For convenience, in this section we will extend $\bar{u}_h$ to Ω by taking $\bar{u}_h(x) = \bar{u}(x)$ for every $x \in \Omega$.

LEMMA 3.9. *Let $\delta > 0$ be as in Theorem 2.12. Then there exists $h_0 > 0$ such that*

$$\frac{\delta}{2}\|\bar{u} - \bar{u}_h\|^2_{L^2(\Omega_h)} \leq (J'(\bar{u}_h) - J'(\bar{u}))(\bar{u}_h - \bar{u}) \quad \forall h < h_0. \tag{3.18}$$

PROOF. Let us set

$$\bar{d}_h(x) = \frac{\partial L}{\partial u}(x, \bar{y}_h(x), \bar{u}_h(x)) + \bar{\varphi}_h(x)$$

and take $\delta > 0$ and $\tau > 0$ as in Theorem 2.12. We know that $\bar{d}_h$ converge uniformly to $\bar{d}$ in Ω, therefore there exists $h_\tau > 0$ such that

$$\|\bar{d} - \bar{d}_h\|_{L^\infty(\Omega)} < \frac{\tau}{4} \quad \forall h \leq h_\tau. \tag{3.19}$$

For every $T \in \mathcal{T}_h$ we define

$$I_T = \int_T \bar{d}_h(x)\, dx.$$

From (3.10) follows

$$\bar{u}_h \,|_T = \begin{cases} \alpha & \text{if } \ I_T > 0 \\ \beta & \text{if } \ I_T < 0. \end{cases}$$

Let us take $0 < h_1 \leq h_\tau$ such that

$$|\bar{d}(x_2) - \bar{d}(x_1)| < \frac{\tau}{4} \ \text{ if } \ |x_2 - x_1| < h_1.$$

This inequality along with (3.19) imply that

$$\text{if } \ \xi \in T \text{ and } \bar{d}(\xi) > \tau \Rightarrow \bar{d}_h(x) > \frac{\tau}{2} \quad \forall x \in T, \quad \forall T \in \hat{\mathcal{T}}_h, \quad \forall h < h_1,$$

hence $I_T > 0$, therefore $\bar{u}_h \,|_T = \alpha$, in particular $\bar{u}_h(\xi) = \alpha$. From (2.12) we also have $\bar{u}(\xi) = \alpha$. Then $(\bar{u}_h - \bar{u})(\xi) = 0$ whenever $\bar{d}(\xi) > \tau$ and $h < h_1$. We can prove the analogous result when $\bar{d}(\xi) < -\tau$. On the other hand, since $\alpha \leq \bar{u}_h(x) \leq \beta$, it is obvious $(\bar{u}_h - \bar{u})(x) \geq 0$ if $\bar{u}(x) = \alpha$ and $(\bar{u}_h - \bar{u})(x) \leq 0$ if $\bar{u}(x) = \beta$. Thus we have proved that $(\bar{u}_h - \bar{u}) \in C^\tau_{\bar{u}}$, remember that $\bar{u} = \bar{u}_h$ in $\Omega \setminus \Omega_h$. Then (2.22) leads to

$$J''(\bar{u})(\bar{u}_h - \bar{u})^2 \geq \delta\|\bar{u}_h - \bar{u}\|^2_{L^2(\Omega)} = \delta\|\bar{u}_h - \bar{u}\|^2_{L^2(\Omega_h)} \quad \forall h < h_1. \tag{3.20}$$

On the other hand, by applying the mean value theorem, we get for some $0 < \theta_h < 1$ that

$$(J'(\bar{u}_h) - J'(\bar{u}))(\bar{u}_h - \bar{u}) = J''(\bar{u} + \theta_h(\bar{u}_h - \bar{u}))(\bar{u}_h - \bar{u})^2 \geq$$

$$(J''(\bar{u} + \theta_h(\bar{u}_h - \bar{u})) - J''(\bar{u}))(\bar{u}_h - \bar{u})^2 + J''(\bar{u})(\bar{u}_h - \bar{u})^2 \geq$$

$$(\delta - \|J''(\bar{u} + \theta_h(\bar{u}_h - \bar{u})) - J''(\bar{u})\|)\,\|\bar{u}_h - \bar{u}\|^2_{L^2(\Omega)}.$$

Finally it is enough to choose $0 < h_0 \leq h_1$ such that

$$\|J''(\bar{u} + \theta_h(\bar{u}_h - \bar{u})) - J''(\bar{u})\| \leq \frac{\delta}{2} \quad \forall h < h_0$$

to deduce (3.18). The last inequality can be obtained easily from the relationship (2.5) thanks to the uniform convergence $(\bar{\varphi}_h, \bar{y}_h, \bar{u}_h) \to (\bar{\varphi}, \bar{y}, \bar{u})$ and hypothesis **(H1)**. □

The next step consists in estimating the convergence of J'_h to J'.

LEMMA 3.10. *There exists a constant $C > 0$ independent of h such that for every u_1, $u_2 \in \mathbb{K}$ and every $v \in L^2(\Omega)$ the following inequalities are fulfilled*

$$|(J'_h(u_2) - J'(u_1))v| \leq C\left\{h + \|u_2 - u_1\|_{L^2(\Omega)}\right\}\|v\|_{L^2(\Omega)}. \tag{3.21}$$

PROOF. By using the expression of the derivatives given by (2.4) and (3.5) along with the inequality (3.1) we get

$$|(J'_h(u_2) - J'(u_1))v| \leq \int_{\Omega\setminus\Omega_h}\left|\frac{\partial L}{\partial u}(x, y_{u_1}, u_1) + \varphi_{u_1}\right||v|\,dx \leq$$

$$\int_{\Omega_h}\left|\left(\frac{\partial L}{\partial u}(x, y_h(u_2), u_2) + \varphi_h(u_2)\right) - \left(\frac{\partial L}{\partial u}(x, y_{u_1}, u_1) + \varphi_{u_1}\right)\right||v|\,dx \leq$$

$$C\left\{h + \|\varphi_h(u_2) - \varphi_{u_1}\|_{L^2(\Omega)} + \|y_h(u_2) - y_{u_1}\|_{L^2(\Omega)}\right\}\|v\|_{L^2(\Omega)}.$$

Now (3.21) follows from the previous inequality and (3.14). □

A key point in the derivation of the error estimate is to get a good approximate of $\bar{u}$ by a discrete control $u_h \in \mathbb{K}_h$ satisfying $J'(\bar{u})\bar{u} = J'(\bar{u})u_h$. Let us define this control u_h and prove that it fulfills the required conditions. For every $T \in \mathcal{T}_h$ let us set

$$I_T = \int_T \bar{d}(x)\,dx.$$

We define $u_h \in U_h$ with $u_{h|T} = u_{hT}$ for every $T \in \mathcal{T}_h$ given by the expression

$$u_{hT} = \begin{cases} \dfrac{1}{I_T}\displaystyle\int_T \bar{d}(x)\bar{u}(x)\,dx & \text{si} \quad I_T \neq 0 \\[2ex] \dfrac{1}{m(T)}\displaystyle\int_T \bar{u}(x)\,dx & \text{si} \quad I_T = 0. \end{cases} \tag{3.22}$$

We extend this function to Ω by taking $u_h(x) = \bar{u}(x)$ for every $x \in \Omega \setminus \Omega_h$. This function u_h satisfies our requirements.

Lemma 3.11. *There exists $h_0 > 0$ such that for every $0 < h < h_0$ the following properties hold*

(1) $u_h \in \mathbb{K}_h$.
(2) $J'(\bar{u})\bar{u} = J'(\bar{u})u_h$.
(3) *There exists $C > 0$ independent of h such that*

$$\|\bar{u} - u_h\|_{L^\infty(\Omega_h)} \leq Ch. \tag{3.23}$$

Proof. Let $\Lambda_{\bar{u}} > 0$ be the Lipschitz constant of $\bar{u}$ and let us take $h_0 = (\beta - \alpha)/(2\Lambda_{\bar{u}})$, then for every $T \in \mathcal{T}_h$ and every $h < h_0$

$$|\bar{u}(\xi_2) - \bar{u}(\xi_1)| \leq \Lambda_{\bar{u}}|\xi_2 - \xi_1| \leq \Lambda_{\bar{u}}h < \frac{\beta - \alpha}{2} \qquad \forall \xi_1, \xi_2 \in T$$

which implies that $\bar{u}$ cannot take the values α and β in a same element T for any $h < h_0$. Therefore the sign of $\bar{d}$ in T must be constant thanks to (2.12). Hence $I_T = 0$ if and only if $\bar{d}(x) = 0$ for all $x \in T$. Moreover if $I_T \neq 0$, then $\bar{d}(x)/I_T \geq 0$ for every $x \in T$. As a first consequence of this we get that $\alpha \leq u_{hT} \leq \beta$, which means that $u_h \in \mathbb{K}_h$. On the other hand

$$\begin{aligned} J'(\bar{u})u_h &= \int_{\Omega\setminus\Omega_h} \bar{d}(x)\bar{u}_h(x)\,dx + \sum_{T\in\mathcal{T}_h}\left(\int_T \bar{d}(x)\,dx\right)u_{hT} \\ &= \int_{\Omega\setminus\Omega_h} \bar{d}(x)\bar{u}(x)\,dx + \sum_{T\in\mathcal{T}_h}\int_T \bar{d}(x)\bar{u}(x)\,dx = J'(\bar{u})\bar{u}. \end{aligned}$$

Finally let us prove (3.23). Since the sign of $\bar{d}(x)/I_T$ is always non negative and $\bar{d}$ is a continuous function, we get for any of the two possible definitions of u_{hT} the existence of a point $\xi^j \in T$ such that $u_{hT} = \bar{u}(\xi_j)$. Hence for all $x \in T$

$$|\bar{u}(x) - u_h(x)| = |\bar{u}(x) - u_{hT}| = |\bar{u}(x) - \bar{u}(\xi^j)| \leq \Lambda_{\bar{u}}|x - \xi^j| \leq \Lambda_{\bar{u}}h,$$

which proves (3.23). $\square$

Finally we get the error estimates.

THEOREM 3.12. *There exists a constant $C > 0$ independent of h such that*

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} \leq Ch. \tag{3.24}$$

PROOF. Taking $u = \bar{u}_h$ in (2.9) we get

$$J'(\bar{u})(\bar{u}_h - \bar{u}) = \int_\Omega \left(\bar{\varphi} + \frac{\partial L}{\partial u}(x, \bar{y}, \bar{u})\right)(\bar{u}_h - \bar{u})\, dx \geq 0. \tag{3.25}$$

From (3.10) with u_h defined by (3.22) it follows

$$J_h'(\bar{u}_h)(u_h - \bar{u}_h) = \int_\Omega \left(\bar{\varphi}_h + \frac{\partial L}{\partial u}(x, \bar{y}_h, \bar{u}_h)\right)(u_h - \bar{u}_h)\, dx \geq 0,$$

then

$$J_h'(\bar{u}_h)(\bar{u} - \bar{u}_h) + J_h'(\bar{u}_h)(u_h - \bar{u}) \geq 0. \tag{3.26}$$

Adding (3.25) and (3.26) and using Lemma 3.11-2, we deduce

$$\left(J'(\bar{u}) - J_h'(\bar{u}_h)\right)(\bar{u} - \bar{u}_h) \leq J_h'(\bar{u}_h)(u_h - \bar{u}) = \left(J_h'(\bar{u}_h) - J'(\bar{u})\right)(u_h - \bar{u}).$$

For h small enough, this inequality along with (3.18) imply

$$\frac{\delta}{2}\|\bar{u} - \bar{u}_h\|^2_{L^2(\Omega)} \leq \left(J'(\bar{u}) - J'(\bar{u}_h)\right)(\bar{u} - \bar{u}_h) \leq$$
$$\left(J_h'(\bar{u}_h) - J'(\bar{u}_h)\right)(\bar{u} - \bar{u}_h) + \left(J'(\bar{u}_h) - J'(\bar{u})\right)(u_h - \bar{u}).$$

Using (3.21) with $u_2 = u_1 = \bar{u}_h$ and $v = \bar{u} - \bar{u}_h$ in the first addend of the previous line and th expression of J' given by (2.4) along with (3.14) for $v = \bar{u}$ and $v_h = \bar{u}_h$, in the second addend, it comes

$$\frac{\delta}{2}\|\bar{u} - \bar{u}_h\|^2_{L^2(\Omega)} \leq C_1 h\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} +$$
$$C_2\left(h^2 + \|\bar{u} - \bar{u}_h\|_{L^2(\Omega)}\right)\|\bar{u} - u_h\|_{L^2(\Omega)}.$$

From (3.23) and by using Young's inequality in the above inequality we deduce

$$\frac{\delta}{4}\|\bar{u} - \bar{u}_h\|^2_{L^2(\Omega_h)} = \frac{\delta}{4}\|\bar{u} - \bar{u}_h\|^2_{L^2(\Omega)} \leq C_3 h^2,$$

which implies (3.24). □

Finally let us prove the error estimates in $L^\infty(\Omega)$.

THEOREM 3.13. *There exists a constant $C > 0$ independent of h such that*

$$\|\bar{u} - \bar{u}_h\|_{L^\infty(\Omega_h)} \leq Ch. \tag{3.27}$$

PROOF. Let ξ_T be defined by (3.17). In the proof of Theorem 3.3 we obtained

$$\|\bar u - \bar u_h\|_{L^\infty(\Omega_h)} \le \|\bar s - \bar s_h\|_{L^\infty(\Omega_h)} \le \Lambda_{\bar s} h +$$
$$\max_{T\in\mathcal{T}_h}\left|\frac{\partial L}{\partial u}(\xi_T, \bar y_h(\xi_T), \bar s(\xi_T)) - \frac{\partial L}{\partial u}(\xi_T, \bar y(\xi_T), \bar s(\xi_T))\right| + |\bar\varphi(\xi_T) - \bar\varphi_h(\xi_T)|.$$

Using the hypothesis **(H1)**, (3.15) and (3.24) we get

$$\|\bar u - \bar u_h\|_{L^\infty(\Omega_h)} \le \Lambda_{\bar s} h + C(\|\bar y - \bar y_h\|_{L^\infty(\Omega)} + \|\bar\varphi - \bar\varphi_h\|_{L^\infty(\Omega)}) \le$$
$$\Lambda_{\bar s} h + C(h + \|\bar u - \bar u_h\|_{L^2(\Omega_h)}) \le Ch.$$

□

REMARK 3.14. Error estimates for problems with pointwise state constraints is an open problem. The reader is referred to Deckelnick and Hinze [**32**] for the linear quadratic case, when one side pointwise state constraints and no control constraints. The case of integral state constraints has been studied by Casas [**17**].

In Casas [**18**], the approximation of the control problem was done by using piecewise linear continuous functions. For these approximations the error estimate can be improved.

The case of Neumann boundary controls has been studied by Casas, Mateos and Tröltzsch [**26**] and Casas and Mateos [**24**]. Casas and Raymond considered the case of Dirichlet controls [**27**].

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Error Estimates for the Numerical Approximation of Boundary Semilinear Elliptic Control Problems*

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Abstract. We study the numerical approximation of boundary optimal control problems governed by semilinear elliptic partial differential equations with pointwise constraints on the control. The analysis of the approximate control problems is carried out. The uniform convergence of discretized controls to optimal controls is proven under natural assumptions by taking piecewise constant controls. Finally, error estimates are established and some numerical experiments, which confirm the theoretical results, are performed.

Keywords: boundary control, semilinear elliptic equation, numerical approximation, error estimates

AMS Subject Classification: 49J20, 49K20, 49M05, 65K10

1. Introduction

With this paper, we continue the discussion of error estimates for the numerical approximation of optimal control problems we have started for semilinear elliptic equations and distributed controls in [1]. The case of distributed control is the easiest one with respect to the mathematical analysis. In [1] it was shown that, roughly speaking, the distance between a locally optimal control $\bar{u}$ and its numerical approximation $\bar{u}_h$ has the order of the mesh size h in the L^2-norm and in the L^∞-norm. This estimate holds for a finite element approximation of the equation by standard piecewise linear elements and piecewise constant control functions.

The analysis for boundary controls is more difficult, since the regularity of the state function is lower than that for distributed controls. Moreover, the internal approximation of the domain causes problems. In the general case, we have to approximate the boundary by a

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polygon. This requires the comparison of the original control that is located at the boundary Γ and the approximate control that is defined on the polygonal boundary Γ_h. Moreover, the regularity of elliptic equations in domains with corners needs special care. To simplify the analysis, we assume here that Ω is a polygonal domain of $\mathbb{R}^2$. Though this makes the things easier, the lower regularity of states in polygonal domains complicates, together with the presence of nonlinearities, the analysis.

Another novelty of our paper is the numerical confirmation of the predicted error estimates. We present two examples, where we know the exact solutions. The first one is of linear-quadratic type, while the second one is semilinear. We are able to verify our error estimates quite precisely.

Let us mention some further papers related to this subject. The case of linear-quadratic elliptic control problems approximated by finite elements was discussed in early papers by Falk [11], Geveci [12] and Malanowski [25], and Arnautu and Neittaanmäki [2], who already proved the optimal error estimate of order h in the L^2-norm. In [25], also the case of piecewise linear control functions is addressed. For some recent research in the case of linear quadratic control problems, the reader is referred to Hinze [17] and Meyer and Rösch [27].

In the paper [8], the case of linear-quadratic elliptic problems was investigated again from a slightly different point of view: It was assumed that only the control is approximated while considering the elliptic equation as exactly solvable. Here, all main variants of elliptic problems have been studied—distributed control, boundary control, distributed observation and boundary observation. Moreover, the case of piecewise linear control functions was studied in domains of dimension 2. Finally, we refer to [7], where error estimates were derived for elliptic problems with integral state constraints.

There is an extensive literature on error estimates for the numerical approximation of optimal control problems for ordinary differential equations and an associated abstract theory of stability analysis. We mention only Hager [14], Dontchev and Hager [9], Dontchev et al. [10] and Malanowski et al. [26]. We also refer to the detailed bibliography in [10] and to the nicely written short survey given by Hager in [15]. One way to perform the error analysis is to apply ideas from this abstract theory to the case of PDEs. In our former paper [1] we have partially done this by adopting a well known perturbation trick that permits to derive optimal error estimates.

Here, we apply a new, quite elegant and completely different technique that essentially shortens the presentation. It does not rely on the available abstract perturbation analysis.

2. The control problem

Throughout the sequel, Ω denotes an open convex bounded polygonal set of $\mathbb{R}^2$ and Γ is the boundary of Ω. In this domain we formulate the following control problem

$$(\mathrm{P})\quad \begin{cases} \inf J(u) = \displaystyle\int_\Omega L(x, y_u(x))\,dx + \int_\Gamma l(x, y_u(x), u(x))\,d\sigma(x) \\ \text{subject to } (y_u, u) \in H^1(\Omega) \times L^\infty(\Gamma), \\ \qquad u \in U^{ad} = \{u \in L^\infty(\Gamma) \mid \alpha \le u(x) \le \beta \text{ a.e. } x \in \Gamma\}, \\ \qquad (y_u, u) \text{ satisfying the state equation (2.1)} \end{cases}$$

$$\begin{cases} -\triangle y_u(x) = a_0(x, y_u(x)) & \text{in } \Omega \\ \partial_\nu y_u(x) = b_0(x, y_u(x)) + u(x) & \text{on } \Gamma, \end{cases} \tag{2.1}$$

where $-\infty < \alpha < \beta < +\infty$. Here u is the control while y_u is said to be the associated state. The following hypotheses are assumed about the functions involved in the control problem (P):

(A1) The function $L : \Omega \times \mathbb{R} \to \mathbb{R}$ is measurable with respect to the first component, of class C^2 with respect to the second, $L(\cdot, 0) \in L^1(\Omega)$ and for all $M > 0$ there exist a function $\psi_{L,M} \in L^p(\Omega)$ $(p > 2)$ and a constant $C_{L,M} > 0$ such that

$$\left|\frac{\partial L}{\partial y}(x, y)\right| \le \psi_{L,M}(x), \quad \left|\frac{\partial^2 L}{\partial y^2}(x, y)\right| \le C_{L,M},$$
$$\left|\frac{\partial^2 L}{\partial y^2}(x, y_2) - \frac{\partial^2 L}{\partial y^2}(x, y_1)\right| \le C_{L,M}|y_2 - y_1|,$$

for a.e. $x, x_i \in \Omega$ and $|y|, |y_i| \le M$, $i = 1, 2$.

(A2) The function $l : \Gamma \times \mathbb{R}^2 \to \mathbb{R}$ is measurable with respect to the first component, of class C^2 with respect to the second and third variables, $l(x, 0, 0) \in L^1(\Gamma)$ and for all $M > 0$ there exists a constant $C_{l,M} > 0$ and a function $\psi_{l,M} \in L^p(\Gamma)$ $(p > 1)$ such that

$$\left|\frac{\partial l}{\partial y}(x, y, u)\right| \le \psi_{l,M}(x), \quad \left\|D^2_{(y,u)} l(x, y, u)\right\| \le C_{l,M},$$
$$\left|\frac{\partial l}{\partial u}(x_2, y, u) - \frac{\partial l}{\partial u}(x_1, y, u)\right| \le C_{l,M}|x_2 - x_1|,$$
$$\left\|D^2_{(y,u)} l(x, y_2, u_2) - D^2_{(y,u)} l(x, y_1, u_1)\right\| \le C_{l,M}(|y_2 - y_1| + |u_2 - u_1|),$$

for a.e. $x, x_i \in \Gamma$ and $|y|, |y_i|, |u|, |u_i| \le M$, $i = 1, 2$, where $D^2_{(y,u)} l$ denotes the second derivative of l with respect to (y, u). Moreover we assume that there exists $m_l > 0$ such that

$$\frac{\partial^2 l}{\partial u^2}(x, y, u) \ge m_l, \quad \text{a.e. } x \in \Gamma \text{ and } (y, u) \in \mathbb{R}^2.$$

Let us remark that this inequality implies the strict convexity of l with respect to the third variable.

(A3) The function $a_0 : \Omega \times \mathbb{R} \to \mathbb{R}$ is measurable with respect to the first variable and of class C^2 with respect to the second,

$$a_0(\cdot, 0) \in L^p(\Omega)\ (p > 2), \quad \frac{\partial a_0}{\partial y}(x, y) \le 0 \quad \text{a.e. } x \in \Omega \text{ and } y \in \mathbb{R}$$

and for all $M > 0$ there exists a constant $C_{a_0,M} > 0$ such that

$$\left|\frac{\partial a_0}{\partial y}(x,y)\right| + \left|\frac{\partial^2 a_0}{\partial y^2}(x,y)\right| \le C_{a_0,M} \quad \text{a.e. } x \in \Omega \text{ and } |y| \le M,$$
$$\left|\frac{\partial^2 a_0}{\partial y^2}(x,y_2) - \frac{\partial^2 a_0}{\partial y^2}(x,y_1)\right| < C_{a_0,M}|y_2 - y_1|$$
$$\text{a.e. } x \in \Omega \text{ and } |y_1|, |y_2| \le M.$$

(A4) The function $b_0 : \Gamma \times \mathbb{R} \to \mathbb{R}$ is Lipschitz with respect to the first variable and of class C^2 with respect to the second, $b_0(\cdot, 0) \in W^{1-1/p,p}(\Gamma)$, with $p > 2$,

$$\frac{\partial b_0}{\partial y}(x,y) \le 0$$

and for all $M > 0$ there exists a constant $C_{b_0,M} > 0$ such that

$$\left|\frac{\partial b_0}{\partial y}(x,y)\right| + \left|\frac{\partial^2 b_0}{\partial y^2}(x,y)\right| \le C_{b_0,M},$$
$$\left|\frac{\partial^2 b_0}{\partial y^2}(x,y_2) - \frac{\partial^2 b_0}{\partial y^2}(x,y_1)\right| \le C_{b_0,M}|y_2 - y_1|.$$

for all $x \in \Gamma$ and $|y|, |y_1|, |y_2| \le M$.

(A5) At least one of the two conditions must hold: either $(\partial a_0/\partial y)(x,y) < 0$ in $E_\Omega \times \mathbb{R}$ with $E_\Omega \subset \Omega$ of positive n-dimensional measure or $(\partial b_0/\partial y)(x,y) < 0$ on $E_\Gamma \times \mathbb{R}$ with $E_\Gamma \subset \Gamma$ of positive $(n-1)$-dimensional measure.

Before finishing this section let us study the state equation (2.1).

Theorem 2.1. *For every $u \in L^2(\Gamma)$ the state equation* (2.1) *has a unique solution $y_u \in H^{3/2}(\Omega)$, that depends continuously on u. Moreover, there exists $p_0 > 2$ depending on the measure of the angles in Γ such that $u \in W^{1-1/p,p}(\Gamma)$ with some $2 \le p \le p_0$ implies $y_u \in W^{2,p}(\Omega)$.*

Proof: Due to the Assumptions (A3)–(A5), it is classical to show the existence of a unique solution $y_u \in H^1(\Omega) \cap L^\infty(\Omega)$. From the Assumptions (A3)–(A4) we also deduce that $a_0(\cdot, y_u(\cdot)) \in L^2(\Omega)$ and $u - b_0(\cdot, y_u(\cdot)) \in L^2(\Gamma)$. In this situation Lemma 2.2 below proves that $y_u \in H^{3/2}(\Omega)$.

Let us verify the $W^{2,p}(\Omega)$ regularity. It is known that $H^{3/2}(\Omega) \subset W^{1,4}(\Omega)$; see for instance Grisvard [13]. Therefore, the trace of y_u belongs to the space $W^{1-1/4,4}(\Gamma)$ [13, Theorem 1.5.13]. From the Lipschitz property of b_0 with respect to x and y, we deduce that $b_0(\cdot, y_u(\cdot)) \in W^{1-1/4,4}(\Gamma)$ too. Now Corollary 4.4.3.8 of Grisvard [13] yields the existence of some $p_0 \in (2, 4]$ depending on the measure of the angles in Γ such that $y_u \in W^{2,p}(\Omega)$ for any $2 \le p \le p_0$ provided that $u \in W^{1-1/p,p}(\Gamma)$. We should remind at this point that we have assumed Ω to be convex. $\square$

Lemma 2.2. *Let us assume that $f \in L^2(\Omega)$ and $g \in L^2(\Gamma)$ satisfy that*

$$\int_\Omega f(x)\,dx + \int_\Gamma g(x)\,d\sigma(x) = 0.$$

Then the problem

$$\begin{cases} -\triangle y = f & \text{in } \Omega \\ \partial_\nu y = g & \text{on } \Gamma \end{cases} \tag{2.2}$$

has a solution $y \in H^{3/2}(\Omega)$ that is unique up to an additive constant.

Proof: It is a consequence of Lax-Milgram Theorem's that (2.2) has a unique solution in $H^1(\Omega)$ up to an additive constant. Let us prove the $H^{3/2}(\Omega)$ regularity. To show this we consider the problem

$$\begin{cases} -\triangle y_1 = f & \text{in } \Omega \\ y_1 = 0 & \text{on } \Gamma. \end{cases}$$

Following Jerison and Kenig [19], this problem has a unique solution $y_1 \in H^{3/2}(\Omega)$. Moreover, from $\triangle y_1 \in L^2(\Omega)$ and $y_1 \in H^{3/2}(\Omega)$ we deduce that $\partial_\nu y_1 \in L^2(\Gamma)$; see Kenig [21].

From the equality

$$\int_\Gamma (g - \partial_\nu y_1)\,d\sigma = -\int_\Omega f\,dx - \int_\Gamma \partial_\nu y_1\,d\sigma = -\int_\Omega f\,dx - \int_\Omega \triangle y_1\,dx = 0$$

we deduce the existence of a unique solution $y_2 \in H^1(\Omega)$ of

$$\begin{cases} -\triangle y_2 = 0 & \text{in } \Omega \\ \partial_\nu y_2 = g - \partial_\nu y_1 & \text{on } \Gamma \\ \displaystyle\int_\Omega y_2\,dx = \int_\Omega (y - y_1)\,dx. \end{cases}$$

Once again following Jerison and Kenig [18] we know that $y_2 \in H^{3/2}(\Omega)$. Now it is easy to check that $y = y_1 + y_2 \in H^{3/2}(\Omega)$. □

Let us note that $H^{3/2}(\Omega) \subset C(\bar{\Omega})$ holds for Lipschitz domains in $\mathbb{R}^2$. As a consequence of the theorem above, we know that the functional J is well defined in $L^2(\Gamma)$.

Remark 2.3. It is important to notice that, regarding to the control, the cost functional J and the state equation are convex and linear respectively. These assumptions are crucial to prove the existence of a solution of Problem (P) as well as to establish the convergence of the discretizations. Indeed using the convexity of l with respect to u, we can prove, as in Casas and Mateos [7], the existence of at least one global solution of (P). The reader is also referred to this paper to check the importance of this structure of (P) to carry out the convergence analysis of the discretizations.

Let us discuss the differentiability properties of J.

Theorem 2.4. *Suppose that assumptions* (A3)–(A4) *are satisfied. Then the mapping* $G : L^\infty(\Gamma) \to H^{3/2}(\Omega)$ *defined by* $G(u) = y_u$ *is of class* C^2. *Moreover, for all* $u, v \in L^\infty(\Gamma)$, $z_v = G'(u)v$ *is the solution of*

$$\begin{cases} -\triangle z_v = \dfrac{\partial a_0}{\partial y}(x, y_u)z_v & \text{in } \Omega \\ \partial_\nu z_v = \dfrac{\partial b_0}{\partial y}(x, y_u)z_v + v & \text{on } \Gamma. \end{cases} \tag{2.3}$$

Finally, for every $v_1, v_2 \in L^\infty(\Omega)$, $z_{v_1 v_2} = G''(u)v_1 v_2$ *is the solution of*

$$\begin{cases} -\triangle z_{v_1 v_2} = \dfrac{\partial a_0}{\partial y}(x, y_u)z_{v_1 v_2} + \dfrac{\partial^2 a_0}{\partial y^2}(x, y_u)z_{v_1} z_{v_2} & \text{in } \Omega \\ \partial_\nu z_{v_1 v_2} = \dfrac{\partial b_0}{\partial y}(x, y_u)z_{v_1 v_2} + \dfrac{\partial^2 b_0}{\partial y^2}(x, y_u)z_{v_1} z_{v_2} & \text{on } \Gamma. \end{cases} \tag{2.4}$$

where $z_{v_i} = G'(u)v_i$, $i = 1, 2$.

This theorem is now standard and can be proved by using the implicit function theorem; see Casas and Mateos [6].

Theorem 2.5. *Under the assumptions* (A1)–(A4), *the functional* $J : L^\infty(\Gamma) \to \mathbb{R}$ *is of class* C^2. *Moreover, for every* $u, v, v_i, v_2 \in L^\infty(\Gamma)$

$$J'(u)v = \int_\Gamma \left(\frac{\partial l}{\partial u}(x, y_u, u) + \varphi_u \right) v \, d\sigma \tag{2.5}$$

and

$$\begin{aligned} J''(u)v_1 v_2 = &\int_\Omega \left[\frac{\partial^2 L}{\partial y^2}(x, y_u)z_{v_1} z_{v_2} + \varphi_u \frac{\partial^2 a_0}{\partial y^2}(x, y_u)z_{v_1} z_{v_2} \right] dx \\ &+ \int_\Gamma \left[\frac{\partial^2 l}{\partial y^2}(x, y_u, u)z_{v_1} z_{v_2} + \frac{\partial^2 l}{\partial y \partial u}(x, y_u, u)\big(z_{v_1} v_2 + z_{v_2} v_1\big) \right. \\ &\left. + \frac{\partial^2 l}{\partial u^2}(x, y_u, u)v_1 v_2 + \varphi_u \frac{\partial^2 b_0}{\partial y^2}(x, y_u)z_{v_1} z_{v_2} \right] d\sigma \end{aligned} \tag{2.6}$$

where $z_{v_i} = G'(u)v_i$, $i = 1, 2$, $y_u = G(u)$, *and the adjoint state* $\varphi_u \in H^{3/2}(\Omega)$ *is the unique solution of the problem*

$$\begin{cases} -\triangle\varphi = \dfrac{\partial a_0}{\partial y}(x, y_u)\varphi + \dfrac{\partial L}{\partial y}(x, y_u) & \text{in } \Omega \\ \partial_\nu\varphi = \dfrac{\partial b_0}{\partial y}(x, y_u)\varphi + \dfrac{\partial l}{\partial y}(x, y_u, u) & \text{on } \Gamma. \end{cases} \tag{2.7}$$

This theorem follows from Theorem 2.4 and the chain rule.

3. First and second order optimality conditions

The first order optimality conditions for Problem (P) follow readily from Theorem 2.5.

Theorem 3.1. *Assume that* $\bar{u}$ *is a local solution of Problem (P). Then there exist* $\bar{y}, \bar{\varphi} \in H^{3/2}(\Omega)$ *such that*

$$\begin{cases} -\triangle\bar{y}(x) = a_0(x, \bar{y}(x)) & \textit{in } \Omega \\ \partial_\nu\bar{y}(x) = b_0(x, \bar{y}(x)) + \bar{u}(x) & \textit{on } \Gamma, \end{cases} \tag{3.1}$$

$$\begin{cases} -\triangle\bar{\varphi} = \dfrac{\partial a_0}{\partial y}(x, \bar{y})\bar{\varphi} + \dfrac{\partial L}{\partial y}(x, \bar{y}) & \textit{in } \Omega \\ \partial_\nu\bar{\varphi} = \dfrac{\partial b_0}{\partial y}(x, \bar{y})\bar{\varphi} + \dfrac{\partial l}{\partial y}(x, \bar{y}, \bar{u}) & \textit{on } \Gamma, \end{cases} \tag{3.2}$$

$$\int_\Gamma \left(\frac{\partial l}{\partial u}(x, \bar{y}, \bar{u}) + \bar{\varphi}\right)(u - \bar{u})\, d\sigma \geq 0 \quad \forall u \in U^{ad}. \tag{3.3}$$

If we define

$$\bar{d}(x) = \frac{\partial l}{\partial u}(x, \bar{y}(x), \bar{u}(x)) + \bar{\varphi}(x),$$

then we deduce from (3.3) that

$$\bar{d}(x) = \begin{cases} 0 & \text{for a.e. } x \in \Gamma \text{ where } \alpha < \bar{u}(x) < \beta, \\ \geq 0 & \text{for a.e. } x \in \Gamma \text{ where } \bar{u}(x) = \alpha, \\ \leq 0 & \text{for a.e. } x \in \Gamma \text{ where } \bar{u}(x) = \beta. \end{cases} \tag{3.4}$$

In order to establish the second order optimality conditions we define the cone of critical directions

$$C_{\bar{u}} = \{v \in L^2(\Gamma) \text{ satisfying (3.5) and } v(x) = 0 \text{ if } |\bar{d}(x)| > 0\},$$

$$v(x) = \begin{cases} \geq 0 & \text{for a.e. } x \in \Gamma \text{ where } \bar{u}(x) = \alpha, \\ \leq 0 & \text{for a.e. } x \in \Gamma \text{ where } \bar{u}(x) = \beta. \end{cases} \tag{3.5}$$

Now we formulate the second order necessary and sufficient optimality conditions.

Theorem 3.2. *If $\bar{u}$ is a local solution of (P), then $J''(\bar{u})v^2 \geq 0$ holds for all $v \in C_{\bar{u}}$. Conversely, if $\bar{u} \in U^{ad}$ satisfies the first order optimality conditions* (3.1)–(3.3) *and the coercivity condition $J''(\bar{u})v^2 > 0$ holds for all $v \in C_{\bar{u}} \setminus \{0\}$, then there exist $\delta > 0$ and $\varepsilon > 0$ such that*

$$J(u) \geq J(\bar{u}) + \delta \|u - \bar{u}\|^2_{L^2(\Gamma)} \tag{3.6}$$

is satisfied for every $u \in U^{ad}$ such that $\|u - \bar{u}\|_{L^\infty(\Omega)} \leq \varepsilon$.

The necessary condition provided in the theorem is quite easy to get. The sufficient conditions are proved by Casas and Mateos [6, Theorem 4.3] for distributed control problems with integral state constraints. The proof can be translated in a straightforward way to the case of boundary controls. The hypothesis $(\partial^2 l/\partial u^2) \geq m_l > 0$ introduced in Assumption (A2) as well as the linearity of u in the state equation is essential to apply the mentioned Theorem 4.3. The same result can be proved by following the approach of Bonnans and Zidani [5].

Remark 3.3. By using the assumption $(\partial^2 l/\partial u^2)(x, y, u) \geq m_l > 0$, we deduce from Casas and Mateos [6, Theorem 4.4] that the following two conditions are equivalent:

(1) $J''(\bar{u})v^2 > 0$ for every $v \in C_{\bar{u}} \setminus \{0\}$.
(2) There exist $\delta > 0$ and $\tau > 0$ such that $J''(\bar{u})v^2 \geq \delta \|v\|^2_{L^2(\Gamma)}$ for every $v \in C^\tau_{\bar{u}}$, where

$$C^\tau_{\bar{u}} = \{v \in L^2(\Gamma) \text{ satisfying (3.5) and } v(x) = 0 \text{ if } |\bar{d}(x)| > \tau\}.$$

It is clear that that $C^\tau_{\bar{u}}$ contains strictly $C_{\bar{u}}$, so the condition (2) seems to be stronger than (1), but in fact they are equivalent.

We finish this section by providing a characterization of the optimal control $\bar{u}$ and deducing from it the Lipschitz regularity of $\bar{u}$ as well as some extra regularity of $\bar{y}$ and $\bar{\varphi}$.

Theorem 3.4. *Suppose that $\bar{u}$ is a local solution of (P), then for all $x \in \Gamma$ the equation*

$$\bar{\varphi}(x) + \frac{\partial l}{\partial u}(x, \bar{y}(x), t) = 0 \tag{3.7}$$

has a unique solution $\bar{t} = \bar{s}(x)$. The mapping $\bar{s} : \Gamma \to \mathbb{R}$ is Lipschitz and it is related with $\bar{u}$ through the formula

$$\bar{u}(x) = Proj_{[\alpha,\beta]}(\bar{s}(x)) = \max\{\alpha, \min\{\beta, \bar{s}(x)\}\}. \tag{3.8}$$

Moreover $\bar{u} \in C^{0,1}(\Gamma)$ and $\bar{y}, \bar{\varphi} \in W^{2,p}(\Omega) \subset C^{0,1}(\bar{\Omega})$ for some $p > 2$.

Proof: Let us remind that $\bar{y}, \bar{\varphi} \in H^{3/2}(\Omega) \subset C(\bar{\Omega})$ because $n = 2$. We fix $x \in \Gamma$ and consider the real function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(t) = \bar{\varphi}(x) + \frac{\partial l}{\partial u}(x, \bar{y}(x), t).$$

From assumption (A2) we have that g is C^1 with $g'(t) \geq m_l > 0$ for every $t \in \mathbb{R}$. Therefore, there exists a unique real number $\bar{t}$ satisfying $g(\bar{t}) = 0$. Consequently $\bar{s}$ is well defined and relation (3.8) is an immediate consequence of (3.4). Let us prove the regularity results. Invoking once again assumption (A2) along with (3.7) and (3.8), we get for every $x_1, x_2 \in \Gamma$

$$\begin{aligned} &|\bar{u}(x_2) - \bar{u}(x_1)| \\ &\quad \leq |\ \bar{s}(x_2) - \bar{s}(x_1)\ | \leq \frac{1}{m_l}\left|\frac{\partial l}{\partial u}(x_2, \bar{y}(x_2), \bar{s}(x_1)) - \frac{\partial l}{\partial u}(x_2, \bar{y}(x_2), \bar{s}(x_1))\right| \\ &\quad \leq \frac{1}{m_l}\left\{|\bar{\varphi}(x_2) - \bar{\varphi}(x_1)| + \left|\frac{\partial l}{\partial u}(x_1, \bar{y}(x_1), \bar{s}(x_1)) - \frac{\partial l}{\partial u}(x_2, \bar{y}(x_2), \bar{s}(x_1))\right|\right\} \\ &\quad \leq C\{|x_2 - x_1| + |\bar{\varphi}(x_2) - \bar{\varphi}(x_1)| + |\bar{y}(x_2) - \bar{y}(x_1)|\}. \end{aligned} \tag{3.9}$$

The embedding $H^{3/2}(\Omega) \subset W^{1,4}(\Omega)$ ensures that the traces of $\bar{y}$ and $\bar{\varphi}$ belong to the space $W^{1-1/4,4}(\Gamma)$. Exploiting that $n = 2$ and taking in this space the norm

$$\|z\|_{W^{1-1/4,4}(\Gamma)} = \left\{\|z\|^4_{L^4(\Gamma)} + \int_\Gamma \int_\Gamma \frac{|z(x_2) - z(x_1)|^4}{|x_2 - x_1|^4}\, d\sigma(x_1)\, d\sigma(x_2)\right\}^{1/4},$$

the regularity $\bar{u}, \bar{s} \in W^{1-1/4,4}(\Gamma) \subset W^{1-1/p,p}(\Gamma)$ $(1 \leq p \leq 4)$ follows from (3.9). Now Theorem 2.1 leads to the regularity $\bar{y} \in W^{2,p}(\Omega)$. The same is also true for $\bar{\varphi}$. Indeed, it is enough to use Corollary 4.4.3.8 of Grisvard [13] as in the proof of Theorem 2.1. Using the embedding $W^{2,p}(\Omega) \subset C^{0,1}(\bar{\Omega})$ and (3.9) we get the Lipschitz regularity of $\bar{u}$ and $\bar{s}$. □

4. Approximation of (P) by finite elements and piecewise constant controls

Here, we define a finite-element based approximation of the optimal control problem (P). To this aim, we consider a family of triangulations $\{\mathcal{T}_h\}_{h>0}$ of $\bar{\Omega}$: $\bar{\Omega} = \cup_{T\in\mathcal{T}_h} T$. This triangulation is supposed to be regular in the usual sense that we state exactly here. With each element $T \in \mathcal{T}_h$, we associate two parameters $\rho(T)$ and $\sigma(T)$, where $\rho(T)$ denotes the diameter of the set T and $\sigma(T)$ is the diameter of the largest ball contained in T. Let us define the size of the mesh by $h = \max_{T\in\mathcal{T}_h} \rho(T)$. The following regularity assumption is assumed. (H)—There exist two positive constants ρ and σ such that

$$\frac{\rho(T)}{\sigma(T)} \leq \sigma, \quad \frac{h}{\rho(T)} \leq \rho$$

hold for all $T \in \mathcal{T}_h$ and all $h > 0$.

For fixed $h > 0$, we denote by $\{T_j\}_{j=1}^{N(h)}$ the family of triangles of $\mathcal{T}_h$ with a side on the boundary of Γ. If the edges of $T_j \cap \Gamma$ are x_Γ^j and x_Γ^{j+1} then $[x_\Gamma^j, x_\Gamma^{j+1}] := T_j \cap \Gamma$, $1 \leq j \leq N(h)$, with $x_\Gamma^{N(h)+1} = x_\Gamma^1$. Associated with this triangulation we set

$$U_h = \left\{u \in L^\infty(\Gamma) \mid u \text{ is constant on every side } \left(x_\Gamma^j, x_\Gamma^{j+1}\right) \text{ for } 1 \leq j \leq N(h)\right\}.$$
$$Y_h = \left\{y_h \in C(\bar{\Omega}) | y_{h|T} \in \mathcal{P}_1, \quad \text{for all } T \in \mathcal{T}_h\right\},$$

where $\mathcal{P}_i$ is the space of polynomials of degree less than or equal to 1. For each $u \in^{L^\infty} (\Gamma)$, we denote by $y_h(u)$ the unique element of Y_h that satisfies

$$a(y_h(u), z_h) = \int_\Omega a_0(x, y_h(u)) z_h\, dx + \int_\Gamma [b_0(x, y_h(u)) + u] z_h\, dx \quad \forall z_h \in Y_h, \quad (4.1)$$

where $a : Y_h \times Y_h \to \mathbb{R}$ is the bilinear form defined by

$$a(y_h, z_h) = \int_\Omega \nabla y_h(x) \nabla z_h(x)\, dx.$$

The existence and uniqueness of a solution of (4.1) follows in the standard way from the monotonicity of a_0 and b_0. For instance, it can be deduced from [24, Lemma 4.3].

The finite dimensional control problem is defined by

$$(\mathrm{P}_h) \begin{cases} \min J_h(u_h) = \displaystyle\int_\Omega L(x, y_h(u_h)(x))\, dx + \int_\Gamma l(x, y_h(u_h)(x), u_h(x))\, d\sigma(x), \\ \text{subject to } (y_h(u_h), u_h) \in Y_h \times U_h^{a1}, \end{cases}$$

where

$$U_h^{ad} = U_h \cap U^{ad} = \{u_h \in U_h | \alpha \leq u_h(x) \leq \beta \text{ for all } x \in \Gamma\}.$$

Since J_h is a continuous function and U_h^{ad} is compact , we get that (P_h) has at least one global solution. The first order optimality conditions can be written as follows:

Theorem 4.1. *Assume that $\bar{u}_h$ is a local optimal solution of (P_h). Then there exist $\bar{y}_h$ and $\bar{\varphi}_h$ in Y_h satisfying*

$$a(\bar{y}_h, z_h) = \int_\Omega a_0(x, \bar{y}_h) z_h\, dx + \int_\Gamma (b_0(x, \bar{y}_h) + \bar{u}_h) z_h\, dx \quad \forall\, z_h \in Y_h, \quad (4.2)$$

$$a(\bar{\varphi}_h, z_h) = \int_\Omega \left(\frac{\partial a_0}{\partial y}(x, \bar{y}_h)\bar{\varphi}_h + \frac{\partial L}{\partial y}(x, \bar{y}_h)\right) z_h\, dx + \int_\Gamma \left(\frac{\partial b_0}{\partial_y}(x, \bar{y}_h)\bar{\varphi}_h + \frac{\partial l}{\partial y}(x, \bar{y}_h, \bar{u}_h)\right) z_h\, d\sigma(x) \quad \forall\, z_h \in Y_h, \quad (4.3)$$

$$\int_\Gamma \left(\bar\varphi_h + \frac{\partial l}{\partial u}(x, \bar y_h, \bar u_h)\right)(u_h - \bar u_h)\, d\sigma(x) \geq 0 \quad \forall u_h \in U_h^{ad}. \tag{4.4}$$

The following result is the counterpart of Theorem 3.4.

Theorem 4.2. *Let us assume that $\bar u_h$ is a local solution of problem (P_h).Then for every $1 \leq j \leq N(h)$, the equation*

$$\int_{x_\Gamma^j}^{x_\Gamma^{j+1}} \left(\bar\varphi_h(x) + \frac{\partial l}{\partial u}(x, \bar y_h(x), t)\right) d\sigma(x) = 0 \tag{4.5}$$

has a unique solution $\bar s_j$. The mapping $\bar s_h \in U_h$, defined by $\bar s_h(x) = \bar s_j$ on every side $(x_\Gamma^j, x_\Gamma^{j+1})$, is related to $\bar u_h$ by the formula

$$\bar u_h(x) = Proj_{[\alpha,\beta]}(\bar s_h(x)) = \min\{\alpha, \max\{\beta, \bar s_h(x)\}\}. \tag{4.6}$$

4.1. Convergence results

Our main aim is to prove the convergence of the local solutions of (P_h) to local solutions of (P) as well as to derive error estimates. Before doing this we need to establish the order of convergence of the solutions of the discrete equation (4.1) to the solution of the state equation (2.1). An analogous result is needed for the adjoint state equation.

Theorem 4.3. *For any $u \in L^2(\Gamma)$ there exists a constants $C = C(\|u\|_{L^2(\Gamma)}) > 0$ independent of h such that*

$$\|y_u - y_h(u)\|_{L^2(\Omega)} + \|\varphi_u - \varphi_h(u)\|_{L^2(\Omega)} \leq Ch, \tag{4.7}$$

where y_u denotes the solution of (2.1) *and φ_u is the solution of* (3.2) *with $(\bar y, \bar u)$ being replaced by (y, u). Moreover, if $u \in W^{1-1/p,p}(\Gamma)$ holds for some $p > 2$ and $u_h \in U_h$ then*

$$\|y_u - y_h(u_h)\|_{H^1(\Omega)} + \|\varphi_u - \varphi_h(u_h)\|_{H^1(\Omega)} \leq C\big\{h + \|u - u_h\|_{L^2(\Gamma)}\big\}. \tag{4.8}$$

Finally, if $u_h \to u$ weakly in $L^2(\Gamma)$, then $y_h(u_h) \to y_u$ and $\varphi_h(u_h) \to \varphi_u$ strongly in $C(\bar\Omega)$.

Proof: Let us prove the theorem for the state y. The corresponding proof for the adjoint state φ follows the same steps. Inequality (4.7) is proved by Casas and Mateos [7]. Let us prove (4.8). The regularity of u implies that $y_u \in H^2(\Omega)$, then

$$\|y_u - y_h(u)\|_{H^1(\Omega)} \leq Ch\|y_u\|_{H^2(\Omega)} = hC\big(\|u\|_{H^{1/2}(\Gamma)}\big);$$

see Casas and Mateos [7].

On the other hand, from the monotonicity of a_0 and b_0 and the assumption (A5) it is easy to get by classical arguments

$$\|y_h(u) - y_h(u_h)\|_{H^1(\Omega)} \leq C\|u - u_h\|_{L^2(\Gamma)}.$$

Combining both inequalities we achieve the desired result for the states . For the proof of the uniform convergence of the states and adjoint states the reader is also referred to [7]. □

Now we can prove the convergence of the discretizations.

Theorem 4.4. *For every $h > 0$ let $\bar{u}_h$ be a global solution of problem (P_h).Then there exist weakly*-converging subsequences of $\{\bar{u}_h\}_{h>0}$ in $L^\infty(\Gamma)$ (still indexed by h). If the subsequence $\{\bar{u}_h\}_{h>0}$ is converging weakly* to $\bar{u}$, then $\bar{u}$ is a solution of (P),*

$$\lim_{h\to 0} J_h(\bar{u}_h) = J(\bar{u}) = \inf(P) \quad \textit{and} \quad \lim_{h\to 0} \|\bar{u} - \bar{u}_h\|_{L^\infty(\Gamma)} = 0. \tag{4.9}$$

Proof: Since $U_h^{ad} \subset U^{ad}$ holds for every $h > 0$ and u^{ad} is bounded in $L^\infty(\Gamma)$, $\{\bar{u}_h\}_{h>0}$ is also bounded in $L^\infty(\Gamma)$. Therefore, there exist weakly*-converging subsequences as claimed in the statement of the theorem. Let $\bar{u}_h$ be the of one of these subsequences. By the definition of U^{ad} it is obvious that $\bar{u}_h \in U^{ad}$. Let us prove that the weak* limit $\bar{u}$ is a solution of (ρ). Let $\tilde{u} \in U^{ad}$ be a solution fo (P) and consider the operator $\Pi_h : L^1(\Gamma) \to U_h$ defined by

$$\Pi_h u|_{(x_\Gamma^j, x_\Gamma^{j+1})} = \frac{1}{\left|x_\Gamma^{j+1} - x_\Gamma^j\right|} \int_{x_\Gamma^j}^{x_\Gamma^{j+1}} u(x)\, d\sigma(x).$$

According to Theorem 3.4 we have that $\tilde{u} \in C^{0,1}(\Gamma)$ and then

$$\|\tilde{u} - \Pi_h \tilde{u}\|_{L^\infty(\Gamma)} \leq Ch\|\tilde{u}\|_{C^{0,1}(\Gamma)}.$$

Remark that $\Pi_h \tilde{u} \in U_h^{ad}$ for every h. Now using the convexity of l with respect to u and the uniform convergence $\bar{y}_h = y_h(\bar{u}_h) \to \bar{y} = y_{\bar{u}}$ and $y_h(\Pi_h \tilde{u}) \to y_{\tilde{u}}$ (Theorem 4.3) along with the assumptions on L and l we get

$$J(\bar{u}) \leq \liminf_{h\to 0} J_h(\bar{u}_h) \leq \limsup_{h\to 0} J_h(\bar{u}_h) \leq \limsup_{h\to 0} J_h(\Pi_h \tilde{u}) = J(\tilde{u}) = \inf(P).$$

This proves that $\bar{u}$ is a solution of (P) as well as the convergence of the optimal costs. Let us verify the uniform convergence of $\{\bar{u}_h\}$ to $\bar{u}$. From (3.8) and (4.6) we obtain

$$\|\bar{u} - \bar{u}_h\|_{L^\infty(\Gamma)} \leq \|\bar{s} - \bar{s}_h\|_{L^\infty(\Gamma)},$$

therefore it is enough to prove the uniform convergence of $\{\bar{s}_h\}_{h>0}$ *to* $\bar{s}$. On the other hand, from (4.5) and the continuity of the integrand with respect to x we deduce the existence of a point $\xi_\Gamma^j \in (x_\Gamma^j, x_\Gamma^{j+1})$ such that

$$\bar{\varphi}_h(\xi_\Gamma^j) + \frac{\partial l}{\partial u}(\xi_\Gamma^j, \bar{y}_h(\xi_\Gamma^j), \bar{s}_h(\xi_\Gamma^j)) = 0. \tag{4.10}$$

Given $x \in \Gamma$, let us take $1 \leq j \leq N(h)$ such that $x \in (x_\Gamma^j, x_\Gamma^{j+1})$. By the fact that $\bar{s}_h$ is constant on each of these intervals we get

$$\begin{aligned}|\bar{s}(x) - \bar{s}_h(x)| &\leq |\bar{s}(x) - \bar{s}(\xi_\Gamma^j)| + |\bar{s}(\xi_\Gamma^j) - \bar{s}_h(\xi_\Gamma^j)|\\ &\leq \Lambda_s|x - \xi_\Gamma^j| + |\bar{s}(\xi_\Gamma^j) - \bar{s}_h(\xi_\Gamma^j)| \leq \Lambda_s h + |\bar{s}(\xi_\Gamma^j) - \bar{s}_h(\xi_\Gamma^j)|,\end{aligned}$$

where Λ_s is the Lipschitz constant of $\bar{s}$. So it remains to prove the convergence $\bar{s}_h(\xi_\Gamma^j) \to \bar{s}(\xi_\Gamma^j)$ for every j. For it we use the strict positivity of the second derivative of l with respect to u (Assumption (A2)) along with the Eqs. (3.7) satisfied by $\bar{s}(x)$ and (4.10) to deduce

$$\begin{aligned}m_l|\bar{s}(\xi_\Gamma^j) - \bar{s}_h(\xi_\Gamma^j)| &\leq \left|\frac{\partial l}{\partial u}(\xi_\Gamma^j, \bar{y}_h(\xi_\Gamma^j), \bar{s}(\xi_\Gamma^j)) - \frac{\partial l}{\partial u}(\xi_\Gamma^j, \bar{y}_h(\xi_\Gamma^j), \bar{s}_h(\xi_\Gamma^j))\right|\\ &\leq \left|\frac{\partial l}{\partial u}(\xi_\Gamma^j, \bar{y}_h(\xi_\Gamma^j), \bar{s}(\xi_\Gamma^j)) - \frac{\partial l}{\partial u}(\xi_\Gamma^j, \bar{y}(\xi_\Gamma^j), \bar{s}(\xi_\Gamma^j))\right|\\ &\quad + \left|\frac{\partial l}{\partial u}(\xi_\Gamma^j, \bar{y}(\xi_\Gamma^j), \bar{s}(\xi_\Gamma^j)) - \frac{\partial l}{\partial u}(\xi_\Gamma^j, \bar{y}_h(\xi_\Gamma^j), \bar{s}_h(\xi_\Gamma^j))\right|\\ &= \left|\frac{\partial l}{\partial u}(\xi_\Gamma^j, \bar{y}_h(\xi_\Gamma^j), \bar{s}(\xi_\Gamma^j)) - \frac{\partial l}{\partial u}(\xi_\Gamma^j, \bar{y}(\xi_\Gamma^j), \bar{s}(\xi_\Gamma^j))\right|\\ &\quad + |\bar{\varphi}(\xi_\Gamma^j) - \bar{\varphi}_h(\xi_\Gamma^j)| \to 0\end{aligned}$$

because of the uniform convergence of $\bar{y}_h \to \bar{y}$ and $\bar{\varphi}_h \to \bar{\varphi}$; see Theorem 4.3. □

The next theorem is a kind of reciprocal result of the previous one. At this point we are wondering if every local minimum $\bar{u}$ of (P) can be approximated by a local minimum of (P_h). The following theorem answers positively this question under the assumption that $\bar{u}$ satisfies the second order sufficient optimality conditions given in Theorem 3.2. In the sequel, $B_\rho(u)$ will denote the open ball of $L^\infty(\Gamma)$ centered at u with radius ρ. By $\bar{B}_\rho(u)$ we denote the corresponding closed ball.

Theorem 4.5. *Let $\bar{u}$ be a local minimum of (P) satisfying the second order sufficient optimality condition given in Theorem* 3.2. *Then there exist $\varepsilon > 0$ and $h_0 > 0$ such that (P_h) has a local minimum $\bar{u}_h \in B_\varepsilon(\bar{u})$ for every $h < h_0$. Furthermore, the convergences* (4.9) *hold.*

Proof: Let $\varepsilon > 0$ be given by Theorem 3.2 and consider the problems

$$(\mathrm{P}_\varepsilon)\begin{cases}\min J(u)\\ \text{subject to } (y_u, u) \in H^1(\Omega) \times (U^{ad} \cap \bar{B}_\varepsilon(\bar{u}))\end{cases}$$

and

$$(\mathrm{P}_{h\varepsilon})\begin{cases}\min J_h(u_h)\\ \text{subject to } (y_h(u_h), u_h) \in Y_h \times \big(U_h^{ad} \cap \bar{B}_\varepsilon(\bar{u})\big).\end{cases}$$

According to Theorem 3.2, $\bar{u}$ is the unique solution of (P_ε). Moreover $\pi_h\bar{u}$ is a feasible control for $(\mathrm{P}_{h\varepsilon})$ for every h shall enough. Therefore $U_h^{ad} \cap \bar{B}_\varepsilon(\bar{u})$ is a non empty compact set and consequently $(\mathrm{P}_{h\varepsilon})$ has at least one solution $\bar{u}_h$. Now we can argue as in the proof of Theorem 4.4 to deduce that $\bar{u}_h \to \bar{u}$ uniformly, hence $\bar{u}_h$ is a local solution of (P_h) in the open ball $B_\varepsilon(\bar{u})$ as required. □

4.2. Error estimates

In this section we denote by $\bar{u}$ a fixed local reference solution of (P) satisfying the second order sufficient optimality conditions and by $\bar{u}_h$ the associated local solution of (P_h) converging uniformly to $\bar{u}$. As usual $\bar{y}$, $\bar{y}_h$ and $\bar{\varphi}$, $\bar{\varphi}_h$ are the state and adjoint states corresponding to $\bar{u}$ and $\bar{u}_h$. The goal is to estimate $\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)}$. Let us start by proving a first estimate for this term.

Lemma 4.6. *Let $\delta > 0$ given as in Remark* 3.3, (2). *Then there exists $h_0 > 0$ such that*

$$\frac{\delta}{2}\|\bar{u} - \bar{u}_h\|^2_{L^2(\Gamma)} \le (J'(\bar{u}_h) - J'(\bar{u}))(\bar{u}_h - \bar{u}) \quad \forall h < h_0. \tag{4.11}$$

Proof: Let us set

$$\bar{d}_h(x) = \frac{\partial l}{\partial u}(x, \bar{y}_h(x), \bar{u}_h(x)) + \bar{\varphi}_h(x)$$

and take $\delta > 0$ and $\tau > 0$ as introduced in Remark 3.3, (2). We know that $\bar{d}_h \to \bar{d}$ uniformly in Γ , therefore there exists $h_\tau > 0$ such that

$$\|\bar{d} - \bar{d}_h\|_{L^\infty(\Gamma)} < \frac{\tau}{4} \quad \forall\, h \le h_\tau. \tag{4.12}$$

For every $1 \le j \le N(h)$ we define

$$I_j = \int_{x_\Gamma^j}^{x_\Gamma^{j+1}} \bar{d}_h(x)\, d\sigma(x).$$

From Theorem (4.1) we deduce by the classical argumentation that

$$\bar{u}_h|_{(x_\Gamma^j, x_\Gamma^{j+1})} = \begin{cases} \alpha & \text{if } I_j > 0 \\ \beta & \text{if } I_j < 0. \end{cases}$$

Let us take $0 < h_1 \leq h_\tau$ such that

$$|\bar{d}(x_2) - \bar{d}(x_1)| < \frac{\tau}{4} \quad \text{if } |x_2 - x_1| < h_1.$$

This inequality along with (4.12) implies that

$$\text{if } \xi \in (x_\Gamma^j, x_\Gamma^{j+1}) \quad \text{and} \quad \bar{d}(\xi) > \tau \Rightarrow \bar{d}_h(x) > \frac{\tau}{2} \quad \forall x \in \big(x_\Gamma^j, x_\Gamma^{j+1}\big), \forall h < h_1,$$

which implies that $I_j > 0$, hence $\bar{u}_h|_{(x_\Gamma^j, x_\Gamma^{j+1})} = \alpha$, in particular $\bar{u}_h(\xi) = \alpha$. From (3.4) we also deduce that $\bar{u}(x) = \alpha$. Therefore $\bar{u}_h - \bar{u}(\xi) = 0$ whenever $\bar{d}(\xi) > \tau$ and $h < h_1$. Analogously we can prove that the same is true when $\bar{d}(\xi) < -\tau$. Moreover since $\alpha \leq \bar{u}_h(x) \leq \beta$, it is obvious that $(\bar{u}_h - \bar{u})(x) \geq 0$ if $\bar{u}(x) = \alpha$ and $(\bar{u}_h - \bar{u})(x) \leq 0$ if $\bar{u}(x) = \beta$. Thus we have proved that $(\bar{u}_h - \bar{u}) \in C_{\bar{u}}^\tau$ and according to Remark 3.3(2) we have

$$J''(\bar{u})(\bar{u}_h - \bar{u})^2 \geq \delta \|\bar{u}_h - \bar{u}\|^2_{L^2(\Gamma)} \quad \forall h < h_1. \tag{4.13}$$

On the other hand, by applying the mean value theorem we get for some $0 < \theta_h < 1$

$$\begin{aligned} (J'(\bar{u}_h) - J'(\bar{u}))(\bar{u}_h - \bar{u}) &= J''(\bar{u} + \theta_h(\bar{u}_h - \bar{u}))(\bar{u}_h - \bar{u})^2 \\ &\geq (J''(\bar{u} + \theta_h(\bar{u}_h - \bar{u})) - J''(\bar{u}))(\bar{u}_h - \bar{u})^2 + J''(\bar{u})(\bar{u}_h - \bar{u})^2 \\ &\geq (\delta - \|J''(\bar{u} + \theta_h(\bar{u}_h - \bar{u})) - J''(\bar{u})\|)\|\bar{u}_h - \bar{u}\|^2_{L^2(\Gamma)}. \end{aligned}$$

Finally it is enough to choose $0 < h_0 \leq h_1$ such that

$$\|J''(\bar{u} + \theta_h(\bar{u}_h - \bar{u})) - J''(\bar{u})\| \leq \frac{\delta}{2} \quad \forall h < h_0$$

to prove (4.11). The last inequality can be obtained easily from the relation (2.6) thanks to the uniform convergence of $(\bar{\varphi}_h, \bar{y}_h, \bar{u}_h) \to (\bar{\varphi}, \bar{y}, \bar{u})$ and the assumptions (A1)–(A4). □

The next step consists of estimating the convergence of J_h' to J'.

Lemma 4.7. *For every $\rho > 0$ there exists $C_\rho > 0$ independent of h such that*

$$|(J_h'(\bar{u}_h) - J'(\bar{u}_h))v| \leq (C_\rho h + \rho\|\bar{u}_h - \bar{u}\|_{L^2(\Gamma)}), \|v\|_{L^2(\Gamma)} \quad \forall v \in L^2(\Gamma). \tag{4.14}$$

Proof: From the hypotheses on l it is readily deduced

$$\begin{aligned}|(J_h'(\bar u_h) - J'(\bar u_h))v| &\le \int_\Gamma \left(|\bar\varphi_h - \varphi_{\bar u_h}| + \left| \frac{\partial l}{\partial_u}(x, \bar y_h, \bar u_h) - \frac{\partial l}{\partial_u}(x, y_{\bar u_h}, \bar u_h) \right| \right) v d\sigma(x) \\ &\le C\big(\|\bar\varphi_h - \varphi_{\bar u_h}\|_{L^2(\Gamma)} + \|\bar y_h - y_{\bar u_h}\|_{L^2(\Gamma)} \big) \|v\|_{L^2(\Gamma)}, \qquad (4.15)\end{aligned}$$

where $y_{\bar u_h}$ and $\varphi_{\bar u_h}$ are the solutions of (2.1) and 2.7) corresponding to $\bar u_h$.

We use the following well known property. For every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\|z\|_{L^2(\Gamma)} \le \varepsilon \|z\|_{H_1(\Omega)} + C_\varepsilon \|z\|_{L^2(\Omega)}$$

Thus we get with the aid of (4.7)

$$\begin{aligned}\|\bar y_h - y_{\bar u_h}\|_{L^2(\Gamma)} &= \|y_h(\bar u_h) - y_{\bar u_h}\|_{L^2(\Gamma)} \le \varepsilon \|y_h(\bar u_h) - y_{\bar u_h}\|_{H^1(\Omega)} \\ &\quad + C_\varepsilon \|y_h(\bar u_h) - y_{\bar u_h}\|_{L^2(\Omega)} \le \varepsilon \|y_h(\bar u_h) - y_{\bar u_h}\|_{H^1(\Omega)} + C_\varepsilon C h \\ &= \varepsilon \|\bar y_h - y_{\bar u_h}\|_{H^1(\Omega)} + C_\varepsilon C h.\end{aligned}$$

Thanks to the monotonicity of a_0 and b_0 and the assumption $(A5)$ we obtain from the state equation in the standard way

$$\|\bar y - y_{\bar u_h}\|_{H^1(\Omega)} \le C \|\bar u - \bar u_h\|_{L^2(\Gamma)}.$$

On the other hand, (4.8) leads to

$$\|\bar y - \bar y_h\|_{H^1(\Omega)} \le C\big(h + \|\bar u - \bar u_h\|_{L^2(\Gamma)}\big).$$

Combining the last three inequalities we deduce

$$\|\bar y_h - y_{\bar u_h}\|_{L^2(\Gamma)} \le C\big(\varepsilon\big(h + \|\bar u - \bar u_h\|_{L^2(\Gamma)}\big) + C_\varepsilon h\big).$$

The same arguments can be applied to the adjoint state, so (4.14) follows from (4.15). Inequality (4.14) is obtained by choosing $C\varepsilon = \rho$ and $C_\rho = C_\varepsilon + C\varepsilon$. □

One key point in the proof of error estimates is to get a discrete control $u_h \in U_h^{ad}$ that approximates $\bar u$ conveniently and satisfies $J'(\bar u)\bar u = J'(\bar u)u_h$. Let us find such a control. Let $\bar d$ be defined as in Section 3 and set I_j for every $1 \le j \le N(h)$ as in the proof of Lemma 4.6

$$I_j = \int_{x_\Gamma^j}^{x_\Gamma^{j+1}} \bar d(x)\, d\sigma(x).$$

Now we define $u_h \in U_h$ with $u_h(x) \equiv u_h^j$ on the intervals $(x_\Gamma^j, x_\Gamma^{j+1})$ by the expression

$$u_h^j = \begin{cases} \dfrac{1}{I_j} \displaystyle\int_{x_\Gamma^j}^{x_\Gamma^{j+1}} \bar{d}(x)\bar{u}(x)\, d\sigma(x) & \text{if } I_j \neq 0 \\[2ex] \dfrac{1}{\left|x_\Gamma^j - x_\Gamma^{j+1}\right|} \displaystyle\int_{x_\Gamma^j}^{x_\Gamma^{j+1}} \bar{u}(x) d\sigma(x) & \text{if } I_j = 0. \end{cases} \tag{4.16}$$

This u_h satisfies our requirements.

Lemma 4.8. *There exists $h_0 > 0$ such that for every $0 < h < h_0$ the following properties hold:*

1. $u_h \in U_h^{ad}$.
2. $J'(\bar{u})\bar{u} = J'(\bar{u})u_h$.
3. *There exists $C > 0$ independent of h such that*

$$\|\bar{u} - u_h\|_{L^\infty(\Gamma)} \leq Ch. \tag{4.17}$$

Proof: Let $L_u > 0$ be the Lipschitz constant of $\bar{u}$ and take $h_0 = (\beta - \alpha)/(2L_u)$, then

$$|\bar{u}(\xi_2) - \bar{u}(\xi_1)| \leq L_u|\xi_2 - \xi_1| \leq L_u h < \frac{\beta - \alpha}{2} \; \forall \xi_1, \xi_2 \in \left[x_\Gamma^j, x_\Gamma^{j+1}\right],$$

which implies that $\bar{u}$ can not admit the values α and β on one segment $[x_\Gamma^j, x_\Gamma^{j+1}]$ for all $h < h_0$. Hence the sign of $\bar{d}$ on $[x_\Gamma^j, x_\Gamma^{j+1}]$ must be constant due to (3.4). Therefore, $I_j = 0$ if and only if $\bar{d}(x) = 0$ for all $x \in [x_\Gamma^j, x_\Gamma^{j+1}]$. Moreover if $I_j \neq 0$, then $\bar{d}(x)/I_j \geq 0$ for every $x \in [x_\Gamma^j, x_\Gamma^{j+1}]$. As a first consequence of this we get that $\alpha \leq u_h^j \leq \beta$, which means that $u_h \in U_h^{ad}$. On the other hand

$$J'(\bar{u})u_h = \sum_{j=1}^{N(h)} \int_{x_\Gamma^j}^{x_\Gamma^{j+1}} \bar{d}(x)\, d\sigma(x) u_h^j = \sum_{j=1}^{N(h)} \int_{x_\Gamma^j}^{x_\Gamma^{j+1}} \bar{d}(x)\bar{u}(x)\, d\sigma(x) = J'(\bar{u})\bar{u}.$$

Finally let us prove (4.17). Since the sign of $\bar{d}(x)/I_j$ is always non negative and $\bar{d}$ is a continuous function, we get for any of the two possible definitions of u_h^j the existence of a point $\xi^j \in [x_\Gamma^j, x_\Gamma^{j+1}]$ such that $u_h^j = \bar{u}(\xi_j)$. Therefore, for any $x \in [x_\Gamma^j, x_\Gamma^{j+1}]$

$$|\bar{u}(x) - u_h(x)| = \left|\bar{u}(x) - u_h^j\right| = |\bar{u}(x) - \bar{u}(\xi^j)| \leq L_u|x - \xi^j| \leq L_u h,$$

which leads to (4.17). □

Finally, we derive the main error estimate.

Theorem 4.9. *There exists a constant $C > 0$ independent of h such that*

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} \leq Ch. \tag{4.18}$$

Proof: Setting $u = \bar{u}_h$ in (3.3) we get

$$J'(\bar{u})(\bar{u}_h - \bar{u}) = \int_\Gamma \left(\frac{\partial l}{\partial u}(x, \bar{y}, \bar{u}) + \bar{\varphi}\right)(\bar{u}_h - \bar{u})d\sigma \geq 0. \tag{4.19}$$

From (4.4) with u_h defined by (4.16) it follows

$$J_h'(\bar{u}_h)(u_h - \bar{u}_h) = \int_\Gamma \left(\bar{\varphi}_h + \frac{\partial l}{\partial u}(x, \bar{y}_h, \bar{u}_h)\right)(u_h - \bar{u}_h)d\sigma(x) \geq 0$$

and then

$$J_h'(\bar{u}_h)(\bar{u} - \bar{u}_h) + J_h'(\bar{u}_h)(u_h - \bar{u}) \geq 0. \tag{4.20}$$

By adding (4.19) and (4.20) and using Lemma 4.8-2, we derive

$$(J'(\bar{u}) - J_h'(\bar{u}_h))(\bar{u} - \bar{u}_h) \leq J_h'(\bar{u}_h)(u_h - \bar{u}) = (J_h'(\bar{u}_h) - J'(\bar{u}))(u_h - \bar{u}).$$

For h small enough, this inequality and (4.11) lead to

$$\begin{aligned}\frac{\delta}{2}\|\bar{u} - \bar{u}_h\|^2_{L^2(\Gamma)} \leq (J'(\bar{u}) - J'(\bar{u}_h))(\bar{u} - \bar{u}_h) &\leq (J_h'(\bar{u}_h) - J'(\bar{u}_h))(\bar{u} - \bar{u}_h)\\ &\quad + (J_h'(\bar{u}_h) - J'(\bar{u}))(u_h - \bar{u}).\end{aligned} \tag{4.21}$$

Arguing as in (4.15) and using (4.8) and (4.17) we get

$$\begin{aligned}|(J_h'(\bar{u}_h) - J'(\bar{u}))(u_h - \bar{u})| &\leq C\big(\|\bar{\varphi}_h - \bar{\varphi}\|_{L^2(\Gamma)} + \|\bar{y}_h - \bar{y}\|_{L^2(\Gamma)}\big)\|u_h - \bar{u}\|_{L^2(\Gamma)}\\ &\leq C\big(h + \|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)}\big)\|u_h - \bar{u}\|_{L^2(\Gamma)}\\ &\leq C\big(h^2 + h\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)}\big).\end{aligned} \tag{4.22}$$

On the other hand, using (4.14)

$$|(J_h'(\bar{u}_h) - J'(\bar{u}_h))(\bar{u} - \bar{u}_h)| \leq \big(C_\rho h + \rho\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)}\big)\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)}.$$

By taking $\rho = \delta/4$, we deduce from this inequality along with (4.21) and (4.22)

$$\frac{\delta}{4}\|\bar{u} - \bar{u}_h\|^2_{L^2(\Gamma)} \leq Ch^2 + (C + C_\rho)h\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)},$$

which proves (4.18) for a convenient constant C independent of h. ☐

5. Numerical confirmation

In this section we shall verify our error estimates by numerical test examples for which we know the exact solution. We report both on a linear-quadratic problem and on a semilinear problem.

5.1. A linear-quadratic problem and primal-dual active set strategy

Let us consider the problem

$$
(E1)\quad \begin{cases} \min J(u) = \dfrac{1}{2}\displaystyle\int_\Omega (y_u(x) - y_\Omega(x))^2\,dx + \dfrac{\mu}{2}\int_\Gamma u(x)^2 d\sigma(x) \\ \qquad\qquad + \displaystyle\int_\Gamma e_u(x)u(x)d\sigma(x) + \int_\Gamma e_y(x)y_u(x)d\sigma(x) \\ \text{subject to } (y_u, u) \in H^1(\Omega) \times L^\infty(\Gamma), \\ \qquad\qquad u \in U_{ad} = \{u \in L^\infty(\Gamma) | 0 \le u(x) \le 1 \text{ a.e. } x \in \Gamma\}, \\ \qquad\qquad (y_u, u) \text{ satisfying the linear state equation (5.1)} \end{cases}
$$

$$
\begin{cases} -\Delta y_u(x) + c(x)y_u(x) = e_1(x) & \text{in } \Omega \\ \partial_\nu y_u(x) + y_u(x) = e_2(x) + u(x) & \text{on } \Gamma. \end{cases} \tag{5.1}
$$

We fix the following data: $\Omega = (0, 1)^2$, $\mu = 1$, $c(x_1, x_2) = 1 + x_1^2 - x_2^2$, $e_y(x_1, x_2) = 1$, $y_\Omega(x_1, x_2) = x_1^2 + x_1x_2$, $e_1(x_1, x_2) = -2 + (1 + x_1^2 - x_2^2)(1 + 2x_1^2 + x_1x_2 - x_2^2)$,

$$
e_u(x_1, x_2) = \begin{cases} -1 - x_1^3 & \text{on } \Gamma_1 \\ -1 - \min\left\{ \begin{array}{l} 8(x_2 - 0.5)^2 + 0.5, \\ 1 - 16x_2(x_2 - 0.25)(x_2 - 0.75)(x_2 - 1) \end{array} \right\} & \text{on } \Gamma_2 \\ -1 - x_1^2 & \text{on } \Gamma_3 \\ -1 + x_2(1 - x_2) & \text{on } \Gamma_4 \end{cases}
$$

and

$$
e_2(x_1, x_2) = \begin{cases} 1 - x_1 + 2x_1^2 - x_1^3 & \text{on } \Gamma_1 \\ 7 + 2x_2 - x_2^2 - \min\{8(x_2 - .5)^2 + .5, 1\} & \text{on } \Gamma_2 \\ -2 + 2x_1 + x_1^2 & \text{on } \Gamma_3 \\ 1 - x_2 - x_2^2 & \text{on } \Gamma_4, \end{cases}
$$

where Γ_1 to Γ_4 are the four sides of the square, starting at the bottom side and turning counterclockwise. This problem has the following solution $(\bar{y}, \bar{u})$ with adjoint state $\bar{\varphi}$:

$\bar{y}(x) = 1 + 2x_1^2 + x_1x_2 - x_2^2$, $\bar{\varphi}(x_1, x_2) = 1$ and

$$\bar{u}(x_1, x_2) = \begin{cases} x_1^3 & \text{on } \Gamma_1 \\ \min\{8(x_2 - .5)^2 + .5, 1\} & \text{on } \Gamma_2 \\ x_1^2 & \text{on } \Gamma_3 \\ 0 & \text{on } \Gamma_4. \end{cases}$$

It is not difficult to check that the state equation (5.1) is satisfied by $(\bar{y}, \bar{u})$. The same refers to the adjoint equation

$$\begin{cases} -\Delta\bar{\varphi}(x) + c(x)\bar{\varphi}(x) = \bar{y}(x) - y_\Omega(x) & \text{in } \Omega \\ \partial_\nu\bar{\varphi}(x) + \bar{\varphi}(x) = e_y & \text{on } \Gamma. \end{cases}$$

In example (E1), the function

$$\bar{d}(x) = \bar{\varphi}(x) + e_u(x) + \bar{u}(x) = \begin{cases} 0 & \text{on } \Gamma_1 \\ \min\{0, 16x_2(x_2 - 0.25)(x_2 - 0.75)(x_2 - 1)\} & \text{on } \Gamma_2 \\ 0 & \text{on } \Gamma_3 \\ x_2(1 - x_2) & \text{on } \Gamma_4 \end{cases}$$

satisfies the relations (3.4) (see figure 1, where each interval $(i - 1, i)$ on the x axis corresponds to Γ_i, $1 \le i \le 4$), hence the first order necessary condition (3.3) is fulfilled. Since (E1) is a convex problem, this condition is also sufficient for $(\bar{y}, \bar{u})$ to be global minimum.

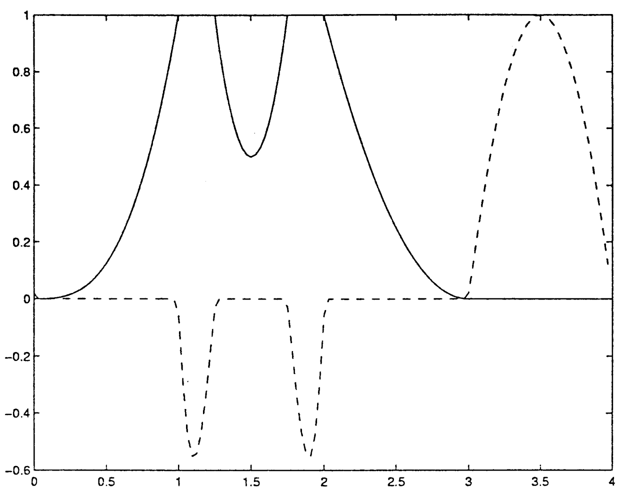

Figure 1. Solid: $\bar{u}(x_1, x_2)$, dashed: $\bar{d}(x_1, x_2)$.

Let us briefly describe how we have performed the optimization. We define the following operators: $S : L^2(\Gamma) \to L^2(\Omega)$, and $\Xi : L^2(\Gamma) \to L^2(\Gamma)$. For $u \in L^2(\Gamma)$, $Su = y$, and $\Xi u = y|_\Gamma$, where

$$\begin{cases} -\Delta y(x) + c(x)y(x) = 0 & \text{in } \Omega \\ \partial_\nu y(x) + y(x) = u(x) & \text{on } \Gamma. \end{cases}$$

If we define y_0 as the state associated to $u(x) = 0$ for all $x \in \Gamma$ and set $y_d(x) = y_\Omega(x) - y_0(x)$ then minimizing $J(u)$ is equivalent to minimize

$$\tilde{J}(u) = \frac{1}{2}(S^*Su + u, u)_{L^2(\Gamma)} + (e_u + \Xi^* e_y - S^* y_d, u)_{L^2(\Gamma)},$$

subject to $u \in U_{ad}$ where $(\cdot, \cdot)_X$ denotes the inner scalar product in the space X.

We perform the discretization in two steps. First we discretize the control and thereafter the state. Let us take $\{e_j\}_{j=1}^{N(h)}$ as a basis of U_h. If $u_h(x) = \sum_{j=1}^{N(h)} u_j e_j(x)$ for $x \in \Gamma$, we must perform the optimization over U_h of

$$\tilde{J}(u_h) = \frac{1}{2} \sum_{i,j=1}^{N(h)} u_i u_j (S^*Se_i + e_i, e_j)_{L^2(\Gamma)} + \sum_{j=1}^{N(h)} u_j (e_j, e_u + \Xi^* e_y - S^* y_d)_{L^2(\Gamma)}$$

subject to $0 \le u_j \le 1$ for $j = 1, \ldots, N(h)$.

If we set $A_{i,j} = (S^*Se_i + e_i, e_j)_{L^2(\Gamma)}$, $b_i = (e_i, e_u + \Xi^* e_y - S^* y_d)_{L^2(\Gamma)}$ and $\vec{u} = (u_1, \ldots, u_{N(h)})^T$, then we must minimize

$$f(\vec{u}) = \frac{1}{2}\vec{u}^T A \vec{u} + \vec{b}^T \vec{u}$$

subject to $0 \le u_j \le 1$ for $j = 1, \ldots, N(h)$. Calculating the matrix A explicitely would require solving $2N(h)$ partial differential equations. and this is numerically too expensive. Therefore usual routines to perform quadratic constrained minimization should not be used. General optimization programs that require only an external routine providing the function and its gradient do not take advantage of the fact that we indeed have a quadratic functional. Therefore, we have implemented our own routine for a primal-dual active set strategy according to Bergounioux and Kunisch [4]; see also Kunisch and Rösch [23]. Let us briefly describe the main steps of this iterative method. First of all, let us define the active sets for a vector $\vec{u} \in \mathbb{R}^{N(h)}$. We choose a parameter $c > 0$ and make

$$A_{h,+}(\vec{u}) = \left\{ j \in \{1, \ldots, N(h)\} \mid u_j - \frac{\partial_{u_j} f(\vec{u})}{c} > 1 \right\}$$

and

$$A_{h,-}(\vec{u}) = \left\{ j \in \{1, \ldots, N(h)\} \mid u_j - \frac{\partial_{u_j} f(\vec{u})}{c} < 0 \right\}.$$

Notice that $\partial_{u_j} f(\vec{u}) = J'(u_h)e_j$.

1. We choose an starting point $\vec{u}_0$ (not necessarily feasible) and fix its active sets $A^0_{h,+} = A_{h,+}(\vec{u}_0)$ and $A^0_{h,-} = A_{h,-}(\vec{u}_0)$. Set $n = 0$.
   At each step, we solve an unconstrained problem to get $\vec{u}_{n+1}$. To do this:
2. We define a vector $\vec{u}^{act}_{n+1}$ that has zeros in all its components, except those belonging to $A^n_{h,+}$, which are set to 1 and those belonging to $A^n_{h,-}$ which are set to the lower bound (which is also zero in this problem).
   Set $m = N(h) - |A^n_{h,+}| - |A^n_{h,-}|$.
3. If $m = 0$, we set $\vec{u}_{n+1} = \vec{u}^{act}_{n+1}$ and go to 5.
4. If $m > 0$, we define a matrix K with $N(h)$ rows and m columns such that row j is the zero vector if $j \in A^n_{h,+} \cup A^n_{h,-}$ and the submatrix formed by the rest of the rows is the identity $m \times m$ matrix. At each iteration we must minimize $f(K\vec{v} + \vec{u}^{act}_{n+1})$, where $\vec{v} \in \mathbb{R}^m$. This is equivalent to minimizing

$$q(\vec{v}) = \frac{1}{2}\vec{v}^T K^T A K \vec{v} + \left(K^T\left(\vec{b} + A\vec{u}^{act}_{n+1}\right)\right)^T \vec{v}$$

   for $\vec{v} \in \mathbb{R}^m$. This is the unconstrained quadratic program. We will call $\vec{v}^{n+1}$ its solution. Now we set $\vec{u}_{n+1} = K\vec{v}^{n+1} + \vec{u}^{act}_{n+1}$
5. We fix the new active sets $A^{n+1}_{h,+} = A_{h,+}(\vec{u}_{n+1})$ and $A^{n+1}_{h,-} = A_{h,-}(\vec{u}_{n+1})$.
6. The solution is achieved if $A^n_{h,+} = A^{n+1}_{h,+}$ and $A^n_{h,-} = A^{n+1}_{h,-}$. If this is not the case, we set $n := n + 1$ and return to 2.

It is shown in Kunisch and Rösch [23, Corollary 4.7] that with an adequate parameter c, the algorithm terminates in finitely many iterations for the discretized problem. In practice, we had no problem to choose $c = 1$.

Let us make a comment on how the unconstrained quadratic optimization in step 4 is performed. Since it is not possible to compute the whole matrix A, we solve this problem by the conjugate gradient method. At each iteraction of this method we must evaluate $A\vec{w}$ for some $\vec{w} \in \mathbb{R}^{N(h)}$. If we define $w = \sum_{j=1}^{N(h)} w_j e_j$, the component i of the vector $A\vec{w}$ is given by $(e_j, \varphi + w)_{L^2(\Gamma)}$, where φ is obtained solving the two partial differential equations

$$\begin{cases} -\Delta y(x) + c(x)y(x) = 0 & \text{in } \Omega \\ \partial_\nu y(x) + y(x) = w(x) & \text{on } \Gamma \end{cases} \quad \text{and} \quad \begin{cases} -\Delta\varphi(x) + c(x)\varphi(x) = y(x) & \text{in } \Omega \\ \partial_\nu\varphi(x) + \varphi(x) = 0 & \text{on } \Gamma. \end{cases}$$

These equations are solved by the finite element method. We have used the MATLAB PDE Toolbox just to get the mesh for Ω, but we have performed the assembling of the mass and stiffness matrices and of the right hand side vector with our own routines to determine all the integrals in an exact way. We had two reasons to do this. First, we have not included the effect of integration errors in our previous research, and secondly, when making a non-exact integration, the approximate adjoint state is possibly not the adjoint state of the approximate state. This fact may negatively affect the convergence. In practice, a low order integration method slows down the convergence.

Observe that the discretization of the state can be done independently of the discretization of the controls. We have performed two tests to show that the bottleneck of the error is the discretization of the controls. In the first test we have chosen the same mesh sizes both for the state and the control. In the second test we have chosen a fixed small mesh size for the state and we have varied the mesh size for the control. These are the results:

Test 1.

h	$\|\bar{y}-\bar{y}_h\|_{L^2(\Omega)}$	$\lvert\bar{y}-\bar{y}_h\rvert_{H^1(\Omega)}$	$\|\bar{u}-\bar{u}_h\|_{L^2(\Gamma)}$	$\|\bar{u}-\bar{u}_h\|_{L^\infty(\Gamma)}$
2^{-4}	$5.617876e-04$	$7.259364e-02$	$4.330776e-02$	$1.146090e-01$
2^{-5}	$1.423977e-04$	$3.635482e-02$	$2.170775e-02$	$5.990258e-02$
2^{-6}	$3.500447e-05$	$1.800239e-02$	$1.086060e-02$	$3.060061e-02$
2^{-7}	$8.971788e-06$	$8.950547e-03$	$5.431141e-03$	$1.546116e-02$

The orders of convergence obtained are h^2 for $\|\bar{y}-\bar{y}_h\|_{L^2(\Omega)}$ and h for the seminorm in the $H^1(\Omega)$, $L^2(\Gamma)$ and $L^\infty(\Gamma)$ norms. Figure 2 compares the error logarithm with $p\ \log(h)$, where p is the order of convergence obtained and the x axis represents the values of $\log(h)$.

The estimates $\lvert\bar{y}-\bar{y}_h\rvert_{H^1(\Omega)} \leq Ch$ and for $\lvert\bar{u}-\bar{u}_h\rvert_{L^2(\Gamma)} \leq Ch$ are the ones expected from inequalities (4.8) and (4.18). The estimate $\lvert\bar{y}-\bar{y}_h\rvert_{L^2(\Omega)} \leq Ch^2$ is indeed better than the one we can expect from inequality (4.7). This cannot only be explained by the information that $\bar{y} \in H^2(\Omega)$ ensures order h^2 for the FEM. Neverheless. the observed order h^2 can be theoretically justified. A forthcoming paper by A. Rösch studies this case.

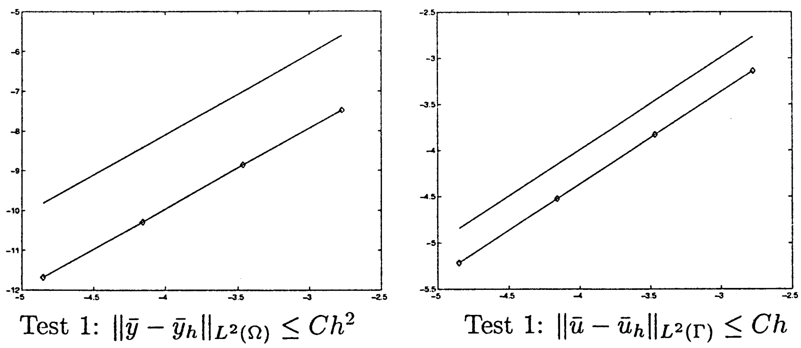

Figure 2. Solid line: p log h. Dotted line: Data from Test 1.

Test 2. We fix now the mesh size for the state to $h_y = 2^{-7}$. This ensures a fairly accurate solution of the partial differential equations.

h	$\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)}$	$\mid \bar{y} - \bar{y}_h \mid_{H^1(\Omega)}$	$\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)}$	$\|\bar{u} - \bar{u}_h\|_{L^\infty(\Gamma)}$
2^{-4}	$1.831053e-04$	$9.837630e-03$	$4.330774e-02$	$1.145890e-01$
2^{-5}	$4.648617e-05$	$9.026588e-03$	$2.170775e-02$	$5.989731e-02$
2^{-6}	$1.424508e-05$	$8.952289e-03$	$1.086060e-02$	$3.059955e-02$
2^{-7}	$8.971788e-06$	$8.950547e-03$	$5.431141e-03$	$1.546116e-02$

The error for the state is very small from the beginning. The order is again h for the last two columns. We observe that refining the mesh for the state does not improve the approximation of the control.

5.2. *A semilinear example*

Let us next consider the problem

$$
\text{(E2)} \quad \begin{cases} \min J(u) = \dfrac{1}{2}\displaystyle\int_\Omega (y_u(x) - y_\Omega(x))^2\,dx + \dfrac{\mu}{2}\int_\Gamma u(x)^2\,d\sigma(x) \\ \qquad\qquad + \displaystyle\int_\Gamma e_u(x)u(x)\,d\sigma(x) + \int_\Gamma e_y(x)y_u(x)\,d\sigma(x) \\ \text{subject to } (y_u, u) \in H^1(\Omega) \times L^\infty(\Gamma), \\ \qquad\qquad u \in U_{ad} = \{u \in L^\infty(\Gamma) | 0 \le u(x) \le 1 \text{ a.e. } x \in \Gamma\}, \\ \qquad\qquad (y_u, u) \text{ satisfying the semilinear state equation (5.2)} \end{cases}
$$

$$
\begin{cases} -\triangle y_u(x) + c(x)y_u(x) = e_1(x) & \text{in } \Omega \\ \partial_\nu y_u(x) + y_u(x) = e_2(x) + u(x) - y(x)|y(x)| & \text{on } \Gamma. \end{cases} \tag{5.2}
$$

The term $y|y|$ stands for y^2 that does not satisfy the assumptions on monotonicity required for our current work. However, in our computations negative values of y never occured so that in fact y^2 was used. This also assures that locally assumption (A4) is satisfied.

We fix: $\Omega = (0, 1)^2$, $\mu = 1$, $c(x_1, x_2) = x_2^2 + x_1x_2$, $e_y(x_1, x_2) = -3 - 2x_1^2 - 2x_1x_2$, $y_\Omega(x_1, x_2) = 1 + (x_1 + x_2)^2$, $e_1(x_1, x_2) = -2 + (1 + x_1^2 + x_1x_2)(x_2^2 + x_1x_2)$,

$$
e_u(x_1, x_2) = \begin{cases} 1 - x_1^3 & \text{on } \Gamma_1 \\ 1 - \min\left\{\begin{array}{l} 8(x_2 - 0.5)^2 + 0.5, \\ 1 - 16x_2(x_2 - 0.25)(x_2 - 0.75)(x_2 - 1) \end{array}\right\} & \text{on } \Gamma_2 \\ 1 - x_1^2 & \text{on } \Gamma_3 \\ 1 + x_2(1 - x_2) & \text{on } \Gamma_4 \end{cases}
$$

and

$$e_2(x_1, x_2) = \begin{cases} 2 - x_1 + 3x_1^2 - x_1^3 + x_1^4 & \text{on } \Gamma_1 \\ 8 + 6x_2 + x_2^2 - \min\{8(x_2 - .5)^2 + .5, 1\} & \text{on } \Gamma_2 \\ 2 + 4x_1 + 3x_1^2 + 2x_1^3 + x_1^4 & \text{on } \Gamma_3 \\ 2 - x_2 & \text{on } \Gamma_4. \end{cases}$$

This problem has the following solution $(\bar{y}, \bar{u})$ with adjoint state $\bar{\varphi}$: $\bar{y}(x) = 1 + 2x_1^2 + x_1x_2$, $\bar{\varphi}(x_1, x_2) = -1$ and $\bar{u}$ is the same as in example (El). Again, it holds $\bar{d}(x) = \bar{\varphi}(x) + e_u(x) + \bar{u}(x)$, which is also the same as in example (El) and satisfies relation (3.4) so that the first order necessary condition (3.3) is fulfilled. The second derivative of $J(\bar{u})$ is, according to (2.6),

$$J''(\bar{u})v^2 = \int_\Omega z_v(x)^2\,dx + \int_\Gamma v(x)^2 d\sigma(x) + \int_\Gamma (-2)\text{sign}(\bar{y}(x))\bar{\varphi}(x)z_v(x)^2 d\sigma(x),$$

where z_v is given by Eq. (2.3). Since $\bar{\varphi}(x) \leq 0$ and $\bar{y}(x) \geq 0$, clearly $J''(\bar{u})v^2 \geq \|v\|^2_{L^2(\Gamma)}$ holds. Therefore the second order sufficient conditions are fulfilled.

For the optimization, a standard SQP method was implemented; see for instance Heinkenschloss and Tröltzsch [16], Kelley and sachs [20]. Kunisch and Sachs [22] and Tröltzsch [28] and the references therein. Given $w_k = (y_k, u_k, \varphi_k)$, at step $k+1$ we have to solve the following linear-quadratic problem to find $(y_{k+1,} u_{k+1})$:

$$(QP)_{k+1} \begin{cases} \min J_{k+1}(u_{k+1}) = \dfrac{1}{2}\displaystyle\int_\Omega (y_{k+1}(x) - y_\Omega(x))^2\,dx + \dfrac{1}{2}\int_\Gamma u_{k+1}(x)^2\,d\sigma(x) \\ \qquad\qquad + \displaystyle\int_\Gamma e_u(x)u_{k+1}(x)\,d\sigma(x) + \int_\Gamma e_y(x)y_{k+1}(x)\,d\sigma(x) \\ \qquad\qquad - \displaystyle\int_\Gamma \text{sign}(y_k(x))\varphi_k(x)(y_{k+1}(x) - y_k(x))^2\,d\sigma(x) \\ \text{subject to } (y_{k+1}, u_{k+1}) \in H^1(\Omega) \times L^\infty(\Gamma), \\ \qquad u_{k+1} \in U_{ad}, \\ \qquad (y_{k+1}, u_{k+1}) \text{ satisfying the linear state equation (5.3)} \end{cases}$$

$$\begin{cases} -\triangle y_{k+1}(x) + c(x)y_{k+1}(x) = e_1(x) & \text{in } \Omega \\ \partial_\nu y_{k+1}(x) + y_{k+1}(x) = e_2(x) + u_{k+1}(x) - y_k(x) \mid y_k(x) \mid & \\ \qquad\qquad - 2 \mid y_k(x) \mid (y_{k+1}(x) - y_k(x)) & \text{on } \Gamma. \end{cases} \tag{5.3}$$

The new iterate φ_{k+1} is the solution of the associated adjoint equation. It is known (see Unger [29]) that the sequence $\{w_k\}$ converges quadratically to $\bar{w} = \{(\bar{y}, \bar{u}, \bar{\varphi})\}$ in the L^∞

norm provided that the initial guess is taken close to $\bar{w}$, where $(\bar{y}, \bar{u})$ is a local solution of (E2) and $\bar{\varphi}$ is the associated adjoint state:

$$\|w_{k+1} - \bar{w}\|_{C(\bar{\Omega})\times L^\infty(\Gamma)\times C(\bar{\Omega})} \le C\|w_k - \bar{w}\|^2_{c(\bar{\Omega})\times L^\infty(\Gamma)\times C(\bar{\Omega})}.$$

To solve each of the linear-quadratic problems $(QP)_k$ we have applied the primal-dual active set strategy explained for (E1). For the semilinear example the same tests were made as for (E1). First we considered the same mesh both for control and state. Next a very fine mesh was taken for the state while refining the meshes for the control.

Test 1.

h	$\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)}$	$\mid \bar{y} - \bar{y}_h \mid_{H^1(\Omega)}$	$\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)}$	$\|\bar{u} - \bar{u}_h\|_{L^\infty(\Gamma)}$
2^{-4}	$3.178397e-04$	$3.547400e-02$	$4.330792e-02$	$1.145619e-01$
2^{-5}	$8.094299e-05$	$1.769994e-02$	$2.170777e-02$	$5.988813e-02$
2^{-6}	$1.983313e-05$	$8.783231e-03$	$1.086060e-02$	$3.059566e-02$
2^{-7}	$4.938929e-06$	$4.365300e-03$	$5.431140e-03$	$1.546130e-02$

The observed orders of convergence are again h^2 for $\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)}$ and h for the other columns.

Test 2. We fix now the mesh size for the state to $h_y = 2^{-7}$. This ensures a fairly accurate solution of the partial differential equations. The order of convergence for the error in the control is again h.

h	$\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)}$	$\mid \bar{y} - \bar{y}_h \mid_{H^1(\Omega)}$	$\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)}$	$\|\bar{u} - \bar{u}_h\|_{L^\infty(\Gamma)}$
2^{-4}	$1.093204e-04$	$5.695770e-03$	$4.330780e-02$	$1.145649e-01$
2^{-5}	$2.782787e-05$	$4.498224e-03$	$2.170776e-02$	$5.988683e-02$
2^{-6}	$8.585435e-06$	$4.367794e-03$	$1.086060e-02$	$3.059585e-02$
2^{-7}	$4.938929e-06$	$8.365300e-03$	$5.431140e-03$	$1.546130e-02$

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Using piecewise linear functions in the numerical approximation of semilinear elliptic control problems ⋆

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This paper is dedicated to Mariano Gasca on the occasion of his 60th birthday

We study the numerical approximation of distributed optimal control problems governed by semilinear elliptic partial differential equations with pointwise constraints on the control. Piecewise linear finite elements are used to approximate the control as well as the state. We prove that the L^2-error estimates are of order $o(h)$, which is optimal according with the $C^{0,1}(\overline{\Omega})$-regularity of the optimal control.

Keywords: optimal control, semilinear elliptic equations, numerical approximation, error estimates.

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1. Introduction

In this paper we study an optimal control problem (P) governed by a semilinear elliptic equation, the control being distributed in the domain Ω. Bound constraints on the control are included in the formulation of the problem. Based on a standard finite element approximation, we set up an approximate optimal control problem (P_h). Our main aim is to estimate the error $\|\bar{u} - \bar{u}_h\|$, where $\bar{u}$ stands for an optimal control of (P) and $\bar{u}_h$ is an associated optimal one of (P_h). Error estimates for this problem were already obtained by Arada, Casas and Tröltzsch [1] by using piecewise constant functions to approximate the control space. With a such discretization the error estimate

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega_h)} \leqslant Ch, \tag{1.1}$$

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was proved. The proof of this estimate was based on the fact that the optimal controls are Lipschitz functions in $\overline{\Omega}$. If we think of the approximation of a Lipschitz function by a piecewise constant function, not necessarily solutions of any control problem, these estimates are optimal in general. If we want to improve the estimates we need more regularity for $\bar{u}$ and more regularity for the discrete functions. In particular, it is well known that

$$\lim_{h\to 0} \frac{1}{h}\|\bar{u} - \bar{u}_h\|_{L^2(\Omega_h)} = 0 \tag{1.2}$$

when $\bar{u}_h$ is a continuous piecewise linear function interpolating the Lipschitz function $\bar{u}$ in the nodes of the triangulation. If $\bar{u}$ belongs to the Sobolev space $H^2(\Omega)$, then order $\mathrm{O}(h^2)$ can be proved for the interpolation. But unfortunately the $H^2(\Omega)$-regularity fails for the optimal controls under the presence of bound constraints. Therefore it is natural to set the question about if the convergence (1.2) remains valid for the usual approximations by continuous piecewise linear functions. The goal of this paper is to prove that the answer is positive.

The estimate (1.1) has been also proved under the presence of a finitely number of equality and inequality integral constraints on the state by Casas [3]. The case of a Neumann boundary control was studied by Casas, Mateos and Tröltzsch [6]. In [1, 3, 6] was crucial the fact that we could obtain a representation of the discrete optimal controls analogous to the ones obtained for the continuous optimal controls, which allowed us to prove the uniform convergence $\bar{u}_h \to \bar{u}$. This representation cannot be obtained for piecewise linear optimal controls and consequently we do not know to deduce the uniform convergence, just we can prove the L^2-convergence. To overcome this difficulty we have followed a different approach in order to prove (1.2).

As far as we know, the only previous paper concerned with the error estimates for piecewise linear approximations of the control is due to Casas and Raymond [7]. In this paper the case of a Dirichlet boundary control was studied. In this case it is necessary to consider the same approximations for the control and the states, therefore piecewise linear approximations were decided as optimal. For the optimal control we established the regularity $\bar{u} \in W^{1-1/p,p}(\Gamma)$ for some $p = 2 + \varepsilon$, which allowed to prove

$$\lim_{h\to 0} \frac{1}{h^{1/2}}\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} = 0. \tag{1.3}$$

The proof of these error estimates given in [7] has inspired the corresponding proof of this paper, but some new ideas have been necessary to achieve the desired result.

There is no many papers in the literature concerning the error estimates for control problems governed by partial differential equations. The pioneer works in the context of linear quadratic control problems are due to Falk [12] and Geveci [13]. For nonlinear equations the situation is more difficult because second-order optimality conditions are required to get the estimates. These conditions for optimality has been obtained in the last years; see Bonnans and Zidani [2], Casas and Mateos [4], Casas and Tröltzsch [8], Raymond and Tröltzsch [23].

In the case of parabolic problems the approximation theory is far from being complete, but some research has been carried out; see Knowles [17], Lasiecka [18, 19], McKnight and Bosarge [21], Tiba and Tröltzsch [24] and Tröltzsch [25–28].

In the context of control problems of ordinary differential equations a great work has been done by Hager [15, 16] and Dontchev and Hager [10, 11]; see also the work by Malanowski et al. [20]. The reader is also referred to the detailed bibliography in [11].

The plan of the paper is as follows. In section 2 the control problem is introduced and the first and second order optimality conditions are recalled. From the first order optimality conditions, the Lipschitz property of the optimal controls is deduced. In section 3 the numerical approximation of (P) is carried out. In this section the first order optimality conditions are also given and the convergence properties of the discretization are established. Finally the error estimate (1.2) is proved in section 4.

2. The control problem

Given an open bounded and convex set $\Omega \subset \mathbb{R}^n$, with $n = 2$ or 3, Γ being its boundary of class $C^{1,1}$, we consider the following Dirichlet boundary value problem in this domain

$$\begin{cases} Ay + f(x, y) = u & \text{in } \Omega, \\ y = 0 & \text{on } \Gamma, \end{cases} \tag{2.1}$$

where

$$Ay = -\sum_{i,j=1}^{n} \partial_{x_j}\big(a_{ij}(x)\partial_{x_i} y(x)\big) + a_0(x)y(x),$$

with $a_{ij} \in C^{0,1}(\overline{\Omega})$ and $a_0 \in L^\infty(\Omega)$ satisfying

$$\begin{cases} \exists m > 0 \quad \text{such that} \quad \displaystyle\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geqslant m|\xi|^2 \quad \forall\xi \in \mathbb{R}^n \text{ and } \forall x \in \Omega, \\ a_0(x) \geqslant 0 \quad \text{a.e. } x \in \Omega, \end{cases}$$

and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a given function. The control is denoted by u and the solution of the above system y_u is the corresponding state. The assumptions will be precise below.

Now we consider the control problem

$$(\mathrm{P}) \quad \begin{cases} \text{Minimize } J(u) = \displaystyle\int_\Omega L\big(x, y_u(x), u(x)\big)\,\mathrm{d}x, \\ u \in \mathbb{K} = \big\{u \in L^\infty(\Omega)\colon\ \alpha \leqslant u(x) \leqslant \beta \text{ a.e. } x \in \Omega\big\}, \end{cases}$$

where $-\infty < \alpha < \beta < +\infty$ are fixed and $L : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is a given function. Let us state the assumptions on the functions involved in the control problem (P).

(A1) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathédory function of class C^2 with respect to the second variable,

$$f(\cdot, 0) \in L^\infty(\Omega), \quad \frac{\partial f}{\partial y}(x, y) \geqslant 0$$

and for all $M > 0$ there exists a constant $C_{f,M} > 0$ such that

$$\left|\frac{\partial f}{\partial y}(x, y)\right| + \left|\frac{\partial^2 f}{\partial y^2}(x, y)\right| \leqslant C_{f,M} \qquad \text{for a.e. } x \in \Omega \text{ and } |y| \leqslant M,$$

$$\left|\frac{\partial^2 f}{\partial y^2}(x, y_2) - \frac{\partial^2 f}{\partial y^2}(x, y_1)\right| < C_{f,M}|y_2 - y_1| \quad \text{for } |y_1|, |y_2| \leqslant M \text{ and } x \in \Omega.$$

(A2) $L : \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a Carathédory function of class C^2 with respect to the second and third variables, $L(\cdot, 0, 0) \in L^1(\Omega)$, and for all $M > 0$ there exist a constant $C_{L,M} > 0$ and a function $\psi_M \in L^p(\Omega)$ $(p > n)$ such that

$$\left|\frac{\partial L}{\partial y}(x, y, u)\right| \leqslant \psi_M(x), \qquad \left\| D^2_{(y,u)} L(x, y, u)\right\| \leqslant C_{L,M},$$

$$\left|\frac{\partial L}{\partial u}(x_2, y, u) - \frac{\partial L}{\partial u}(x_1, y, u)\right| \leqslant C_{L,M}|x_2 - x_1|,$$

$$\left\| D^2_{(y,u)} L(x, y_2, u_2) - D^2_{(y,u)} L(x, y_1, u_1)\right\| \leqslant C_{L,M}\big(|y_2 - y_1| + |u_2 - u_1|\big),$$

for a.e. $x, x_i \in \Omega$ and $|y|, |y_i|, |u|, |u_i| \leqslant M$, $i = 1, 2$, where $D^2_{(y,u)} L$ denotes the second derivative of L with respect to (y, u). Moreover we assume that there exists $\Lambda > 0$ such that

$$\frac{\partial^2 L}{\partial u^2}(x, y, u) \geqslant \Lambda, \quad \text{a.e. } x \in \Omega \text{ and } (y, u) \in \mathbb{R}^2.$$

It is well known that the state equation (2.1) has a unique solution $y_u \in H_0^1(\Omega) \cap W^{2,p}(\Omega)$ for any $1 \leqslant p < +\infty$; see Grisvard [14] for the $W^{2,p}(\Omega)$ regularity results.

Under the previous assumptions it is easy to prove the existence of a solution of problem (P). In the proof it is essential the convexity of L with respect to the control. In (A2) we have assume that L is strictly convex with respect to u, which will be useful to prove the strong convergence of the discretizations. Therefore this strong convexity is not a too restrictive assumption if we want to have a well posed problem in the sense that it has at least one solution. However, there is a situation which is interesting in practice and it is not included in our formulation. This is the case of a function L depending only on the variables (x, y), but not on u. The optimal control problem is typically bang-bang in this situation. It is an open problem for us the derivation of the error estimates in the bang-bang case.

Among the functionals included in our problem, we can consider those of the type $L(x, y, u) = g(x, y) + h(u)$, with $h''(u) \geqslant \Lambda$. For instance, the classical example $L(x, y, u) = (y - y_d(x))^2 + \Lambda u^2$, with $\Lambda > 0$ is of this type.

Before taking a decision about the type of finite elements we are going to choose in order to formulate a discrete version of the control problem (P), the regularity of the optimal controls $\bar{u}$ must be investigated. These regularity properties can be deduced from the first order optimality conditions. On the other hand, the proof of the error estimates require the sufficient second order conditions. The rest of this section is devoted to the formulation of these optimality conditions and to the study of the regularity of the optimal controls. As a first step, let us recalling the differentiability properties of the functionals involve in the control problem. For the detailed proofs the reader is referred to Casas and Mateos [4].

Theorem 2.1. For every $u \in L^\infty(\Omega)$, the state equation (2.1) has a unique solution y_u in the space $W^{2,p}(\Omega)$ and the mapping $G : L^\infty(\Omega) \to W^{2,p}(\Omega)$, defined by $G(u) = y_u$ is of class C^2. Moreover for all $v, u \in L^\infty(\Omega)$, $z_v = G'(u)v$ is defined as the solution of

$$\begin{cases} Az_v + \dfrac{\partial f}{\partial y}(x, y_u)z_v = v & \text{in } \Omega, \\ z_v = 0 & \text{on } \Gamma. \end{cases} \tag{2.2}$$

Finally, for every $v_1, v_2 \in L^\infty(\Omega)$, $z_{v_1v_2} = G''(u)v_1v_2$ is the solution of

$$\begin{cases} Az_{v_1v_2} + \dfrac{\partial f}{\partial y}(x, y_u)z_{v_1v_2} + \dfrac{\partial^2 f}{\partial y^2}(x, y_u)z_{v_1}z_{v_2} = 0 & \text{in } \Omega, \\ z_{v_1v_2} = 0 & \text{on } \Gamma, \end{cases} \tag{2.3}$$

where $z_{v_i} = G'(u)v_i$, $i = 1, 2$.

The value of p in the previous theorem is that one considered in assumption (A2) for the regularity of ψ_M.

Theorem 2.2. The functional $J : L^\infty(\Omega) \to \mathbb{R}$ is of class C^2. Moreover, for every $u, v, v_1, v_2 \in L^\infty(\Omega)$

$$J'(u)v = \int_\Omega \left(\frac{\partial L}{\partial u}(x, y_u, u) + \varphi_u\right) v \, \mathrm{d}x \tag{2.4}$$

and

$$\begin{aligned} J''(u)v_1v_2 = \int_\Omega \bigg[&\frac{\partial^2 L}{\partial y^2}(x, y_u, u)z_{v_1}z_{v_2} + \frac{\partial^2 L}{\partial y \partial u}(x, y_u, u)(z_{v_1}v_2 + z_{v_2}v_1) \\ &+ \frac{\partial^2 L}{\partial u^2}(x, y_u, u)v_1v_2 - \varphi_u \frac{\partial^2 f}{\partial y^2}(x, y_u)z_{v_1}z_{v_2} \bigg] \mathrm{d}x, \end{aligned} \tag{2.5}$$

where $y_u = G(u)$, $\varphi_u \in W^{2,p}(\Omega)$ is the unique solution of the problem

$$\begin{cases} A^*\varphi + \dfrac{\partial f}{\partial y}(x, y_u)\varphi = \dfrac{\partial L}{\partial y}(x, y_u, u) & \text{in } \Omega, \\ \varphi = 0 \quad \text{on } \Gamma, \end{cases} \tag{2.6}$$

where A^* is the adjoint operator of A and $z_{v_i} = G'(u)v_i$, $i = 1, 2$.

From our assumptions (A1) and (A2) it is easy to check that $J''(u)$ can be extended to a continuous quadratic function in $L^2(\Omega)$. Indeed it is enough to verify that the integrals of (2.5) are well defined for any function $v \in L^2(\Omega)$ and they are continuous with respect to the topology of $L^2(\Omega)$. This property will be used later.

Now the first order optimality conditions can be easily deduced from the above theorem by the classical procedure:

Theorem 2.3. Let $\bar{u}$ be a local minimum of (P). Then there exist $\bar{y}, \overline{\varphi} \in H_0^1(\Omega) \cap W^{2,p}(\Omega)$ such that the following relations hold:

$$\begin{cases} A\bar{y} + f(x, \bar{y}) = \bar{u} & \text{in } \Omega, \\ \bar{y} = 0 \quad \text{on } \Gamma, \end{cases} \tag{2.7}$$

$$\begin{cases} A^*\overline{\varphi} + \dfrac{\partial f}{\partial y}(x, \bar{y})\overline{\varphi} = \dfrac{\partial L}{\partial y}(x, \bar{y}, \bar{u}) & \text{in } \Omega, \\ \overline{\varphi} = 0 \quad \text{on } \Gamma, \end{cases} \tag{2.8}$$

$$\int_\Omega \left\{ \overline{\varphi}(x) + \frac{\partial L}{\partial u}\big(x, \bar{y}(x), \bar{u}(x)\big) \right\} \big(u(x) - \bar{u}(x)\big)\, \mathrm{d}x \geqslant 0 \quad \forall u \in \mathbb{K}. \tag{2.9}$$

From this theorem we deduce the regularity of $\bar{u}$; see Arada, Casas and Tröltzsch [1] for the proof.

Theorem 2.4. Let $\bar{u}$ be a local minimum of (P). Then for every $x \in \overline{\Omega}$, the equation

$$\overline{\varphi}(x) + \frac{\partial L}{\partial u}\big(x, \bar{y}(x), t\big) = 0 \tag{2.10}$$

has a unique solution $\bar{t} = \bar{s}(x)$, where $\bar{y}$ is the state associated to $\bar{u}$ and $\overline{\varphi}$ is the adjoint state defined by (2.8). The function $\bar{s} : \overline{\Omega} \to \mathbb{R}$ is Lipschitz. Moreover $\bar{u}$ and $\bar{s}$ are related by the formula

$$\bar{u}(x) = \mathrm{Proj}_{[\alpha,\beta]}\big(\bar{s}(x)\big) = \max\big(\alpha, \min\big(\beta, \bar{s}(x)\big)\big), \tag{2.11}$$

and $\bar{u}$ is also a Lipschitz function.

Let us finish this section by formulating the necessary and sufficient second-order conditions for optimality. Let $\bar{u}$ be a local minimum of (P), with $\bar{y}$ and $\overline{\varphi}$ the associated state and adjoint state respectively. To simplify the notation let us introduce the function

$$\bar{d}(x) = \frac{\partial L}{\partial u}\big(x, \bar{y}(x), \bar{u}(x)\big) + \overline{\varphi}(x). \tag{2.12}$$

From (2.9) we get

$$\bar{d}(x) = \begin{cases} 0 & \text{if } \alpha < \bar{u}(x) < \beta, \\ \geqslant 0 & \text{if } \bar{u}(x) = \alpha, \\ \leqslant 0 & \text{if } \bar{u}(x) = \beta. \end{cases} \tag{2.13}$$

Now we define the *cone of critical directions*

$$C_{\bar{u}} = \big\{v \in L^2(\Omega) \text{ satisfying (2.14) and } v(x) = 0 \text{ if } \bar{d}(x) \neq 0\big\},$$
$$v(x) = \begin{cases} \geqslant 0 & \text{a.e. } x \in \Omega \text{ if } \bar{u}(x) = \alpha, \\ \leqslant 0 & \text{a.e. } x \in \Omega \text{ if } \bar{u}(x) = \beta. \end{cases} \tag{2.14}$$

Now we are ready to state the second-order necessary and sufficient optimality conditions.

Theorem 2.5. If $\bar{u}$ is a local minimum of (P), then

$$J''(\bar{u})v^2 \geqslant 0 \quad \forall v \in C_{\bar{u}}. \tag{2.15}$$

Reciprocally, if $\bar{u} \in \mathbb{K}$ satisfies the first-order optimality conditions (2.7)–(2.9) and the second order condition

$$J''(\bar{u})v^2 > 0 \quad \forall v \in C_{\bar{u}} \setminus \{0\}, \tag{2.16}$$

then there exist $\delta > 0$ and $\varepsilon > 0$ such that

$$J(u) \geqslant J(\bar{u}) + \frac{\delta}{2}\|u - \bar{u}\|^2_{L^2(\Omega)} \quad \forall u \in \mathbb{K} \cap \overline{B}_\varepsilon(\bar{u}), \tag{2.17}$$

where $\overline{B}_\varepsilon(\bar{u})$ denotes the closed ball of $L^\infty(\Omega)$ with center at $\bar{u}$ and radius ε.

Sufficient optimality conditions (2.16) can be formulated in an equivalent form, which is more convenient for us to prove the error estimates of the numerical discretizations of (P).

Theorem 2.6. Let $\bar{u}$ be an element of $\mathbb{K}$ satisfying (2.9), then the following statements are equivalent:

$$J''(\bar{u})v^2 > 0 \quad \forall v \in C_{\bar{u}} \setminus \{0\} \tag{2.18}$$

and

$$\exists \delta > 0 \text{ and } \exists \tau > 0 \quad \text{such that} \quad J''(\bar{u})v^2 \geqslant \delta\|v\|^2_{L^2(\Omega)} \quad \forall v \in C^\tau_{\bar{u}}, \tag{2.19}$$

where

$$C^{\tau}_{\bar u} = \left\{v \in L^2(\Omega) \text{ satisfying (2.14) and } v(x) = 0 \text{ if } \left|\bar d(x)\right| > \tau\right\}.$$

3. Numerical approximation by using piecewise linear functions

Here we define a finite-element based approximation of the optimal control problem (P). To this aim, we consider a family of triangulations $\{\mathcal{T}_h\}_{h>0}$ of $\overline{\Omega}$. This triangulation is supposed to be regular in the usual sense that we state exactly here. With each element $T \in \mathcal{T}_h$, we associate two parameters $\rho(T)$ and $\sigma(T)$, where $\rho(T)$ denotes the diameter of the set T and $\sigma(T)$ is the diameter of the largest ball contained in T. Define the size of the mesh by $h = \max_{T\in\mathcal{T}_h} \rho(T)$. We suppose that the following regularity assumptions are satisfied.

(i) There exist two positive constants ρ and σ such that

$$\frac{\rho(T)}{\sigma(T)} \leqslant \sigma, \qquad \frac{h}{\rho(T)} \leqslant \rho$$

hold for all $T \in \mathcal{T}_h$ and all $h > 0$.

(ii) Let us take $\overline{\Omega}_h = \bigcup_{T\in\mathcal{T}_h} T$, and let Ω_h and Γ_h denote its interior and its boundary, respectively. We assume that $\overline{\Omega}_h$ is convex and that the vertices of $\mathcal{T}_h$ placed on the boundary of Γ_h are points of Γ. From [22, estimate (5.2.19)] we know

$$|\Omega \setminus \Omega_h| \leqslant Ch^2, \tag{3.1}$$

where $|B|$ denotes the Lebesgue measure of a measurable set $B \subset \mathbb{R}^n$. Let us set

$$U_h = \left\{u \in C\left(\overline{\Omega}_h\right) \mid u_{|T} \in \mathcal{P}_1, \text{ for all } T \in \mathcal{T}_h\right\},$$
$$Y_h = \left\{y_h \in C\left(\overline{\Omega}\right) \mid y_{h|T} \in \mathcal{P}_1, \text{ for all } T \in \mathcal{T}_h, \text{ and } y_h = 0 \text{ on } \overline{\Omega} \setminus \Omega_h\right\},$$

where $\mathcal{P}_1$ is the space of polynomials of degree less or equal than 1. Let us denote by $\{x_j\}_{j=1}^{N(h)}$ the nodes of the triangulation $\mathcal{T}_h$. A basis of U_h is formed by the functions $\{e_j\}_{j=1}^{N(h)} \subset U_h$ defined by their values on the nodes x_j

$$e_j(x_i) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

In the sequel we will follow the notation $u_{hj} = u_h(x_j)$ for any function $u_h \in U_h$, so that

$$u_h = \sum_{j=1}^{N(h)} u_{hj} e_j.$$

For each $u \in L^\infty(\Omega_h)$, we denote by $y_h(u)$ the unique element of Y_h that satisfies

$$a_h\big(y_h(u), z_h\big) + \int_{\Omega_h} f\big(x, y_h(u)\big) z_h(x)\,\mathrm{d}x = \int_{\Omega_h} u(x) z_h(x)\,\mathrm{d}x \quad \forall z_h \in Y_h, \tag{3.2}$$

where $a_h : Y_h \times Y_h \to \mathbb{R}$ is the bilinear form defined by

$$a_h(y_h, z_h) = \int_{\Omega_h} \left(\sum_{i,j=1}^{n} a_{ij}(x) \partial_{x_i} y_h(x) \partial_{x_j} z_h(x) + a_0(x) y_h(x) z_h(x) \right) \mathrm{d}x.$$

In other words, $y_h(u)$ is the approximate state associated with u. Notice that $y_h = z_h = 0$ in $\Omega \setminus \Omega_h$, therefore the previous integrals are equivalent to the integration on Ω. The finite dimensional approximation of the optimal control problem is defined by

$$(\mathrm{P}_h) \quad \begin{cases} \min J_h(u_h) = \displaystyle\int_{\Omega_h} L\big(x, y_h(u_h)(x), u_h(x)\big)\,\mathrm{d}x, \\ \text{subject to} \quad \big(y_h(u_h), u_h\big) \in Y_h \times U_h, \\ \qquad\qquad\quad u_h \in \mathbb{K}_h = \big\{u_h \in U_h \colon \alpha \leqslant u_{hj} \leqslant \beta \quad 1 \leqslant j \leqslant N(h)\big\}. \end{cases}$$

We start the study of problem (P_h) by analyzing the differentiability of the functions involved in the control problem. Let us collect the differentiability results analogous to those of section 2.

Proposition 3.1. For every $u \in L^\infty(\Omega_h)$, problem (3.2) has a unique solution $y_h(u) \in Y_h$, the mapping $G_h : L^\infty(\Omega_h) \to Y_h$, defined by $G_h(u) = y_h(u)$, is of class C^2 and for all $v, u \in L^\infty(\Omega_h)$, $z_h(v) = G_h'(u)v$ is the solution of

$$a_h\big(z_h(v), q_h\big) + \int_\Omega \frac{\partial f}{\partial y}\big(x, y_h(u)\big) z_h(v) q_h\,\mathrm{d}x = \int_\Omega v q_h\,\mathrm{d}x \quad \forall q_h \in Y_h. \tag{3.3}$$

Finally, for all $v_1, v_2 \in L^\infty(\Omega)$, $z_h(v_1, v_2) = G''(u)v_1 v_2 \in Y_h$ is the solution of the variational equation

$$a_h(z_h, q_h) + \int_\Omega \frac{\partial f}{\partial y}\big(x, y_h(u)\big) z_h q_h\,\mathrm{d}x + \int_\Omega \frac{\partial^2 f}{\partial y^2}\big(x, y_h(u)\big) z_{h1} z_{h2} q_h\,\mathrm{d}x = 0 \quad \forall q_h \in Y_h, \tag{3.4}$$

where $z_{hi} = G_h'(u)v_i$, $i = 1, 2$.

Proposition 3.2. The functional $J_h : L^\infty(\Omega_h) \to \mathbb{R}$ is of class C^2. Moreover for all $u, v, v_1, v_2 \in L^\infty(\Omega_h)$

$$J_h'(u)v = \int_{\Omega_h} \left(\frac{\partial L}{\partial u}\big(x, y_h(u), u\big) + \varphi_h(u) \right) v\,\mathrm{d}x \tag{3.5}$$

and

$$J_h''(u)v_1v_2 = \int_{\Omega_h}\Big[\frac{\partial^2 L}{\partial y^2}\big(x, y_h(u), u\big)z_h(v_1)z_h(v_2) + \frac{\partial^2 L}{\partial y\partial u}\big(x, y_h(u), u\big)\big[z_h(v_1)v_2 + z_h(v_2)v_1\big] + \frac{\partial^2 L}{\partial u^2}\big(x, y_h(u), u\big)v_1v_2 - \varphi_h(u)\frac{\partial^2 f}{\partial y^2}\big(x, y_h(u)\big)z_{h1}z_{h2}\Big]\,\mathrm{d}x, \quad (3.6)$$

where $y_h(u) = G_h(u)$ and $\varphi_h(u) \in Y_h$ is the unique solution of the problem

$$a_h\big(q_h, \varphi_h(u)\big) + \int_\Omega \frac{\partial f}{\partial y}\big(x, y_h(u)\big)\varphi_h(u)q_h\,\mathrm{d}x = \int_\Omega \frac{\partial L}{\partial y}\big(x, y_h(u), u\big)q_h\,\mathrm{d}x \quad \forall q_h \in Y_h, \quad (3.7)$$

with $z_{hi} = G_h'(u)v_i$, $i = 1, 2$.

Let us conclude this section by writing the first-order optimality conditions for (P_h).

Theorem 3.3. For every $h > 0$ problem (P_h) has at least one solution. Moreover, if $\bar u_h$ is a local minimum of (P_h), then there exist $\bar y_h, \overline{\varphi}_h \in Y_h$ such that

$$a(\bar y_h, q_h) + \int_\Omega f(x, \bar y_h)q_h(x)\,\mathrm{d}x = \int_\Omega \bar u_h(x)q_h(x)\,\mathrm{d}x \quad \forall q_h \in Y_h, \quad (3.8)$$

$$a(q_h, \overline{\varphi}_h) + \int_\Omega \frac{\partial f}{\partial y}(x, \bar y_h)\overline{\varphi}_h q_h\,\mathrm{d}x = \int_\Omega \frac{\partial L}{\partial y}(x, \bar y_h, \bar u_h)q_h\,\mathrm{d}x \quad \forall q_h \in Y_h, \quad (3.9)$$

$$\int_{\Omega_h}\Big\{\overline{\varphi}_h + \frac{\partial L}{\partial u}(x, \bar y_h, \bar u_h)\Big\}(u_h - \bar u_h)\,\mathrm{d}x \geqslant 0 \quad \forall u_h \in \mathbb{K}_h. \quad (3.10)$$

Proof. The existence of a solution of (P_h) is an immediate consequence of the compactness of $\mathbb{K}_h$ in U_h an the continuity of J_h in $\mathbb{K}_h$. The optimality system (3.8)–(3.10) is obtained by classical arguments with the help of proposition 3.2. □

We finish this section by proving the convergence of the solutions of (P_h) toward the solutions of (P). But first we are going to summarize some estimates and properties of $y_h(u_h) - y_u$ and $\varphi_h(u_h) - \varphi(u)$. The properties we need are collected in two lemmas whose proofs can be obtained in Arada, Casas and Tröltzsch [1] and Casas and Mateos [5].

Lemma 3.4. Let $v_2, v_1 \in L^\infty(\Omega)$ satisfy $\|v_i\|_{L^\infty(\Omega)} \leqslant M$, $i = 1, 2$, for some $M < \infty$. Let us suppose that y_{v_2} and $y_h(v_1)$ are the solutions of (2.1) and (3.2) corresponding to v_2 and v_1, respectively. Moreover let φ_{v_2} and $\varphi_h(v_1)$ be the solutions of (2.6) and (3.7) also corresponding to v_2 and v_1 respectively. Then the following estimates hold:

$$\|y_{v_2} - y_h(v_1)\|_{H^1(\Omega)} + \|\varphi_{v_2} - \varphi_h(v_1)\|_{H^1(\Omega)} \leqslant C\big(h + \|v_2 - v_1\|_{L^2(\Omega)}\big), \tag{3.11}$$

$$\|y_{v_2} - y_h(v_1)\|_{L^2(\Omega)} + \|\varphi_{v_2} - \varphi_h(v_1)\|_{L^2(\Omega)} \leqslant C\big(h^2 + \|v_2 - v_1\|_{L^2(\Omega)}\big), \tag{3.12}$$

$$\|y_{v_2} - y_h(v_1)\|_{L^\infty(\Omega)} + \|\varphi_{v_2} - \varphi_h(v_1)\|_{L^\infty(\Omega)} \leqslant C\big(h^\sigma + \|v_2 - v_1\|_{L^2(\Omega)}\big), \tag{3.13}$$

where $C \equiv C(\Omega, n, M)$ is independent of h, and $\sigma = 1$ if $n = 2$ or the triangulation is of nonnegative type and $\sigma = 1/2$ otherwise.

This result follows from [1, theorem 4.2].

The reader is referred to Ciarlet [9] for the definition and properties of a nonnegative type triangulation.

Lemma 3.5. Let $\{u_h\}_{h>0}$ be a sequence, with $u_h \in \mathbb{K}_h$ and $u_h \rightharpoonup u$ weakly in $L^1(\Omega)$, then $y_h(u_h) \to y_u$ and $\varphi_h(u_h) \to \varphi_u$ in $H_0^1(\Omega) \cap C(\overline{\Omega})$ strongly. Moreover $J(u) \leqslant \liminf_{h\to 0} J_h(u_h)$.

For the proof the reader can consult [5, theorem 9 and lemma 11]. The following theorem is also proved in [5, theorems 11 and 12].

Theorem 3.6. For every $h > 0$ let $\bar{u}_h$ be a solution of (P$_h$). Then there exist subsequences $\{\bar{u}_h\}_{h>0}$ converging in the weak* topology of $L^\infty(\Omega)$, that will be denoted in the same way. If $\bar{u}_h \rightharpoonup \bar{u}$ in the mentioned topology, then $\bar{u}$ is a solution of (P) and we have

$$\lim_{h\to 0} J_h(\bar{u}_h) = J(\bar{u}) = \inf(P) \quad \text{and} \quad \lim_{h\to 0} \|\bar{u} - \bar{u}_h\|_{L^2(\Omega_h)} = 0. \tag{3.14}$$

4. Error estimates

The goal of this section is to prove the following theorem.

Theorem 4.1. Let us assume that $\bar{u}$ is a local solution of (P) satisfying the sufficient second-order optimality conditions provided in theorem 2.5 and let $\bar{u}_h$ be a local solution of (P$_h$) such that

$$\lim_{h\to 0} \|\bar{u} - \bar{u}_h\|_{L^2(\Omega_h)} = 0; \tag{4.1}$$

see theorem 3.6. Then the following identity holds:

$$\lim_{h\to 0} \frac{1}{h}\|\bar{u} - \bar{u}_h\|_{L^2(\Omega_h)} = 0. \tag{4.2}$$

We will prove the theorem arguing by contradiction. If (4.2) is false, then

$$\limsup_{h\to 0} \frac{1}{h}\|\bar{u} - \bar{u}_h\|_{L^2(\Omega_h)} > 0,$$

eventually $+\infty$, therefore there exists a constant $C > 0$ and a sequence of h, denoted in the same way, such that

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega_h)} \geqslant Ch \quad \forall h > 0. \tag{4.3}$$

We will obtain a contradiction for this sequence. In the sequel we consider the extension of $\bar{u}_h$ to Ω, keeping the same notation, as follows

$$\bar{u}_h(x) = \begin{cases} \bar{u}_h(x) & \text{if } x \in \overline{\Omega}_h, \\ \bar{u}(x) & \text{otherwise.} \end{cases} \tag{4.4}$$

The interpolation operator $\Pi_h : C(\overline{\Omega}) \to U_h$ is defined by

$$\Pi_h u = \sum_{j=1}^{N(h)} u(x_j)e_j, \tag{4.5}$$

where $\{e_j\}_{j=1}^{N(h)}$ is the basis of U_h introduced in section 3. It is well known that

$$\lim_{h\to 0} \frac{1}{h}\|u - \Pi_h u\|_{L^p(\Omega_h)} = 0 \quad \forall u \in W^{1,p}(\Omega) \text{ and } n < p < +\infty. \tag{4.6}$$

As above, we also consider the extension of $\Pi_h\bar{u}$ to $\overline{\Omega}$ by setting $\Pi_h\bar{u} = \bar{u}$ on $\overline{\Omega} \setminus \overline{\Omega}_h$.

For the proof of theorem 4.1 we need some lemmas.

Lemma 4.2. Let us assume that (4.2) is false. Let $\delta > 0$ and Λ be the parameters introduced in assumption (2.19) and (A2), respectively. Then there exist $h_0 > 0$ and $\mu > 0$ independent of h such that

$$\mu\|\bar{u} - \bar{u}_h\|^2_{L^2(\Omega_h)} \leqslant \big(J'(\bar{u}_h) - J'(\bar{u})\big)(\bar{u}_h - \bar{u}) \quad \forall h < h_0. \tag{4.7}$$

Proof. By applying the mean value theorem we get for some $\hat{u}_h = \bar{u} + \theta_h(\bar{u}_h - \bar{u})$

$$\big(J'(\bar{u}_h) - J'(\bar{u})\big)(\bar{u}_h - \bar{u}) = J''(\hat{u}_h)(\bar{u}_h - \bar{u})^2. \tag{4.8}$$

Let us take

$$v_h = \frac{1}{\|\bar{u}_h - \bar{u}\|_{L^2(\Omega)}}(\bar{u}_h - \bar{u}).$$

Let $\{h_k\}_{k=1}^{\infty}$ be a sequence converging to 0 such that

$$\lim_{k\to\infty} J''_{h_k}(\hat{u}_{h_k})v^2_{h_k} = \liminf_{h\to 0} J''_h(\hat{u}_h)v^2_h, \quad \bar{u}_{h_k}(x) \to \bar{u}(x) \text{ a.e. } x \in \Omega, \quad v_{h_k} \rightharpoonup v \text{ in } L^2(\Omega).$$

The goal is to prove that

$$\liminf_{h\to 0} J''_h(\hat{u})v^2_h \geqslant \begin{cases} \Lambda & \text{if } v = 0, \\ \delta\|v\|^2_{L^2(\Omega)} & \text{if } v \neq 0, \end{cases} \tag{4.9}$$

where Λ is introduced in assumption (A2). Then (4.7) follows from (4.8) and this inequality by taking $\mu = \Lambda/2$ if $v = 0$ and $\mu = (\delta/2)\|v\|^2_{L^2(\Omega)}$ otherwise.

In order to simplify the notation we will replace h_k by h, but we must have in mind that $\{h\}$ is the sequence $\{h_k\}_{k=1}^{\infty}$. Let us prove that v belongs to the critical cone $C_{\bar{u}}$ defined in section 2. First of all remark that every v_h satisfies the sign condition (2.14), hence v also does. Let us prove that $v(x) = 0$ if $\bar{d}(x) \neq 0$, $\bar{d}$ being defined by (2.12). We will use the interpolation operator Π_h defined by (4.5). Since $\bar{u} \in \mathbb{K}$ it is obvious that $\Pi_h \bar{u} \in \mathbb{K}_h$. Let us define

$$\bar{d}_h(x) = \overline{\varphi}_h(x) + \frac{\partial L}{\partial u}\big(x, \bar{y}_h(x), \bar{u}_h(x)\big). \tag{4.10}$$

From assumption (A2), (4.1), (4.4) and lemma 3.5 we deduce that $\bar{d}_h \to \bar{d}$ in $L^2(\Omega)$. Now we have

$$\begin{aligned}\int_\Omega \bar{d}(x)v(x)\,\mathrm{d}x &= \lim_{h\to 0}\int_{\Omega_h} \bar{d}_h(x)v_h(x)\,\mathrm{d}x\\ &= \lim_{h\to 0}\frac{1}{\|u_h-\bar{u}\|_{L^2(\Omega_h)}}\left\{\int_{\Omega_h}\bar{d}_h(\Pi_h\bar{u}-\bar{u})\,\mathrm{d}x + \int_{\Omega_h}\bar{d}_h(\bar{u}_h-\Pi_h\bar{u})\,\mathrm{d}x\right\}.\end{aligned}$$

Using the regularity of $\bar{u}$ proved in theorem 2.4 and (3.10), (4.3) and (4.6) we deduce

$$\begin{aligned}\int_\Omega \bar{d}(x)v(x)\,\mathrm{d}x &\leqslant \lim_{h\to 0}\frac{1}{\|u_h-\bar{u}\|_{L^2(\Omega_h)}}\int_{\Omega_h}\bar{d}_h(x)\big(\Pi_h\bar{u}(x)-\bar{u}(x)\big)\,\mathrm{d}x\\ &\leqslant C\lim_{h\to 0}\frac{1}{h}\|\bar{u}-\Pi_h\bar{u}\|_{L^2(\Omega_h)} = 0.\end{aligned}$$

Since v satisfies the sign condition (2.14), then $\bar{d}(x)v(x) \geqslant 0$, hence the above inequality proves that v is zero whenever $\bar{d} \neq 0$, which allows us to conclude that $v \in C_{\bar{u}}$.

Let us assume first that $v = 0$, then from the definition of v_h, (2.5), (2.19) and assumption (A2) we get

$$\begin{aligned}\liminf_{h\to 0} J''(\hat{u}_h)v_h^2 &= \liminf_{h\to 0}\left\{\int_\Omega\left[\frac{\partial^2 L}{\partial y^2}(x, y_{\hat{u}_h}, \hat{u}_h) - \varphi_{\hat{u}_h}\frac{\partial^2 f}{\partial y^2}(x, y_{\hat{u}_h})\right]z_{v_h}^2\,\mathrm{d}x\right.\\ &\qquad\left. + 2\int_\Omega\frac{\partial^2 L}{\partial y\partial u}(x, y_{\hat{u}_h}, \hat{u}_h)z_{v_h}v_h\,\mathrm{d}x + \int_\Omega\frac{\partial^2 L}{\partial u^2}(x, y_{\hat{u}_h}, \hat{u}_h)v_h^2\,\mathrm{d}x\right\}\\ &\geqslant \int_\Omega\left[\frac{\partial^2 L}{\partial y^2}(x, \bar{y}, \bar{u}) - \overline{\varphi}\frac{\partial^2 f}{\partial y^2}(x, \bar{y})\right]z_v^2\,\mathrm{d}x\\ &\qquad + 2\int_\Omega\frac{\partial^2 L}{\partial y\partial u}(x, \bar{y}, \bar{u})z_v v\,\mathrm{d}x + \Lambda = \Lambda,\end{aligned}$$

which implies (4.9) for $v = 0$.

Finally, let us consider the case $v \neq 0$. Arguing as above we get

$$\begin{aligned}\liminf_{h\to 0} J''(\hat{u}_h)v_h^2 &= \int_\Omega\left[\frac{\partial^2 L}{\partial y^2}(x, \bar{y}, \bar{u}) - \overline{\varphi}\frac{\partial^2 f}{\partial y^2}(x, \bar{y})\right]z_v^2\,\mathrm{d}x\\ &\qquad + 2\int_\Omega\frac{\partial^2 L}{\partial y\partial u}(x, \bar{y}, \bar{u})z_v v\,\mathrm{d}x + \liminf_{h\to 0}\int_\Omega\frac{\partial^2 L}{\partial u^2}(x, y_{\hat{u}_h}, \hat{u}_h)v_h^2\,\mathrm{d}x.\end{aligned}$$

Now thanks to Lusin's theorem, for any $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset \Omega$ such that

$$\lim_{h\to 0} \|\bar{u} - \bar{u}_h\|_{L^\infty(K_\varepsilon)} = 0 \quad \text{and} \quad |\Omega \setminus K_\varepsilon| < \varepsilon.$$

Combining this fact with the above inequality and remembering that $v_h \rightharpoonup v$ weakly in $L^2(\Omega)$ and $\|v_h\|_{L^2(\Omega)} = 1$, we deduce

$$\begin{aligned}
\liminf_{h\to 0} J''(\hat{u}_h)v_h^2 \geqslant{}& \int_\Omega \left[\frac{\partial^2 L}{\partial y^2}(x, \bar{y}, \bar{u}) - \bar{\varphi}\frac{\partial^2 f}{\partial y^2}(x, \bar{y})\right] z_v^2 \,\mathrm{d}x + 2\int_\Omega \frac{\partial^2 L}{\partial y \partial u}(x, \bar{y}, \bar{u}) z_v v \,\mathrm{d}x \\
&+ \liminf_{h\to 0} \int_{K_\varepsilon} \frac{\partial^2 L}{\partial u^2}(x, \bar{y}, \bar{u}) v_h^2 \,\mathrm{d}x \\
&+ \liminf_{h\to 0} \int_{K_\varepsilon} \left[\frac{\partial^2 L}{\partial u^2}(x, y_{\hat{u}_h}, \hat{u}_h) - \frac{\partial^2 L}{\partial u^2}(x, \bar{y}, \bar{u})\right] v_h^2 \,\mathrm{d}x \\
&+ \liminf_{h\to 0} \int_{\Omega\setminus K_\varepsilon} \frac{\partial^2 L}{\partial u^2}(x, y_{\hat{u}_h}, \hat{u}_h) v_h^2 \,\mathrm{d}x \\
\geqslant{}& \int_\Omega \left[\frac{\partial^2 L}{\partial y^2}(x, \bar{y}, \bar{u}) - \bar{\varphi}\frac{\partial^2 f}{\partial y^2}(x, \bar{y})\right] z_v^2 \,\mathrm{d}x + 2\int_\Omega \frac{\partial^2 L}{\partial y \partial u}(x, \bar{y}, \bar{u}) z_v v \,\mathrm{d}x \\
&+ \int_{K_\varepsilon} \frac{\partial^2 L}{\partial u^2}(x, \bar{y}, \bar{u}) v^2 \,\mathrm{d}x + \liminf_{h\to 0} \Lambda \int_{\Omega\setminus K_\varepsilon} v_h^2 \,\mathrm{d}x \\
\geqslant{}& J''(\bar{u})v^2 - \int_{\Omega\setminus K_\varepsilon} \frac{\partial^2 L}{\partial u^2}(x, \bar{y}, \bar{u}) v^2 \,\mathrm{d}x + \Lambda \int_{\Omega\setminus K_\varepsilon} v^2 \,\mathrm{d}x.
\end{aligned}$$

Finally using (2.19) and making $\varepsilon \to 0$ we deduce

$$\liminf_{h\to 0} J''(\hat{u}_h)v_h^2 \geqslant \delta\|v\|^2_{L^2(\Omega)},$$

which concludes the proof. □

Lemma 4.3. There exist a constant $C > 0$ independent of h such that for every u_1, $u_2 \in \mathbb{K}$ and all $v \in L^2(\Omega)$, with $v = 0$ in $\Omega \setminus \Omega_h$,

$$\left|\big(J_h'(u_2) - J'(u_1)\big)v\right| \leqslant C\big\{h^2 + \|u_2 - u_1\|_{L^2(\Omega)}\big\}\|v\|_{L^2(\Omega)}. \tag{4.11}$$

Proof. From (2.4) and (3.5) we obtain

$$\begin{aligned}
&\left|\big(J_h'(u_2) - J'(u_1)\big)v\right| \\
&\quad\leqslant \int_{\Omega\setminus\Omega_h} \left|\frac{\partial L}{\partial u}(x, y_{u_1}, u_1) + \varphi_{u_1}\right| |v| \,\mathrm{d}x \\
&\qquad+ \int_{\Omega_h} \left|\left(\frac{\partial L}{\partial u}\big(x, y_h(u_2), u_2\big) + \varphi_h(u_2)\right) - \left(\frac{\partial L}{\partial u}(x, y_{u_1}, u_1) + \varphi_{u_1}\right)\right| |v| \,\mathrm{d}x \\
&\quad\leqslant C\big\{\|u_2 - u_1\|_{L^2(\Omega)} + \|\varphi_h(u_2) - \varphi_{u_1}\|_{L^2(\Omega)} + \|y_h(u_2) - y_{u_1}\|_{L^2(\Omega)}\big\}\|v\|_{L^2(\Omega)}.
\end{aligned}$$

Now (4.11) follows from this inequality and (3.12). □

Lemma 4.4. The following identity holds:

$$\lim_{h\to 0}\frac{1}{h^2}\left|J'(\bar u)(\Pi_h\bar u-\bar u)\right|=0. \tag{4.12}$$

Proof. We will distinguish two different elements in every triangulation $\mathcal{T}_h$

$$\mathcal{T}_h^+=\left\{T\in\mathcal{T}_h\colon\ \left|\bar d(x)\right|>0\ \forall x\in T\right\},$$
$$\mathcal{T}_h^0=\left\{T\in\mathcal{T}_h\colon\ \exists\xi_T\in T \text{ such that } \bar d(\xi_T)=0\right\}.$$

Since $\bar d$ is a continuous function, then $\bar d(x)>0\ \forall x\in T$ or $\bar d(x)<0\ \forall x\in T$, for any $T\in\mathcal{T}_h^+$. Now, according to (2.13) $\bar u(x)=\alpha$ for every $x\in T$ or $\bar u(x)=\beta$ for every $x\in T$. This implies that $\Pi_h\bar u(x)=\bar u(x)$ for every $x\in T$ and every $T\in\mathcal{T}_h^+$. On the other part, let us remind that $\Pi_h\bar u$ has been defined in $\overline{\Omega}\setminus\overline{\Omega}_h$ by $\bar u$. Therefore

$$\begin{aligned}\left|J'(\bar u)(\Pi_h\bar u-\bar u)\right|&=\left|\sum_{T\in\mathcal{T}_h}\int_T\bar d(x)\big(\Pi_h\bar u(x)-\bar u(x)\big)\,\mathrm{d}x\right|\\&=\left|\sum_{T\in\mathcal{T}_h^0}\int_T\bar d(x)\big(\Pi_h\bar u(x)-\bar u(x)\big)\,\mathrm{d}x\right|\\&\leqslant\sum_{T\in\mathcal{T}_h^0}\int_T\left|\bar d(x)-\bar d(\xi_T)\right|\left|\Pi_h\bar u(x)-\bar u(x)\right|\mathrm{d}x\\&\leqslant\Lambda_{\bar d}h\int_{\Omega_h}\left|\Pi_h\bar u(x)-\bar u(x)\right|\mathrm{d}x,\end{aligned}$$

where $\Lambda_{\bar d}$ is the Lipschitz constant of $\bar d$. This Lipschitz property follows from assumption (A2) and the fact that $\bar y,\overline{\varphi}\in W^{2,p}(\Omega)\subset C^{0,1}(\overline{\Omega})$. Finally, the last inequality along with (4.6) leads to (4.12). □

Proof of theorem 4.1. Taking $u=\bar u_h$ in (2.9), with $\bar u_h$ extended to Ω by (4.4) we get

$$J'(\bar u)(\bar u_h-\bar u)=\int_\Omega\left(\overline{\varphi}+\frac{\partial L}{\partial u}(x,\bar y,\bar u)\right)(\bar u_h-\bar u)\,\mathrm{d}x\geqslant 0. \tag{4.13}$$

From (3.10) and the fact that $\Pi_h\bar u=\bar u_h=\bar u$ on $\Omega\setminus\overline{\Omega}_h$ it comes

$$J_h'(\bar u_h)(\Pi_h\bar u-\bar u_h)=\int_\Omega\left(\overline{\varphi}_h+\frac{\partial L}{\partial u}(x,\bar y_h,\bar u_h)\right)(\Pi_h\bar u-\bar u_h)\,\mathrm{d}x\geqslant 0,$$

hence

$$J_h'(\bar u_h)(\bar u-\bar u_h)+J_h'(\bar u_h)(\Pi_h\bar u-\bar u)\geqslant 0. \tag{4.14}$$

Adding (4.13) and (4.14) we obtain

$$\begin{aligned}\big(J'(\bar u)-J_h'(\bar u_h)\big)(\bar u-\bar u_h)&\leqslant J_h'(\bar u_h)(\Pi_h\bar u-\bar u)\\&=\big(J_h'(\bar u_h)-J'(\bar u)\big)(\Pi_h\bar u-\bar u)+J'(\bar u)(\Pi_h\bar u-\bar u).\end{aligned}$$

For every h small enough, this inequality and (4.7) imply

$$\begin{aligned}\mu\|\bar{u}-\bar{u}_h\|^2_{L^2(\Omega_h)} &\leqslant \big(J'(\bar{u})-J'(\bar{u}_h)\big)(\bar{u}-\bar{u}_h)\\ &\leqslant \big(J_h'(\bar{u}_h)-J'(\bar{u}_h)\big)(\bar{u}-\bar{u}_h)+\big(J_h'(\bar{u}_h)-J'(\bar{u})\big)(\Pi_h\bar{u}-\bar{u})\\ &\quad + J'(\bar{u})(\Pi_h\bar{u}-\bar{u}).\end{aligned}$$

Now from (4.11) with $u_2 = u_1 = \bar{u}_h$ in the first summand of the previous line and $u_2 = \bar{u}_h$ and $u_1 = \bar{u}$ in the second, we get

$$\begin{aligned}\mu\|\bar{u}-\bar{u}_h\|^2_{L^2(\Omega_h)} &\leqslant C_1h^2\|\bar{u}-\bar{u}_h\|_{L^2(\Omega)} + C_2\big(h^2+\|\bar{u}-\bar{u}_h\|_{L^2(\Omega)}\big)\|\bar{u}-\Pi_h\bar{u}\|_{L^2(\Omega)}\\ &\quad + J'(\bar{u})(\Pi_h\bar{u}-\bar{u}).\end{aligned}$$

Using Young's inequality and reminding that $\bar{u} = \bar{u}_h = \Pi_h\bar{u}$ on $\Omega \setminus \Omega_h$ we deduce

$$\frac{\mu}{2}\|\bar{u}-\bar{u}_h\|^2_{L^2(\Omega_h)} \leqslant C_3\big(h^4+\|\bar{u}-\Pi_h\bar{u}\|^2_{L^2(\Omega_h)}\big) + J'(\bar{u})(\Pi_h\bar{u}-\bar{u}).$$

Finally, (4.6) and (4.12) imply that

$$\lim_{h\to 0}\frac{1}{h^2}\|\bar{u}-\bar{u}_h\|^2_{L^2(\Omega_h)} = 0,$$

which contradicts (4.3). □

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ERROR ESTIMATES FOR THE NUMERICAL APPROXIMATION OF NEUMANN CONTROL PROBLEMS. *

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Abstract. We continue the discussion of error estimates for the numerical analysis of Neumann boundary control problems we started in [6]. In that paper piecewise constant functions were used to approximate the control and a convergence of order $O(h)$ was obtained. Here, we use continuous piecewise linear functions to discretize the control and obtain the rates of convergence in $L^2(\Gamma)$. Error estimates in the uniform norm are also obtained. We also discuss the approach suggested by Hinze [9] as well as the improvement of the error estimates by making an extra assumption over the set of points corresponding to the active control constraints. Finally, numerical evidence of our estimates is provided.

Key words. Boundary control, semilinear elliptic equation, numerical approximation, error estimates

AMS subject classifications. 49J20, 49K20, 49M05, 65K10

1. Introduction. This paper continues a series of works about error estimates for the numerical analysis of control problems governed by semilinear elliptic partial differential equations. In [1] a distributed problem approximated by piecewise constant controls was studied. In [6] the control appears in the boundary. This makes the task more difficult since the states are now less regular than in the distributed case. Piecewise constant approximations were used in that reference. The advantage of these is that we have a pointwise expression both for the control and its approximation, which we can compare to get uniform convergence. The reader is addressed to these papers for further references about error estimates for the approximation of linear-quadratic problems governed by partial differential equations and for the approximation of ordinary differential equations.

In the case of continuous piecewise linear approximations of the control, there exists not such a pointwise formula in general. If the functional is quadratic with respecto to the control, recent results in [7] about the stability of L^2 projections in Sobolev $W^{s,p}(\Gamma)$ spaces allow us to obtain uniform convergence and adapt the proofs. The general case is more delicate. Results for distributed control problems can be found in [3]. The main purpose of this paper is to obtain similar results for Neumann boundary controls. This is done in Theorem 4.6.

We also refer to the works for distributed linear-quadratic problems about semidiscretization [9] and postprocessing [11]. The first proposes only discretizing the state, and not the control. The solution can nevertheless be expressed with a finite number of parameters via the adjoint-state and the problem can be solved with a computer with a slightly changed optimization code. The second one proposes solving a completely discretized problem with piecewise constant approximations of the control and finally construct a new control using the pointwise projection of the discrete adjoint state. We are able to reproduce the first scheme for Neumann boundary controls, a general functional and a semilinear equation.

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The rest of the paper is as follows. In the next section, we define precisely the problem. In Section 3 we recall several results about this control problem. Section 4 contains the main results of this paper: we discretize the problem and obtain error estimates for the solutions. In Section 5 we investigate what happens when we only discretize the state, and not the control. Section 6 is devoted to the special case of an objective function quadratic with respect to the control. Numerical evidence of our results is presented in Section 7. Finally, in an appendix, we include the proof of a finite element error estimate in the boundary.

2. Statement of the problem. Throughout the sequel, Ω denotes an open convex bounded polygonal set of $\mathbb{R}^2$ and Γ is the boundary of Ω. We will also take $p > 2$. In this domain we formulate the following control problem

$$
(\mathrm{P}) \left\{ \begin{array}{l}
\inf \quad J(u) = \displaystyle\int_\Omega L(x, y_u(x))\,dx \ + \int_\Gamma l(x, y_u(x), u(x))\,d\sigma(x) \\
\\
\text{subject to } (y_u, u) \in H^1(\Omega) \times L^\infty(\Gamma), \\
u \in U^{ad} = \{u \in L^\infty(\Gamma) \mid \alpha \le u(x) \le \beta \ \text{ a.e. } x \in \Gamma\}, \\
(y_u, u) \ \text{ satisfying the state equation (2.1)}
\end{array} \right.
$$

$$
\left\{ \begin{array}{rcll}
-\Delta y_u(x) & = & a_0(x, y_u(x)) & \text{in} \quad \Omega \\
\partial_\nu y_u(x) & = & b_0(x, y_u(x)) + u(x) & \text{on} \quad \Gamma,
\end{array} \right. \tag{2.1}
$$

where $-\infty < \alpha < \beta < +\infty$. Here u is the control while y_u is said to be the associated state. The following hypotheses are assumed about the functions involved in the control problem (P):

(A1) The function $L : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is measurable with respect to the first component, of class C^2 with respect to the second, $L(\cdot, 0) \in L^1(\Omega)$, $\dfrac{\partial L}{\partial y}(\cdot, 0) \in L^p(\Omega)$ $\dfrac{\partial^2 L}{\partial y^2}(\cdot, 0) \in L^\infty(\Omega)$ and for all $M > 0$ there exists a constant $C_{L,M} > 0$ such that

$$
\left| \frac{\partial^2 L}{\partial y^2}(x, y_2) - \frac{\partial^2 L}{\partial y^2}(x, y_1) \right| \le C_{L,M} |y_2 - y_1|,
$$

for a.e. $x \in \Omega$ and $|y|, |y_i| \le M$, $i = 1, 2$.

(A2) The function $l : \Gamma \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is Lipschitz with respect to the first component, of class C^2 with respect to the second and third variables, $l(\cdot, 0, 0) \in L^1(\Gamma)$, $D^2_{(y,u)} l(\cdot, 0, 0) \in L^\infty(\Gamma)$ and for all $M > 0$ there exists a constant $C_{l,M} > 0$ such that

$$
\left| \frac{\partial l}{\partial y}(x_2, y, u) - \frac{\partial l}{\partial y}(x_1, y, u) \right| + \left| \frac{\partial l}{\partial u}(x_2, y, u) - \frac{\partial l}{\partial u}(x_1, y, u) \right| \le C_{l,M} |x_2 - x_1|,
$$

$$
\| D^2_{(y,u)} l(x, y_2, u_2) - D^2_{(y,u)} l(x, y_1, u_1) \| \le C_{l,M} (|y_2 - y_1| + |u_2 - u_1|),
$$

for a.e. $x, x_i \in \Gamma$ and $|y|, |y_i|, |u|, |u_i| \le M$, $i = 1, 2$, where $D^2_{(y,u)} l$ denotes the second derivative of l with respect to (y, u). Moreover we assume that there exists $\Lambda > 0$ such that

$$
\frac{\partial^2 l}{\partial u^2}(x, y, u) \ge \Lambda, \quad \text{a.e. } x \in \Gamma \text{ and } (y, u) \in \mathbb{R}^2. \tag{2.2}
$$

Let us remark that this inequality implies the strict convexity of l with respect to the third variable.

(A3) The function $a_0 : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is measurable with respect to the first variable and of class C^2 with respect to the second, $a_0(\cdot, 0) \in L^p(\Omega)$, $\dfrac{\partial a_0}{\partial y}(\cdot, 0) \in L^\infty(\Omega)$, $\dfrac{\partial^2 a_0}{\partial y^2}(\cdot, 0) \in L^\infty(\Omega)$,

$$\frac{\partial a_0}{\partial y}(x, y) \leq 0 \quad \text{a.e. } x \in \Omega \text{ and } y \in \mathbb{R}$$

and for all $M > 0$ there exists a constant $C_{a_0,M} > 0$ such that

$$\left| \frac{\partial^2 a_0}{\partial y^2}(x, y_2) - \frac{\partial^2 a_0}{\partial y^2}(x, y_1) \right| < C_{a_0,M} |y_2 - y_1| \quad \text{a.e. } x \in \Omega \text{ and } |y_1|, |y_2| \leq M.$$

(A4) The function $b_0 : \Gamma \times \mathbb{R} \longrightarrow \mathbb{R}$ is Lipschitz with respect to the first variable and of class C^2 with respect to the second, $b_0(\cdot, 0) \in W^{1-1/p,p}(\Gamma)$, $\dfrac{\partial^2 b_0}{\partial y^2}(\cdot, 0) \in L^\infty(\Gamma)$,

$$\frac{\partial b_0}{\partial y}(x, y) \leq 0$$

and for all $M > 0$ there exists a constant $C_{b_0,M} > 0$ such that

$$\left| \frac{\partial b_0}{\partial y}(x_2, y) - \frac{\partial b_0}{\partial y}(x_1, y) \right| \leq C_{b_0,M} |x_2 - x_1|,$$

$$\left| \frac{\partial^2 b_0}{\partial y^2}(x, y_2) - \frac{\partial^2 b_0}{\partial y^2}(x, y_1) \right| \leq C_{b_0,M} |y_2 - y_1|.$$

for a.e. $x, x_1, x_2 \in \Gamma$ and $|y|, |y_1|, |y_2| \leq M$.

(A5) At least one of the two conditions must hold: either $(\partial a_0/\partial y)(x, y) < 0$ in $E_\Omega \times \mathbb{R}$ with $E_\Omega \subset \Omega$ of positive n-dimensional measure or $(\partial b_0/\partial y)(x, y) < 0$ on $E_\Gamma \times \mathbb{R}$ with $E_\Gamma \subset \Gamma$ of positive $(n-1)$-dimensional measure.

3. Analysis of the control problem. Let us briefly state some useful results known for this control problem. The proofs can be found in [6].

THEOREM 3.1. *For every $u \in L^2(\Gamma)$ the state equation (2.1) has a unique solution $y_u \in H^{3/2}(\Omega)$, that depends continuously on u. Moreover, there exists $p_0 > 2$ depending on the measure of the angles in Γ such that if $u \in W^{1-1/p,p}(\Gamma)$ for some $2 \leq p \leq p_0$, then $y_u \in W^{2,p}(\Omega)$.*

Let us note that the inclusion $H^{3/2}(\Omega) \subset C(\bar{\Omega})$ holds for Lipschitz domains in $\mathbb{R}^2$. As a consequence of the theorem above, we know that the functional J is well defined in $L^2(\Gamma)$.

Let us discuss the differentiability properties of J.

THEOREM 3.2. *Suppose that assumptions (A3)–(A4) are satisfied. Then the mapping $G : L^\infty(\Gamma) \longrightarrow H^{3/2}(\Omega)$ defined by $G(u) = y_u$ is of class C^2. Moreover, for* *all* $u, v \in L^\infty(\Gamma)$, $z_v = G'(u)v$ *is the solution of*

$$\left\{\begin{array}{rcll} -\Delta z_v & = & \dfrac{\partial a_0}{\partial y}(x, y_u) z_v & \text{in} \quad \Omega \\ \partial_\nu z_v & = & \dfrac{\partial b_0}{\partial y}(x, y_u) z_v + v & \text{on} \quad \Gamma. \end{array}\right. \tag{3.1}$$

Finally, for every $v_1, v_2 \in L^\infty(\Omega)$, $z_{v_1 v_2} = G''(u) v_1 v_2$ *is the solution of*

$$\left\{\begin{array}{rcll} -\Delta z_{v_1 v_2} & = & \dfrac{\partial a_0}{\partial y}(x, y_u) z_{v_1 v_2} + \dfrac{\partial^2 a_0}{\partial y^2}(x, y_u) z_{v_1} z_{v_2} & \text{in} \quad \Omega \\ \partial_\nu z_{v_1 v_2} & = & \dfrac{\partial b_0}{\partial y}(x, y_u) z_{v_1 v_2} + \dfrac{\partial^2 b_0}{\partial y^2}(x, y_u) z_{v_1} z_{v_2} & \text{on} \quad \Gamma, \end{array}\right. \tag{3.2}$$

where $z_{v_i} = G'(u)v_i$, $i = 1, 2$.

THEOREM 3.3. *Under the assumptions (A1)–(A4), the functional* $J : L^\infty(\Gamma) \to \mathbb{R}$ *is of class* C^2*. Moreover, for every* $u, v, v_1, v_2 \in L^\infty(\Gamma)$

$$J'(u)v = \int_\Gamma \left(\frac{\partial l}{\partial u}(x, y_u, u) + \varphi_u \right) v \, d\sigma \tag{3.3}$$

and

$$\begin{aligned} J''(u) v_1 v_2 = & \int_\Omega \left[\frac{\partial^2 L}{\partial y^2}(x, y_u) z_{v_1} z_{v_2} + \varphi_u \frac{\partial^2 a_0}{\partial y^2}(x, y_u) z_{v_1} z_{v_2} \right] dx \\ + & \int_\Gamma \left[\frac{\partial^2 l}{\partial y^2}(x, y_u, u) z_{v_1} z_{v_2} + \frac{\partial^2 l}{\partial y \partial u}(x, y_u, u)(z_{v_1} v_2 + z_{v_2} v_1) \right. \\ & \left. + \frac{\partial^2 l}{\partial u^2}(x, y_u, u) v_1 v_2 + \varphi_u \frac{\partial^2 b_0}{\partial y^2}(x, y_u) z_{v_1} z_{v_2} \right] d\sigma \end{aligned} \tag{3.4}$$

where $z_{v_i} = G'(u)v_i$, $i = 1, 2$, $y_u = G(u)$, *and the adjoint state* $\varphi_u \in H^{3/2}(\Omega)$ *is the unique solution of the problem*

$$\left\{\begin{array}{l} -\Delta \varphi = \dfrac{\partial a_0}{\partial y}(x, y_u)\varphi + \dfrac{\partial L}{\partial y}(x, y_u) \quad \text{in } \Omega \\ \partial_\nu \varphi = \dfrac{\partial b_0}{\partial y}(x, y_u)\varphi + \dfrac{\partial l}{\partial y}(x, y_u, u) \quad \text{on } \Gamma. \end{array}\right. \tag{3.5}$$

The existence of a solution for problem (P) follows easily from our assumptions (A1)–(A5). In particular, we underline the important fact that the function l is convex with respect to the third variable. See (2.2). The first order optimality conditions for Problem (P) follow readily from Theorem 3.3.

THEOREM 3.4. *Assume that* $\bar{u}$ *is a local solution of Problem (P). Then there exist* $\bar{y}, \bar{\varphi} \in H^{3/2}(\Omega)$ *such that*

$$\left\{\begin{array}{rcll} -\Delta \bar{y}(x) & = & a_0(x, \bar{y}(x)) & \text{in} \quad \Omega \\ \partial_\nu \bar{y}(x) & = & b_0(x, \bar{y}(x)) + \bar{u}(x) & \text{on} \quad \Gamma, \end{array}\right. \tag{3.6}$$

$$\left\{\begin{array}{rcll} -\Delta \bar{\varphi} & = & \dfrac{\partial a_0}{\partial y}(x, \bar{y})\bar{\varphi} + \dfrac{\partial L}{\partial y}(x, \bar{y}) & \text{in} \quad \Omega \\ \partial_\nu \bar{\varphi} & = & \dfrac{\partial b_0}{\partial y}(x, \bar{y})\bar{\varphi} + \dfrac{\partial l}{\partial y}(x, \bar{y}, \bar{u}) & \text{on} \quad \Gamma, \end{array}\right. \tag{3.7}$$

$$\int_\Gamma \left(\frac{\partial l}{\partial u}(x, \bar y, \bar u) + \bar\varphi \right) (u - \bar u)\, d\sigma(x) \ge 0 \quad \forall u \in U^{ad}. \tag{3.8}$$

If we define

$$\bar d(x) = \frac{\partial l}{\partial u}(x, \bar y(x), \bar u(x)) + \bar\varphi(x),$$

then we deduce from (3.8) that

$$\bar d(x) = \begin{cases} 0 & \text{for a.e. } x \in \Gamma \text{ where } \alpha < \bar u(x) < \beta, \\ \ge 0 & \text{for a.e. } x \in \Gamma \text{ where } \bar u(x) = \alpha, \\ \le 0 & \text{for a.e. } x \in \Gamma \text{ where } \bar u(x) = \beta. \end{cases} \tag{3.9}$$

In order to establish the second order optimality conditions we define the cone of critical directions

$$C_{\bar u} = \{v \in L^2(\Gamma) \text{ satisfying (3.10) and } v(x) = 0 \text{ if } |\bar d(x)| > 0\},$$

$$v(x) = \begin{cases} \ge 0 & \text{for a.e. } x \in \Gamma \text{ where } \bar u(x) = \alpha, \\ \le 0 & \text{for a.e. } x \in \Gamma \text{ where } \bar u(x) = \beta. \end{cases} \tag{3.10}$$

Now we formulate the second order necessary and sufficient optimality conditions.

THEOREM 3.5. *If $\bar u$ is a local solution of (P), then $J''(\bar u)v^2 \ge 0$ holds for all $v \in C_{\bar u}$. Conversely, if $\bar u \in U^{ad}$ satisfies the first order optimality conditions* (3.6)–(3.8) *and the coercivity condition $J''(\bar u)v^2 > 0$ holds for all $v \in C_{\bar u} \setminus \{0\}$, then there exist $\delta > 0$ and $\varepsilon > 0$ such that*

$$J(u) \ge J(\bar u) + \delta \|u - \bar u\|^2_{L^2(\Gamma)} \tag{3.11}$$

is satisfied for every $u \in U^{ad}$ such that $\|u - \bar u\|_{L^\infty(\Omega)} \le \varepsilon$.

REMARK 3.6. *By using the assumption $(\partial^2 l/\partial u^2)(x, y, u) \ge \Lambda > 0$, we deduce from Casas and Mateos [4, Theorem 4.4] that the following two conditions are equivalent:*

(1) $J''(\bar u)v^2 > 0$ for every $v \in C_{\bar u} \setminus \{0\}$.

(2) There exist $\delta > 0$ and $\tau > 0$ such that $J''(\bar u)v^2 \ge \delta \|v\|^2_{L^2(\Gamma)}$ for every $v \in C^\tau_{\bar u}$, where

$$C^\tau_{\bar u} = \{v \in L^2(\Gamma) \text{ satisfying (3.10) and } v(x) = 0 \text{ if } |\bar d(x)| > \tau\}.$$

It is clear that that $C^\tau_{\bar u}$ contains strictly $C_{\bar u}$, so the condition (2) seems to be stronger than (1), but in fact they are equivalent.

THEOREM 3.7. *Suppose that $\bar u$ is a local solution of (P), then for all $x \in \Gamma$ the equation*

$$\bar\varphi(x) + \frac{\partial l}{\partial u}(x, \bar y(x), t) = 0 \tag{3.12}$$

has a unique solution $\bar t = \bar s(x)$. The mapping $\bar s : \Gamma \longrightarrow \mathbb{R}$ is Lipschitz and it is related with $\bar u$ through the formula

$$\bar u(x) = Proj_{[\alpha,\beta]}(\bar s(x)) = \max\{\alpha, \min\{\beta, \bar s(x)\}\}. \tag{3.13}$$

Moreover $\bar u \in C^{0,1}(\Gamma)$ and $\bar y, \bar\varphi \in W^{2,p}(\Omega) \subset C^{0,1}(\bar\Omega)$ for some $p > 2$.

4. Full discretization. Here, we define a finite-element based approximation of the optimal control problem (P). To this aim, we consider a family of triangulations $\{\mathcal{T}_h\}_{h>0}$ of $\bar\Omega$: $\bar\Omega = \cup_{T\in\mathcal{T}_h} T$. This triangulation is supposed to be regular in the usual sense that we state exactly here. With each element $T \in \mathcal{T}_h$, we associate two parameters $\rho(T)$ and $\sigma(T)$, where $\rho(T)$ denotes the diameter of the set T and $\sigma(T)$ is the diameter of the largest ball contained in T. Let us define the size of the mesh by $h = \max_{T\in\mathcal{T}_h} \rho(T)$. The following regularity assumption is assumed.

(H) - There exist two positive constants ρ and σ such that

$$\frac{\rho(T)}{\sigma(T)} \le \sigma, \qquad \frac{h}{\rho(T)} \le \rho$$

hold for all $T \in \mathcal{T}_h$ and all $h > 0$.

For fixed $h > 0$, we denote by $\{T_j\}_{j=1}^{N(h)}$ the family of triangles of $\mathcal{T}_h$ with a side on the boundary of Γ. If the edges of $T_j \cap \Gamma$ are x_Γ^j and x_Γ^{j+1} then $[x_\Gamma^j, x_\Gamma^{j+1}] := T_j \cap \Gamma$, $1 \le j \le N(h)$, with $x_\Gamma^{N(h)+1} = x_\Gamma^1$.

4.1. Discretization of the state equation. Associated with this triangulation we set

$$Y_h = \{y_h \in C(\bar\Omega) \mid y_{h|T} \in \mathcal{P}_1, \text{ for all } T \in \mathcal{T}_h\},$$

where $\mathcal{P}_1$ is the space of polynomials of degree less than or equal to 1. For each $u \in L^\infty(\Gamma)$, we denote by $y_h(u)$ the unique element of Y_h that satisfies

$$a(y_h(u), z_h) = \int_\Omega a_0(x, y_h(u)) z_h\,dx + \int_\Gamma [b_0(x, y_h(u)) + u] z_h\,dx \quad \forall z_h \in Y_h, \tag{4.1}$$

where $a : Y_h \times Y_h \longrightarrow \mathbb{R}$ is the bilinear form defined by

$$a(y_h, z_h) = \int_\Omega \nabla y_h(x) \nabla z_h(x)\,dx.$$

The existence and uniqueness of a solution of (4.1) follows in the standard way from the monotonicity of a_0 and b_0 (see [6]).

Let us now introduce the approximate adjoint state associated to a control. To every $u \in U_{ad}$ we relate $\varphi_h(u) \in Y_h$, the unique function satisfying

$$a(\varphi_h(u), z_h) = \int_\Omega \left(\frac{\partial a_0}{\partial y}(x, y_h(u))\varphi_h(u) + \frac{\partial L}{\partial y}(x, y_h(u)) \right) z_h\,dx +$$

$$\int_\Gamma \left(\frac{\partial b_0}{\partial y}(x, y_h(u))\varphi_h(u) + \frac{\partial l}{\partial y}(x, y_h(u), u) \right) z_h\,d\sigma(x) \quad \forall z_h \in Y_h.$$

We will make intensive use of the following approximation properties.

THEOREM 4.1. *(i) For every $u \in H^{1/2}(\Gamma)$ there exists $C > 0$, depending continuously on $\|u\|_{H^{1/2}(\Gamma)}$, such that*

$$\|y_u - y_h(u)\|_{H^s(\Omega)} + \|\varphi_u - \varphi_h(u)\|_{H^s(\Omega)} \le Ch^{2-s} \textit{ for all } 0 \le s \le 1, \tag{4.2}$$

and

$$\|y_u - y_h(u)\|_{L^2(\Gamma)} + \|\varphi_u - \varphi_h(u)\|_{L^2(\Gamma)} \le Ch^{3/2}. \tag{4.3}$$

(ii) For every $u \in L^2(\Gamma)$ there exists $C_0 > 0$, depending continuously on $\|u\|_{L^2(\Gamma)}$, such that

$$\|y_u - y_h(u)\|_{H^s(\Omega)} + \|\varphi_u - \varphi_h(u)\|_{H^s(\Omega)} \le C_0 h^{3/2-s} \text{ for all } 0 \le s \le 1. \tag{4.4}$$

(iii) For every $u_1, u_2 \in L^2(\Gamma)$ there exists a constant $C > 0$ such that

$$\|y_{u_1} - y_{u_2}\|_{H^1(\Omega)} + \|y_h(u_1) - y_h(u_2)\|_{H^1(\Omega)} +$$

$$\|\varphi_{u_1} - \varphi_{u_2}\|_{H^1(\Omega)} + \|\varphi_h(u_1) - \varphi_h(u_2)\|_{H^1(\Omega)} \le C\|u_1 - u_2\|_{L^2(\Gamma)}.$$

(iv) Moreover, if $u_h \rightharpoonup u$ weakly in $L^2(\Gamma)$, then $y_h(u_h) \to y_u$ and $\varphi_h(u_h) \to \varphi_u$ strongly in $C(\bar\Omega)$.

Proof. (i) If $u \in H^{1/2}(\Gamma)$ then both the state and the adjoint state are in $H^2(\Omega)$. Then we can use the results proved in [5] to deduce (4.2) for $s = 0$ and $s = 1$. For $s \in (0,1)$, the estimate can be deduced by real interpolation methods (see Brenner and Scott [2, Section 12.3]. Inequality (4.3) is proved in an appendix.

(ii) Now, we only can assure that the state and the adjoint state are functions of $H^{3/2}(\Omega)$. Again the result follows by real interpolation. See [2, Theorem (12.3.5)].

(iii) This is obtained in a standard way from the monotonicity of a_0 and b_0 and **(A5)**.

(iv) See [5, 6] □

4.2. Discrete optimal control problem. Now we are going to approximate problem (P) by a finite dimensional problem. Set

$$U_h = \{u \in C(\Gamma) \mid u_{|(x_\Gamma^j, x_\Gamma^{j+1})} \in \mathcal{P}_1 \text{ for } 1 \le j \le N(h)\}.$$

The approximated control problem is

$$(\mathrm{P}_h)\begin{cases} \min J_h(u_h) = \displaystyle\int_\Omega L(x, y_h(u_h)(x))\,dx + \int_\Gamma l(x, y_h(u_h)(x), u_h(x))\,d\sigma(x), \\ \text{subject to } (y_h(u_h), u_h) \in Y_h \times U_h^{ad} \text{ satysfying (4.1)}, \end{cases}$$

where $U_h^{ad} = U_h \cap U^{ad}$.

Since J_h is a continuous function and U_h^{ad} is compact, we get that (P$_h$) has at least one global solution. The first order optimality conditions can be written as follows:

Theorem 4.2. *Assume that $\bar u_h$ is a local optimal solution of (P$_h$). Then there exist $\bar y_h$ and $\bar\varphi_h$ in Y_h satisfying*

$$a(\bar y_h, z_h) = \int_\Omega a_0(x, \bar y_h) z_h\,dx + \int_\Gamma (b_0(x, \bar y_h) + \bar u_h) z_h\,dx \quad \forall z_h \in Y_h, \tag{4.5}$$

$$a(\bar\varphi_h, z_h) = \int_\Omega \left(\frac{\partial a_0}{\partial y}(x, \bar y_h)\bar\varphi_h + \frac{\partial L}{\partial y}(x, \bar y_h)\right) z_h\,dx +$$

$$\int_\Gamma \left(\frac{\partial b_0}{\partial y}(x, \bar y_h)\bar\varphi_h + \frac{\partial l}{\partial y}(x, \bar y_h, \bar u_h)\right) z_h\,d\sigma(x) \quad \forall z_h \in Y_h, \tag{4.6}$$

$$\int_\Gamma \left(\bar\varphi_h + \frac{\partial l}{\partial u}(x, \bar y_h, \bar u_h)\right)(u_h - \bar u_h)\, d\sigma(x) \geq 0 \quad \forall u_h \in U_h^{ad}. \tag{4.7}$$

We will denote

$$\bar d_h(x) = \bar\varphi_h(x) + \frac{\partial l}{\partial u}(x, \bar y_h(x), \bar u_h(x)).$$

REMARK 4.3. *At this point, we can show the difficulty introduced by the fact that U_h is formed by continuous piecewise linear functions instead of piecewise constant functions. To make a clear presentation, let us assume for a while that $l(x,y,u) = \ell(x,y) + \frac{\Lambda}{2}u^2$. In the case where U_h is formed by piecewise constant functions, we get from* (4.7) *that*

$$\bar u_{h|(x_\Gamma^j, x_\Gamma^{j+1})} = Proj_{[\alpha,\beta]}\left(-\frac{1}{\Lambda}\int_{x_\Gamma^j}^{x_\Gamma^{j+1}} \bar\varphi_h(x) d\sigma(x)\right).$$

Comparing this representation of $\bar u_h$ with (3.13) *we can prove that $\bar u_h \to \bar u$ strongly in $L^\infty(\Gamma)$; see [6].*

Since we are considering piecewise linear controls in the present paper, no such pointwise projection formula can be deduced. We only can say that $\bar u_h$ is the convex projection of $-\frac{1}{\Lambda}\bar\varphi_h(x)$. More precisely, $\bar u_h$ is the solution of problem

$$\min_{v_h \in U_h} \|\bar\varphi_h + \Lambda v_h\|^2_{L^2(\Gamma)}. \tag{4.8}$$

This makes the analysis of the convergence more difficult than in [6]. In particular, we can prove that $\bar u_h \to \bar u$ strongly in $L^2(\Gamma)$, but this convergence cannot be obtained in $L^\infty(\Gamma)$ in an easy way as done in [6]; see Section 6 for further discussion on this particular case. The reader is also referred to [7] for the study of problem (4.8).

THEOREM 4.4. *For every $h > 0$ let $\bar u_h$ be a solution of (P_h). Then there exist subsequences $\{\bar u_h\}_{h>0}$ converging in the weak* topology of $L^\infty(\Gamma)$ that will be denoted in the same way. If $\bar u_h \rightharpoonup \bar u$ in the mentioned topology, then $\bar u$ is a solution of (P) and*

$$\lim_{h\to 0} J_h(\bar u_h) = J(\bar u) \quad \textit{and} \quad \lim_{h\to 0} \|\bar u - \bar u_h\|_{L^2(\Gamma)} = 0.$$

Proof. Since $U_h^{ad} \subset U^{ad}$ holds for every $h > 0$ and U^{ad} is bounded in $L^\infty(\Gamma)$, $\{\bar u_h\}_{h>0}$ is also bounded in $L^\infty(\Gamma)$. Therefore, there exist weakly*-converging subsequences as claimed in the statement of the theorem. Let $\bar u_h$ be the of one of these subsequences. By the definition of U^{ad} it is obvious that $\bar u_h \in U^{ad}$. Let us prove that the weak* limit $\bar u$ is a solution of (P). Let $\tilde u \in U^{ad}$ be a solution of (P) and consider the operator $\Pi_h : C(\Gamma) \to U_h$ defined by

$$\Pi_h u(x_\Gamma^j) = u(x_\Gamma^j) \text{ for } j = 1, \ldots, N(h).$$

According to Theorem 3.7 we have that $\tilde u \in C^{0,1}(\Gamma)$ and then

$$\|\tilde u - \Pi_h \tilde u\|_{L^\infty(\Gamma)} \leq Ch\|\tilde u\|_{C^{0,1}(\Gamma)};$$

see Lemma 4.5 below. Remark that $\Pi_h\tilde{u} \in U_h^{ad}$ for every h. Now using the convexity of $l(x, y, u)$ with respect to u and the uniform convergence $\bar{y}_h = y_h(\bar{u}_h) \to \bar{y} = y_{\bar{u}}$ and $y_h(\Pi_h\tilde{u}) \to y_{\tilde{u}}$ (Theorem 4.1(iv)) along with the assumptions on L and l we get

$$J(\bar{u}) \leq \liminf_{h\to 0} J_h(\bar{u}_h) \leq \limsup_{h\to 0} J_h(\bar{u}_h) \leq \limsup_{h\to 0} J_h(\Pi_h\tilde{u}) = J(\tilde{u}) = \inf(P).$$

This proves that $\bar{u}$ is a solution of (P) as well as the convergence of the optimal costs. The L^2 convergence of $\{\bar{u}_h\}$ to $\bar{u}$ follows now in a standard way from the convergence $J(\bar{u}_h) \to J(\bar{u})$ together with assumptions **(A1)** and **(A2)**, the weak convergence $\bar{u}_h \rightharpoonup \bar{u}$ in $L^2(\Gamma)$, the strong convergence $\bar{y}_h \to \bar{y}$ in $C(\bar{\Omega})$ and the condition (2.2) on the strict positivity of the second derivative of l with respect to the third variable. □

The following interpolation results are well know; see for instance [2].

Lemma 4.5. *For all $u \in C^{0,1}(\Gamma)$, $1 \leq q < +\infty$ there exists $C > 0$*

$$\lim_{h\to 0} \frac{\|u - \Pi_h u\|_{L^q(\Gamma)}}{h} = 0, \; \|u - \Pi_h u\|_{L^\infty(\Gamma)} \leq Ch\|u\|_{C^{0,1}(\Gamma)} \tag{4.9}$$

and for all $\varphi \in H^{3/2}(\Gamma)$ there exists $C > 0$ such that

$$\|\varphi - \Pi_h\varphi\|_{L^2(\Gamma)} \leq Ch^{3/2}\|\varphi\|_{H^{3/2}(\Gamma)}. \tag{4.10}$$

4.3. Error estimates. The main result of the paper is the following.

Theorem 4.6. *Let $\bar{u}$ be a solution of problem (P) such that $J''(\bar{u})v^2 > 0$ holds for all $v \in C_{\bar{u}} \setminus \{0\}$ and $\bar{u}_h$ a sequence of solutions of (P_h) converging in $L^2(\Gamma)$ to $\bar{u}$. Then*

$$\lim_{h\to 0} \frac{\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)}}{h} = 0.$$

To prove this we will suppose it is false and finally we will get a contradiction. Indeed, we will suppose that there exists a constant $\bar{c} > 0$ and a sequence of h, denoted in the same way, such that

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} \geq \bar{c}h \quad \forall h > 0. \tag{4.11}$$

We will state four auxiliary lemmas. Through the rest of this section, $\bar{u}$ and $\bar{u}_h$ will be the ones given in the assumptions of Theorem 4.6.

Lemma 4.7. *There exists $\nu > 0$ and $h_1 > 0$ such that for all* $0 < h < h_1$

$$\nu\|\bar{u} - \bar{u}_h\|^2_{L^2(\Gamma)} \leq (J'(\bar{u}_h) - J'(\bar{u}))(\bar{u}_h - \bar{u}).$$

Proof. By applying the mean value theorem we get for some $\hat{u}_h = \bar{u} + \theta_h(\bar{u}_h - \bar{u})$

$$(J'(\bar{u}_h) - J'(\bar{u}))(\bar{u}_h - \bar{u}) = J''(\hat{u}_h)(\bar{u}_h - \bar{u})^2. \tag{4.12}$$

Let us take

$$v_h = \frac{\bar{u}_h - \bar{u}}{\|\bar{u}_h - \bar{u}\|_{L^2(\Gamma)}}.$$

Let $\{h_k\}_{k=1}^{\infty}$ be a sequence converging to 0 such that

$$\lim_{k\to\infty} J''_{h_k}(\hat{u}_{h_k})v_{h_k}^2 = \liminf_{h\to 0} J''_h(\hat{u}_h)v_h^2,\ \bar{u}_{h_k} \to \bar{u} \text{ a.e. } x \in \Gamma,\ v_{h_k} \rightharpoonup v \in L^2(\Gamma).$$

The goal is to prove that

$$\liminf_{h\to 0} J''(\hat{u}_h)v_h^2 \geq \left\{ \begin{array}{ll} \Lambda & \text{if } v = 0, \\ \delta\|v\|^2_{L^2(\Gamma)} & \text{if } v \neq 0. \end{array} \right.$$

Then the result follows by (4.12) and this inequality by taking $\nu = \Lambda/2$ if $v = 0$ and $\nu = (\delta/2)\|v\|^2_{L^2(\Gamma)}$ otherwise.

To simplify the notation we will write h instead of h_k. Let us check that $v \in C_{\bar{u}}$. The sign condition (3.13) is satisfied by the v_h and obviously also by v. Let us see that $\bar{d}(x) \neq 0$ implies $v = 0$. Since v satisfies the sign condition and $\bar{d}_h \to \bar{d}$ strongly in $L^2(\Gamma)$, we have

$$\int_\Gamma |\bar{d}(x)v(x)|d\sigma(x) = \int_\Gamma \bar{d}(x)v(x)d\sigma(x) =$$

$$= \lim_{h\to 0}\int_\Gamma \bar{d}_h(x)v_h(x)d\sigma(x) = \lim_{h\to 0}\int_\Gamma \bar{d}_h(x)\frac{\bar{u}_h - \bar{u}}{\|\bar{u}_h - \bar{u}\|_{L^2(\Gamma)}}d\sigma(x) =$$

$$= \lim_{h\to 0}\int_\Gamma \bar{d}_h(x)\frac{\bar{u}_h - \Pi_h\bar{u}}{\|\bar{u}_h - \bar{u}\|_{L^2(\Gamma)}}d\sigma(x) + \lim_{h\to 0}\int_\Gamma \bar{d}_h(x)\frac{\Pi_h\bar{u} - \bar{u}}{\|\bar{u}_h - \bar{u}\|_{L^2(\Gamma)}}d\sigma(x).$$

First order optimality conditions for problem (P$_h$) state that the first integral is less or equal than 0. Using Cauchy inequality, we get

$$\int_\Gamma |\bar{d}(x)v(x)|d\sigma(x) \leq \lim_{h\to 0}\|\bar{d}_h\|_{L^2(\Gamma)}\frac{\|\Pi_h\bar{u} - \bar{u}\|_{L^2(\Gamma)}}{\|\bar{u}_h - \bar{u}\|_{L^2(\Gamma)}}.$$

Taking into account (4.9) and (4.11), we can pass to the limit when $h \to 0$ to get that

$$\int_\Gamma |\bar{d}(x)||v(x)|d\sigma(x) = 0.$$

Therefore, if $\bar{d}(x) \neq 0$, then $v(x) = 0$ and $v \in C^\tau_{\bar{u}}$ for all $\tau \geq 0$.

(i) $v = 0$. In this case weak convergence of $v_h \rightharpoonup v = 0$ in $L^2(\Gamma)$ is enough to obtain strong convergence of $z_{v_h} \to z_v = 0$ in $C(\bar{\Omega})$ and we have

$$\liminf_{h\to 0} J''(\hat{u}_h)v_h^2 = \liminf_{h\to 0}\left\{\int_\Omega \left[\frac{\partial^2 L}{\partial y^2}(x, y_{\hat{u}_h}) + \varphi_{\hat{u}_h}\frac{\partial^2 a_0}{\partial y^2}(x, y_{\hat{u}_h})\right] z^2_{v_h}\,dx\right.$$

$$+\int_\Gamma \left[\left(\frac{\partial^2 l}{\partial y^2}(x, y_{\hat{u}_h}, \hat{u}_h) + \varphi_{\hat{u}_h}\frac{\partial^2 b_0}{\partial y^2}(x, y_{\hat{u}_h})\right) z^2_{v_h} + 2\frac{\partial^2 l}{\partial y\partial u}(x, y_{\hat{u}_h}, \hat{u}_h)z_{v_h}v_h\right] d\sigma(x)$$

$$\left.+\int_\Gamma \frac{\partial^2 l}{\partial u^2}(x, y_{\hat{u}_h}, \hat{u}_h)v_h^2\,d\sigma(x)\right\} \geq$$

$$\liminf_{h\to 0}\left\{\int_\Omega\left[\frac{\partial^2 L}{\partial y^2}(x,y_{\hat u_h})+\varphi_{\hat u_h}\frac{\partial^2 a_0}{\partial y^2}(x,y_{\hat u_h})\right]z_{v_h}^2\,dx\right.$$

$$+\int_\Gamma\left[\left(\frac{\partial^2 l}{\partial y^2}(x,y_{\hat u_h},\hat u_h)+\varphi_{\hat u_h}\frac{\partial^2 b_0}{\partial y^2}(x,y_{\hat u_h})\right)z_{v_h}^2+2\frac{\partial^2 l}{\partial y\partial u}(x,y_{\hat u_h},\hat u_h)z_{v_h}v_h\right]d\sigma(x)$$

$$\left.+\Lambda\int_\Gamma v_h^2\,d\sigma(x)\right\}=$$

$$\int_\Omega\left[\frac{\partial^2 L}{\partial y^2}(x,\bar y)+\bar\varphi\frac{\partial^2 a_0}{\partial y^2}(x,\bar y)\right]z_v^2\,dx$$

$$+\int_\Gamma\left[\left(\frac{\partial^2 l}{\partial y^2}(x,\bar y,\bar u)+\bar\varphi\frac{\partial^2 b_0}{\partial y^2}(x,\bar y)\right)z_v^2+2\frac{\partial^2 l}{\partial y\partial u}(x,\bar y,\bar u)z_v v\right]d\sigma(x)+\Lambda=\Lambda$$

(ii) $v\neq 0$. Arguing as above

$$\liminf_{h\to 0}J''(\hat u_h)v_h^2=\int_\Omega\left[\frac{\partial^2 L}{\partial y^2}(x,\bar y)+\bar\varphi\frac{\partial^2 a_0}{\partial y^2}(x,\bar y)\right]z_v^2\,dx$$

$$+\int_\Gamma\left[\left(\frac{\partial^2 l}{\partial y^2}(x,\bar y,\bar u)+\bar\varphi\frac{\partial^2 b_0}{\partial y^2}(x,\bar y)\right)z_v^2+2\frac{\partial^2 l}{\partial y\partial u}(x,\bar y,\bar u)z_v v\right]d\sigma(x)+$$

$$+\liminf_{h\to 0}\int_\Gamma\frac{\partial^2 l}{\partial u^2}(x,y_{\hat u_h},\hat u_h)v_h^2\,d\sigma(x).$$

Now we use Lusin's theorem. For any $\varepsilon>0$ there exists a compact set $K_\varepsilon\subset\Gamma$ such that

$$\lim_{h\to 0}\|\bar u-\bar u_h\|_{L^\infty(K_\varepsilon)}=0 \text{ and } |\Gamma\setminus K_\varepsilon|<\varepsilon.$$

So we have

$$\liminf_{h\to 0}J''(\hat u_h)v_h^2=\int_\Omega\left[\frac{\partial^2 L}{\partial y^2}(x,\bar y)+\bar\varphi\frac{\partial^2 a_0}{\partial y^2}(x,\bar y)\right]z_v^2\,dx$$

$$+\int_\Gamma\left[\left(\frac{\partial^2 l}{\partial y^2}(x,\bar y,\bar u)+\bar\varphi\frac{\partial^2 b_0}{\partial y^2}(x,\bar y)\right)z_v^2+2\frac{\partial^2 l}{\partial y\partial u}(x,\bar y,\bar u)z_v v\right]d\sigma(x)+$$

$$+\liminf_{h\to 0}\int_{K_\varepsilon}\frac{\partial^2 l}{\partial u^2}(x,\bar y,\bar u)v_h^2\,d\sigma(x)+\liminf_{h\to 0}\int_{\Gamma\setminus K_\varepsilon}\frac{\partial^2 l}{\partial u^2}(x,y_{\hat u_h},\hat u_h)v_h^2\,d\sigma(x)$$

$$+\liminf_{h\to 0}\int_{K_\varepsilon}\left(\frac{\partial^2 l}{\partial u^2}(x,y_{\hat u_h},\hat u_h)-\frac{\partial^2 l}{\partial u^2}(x,\bar y,\bar u)\right)v_h^2\,d\sigma(x)\geq$$

$$\int_\Omega \left[\frac{\partial^2 L}{\partial y^2}(x,\bar y) + \bar\varphi \frac{\partial^2 a_0}{\partial y^2}(x,\bar y)\right] z_v^2\, dx$$

$$+ \int_\Gamma \left[\left(\frac{\partial^2 l}{\partial y^2}(x,\bar y,\bar u) + \bar\varphi \frac{\partial^2 b_0}{\partial y^2}(x,\bar y)\right) z_v^2 + 2\frac{\partial^2 l}{\partial y \partial u}(x,\bar y,\bar u) z_v v\right] d\sigma(x) +$$

$$+ \int_{K_\varepsilon} \frac{\partial^2 l}{\partial u^2}(x,\bar y,\bar u) v^2\, d\sigma(x) + \liminf_{h\to 0} \Lambda \int_{\Gamma\setminus K_\varepsilon} v_h^2 d\sigma(x)\, d\sigma(x) \geq$$

$$J''(\bar u)v^2 - \int_{\Gamma\setminus K_\varepsilon} \frac{\partial^2 l}{\partial u^2}(x,\bar y,\bar u) v^2\, d\sigma(x) + \Lambda \int_{\Gamma\setminus K_\varepsilon} v^2 d\sigma(x).$$

Using second order sufficient conditions as stated in Remark 3.6(2) and making $\varepsilon \to 0$ we deduce that

$$\liminf_{h\to 0} J''(\hat u_h) v_h^2 \geq \delta \|v\|^2_{L^2(\Gamma)}.$$

□

LEMMA 4.8. *There exists a constant $C > 0$ such that*

$$(J_h'(\bar u_h) - J'(\bar u))v \leq C(h^{3/2} + \|\bar u_h - \bar u\|_{L^2(\Gamma)})\|v\|_{L^2(\Gamma)}.$$

Proof. The proof is straight forward. First, we apply the expressions for the derivatives of J and J_h, the trace theorem and Theorem 4.1(iii).

$$(J_h'(\bar u_h) - J'(\bar u))v = \int_\Gamma (\bar\varphi_h - \bar\varphi) v d\sigma(x) +$$

$$\int_\Gamma \left(\frac{\partial l}{\partial u}(x,\bar y_h,\bar u_h) - \frac{\partial l}{\partial u}(x,\bar y,\bar u)\right) v d\sigma(x) =$$

$$\int_\Gamma (\bar\varphi_h - \varphi_h(\bar u)) v d\sigma(x) + \int_\Gamma (\varphi_h(\bar u) - \bar\varphi) v d\sigma(x) +$$

$$\int_\Gamma \left(\frac{\partial l}{\partial u}(x,\bar y_h,\bar u_h) - \frac{\partial l}{\partial u}(x,\bar y_h,\bar u)\right) v d\sigma(x) +$$

$$\int_\Gamma \left(\frac{\partial l}{\partial u}(x,\bar y_h,\bar u) - \frac{\partial l}{\partial u}(x,y_h(\bar u),\bar u)\right) v d\sigma(x) +$$

$$\int_\Gamma \left(\frac{\partial l}{\partial u}(x,y_h(\bar u),\bar u) - \frac{\partial l}{\partial u}(x,\bar y,\bar u)\right) v d\sigma(x) \leq$$

$$C\big(\|\bar\varphi_h - \varphi_h(\bar u)\|_{L^2(\Gamma)} + \|\varphi_h(\bar u) - \bar\varphi\|_{L^2(\Gamma)} +$$

$$\|\bar u_h - \bar u\|_{L^2(\Gamma)} + \|\bar y_h - y_h(\bar u)\|_{L^2(\Gamma)} + \|y_h(\bar u) - \bar y\|_{L^2(\Gamma)}\big)\|v\|_{L^2(\Gamma)} \le$$

$$\le C\big(\|\varphi_h(\bar u) - \bar\varphi\|_{L^2(\Gamma)} + \|y_h(\bar u) - \bar y\|_{L^2(\Gamma)} + \|\bar u_h - \bar u\|_{L^2(\Gamma)}\big)\|v\|_{L^2(\Gamma)}.$$

Finally, using that $\bar u \in H^{1/2}(\Gamma)$, we can apply (4.3) to get that the last expression is bounded by

$$C(h^{3/2} + \|\bar u_h - \bar u\|_{L^2(\Gamma)})\|v\|_{L^2(\Gamma)}.$$

□

LEMMA 4.9. *For every $\rho > 0$ and every $0 < \varepsilon \le 1/2$ there exists $C_{\varepsilon,\rho} > 0$ independent of h such that*

$$|(J_h'(\bar u_h) - J'(\bar u_h))v| \le \Big(C_{\rho,\varepsilon}h^{3/2-\varepsilon} + \rho\|\bar u_h - \bar u\|_{L^2(\Gamma)}\Big)\|v\|_{L^2(\Gamma)} \quad \forall v \in L^2(\Gamma). \tag{4.13}$$

Proof. From the hypotheses on l it is readily deduced

$$|(J_h'(\bar u_h) - J'(\bar u_h))v| \le \int_\Gamma \left(|\bar\varphi_h - \varphi_{\bar u_h}| + \left|\frac{\partial l}{\partial u}(x, \bar y_h, \bar u_h) - \frac{\partial l}{\partial u}(x, y_{\bar u_h}, \bar u_h)\right|\right) v\, d\sigma(x) \le$$

$$C\left(\|\bar\varphi_h - \varphi_{\bar u_h}\|_{L^2(\Gamma)} + \|\bar y_h - y_{\bar u_h}\|_{L^2(\Gamma)}\right)\|v\|_{L^2(\Gamma)}, \tag{4.14}$$

where $y_{\bar u_h}$ and $\varphi_{\bar u_h}$ are the solutions of (2.1) and (3.5) corresponding to $\bar u_h$.

Here we cannot apply (4.3) because we do not know if $\{\bar u_h\}$ is bounded in $H^{1/2}(\Gamma)$. If we try to apply estimate (4.4) and the trace theorem we would get that

$$|(J_h'(\bar u_h) - J'(\bar u_h))v| \le C_\varepsilon h^{1-\varepsilon}.$$

This result is not enough to get the desired order of convergence for $\|\bar u - \bar u_h\|_{L^2(\Gamma)}$. See (4.16). We would get an order of convergence for $\|\bar u - \bar u_h\|_{L^2(\Gamma)}$ even worse than h. We will make a small turnround. Fix $0 < \varepsilon \le 1/2$. From the trace theorem and Theorem 1.4.3.3 in Grisvard [8] (taking $p = 2$, $s' = 1/2 + \varepsilon$, $s'' = 1/2 + \varepsilon/2$ and $s''' = 0$), we have that there exist $C_\varepsilon > 0$ and $K_\varepsilon > 0$ such that for every $\sigma > 0$

$$\|z\|_{L^2(\Gamma)} \le C_\varepsilon\|z\|_{H^{1/2+\varepsilon/2}(\Omega)} \le C_\varepsilon\left(\sigma\|z\|_{H^{1/2+\varepsilon}(\Omega)} + K_\varepsilon\sigma^{-(1+1/\varepsilon)}\|z\|_{L^2(\Omega)}\right).$$

If we name $K_{\sigma,\varepsilon} = C_\varepsilon K_\varepsilon\sigma^{-(1+1/\varepsilon)}$, we get, using estimate (4.4) and the fact that $\{\bar u_h\}_{h>0}$ is bounded in $L^2(\Gamma)$,

$$\|\bar y_h - y_{\bar u_h}\|_{L^2(\Gamma)} \le C_\varepsilon\sigma\|\bar y_h - y_{\bar u_h}\|_{H^{1/2+\varepsilon}(\Omega)} + K_{\sigma,\varepsilon}\|\bar y_h - y_{\bar u_h}\|_{L^2(\Omega)} \le$$

$$C_\varepsilon\sigma\|\bar y_h - y_{\bar u_h}\|_{H^{1/2+\varepsilon}(\Omega)} + K_{\sigma,\varepsilon}C_0h^{3/2}.$$

From Theorem 4.1(iii) we obtain

$$\|\bar y - y_{\bar u_h}\|_{H^{1/2+\varepsilon}(\Omega)} \le C\|\bar u - \bar u_h\|_{L^2(\Gamma)}.$$

On the other hand, using estimate (4.2) and again Theorem 4.1(iii)

$$\|\bar y - \bar y_h\|_{H^{1/2+\varepsilon}(\Omega)} \le \|\bar y - y_h(\bar u)\|_{H^{1/2+\varepsilon}(\Omega)} + \|y_h(\bar u) - \bar y_h\|_{H^{1/2+\varepsilon}(\Omega)} \le$$

$$C\left(h^{3/2-\varepsilon}+\|\bar u-\bar u_h\|_{L^2(\Gamma)}\right).$$

Combining the last three inequalities we deduce

$$\|\bar y_h - y_{\bar u_h}\|_{L^2(\Gamma)} \le CC_\varepsilon\sigma\left(h^{3/2-\varepsilon}+\|\bar u-\bar u_h\|_{L^2(\Gamma)}\right)+K_{\sigma,\varepsilon}C_0h^{3/2}.$$

The same arguments can be applied to the adjoint states, so (4.13) follows from (4.14). Inequality (4.13) is obtained by choosing $\sigma=\rho/(CC_\varepsilon)$ and $C_{\rho,\varepsilon}=K_{\sigma,\varepsilon}C_0+\rho$. ∎

LEMMA 4.10.

$$\lim_{h\to 0}\frac{1}{h^2}|J'(\bar u)(\Pi_h\bar u-\bar u)|=0.$$

Proof. Let us distinguish two kinds of elements on Γ:

$$\tau_h^+=\{j:\ \bar d(x)\neq 0\ \forall x\in(x_\Gamma^j,x_\Gamma^{j+1})\},$$

$$\tau_h^0=\{j:\ \exists\xi_j\in(x_\Gamma^j,x_\Gamma^{j+1})\text{ such that }\bar d(\xi_j)=0\}.$$

Since $\bar d$ is continuous, for h small enough, its sign is constant on the elements corresponding to indices in τ_h^+, and hence either $\bar u(x)=\alpha$ or $\bar u(x)=\beta$ and $\bar u(x)=\Pi_h\bar u$ on each of these elements. So, taking into account that $\bar d$ is a Lipschitz function of constant, say λ, we obtain

$$|J'(\bar u)(\Pi_h\bar u-\bar u)|=\left|\sum_{j\in\tau_h^0}\int_{x_\Gamma^j}^{x_\Gamma^{j+1}}\bar d(x)(\Pi_h\bar u(x)-\bar u(x))d\sigma(x)\right|\le$$

$$\sum_{j\in\tau_h^0}\int_{x_\Gamma^j}^{x_\Gamma^{j+1}}|\bar d(x)-\bar d(\xi_j)||\Pi_h\bar u(x)-\bar u(x)|d\sigma(x)\le$$

$$\lambda h\sum_{j\in\tau_h^0}\int_{x_\Gamma^j}^{x_\Gamma^{j+1}}|\Pi_h\bar u(x)-\bar u(x)|d\sigma(x). \tag{4.15}$$

So we have that

$$|J'(\bar u)(\Pi_h\bar u-\bar u)|\le\lambda h\|\bar u-\Pi_h\bar u\|_{L^1(\Gamma)}$$

and the result follows taking into account the interpolation error stated in (4.9). ∎

Proof of Theorem 4.6. First order optimality conditions for (P) and (P$_h$) imply that

$$J'(\bar u)(\bar u_h-\bar u)\ge 0$$

$$J_h'(\bar u_h)(\Pi_h\bar u-\bar u_h)\ge 0\Rightarrow J_h'(\bar u_h)(\Pi_h\bar u-\bar u)+J_h'(\bar u_h)(\bar u-\bar u_h)\ge 0$$

Making the sum

$$J'(\bar{u})(\bar{u}_h - \bar{u}) + J_h'(\bar{u}_h)(\Pi_h \bar{u} - \bar{u}) + J_h'(\bar{u}_h)(\bar{u} - \bar{u}_h) \geq 0$$

or equivalently

$$(J'(\bar{u}) - J_h'(\bar{u}_h))(\bar{u} - \bar{u}_h) \leq J_h'(\bar{u}_h)(\Pi_h \bar{u} - \bar{u}).$$

Now applying Lemma 4.7 and the previous inequality

$$\nu \|\bar{u}_h - \bar{u}\|^2_{L^2(\Gamma)} \leq (J'(\bar{u}) - J'(\bar{u}_h))(\bar{u} - \bar{u}_h) =$$

$$(J'(\bar{u}) - J_h'(\bar{u}_h))(\bar{u} - \bar{u}_h) + (J_h'(\bar{u}_h) - J'(\bar{u}_h))(\bar{u} - \bar{u}_h) \leq$$

$$J_h'(\bar{u}_h)(\Pi_h \bar{u} - \bar{u}) + (J_h'(\bar{u}_h) - J'(\bar{u}_h))(\bar{u} - \bar{u}_h) =$$

$$(J_h'(\bar{u}_h) - J'(\bar{u}))(\Pi_h \bar{u} - \bar{u}) + J'(\bar{u})(\Pi_h \bar{u} - \bar{u}) + (J_h'(\bar{u}_h) - J'(\bar{u}_h))(\bar{u} - \bar{u}_h) \tag{4.16}$$

The first term is estimated using Lemma 4.8, and the third one with Lemma 4.9.

$$\nu \|\bar{u} - \bar{u}_h\|^2_{L^2(\Gamma)} \leq C(h^{3/2} + \|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)})\|\Pi_h \bar{u} - \bar{u}\|_{L^2(\Gamma)} + J'(\bar{u})(\Pi_h \bar{u} - \bar{u}) +$$

$$(C_{\rho,\varepsilon} h^{3/2-\varepsilon} + \rho \|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)})\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)}.$$

Now we have just to take $\rho = \nu/2$ and use Young's inequality to get that for all $0 < \varepsilon \leq 1/2$ there exists $C_\varepsilon > 0$

$$\|\bar{u} - \bar{u}_h\|^2_{L^2(\Gamma)} \leq C_\varepsilon (h^{3/2} \|\Pi_h \bar{u} - \bar{u}\|_{L^2(\Gamma)} + \|\Pi_h \bar{u} - \bar{u}\|^2_{L^2(\Gamma)} + J'(\bar{u})(\Pi_h \bar{u} - \bar{u}) + h^{3-2\varepsilon}).$$

Fixing $0 < \varepsilon < 1/2$, dividing by h^2 and taking into account the interpolation error estimate (4.9) and Lemma 4.10, we can pass to the limit and obtain

$$\lim_{h \to 0} \frac{\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)}}{h} = 0,$$

and we have achieved a contradiction. □

As a consequence, we have uniform convergence and even an estimate for the error in $L^\infty(\Gamma)$. We will use the following inverse inequality. For every $u_h \in U_h$

$$\|u_h\|_{L^\infty(\Gamma)} \leq C h^{-1/2} \|u_h\|_{L^2(\Gamma)} \tag{4.17}$$

THEOREM 4.11. *Let $\bar{u}$ and $\bar{u}_h$ be the ones of Theorem 4.6. Then*

$$\lim_{h \to 0} \frac{\|\bar{u} - \bar{u}_h\|_{L^\infty(\Gamma)}}{h^{1/2}} = 0.$$

Proof. Using the triangular inequality, we obtain

$$\|\bar u - \bar u_h\|_{L^\infty(\Gamma)} \le \|\bar u - \Pi_h \bar u\|_{L^\infty(\Gamma)} + \|\Pi_h \bar u - \bar u_h\|_{L^\infty(\Gamma)}.$$

Since the optimal control is Lipschitz, applying (4.9) we have for the first term

$$\frac{\|\bar u - \Pi_h \bar u\|_{L^\infty(\Gamma)}}{h^{1/2}} \le C h^{1/2}.$$

For the second term, we can apply the inverse inequality (4.17):

$$\frac{\|\Pi_h \bar u - \bar u_h\|_{L^\infty(\Gamma)}}{h^{1/2}} \le \frac{C h^{-1/2}\|\Pi_h \bar u - \bar u_h\|_{L^2(\Gamma)}}{h^{1/2}} \le C \frac{\|\Pi_h \bar u - \bar u\|_{L^2(\Gamma)} + \|\bar u - \bar u_h\|_{L^2(\Gamma)}}{h}$$

and the result follows from (4.9) and Theorem 4.6. □

5. Semidiscretization. In this section we will follow the schema proposed by Hinze in [9] for linear quadratic distributed problems. The idea is to discretize the state and solve the corresponding infinite dimensional optimization problem. The new control problem is now defined by

$$(\mathrm{Q}_h)\left\{\begin{array}{l} \min J_h(u) = \displaystyle\int_\Omega L(x, y_h(u)(x))\,dx + \int_\Gamma l(x, y_h(u)(x), u(x))d\sigma(x), \\ \text{subject to } \ (y_h(u), u) \in Y_h \times U^{ad} \text{ satisfying (4.1).} \end{array}\right.$$

The first order optimality conditions can be written as follows:

THEOREM 5.1. *Assume that $\bar u_h$ is a local optimal solution of (Q_h). Then there exist $\bar y_h$ and $\bar\varphi_h$ in Y_h satisfying*

$$a(\bar y_h, z_h) = \int_\Omega a_0(x, \bar y_h) z_h\,dx + \int_\Gamma (b_0(x, \bar y_h) + \bar u_h) z_h\,dx \quad \forall z_h \in Y_h, \tag{5.1}$$

$$a(\bar\varphi_h, z_h) = \int_\Omega \left(\frac{\partial a_0}{\partial y}(x, \bar y_h)\bar\varphi_h + \frac{\partial L}{\partial y}(x, \bar y_h)\right) z_h\,dx +$$

$$\int_\Gamma \left(\frac{\partial b_0}{\partial y}(x, \bar y_h)\bar\varphi_h + \frac{\partial l}{\partial y}(x, \bar y_h, \bar u_h)\right) z_h\,d\sigma(x) \quad \forall z_h \in Y_h, \tag{5.2}$$

$$\int_\Gamma \left(\bar\varphi_h + \frac{\partial l}{\partial u}(x, \bar y_h, \bar u_h)\right)(u - \bar u_h)\,d\sigma(x) \ge 0 \ \ \forall u \in U^{ad}. \tag{5.3}$$

The following result is the counterpart of Theorem 3.7.

THEOREM 5.2. *Assume that $\bar u_h$ is a local optimal solution of (Q_h). Then for all $x \in \Gamma$ the equation*

$$\bar\varphi_h(x) + \frac{\partial l}{\partial u}(x, \bar y_h(x), t) = 0 \tag{5.4}$$

has a unique solution $\bar t = \bar s_h(x)$. The mapping $\bar s_h : \Gamma \longrightarrow \mathbb{R}$ is Lipschitz and it is related with $\bar u_h$ through the formula

$$\bar u_h(x) = Proj_{[\alpha,\beta]}(\bar s_h(x)) = \max\{\alpha, \min\{\beta, \bar s_h(x)\}\}. \tag{5.5}$$

Notice that in general $\bar{u}_h(x) \notin U_h$.

LEMMA 5.3. *For every $h > 0$ let $\bar{u}_h$ be a solution of (Q_h). Then there exist subsequences $\{\bar{u}_h\}_{h>0}$ converging in the weak* topology of $L^\infty(\Gamma)$ that will be denoted in the same way. If $\bar{u}_h \rightharpoonup \bar{u}$ in the mentioned topology, then $\bar{u}$ is a solution of (P) and*

$$\lim_{h\to 0} J_h(\bar{u}_h) = J(\bar{u}) \quad \textit{and} \quad \lim_{h\to 0} \|\bar{u}_h - \bar{u}\|_{L^\infty(\Gamma)} = 0.$$

Proof. The first part is as in the proof of Theorem 4.4. Let us check the second part. Take $x \in \Gamma$.

$$|\bar{u}(x) - \bar{u}_h(x)| \le |\bar{s}(x) - \bar{s}_h(x)|.$$

Due to assumption (A2)

$$\Lambda|\bar{s}(x) - \bar{s}_h(x)| \le \left|\frac{\partial l}{\partial u}(x, \bar{y}_h(x), \bar{s}(x)) - \frac{\partial l}{\partial u}(x, \bar{y}_h(x), \bar{s}_h(x))\right| \le$$

$$\left|\frac{\partial l}{\partial u}(x, \bar{y}_h(x), \bar{s}(x)) - \frac{\partial l}{\partial u}(x, \bar{y}(x), \bar{s}(x))\right| + \left|\frac{\partial l}{\partial u}(x, \bar{y}(x), \bar{s}(x)) - \frac{\partial l}{\partial u}(x, \bar{y}_h(x), \bar{s}_h(x))\right| =$$

$$\left|\frac{\partial l}{\partial u}(x, \bar{y}_h(x), \bar{s}(x)) - \frac{\partial l}{\partial u}(x, \bar{y}(x), \bar{s}(x))\right| + |\bar{\varphi}(x) - \bar{\varphi}_h(x)| \le$$

$$|\bar{y}_h(x) - y(x)| + |\bar{\varphi}(x) - \bar{\varphi}_h(x)|.$$

The proof concludes thanks to the uniform convergence $\bar{y}_h \to \bar{y}$ and $\bar{\varphi}_h \to \bar{\varphi}$ (see Theorem 4.1(iv)). □

THEOREM 5.4. *Let $\bar{u}$ be a solution of problem (P) such that $J''(\bar{u})v^2 > 0$ holds for all $v \in C_{\bar{u}} \setminus \{0\}$ and $\bar{u}_h$ a sequence of solutions of (Q_h) converging in $L^\infty(\Gamma)$ to $\bar{u}$. Then for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that*

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} \le C_\varepsilon h^{3/2-\varepsilon}.$$

We will make a direct proof of this theorem, not following an argument by contradiction as in the proof of Theorem 4.6; see (4.11). Through the rest of the section $\bar{u}$ and $\bar{u}_h$ will be the ones of Theorem 5.4.

LEMMA 5.5. *There exists $\nu > 0$ and $h_1 > 0$ such that for all $0 < h < h_1$*

$$\nu\|\bar{u} - \bar{u}_h\|^2_{L^2(\Gamma)} \le (J'(\bar{u}_h) - J'(\bar{u}))(\bar{u}_h - \bar{u}).$$

Proof. Take $\hat{u}_h$, v_h and v as in the proof of Lemma 4.7. The only point where we used (4.11) was to state that $v \in C_{\bar{u}}$. Now we can proceed as follows.

$$\int_\Gamma |\bar{d}(x)v(x)|d\sigma(x) = \int_\Gamma \bar{d}(x)v(x)d\sigma(x) =$$

$$= \lim_{h\to 0}\int_\Gamma \bar d_h(x)v_h(x)d\sigma(x) = \lim_{h\to 0}\int_\Gamma \bar d_h(x)\frac{\bar u_h-\bar u}{\|\bar u_h-\bar u\|_{L^2(\Gamma)}}d\sigma(x)\le 0$$

since $\bar u$ is an admissible control for (Q_h) and we can apply first order optimality condition (5.3). The rest of the proof is an in Lemma 4.7. □

Proof of Theorem 5.4. We can repeat the proof of Theorem 4.6. Since $\bar u$ is admissible for (Q_h), we can take $\bar u$ instead of $\Pi_h u$ in (4.16) and we have that for every $\rho>0$ and every $0<\varepsilon<1/2$ there exists $C_{\rho,\varepsilon}$ such that

$$\nu\|\bar u_h-\bar u\|^2_{L^2(\Gamma)} \le (J_h'(\bar u_h)-J'(\bar u_h))(\bar u-\bar u_h)\le$$

$$(C_{\rho,\varepsilon}h^{3/2-\varepsilon}+\rho\|\bar u-\bar u_h\|_{L^2(\Gamma)})\|\bar u-\bar u_h\|_{L^2(\Gamma)}.$$

Now we have just to take $\rho=\nu/2$ to get that

$$\|\bar u-\bar u_h\|_{L^2(\Gamma)}\le C_\varepsilon h^{3/2-\varepsilon}$$

for all $0<\varepsilon<1/2$. □

Remark 5.6. *This semidiscretization procedure is interesting when $l(x,y,u)=\ell(x,y)+\frac{\Lambda}{2}u^2$ because in that situation,* (5.5) *leads to*

$$\bar u_h(x) = Proj_{[\alpha,\beta]}\left(-\frac{1}{\Lambda}\bar\varphi_h(x)\right).$$

This expression shows that $\bar u_h$ is piecewise linear. Though $\bar u_h$ can have more corner points than those corresponding to the boundary nodes of the triangulation, the amount of these points is finite. Therefore $\bar u_h$ can be handled by the computer.

6. Objective function quadratic with respect to the control. In many practical cases when we make the full discretization, the order of convergence observed for the controls in $L^2(\Gamma)$ is $h^{3/2}$ and in $L^\infty(\Gamma)$ is h. Let us show why. We will make two assumptions that are fulfilled in many situations:

(Q1) $l(x,y,u)=\ell(x,y)+e(x)u+\dfrac{\Lambda}{2}u^2$, where $\Lambda>0$ and

- the function $\ell:\Gamma\times\mathbb{R}\longrightarrow\mathbb{R}$ is Lipschitz with respect to the first component, of class C^2 with respect to the second variable, $\ell(\cdot,0)\in L^1(\Gamma)$, $\dfrac{\partial^2\ell}{\partial y^2}(\cdot,0)\in L^\infty(\Gamma)$ and for all $M>0$ there exists a constant $C_{\ell,M}>0$ such that

$$\left|\frac{\partial\ell}{\partial y}(x_2,y)-\frac{\partial\ell}{\partial y}(x_1,y)\right|\le C_{\ell,M}|x_2-x_1|,$$

$$\left|\frac{\partial^2\ell}{\partial y^2}(x,y_2)-\frac{\partial^2\ell}{\partial y^2}(x,y_1)\right|\le C_{\ell,M}|y_2-y_1|,$$

  for a.e. $x,x_i\in\Gamma$ and $|y|,|y_i|\le M$, $i=1,2$;
- the function $e:\Gamma\to\mathbb{R}$ is Lipschitz and satisfies the following approximation property: there exists $C_e>0$ such that

$$\|e-\Pi_h e\|_{L^2(\Gamma)}\le C_e h^{3/2}.$$

  This assumption is not very constraining. Although it is not true for Lipschitz functions in general, it is true for a very wide class of functions. For instance for Lipschitz functions that are piecewise in $H^{3/2}(\Gamma)$.

(Q2) If we name $\Gamma_s = \{x \in \Gamma : \bar{u}(x) = \alpha \text{ or } \bar{u}(x) = \beta\}$, then the number of points in $\partial\Gamma_s$ –the boundary of Γ_s in the topology of Γ– is finite. Let us name $\mathcal{N}$ that number.

Through the rest of this section, $\bar{u}$ and $\bar{u}_h$ will be the ones given in the assumptions of Theorem 4.6. (Q1) and (3.6) imply that for every $x \in \Gamma$

$$\bar{u}(x) = Proj_{[\alpha,\beta]}\left(-\frac{1}{\Lambda}(\bar{\varphi}(x) + e(x))\right)$$

Now we can state uniform convergence *a priori*.

LEMMA 6.1. *The sequence* $\{\bar{u}_h\}_{h>0}$ *is bounded in* $W^{1-1/p,p}(\Gamma)$ *and the following convergence property holds:*

$$\|\bar{u} - \bar{u}_h\|_{L^\infty(\Gamma)} = 0.$$

Proof. The control does not appear explicitly in the adjoint equation, and hence $\varphi_{\bar{u}_h}$ is uniformly bounded in $W^{2,p}(\Omega)$. Usual finite element estimates then give us that $\bar{\varphi}_h$ is uniformly bounded in $W^{1,p}(\Omega)$, and therefore their traces are uniformly bounded in $W^{1-1/p,p}(\Gamma)$. Since $\bar{u}_h$ is the $L^2(\Gamma)$ projection of $\frac{-1}{\Lambda}(\bar{\varphi}_h + e_u)$ in $U_h \cap U^{ad}$, and this projection is stable for Sobolev norms (see Casas and Raymond [7]), then the discrete controls are bounded in $W^{1-1/p,p}(\Gamma)$.

Finally, Theorem 4.4 and the compactness of the embedding $W^{1-1/p,p}(\Gamma) \subset L^\infty(\Gamma)$ leads to the desired convergence result. □

This boundness can be taken into account to improve Lemma 4.9.

LEMMA 6.2. *There exists* $C > 0$ *such that*

$$|(J_h'(\bar{u}_h) - J'(\bar{u}_h))v| \leq Ch^{3/2}\|v\|_{L^2(\Gamma)} \quad \forall v \in L^2(\Gamma). \tag{6.1}$$

Proof. Since the controls are uniformly bounded in $H^{1/2}(\Gamma)$, we can use estimate (4.3). From the hypotheses on l it is readily deduced

$$|(J_h'(\bar{u}_h) - J'(\bar{u}_h))v| \leq \int_\Gamma (|\bar{\varphi}_h - \varphi_{\bar{u}_h}|)\, v\, d\sigma(x) \leq$$

$$C\|\bar{\varphi}_h - \varphi_{\bar{u}_h}\|_{L^2(\Gamma)}\|v\|_{L^2(\Gamma)} \leq Ch^{3/2}\|v\|_{L^2(\Gamma)}.$$

□

Remember that Lemma 4.7 was proved using assumption (4.11) and Lemma 5.5 was proved using that $\bar{u}$ was an admissible control for (Q_h). Let us show that in this case the result is still true.

LEMMA 6.3. *There exists* $\nu > 0$ *and* $h_1 > 0$ *such that for all* $0 < h < h_1$

$$\nu\|\bar{u} - \bar{u}_h\|^2_{L^2(\Gamma)} \leq (J'(\bar{u}_h) - J'(\bar{u}))(\bar{u}_h - \bar{u}).$$

Proof. Let us take $\tau > 0$ as in Remark 3.6-(2). Let us prove that $\bar{u}_h - \bar{u} \in C^\tau_{\bar{u}}$ for h small enough. The sign condition (3.13) is trivial. From the uniform convergence $\bar{u}_h \to \bar{u}$, we can deduce that for h small enough $\|\bar{d} - \bar{d}_h\|_{L^\infty(\Gamma)} < \tau/4$. Take $\xi \in [x^j_\Gamma, x^{j+1}_\Gamma]$ where $\bar{d}(\xi) > \tau$. On one side, we have that $\bar{u}(\xi) = \alpha$. On the other hand, since $\bar d$ is a Lipschitz function, for h small enough $\bar d(x) > \tau/2$ for every $x \in [x_\Gamma^j, x_\Gamma^{j+1}]$, and hence $\bar d_h(x) > \tau/4$ for every $x \in [x_\Gamma^j, x_\Gamma^{j+1}]$. First order optimality conditions for problem (P$_h$) imply that $\bar u_h(x) = \alpha$ for every $x \in [x_\Gamma^j, x_\Gamma^{j+1}]$, and then $\bar u_h(\xi) - \bar u(\xi) = 0$. The same is applicable when $\bar d(\xi) < \tau$.

Therefore $\bar u_h - \bar u \in C_{\bar u}^\tau$ and there exists $\delta > 0$ such that

$$\delta\|\bar u_h - \bar u\|_{L^2(\Gamma)}^2 \le J''(\bar u)(\bar u_h - \bar u)^2.$$

By applying the mean value theorem we get for some $\hat u_h = \bar u + \theta_h(\bar u_h - \bar u)$

$$(J'(\bar u_h) - J'(\bar u))(\bar u_h - \bar u) = J''(\hat u_h)(\bar u_h - \bar u)^2.$$

So we can write that

$$\delta\|\bar u_h - \bar u\|_{L^2(\Gamma)}^2 \le (J'(\bar u_h) - J'(\bar u))(\bar u_h - \bar u) + [J''(\bar u) - J''(\hat u_h)](\bar u_h - \bar u)^2.$$

Finally, the uniform convergence of $\bar u_h \to \bar u$ and assumptions (A1)–(A4) allow us to estimate the last term by $\frac{\delta}{2}\|\bar u_h - \bar u\|_{L^2(\Gamma)}^2$ for h small enough. □

LEMMA 6.4. *Under (Q2)*

$$|J'(\bar u)(\Pi_h \bar u - \bar u)| \le Ch^3.$$

Proof. Now we will distinguish three kind of elements:

$$\tau_h^1 = \{j \in \{1, \ldots, N(h)\} : (x_\Gamma^j, x_\Gamma^{j+1}) \subset \Gamma_s\},$$

$$\tau_h^2 = \{j \in \{1, \ldots, N(h)\} : \alpha < \bar u(x) < \beta \ \forall x \in (x_\Gamma^j, x_\Gamma^{j+1})\},$$

and

$$\tau_h^3 = \{j \in \{1, \ldots, N(h)\} : (x_\Gamma^j, x_\Gamma^{j+1}) \cap \partial\Gamma_s \neq \emptyset\}.$$

Notice that

1) $\tau_h^1 \cup \tau_h^2 \cup \tau_h^3 = \{1, \ldots, N(h)\}$ and $\tau_h^i \cap \tau_h^j = \emptyset$ if $i \neq j$.

2) If $j \in \tau_h^1$ then $\bar u(x) = \Pi_h \bar u(x)$ for all $x \in (x_\Gamma^j, x_\Gamma^{j+1})$. Both are either α or β on the segment.

3) If $j \in \tau_h^2$ then $\bar d(x) = 0$ for all $x \in (x_\Gamma^j, x_\Gamma^{j+1})$ (see 3.9).

4) The number of elements of τ_h^3 is less or equal than $\mathcal N$ and if $j \in \tau_h^3$ then there exists $\xi_j \in (x_\Gamma^j, x_\Gamma^{j+1})$ such that $\bar d(\xi_j) = 0$.

Taking into account these considerations, the Lipschitz continuity of $\bar d$ and $\bar u$ we have that

$$|J'(\bar u)(\Pi_h \bar u - \bar u)| = \sum_{j \in \tau_h^3} \int_{x_\Gamma^j}^{x_\Gamma^{j+1}} |\bar d(x) - \bar d(\xi_j)|\, |\Pi_h \bar u - \bar u| d\sigma(x) \le$$

$$\mathcal N h \lambda h \|\bar u\|_{C^{0,1}(\Gamma)} h = Ch^3.$$

□

LEMMA 6.5. *Under (Q1) and (Q2)*

$$\|\Pi_h\bar{u} - \bar{u}\|_{L^2(\Gamma)} \leq Ch^{3/2}.$$

Proof. Take τ_h^i, $i = 1, 2, 3$, as in the previous proof. Notice that if $j \in \tau_h^2$ then $\bar{u}(x) = \frac{1}{\Lambda}(\bar{\varphi}(x) + e(x)$ for all $x \in (x_\Gamma^j, x_\Gamma^{j+1})$ and $\bar{\varphi} \in H^{3/2}(\Gamma)$. So, taking into account Lemma 4.5 and assumption (Q1), we have that

$$\int_\Gamma (\bar{u}(x) - \Pi_h\bar{u}(x))^2 d\sigma(x) \leq \frac{2}{\Lambda} \sum_{j\in\tau_h^2} \int_{x_\Gamma^j}^{x_\Gamma^{j+1}} |\Pi_h\bar{\varphi}(x) - \bar{\varphi}(x)|^2 d\sigma(x) +$$

$$\frac{2}{\Lambda} \sum_{j\in\tau_h^2} \int_{x_\Gamma^j}^{x_\Gamma^{j+1}} |\Pi_h e(x) - e(x)|^2 d\sigma(x) + \sum_{j\in\tau_h^3} \int_{x_\Gamma^j}^{x_\Gamma^{j+1}} |\Pi_h\bar{u}(x) - \bar{u}(x)|^2 d\sigma(x) \leq$$

$$\frac{2}{\Lambda}\left(\|\Pi_h\bar{\varphi} - \bar{\varphi}\|^2_{L^2(\Gamma)} + \|\Pi_h e - e\|^2_{L^2(\Gamma)}\right) + \mathcal{N}\|\bar{u}\|^2_{C^{0,1}(\Gamma)}h^3 \leq Ch^3.$$

□

THEOREM 6.6. *Let $\bar{u}$ be a solution of problem (P) such that $J''(\bar{u})v^2 > 0$ holds for all $v \in C_{\bar{u}} \setminus \{0\}$ and $\bar{u}_h$ a sequence of solutions of (P_h) converging in $L^2(\Gamma)$ to $\bar{u}$. Then there exists $C > 0$ such that*

$$\|\bar{u}_h - \bar{u}\|_{L^2(\Gamma)} \leq Ch^{3/2}$$

Proof. We repeat the steps to get (4.16) and apply the previous lemmas:

$$\nu\|\bar{u}_h - \bar{u}\|^2_{L^2(\Gamma)} \leq$$

$$(J_h'(\bar{u}_h) - J'(\bar{u}))(\Pi_h\bar{u} - \bar{u}) + J'(\bar{u})(\Pi_h\bar{u} - \bar{u}) + (J_h'(\bar{u}_h) - J'(\bar{u}_h))(\bar{u} - \bar{u}_h) \leq$$

$$C(h^{3/2} + \|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)})\|\Pi_h\bar{u} - \bar{u}\|_{L^2(\Gamma)} + Ch^{3/2}\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} + Ch^3.$$

By Young's inequality we get that

$$\|\bar{u} - \bar{u}_h\|^2_{L^2(\Gamma)} \leq C(h^{3/2}\|\Pi_h\bar{u} - \bar{u}\|_{L^2(\Gamma)} + \|\Pi_h\bar{u} - \bar{u}\|^2_{L^2(\Gamma)} + h^3),$$

and the result follows from Lemma 6.5. □

Arguing as in the proof of Theorem 4.11 we obtain the following result.

THEOREM 6.7. *Under the assumptions of Theorem 6.6*

$$\|\bar{u} - \bar{u}_h\|_{L^\infty(\Gamma)} \leq Ch.$$

Notice that for a function $\varphi \in H^{3/2}(\Gamma)$, the interpolation error $\|\varphi - \Pi_h\varphi\|_{L^\infty(\Gamma)} \leq Ch$ (see [2, Eq. (4.4.22)]) and this cannot be improved in general, so we have again the best possible result.

7. Numerical confirmation. In this section we shall verify our error estimates by numerical test examples for which we know the exact solution. We report both on a linear-quadratic problem and on a semilinear problem. A detailed explanation about the optimization procedure used can be found in [6].

7.1. A linear-quadratic problem. Let us consider the problem

$$\text{(E1)}\left\{\begin{array}{l}\min J(u) = \dfrac{1}{2}\displaystyle\int_\Omega (y_u(x) - y_\Omega(x))^2 dx + \dfrac{\mu}{2}\int_\Gamma u(x)^2 d\sigma(x) + \\ + \displaystyle\int_\Gamma e_u(x)u(x)d\sigma(x) + \int_\Gamma e_y(x)y_u(x)d\sigma(x) \\ \\ \text{subject to } (y_u, u) \in H^1(\Omega) \times L^\infty(\Omega), \\ u \in U_{ad} = \{u \in L^\infty(\Gamma) \,|\, 0 \le u(x) \le 1 \text{ a.e. } x \in \Gamma\}, \\ (y_u, u) \text{ satisfying the linear state equation (7.1)}\end{array}\right.$$

$$\left\{\begin{array}{rcllc} -\Delta y_u(x) + c(x)y_u(x) & = & e_1(x) & \text{in} & \Omega \\ \partial_\nu y_u(x) + y_u(x) & = & e_2(x) + u(x) & \text{on} & \Gamma. \end{array}\right. \tag{7.1}$$

We fix the following data: $\Omega = (0,1)^2$, $\mu = 1$, $c(x_1, x_2) = 1 + x_1^2 - x_2^2$, $e_y(x_1, x_2) = 1$, $y_\Omega(x_1, x_2) = x_1^2 + x_1x_2$, $e_1(x_1, x_2) = -2 + (1 + x_1^2 - x_2^2)(1 + 2x_1^2 + x_1x_2 - x_2^2)$,

$$e_u(x_1, x_2) = \left\{\begin{array}{ll} -1 - x_1^3 & \text{on } \Gamma_1 \\ -1 - \min\left\{\begin{array}{l} 8(x_2 - 0.5)^2 + 0.58, \\ 1 - 16x_2(x_2 - y_1^*)(x_2 - y_2^*)(x_2 - 1) \end{array}\right\} & \text{on } \Gamma_2 \\ -1 - x_1^2 & \text{on } \Gamma_3 \\ -1 + x_2(1 - x_2) & \text{on } \Gamma_4 \end{array}\right.$$

and

$$e_2(x_1, x_2) = \left\{\begin{array}{ll} 1 - x_1 + 2x_1^2 - x_1^3 & \text{on } \Gamma_1 \\ 7 + 2x_2 - x_2^2 - \min\{8(x_2 - .5)^2 + .58, 1\} & \text{on } \Gamma_2 \\ -2 + 2x_1 + x_1^2 & \text{on } \Gamma_3 \\ 1 - x_2 - x_2^2 & \text{on } \Gamma_4, \end{array}\right.$$

where Γ_1 to Γ_4 are the four sides of the square, starting at the bottom side and going counterclockwise,

$$y_1^* = 0.5 - \frac{\sqrt{21}}{20} \text{ and } y_2^* = 0.5 + \frac{\sqrt{21}}{20}.$$

This problem has the following solution $(\bar{y}, \bar{u})$ with adjoint state $\bar{\varphi}$: $\bar{y}(x) = 1 + 2x_1^2 + x_1x_2 - x_2^2$, $\bar{\varphi}(x_1, x_2) = 1$ and

$$\bar{u}(x_1, x_2) = \left\{\begin{array}{ll} x_1^3 & \text{on } \Gamma_1 \\ \min\{8(x_2 - .5)^2 + .58, 1\} & \text{on } \Gamma_2 \\ x_1^2 & \text{on } \Gamma_3 \\ 0 & \text{on } \Gamma_4. \end{array}\right.$$

It is not difficult to check that the state equation (7.1) is satisfied by $(\bar{y}, \bar{u})$. So is the adjoint equation

$$\left\{\begin{array}{rcllc} -\Delta\bar{\varphi}(x) + c(x)\bar{\varphi}(x) & = & \bar{y}(x) - y_\Omega(x) & \text{in} & \Omega \\ \partial_\nu\bar{\varphi}(x) + \bar{\varphi}(x) & = & e_y & \text{on} & \Gamma. \end{array}\right.$$

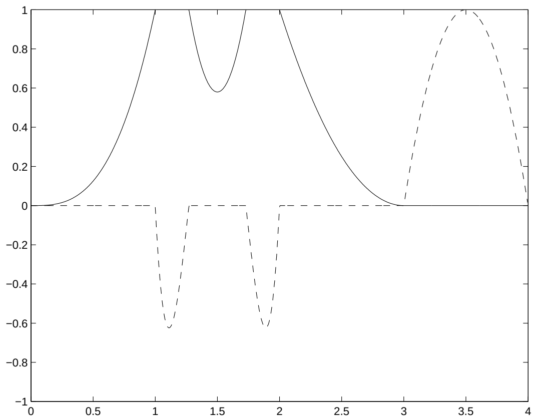

FIG. 7.1. *solid:* $\bar{u}(x_1, x_2)$*, dashed:* $\bar{d}(x_1, x_2)$

In example (E1)

$$\bar{d}(x) = \bar{\varphi}(x) + e_u(x) + \bar{u}(x) = \begin{cases} 0 & \text{on } \Gamma_1 \\ \min\{0, 16x_2(x_2 - y_1^*)(x_2 - y_2^*)(x_2 - 1)\} & \text{on } \Gamma_2 \\ 0 & \text{on } \Gamma_3 \\ x_2(1 - x_2) & \text{on } \Gamma_4, \end{cases}$$

and it satisfies the relations (3.9) (see figure 7.1), so the first order necessary condition (3.8) is fulfilled. Since (E1) is a convex problem, this condition is also sufficient for $(\bar{y}, \bar{u})$ to be global minimum.

Observe that the control attains its constraints on Γ_2 at the points $(1, y_1^*)$ and $(1, y_2^*)$ which are not going to be node points of the control mesh (unless we force it or we have a lot of luck, which would not be natural).

Test 1.

h	$\Vert\bar{y} - \bar{y}_h\Vert_{L^2(\Omega)}$	$\vert\bar{y} - \bar{y}_h\vert_{H^1(\Omega)}$	$\Vert\bar{u} - \bar{u}_h\Vert_{L^2(\Gamma)}$	$\Vert\bar{u} - \bar{u}_h\Vert_{L^\infty(\Gamma)}$
2^{-4}	$5.3e-04$	$7.3e-02$	$8.5e-03$	$4.1e-02$
2^{-5}	$1.3e-04$	$3.6e-02$	$3.0e-03$	$1.5e-02$
2^{-6}	$3.4e-05$	$1.8e-02$	$1.1e-03$	$1.1e-02$
2^{-7}	$8.0e-06$	$9.0e-03$	$3.7e-04$	$3.8e-03$
2^{-8}	$2.1e-06$	$4.5e-03$	$1.4e-04$	$2.7e-03$

The orders of convergence obtained are h^2 for $\|\bar{y} - \bar{y}_h\|_{L^2(\Omega)}$, h for the seminorm $|\bar{y}-\bar{y}_h|_{H^1(\Omega)}$, $h^{1.5}$ for the $L^2(\Gamma)$ norm of the control and h for the $L^\infty(\Gamma)$ norm. Notice, nevertheless, that in the last column when we divide h by 2, the error is not divided into two, but when we divide h by 4 the error is divided into 4. For the subsequences corresponding to values of h even and odd powers of 1/2, $\|\bar{u} - \bar{u}_h\|_{L^\infty(\Gamma)} \leq 0.5h$ for $h = 2^{-2k+1}$ and $\|\bar{u} - \bar{u}_h\|_{L^\infty(\Gamma)} \leq 0.7h$. for $h = 2^{-2k}$.

7.2. Semilinear example. Let us next consider the problem

$$\text{(E2)}\left\{\begin{array}{l} \min J(u) = \dfrac{1}{2}\displaystyle\int_\Omega (y_u(x) - y_\Omega(x))^2 dx + \dfrac{\mu}{2}\int_\Gamma u(x)^2 d\sigma(x) + \\ +\displaystyle\int_\Gamma e_u(x)u(x)d\sigma(x) + \int_\Gamma e_y(x)y_u(x)d\sigma(x) \\ \text{subject to } (y_u, u) \in H^1(\Omega)\times L^\infty(\Omega), \\ u \in U_{ad} = \{u \in L^\infty(\Omega) \mid 0 \le u(x) \le 1 \text{ a.e. } x \in \Gamma\}, \\ (y_u, u) \text{ satisfying the semilinear state equation (7.2)} \end{array}\right.$$

$$\left\{\begin{array}{rcll} -\Delta y_u(x) + c(x)y_u(x) &=& e_1(x) & \text{in } \Omega \\ \partial_\nu y_u(x) + y_u(x) &=& e_2(x) + u(x) - y(x)|y(x)| & \text{on } \Gamma. \end{array}\right. \tag{7.2}$$

The term $y|y|$ stands for y^2 that does not satisfy the assumptions on monotonicity required for our current work. However, in our computations negative values of y never occured so that in fact y^2 was used. This also assures that locally assumption **(A4)** is satisfied.

We fix: $\Omega = (0,1)^2$, $\mu = 1$, $c(x_1,x_2) = x_2^2 + x_1x_2$, $e_y(x_1,x_2) = -3 - 2x_1^2 - 2x_1x_2$, $y_\Omega(x_1,x_2) = 1 + (x_1+x_2)^2$, $e_1(x_1,x_2) = -2 + (1 + x_1^2 + x_1x_2)(x_2^2 + x_1x_2)$,

$$e_u(x_1,x_2) = \left\{\begin{array}{ll} 1 - x_1^3 & \text{on } \Gamma_1 \\ 1 - \min\left\{\begin{array}{l} 8(x_2 - 0.5)^2 + 0.58, \\ 1 - 16x_2(x_2 - y_1^*)(x_2 - y_2^*)(x_2 - 1) \end{array}\right\} & \text{on } \Gamma_2 \\ 1 - x_1^2 & \text{on } \Gamma_3 \\ 1 + x_2(1 - x_2) & \text{on } \Gamma_4 \end{array}\right.$$

and

$$e_2(x_1,x_2) = \left\{\begin{array}{ll} 2 - x_1 + 3x_1^2 - x_1^3 + x_1^4 & \text{on } \Gamma_1 \\ 8 + 6x_2 + x_2^2 - \min\{8(x_2 - .5)^2 + .58, 1\} & \text{on } \Gamma_2 \\ 2 + 4x_1 + 3x_1^2 + 2x_1^3 + x_1^4 & \text{on } \Gamma_3 \\ 2 - x_2 & \text{on } \Gamma_4. \end{array}\right.$$

This problem has the following solution $(\bar y, \bar u)$ with adjoint state $\bar\varphi$: $\bar y(x) = 1 + 2x_1^2 + x_1x_2$, $\bar\varphi(x_1,x_2) = -1$ and $\bar u$ is the same as in example (E1). Again $\bar d(x) = \bar\varphi(x) + e_u(x) + \bar u(x)$, which is also the same as in example (E1) and satisfies relation (3.9) so that the first order necessary condition (3.8) is fulfilled. The second derivative of $J(\bar u)$ is, according to (3.4),

$$J''(\bar u)v^2 = \int_\Omega z_v(x)^2 dx + \int_\Gamma v(x)^2 d\sigma(x) + \int_\Gamma (-2)\text{sign}(\bar y(x))\bar\varphi(x)z_v(x)^2 d\sigma(x),$$

where z_v is given by equation (3.1). Since $\bar\varphi(x) \le 0$ and $\bar y(x) \ge 0$, clearly $J''(\bar u)v^2 \ge \|v\|^2_{L^2(\Gamma)}$ holds. Therefore the second order sufficient conditions are fulfilled.

The tests show the same orders of convergence as for the linear example.

Test 2.

h	$\|\bar y - \bar y_h\|_{L^2(\Omega)}$	$\|\bar y - \bar y_h\|_{H^1(\Omega)}$	$\|\bar u - \bar u_h\|_{L^2(\Gamma)}$	$\|\bar u - \bar u_h\|_{L^\infty(\Gamma)}$
2^{-4}	$2.6e-04$	$3.5e-02$	$8.5e-03$	$4.1e-02$
2^{-5}	$7.0e-05$	$1.8e-02$	$3.0e-03$	$1.5e-02$
2^{-6}	$1.6e-05$	$8.8e-03$	$1.1e-03$	$1.1e-02$
2^{-7}	$4.3e-06$	$4.4e-03$	$3.8e-04$	$3.8e-03$
2^{-8}	$1e-06$	$2.2e-03$	$1.4e-04$	$2.7e-03$

8. Appendix. Proof of estimate (4.3)**.** To simplify the notation, we will drop the subindex u in this reasoning and we will make it only for the state. Let us define

$$\mathbb{A}(x) = \begin{cases} \dfrac{a_0(x,y(x)) - a_0(x,y_h(x))}{y(x)-y_h(x)} & \text{if } y(x)-y_h(x) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathbb{B}(x) = \begin{cases} \dfrac{b_0(x,y(x)) - b_0(x,y_h(x))}{y(x)-y_h(x)} & \text{if } y(x)-y_h(x) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $\mathbb{A}(x) \leq 0$ for a.e. $x \in \Omega$ and $\mathbb{B}(x) \leq 0$ for a.e. $x \in \Gamma$, and due to **(A5)** either $\mathbb{A}(x) < 0$ or $\mathbb{B}(x) < 0$ on a subset of positive measure of Ω or Γ.

For every $g \in L^2(\Gamma)$ there exists a unique $\phi \in H^{3/2}(\Omega)$ (see [10]) solution of

$$\int_\Omega \nabla z(x)\nabla\phi(x)\,dx = \int_\Omega \mathbb{A}(x)\phi(x)z(x)\,dx + \int_\Gamma \mathbb{B}(x)\phi(x)z(x)\,d\sigma(x) +$$

$$\int_\Gamma g(x)z(x)\,d\sigma(x)\ \forall z \in H^1(\Omega)$$

and there exists $C > 0$ such that $\|\phi\|_{H^{3/2}(\Omega)} \leq C\|g\|_{L^2(\Gamma)}$.

There exists also a unique $\phi_h \in Y_h$ solution of

$$\int_\Omega \nabla z_h(x)\nabla\phi_h(x)\,dx = \int_\Omega \mathbb{A}(x)\phi_h(x)z_h(x)\,dx + \int_\Gamma \mathbb{B}(x)\phi_h(x)z_h(x)\,d\sigma(x) +$$

$$\int_\Gamma g(x)z_h(x)\,d\sigma(x)\ \forall z_h \in Y_h$$

and $\|\phi - \phi_h\|_{H^1(\Omega)} \leq Ch^{1/2}\|\phi\|_{H^{3/2}(\Omega)}$. (See [2, Theorem (12.3.5)].)

Take $g \in L^2(\Gamma)$ with $\|g\|_{L^2(\Gamma)} = 1$ and denote $M = \max\{\|y\|_{C(\bar\Omega)}, \|y_h\|_{C(\bar\Omega)}\}$. Now apply the equation satisfied by ϕ, introduce ϕ_h, apply the equations satisfied by y and y_h, the definition of $\mathbb{A}$ and $\mathbb{B}$, assumptions on a_0, b_0, Hölders inequality and the trace theorem to get

$$\int_\Gamma g(x)(y(x)-y_h(x))\,d\sigma(x) = \int_\Omega \nabla(y(x)-y_h(x))\nabla\phi(x)\,dx -$$

$$\int_\Omega \mathbb{A}(x)\phi(x)(y(x)-y_h(x))\,dx - \int_\Gamma \mathbb{B}(x)\phi(x)(y(x)-y_h(x))\,d\sigma(x) =$$

$$= \int_\Omega \nabla(y(x)-y_h(x))\nabla(\phi(x)-\phi_h(x))\,dx + \int_\Omega \nabla(y(x)-y_h(x))\nabla\phi_h(x)\,dx -$$

$$\int_\Omega \mathbb{A}(x)\phi(x)(y(x)-y_h(x))\,dx - \int_\Gamma \mathbb{B}(x)\phi(x)(y(x)-y_h(x))\,d\sigma(x) =$$

$$= \int_\Omega \nabla(y(x) - y_h(x))\nabla(\phi(x) - \phi_h(x))\,dx + \int_\Omega (a_0(x,y(x)) - a_0(x,y_h(x)))\phi_h(x)\,dx +$$

$$\int_\Gamma (b_0(x,y(x)) - b_0(x,y_h(x)))\phi_h(x)\,d\sigma(x) -$$

$$\int_\Omega \mathbb{A}(x)\phi(x)(y(x) - y_h(x))\,dx - \int_\Gamma \mathbb{B}(x)\phi(x)(y(x) - y_h(x))\,d\sigma(x) =$$

$$= \int_\Omega \nabla(y(x) - y_h(x))\nabla(\phi(x) - \phi_h(x))\,dx +$$

$$\int_\Omega (a_0(x,y(x)) - a_0(x,y_h(x)))(\phi_h(x) - \phi(x))\,dx +$$

$$\int_\Gamma (b_0(x,y(x)) - b_0(x,y_h(x)))(\phi_h(x) - \phi(x))\,d\sigma(x) \leq$$

$$\|y - y_h\|_{H^1(\Omega)}\|\phi - \phi_h\|_{H^1(\Omega)} + \int_\Omega C_{a_0,M}|y(x)) - y_h(x)|\,|\phi_h(x) - \phi(x)|\,dx +$$

$$\int_\Gamma C_{b_0,M}|y(x)) - y_h(x)|\,|\phi_h(x) - \phi(x)|\,d\sigma(x) \leq$$

$$\|y - y_h\|_{H^1(\Omega)}\|\phi - \phi_h\|_{H^1(\Omega)} + C_{a_0,M}\|y - y_h\|_{L^2(\Omega)}\|\phi_h - \phi\|_{L^2(\Omega)} +$$

$$C_{b_0,M}\|y - y_h\|_{L^2(\Gamma)}\|\phi_h - \phi\|_{L^2(\Gamma)} \leq \|y - y_h\|_{H^1(\Omega)}\|\phi - \phi_h\|_{H^1(\Omega)} \leq$$

$$Ch\|y\|_{H^2(\Omega)}h^{1/2}\|\phi\|_{H^{3/2}(\Omega)} \leq Ch^{3/2}(\|u\|_{H^{1/2}(\Gamma)} + 1)\|g\|_{L^2(\Gamma)}.$$

Taking into account that $\|g\|_{L^2(\Gamma)} = 1$, we can write that

$$\|y - y_h\|_{L^2(\Omega)} = \sup_{\|g\|_{L^2(\Gamma)}=1} \int_\Gamma g(x)(y(x) - y_h(x))\,d\sigma(x) \leq Ch^{3/2}(\|u\|_{H^{1/2}(\Gamma)} + 1).$$

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Figure
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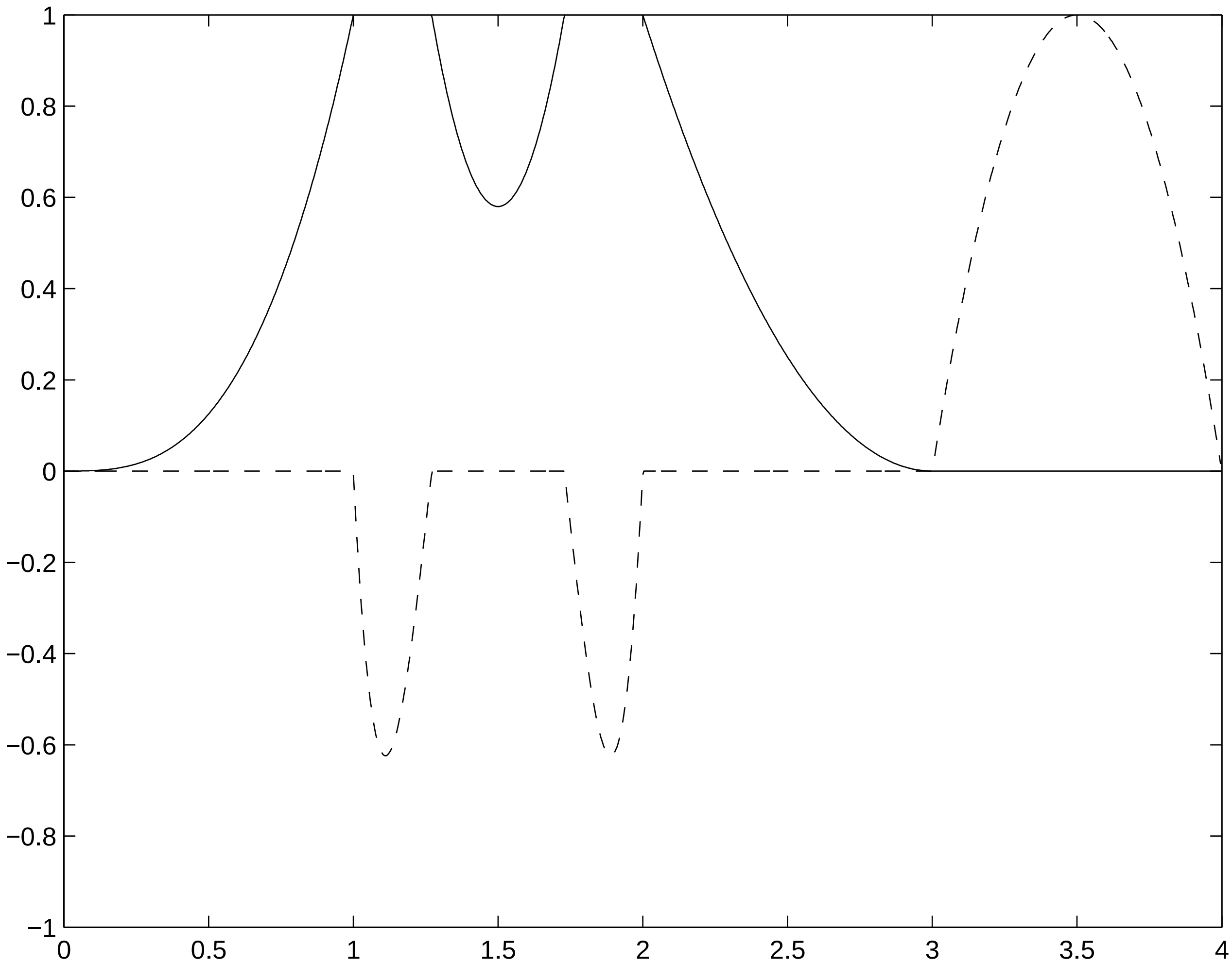

ERROR ESTIMATES FOR THE NUMERICAL APPROXIMATION OF DIRICHLET BOUNDARY CONTROL FOR SEMILINEAR ELLIPTIC EQUATIONS *

EDUARDO CASAS† AND JEAN-PIERRE RAYMOND‡

Abstract. We study the numerical approximation of boundary optimal control problems governed by semilinear elliptic partial differential equations with pointwise constraints on the control. The control is the trace of the state on the boundary of the domain, which is assumed to be a convex, polygonal, open set in $\mathbb{R}^2$. Piecewise linear finite elements are used to approximate the control as well as the state. We prove that the error estimates are of order $O(h^{1-1/p})$ for some $p > 2$, which is consistent with the $W^{1-1/p,p}(\Gamma)$-regularity of the optimal control.

Key words. Dirichlet control, semilinear elliptic equation, numerical approximation, error estimates

AMS subject classifications. 65N30, 65N15, 49M05, 49M25

1. Introduction. In this paper we study an optimal control problem governed by a semilinear elliptic equation. The control is the Dirichlet datum on the boundary of the domain. Bound constraints are imposed on the control and the cost functional involves the control in a quadratic form, and the state in a general way. The goal is to derive error estimates for the discretization of the control problem.

There is not many papers devoted to the derivation of error estimates for the discretization of control problems governed by partial differential equations; see the pioneer works by Falk [19] and Geveci [21]. However recently some papers have appeared providing new methods and ideas. Arada et al. [1] derived error estimates for the controls in the L^∞ and L^2 norms for distributed control problems. Similar results for an analogous problem, but also including integral state constraints, were obtained by Casas [8]. The case of a Neumann boundary control problem has been studied by Casas et al. [11]. The novelty of our paper with respect to the previous ones is double. First of all, here we deal with a Dirichlet problem, the control being the value of the state on the boundary. Second we consider piecewise linear continuous functions to approximate the optimal control, which is necessary because of the Dirichlet nature of the control, but it introduces some new difficulties. In the previous papers the controls were always approximated by piecewise constant functions. In the present situation we have developed new methods, which can be used in the framework of distributed or Neumann controls to consider piecewise linear approximations. This could lead to better error estimates than those ones deduced for piecewise controls.

As far as we know there is another paper dealing with the numerical approximation of a Dirichlet control problem of Navier-Stokes equations by Gunzburger, Hou and Svobodny [23]. Their procedure of proof does not work when the controls are subject to bound constraints, as considered in our problem. To deal with this difficulty we assume that sufficient second order optimality conditions are satisfied. We also see that the gap between the necessary and sufficient optimality conditions of second order is very narrow, the same as in finite dimension.

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Let us mention some recent papers providing some new ideas to derive optimal error estimates. Hinze [26] suggested to discretize the state equation but not the control space. In some cases, including the case of semilinear equations, it is possible to solve the non completely discretized problem in the computer. However we believe there is no advantages of this process for our problem because the discretization of the states forces the discretization of the controls. Another idea, due to Meyer and Rösch [33], works for linear-quadratic control problems in the distributed case, but we do not know if it is possible to adapt it to the general case.

In the case of parabolic problems the theory is far from being complete, but some research has been carried out; see Knowles [27], Lasiecka [28], [29], McKnight and Bosarge [32], Tiba and Tröltzsch [36] and Tröltzsch [38], [39], [40], [41].

In the context of control problems of ordinary differential equations a great work has been done by Hager [24], [25] and Dontchev and Hager [16], [17]; see also the work by Malanowski et al. [31]. The reader is also referred to the detailed bibliography in [17].

The plan of the paper is as follows. In §2 we set the optimal control problem and we establish the results we need for the state equation. In §3 we write the first and second order optimality conditions. The first order conditions allow to deduce some regularity results of the optimal control, which are necessary to derive the error estimates of the discretization. The second order conditions are also essential to prove the error estimates. The discrete optimal control problem is formulated in §4 and the first order optimality conditions are given. To write these conditions we have defined a discrete normal derivative for piecewise linear functions which are solutions of some discrete equation. Sections §6 and §7 are devoted to the analysis of the convergence of the solutions of the discrete optimal control problems and to the proof of error estimates. The main result is Theorem 7.1, where we establish $\|\bar u - \bar u_h\|_{L^2(\Gamma)} = O(h^{1-1/p})$.

The numerical tests we have performed confirm our theoretical estimates. For a detailed report we refer to [12]. A simple example is reported in §8.

2. The Control Problem. Throughout this paper, Ω denotes an open convex bounded polygonal set of $\mathbb{R}^2$ and Γ its boundary. In this domain we formulate the following control problem

$$
\text{(P)}\left\{\begin{array}{l}
\inf \quad J(u) = \displaystyle\int_\Omega L(x, y_u(x))\,dx \; + \frac{N}{2}\int_\Gamma u^2(x)\,dx \\
\text{subject to } (y_u, u) \in L^\infty(\Omega) \times L^\infty(\Gamma), \\
u \in U^{ad} = \{u \in L^\infty(\Gamma) \mid \alpha \le u(x) \le \beta \;\; \text{a.e. } x \in \Gamma\}, \\
(y_u, u) \;\; \text{satisfying the state equation (2.1)},
\end{array}\right.
$$

$$
-\Delta y_u(x) = f(x, y_u(x)) \;\; \text{in } \Omega, \qquad y_u(x) = u(x) \;\; \text{on } \Gamma, \tag{2.1}
$$

where $-\infty < \alpha < \beta < +\infty$ and $N > 0$. Here u is the control while y_u is the associated state. The following hypotheses are assumed about the functions involved in the control problem (P).

(A1) The function $L : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is measurable with respect to the first component, of class C^2 with respect to the second one, $L(\cdot, 0) \in L^1(\Omega)$ and for all $M > 0$ there exist a function $\psi_{L,M} \in L^{\bar p}(\Omega)$ ($\bar p > 2$) and a constant $C_{L,M} > 0$ such that

$$\left|\frac{\partial L}{\partial y}(x,y)\right| \le \psi_{L,M}(x), \quad \left|\frac{\partial^2 L}{\partial y^2}(x,y)\right| \le C_{L,M},$$
$$\left|\frac{\partial^2 L}{\partial y^2}(x,y_2) - \frac{\partial^2 L}{\partial y^2}(x,y_1)\right| \le C_{L,M}|y_2 - y_1|,$$

for a.e. $x \in \Omega$ and $|y|, |y_i| \le M$, $i = 1, 2$.

(A2) The function $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is measurable with respect to the first variable and of class C^2 with respect to the second one,

$$f(\cdot,0) \in L^{\bar p}(\Omega) \ \ (\bar p > 2), \quad \frac{\partial f}{\partial y}(x,y) \le 0 \ \text{ a.e. } x \in \Omega \text{ and } y \in \mathbb{R}.$$

For all $M > 0$ there exists a constant $C_{f,M} > 0$ such that

$$\left|\frac{\partial f}{\partial y}(x,y)\right| + \left|\frac{\partial^2 f}{\partial y^2}(x,y)\right| \le C_{f,M} \text{ a.e. } x \in \Omega \text{ and } |y| \le M,$$
$$\left|\frac{\partial^2 f}{\partial y^2}(x,y_2) - \frac{\partial^2 f}{\partial y^2}(x,y_1)\right| < C_{f,M}|y_2 - y_1| \ \text{ a.e. } x \in \Omega \text{ and } |y_1|, |y_2| \le M.$$

Let us finish this section by proving that problem (P) is well defined. We will say that an element $y_u \in L^\infty(\Omega)$ is a solution of (2.1) if

$$\int_\Omega -\Delta w\, y\, dx = \int_\Omega f(x,y(x))w(x)dx - \int_\Gamma u(x)\partial_\nu w(x)dx \quad \forall w \in H^2(\Omega) \cap H_0^1(\Omega), \tag{2.2}$$

where ∂_ν denotes the normal derivative on the boundary Γ. This is the classical definition in the transposition sense. To study equation (2.1), we state an estimate for the linear equation

$$-\Delta z(x) = b(x)z(x) \ \text{ in } \Omega, \qquad z(x) = u(x) \ \text{ on } \Gamma, \tag{2.3}$$

where b is a nonpositive function belonging to $L^\infty(\Omega)$.

LEMMA 2.1. *For every $u \in L^\infty(\Gamma)$ the linear equation* (2.3) *has a unique solution $z \in L^\infty(\Omega)$ (defined in the transposition sense), and it satisfies*

$$\|z\|_{L^2(\Omega)} \le C\|u\|_{H^{-1/2}(\Gamma)}, \ \|z\|_{H^{1/2}(\Omega)} \le C\|u\|_{L^2(\Gamma)} \text{ and } \|z\|_{L^\infty(\Omega)} \le \|u\|_{L^\infty(\Gamma)}. \tag{2.4}$$

The proof is standard, the first inequality is obtained by using the transposition method, see J.L. Lions and E. Magenes [30]; the second inequality is deduced by interpolation and the last one is obtained by applying the maximum principle.

THEOREM 2.2. *For every $u \in L^\infty(\Gamma)$ the state equation* (2.1) *has a unique solution $y_u \in L^\infty(\Omega) \cap H^{1/2}(\Omega)$. Moreover the following Lipschitz properties hold*

$$\begin{aligned} &\|y_u - y_v\|_{L^\infty(\Omega)} \le \|u - v\|_{L^\infty(\Gamma)} \\ &\|y_u - y_v\|_{H^{1/2}(\Omega)} \le C\|u - v\|_{L^2(\Gamma)} \quad \forall u, v \in L^\infty(\Gamma). \end{aligned} \tag{2.5}$$

Finally if $u_n \rightharpoonup u$ weakly* in $L^\infty(\Gamma)$, then $y_{u_n} \to y_u$ strongly in $L^r(\Omega)$ for all $r < +\infty$.

Proof. Let us introduce the following problems

$$-\Delta z = 0 \ \text{ in } \Omega, \qquad z = u \ \text{ on } \Gamma, \tag{2.6}$$

and

$$-\Delta \zeta = g(x,\zeta) \ \text{ in } \Omega, \qquad \zeta = 0 \ \text{ on } \Gamma, \tag{2.7}$$

where $g : \Omega \times \mathbb{R} \mapsto \mathbb{R}$ is given by $g(x,t) = f(x, z(x) + t)$, z being the solution of (2.6). Lemma 2.1 implies that (2.6) has a unique solution in $L^\infty(\Omega) \cap H^{1/2}(\Omega)$. It is obvious that Assumption **(A2)** is fulfilled by g and (2.7) is a classical well set problem having a unique solution in $H_0^1(\Omega) \cap L^\infty(\Omega)$. Moreover, since Ω is convex, we know that $\zeta \in H^2(\Omega)$; see Grisvard [22]. Finally the solution y_u of (2.1) can be written as $y_u = z + \zeta$. Estimates (2.5) follow from Lemma 2.1; see Arada and Raymond [2] for a detailed proof in the parabolic case. The continuous dependence in $L^r(\Omega)$ follows in a standard way by using (2.5) and the compactness of the inclusion $H^{1/2}(\Omega) \subset L^2(\Omega)$ along with the fact that $\{y_{u_n}\}$ is bounded in $L^\infty(\Omega)$ as deduced from the first inequality of (2.5). □

Now the following theorem can be proved by standard arguments.

THEOREM 2.3. *Problem (P) has at least one solution.*

3. Optimality Conditions. Before writing the optimality conditions for (P) let us state the differentiability properties of J.

THEOREM 3.1. *The mapping $G : L^\infty(\Gamma) \longrightarrow L^\infty(\Omega) \cap H^{1/2}(\Omega)$ defined by $G(u) = y_u$ is of class C^2. Moreover, for all $u, v \in L^\infty(\Gamma)$, $z_v = G'(u)v$ is the solution of*

$$-\Delta z_v = \frac{\partial f}{\partial y}(x, y_u) z_v \ \text{ in } \ \Omega, \qquad z_v = v \ \text{on} \ \ \Gamma, \tag{3.1}$$

and for every $v_1, v_2 \in L^\infty(\Omega)$, $z_{v_1 v_2} = G''(u)v_1 v_2$ is the solution of

$$\left\{ \begin{array}{rclll} -\Delta z_{v_1 v_2} & = & \dfrac{\partial f}{\partial y}(x, y_u) z_{v_1 v_2} + \dfrac{\partial^2 f}{\partial y^2}(x, y_u) z_{v_1} z_{v_2} & \text{in} & \Omega, \\ z_{v_1 v_2} & = & 0 & \text{on} & \Gamma, \end{array} \right. \tag{3.2}$$

where $z_{v_i} = G'(u)v_i$, $i = 1, 2$.

Proof. Let us define the space

$$V = \{y \in H^{1/2}(\Omega) \cap L^\infty(\Omega) : \Delta y \in L^2(\Omega)\}$$

endowed with the natural graph norm. Now we consider the function $F : L^\infty(\Gamma) \times V \longrightarrow L^\infty(\Gamma) \times L^2(\Omega)$ defined by $F(u, y) = (y|_\Gamma - u, \Delta y + f(x, y))$. It is obvious that F is of class C^2 and that for every pair (u, y) satisfying (2.1) we have $F(u, y) = (0, 0)$. Furthermore

$$\frac{\partial F}{\partial y}(u, y) \cdot z = \left(z|_\Gamma, \Delta z + \frac{\partial f}{\partial y}(x, y) z \right).$$

By using Lemma 2.1 we deduce that $(\partial F/\partial y)(u, y) : V \longrightarrow L^\infty(\Gamma) \times L^2(\Omega)$ is an isomorphism. Then the Implicit Function Theorem allows us to conclude that G is of class C^2 and now the rest of the theorem follows easily. □

Theorem 3.1 along with the chain rule lead to the following result.

THEOREM 3.2. *The functional $J : L^\infty(\Gamma) \to \mathbb{R}$ is of class C^2. Moreover, for every $u, v, v_1, v_2 \in L^\infty(\Gamma)$*

$$J'(u)v = \int_\Gamma (Nu - \partial_\nu \phi_u)\, v\, dx \tag{3.3}$$

and

$$J''(u)v_1v_2 = \int_\Omega \left[\frac{\partial^2 L}{\partial y^2}(x, y_u) z_{v_1} z_{v_2} + \phi_u \frac{\partial^2 f}{\partial y^2}(x, y_u) z_{v_1} z_{v_2}\right] dx + \int_\Gamma N v_1 v_2\, dx, \tag{3.4}$$

where $z_{v_i} = G'(u)v_i$, $i = 1, 2$, $y_u = G(u)$, and the adjoint state $\phi_u \in H^2(\Omega)$ is the unique solution of the problem

$$-\Delta\phi = \frac{\partial f}{\partial y}(x, y_u)\phi + \frac{\partial L}{\partial y}(x, y_u) \quad \text{in } \Omega, \qquad \phi = 0 \quad \text{on } \Gamma. \tag{3.5}$$

The first order optimality conditions for Problem (P) follow readily from Theorem 3.2.

THEOREM 3.3. *Assume that $\bar{u}$ is a local solution of Problem (P) and let $\bar{y}$ be the corresponding state. Then there exists $\bar{\phi} \in H^2(\Omega)$ such that*

$$-\Delta\bar{\phi} = \frac{\partial f}{\partial y}(x, \bar{y})\bar{\phi} + \frac{\partial L}{\partial y}(x, \bar{y}) \quad \text{in} \quad \Omega, \qquad \bar{\phi} = 0 \text{ on } \quad \Gamma, \tag{3.6}$$

and

$$\int_\Gamma \left(N\bar{u} - \partial_\nu\bar{\phi}\right)(u - \bar{u})\, dx \geq 0 \quad \forall u \in U^{ad}, \tag{3.7}$$

which is equivalent to

$$\bar{u}(x) = \mathrm{Proj}_{[\alpha,\beta]}\Big(\frac{1}{N}\partial_\nu\bar{\phi}(x)\Big) = \max\Big\{\alpha, \min\Big\{\beta, \frac{1}{N}\partial_\nu\bar{\phi}(x)\Big\}\Big\}. \tag{3.8}$$

THEOREM 3.4. *Assume that $\bar{u}$ is a local solution of Problem (P) and let $\bar{y}$ and $\bar{\phi}$ be the corresponding state and adjoint state. Then there exists $p \in (2, \bar{p}]$ ($\bar{p} > 2$ introduced in assumptions (A1) and (A2)) depending on the measure of the angles of the polygon Ω such that $\bar{y} \in W^{1,p}(\Omega)$, $\bar{\phi} \in W^{2,p}(\Omega)$ and $\bar{u} \in W^{1-1/p,p}(\Gamma) \subset C(\Gamma)$.*

Proof. From assumption (A1) and using elliptic regularity results it follows that $\bar{\phi}$ belongs to $W^{2,p}(\Omega)$ for some $p \in (2, \bar{p}]$ depending on the measure of the angles of Γ; see Grisvard [22, Chapter 4]. To prove that $\bar{u}$ belongs to $W^{1-1/p,p}(\Gamma)$ we recall the norm in this space

$$\|\bar{u}\|_{W^{1-1/p,p}(\Gamma)} = \left\{\int_\Gamma |\bar{u}(x)|^p dx + \int_\Gamma\int_\Gamma \frac{|\bar{u}(x) - \bar{u}(\xi)|^p}{|x - \xi|^p}\, dx\, d\xi\right\}^{1/p},$$

where we have used the fact that $\Omega \subset \mathbb{R}^2$. Now it is enough to take into account that $\partial_\nu\bar{\phi} \in W^{1-1/p,p}(\Gamma)$, the relation (3.8) and

$$\left|\mathrm{Proj}_{[\alpha,\beta]}\Big(\frac{1}{N}\partial_\nu\bar{\phi}(x)\Big) - \mathrm{Proj}_{[\alpha,\beta]}\Big(\frac{1}{N}\partial_\nu\bar{\phi}(\xi)\Big)\right| \leq \frac{1}{N}|\partial_\nu\bar{\phi}(x) - \partial_\nu\bar{\phi}(\xi)|,$$

to deduce that the integrals in the above norm are finite.

Finally, decomposing (2.1) into two problems as in the proof of Theorem 2.3, we get that $\bar{y} = \bar{z} + \bar{\zeta}$, with $\bar{\zeta} \in H^2(\Omega)$ and $\bar{z} \in W^{1,p}(\Omega)$, which completes the proof. □

In order to establish the second order optimality conditions we define the cone of critical directions

$$C_{\bar{u}} = \{v \in L^2(\Gamma) \text{ satisfying (3.9) and } v(x) = 0 \text{ if } |\bar{d}(x)| > 0\},$$

$$v(x) = \begin{cases} \geq 0 & \text{where } \bar{u}(x) = \alpha, \\ \leq 0 & \text{where } \bar{u}(x) = \beta, \end{cases} \quad \text{for a.e. } x \in \Gamma, \tag{3.9}$$

where $\bar{d}$ denotes the derivative $J'(\bar{u})$

$$\bar{d}(x) = N\bar{u}(x) - \partial_\nu \bar{\phi}(x). \tag{3.10}$$

Now we formulate the second order necessary and sufficient optimality conditions.

THEOREM 3.5. *If $\bar{u}$ is a local solution of (P), then $J''(\bar{u})v^2 \geq 0$ holds for all $v \in C_{\bar{u}}$. Conversely, if $\bar{u} \in U^{ad}$ satisfies the first order optimality conditions provided by Theorem 3.3 and the coercivity condition*

$$J''(\bar{u})v^2 > 0 \quad \forall v \in C_{\bar{u}} \setminus \{0\}, \tag{3.11}$$

then there exist $\mu > 0$ and $\varepsilon > 0$ such that $J(u) \geq J(\bar{u}) + \mu\|u - \bar{u}\|^2_{L^2(\Gamma)}$ is satisfied for every $u \in U^{ad}$ obeying $\|u - \bar{u}\|_{L^\infty(\Omega)} \leq \varepsilon$.

The necessary condition provided in the theorem is quite easy to get. The sufficient conditions are proved by Casas and Mateos [9, Theorem 4.3] for distributed control problems with integral state constraints. The proof can be translated in a straightforward way to the case of boundary controls; see also Bonnans and Zidani [4].

REMARK 3.6. *It can be proved (see Casas and Mateos [9, Theorem 4.4]) that the following two conditions are equivalent:*

(1) $J''(\bar{u})v^2 > 0$ for every $v \in C_{\bar{u}} \setminus \{0\}$.

(2) There exist $\delta > 0$ and $\tau > 0$ such that $J''(\bar{u})v^2 \geq \delta\|v\|^2_{L^2(\Gamma)}$ for every $v \in C^\tau_{\bar{u}}$, where

$$C^\tau_{\bar{u}} = \{v \in L^2(\Gamma) \textit{ satisfying } (3.9) \textit{ and } v(x) = 0 \textit{ if } \ |\bar{d}(x)| > \tau\}.$$

It is clear that $C^\tau_{\bar{u}}$ contains strictly $C_{\bar{u}}$, so the condition (2) seems to be stronger than (1), but in fact they are equivalent. For the proof of this equivalence it is used the fact that u appears linearly in the state equation and quadratically in the cost functional.

4. Numerical Approximation of (P). Let us consider a family of triangulations $\{\mathcal{T}_h\}_{h>0}$ of $\bar{\Omega}$: $\bar{\Omega} = \cup_{T\in\mathcal{T}_h} T$. With each element $T \in \mathcal{T}_h$, we associate two parameters $\rho(T)$ and $\sigma(T)$, where $\rho(T)$ denotes the diameter of the set T and $\sigma(T)$ is the diameter of the largest ball contained in T. Let us define the size of the mesh by $h = \max_{T\in\mathcal{T}_h} \rho(T)$. For fixed $h > 0$, we denote by $\{T_j\}_{j=1}^{N(h)}$ the family of triangles of $\mathcal{T}_h$ with a side on the boundary of Γ. If the vertices of $T_j \cap \Gamma$ are x^j_Γ and x^{j+1}_Γ then $[x^j_\Gamma, x^{j+1}_\Gamma] := T_j \cap \Gamma$, $1 \leq j \leq N(h)$, with $x^{N(h)+1}_\Gamma = x^1_\Gamma$. We will also follow the notation $x_\Gamma^0 = x_\Gamma^{N(h)}$. We assume that every vertex of the polygon Ω is one of these boundary points x_Γ^j of the triangulation and the numbering of the nodes $\{x_\Gamma^j\}_{j=1}^{N(h)}$ is made counterclockwise. The length of the interval $[x_\Gamma^j, x_\Gamma^{j+1}]$ is denoted by $h_j = |x_\Gamma^{j+1} - x_\Gamma^j|$. The following hypotheses on the triangulation are also assumed.

$(H1)$ - There exists a constant $\rho > 0$ such that $h/\rho(T) \leq \rho$ for all $T \in \mathcal{T}_h$ and $h > 0$.

$(H2)$ - All the angles of all triangles are less than or equal to $\pi/2$.

The first assumption is not a restriction in practice and it is the usual one. The second assumption is going to allow us to use the discrete maximum principle and it is actually not too restrictive.

Given two points ξ_1 and ξ_2 of Γ, we denote by $[\xi_1, \xi_2]$ the part of Γ obtained by running the boundary from ξ_1 to ξ_2 counterclockwise. With this convention we have $(\xi_2, \xi_1) = \Gamma \setminus [\xi_1, \xi_2]$. According to this notation

$$\int_{\xi_1}^{\xi_2} u(x)\,dx \quad \text{and} \quad \int_{\xi_2}^{\xi_1} u(x)\,dx$$

denote the integrals of a function $u \in L^1(\Gamma)$ on the parts of Γ defined by $[\xi_1, \xi_2]$ and $[\xi_2, \xi_1]$ respectively. In particular we have

$$\int_{\xi_1}^{\xi_2} u(x)\,dx = \int_\Gamma u(x)\,dx - \int_{\xi_2}^{\xi_1} u(x)\,dx.$$

Associated with this triangulation we set

$$\begin{aligned} U_h &= \Big\{u_h \in C(\Gamma) : u_h|_{[x_\Gamma^j, x_\Gamma^{j+1}]} \in \mathcal{P}_1, \text{ for } 1 \leq j \leq N(h)\Big\}, \\ Y_h &= \Big\{y_h \in C(\bar\Omega) : y_h|_T \in \mathcal{P}_1, \text{ for all } T \in \mathcal{T}_h\Big\}, \\ Y_{h0} &= \Big\{y_h \in Y_h : y_h|_\Gamma = 0\Big\}, \end{aligned}$$

where $\mathcal{P}_1$ is the space of polynomials of degree less than or equal to 1. The space U_h is formed by the restrictions to Γ of the functions of Y_h.

Let us consider the projection operator $\Pi_h : L^2(\Gamma) \longmapsto U_h$

$$(\Pi_h v, u_h)_{L^2(\Gamma)} = (v, u_h)_{L^2(\Gamma)} \quad \forall u_h \in U_h.$$

The following approximation property of Π_h is well known (see for instance [20, Lemma 3.1])

$$\|y - \Pi_h y\|_{L^2(\Gamma)} + h^{1/2}\|y - \Pi_h y\|_{H^{1/2}(\Gamma)} \leq C h^{s-1/2}\|y\|_{H^s(\Omega)} \quad \forall y \in H^s(\Omega)$$

and for every $1 \leq s \leq 2$. Observing that, for $1/2 < s \leq 3/2$,

$$u \longmapsto \inf_{y|_\Gamma = u} \|y\|_{H^s(\Omega)}$$

is a norm equivalent to the usual one of $H^{s-1/2}(\Gamma)$, we deduce from the above inequality

$$\|u - \Pi_h u\|_{L^2(\Gamma)} + h^{1/2}\|u - \Pi_h u\|_{H^{1/2}(\Gamma)} \leq C h^s \|u\|_{H^s(\Gamma)} \quad \forall u \in H^s(\Gamma) \tag{4.1}$$

and for every $1/2 < s \leq 3/2$.

Let $a : Y_h \times Y_h \longmapsto \mathbb{R}$ be the bilinear form given by

$$a(y_h, z_h) = \int_\Omega \nabla y_h(x) \nabla z_h(x)\, dx.$$

For all $u \in L^\infty(\Gamma)$, we consider the problem

$$\left\{\begin{array}{l} \text{Find } y_h(u) \in Y_h \text{ such that } y_h = \Pi_h u \text{ on } \Gamma, \text{ and} \\ a(y_h(u), w_h) = \displaystyle\int_\Omega f(x, y_h(u)) w_h\, dx \quad \forall w_h \in Y_{h0}, \end{array}\right. \tag{4.2}$$

PROPOSITION 4.1. *For every $u \in L^\infty(\Gamma)$, the equation* (4.2) *admits a unique solution $y_h(u)$.*

Proof. Let z_h be the unique element in Y_h satisfying $z_h = \Pi_h u$ on Γ, and $z_h(x_i) = 0$ for all vertex x_i of the triangulation $\mathcal{T}_h$ not belonging to Γ. The equation

$$\zeta_h \in Y_{h0}, \quad a(\zeta_h, w_h) = -a(z_h, w_h) + \int_\Omega f(x, z_h + \zeta_h) w_h\, dx \quad \forall w_h \in Y_{h0},$$

admits a unique solution (it is a consequence of the Minty-Browder Theorem [7]). The function $z_h + \zeta_h$ is clearly a solution of equation (4.2). The uniqueness of solution to equation (4.2) also follows from the Minty-Browder Theorem. $\square$

Due to Proposition 4.1, we can define a functional J_h in $L^\infty(\Gamma)$ by:

$$J_h(u) = \int_\Omega L(x, y_h(u)(x))\, dx + \frac{N}{2} \int_\Gamma u^2(x)\, dx.$$

The finite dimensional control problem approximating (P) is

$$(\mathrm{P}_h) \left\{\begin{array}{l} \min J_h(u_h) = \displaystyle\int_\Omega L(x, y_h(u_h)(x))\, dx + \frac{N}{2} \int_\Gamma u_h^2(x)\, dx, \\ \text{subject to } u_h \in U_h^{ad}, \end{array}\right.$$

where

$$U_h^{ad} = U_h \cap U^{ad} = \{u_h \in U_h \mid \alpha \leq u_h(x) \leq \beta \text{ for all } x \in \Gamma\}.$$

The existence of a solution of (P_h) follows from the continuity of J_h in U_h and the fact that U_h^{ad} is a nonempty compact subset of U_h. Our next goal is to write the conditions for optimality satisfied by any local solution $\bar{u}_h$. First we have to obtain an expression for the derivative of $J_h : L^\infty(\Gamma) \to \mathbb{R}$ analogous to the one of J given by the formula (3.3). Given $u \in L^\infty(\Gamma)$ we consider the adjoint state $\phi_h(u) \in Y_{h0}$ solution of the equation

$$a(w_h, \phi_h(u)) = \int_\Omega \left[\frac{\partial f}{\partial y}(x, y_h(u))\phi_h(u) + \frac{\partial L}{\partial y}(x, y_h(u))\right] w_h\, dx \quad \forall w_h \in Y_{h0}. \tag{4.3}$$

To obtain the analogous expression to (3.3) we have to define a discrete normal derivative $\partial_\nu^h \phi_h(u)$.

PROPOSITION 4.2. *Let u belong to $L^\infty(\Gamma)$ and let $\phi_h(u)$ be the solution of equation 4.3. There exists a unique element $\partial_\nu^h \phi_h(u) \in U_h$ verifying*

$$\begin{aligned} &(\partial_\nu^h \phi_h(u), w_h)_{L^2(\Gamma)} = a(w_h, \phi_h(u)) \\ &- \int_\Omega \left[\frac{\partial f}{\partial y}(x, y_h(u))\phi_h(u) + \frac{\partial L}{\partial y}(x, y_h(u))\right] w_h\, dx \quad \forall w_h \in Y_h. \end{aligned} \tag{4.4}$$

Proof. The trace mapping is a surjective mapping from Y_h on U_h, therefore the linear form

$$L(w_h) = a(w_h, \phi_h(u)) - \int_\Omega \left[\frac{\partial f}{\partial y}(x, y_h(u))\phi_h(u) + \frac{\partial L}{\partial y}(x, y_h(u))\right] w_h\, dx$$

is well defined on U_h, and it is continuous on U_h. Let us remark that if in (4.4) the trace of w_h on Γ is zero, then (4.3) leads to

$$L(w_h) = 0.$$

Hence L can be identified with a unique element of U_h, which proves the above proposition. □

Now the function G introduced in Theorem 3.1 is approximated by the function $G_h : L^\infty(\Gamma) \longmapsto Y_h$ defined by $G_h(u) = y_h(u)$. We can easily verify that G_h is of class C^2, and that for $u, v \in L^\infty(\Gamma)$, the derivative $z_h = G'_h(u)v \in Y_h$ is the unique solution of

$$\begin{cases} a(z_h, w_h) = \displaystyle\int_\Omega \frac{\partial f}{\partial y}(x, y_h(u)) z_h w_h\, dx \quad \forall w_h \in Y_{h0}, \\ z_h = \Pi_h v \quad \text{on } \Gamma. \end{cases} \tag{4.5}$$

From here we deduce

$$J'_h(u)v = \int_\Omega \frac{\partial L}{\partial y}(x, y_h(u)) z_h\, dx + N \int_\Gamma uv\, dx.$$

Now (4.4) and the definition of Π_h lead to

$$J'_h(u)v = N\int_\Gamma uv\, dx - \int_\Gamma \partial_\nu^h \phi_h(u) \Pi_h v\, dx = \int_\Gamma (Nu - \partial_\nu^h \phi_h(u)) v\, dx, \tag{4.6}$$

for all $u, v \in L^\infty(\Gamma)$.

Finally we can write the first order optimality conditions.

THEOREM 4.3. *Let us assume that $\bar u_h$ is a local solution of (P_h) and $\bar y_h$ the corresponding state, then there exists $\bar\phi_h \in Y_{h0}$ such that*

$$a(w_h, \bar\phi_h) = \int_\Omega \left[\frac{\partial f}{\partial y}(x, \bar y_h)\bar\phi_h + \frac{\partial L}{\partial y}(x, \bar y_h)\right] w_h\, dx \quad \forall w_h \in Y_{h0}, \tag{4.7}$$

and

$$\int_\Gamma (N\bar u_h - \partial_\nu^h \bar\phi_h)(u_h - \bar u_h)\, dx \ge 0 \quad \forall u_h \in U_h^{ad}. \tag{4.8}$$

This theorem follows readily from (4.6).

REMARK 4.4. *The reader could think that a projection property for $\bar u_h$ similar to that one obtained for $\bar u$ in* (3.8) *can be deduced from* (4.8)*. Unfortunately this property does not hold because $u_h(x)$ cannot be taken arbitrarily in $[\alpha, \beta]$. Functions $u_h \in U_h$ are determined by their values at the nodes $\{x_\Gamma^j\}_{j=1}^{N(h)}$. If we consider the basis of U_h $\{e_j\}_{j=1}^{N(h)}$ defined by $e_j(x_\Gamma^i) = \delta_{ij}$, then we have*

$$u_h = \sum_{j=1}^{N(h)} u_{h,j} e_j, \qquad \textit{with} \quad u_{h,j} = u_h(x_\Gamma^j), \quad 1 \le j \le N(h).$$

Now (4.8) *can be written*

$$\sum_{j=1}^{N(h)} \int_\Gamma (N\bar{u}_h - \partial_\nu^h \bar{\phi}_h) e_j \, dx (u_{h,j} - \bar{u}_{h,j}) \geq 0 \quad \forall \{u_{h,j}\}_{j=1}^{N(h)} \subset [\alpha, \beta], \tag{4.9}$$

where $\bar{u}_{h,j} = \bar{u}_h(x_\Gamma^j)$. *Then* (4.9) *leads to*

$$\bar{u}_{h,j} = \begin{cases} \alpha & \text{if } \int_\Gamma (N\bar{u}_h - \partial_\nu^h \bar{\phi}_h) e_j \, dx > 0 \\ \beta & \text{if } \int_\Gamma (N\bar{u}_h - \partial_\nu^h \bar{\phi}_h) e_j \, dx < 0. \end{cases} \tag{4.10}$$

In order to characterize $\bar{u}_h$ *as the projection of* $\partial_\nu^h \bar{\phi}_h / N$, *let us introduce the operator* $\mathrm{Proj}_h : L^2(\Gamma) \longmapsto U_h^{ad}$ *as follows. Given* $u \in L^2(\Gamma)$, $\mathrm{Proj}_h u$ *denotes the unique solution of the problem*

$$\inf_{v_h \in U_h^{ad}} \|u - v_h\|_{L^2(\Gamma)},$$

which is characterized by the relation

$$\int_\Gamma (u(x) - \mathrm{Proj}_h u(x))(v_h(x) - \mathrm{Proj}_h u(x)) \, dx \leq 0 \quad \forall v_h \in U_h^{ad}. \tag{4.11}$$

Then (4.8) *is equivalent to*

$$\bar{u}_h = \mathrm{Proj}_h \Big(\frac{1}{N} \partial_\nu^h \bar{\phi}_h \Big). \tag{4.12}$$

Let us recall the result in [13, Lemma 3.3], where a chracteriztation of $\mathrm{Proj}_h(u_h)$ *is stated. Given* $u_h \in U_h$ *and* $\bar{u}_h = \mathrm{Proj}_h(u_h)$, *then* $\bar{u}_h$ *is characterized by the inequalities*

$$\begin{aligned} &h_{j-1}[(u_{h,j-1} - \bar{u}_{h,j-1}) + 2(u_{h,j} - \bar{u}_{h,j})](t - \bar{u}_{h,j}) \\ &+ h_j[2(u_{h,j} - \bar{u}_{h,j}) + (u_{h,j+1} - \bar{u}_{h,j+1})](t - \bar{u}_{h,j}) \leq 0 \end{aligned}$$

for all $t \in [\alpha, \beta]$ *and* $1 \leq j \leq N(h)$.

5. Numerical Analysis of the State and Adjoint Equations. Throughout the following the operator $I_h \in \mathcal{L}(W^{1,p}(\Omega), Y_h)$ denotes the classical interpolation operator [6]. We also need the interpolation operator $I_h^\Gamma \in \mathcal{L}(W^{1-1/p,p}(\Gamma), U_h)$. Since we have

$$I_h^\Gamma(y|_\Gamma) = (I_h y)|_\Gamma \quad \text{for all } y \in W^{1,p}(\Omega),$$

we shall use the same notation for both interpolation operators. The reader can observe that this abuse of notation does not lead to any confusion.

The goal of this section is to obtain the error estimates of the approximations $y_h(u)$ given by (4.2) to the solution y_u of (2.1). In order to carry out this analysis we decompose (2.1) in two problems as in the proof of Theorem 2.3. We take $z \in H^{1/2}(\Omega) \cap L^\infty(\Omega)$ and $\zeta \in H_0^1(\Omega) \cap H^2(\Omega)$ as the solutions of (2.6) and (2.7) respectively. Then we have $y_u = z + \zeta$.

Let us consider now the discretizations of (2.6) and (2.7).

$$\begin{cases} \text{Find } z_h \in Y_h \text{ such that } z_h = \Pi_h u \text{ on } \Gamma \text{ and} \\ a(z_h, w_h) = 0 \quad \forall w_h \in Y_{h0}, \end{cases} \tag{5.1}$$

$$\left\{\begin{array}{l}\text{Find } \zeta_h \in Y_{h0} \text{ such that}\\ a(\zeta_h, w_h) = \displaystyle\int_\Omega g_h(x, \zeta_h(x)) w_h(x)\,dx \quad \forall w_h \in Y_{h0},\end{array}\right. \tag{5.2}$$

where $g_h(x,t) = f(x, z_h(x) + t)$. Now the solution $y_h(u)$ of (4.2) is decomposed as follows $y_h(u) = z_h + \zeta_h$. The following lemma provides the estimates for $z - z_h$.

LEMMA 5.1. *Let $u \in U^{ad}$ and let z and z_h be the solutions of* (2.6) *and* (5.1) *respectively, then*

$$\|z_h\|_{L^\infty(\Omega)} \le \|\Pi_h u\|_{L^\infty(\Gamma)} \le C(\alpha,\beta) \text{ and } \|z_h\|_{W^{1,r}(\Omega)} \le C\|\Pi_h u\|_{W^{1-1/r,r}(\Gamma)} \tag{5.3}$$

$$\|z_h\|_{L^2(\Omega)} \le C\|\Pi_h u\|_{H^{-1/2}(\Gamma)}, \tag{5.4}$$

where $1 < r \le p$ is arbitrary, p being given in Theorem 3.4. If in addition $u \in H^s(\Gamma) \cap U^{ad}$, with $0 \le s \le 1$, then we also have

$$\|z - z_h\|_{L^2(\Omega)} \le Ch^{s+1/2}\|u\|_{H^s(\Gamma)} \quad \forall h > 0 \text{ and } 0 \le s \le 1. \tag{5.5}$$

Proof. The first inequality of (5.3) is proved in Ciarlet and Raviart [14], we only have to notice that

$$\|\Pi_h u\|_{L^\infty(\Gamma)} \le C\|u\|_{L^\infty(\Gamma)} \le C(\alpha,\beta), \tag{5.6}$$

where C is independent of h and $u \in U^{ad}$; see Douglas et al. [18].

Inequality (5.5) can be found in French and King [20, Lemma 3.3] just taking into account that

$$\|z\|_{H^{s+1/2}(\Omega)} \le C\|u\|_{H^s(\Gamma)}.$$

The second inequality of (5.3) is established in Bramble et al. [5, Lemma 3.2] for $r = 2$. Let us prove it for all r in the range $(1, p]$. Let us consider $z^h \in H^1(\Omega)$ solution of the problem

$$-\Delta z^h = 0 \quad \text{in } \Omega, \qquad z^h = \Pi_h u \quad \text{on } \Gamma.$$

This is a standard Dirichlet problem with the property (see M. Dauge [15])

$$\|z^h\|_{W^{1,r}(\Omega)} \le C\|\Pi_h u\|_{W^{1-1/r,r}(\Gamma)}.$$

Let us denote by $\hat{I}_h : W^{1,r}(\Omega) \longmapsto Y_h$ the generalized interpolation operator due to Scott and Zhang [35] that preserves piecewise-affine boundary conditions. More precisely, it has the properties: $\hat{I}_h(y_h) = y_h$ for all $y_h \in Y_h$ and $\hat{I}_h(W_0^{1,r}(\Omega)) \subset Y_{h0}$. This properties imply that $\hat{I}_h(z^h) = \Pi_h u$ on Γ. Thus we have

$$-\Delta(z^h - \hat{I}_h(z^h)) = \Delta \hat{I}_h(z^h) \quad \text{in } \Omega, \qquad z^h - \hat{I}_h(z^h) = 0 \quad \text{on } \Gamma$$

and $z_h - \hat{I}_h(z^h) \in Y_{h0}$ satisfies

$$a(z_h - \hat{I}_h(z^h), w_h) = -a(\hat{I}_h(z^h), w_h) \quad \forall w_h \in Y_{h0}.$$

Then by using the L^p estimates (see, for instance, Brenner and Scott [6, Theorem 7.5.3]) we get

$$\begin{aligned}&\|z_h - \hat{I}_h(z^h)\|_{W^{1,r}(\Omega)} \le C\|z^h - \hat{I}_h(z^h)\|_{W^{1,r}(\Omega)}\\ &\le C(\|z^h\|_{W^{1,r}(\Omega)} + \|\hat{I}_h(z^h)\|_{W^{1,r}(\Omega)}) \le C\|z^h\|_{W^{1,r}(\Omega)} \le C\|\Pi_h u\|_{W^{1-1/r,r}(\Gamma)}.\end{aligned}$$

Then we conclude the proof as follows

$$\|z_h\|_{W^{1,r}(\Omega)} \leq \|\hat{I}_h(z^h)\|_{W^{1,r}(\Omega)} + \|z_h - \hat{I}_h(z^h)\|_{W^{1,r}(\Omega)} \leq C\|\Pi_h u\|_{W^{1-1/r,r}(\Gamma)}.$$

Finally let us prove (5.4). Using (5.5) with $s = 0$, (2.4), and an inverse inequality we get

$$\begin{aligned}\|z_h\|_{L^2(\Omega)} &\leq \|z^h - z_h\|_{L^2(\Omega)} + \|z^h\|_{L^2(\Omega)}\\ &\leq C(h^{1/2}\|\Pi_h u\|_{L^2(\Gamma)} + \|\Pi_h u\|_{H^{-1/2}(\Gamma)}) \leq C\|\Pi_h u\|_{H^{-1/2}(\Gamma)}.\end{aligned}$$

□

REMARK 5.2. *The inverse estimate used in the proof*

$$\|u\|_{L^2(\Gamma)} \leq Ch^{-1/2}\|u\|_{H^{-1/2}(\Gamma)} \quad \textit{for all } u \in U_h,$$

can be derived from the well known inverse estimate [3]

$$\|u\|_{H^{1/2}(\Gamma)} \leq Ch^{-1/2}\|u\|_{L^2(\Gamma)} \quad \textit{for all } u \in U_h,$$

and from the equality

$$\|u\|^2_{L^2(\Gamma)} = \|u\|_{H^{1/2}(\Gamma)}\|u\|_{H^{-1/2}(\Gamma)}.$$

Now we obtain the estimates for $\zeta - \zeta_h$.

LEMMA 5.3. *There exist constants $C_i = C_i(\alpha, \beta) > 0$ $(i = 1, 2)$ such that, for all $u \in U^{ad} \in H^s(\Gamma)$, the following estimates hold*

$$\|\zeta_h\|_{L^\infty(\Omega)} \leq C_1 \quad \forall h > 0 \text{ and } s = 0, \tag{5.7}$$

$$\|\zeta - \zeta_h\|_{L^2(\Omega)} \leq C_2 h^{s+1/2}(1 + \|u\|_{H^s(\Gamma)}) \quad \forall h > 0 \text{ and } 0 \leq s \leq 1, \tag{5.8}$$

where ζ and ζ_h are the solutions of (2.7) *and* (5.2) *respectively.*

Proof. We are going to introduce an intermediate function $\zeta^h \in H^2(\Omega)$ satisfying

$$-\Delta\zeta^h = g_h(x, \zeta^h(x)) \quad \text{in } \Omega, \qquad \zeta^h = 0 \quad \text{on } \Gamma. \tag{5.9}$$

By using classical methods, see for instance Stampacchia [34], we get the boundedness of ζ and ζ^h in $L^\infty(\Omega)$ for some constants depending on $\|u\|_{L^\infty(\Gamma)}$ and $\|\Pi_h u\|_{L^\infty(\Gamma)}$, which are uniformly estimated by a constant only depending on α and β; see (5.6). On the other hand from (2.7), (5.9) and the assumption **(A2)** we deduce

$$\begin{aligned}&C_1\|\zeta - \zeta^h\|^2_{H^1(\Omega)} \leq a(\zeta - \zeta^h, \zeta - \zeta^h)\\ &= \int_\Omega [g(x, \zeta(x)) - g_h(x, \zeta^h(x))](\zeta(x) - \zeta^h(x))\,dx\\ &= \int_\Omega [g(x, \zeta(x)) - g(x, \zeta^h(x))](\zeta(x) - \zeta^h(x))\,dx\\ &\qquad + \int_\Omega [g(x, \zeta^h(x)) - g_h(x, \zeta^h(x))](\zeta(x) - \zeta^h(x))\,dx\\ &\leq \int_\Omega [g(x, \zeta^h(x)) - g_h(x, \zeta^h(x))](\zeta(x) - \zeta^h(x))\,dx \leq C_2\|z - z_h\|_{L^2(\Omega)}\|\zeta - \zeta^h\|_{L^2(\Omega)}\\ &\leq C_3\|z - z_h\|^2_{L^2(\Omega)} + \frac{C_1}{2}\|\zeta - \zeta^h\|^2_{L^2(\Omega)}.\end{aligned}$$

This inequality along with (5.5) implies

$$\|\zeta - \zeta^h\|_{H^1(\Omega)} \leq Ch^{s+1/2}\|u\|_{H^s(\Gamma)}. \tag{5.10}$$

Thanks to the convexity of Ω, ζ^h belongs to $H^2(\Omega)$ (see Grisvard [22]) and

$$\|\zeta^h\|_{H^2(\Omega)} \leq C\|g_h(x,\zeta^h)\|_{L^2(\Omega)} = C(\|u\|_{L^\infty(\Gamma)}, \|\Pi_h u\|_{L^\infty(\Gamma)}).$$

Now using the results of Casas and Mateos [10, Lemma 4 and Theorem 1] we deduce that

$$\|\zeta^h - \zeta_h\|_{L^2(\Omega)} \leq Ch^2, \tag{5.11}$$

$$\|\zeta^h - \zeta_h\|_{L^\infty(\Omega)} \leq Ch. \tag{5.12}$$

Finally (5.8) follows from (5.10) and (5.11), and (5.7) is a consequence of the boundedness of $\{\zeta^h\}_{h>0}$ and (5.12). ◻

THEOREM 5.4. *There exist constants $C_i = C_i(\alpha,\beta) > 0$ $(i = 1,2)$ such that for every $u \in U^{ad} \cap H^s(\Gamma)$, with $0 \leq s \leq 1$, the following inequalities hold*

$$\|y_h(u)\|_{L^\infty(\Omega)} \leq C_1 \quad \forall h > 0 \text{ and } s = 0, \tag{5.13}$$

$$\|y_u - y_h(u)\|_{L^2(\Omega)} \leq C_2 h^{s+1/2}(1 + \|u\|_{H^s(\Gamma)}) \quad \forall h > 0 \text{ and } 0 \leq s \leq 1. \tag{5.14}$$

Furthermore if $u_h \rightharpoonup u$ weakly in $L^2(\Gamma)$, $\{u_h\}_{h>0} \subset U^{ad}$, then $y_h(u_h) \to y_u$ strongly in $L^r(\Omega)$ for every $r < +\infty$.

Proof. Remembering that $y_u = z + \zeta$ and $y_h(u) = z_h + \zeta_h$, (5.3), (5.5), (5.7) and (5.8) lead readily to the inequalities (5.13) and (5.14). To prove the last part of theorem it is enough to use Theorem 2.2 and (5.14) with $s = 0$ as follows

$$\|y_u - y_h(u_h)\|_{L^2(\Omega)} \leq \|y_u - y_{u_h}\|_{L^2(\Omega)} + \|y_{u_h} - y_h(u_h)\|_{L^2(\Omega)} \longrightarrow 0 \quad \text{as } h \longrightarrow 0.$$

The convergence in $L^r(\Omega)$ follows from (5.13). ◻

COROLLARY 5.5. *There exists a constant $C = C(\alpha,\beta) > 0$ such that, for all $u \in U^{ad}$ and $v \in U^{ad} \cap H^s(\Gamma)$, with $0 \leq s \leq 1$, we have*

$$\|y_u - y_h(v)\|_{L^2(\Omega)} \leq C\Big\{\|u - v\|_{L^2(\Gamma)} + h^{s+1/2}(1 + \|v\|_{H^s(\Gamma)})\Big\}. \tag{5.15}$$

This corollary is an immediate consequence of the second estimate in (2.5) and of (5.14).

Let us finish this section by establishing some estimates for the adjoint states.

THEOREM 5.6. *Given $u, v \in U^{ad}$, let ϕ_u and $\phi_h(v)$ be the solutions of* (3.5) *and* (4.3) *with u replaced by v in the last equation. Then there exist some constants $C_i = C_i(\alpha,\beta) > 0$ $(1 \leq i \leq 3)$ such that*

$$\|\phi_h(v)\|_{L^\infty(\Omega)} \leq C_1 \quad \forall h > 0, \tag{5.16}$$

$$\|\phi_u - \phi_h(v)\|_{L^2(\Omega)} \leq C_2(\|u - v\|_{L^2(\Gamma)} + h^2), \tag{5.17}$$

$$\|\phi_u - \phi_h(v)\|_{L^\infty(\Omega)} + \|\phi_u - \phi_h(v)\|_{H^1(\Omega)} \leq C_3(\|u - v\|_{L^2(\Gamma)} + h). \tag{5.18}$$

Proof. All the inequalities follow from the results of Casas and Mateos [10] just by taking into account that

$$\|\phi_u - \phi_h(v)\|_X \leq \|\phi_u - \phi_v\|_X + \|\phi_v - \phi_h(v)\|_X \leq C(\|y_u - y_v\|_{L^2(\Omega)} + \|\phi_v - \phi_h(v)\|_X),$$

with X equal to $L^\infty(\Omega)$, $L^2(\Omega)$ and $H^1(\Omega)$ respectively. □

Now we provide an error estimate for the discrete normal derivative of the adjoint state defined by Proposition 4.2.

THEOREM 5.7. *There exists a constant $C = C(\alpha,\beta) > 0$ such that the following estimate holds*

$$\|\partial_\nu\phi_u - \partial_\nu^h\phi_h(u)\|_{L^2(\Gamma)} \le \begin{cases} Ch^{1/2} \quad \forall u\in U^{ad}, \\ C(\|u\|_{H^{1/2}(\Gamma)}+1)h^{1-1/p} \quad \forall u \in U^{ad}\cap H^{1/2}(\Gamma). \end{cases} \tag{5.19}$$

Proof. First of all let us remind that $\phi_u \in H^2(\Omega)$ and therefore $\partial_\nu\phi_u \in H^{1/2}(\Gamma)$. Observe that the definition of the projection operator Π_h leads to

$$\int_\Gamma \Big|\partial_\nu\phi_u - \partial_\nu^h\phi_h(u)\Big|^2 = \int_\Gamma \Big|\partial_\nu\phi_u - \Pi_h\partial_\nu\phi_u\Big|^2 + \int_\Gamma \Big|\Pi_h\partial_\nu\phi_u - \partial_\nu^h\phi_h(u)\Big|^2 = I_1 + I_2.$$

Since $\partial_\nu^h\phi_h(u)$ belongs to U_h, we can write

$$I_2 = \int_\Gamma (\partial_\nu\phi_u - \partial_\nu^h\phi_h(u))(\Pi_h\partial_\nu\phi_u - \partial_\nu^h\phi_h(u)).$$

Let us introduce $z_h \in Y_h$ as the solution to the variational equation

$$\begin{cases} a(z_h, w_h) = 0 \quad \forall w_h \in Y_{h0} \\ z_h = \Pi_h\partial_\nu\phi_u - \partial_\nu^h\phi_h(u) \quad \text{on } \Gamma. \end{cases}$$

From (5.3) it follows that

$$\|z_h\|_{H^1(\Omega)} \le C\|\Pi_h\partial_\nu\phi_u - \partial_\nu^h\phi_h(u)\|_{H^{1/2}(\Gamma)}. \tag{5.20}$$

Now using the definition of $\partial_\nu^h\phi_h(u)$ stated in Proposition 4.2 and a Green formula for ϕ_u, we can write

$$\begin{aligned} I_2 = a(z_h, \phi_u - \phi_h(u)) &+ \int_\Omega \Big(\frac{\partial f}{\partial y}(x, y_h(u))\phi_h(u) - \frac{\partial f}{\partial y}(x,y_u)\phi_u\Big) z_h \\ &+ \int_\Omega \Big(\frac{\partial L}{\partial y}(x, y_h(u)) - \frac{\partial L}{\partial y}(x, y_u)\Big) z_h. \end{aligned} \tag{5.21}$$

Due to the equation satisfied by z_h

$$a(z_h, I_h\phi_u) = a(z_h, \phi_h(u)) = 0,$$

we also have

$$\begin{aligned} I_2 &= a(z_h, \phi_u - I_h\phi_u) + \int_\Omega \Big(\frac{\partial f}{\partial y}(x, y_h(u)) - \frac{\partial f}{\partial y}(x,y_u)\Big)\phi_u z_h \\ &+ \int_\Omega \frac{\partial f}{\partial y}(x, y_h(u))(\phi_h(u) - \phi_u) z_h + \int_\Omega \Big(\frac{\partial L}{\partial y}(x, y_h(u)) - \frac{\partial L}{\partial y}(x, y_u)\Big) z_h. \end{aligned} \tag{5.22}$$

From well known interpolation estimates, the second inequality of (5.3) and an inverse inequality it follows that

$$\begin{aligned} &a(z_h, \phi_u - I_h\phi_u) \le \|z_h\|_{W^{1,p'}(\Omega)}\|\phi_u - I_h\phi_u\|_{W^{1,p}(\Omega)} \\ &\le Ch\|\phi_u\|_{W^{2,p}(\Omega)}\|z_h|_\Gamma\|_{W^{1-1/p',p'}(\Gamma)} \le Ch\|z_h|_\Gamma\|_{H^{1-1/p'}(\Gamma)} \\ &\le Ch^{1/p'}\|z_h|_\Gamma\|_{L^2(\Gamma)} = Ch^{1/p'}\sqrt{I_2}, \end{aligned} \tag{5.23}$$

where $p' = p/(p-1)$.

From assumptions **(A1)** and **(A2)** and inequalities (5.13), (5.14) with $s = 0$, (5.16) and (5.17), we get

$$\Big|\int_\Omega \Big(\frac{\partial f}{\partial y}(x, y_h(u)) - \frac{\partial f}{\partial y}(x, y_u)\Big)\phi_u z_h\Big| \leq Ch^{1/2}\|z_h\|_{L^2(\Omega)}, \tag{5.24}$$

$$\begin{aligned}&\Big|\int_\Omega \frac{\partial f}{\partial y}(x, y_h(u))(\phi_h(u) - \phi_u) z_h\Big| \leq C\|\phi_h(u) - \phi_u\|_{L^2(\Omega)}\|z_h\|_{L^2(\Omega)}\\ &\leq Ch^2\|z_h\|_{L^2(\Omega)},\end{aligned} \tag{5.25}$$

and

$$\Big|\int_\Omega \Big(\frac{\partial L}{\partial y}(x, y_h(u)) - \frac{\partial L}{\partial y}(x, y_u)\Big) z_h\Big| \leq Ch^{1/2}\|z_h\|_{L^2(\Omega)}. \tag{5.26}$$

Collecting together the estimates (5.23)-(5.26) and using (5.20) and the fact that $p' < 2$, we obtain

$$\begin{aligned}&I_2 \leq Ch^{1/p'}\sqrt{I_2} + Ch^{1/2}\|z_h\|_{L^2(\Omega)}\\ &\leq C(h^{1/p'}\sqrt{I_2} + h^{1/2}\|\Pi_h\partial_\nu\phi_u - \partial_\nu^h\phi_h(u)\|_{L^2(\Gamma)}) \leq Ch^{1/2}\sqrt{I_2},\end{aligned} \tag{5.27}$$

which implies that

$$I_2 \leq Ch. \tag{5.28}$$

Using again that $\phi_u \in W^{2,p}(\Omega)$, we get that $\partial_\nu\phi_u \in W^{1-1/p,p}(\Gamma) \subset H^{1-1/p}(\Gamma)$. Hence from (4.1) with $s = 1 - 1/p$, we can derive

$$I_1 \leq Ch\|\partial_\nu\phi_u\|^2_{H^{1/2}(\Gamma)} \leq Ch\|\phi_u\|^2_{H^2(\Omega)} \leq Ch^{2(1-1/p)}. \tag{5.29}$$

So the first estimate in (5.19) is proved.

To complete the proof let us assume that $u \in H^{1/2}(\Gamma)$, then we can use (5.14) with $s = 1/2$ to estimate $y_u - y_h(u)$ in $L^2(\Omega)$ by Ch. This allows us to change $h^{1/2}$ in (5.24) and (5.26) by h. Therefore (5.27) can be replaced by $I_2 \leq Ch^{1/p'} = Ch^{1-1/p}$, thus $I_2 \leq Ch^{2(1-1/p)}$. So the second estimate in (5.19) is proved. □

Corollary 5.8. *There exists a constant C independent of h such that*

$$\left\{\begin{aligned}&\|\partial_\nu^h\phi_h(u)\|_{H^{1/2}(\Gamma)} \leq C \quad \forall u \in U^{ad},\\ &\|\partial_\nu^h\phi_h(u)\|_{W^{1-1/p,p}(\Gamma)} \leq C(\|u\|_{H^{1/2}(\Gamma)} + 1) \quad \forall u \in U^{ad} \cap H^{1/2}(\Gamma),\\ &\|\partial_\nu\phi_u - \partial_\nu^h\phi_h(v)\|_{L^2(\Gamma)} \leq C\left\{\|u - v\|_{L^2(\Gamma)} + h^\kappa\right\} \quad \forall u, v \in U^{ad},\end{aligned}\right. \tag{5.30}$$

where $\kappa = 1 - 1/p$ if $v \in H^{1/2}(\Gamma)$ and $\kappa = 1/2$ otherwise.

Proof. Let us make the proof in the case where $u \in U^{ad} \cap H^{1/2}(\Gamma)$. The case where $u \in U^{ad}$ can be treated similarly. We know that

$$\|\partial_\nu\phi_u\|_{W^{1-1/p,p}(\Gamma)} \leq C\|\phi_u\|_{W^{2,p}(\Omega)} \leq C \quad \forall u \in U^{ad}.$$

On the other hand, the projection operator Π_h is stable in the Sobolev spaces $W^{s,q}(\Gamma)$, for $1 \leq q \leq \infty$ and $0 \leq s \leq 1$, see Casas and Raymond [13], therefore

$$\|\Pi_h\partial_\nu\phi_u\|_{W^{1-1/p,p}(\Gamma)} \leq C\|\partial_\nu\phi_u\|_{W^{1-1/p,p}(\Gamma)}.$$

Finally, with an inverse inequality and the estimate $I_2 \le Ch^{2-2/p}$ obtained in the previous proof we deduce

$$\begin{aligned}
&\|\partial_\nu^h \phi_h(u)\|_{W^{1-1/p,p}(\Gamma)} \le \|\Pi_h \partial_\nu \phi_u - \partial_\nu^h \phi_h(u)\|_{W^{1-1/p,p}(\Gamma)} + \|\Pi_h \partial_\nu \phi_u\|_{W^{1-1/p,p}(\Gamma)} \\
&\le C\|\Pi_h \partial_\nu \phi_u - \partial_\nu^h \phi_h(u)\|_{H^{1-1/p}(\Gamma)} + \|\Pi_h \partial_\nu \phi_u\|_{W^{1-1/p,p}(\Gamma)} \\
&\le Ch^{-1+1/p}\|\Pi_h \partial_\nu \phi_u - \partial_\nu^h \phi_h(u)\|_{L^2(\Gamma)} + \|\partial_\nu \phi_u\|_{W^{1-1/p,p}(\Gamma)} \le C.
\end{aligned}$$

The third inequality of (5.30) is an immediate consequence of Theorem 5.7.

□

6. Convergence Analysis for (P$_h$). In this section we will prove the strong convergence in $L^2(\Gamma)$ of the solutions $\bar u_h$ of discrete problems (P$_h$) to the solutions of (P). Moreover we will prove that $\{\bar u_h\}_h$ remains bounded in $H^{1/2}(\Gamma)$, and next that it is also bounded in $W^{1-1/p,p}(\Gamma)$. Finally we will prove the strong convergence of the solutions $\bar u_h$ of discrete problems (P$_h$) to the solutions of (P) in $C(\Gamma)$.

THEOREM 6.1. *For every $h > 0$ let $\bar u_h$ be a global solution of problem (P$_h$). Then there exist weakly*-converging subsequences of $\{\bar u_h\}_{h>0}$ in $L^\infty(\Gamma)$ (still indexed by h). If the subsequence $\{\bar u_h\}_{h>0}$ is converging weakly* in $L^\infty(\Gamma)$ to some $\bar u$, then $\bar u$ is a solution of* (P),

$$\lim_{h\to 0} J_h(\bar u_h) = J(\bar u) = \inf(P) \quad \text{and} \quad \lim_{h\to 0} \|\bar u - \bar u_h\|_{L^2(\Gamma)} = 0. \tag{6.1}$$

Proof. Since $U_h^{ad} \subset U^{ad}$ holds for every $h > 0$ and U^{ad} is bounded in $L^\infty(\Gamma)$, $\{\bar u_h\}_{h>0}$ is also bounded in $L^\infty(\Gamma)$. Therefore, there exist weakly*-converging subsequences as claimed in the statement of the theorem. Let $\{\bar u_h\}$ be one of these subsequences and let $\bar u$ be the weak* limit. It is obvious that $\bar u \in U^{ad}$. Let us prove that $\bar u$ is a solution of (P). Let us take a solution of (P), $\tilde u \in U^{ad}$, therefore $\tilde u \in W^{1-1/p,p}(\Gamma)$ for some $p > 2$; see Theorem 3.4. Let us take $u_h = I_h \tilde u$. Then $u_h \in U_h^{ad}$ and $\{u_h\}_h$ tends to $\tilde u$ in $L^\infty(\Gamma)$; see Brenner and Scott [6]. By taking $u = \tilde u$, $v = u_h$ and $s = 0$ in (5.15) we deduce that $y_h(u_h) \to y_{\tilde u}$ in $L^2(\Gamma)$. Moreover (5.13) implies that $\{y_h(u_h)\}_{h>0}$ is bounded in $L^\infty(\Omega)$. On the other hand, Theorem 5.4 implies that $\bar y_h = y_h(\bar u_h) \to \bar y = y_{\bar u}$ strongly in $L^2(\Omega)$ and $\{\bar y_h\}_{h>0}$ is also bounded in $L^\infty(\Omega)$. Then we have

$$J(\bar u) \le \liminf_{h\to 0} J_h(\bar u_h) \le \limsup_{h\to 0} J_h(\bar u_h) \le \limsup_{h\to 0} J_h(I_h \tilde u) = J(\tilde u) = \inf(P).$$

This proves that $\bar u$ is a solution of (P) as well as the convergence of the optimal costs, which leads to $\|\bar u_h\|_{L^2(\Gamma)} \longrightarrow \|\bar u\|_{L^2(\Gamma)}$, hence we deduce the strong convergence of the controls in $L^2(\Gamma)$. □

THEOREM 6.2. *Let $p > 2$ be as in Theorem 3.4 and for every h let $\bar u_h$ denote a local solution of (P$_h$). Then there exists a constant $C > 0$ independent of h such that*

$$\|\bar u_h\|_{W^{1-1/p,p}(\Gamma)} \le C \quad \forall h > 0. \tag{6.2}$$

Moreover the convergence of $\{\bar u_h\}_{h>0}$ to $\bar u$ stated in Theorem 6.1 holds in $C(\Gamma)$.

Proof. By using the stability in $H^{1/2}(\Gamma)$ of the $L^2(\Gamma)$-projections on the sets U_h^{ad} (see Casas and Raymond [13]) along with (4.12) and the first inequality of (5.30), we get that $\{\bar u_h\}_{h>0}$ is uniformly bounded in $H^{1/2}(\Gamma)$. Using now the second inequality of (5.30) and the stability of Π_h in $W^{1-1/p,p}(\Gamma)$ we deduce (6.2). Finally the convergence is a consequence of the compactness of the imbedding $W^{1-1/p,p}(\Gamma) \subset C(\Gamma)$ for $p > 2$.

□

7. Error estimates. The goal is to prove the following theorem.

THEOREM 7.1. *Let us assume that $\bar u$ is a local solution of (P) satisfying the sufficient second order optimality conditions provided in Theorem 3.5 and let $\bar u_h$ be a local solution of (P_h) such that $\bar u_h \to \bar u$ in $L^2(\Gamma)$; see Theorem 6.1. Then the following inequality holds*

$$\|\bar u - \bar u_h\|_{L^2(\Gamma)} \le Ch^{1-1/p}, \tag{7.1}$$

where $p > 2$ is given by Theorem 3.4.

We will prove the theorem arguing by contradiction. The statement of the theorem can be sated as follows. There exists a positive constant C such that for all $0 < h < 1/C$, we have

$$\frac{\|\bar u - \bar u_h\|_{L^2(\Gamma)}}{h^{1-1/p}} \le C.$$

Thus if (7.1) is false, for all $k > 0$, there exists $0 < h_k < 1/k$ such that

$$\frac{\|\bar u - \bar u_{h_k}\|_{L^2(\Gamma)}}{h_k^{1-1/p}} > k.$$

Therefore there exists a sequence of h such that

$$\lim_{h\to 0} \frac{1}{h^{1-1/p}}\|\bar u - \bar u_h\|_{L^2(\Gamma)} = +\infty. \tag{7.2}$$

We will obtain a contradiction for this sequence. For the proof of this theorem we need some lemmas.

LEMMA 7.2. *Let us assume that* (7.1) *is false. Let $\delta > 0$ given by Remark 3.6-(2). Then there exists $h_0 > 0$ such that*

$$\frac{1}{2}\min\{\delta, N\}\|\bar u - \bar u_h\|^2_{L^2(\Gamma)} \le (J'(\bar u_h) - J'(\bar u))(\bar u_h - \bar u) \quad \forall h < h_0. \tag{7.3}$$

Proof. Let $\{\bar u_h\}_h$ be a sequence satisfying (7.2). By applying the mean value theorem we get for some $\hat u_h = \bar u + \theta_h(\bar u_h - \bar u)$

$$(J'(\bar u_h) - J'(\bar u))(\bar u_h - \bar u) = J''(\hat u_h)(\bar u_h - \bar u)^2. \tag{7.4}$$

Let us take

$$v_h = \frac{1}{\|\bar u_h - \bar u\|_{L^2(\Gamma)}}(\bar u_h - \bar u).$$

Taking a subsequence if necessary we can assume that $v_h \rightharpoonup v$ in $L^2(\Gamma)$. Let us prove that v belongs to the critical cone $C_{\bar u}$ defined in §3. First of all remark that every v_h satisfies the sign condition (3.9), hence v also does. Let us prove that $v(x) = 0$ if $\bar d(x) \ne 0$, $\bar d$ being defined by (3.10). We will use the interpolation operator $I_h \in \mathcal{L}(W^{1-1/p,p}(\Gamma), U_h)$, with $p > 2$ given in Theorem 3.4. Since $\bar u \in U^{ad}$ it is obvious that $I_h \bar u \in U_h^{ad}$. Given $y \in W^{1,p}(\Omega)$ such that $y|_\Gamma = \bar u$. It is obvious that $I_h\bar u$ is the trace of $I_h y$ (see the beginning of section 5). Now, by using a result by Grisvard [22, Chapter 1] we get

$$\|\bar u - I_h\bar u\|^p_{L^p(\Gamma)} \le C\left(\varepsilon^{1-1/p}\|y - I_h y\|^p_{W^{1,p}(\Omega)} + \varepsilon^{-1/p}\|y - I_h y\|^p_{L^p(\Omega)}\right),$$

for every $\varepsilon > 0$ and for some constant $C > 0$ independent of ε and y. Setting $\varepsilon = h^p$ and using that (see for instance Brenner and Scott [6])

$$\|y - I_h y\|_{L^p(\Omega)} \leq C_1 h \|y\|_{W^{1,p}(\Omega)}, \quad \|I_h y\|_{W^{1,p}(\Omega)} \leq C_2 \|y\|_{W^{1,p}(\Omega)}$$

and

$$\inf_{y|_\Gamma = \bar{u}} \|y\|_{W^{1,p}(\Omega)} \leq C_3 \|\bar{u}\|_{W^{1-1/p}(\Gamma)},$$

we conclude that

$$\|\bar{u} - I_h \bar{u}\|_{L^2(\Gamma)} \leq |\Gamma|^{\frac{p-2}{2p}} \|\bar{u} - I_h \bar{u}\|_{L^p(\Gamma)} \leq C h^{1-1/p} \|\bar{u}\|_{W^{1-1/p,p}(\Gamma)}. \tag{7.5}$$

Let us define

$$\bar{d}_h(x) = N \bar{u}_h(x) - \partial_\nu^h \bar{\phi}_h(x). \tag{7.6}$$

The third inequality of (5.30) implies that $\bar{d}_h \to \bar{d}$ in $L^2(\Gamma)$. Now we have

$$\begin{aligned}
&\int_\Gamma \bar{d}(x) v(x)\, dx = \lim_{h \to 0} \int_\Gamma \bar{d}_h(x) v_h(x)\, dx \\
&= \lim_{h \to 0} \frac{1}{\|\bar{u}_h - \bar{u}\|_{L^2(\Gamma)}} \left\{ \int_\Gamma \bar{d}_h (I_h \bar{u} - \bar{u})\, dx + \int_\Gamma \bar{d}_h (\bar{u}_h - I_h \bar{u})\, dx \right\}.
\end{aligned}$$

From (4.8), (7.2) and (7.5) we deduce

$$\begin{aligned}
&\int_\Gamma \bar{d}(x) v(x)\, dx \leq \lim_{h \to 0} \frac{1}{\|\bar{u}_h - \bar{u}\|_{L^2(\Gamma)}} \int_\Gamma \bar{d}_h(x) (I_h \bar{u}(x) - \bar{u}(x))\, dx \\
&\leq \lim_{h \to 0} \frac{C h^{1-1/p}}{\|\bar{u}_h - \bar{u}\|_{L^2(\Gamma)}} = 0.
\end{aligned}$$

Since v satisfies the sign condition (3.9), then $\bar{d}(x) v(x) \geq 0$, hence the above inequality proves that v is zero whenever $\bar{d}$ is not, which allows us to conclude that $v \in C_{\bar{u}}$. Now from the definition of v_h, (3.4) and (3.11) we get

$$\begin{aligned}
&\lim_{h \to 0} J''(\hat{u}_h) v_h^2 = \lim_{h \to 0} \left\{ \int_\Omega \left[\frac{\partial^2 L}{\partial y^2}(x, y_{\hat{u}_h}) + \phi_{\hat{u}_h} \frac{\partial^2 f}{\partial y^2}(x, y_{\hat{u}_h}) \right] z_{v_h}^2\, dx + N \right\} \\
&= \int_\Omega \left[\frac{\partial^2 L}{\partial y^2}(x, \bar{y}) + \bar{\phi} \frac{\partial^2 f}{\partial y^2}(x, \bar{y}) \right] z_v^2\, dx + N \\
&= J''(\bar{u}) v^2 + N(1 - \|v\|_{L^2(\Gamma)}^2) \geq N + (\delta - N) \|v\|_{L^2(\Gamma)}^2.
\end{aligned}$$

Taking into account that $\|v\|_{L^2(\Gamma)} \leq 1$, these inequalities lead to

$$\lim_{h \to 0} J''(\hat{u}_h) v_h^2 \geq \min\{\delta, N\} > 0,$$

which proves the existence of $h_0 > 0$ such that

$$J''(\hat{u}_h) v_h^2 \geq \frac{1}{2} \min\{\delta, N\} \quad \forall h < h_0.$$

From this inequality, the definition of v_h and (7.4) we deduce (7.3). $\square$

LEMMA 7.3. *There exists a constant $C > 0$ independent of h such that for every $v \in L^\infty(\Gamma)$*

$$|(J_h'(\bar u_h) - J'(\bar u_h))v| \le Ch^{1-1/p}\|v\|_{L^2(\Gamma)}. \tag{7.7}$$

Proof. From (3.3), (4.6), (7.6), (6.2) and Theorem 5.7 we get

$$(J_h'(\bar u_h) - J'(\bar u_h))v = \int_\Gamma (\partial_\nu \phi_{\bar u_h} - \partial_\nu^h \bar\phi_h)v\,dx \le \|\partial_\nu \phi_{\bar u_h} - \partial_\nu^h \bar\phi_h\|_{L^2(\Gamma)}\|v\|_{L^2(\Gamma)}$$
$$\le C(\|\bar u_h\|_{H^{1/2}(\Gamma)} + 1)h^{(1-1/p)}\|v\|_{L^2(\Gamma)} \le Ch^{(1-1/p)}\|v\|_{L^2(\Gamma)}.$$

□

LEMMA 7.4. *There exists a constant $C > 0$ independent of h such that for every $v \in L^\infty(\Gamma)$*

$$|(J_h'(\bar u_h) - J'(\bar u))v| \le \Big(N\|\bar u - \bar u_h\|_{L^2(\Gamma)} + Ch^{1-1/p}\Big)\|v\|_{L^2(\Gamma)}. \tag{7.8}$$

Proof. Arguing in a similar way to the previous proof and using (5.30) and (6.2) we have

$$(J_h'(\bar u_h) - J'(\bar u))v = \int_\Gamma \Big(N\bar u_h - \partial_\nu^h \bar\phi_h\Big)\Pi_h v\,dx - \int_\Gamma \Big(N\bar u - \partial_\nu \bar\phi\Big)v\,dx$$
$$= N\int_\Gamma (\bar u_h - \bar u)v\,dx + \int_\Gamma \Big(\partial_\nu \bar\phi - \partial_\nu^h \bar\phi_h\Big)v\,dx$$
$$\le \Big(N\|\bar u_h - \bar u\|_{L^2(\Gamma)} + Ch^{(1-1/p)}\Big)\|v\|_{L^2(\Gamma)}.$$

□

One key point in the proof of error estimates is to get a discrete control $u_h \in U_h^{ad}$ that approximates $\bar u$ conveniently and satisfies $J'(\bar u)\bar u = J'(\bar u)u_h$. Let us find such a control. Let us set I_j for every $1 \le j \le N(h)$

$$I_j = \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} \bar d(x)e_j(x)\,dx.$$

Now we define $u_h \in U_h$ with $u_h(x_\Gamma^j) = u_{h,j}$ for every node $x_\Gamma^j \in \Gamma$ by the expression

$$u_{h,j} = \begin{cases} \dfrac{1}{I_j}\displaystyle\int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} \bar d(x)\bar u(x)e_j(x)\,dx & \text{if } I_j \ne 0, \\[2ex] \dfrac{1}{h_{j-1}+h_j}\displaystyle\int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} \bar u(x)\,dx & \text{if } I_j = 0. \end{cases} \tag{7.9}$$

Remind that the measure of $[x_\Gamma^{j-1}, x_\Gamma^{j+1}]$ is $h_{j-1} + h_j = |x_\Gamma^j - x_\Gamma^{j-1}| + |x_\Gamma^{j+1} - x_\Gamma^j|$, which coincides with $|x_\Gamma^{j+1} - x_\Gamma^{j-1}|$ if x_Γ^j is not a vertex of Ω.

In the following lemma, we state that the function u_h defined by (7.9) satisfies our requirements.

LEMMA 7.5. *There exists $h_0 > 0$ such that, for every $0 < h < h_0$, the element $u_h \in U_h$ defined by* (7.9) *obeys the following properties:*

1. $u_h \in U_h^{ad}$.
2. $J'(\bar u)\bar u = J'(\bar u)u_h$.
3. The approximation property

$$\|\bar u - u_h\|_{L^2(\Gamma)} \le Ch^{1-1/p} \tag{7.10}$$

is fulfilled for some constant $C > 0$ independent of h.

Proof. Since $\bar u$ is continuous on Γ, there exists $h_0 > 0$ such that

$$|\bar u(\xi_2) - \bar u(\xi_1)| \le \frac{\beta-\alpha}{2} \quad \forall h < h_0, \quad \forall \xi_1, \xi_2 \in [x_\Gamma^{j-1}, x_\Gamma^{j+1}],\ 1 \le j \le N(h),$$

which implies that $\bar u$ cannot admit both the values α and β on one segment $[x_\Gamma^{j-1}, x_\Gamma^{j+1}]$ for any $h < h_0$. Hence the sign of $\bar d$ on $[x_\Gamma^{j-1}, x_\Gamma^{j+1}]$ must be constant due to (3.7). Therefore, $I_j = 0$ if and only if $\bar d(x) = 0$ for all $x \in [x_\Gamma^{j-1}, x_\Gamma^{j+1}]$. Moreover if $I_j \neq 0$, then $\bar d(x)/I_j \ge 0$ for every $x \in [x_\Gamma^{j-1}, x_\Gamma^{j+1}]$. As a first consequence of this we get that $\alpha \le u_{h,j} \le \beta$, which means that $u_h \in U_h^{ad}$. On the other hand

$$J'(\bar u)u_h = \sum_{j=1}^{N(h)} \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} \bar d(x)e_j(x)\,dx\,u_{h,j} = \sum_{j=1}^{N(h)} \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} \bar d(x)\bar u(x)e_j(x)\,dx = J'(\bar u)\bar u.$$

Finally let us prove (7.10). Let us remind that $\bar u \in W^{1-1/p,p}(\Gamma) \subset H^{1-1/p}(\Gamma)$ and $p > 2$. Remind that the norm in $H^s(\Gamma)$, $0 < s < 1$, is given by

$$\|u\|_{H^s(\Gamma)} = \Big(\|u\|_{L^2(\Gamma)}^2 + \int_\Gamma\int_\Gamma \frac{|u(x)-u(\xi)|^2}{|x-\xi|^{1+2s}}\,dx\,d\xi\Big)^{1/2}. \tag{7.11}$$

Using that $\sum_{j=1}^{N(h)} e_j(x) = 1$ and $0 \le e_j(x) \le 1$ we get

$$\begin{aligned}\|\bar u - u_h\|_{L^2(\Gamma)}^2 &= \int_\Gamma \Big|\sum_{j=1}^{N(h)} (\bar u(x) - u_{h,j})e_j(x)\Big|^2 dx \le \\ &\le \sum_{j=1}^{N(h)} \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} |\bar u(x) - u_{h,j}|^2 e_j(x)\,dx \le \sum_{j=1}^{N(h)} \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} |\bar u(x) - u_{h,j}|^2\,dx.\end{aligned} \tag{7.12}$$

Let us estimate every term of the sum.

Let us start by assuming that $I_j = 0$ so that $u_{h,j}$ is defined by the second relation in (7.9). Then we have

$$\begin{aligned}&\int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} |\bar u(x) - u_{h,j}|^2\,dx = \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} \Big|\frac{1}{h_{j-1}+h_j}\int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} (\bar u(x) - \bar u(\xi))\,d\xi\Big|^2 dx \\ &\le \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} \frac{1}{h_{j-1}+h_j}\int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} |\bar u(x) - \bar u(\xi)|^2\,d\xi\,dx \\ &\le (h_{j-1}+h_j)^{2(1-1/p)} \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}}\int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} \frac{|\bar u(x)-\bar u(\xi)|^2}{|x-\xi|^{1+2(1-1/p)}}\,dx\,d\xi \\ &\le (2h)^{2(1-1/p)}\|\bar u\|_{H^{1-1/p}(x_\Gamma^{j-1},x_\Gamma^{j+1})}^2.\end{aligned} \tag{7.13}$$

Now let us consider the case $I_j \neq 0$.

$$
\begin{aligned}
&\int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} |\bar u(x) - u_{h,j}|^2\,dx = \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} \Big|\frac{1}{I_j}\int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} \bar d(\xi) e_j(\xi)(\bar u(x) - \bar u(\xi))\,d\xi\Big|^2\,dx\\
&\le \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} \Big|\int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} \sqrt{\frac{\bar d(\xi) e_j(\xi)}{I_j}}|\bar u(x) - \bar u(\xi)|\sqrt{\frac{\bar d(\xi) e_j(\xi)}{I_j}}\,d\xi\Big|^2\,dx\\
&\le \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} |\bar u(x) - \bar u(\xi)|^2 \frac{\bar d(\xi) e_j(\xi)}{I_j}\,d\xi\,dx\\
&\le \Big(\int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} \frac{\bar d(\xi) e_j(\xi)}{I_j}\,d\xi\Big) \sup_{\xi\in[x_\Gamma^{j-1}, x_\Gamma^{j+1}]} \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} |\bar u(x) - \bar u(\xi)|^2\,dx\\
&= \sup_{\xi\in[x_\Gamma^{j-1}, x_\Gamma^{j+1}]} \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} |\bar u(x) - \bar u(\xi)|^2\,dx.
\end{aligned}
\tag{7.14}
$$

To obtain the estimate for the last term we are going to use Lemma 7.6 stated below with

$$f(\xi) = \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} |\bar u(x) - \bar u(\xi)|^2\,dx.$$

Since $H^{1-1/p}(\Gamma) \subset C^{0,\theta}(\Gamma)$ for $\theta = 1/2 - 1/p$ (see e.g. [37, Theorem 2.8.1]),it is easy to check that

$$
\begin{aligned}
&|f(\xi_2) - f(\xi_1)| \le \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} \Big|[\bar u(x) - \bar u(\xi_1)] + [\bar u(x) - \bar u(\xi_2)]\Big|\,\Big|\bar u(\xi_2) - \bar u(\xi_1)\Big|\,dx\\
&\le 2(h_{j-1} + h_j)^{1+2\theta} C_{\theta,p} \|\bar u\|^2_{H^{1-1/p}(x_\Gamma^{j-1}, x_\Gamma^{j+1})}.
\end{aligned}
$$

On the other hand we have

$$
\begin{aligned}
&\int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} f(\xi)\,d\xi = \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} \frac{|\bar u(x) - \bar u(\xi)|^2}{|x-\xi|^{1+2(1-1/p)}} |x-\xi|^{1+2(1-1/p)}\,dx\,d\xi\\
&\le (h_{j-1} + h_j)^{1+2(1-1/p)} \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} \int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} \frac{|\bar u(x) - \bar u(\xi)|^2}{|x-\xi|^{1+2(1-1/p)}}\,dx\,d\xi\\
&\le (h_{j-1} + h_j)^{2+(1-2/p)} \|\bar u\|^2_{H^{1-1/p}(x_\Gamma^{j-1}, x_\Gamma^{j+1})}.
\end{aligned}
$$

Then we can apply Lemma 7.6 to the function f with

$$M = (h_{j-1} + h_j)^{2\theta} \max\{4C_{\theta,p}, 1\} \|\bar u\|^2_{H^{1-1/p}(x_\Gamma^{j-1}, x_\Gamma^{j+1})} \le C h^{2\theta} \|\bar u\|^2_{H^{1-1/p}(x_\Gamma^{j-1}, x_\Gamma^{j+1})},$$

to deduce that

$$f(\xi) \le C \|\bar u\|^2_{H^{1-1/p}(x_\Gamma^{j-1}, x_\Gamma^{j+1})} h^{1+2\theta}. \tag{7.15}$$

This inequality along with (7.14) leads to

$$\int_{x_\Gamma^{j-1}}^{x_\Gamma^{j+1}} |\bar u(x) - u_{h,j}|^2\,dx \le C \|\bar u\|^2_{H^{1-1/p}(x_\Gamma^{j-1}, x_\Gamma^{j+1})} h^{1+2\theta}, \tag{7.16}$$

in the case where $I_j \neq 0$.

Since

$$\sum_{j=1}^{N(h)} \|\bar{u}\|^2_{H^{1-1/p}(x_\Gamma^{j-1}, x_\Gamma^{j+1})} \leq 2\|\bar{u}\|^2_{H^{1-1/p}(\Gamma)},$$

inequality (7.10) follows from (7.12), (7.13), (7.16) and the fact that $1+2\theta = 2(1-1/p)$. ∎

Lemma 7.6. *Given $-\infty < a < b < +\infty$ and $f : [a,b] \longmapsto \mathbb{R}^+$ a function satisfying*

$$|f(x_2) - f(x_1)| \leq \frac{M}{2}(b-a) \quad \text{and} \quad \int_a^b f(x)\,dx \leq M(b-a)^2,$$

then $f(x) \leq 2M(b-a) \quad \forall x \in [a,b]$.

Proof. We argue by contradiction and we assume that there exists a point $\xi \in [a,b]$ such that $f(\xi) > 2M(b-a)$, then

$$\int_a^b f(x)\,dx = \int_a^b \{[f(x) - f(\xi)] + f(\xi)\}\,dx > -\frac{M}{2}(b-a)^2 + 2M(b-a)^2 = \frac{3M}{2}(b-a)^2,$$

which contradicts the second assumption on f. ∎

Proof of Theorem 7.1. Setting $u = \bar{u}_h$ in (3.7) we get

$$J'(\bar{u})(\bar{u}_h - \bar{u}) = \int_\Gamma \left(N\bar{u} - \partial_\nu \bar{\phi}\right)(\bar{u}_h - \bar{u})\,dx \geq 0. \tag{7.17}$$

From (4.8) with u_h defined by (7.9) it follows

$$J_h'(\bar{u}_h)(u_h - \bar{u}_h) = \int_\Gamma \left(N\bar{u}_h - \partial_\nu^h \bar{\phi}_h\right)(u_h - \bar{u}_h)\,dx \geq 0$$

and then

$$J_h'(\bar{u}_h)(\bar{u} - \bar{u}_h) + J_h'(\bar{u}_h)(u_h - \bar{u}) \geq 0. \tag{7.18}$$

By adding (7.17) and (7.18) and using Lemma 7.5-2, we derive

$$\left(J'(\bar{u}) - J_h'(\bar{u}_h)\right)(\bar{u} - \bar{u}_h) \leq J_h'(\bar{u}_h)(u_h - \bar{u}) = \left(J_h'(\bar{u}_h) - J'(\bar{u})\right)(u_h - \bar{u}).$$

For $h < h_0$, this inequality and (7.3) lead to

$$\begin{aligned} &\tfrac{1}{2}\min\{N,\delta\}\|\bar{u} - \bar{u}_h\|^2_{L^2(\Gamma)} \leq \left(J'(\bar{u}) - J'(\bar{u}_h)\right)(\bar{u} - \bar{u}_h) \\ &\leq \left(J_h'(\bar{u}_h) - J'(\bar{u}_h)\right)(\bar{u} - \bar{u}_h) + \left(J_h'(\bar{u}_h) - J'(\bar{u})\right)(u_h - \bar{u}). \end{aligned} \tag{7.19}$$

Now from (7.7) and Young's inequality we obtain

$$\begin{aligned} &|(J_h'(\bar{u}_h) - J'(\bar{u}_h))(\bar{u} - \bar{u}_h)| \leq Ch^{1-1/p}\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} \\ &\leq Ch^{2(1-1/p)} + \tfrac{1}{8}\min\{N,\delta\}\|\bar{u} - \bar{u}_h\|^2_{L^2(\Gamma)}. \end{aligned} \tag{7.20}$$

On the other hand, using again Young's inequality, (7.8) and (7.10) we deduce

$$\begin{aligned} |(J_h'(\bar{u}_h) - J'(\bar{u}))(u_h - \bar{u})| &\leq \Big(N\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} + Ch^{1-1/p}\Big)\|\bar{u} - u_h\|_{L^2(\Gamma)} \\ &\leq \Big(N\|\bar{u} - \bar{u}_h\|_{L^2(\Gamma)} + Ch^{1-1/p}\Big)h^{1-1/p} \\ &\leq \tfrac{1}{8}\min\{N,\delta\}\|\bar{u} - \bar{u}_h\|^2_{L^2(\Gamma)} + Ch^{2(1-1/p)}, \end{aligned} \tag{7.21}$$

From (7.19)–(7.21) it comes

$$\frac{1}{4}\min\{N,\delta\}\|\bar{u}-\bar{u}_h\|^2_{L^2(\Gamma)} \le Ch^{2(1-1/p)},$$

which contradicts (7.2). ∎

8. Numerical tests. In this section we present some numerical tests which illustrate our theoretical results. Let Ω be the unit square $(0,1)^2$. Consider

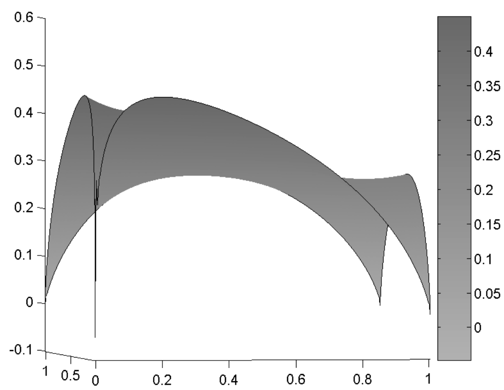
$$y_d(x_1,x_2) = \frac{1}{(x_1^2+x_2^2)^{1/3}}.$$

We are going to solve the following problem

$$(\mathrm{P})\left\{\begin{array}{l} \mathrm{Min}\ J(u) = \dfrac{1}{2}\displaystyle\int_\Omega (y_u(x)-y_d(x))^2dx + \dfrac{1}{2}\int_\Gamma u(x)^2dx \\ u \in U_{ad} = \{u \in L^2(\Gamma):\ -1 \le u(x) \le 2 \text{ a.e. } x \in \Gamma\}, \\ -\Delta y_u = 0 \text{ in } \Omega,\ y_u = u \text{ on } \Gamma. \end{array}\right.$$

Remark that $y_d \in L^p(\Omega)$ for all $p < 3$, but $y_d \notin L^3(\Omega)$, therefore the optimal adjoint state $\bar{\varphi}$ is actually in $W^{2,p}(\Omega)$ for $p < 3$. Consequently we can deduce that the optimal control belongs to $W^{1-1/p,p}(\Gamma)$, but $W^{1-1/p,p}(\Gamma)$ is not included in $H^1(\Gamma)$. There is no reason that the normal derivative $\partial_\nu\bar{\varphi}$ be more regular than $W^{1-1/p,p}(\Gamma)$. For our problem, the plot shows that the optimal control has a singularity in the corner at the origin, and it seems that $\bar{u} \notin H^1(\Gamma)$. So we cannot hope to have a convergence order of $O(h)$. Instead of that we have a convergence of order $O(h^{1-1/p})$ for some $p > 2$, as predicted by the theory.

Since we do not have an exact solution for (P), we have solved it numerically for $h = 2^{-9}$ and we have used this solution to compare with other solutions for bigger values of h. We have solved it using an active set strategy, as is explained in [11]. Here is a plot of the optimal solution.



The control constraints are not active at the optimal control. In the following table we show the norm in $L^2(\Gamma)$ of the error of the control and the order of convergence step by step. This is measured as

$$o_i = \frac{\log(\|\bar{u}_{h_i}-\bar{u}\|_{L^2(\Gamma)}) - \log(\|\bar{u}_{h_{i-1}}-\bar{u}\|_{L^2(\Gamma)})}{\log(h_i)-\log(h_{i-1})}$$

$h_i/\sqrt{2}$	$\|\bar{u}_{h_i} - \bar{u}\|_{L^2(\Gamma)}$	o_i
2^{-3}	0.1055	–
2^{-4}	0.0652	0.6944
2^{-5}	0.0393	0.7302
2^{-6}	0.0237	0.7314
2^{-7}	0.0146	0.7008
2^{-8}	0.0093	0.6493

Let us remark that $1 - 1/p < 2/3$ for $p < 3$. The values o_i are around 2/3. We believe that the order of convergence could be closer to 2/3 if we could compare the computed controls with the true optimal control instead with its numerical approximation. We refer to [12] for more details and numerical tests.

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