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Controllability of Partial Differential Equations

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Contents

Chapter 1

Control Theory: History, Mathematical Achievements and Perspectives *(E. Fernández-Cara and E. Zuazua)*

joint work with E. Fernández-Cara, Universidad de Sevilla, Spain, in Bol. SEMA (Sociedad Española de Matemática Aplicada), **26***, 2003, 79-140.*

1.1 Introduction

This article is devoted to present some of the mathematical milestones of Control Theory. We will focus on systems described in terms of ordinary differential equations. The control of (deterministic and stochastic) partial differential equations remains out of our scope. However, it must be underlined that most ideas, methods and results presented here do extend to this more general setting, which leads to very important technical developments.

The underlying idea that motivated this article is that Control Theory is certainly, at present, one of the most interdisciplinary areas of research. Control Theory arises in most modern applications. The same could be said about the very first technological discoveries of the industrial revolution. On the other hand, Control Theory has been a discipline where many mathematical ideas and methods have melt to produce a new body of important Mathematics. Accordingly, it is nowadays a rich crossing point of Engineering and Mathematics.

Along this paper, we have tried to avoid unnecessary technical difficulties, to make the text accessible to a large class of readers. However, in order to introduce some of the main achievements in Control Theory, a minimal body of basic mathematical concepts and results is needed. We develop this material to make the text self-contained.

These notes contain information not only on the main mathematical results in Control Theory, but also about its origins, history and the way applications and interactions of Control Theory with other Sciences and Technologies have conducted the development of the discipline.

The plan of the paper is the following. Section 2 is concerned with the origins and most basic concepts. In Section 3 we study a simple but very interesting example: the *pendulum*. As we shall see, an elementary analysis of this simple but important mechanical system indicates that the fundamental ideas of Control Theory are extremely meaningful from a physical viewpoint.

In Section 4 we describe some relevant historical facts and also some important contemporary applications. There, it will be shown that Control Theory is in fact an interdisciplinary subject that has been strongly involved in the development of the contemporary society.

In Section 5 we describe the two main approaches that allow to give rigorous formulations of control problems: controllability and optimal control. We also discuss their mutual relations, advantages and drawbacks.

In Sections 6 and 7 we present some basic results on the controllability of linear and nonlinear finite dimensional systems. In particular, we revisit the Kalman approach to the controllability of linear systems, and we recall the use of Lie brackets in the control of nonlinear systems, discussing a simple example of a planar moving square car.

In Section 8 we discuss how the complexity of the systems arising in modern technologies affects Control Theory and the impact of numerical approximations and discrete modelling, when compared to the classical modelling in the context of Continuum Mechanics.

In Section 9 we describe briefly two beautiful and extremely important challenging applications for Control Theory in which, from a mathematical viewpoint, almost all remains to be done: laser molecular control and the control of floods.

In Section 10 we present a list of possible future applications and lines of development of Control Theory: large space structures, Robotics, biomedical research, etc.

Finally, we have included two Appendices, where we recall briefly two of the main principles of modern Control Theory, namely *Pontryagin's maximum principle* and *Bellman's dynamical programming principle*.

1.2 Origins and basic ideas, concepts and ingredients

The word *control* has a double meaning. First, controlling a system can be understood simply as testing or checking that its behavior is satisfactory. In a deeper sense, to control is also to act, to put things in order to guarantee that the system behaves as desired.

S. Bennet starts the first volume of his book [16] on the history of *Control Engineering* quoting the following sentence of Chapter 3, Book 1, of the monograph "Politics" by Aristotle:

> "...if every instrument could accomplish its own work, obeying or anticipating the will of others ...if the shuttle weaved and the pick touched the lyre without a hand to guide them, chief workmen would not need servants, nor masters slaves."

This sentence by Aristotle describes in a rather transparent way the guiding goal of *Control Theory:* the need of automatizing processes to let the human being gain in liberty, freedom, and quality of life.

Let us indicate briefly how control problems are stated nowadays in mathematical terms. To fix ideas, assume we want to get a good behavior of a physical system governed by the *state equation*

$$A(y) = f(v). \tag{1.1}$$

Here, y is the *state*, the unknown of the system that we are willing to control. It belongs to a vector space Y. On the other hand, v is the *control*. It belongs to the *set of admissible controls* $\mathcal{U}_{\rm ad}$. This is the variable that we can choose freely in $\mathcal{U}_{\rm ad}$ to act on the system.

Let us assume that $A : D(A) \subset Y \mapsto Y$ and $f : \mathcal{U}_{\rm ad} \mapsto Y$ are two given (linear or nonlinear) mappings. The operator A determines the equation that must be satisfied by the state variable y, according to the laws of Physics. The function f indicates the way the control v acts on the system governing the state. For simplicity, let us assume that, for each $v \in \mathcal{U}_{\rm ad}$, the state equation (1.1) possesses exactly one solution $y = y(v)$ in Y. Then, roughly speaking, to control (1.1) is to find $v \in \mathcal{U}_{\rm ad}$ such that the solution to (1.1) gets close to the desired prescribed state. The "best" among all the existing controls achieving the desired goal is frequently referred to as the *optimal control*.

This mathematical formulation might seem sophisticated or even obscure for readers not familiar with this topic. However, it is by now standard and it has been originated naturally along the history of this rich discipline. One of the main advantages of such a general setting is that many problems of very different nature may fit in it, as we shall see along this work.

As many other fields of human activities, the discipline of Control existed much earlier than it was given that name. Indeed, in the world of living species, organisms are endowed with sophisticated mechanisms that regulate the various tasks they develop. This is done to guarantee that the essential variables are kept in optimal regimes to keep the species alive allowing them to grow, develop and reproduce.

Thus, although the mathematical formulation of control problems is intrinsically complex, the key ideas in Control Theory can be found in Nature, in the evolution and behavior of living beings.

The first key idea is the *feedback* concept. This term was incorporated to Control Engineering in the twenties by the engineers of the "Bell Telephone Laboratory" but, at that time, it was already recognized and consolidated in other areas, such as Political Economics.

Essentially, a feedback process is the one in which the state of the system determines the way the control has to be exerted at any time. This is related to the notion of *real time control*, very important for applications. In the framework of (1.1), we say that the control u is given by a *feedback law* if we are able to provide a mapping $G : Y \mapsto \mathcal{U}_{\rm ad}$ such that

$$u = G(y), \quad \text{where } y = y(u), \tag{1.2}$$

i.e. y solves (1.1) with v replaced by u.

Nowadays, feedback processes are ubiquitous not only in Economics, but also in Biology, Psychology, etc. Accordingly, in many different related areas, the *cause-effect principle* is not understood as a static phenomenon any more, but it is rather being viewed from a dynamical perspective. Thus, we can speak of the *cause-effect-cause principle*. See [162] for a discussion on this and other related aspects.

The second key idea is clearly illustrated by the following sentence by H.R. Hall in [102] in 1907 and that we have taken from [16]:

> "It is a curious fact that, while political economists recognize that for the proper action of the law of supply and demand there must be fluctuations, it has not generally been recognized by mechanicians in this matter of the steam engine governor. The aim of the mechanical economist, as is that of the political economist, should be not to do away with these fluctuations all together (for then he does away with the principles of self-regulation), but to diminish them as much as possible, still leaving them large enough to have sufficient regulating power."

The need of having room for fluctuations that this paragraph evokes is related to a basic principle that we apply many times in our daily life. For instance, when driving a car at a high speed and needing to brake, we usually try to make it intermittently, in order to keep the vehicle under control at any moment. In the context of human relationships, it is also clear that insisting permanently in the same idea might not be precisely the most convincing strategy.

The same rule applies for the control of a system. Thus, to control a system arising in Nature or Technology, we do not have necessarily to stress the system and drive it to the desired state immediately and directly. Very often, it is much more efficient to control the system letting it fluctuate, trying to find a harmonic dynamics that will drive the system to the desired state without forcing it too much. An excess of control may indeed produce not only an inadmissible cost but also irreversible damages in the system under consideration.

Another important underlying notion in Control Theory is *Optimization*. This can be regarded as a branch of Mathematics whose goal is to improve a variable in order to maximize a benefit (or minimize a cost). This is applicable to a lot of practical situations (the variable can be a temperature, a velocity field, a measure of information, etc.). Optimization Theory and its related techniques are such a broad subject that it would be impossible to make a unified presentation. Furthermore, a lot of recent developments in *Informatics* and *Computer Science* have played a crucial role in Optimization. Indeed, the complexity of the systems we consider interesting nowadays makes it impossible to implement efficient control strategies without using appropriate (and sophisticated) software.

In order to understand why Optimization techniques and Control Theory are closely related, let us come back to (1.1). Assume that the set of admissible controls $\mathcal{U}_{\rm ad}$ is a subset of the Banach space $\mathcal{U}$ (with norm $\|\cdot\|_{\mathcal{U}}$) and the state space Y is another Banach space (with norm $\|\cdot\|_Y$). Also, assume that the state $y_d \in Y$ is the preferred state and is chosen as a target for the state of the system. Then, the control problem consists in finding controls v in $\mathcal{U}_{\rm ad}$ such that the associated solution coincides or gets close to y_d.

It is then reasonable to think that a fruitful way to choose a good control v is by minimizing a *cost function* of the form

$$J(v) = \frac{1}{2}\|y(v) - y_d\|_Y^2 \quad \forall v \in \mathcal{U}_{\rm ad} \tag{1.3}$$

or, more generally,

$$J(v) = \frac{1}{2}\|y(v) - y_d\|_Y^2 + \frac{\mu}{2}\|v\|_{\mathcal{U}}^2 \quad \forall v \in \mathcal{U}_{\rm ad}\,, \tag{1.4}$$

where $\mu \geq 0$.

These are (constrained) extremal problems whose analysis corresponds to Optimization Theory.

It is interesting to analyze the two terms arising in the functional J in (1.4) when $\mu > 0$ separately, since they play complementary roles. When minimizing the functional in (1.4), we are minimizing the balance between these two terms. The first one requires to get close to the target y_d while the second one penalizes using too much costly control. Thus, roughly speaking, when minimizing J we are trying to drive the system to a state close to the target y_d without too much effort.

We will give below more details of the connection of Control Theory and Optimization below.

So far, we have mentioned three main ingredients arising in Control Theory: the notion of feedback, the need of fluctuations and Optimization. But of course in the development of Control Theory many other concepts have been important.

One of them is *Cybernetics*. The word "cybernétique" was proposed by the French physicist A.-M. Ampère in the XIX Century to design the nonexistent science of process controlling. This was quickly forgotten until 1948, when N. Wiener chose "Cybernetics" as the title of his book.

Wiener defined Cybernetics as "the science of control and communication in animals and machines". In this way, he established the connection between Control Theory and Physiology and anticipated that, in a desirable future, engines would obey and imitate human beings.

At that time this was only a dream but now the situation is completely different, since recent developments have made possible a large number of new applications in *Robotics*, *Computer-Aided Design*, etc. (see [199] for an overview). Today, Cybernetics is not a dream any more but an ubiquitous reality. On the other hand, Cybernetics leads to many important questions that are relevant for the development of our society, very often in the borderline of *Ethics* and *Philosophy*. For instance,

> *Can we be inspired by Nature to create better engines and machines ?*

Or

> *Is the animal behavior an acceptable criterium to judge the performance of an engine ?*

Many movies of science fiction describe a world in which machines do not obey any more to humans and humans become their slaves. This is the opposite situation to the one Control Theory has been and is looking for. The development of Science and Technology is obeying very closely to the predictions made fifty years ago. Therefore, it seems desirable to deeply consider and revise our position towards Cybernetics from now on, many years ahead, as we do permanently in what concerns, for instance, Genetics and the possibilities it provides to intervene in human reproduction.

1.3 The pendulum

We will analyze in this Section a very simple and elementary control problem related to the dynamics of *the pendulum.*

The analysis of this model will allow us to present the most relevant ideas in the control of finite dimensional systems, that, as we said above, are essential for more sophisticated systems too. In our presentation, we will closely follow the book by E. Sontag [206].

The problem we discuss here, far from being purely academic, arises in many technological applications and in particular in Robotics, where the goal is to control a *gyratory arm* with a motor located at one extreme connecting the arm to the rest of the structure.

In order to model this system, we assume that the total mass m of the arm is located at the free extreme and the bar has unit length. Ignoring the effect of friction, we write

$$m\ddot{\theta}(t) = -mg\sin\theta(t) + v(t), \tag{1.5}$$

which is a direct consequence of *Newton's law.* Here, $\theta = \theta(t)$ is the angle of the arm with respect to the vertical axis measured counterclockwise, g is the acceleration due to gravity and u is the applied external *torsional momentum.* The state of the system is $(\theta, \dot{\theta})$, while $v = v(t)$ is the control.

To simplify our analysis, we also assume that $m = g = 1$. Then, (1.5) becomes:

$$\ddot{\theta}(t) + \sin\theta(t) = v(t). \tag{1.6}$$

The vertical stationary position $(\theta = \pi, \dot{\theta} = 0)$ is an equilibrium configuration in the absence of control, i.e. with $v \equiv 0$. But, obviously, this is an *unstable* equilibrium. Let us analyze the system around this configuration, to understand how this instability can be compensated by means of the applied control force v.

Taking into account that $\sin\theta \sim \pi - \theta$ near $\theta = \pi$, at first approximation, the linearized system with respect to the variable $\varphi = \theta - \pi$ can be written in the form

$$\ddot{\varphi} - \varphi = v(t). \tag{1.7}$$

The goal is then to drive $(\varphi, \dot{\varphi})$ to the desired state $(0, 0)$ for all small initial data, without making the angle and the velocity too large along the controlled trajectory.

The following control strategy is in agreement with common sense: when the system is to the left of the vertical line, i.e. when $\varphi = \theta - \pi > 0$, we push the system towards the right side, i.e. we apply a force v with negative sign; on the other hand, when $\varphi < 0$, it seems natural to choose $v > 0$.

This suggests the following *feedback law,* in which the control is proportional to the state:

$$v = -\alpha\varphi, \qquad \text{with } \alpha > 0. \tag{1.8}$$

In this way, we get the *closed loop system*

$$\ddot{\varphi} + (\alpha - 1)\varphi = 0. \tag{1.9}$$

It is important to understand that, solving (1.9), we simultaneously obtain the state $(\varphi, \dot{\varphi})$ and the control $v = -\alpha\varphi$. This justifies, at least in this case, the relevance of a feedback law like (1.8).

The roots of the characteristic polynomial of the linear equation (1.9) are $z = \pm\sqrt{1-\alpha}$. Hence, when $\alpha > 1$, the nontrivial solutions of this differential equation are *oscillatory*. When $\alpha < 1$, all solutions diverge to $\pm\infty$ as $t \to \pm\infty$, except those satisfying

$$\dot{\varphi}(0) = -\sqrt{1-\alpha}\;\varphi(0).$$

Finally, when $\alpha = 1$, all nontrivial solutions satisfying $\dot{\varphi}(0) = 0$ are constant.

Thus, the solutions to the linearized system (1.9) do not reach the desired configuration $(0, 0)$ in general, independently of the constant α we put in (1.8).

This can be explained as follows. Let us first assume that $\alpha < 1$. When $\varphi(0)$ is positive and small and $\dot{\varphi}(0) = 0$, from equation (1.9) we deduce that $\ddot{\varphi}(0) > 0$. Thus, φ and $\dot{\varphi}$ grow and, consequently, the pendulum goes away from the vertical line. When $\alpha > 1$, the control acts on the correct direction but with too much inertia.

The same happens to be true for the nonlinear system (1.6).

The most natural solution is then to keep $\alpha > 1$, but introducing an additional term to diminish the oscillations and penalize the velocity. In this way, a new feedback law can be proposed in which the control is given as a linear combination of φ and $\dot{\varphi}$:

$$v = -\alpha\varphi - \beta\dot{\varphi}, \quad \text{with } \alpha > 1 \text{ and } \beta > 0. \tag{1.10}$$

The new closed loop system is

$$\ddot{\varphi} + \beta\dot{\varphi} + (\alpha - 1)\varphi = 0, \tag{1.11}$$

whose characteristic polynomial has the following roots

$$\frac{-\beta \pm \sqrt{\beta^2 - 4(\alpha - 1)}}{2}\,. \tag{1.12}$$

Now, the real part of the roots is negative and therefore, all solutions converge to zero as $t \to +\infty$. Moreover, if we impose the condition

$$\beta^2 > 4(\alpha - 1), \tag{1.13}$$

we see that solutions tend to zero monotonically, without oscillations.

This simple model is rich enough to illustrate some systematic properties of control systems:

- Linearizing the system is an useful tool to address its control, although the results that can be obtained this way are only of local nature.
- One can obtain feedback controls, but their effects on the system are not necessarily in agreement with the very first intuition. Certainly, the (asymptotic) stability properties of the system must be taken into account.
- Increasing dissipation one can eliminate the oscillations, as we have indicated in (1.13).

In connection with this last point, notice however that, as dissipation increases, trajectories converge to the equilibrium more slowly. Indeed, in (1.10), for fixed $\alpha > 1$, the value of β that minimizes the largest real part of a root of the characteristic polynomial (1.11) is

$$\beta = 2\sqrt{\alpha - 1}.$$

With this value of β, the associated real part is

$$\sigma^* = -\sqrt{\alpha - 1}$$

and, increasing β, the root corresponding to the plus sign increases and converges to zero:

$$\frac{-\beta + \sqrt{\beta^2 - 4(\alpha - 1)}}{2} > -\sqrt{\alpha - 1} \quad \forall \beta > 2\sqrt{\alpha - 1} \tag{1.14}$$

and

$$\frac{-\beta + \sqrt{\beta^2 - 4(\alpha - 1)}}{2} \to 0^- \quad \text{as } \beta \to +\infty. \tag{1.15}$$

This phenomenon is known as *overdamping* in Engineering and has to be taken into account systematically when designing feedback mechanisms.

At the practical level, implementing the control (1.10) is not so simple, since the computation of v requires knowing the position φ and the velocity $\dot{\varphi}$ at every time.

Let us now describe an interesting alternative. The key idea is to evaluate φ and $\dot{\varphi}$ only on a discrete set of times

$$0,\ \delta,\ 2\delta,\ \ldots,\ k\delta,\ \ldots$$

and modify the control at each of these values of t. The control we get this way is kept constant along each interval $[k\delta, (k+1)\delta]$.

Computing the solution to system (1.7), we see that the result of applying the constant control v_k in the time interval $[k\delta, (k+1)\delta]$ is as follows:

$$\begin{pmatrix} \varphi(k\delta + \delta) \\ \dot{\varphi}(k\delta + \delta) \end{pmatrix} = A \begin{pmatrix} \varphi(k\delta) \\ \dot{\varphi}(k\delta) \end{pmatrix} + v_k\, b,$$

where

$$A = \begin{pmatrix} \cos h\delta & \sin h\delta \\ \sin h\delta & \cos h\delta \end{pmatrix}, \quad b = \begin{pmatrix} \cos h\delta - 1 \\ \sin h\delta \end{pmatrix}.$$

Thus, we obtain a discrete system of the form

$$x_{k+1} = (A + bf^t)x_k \,,$$

where f is the vector such that

$$v_k = f^t x_k \,.$$

Observe that, if f is such that the matrix $A + bf^t$ is nilpotent, i.e.

$$[A + bf^t]^2 = 0,$$

then we reach the equilibrium in two steps. A simple computation shows that this property holds if $f^t = (f_1, f_2)$, with

$$f_1 = \frac{1 - 2\cos h\delta}{2(\cos h\delta - 1)} \,, \quad f_2 = -\frac{1 + 2\cos h\delta}{2\sin h\delta} \,. \tag{1.16}$$

The main advantage of using controllers of this form is that we get the stabilization of the trajectories in finite time and not only asymptotically, as $t \to +\infty$. The controller we have designed is a *digital control* and it is extremely useful because of its robustness and the ease of its implementation.

The digital controllers we have built are similar and closely related to the *bang-bang* controls we are going to describe now.

Once $\alpha > 1$ is fixed, for instance $\alpha = 2$, we can assume that

$$v = -2\varphi + w, \tag{1.17}$$

so that (1.7) can be written in the form

$$\ddot{\varphi} + \varphi = w. \tag{1.18}$$

This is Newton's law for the vibration of a spring.

This time, we look for controls below an admissible cost. For instance, we impose

$$|w(t)| \leq 1 \quad \forall t.$$

The function $w = w(t)$ that, satisfying this constraint, controls the system in minimal time, i.e. *the optimal control*, is necessarily of the form

$$w(t) = \mathrm{sgn}(p(t)),$$

where η is a solution of

$$\ddot{p} + p = 0.$$

This is a consequence of *Pontryagin's maximum principle* (see Appendix 1 for more details).

Therefore, the optimal control takes only the values ± 1 and, in practice, it is sufficient to determine the *switching times* at which the sign of the optimal control changes.

In order to compute the optimal control, let us first compute the solutions corresponding to the extremal controllers ± 1. Using the new variables x_1 and x_2 with $x_1 = \varphi$ and $x_2 = \dot{\varphi}$, this is equivalent to solve the systems

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + 1 \end{cases} \tag{1.19}$$

and

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - 1. \end{cases} \tag{1.20}$$

The solutions can be identified to the circumferences in the plane (x_1, x_2) centered at $(1, 0)$ and $(-1, 0)$, respectively. Consequently, in order to drive (1.18) to the final state $(\varphi, \dot{\varphi})(T) = (0, 0)$, we must follow these circumferences, starting from the prescribed initial state and switching from one to another appropriately.

For instance, assume that we start from the initial state $(\varphi, \dot{\varphi})(0) = (\varphi^0, \varphi^1)$, where φ^0 and φ^1 are positive and small. Then, we first take $w(t) = 1$ and solve (1.19) for $t \in [0, T_1]$, where T_1 is such that $x_2(T_1) = 0$, i.e. we follow counterclockwise the arc connecting the points (φ^0, φ^1) and $(x_1(T_1), 0)$ in the (x_1, x_2) plane. In a second step, we take $w(t) = -1$ and solve (1.20) for $t \in [T_1, T_2]$, where T_2 is such that $(1 - x_1(T_2))^2 + x_2(T_2)^2 = 1$. We thus follow (again counterclockwise) the arc connecting the points $(x_1(T_1), 0)$ and $(x_1(T_2), x_2(T_2))$. Finally, we take $w(t) = 1$ and solve (1.19) for $t \in [T_2, T_3]$, with T_3 such that $x_1(T_3 = x_2(T_3) = 0$.

Similar constructions of the control can be done when $\varphi^0 \leq 1$ or $\varphi^1 \leq 0$.

In this way, we reach the equilibrium $(0, 0)$ in finite time and we obtain a feedback mechanism

$$\ddot{\varphi} + \varphi = F(\varphi, \dot{\varphi}),$$

where F is the function taking the value -1 above the switching curve and $+1$ below. In what concerns the original system (1.7), we have

$$\ddot{\varphi} - \varphi = -2\varphi + F(\varphi, \dot{\varphi}).$$

The action of the control in this example shows clearly the suitability of self-regulation mechanisms. If we want to lead the system to rest in a minimal time, it is advisable to do it following a somewhat indirect path, allowing the system to evolve naturally and avoiding any excessive forcing.

Bang-bang controllers are of high interest for practical purposes. Although they might seem irregular and unnatural, they have the advantages of providing minimal time control and being easy to compute.

As we said above, although the problem we have considered is very simple, it leads naturally to some of the most relevant ideas of Control Theory: feedback laws, overdamping, digital and bang-bang controls, etc.

1.4 History and contemporary applications

In this paper, we do not intend to make a complete overview of the history of Control Theory, nor to address its connections with the philosophical questions we have just mentioned. Without any doubt, this would need much more space. Our intention is simply to recall some classical and well known results that have to some extent influenced the development of this discipline, pointing out several facts that, in our opinion, have been relevant for the recent achievements of Control Theory.

Let us go back to the origins of Control Engineering and Control Theory and let us describe the role this discipline has played in History.

Going backwards in time, we will easily conclude that Romans did use some elements of Control Theory in their aqueducts. Indeed, ingenious systems of regulating valves were used in these constructions in order to keep the water level constant.

Some people claim that, in the ancient Mesopotamia, more than 2000 years B.C., the control of the irrigation systems was also a well known art.

On the other hand, in the ancient Egypt the "harpenodaptai" (string stretchers), were specialized in stretching very long strings leading to long straight segments to help in large constructions. Somehow, this is an evidence of the fact that in the ancient Egypt the following two assertions were already well understood:

- The shortest distance between two points is the straight line (which can be considered to be the most classical assertion in Optimization and Calculus of Variations);
- This is equivalent to the following dual property: among all the paths of a given length the one that produces the longest distance between its extremes is the straight line as well.

The task of the "harpenodaptai" was precisely to build these "optimal curves".

The work by Ch. Huygens and R. Hooke at the end of the XVII Century on the *oscillations of the pendulum* is a more modern example of development in Control Theory. Their goal was to achieve a precise measurement of time and location, so precious in navigation.

These works were later adapted to regulate the velocity of windmills. The main mechanism was based on a system of balls rotating around an axis, with a velocity proportional to the velocity of the windmill. When the rotational velocity increased, the balls got farther from the axis, acting on the wings of the mill through appropriate mechanisms.

J. Watt adapted these ideas when he invented the *steam engine* and this constituted a magnificent step in the industrial revolution. In this mechanism, when the velocity of the balls increases, one or several valves open to let the vapor scape. This makes the pressure diminish. When this happens, i.e. when the pressure inside the boiler becomes weaker, the velocity begins to go down. The goal of introducing and using this mechanism is of course to keep the velocity as close as possible to a constant.

The British astronomer G. Airy was the first scientist to analyze mathematically the regulating system invented by Watt. But the first definitive mathematical description was given only in the works by J.C. Maxwell, in 1868, where some of the erratic behaviors encountered in the steam engine were described and some control mechanisms were proposed.

The central ideas of Control Theory gained soon a remarkable impact and, in the twenties, engineers were already preferring the continuous processing and using semi-automatic or automatic control techniques. In this way, Control Engineering germinated and got the recognition of a distinguished discipline.

In the thirties important progresses were made on automatic control and design and analysis techniques. The number of applications increased covering *amplifiers* in telephone systems, distribution systems in electrical plants, stabilization of aeroplanes, electrical mechanisms in paper production, Chemistry, petroleum and steel Industry, etc.

By the end of that decade, two emerging and clearly different methods or approaches were available: a first method based on the use of differential equations and a second one, of frequential nature, based on the analysis of amplitudes and phases of "inputs" and "outputs".

By that time, many institutions took conscience of the relevance of automatic control. This happened for instance in the American ASME (American Society of Mechanical Engineers) and the British IEE (Institution of Electrical Engineers). During the Second World War and the following years, engineers and scientists improved their experience on the control mechanisms of plane tracking and ballistic missiles and other designs of anti-aircraft batteries. This produced an important development of frequential methods.

After 1960, the methods and ideas mentioned above began to be considered as part of "classical" Control Theory. The war made clear that the models considered up to that moment were not accurate enough to describe the complexity of the real word. Indeed, by that time it was clear that *true systems* are often *nonlinear* and *nondeterministic*, since they are affected by "noise".

This generated important new efforts in this field.

The contributions of the U.S. scientist R. Bellman in the context of *dynamic programming*, R. Kalman in *filtering techniques* and the algebraic approach to linear systems and the Russian L. Pontryagin with the *maximum principle* for nonlinear optimal control problems established the foundations of modern Control Theory.

We shall describe in Section 6 the approach by Kalman to the controllability of linear finite dimensional systems. Furthermore, at the end of this paper we give two short Appendices where we have tried to present, as simply as possible, the central ideas of Bellman's and Pontryagin's works.

As we have explained, the developments of Industry and Technology had a tremendous impact in the history of Control Engineering. But the development of Mathematics had a similar effect.

Indeed, we hav already mentioned that, in the late thirties, two emerging strategies were already established. The first one was based on the use of differential equations and, therefore, the contributions made by the most celebrated mathematicians between the XVIIth and the XIXth Centuries played a fundamental role in that approach. The second one, based on a frequential approach, was greatly influenced by the works of J. Fourier.

Accordingly, Control Theory may be regarded nowadays from two different and complementary points of view: as a theoretical support to *Control Engineering* (a part of *System Engineering*) and also as a mathematical discipline. In practice, the frontiers between these two subworlds are extremely vague. In fact, Control Theory is one of the most interdisciplinary areas of Science nowadays, where Engineering and Mathematics melt perfectly and enrich each other.

Mathematics is currently playing an increasing role in Control Theory. Indeed, the degree of sophistication of the systems that Control Theory has to deal with increases permanently and this produces also an increasing demand of Mathematics in the field.

Along these notes, it will become clear that Control Theory and Calculus of Variations have also common roots. In fact, these two disciplines are very often hard to distinguish.

The history of the Calculus of Variations is also full of mathematical achievements. We shall now mention some of them.

As we said above, one can consider that the starting point of the Calculus of Variations is the understanding that the straight line is the shortest path between two given points. In the first Century, Heron of Alexandria showed in his work "La Catoptrique" that the law of reflection of light (the fact that the incidence and reflection angles are identical) may be obtained as a consequence of the variational principle that light minimizes distance along the preferred path.

In the XVII Century, P. De Fermat generalized this remark by Heron and formulated the following minimum principle:

> *Light in a medium with variable velocity prefers the path that guarantees the minimal time.*

Later Leibnitz and Huygens proved that the law of refraction of light may be obtained as a consequence of Fermat's principle.

The refraction law had been discovered by G. Snell in 1621, although it remained unpublished until 1703, as Huygens published his *Dioptrica*.

It is interesting to observe that, in account of this principle, a ray of light may be unable to propagate from a *slow* medium to a *fast* medium. Indeed, if $n_1 > n_2$, there exists a critical angle θ_c such that, when $\theta_1 > \theta_c$, Snell's law cannot be satisfied whatever θ_2 is.

Contrarily, the light can always propagate from a fast to a slow medium.

Here we have denoted by n_i the *index of refraction* of the i-th medium. By definition, we have $n_i = c/v_i$, where c and v_i are the speeds of propagation of light in the vacuum and the i-th medium, respectively.

In 1691, J. Bernoulli proved that the *catenary* is the curve which provides the shape of a string of a given length and constant density with fixed ends under the action of gravity. Let us also mention that the problem of the *bachistocrone*, formulated by Bernoulli in 1696, is equivalent to finding the rays of light in the upper half-plane $y \geq 0$ corresponding to a light velocity c given by the formula $c(x, y) = \sqrt{y}$ (Newton proved in 1697 that the solution is the *cycloid*). The reader interested in these questions may consult the paper by H. Sussmann [209].

R. Kalman, one of the greatest protagonists of modern Control Theory, said in 1974 that, in the future, the main advances in Control and Optimization of systems would come more from mathematical progress than from the technological development. Today, the state of the art and the possibilities that Technology offers are so impressive that maintaining that statement is probably very risky. But, without any doubt, the development of Control Theory will require deep contributions coming from both fields.

In view of the rich history of Control Theory and all the mathematical achievements that have been undertaken in its domain of influence, one could ask whether the field has reached its end. But this is far from reality. Our society provides every day new problems to Control Theory and this fact is stimulating the creation of new Mathematics.

Indeed, the range of applications of Control Theory goes from the simplest mechanisms we manipulate in everyday life to the most sophisticated ones, emerging in new technologies.

The book edited by W.S. Levine [138] provides a rather complete description of this variety of applications.

One of the simplest applications of Control Theory appears in such an apparently simple machine as the tank of our bathroom. There are many variants of tanks and some of the licences go back to 1886 and can be found in [136]. But all them work under the same basic principles: the tank is supplied of regulating valves, security mechanisms that start the control process, feedback mechanisms that provide more or less water to the tank depending of the level of water in its interior and, finally, mechanisms that avoid the unpleasant flooding in case that some of the other components fail.

The systems of heating, ventilation and air conditioning in big buildings are also very efficient large scale control systems composed of interconnected thermo-fluid and electro-mechanical subsystems. The main goal of these systems is to keep a comfortable and good quality air under any circumstance, with a low operational cost and a high degree of reliability. The relevance of a proper and efficient functioning of these systems is crucial from the viewpoint of the impact in Economical and Environmental Sciences. The predecessor of these sophisticated systems is the classical *thermostat* that we all know and regulates temperature at home.

The list of applications of Control Theory in Industry is endless. We can mention, for instance, the pH control in chemical reactions, the paper and automobile industries, nuclear security, defense, etc.

The control of *chaos* is also being considered by many researchers nowadays. The chaotic behavior of a system may be an obstacle for its control; but it may also be of help. For instance, the control along unstable trajectories is of great use in controlling the dynamics of fight aircrafts. We refer to [168] for a description of the state of the art of *active control* in this area.

Space structures, optical reflectors of large dimensions, satellite communication systems, etc. are also examples of modern and complex control systems. The control of *robots*, ranging from the most simple engines to the *bipeds* that simulate the locomotive ability of humans is also another emerging area of Control Theory.

For instance, see the web page `http://www.inrialpes.fr/bipop/` of the French Institute I.N.R.I.A. (Institut National de Recherche en Informatique et Automatique), where illustrating images and movies of the antropomorphic biped BIP2000 can be found.

Compact disk players is another area of application of modern control systems. A CD player is endowed with an optical mechanism allowing to interpret the registered code and produce an acoustic signal. The main goal when designing CD players is to reach higher velocities of rotation, permitting a faster reading, without affecting the stability of the disk. The control mechanisms have to be even more robust when dealing with portable equipments.

Electrical plants and distribution networks are other modern applications of Control Theory that influence significantly our daily life. There are also many relevant applications in Medicine ranging from artificial organs to mechanisms for insulin supply, for instance.

We could keep quoting other relevant applications. But those we have mentioned and some others that will appear later suffice to prove the ubiquity of control mechanisms in the real world. The underlying mathematical theory is also impressive. The reader interested in an introduction to the classical and basic mathematical techniques in Control Engineering is referred to [68] and [175].

1.5 Controllability versus optimization

As already mentioned, for systems of the form (1.1), the main goal of Control Theory is to find *controls* v leading the *associated states* $y(v)$, i.e. the solutions of the corresponding controlled systems, to a desired situation.

There are however (at least) two ways of specifying a "desired prescribed situation":

- To fix a desired state y_d and require

$$y(v) = y_d \tag{1.21}$$

  or, at least,

$$y(v) \sim y_d \tag{1.22}$$

  in some sense. This is the *controllability* viewpoint.

  The main question is then the existence of an admissible control v so that the corresponding state $y(v)$ satisfies (1.21) or (1.22). Once the existence of such a control v is established, it is meaningful to look for an optimal control, for instance, a control of *minimal size*. Other important questions arise in this context too. For instance, the existence of "bang-bang" controls, the minimal time of control, etc.

  As we shall see, this problem may be difficult (or even very difficult) to solve. In recent years, an important body of beautiful Mathematics has been developed in connection with these questions.

- To fix a *cost function* $J = J(v)$ like for instance (1.3) or (1.4) and to look for a *minimizer* u of J. This is the *optimization* or *optimal control* viewpoint.

  As in (1.3) and (1.4), J is typically related to the "distance" to a prescribed state. Both approaches have the same ultimate goal, to bring the state close to the desired target but, in some sense, the second one is more realistic and easier to implement.

The optimization viewpoint is, at least apparently, humble in comparison with the controllability approach. But it is many times much more realistic. In practice, it provides satisfactory results in many situations and, at the same time, it requires simpler mathematical tools.

To illustrate this, we will discuss now a very simple example. It is trivial in the context of Linear Algebra but it is of great help to introduce some of the basic tools of Control Theory.

We will assume that the state equation is

$$Ay = b, \tag{1.23}$$

where A is a $n \times n$ real matrix and the state is a column vector

$$y = (y_1, y_2, \dots, y_n)^t \in \mathbf{R}^n.$$

To simplify the situation, let us assume that A is nonsingular. The control vector is $b \in \mathbf{R}^n$. Obviously, we can rewrite (1.23) in the form $y = A^{-1}b$, but we do not want to do this. In fact, we are mainly interested in those cases in which (1.23) can be difficult to solve.

Let us first adopt the controllability viewpoint. To be specific, let us impose as an objective to make the first component y_1 of y coincide with a prescribed value y_1^* :

$$y_1 = y_1^* \, . \tag{1.24}$$

This is the sense we are giving to (1.22) in this particular case. So, we are consider the following controllability problem:

Problem 0: *To find $b \in \mathbf{R}^n$ such that the solution of* (1.23) *satisfies* (1.24).

Roughly speaking, we are addressing here a *partial controllability* problem, in the sense that we are controlling only one component, y_1 , of the state.

Obviously, such controls b exist. For instance, it suffices to take $y^* = (y_1^*, 0, \cdots, 0)^t$ and then choose $b = Ay^*$. But this argument, by means of which we find the state directly without previously determining the control, is frequently impossible to implement in practice. Indeed, in most real problems, we have first to find the control and, only then, we can compute the state by solving the state equation.

The number of control parameters (the n components of b) is greater or equal than the number of state components we have to control. But, what happens if we stress our own possibilities ? What happens if, for instance, $b_1, \dots, b_{n-1}$ are fixed and we only have at our disposal b_n to control the system ?

From a mathematical viewpoint, the question can be formulated as follows. In this case,

$$Ay = c + be \tag{1.25}$$

where $c \in \mathbf{R}^n$ is a prescribed column vector, e is the unit vector $(0, \ldots, 0, 1)^t$ and b is a scalar control parameter. The corresponding controllability problem is now the following:

Problem 1: *To find* $b \in \mathbf{R}$ *such that the solution of* (1.25) *satisfies* (1.24).

This is a less obvious question. However, it is not too difficult to solve. Note that the solution y to (1.25) can be decomposed in the following way:

$$y = x + z, \tag{1.26}$$

where

$$x = A^{-1}c \tag{1.27}$$

and z satisfies

$$Az = be, \quad \text{i.e.} \quad z = bz^* \quad z^* = A^{-1}e. \tag{1.28}$$

To guarantee that y_1 can take *any* value in $\mathbf{R}$, as we have required in (1.24), it is necessary and sufficient to have $z_1^* \neq 0$, z_1^* being the first component of $z^* = A^{-1}e$.

In this way, we have a precise answer to this second controllability problem:

> *The problem above can be solved for any* y_1^* *if and only if the first component of* $A^{-1}e$ *does not vanish.*

Notice that, when the first component of $A^{-1}e$ vanishes, whatever the control b is, we always have $y_1 = x_1$, x_1 being the first component of the fixed vector x in (1.27). In other words, y_1 *is not sensitive* to the control b_n. In this degenerate case, the set of values taken by y_1 is a singleton, a 0-dimensional manifold. Thus, we see that the state is confined in a "space" of low dimension and controllability is lost in general.

But, is it really frequent in practice to meet degenerate situations like the previous one, where some components of the system are insensitive to the control ?

Roughly speaking, it can be said that systems are generically not degenerate. In other words, in examples like the one above, it is actually rare that z_1^* vanishes.

There are however a few remarks to do. When z_1^* does not vanish but is very small, even though controllability holds, the control process is very unstable in the sense that one needs very large controls in order to get very small variations of the state. In practice, this is very important and must be taken into account (one needs the system not only to be controllable but this to happen with realistic and feasible controls).

On the other hand, it can be easily imagined that, when systems under consideration are complex, i.e. many parameters are involved, it is difficult to know *a priori* whether or not there are components of the state that are insensitive to the control[1].

Let us now turn to the optimization approach. Let us see that the difficulties we have encountered related to the possible degeneracy of the system disappear (which confirms the fact that this strategy leads to easier questions).

For example, let us assume that $k > 0$ is a reasonable bound of the control b that we can apply. Let us put

$$J(b_n) = \frac{1}{2}|y_1 - y_1^*|^2 \qquad \forall b_n \in \mathbf{R}, \tag{1.29}$$

where y_1 is the first component of the solution to (1.25). Then, it is reasonable to admit that the best response is given by the solution to the following problem:

Problem $1'$**:** *To find* $b_n^k \in [-k, k]$ *such that*

$$J(b_n^k) \leq J(b_n) \quad \forall b_n \in [-k, k]. \tag{1.30}$$

Since $b_n \mapsto J(b_n)$ is a continuous function, it is clear that this problem possesses a solution $b_n^k \in I_k$ for each $k > 0$. This confirms that the considered optimal control problem is simpler.

On the other hand, this point of view is completely natural and agrees with common sense. According to our intuition, most systems arising in real life should possess an optimal strategy or configuration. At this respect L. Euler said:

> "Universe is the most perfect system, designed by the most wise Creator. Nothing will happen without emerging, at some extent, a maximum or minimum principle".

Let us analyze more closely the similarities and differences arising in the two previous formulations of the control problem.

- Assume the controllability property holds, that is, **Problem** 1 is solvable for any y_1^*. Then, if the target y_1^* is given and k is sufficiently large, the solution to **Problem** $1'$ coincides with the solution to **Problem** 1.

- On the other hand, when there is no possibility to attain y_1^* exactly, the optimization viewpoint, i.e. **Problem** $1'$, furnishes the best response.

[1] In fact, it is a very interesting and non trivial task to design strategies guaranteeing that we do not fall in a degenerate situation.

- To investigate whether the controllability property is satisfied, it can be appropriate to solve **Problem** $1'$ for each $k > 0$ and analyze the behavior of the *cost*
$$J_k = \min_{b_n \in [-k,k]} J(b_n) \tag{1.31}$$
as k grows to infinity. If J_k stabilizes near a positive constant as k grows, we can suspect that y_1^* cannot be attained exactly, i.e. that **Problem** 1 does not have a solution for this value of y_1^*.

In view of these considerations, it is natural to address the question of whether it is actually necessary to solve controllability problems like **Problem** 1 or, by the contrary, whether solving a related optimal control problem (like **Problem** $1'$) suffices.

There is not a generic and systematic answer to this question. It depends on the level of precision we require to the control process and this depends heavily on the particular application one has in mind. For instance, when thinking of technologies used to stabilize buildings, or when controlling space vehicles, etc., the efficiency of the control that is required demands much more than simply choosing the best one with respect to a given criterion. In those cases, it is relevant to know how close the control will drive the state to the prescribed target. There are, consequently, a lot of examples for which simple optimization arguments as those developed here are insufficient.

In order to choose the appropriate control we need first to develop a rigorous modelling (in other words, we have to put equations to the real life system). The choice of the control problem is then a second relevant step in modelling.

Let us now recall and discuss some mathematical techniques allowing to handle the minimization problems arising in the optimization approach (in fact, we shall see that these techniques are also relevant when the controllability point of view is adopted).

These problems are closely related to the Calculus of Variations. Here, we do not intend to provide a survey of the techniques in this field but simply to mention some of the most common ideas.

For clarity, we shall start discussing *Mathematical Programming*. In the context of Optimization, *Programming* is not the art of writing computer codes. It was originated by the attempt to *optimize* the planning of the various tasks or activities in an organized system (a plant, a company, etc.). The goal is then to find what is known as an *optimal planning* or *optimal programme*.

The simplest problem of *assignment* suffices to exhibit the need of a mathematical theory to address these issues.

Assume that we have 70 workers in a plant. They have different qualifications and we have to assign them 70 different tasks. The total number of possible distributions is $70\,!$, which is of the order of 10^{100}. Obviously, in order to be able to solve rapidly a problem like this, we need a mathematical theory to provide a good strategy.

This is an example of assignment problem. Needless to say, problems of this kind are not only of academic nature, since they appear in most human activities.

In the context of *Mathematical Programming*, we first find *linear programming techniques*. As their name indicates, these are concerned with those optimization problems in which the involved functional is linear.

Linear Programming was essentially unknown before 1947, even though Joseph Fourier had already observed in 1823 the relevance of the questions it deals with. L.V. Kantorovich, in a monograph published in 1939, was the first to indicate that a large class of different planning problems could be covered with the same formulation. The *method of simplex,* that we will recall below, was introduced in 1947 and its efficiency turned out to be so impressive that very rapidly it became a common tool in Industry.

There has been a very intense research in these topics that goes beyond Linear Programming and the method of simplex. We can mention for instance *nonlinear programming methods*, inspired by the *method of descent*. This was formally introduced by the French mathematician A.L. Cauchy in the XIX Century. It relies on the idea of solving a nonlinear equation by searching the critical points of the corresponding primitive function.

Let us now give more details on Linear Programming. At this point, we will follow a presentation similar to the one by G. Strang in [208].

The problems that one can address by means of linear programming involve the minimization of linear functions subject to linear constraints. Although they seem extremely simple, they are ubiquitous and can be applied in a large variety of areas such as the control of traffic, Game Theory, Economics, etc. Furthermore, they involve in practice a huge quantity of unknowns, as in the case of the optimal planning problems we have presented before.

The simplest problem in this field can be formulated in the following way:

> *Given a real matrix A of order $M \times N$ (with $M \le N$), and given a column vector b of M components and a column vector c with N components, to minimize the linear function*
>
> $$\langle c, x\rangle = c_1 x_1 + \cdots + c_N x_N$$
>
> *under the restrictions*
>
> $$Ax = b, \quad x \ge 0.$$

Here and in the sequel, we use $\langle \cdot\,, \cdot \rangle$ to denote the usual Euclidean scalar products in $\mathbf{R}^N$ and $\mathbf{R}^M$. The associated norm will be denoted by $|\cdot|$.

Of course, the second restriction has to be understood in the following way:

$$x_j \geq 0, \quad j = 1, \ldots, N.$$

In general, the solution to this problem is given by a unique vector x with the property that $N - M$ components vanish. Accordingly, the problem consists in finding out which are the $N - M$ components that vanish and, then, computing the values of the remaining M components.

The method of simplex leads to the correct answer after a finite number of steps. The procedure is as follows:

- Step 1: We look for a vector x with $N - M$ zero components and satisfying $Ax = b$, in addition to the unilateral restriction $x \geq 0$. Obviously, this first choice of x will not provide the optimal answer in general.
- Step 2: We modify appropriately this first choice of x allowing one of the zero components to become positive and vanishing one of the positive components and this in such a way that the restrictions $Ax = b$ and $x \geq 0$ are kept.

After a finite number of steps like Step 2, the value of $\langle c, x \rangle$ will have been tested at all possible minimal points. Obviously, the solution to the problem is obtained by choosing, among these points x, that one at which the minimum of $\langle c, x \rangle$ is attained.

Let us analyze the geometric meaning of the simplex method with an example.

Let us consider the problem of minimizing the function

$$10x_1 + 4x_2 + 7x_3$$

under the constraints

$$2x_1 + x_2 + x_3 = 1, \quad x_1, x_2, x_3 \geq 0.$$

In this case, the set of admissible triplets (x_1, x_2, x_3), i.e. those satisfying the constraints is the triangle in $\mathbf{R}^3$ of vertices $(0,0,1)$, $(0,1,0)$ and $(1/2,0,0)$ (a face of a tetrahedron). It is easy to see that the minimum is achieved at $(0,1,0)$, where the value is 4.

Let us try to give a geometrical explanation to this fact. Since $x_1, x_2, x_3 \geq 0$ for any admissible triplet, the minimum of the function $10x_1 + 4x_2 + 7x_3$ has necessarily to be nonnegative. Moreover, the minimum cannot be zero since the hyperplane

$$10x_1 + 4x_2 + 7x_3 = 0$$

has an empty intersection with the triangle of admissible states. When increasing the cost $10x_1 + 4x_2 + 7x_3$, i.e. when considering level sets of the form

$10x_1 + 4x_2 + 7x_3 = c$ with increasing $c > 0$, we are considering planes parallel to $10x_1 + 4x_2 + 7x_3 = 0$ that are getting away from the origin and closer to the triangle of admissible states. The first value of c for which the level set intersects the admissible triangle provides the minimum of the cost function and the point of contact is the minimizer.

It is immediate that this point is the vertex $(0, 1, 0)$.

These geometrical considerations indicate the relevance of the convexity of the set where the minimum is being searched. Recall that, in a linear space E, a set K is *convex* if it satisfies the following property:

$$x, y \in K, \quad \lambda \in [0, 1] \Rightarrow \lambda x + (1 - \lambda) y \in K.$$

The crucial role played by convexity will be also observed below, when considering more sophisticated problems.

The method of simplex, despite its simplicity, is very efficient. There are many variants, adapted to deal with particular problems. In some of them, when looking for the minimum, one runs across the convex set and not only along its boundary. For instance, this is the case of *Karmakar's method*, see [208]. For more information on Linear Programming, the method of simplex and its variants, see for instance [186].

As the reader can easily figure out, many problems of interest in Mathematical Programming concern the minimization of *nonlinear* functions. At this respect, let us recall the following fundamental result whose proof is the basis of the so called *Direct Method of the Calculus of Variations* (DMCV):

Theorem 1.5.1 *If H is a Hilbert space with norm $\left| \cdot \right|_H$ and the function $J : H \mapsto \mathbf{R}$ is continuous, convex and coercive in H, i.e. it satisfies*

$$J(v) \to +\infty \quad \text{as} \quad \left| v \right|_H \to +\infty, \tag{1.32}$$

then J attains its minimum at some point $u \in H$. If, moreover, J is strictly convex, this point is unique.

If, in the previous result, J is a C^1 function, any minimizer u necessarily satisfies

$$J'(u) = 0, \quad u \in H. \tag{1.33}$$

Usually, (1.33) is known as the *Euler equation* of the minimization problem

$$\text{Minimize } J(v) \text{ subject to } v \in H. \tag{1.34}$$

Consequently, if J is C^1, Theorem 1.5.1 serves to prove that the (generally nonlinear) Euler equation (1.33) possesses at least one solution.

Many systems arising in Continuum Mechanics can be viewed as the Euler equation of a minimization problem. Conversely, one can associate Euler equations to many minimization problems. This mutual relation can be used in both directions: either to solve differential equations by means of minimization techniques, or to solve minimization problems through the corresponding Euler equations.

In particular, this allows proving existence results of equilibrium configurations for many problems in Continuum Mechanics.

Furthermore, combining these ideas with the approximation of the space H where the minimization problem is formulated by means of finite dimensional spaces and increasing the dimension to cover in the limit the whole space H, one obtains *Galerkin's approximation method*. Suitable choices of the approximating subspaces lead to the *finite element methods*.

In order to illustrate these statements and connect them to Control Theory, let us consider the example

$$\begin{cases} \dot{x} = Ax + Bv, & t \in [0,T], \\ x(0) = x^0, \end{cases} \tag{1.35}$$

in which the *state* $x = (x_1(t), \dots, x_N(t))^t$ is a vector in $\mathbf{R}^N$ depending on t (the time variable) and the *control* $v = (v_1(t), \dots, v_M(t))^t$ is a vector with M components that also depends on time.

In (1.35), we will assume that A is a square, constant coefficient matrix of dimension $N \times N$, so that the underlying system is *autonomous*, i.e. invariant with respect to translations in time. The matrix B has also constant coefficients and dimension $N \times M$.

Let us set

$$J(v) = \frac{1}{2}|x(T) - x^1|^2 + \frac{\mu}{2}\int_0^T |v(t)|^2 \, dt \quad \forall v \in L^2(0,T;\mathbf{R}^M), \tag{1.36}$$

where $x^1 \in \mathbf{R}^N$ is given, $x(T)$ is the final value of the solution of (1.35) and $\mu > 0$.

It is not hard to prove that $J : L^2(0,T;\mathbf{R}^M) \mapsto \mathbf{R}$ is well defined, continuous, coercive and strictly convex. Consequently, J has a unique minimizer in $L^2(0,T;\mathbf{R}^M)$. This shows that the control problem (1.35)–(1.36) has a unique solution.

With the DMCV, the existence of minimizers for a large class of problems can be proved. But there are many other interesting problems that do not enter in this simple framework, for which minimizers do not exist.

Indeed, let us consider the simplest and most classical problem in the Calculus of Variations: to show that the shortest path between two given points is the straight line segment.

Of course, it is very easy to show this by means of geometric arguments. However,

What happens if we try to use the DMCV ?

The question is now to minimize the functional

$$\int_0^1 |\dot{x}(t)|\,dt$$

in the class of curves $x : [0,1] \mapsto \mathbf{R}^2$ such that $x(0) = P$ and $x(1) = Q$, where P and Q are two given points in the plane.

The natural functional space for this problem is not a Hilbert space. It can be the Sobolev space $W^{1,1}(0,1)$ constituted by all functions $x = x(t)$ such that x and its time derivative $\dot{x}$ belong to $L^1(0,1)$. It can also be the more sophisticated space $BV(0,1)$ of functions of bounded variation. But these are not Hilbert spaces and solving the problem in any of them, preferably in $BV(0,1)$, becomes much more subtle.

We have described the DMCV in the context of problems without constraints. Indeed, up to now, the functional has been minimized in the whole space. But in most realistic situations the nature of the problem imposes restrictions on the control and/or the state. This is the case for instance for the linear programming problems we have considered above.

As we mentioned above, convexity plays a key role in this context too:

Theorem 1.5.2 *Let H be a Hilbert space, $K \subset H$ a closed convex set and $J : K \mapsto \mathbf{R}$ a convex continuous function. Let us also assume that either K is bounded or J is coercive in K, i.e.*

$$J(v) \to +\infty \quad as v \in K, \quad \left|v\right|_H \to +\infty.$$

Then, there exists a point $u \in K$ where J reaches its minimum over K.

Furthermore, if J is strictly convex, the minimizer is unique.

In order to illustrate this result, let us consider again the system (1.35) and the functional

$$J(v) = \frac{1}{2}|x(T) - x^1|^2 + \frac{\mu}{2}\int_0^T |v(t)|^2\,dt \quad \forall v \in K, \tag{1.37}$$

where $\mu \geq 0$ and $K \subset L^2(0,T;\mathbf{R}^M)$ is a closed convex set. In view of Theorem 1.5.2, we see that, if $\mu > 0$, the optimal control problem determined by (1.35) and (1.37) has a unique solution. If $\mu = 0$ and K is bounded, this problem possesses at least one solution.

Let us discuss more deeply the application of these techniques to the analysis of the control properties of the linear finite dimensional system (1.35).

Let $J : H \mapsto \mathbf{R}$ be, for instance, a functional of class C^1. Recall again that, at each point u where J reaches its minimum, one has

$$J'(u) = 0, \quad u \in H. \tag{1.38}$$

It is also true that, when J is convex and C^1, if u solves (1.38) then u is a global minimizer of J in H. Equation (1.38) is the *Euler equation* of the corresponding minimization problem.

More generally, in a convex minimization problem, if the function to be minimized is of class C^1, an *Euler inequality* is satisfied by each minimizer. Thus, u is a minimizer of the convex functional J in the convex set K of the Hilbert space H if and only if

$$(J'(u), v - u)_H \geq 0 \quad \forall v \in K, \quad u \in K. \tag{1.39}$$

Here, $(\cdot\,,\cdot)_H$ stands for the scalar product in H.

In the context of Optimal Control, this characterization of u can be used to deduce the corresponding *optimality conditions*, also called the *optimality system*.

For instance, this can be made in the case of problem (1.35),(1.37). Indeed, it is easy to see that in this case (1.39) reduces to

$$\left\{ \begin{array}{l} \mu \displaystyle\int_0^T \langle u(t), v(t) - u(t)\rangle \, dt + \langle x(T) - x^1, z_v(T) - z_u(T)\rangle \geq 0 \\ \forall v \in K, \quad u \in K, \end{array} \right. \tag{1.40}$$

where, for each $v \in L^2(0, T; \mathbf{R}^M)$, $z_v = z_v(t)$ is the solution of

$$\left\{ \begin{array}{l} \dot{z}_v = Az_v + Bv, \quad t \in [0, T], \\ z_v(0) = 0 \end{array} \right.$$

(recall that $\langle \cdot\,, \cdot \rangle$ stands for the Euclidean scalar products in $\mathbf{R}^M$ and $\mathbf{R}^N$).

Now, let $p = p(t)$ be the solution of the backward in time differential problem

$$\left\{ \begin{array}{ll} -\dot{p} = A^t p, & t \in [0, T], \\ p(T) = x(T) - x^1. & \end{array} \right. \tag{1.41}$$

Then

$$\langle x(T) - x^1, z_v(T) - z_u(T)\rangle = \langle p(T), z_v(T) - z_u(T)\rangle = \int_0^T \langle p(t), B(v(t) - u(t))\rangle \, dt$$

and (1.40) can also be written in the form:

$$\begin{cases} \displaystyle\int_0^T \langle \mu u(t) + B^t p(t), v(t) - u(t) \rangle \, dt \geq 0 \\ \forall v \in K, \quad u \in K. \end{cases} \tag{1.42}$$

The system constituted by the state equation (1.35) for $v = u$, i.e.

$$\begin{cases} \dot{x} = Ax + Bu, \quad t \in [0, T], \\ x(0) = x^0, \end{cases} \tag{1.43}$$

the *adjoint state equation* (1.41) and the inequalities (1.42) is referred to as the *optimality system*. This system provides, in the case under consideration, a characterization of the optimal control.

The function $p = p(t)$ is the *adjoint state*. As we have seen, the introduction of p leads to a rewriting of (1.40) that is more explicit and easier to handle.

Very often, when addressing optimization problems, we have to deal with *restrictions* or *constraints* on the controls and/or state. *Lagrange multipliers* then play a fundamental role and are needed in order to write the equations satisfied by the minimizers: the so called *Euler-Lagrange equations*.

To do that, we must introduce the associated *Lagrangian* and, then, we must analyze its *saddle points*. The determination of saddle points leads to two equivalent extremal problems of dual nature.

This is a surprising fact in this theory that can be often used with efficiency: the original minimization problem being difficult to solve, one may often write a *dual minimization problem* (passing through the Lagrangian); it may well happen to the second problem to be simpler than the original one.

Saddle points arise naturally in many optimization problems. But they can also be viewed as the solutions of *minimax problems*. Minimax problems arise in many contexts, for instance:

- In *Differential Game Theory*, where two or more players *compete* trying to maximize their profit and minimize the one of the others.

- In the characterization of the proper vibrations of elastic bodies. Indeed, very often these can be characterized as eigenvalues of a self-adjoint compact operator in a Hilbert space through a minimax principle related to the *Rayleigh quotient*.

One of the most relevant contributions in this field was the one by J. Von Neumann in the middle of the XX Century, proving that the existence of a minimax is guaranteed under very weak conditions.

In the last three decades, these results have been used systematically for solving nonlinear differential problems, in particular with the help of the *Mountain Pass Lemma* (for instance, see [121]). At this respect, it is worth mentioning that a mountain pass is indeed a beautiful example of saddle point provided by Nature. A mountain pass is the location one chooses to cross a mountain chain: this point must be of minimal height along the mountain chain but, on the contrary, it is of maximal height along the crossing path we follow.

The reader interested in learning more about Convex Analysis and the related duality theory is referred to the books [70] and [191], by I. Ekeland and R. Temam and R.T. Rockafellar, respectively. The lecture notes by B. Larrouturou and P.L. Lions [129] contain interesting introductions to these and other related topics, like mathematical modelling, the theory of partial differential equations and numerical approximation techniques.

1.6 Controllability of linear finite dimensional systems

We will now be concerned with the controllability of ordinary differential equations. We will start by considering linear systems.

As we said above, Control Theory is full of interesting mathematical results that have had a tremendous impact in the world of applications (most of them are too complex to be reproduced in these notes). One of these important results, simple at the same time, is a theorem by R.E. Kalman which characterizes the linear systems that are controllable.

Let us consider again the linear system

$$\left\{\begin{array}{l} \dot{x} = Ax + Bv, \quad t > 0, \\ x(0) = x^0, \end{array}\right. \tag{1.44}$$

with state $x = (x_1(t), \ldots, x_N(t))^t$ and control $v = (v_1(t), \ldots, v_M(t))^t$. The matrices A and B have constant coefficients and dimensions $N \times N$ and $N \times M$, respectively.

Assume that $N \geq M \geq 1$. In practice, the cases where M is much smaller than N are especially significant. Of course, the most interesting case is that in which $M = 1$ and, simultaneously, N is very large. We then dispose of a single scalar control to govern the behavior of a very large number N of components of the state.

System (1.44) is said to be controllable at time $T > 0$ if, for every initial state $x^0 \in \mathbf{R}^N$ and every final state $x^1 \in \mathbf{R}^N$, there exists at least one control $u \in C^0([0,T]; \mathbf{R}^M)$ such that the associated solution satisfies

$$x(T) = x^1. \tag{1.45}$$

The following result, due to Kalman, characterizes the controllability of (1.44) (see for instance [136]):

Theorem 1.6.1 *A necessary and sufficient condition for system* (1.44) *to be controllable at some time* $T > 0$ *is that*

$$\text{rank } \left[B \,|\, AB \,|\, \cdots \,|\, A^{N-1}B \right] = N. \tag{1.46}$$

Moreover, if this is satisfied, the system is controllable for all $T > 0$.

When the rank of this matrix is k, *with* $1 \leq k \leq N-1$, *the system is not controllable and, for each* $x^0 \in \mathbf{R}^N$ *and each* $T > 0$, *the set of solutions of* (1.44) *at time* $T > 0$ *covers an affine subspace of* $\mathbf{R}^N$ *of dimension* k.

The following remarks are now in order:

- The degree of controllability of a system like (1.44) is completely determined by the rank of the corresponding matrix in (1.46). This rank indicates how many components of the system are sensitive to the action of the control.
- The matrix in (1.46) is of dimension $(N \times M) \times N$ so that, when we only have one control at our disposal (i.e. $M = 1$), this is a $N \times N$ matrix. It is obviously in this case when it is harder to the rank of this matrix to be N. This is in agreement with common sense, since the system should be easier to control when the number of controllers is larger.
- The system is controllable at some time if and only if it is controllable at any positive time. In some sense, this means that, in (1.44), *information propagates at infinite speed.* Of course, this property is not true in general in the context of partial differential equations.

As we mentioned above, the concept of *adjoint system* plays an important role in Control Theory. In the present context, the adjoint system of (1.44) is the following:

$$\left\{ \begin{array}{ll} -\dot{\varphi} = A^t \varphi, & t < T, \\ \varphi(T) = \varphi^0. \end{array} \right. \tag{1.47}$$

Let us emphasize the fact that (1.47) is a backward (in time) system. Indeed, in (1.47) the sense of time has been reversed and the differential system has been completed with a *final condition* at time $t = T$.

The following result holds:

Theorem 1.6.2 *The rank of the matrix in* (1.46) *is* N *if and only if, for every* $T > 0$, *there exists a constant* $C(T) > 0$ *such that*

$$|\varphi^0|^2 \leq C(T) \int_0^T |B^t \varphi|^2 \, dt \tag{1.48}$$

for every solution of (1.47).

The inequality (1.48) is called an *observability inequality*. It can be viewed as the *dual* version of the controllability property of system (1.44).

This inequality guarantees that the adjoint system can be "observed" through $B^t\varphi$, which provides M linear combinations of the adjoint state. When (1.48) is satisfied, we can affirm that, from the controllability viewpoint, B^t captures appropriately all the components of the adjoint state φ. This turns out to be equivalent to the controllability of (1.44) since, in this case, the control u acts efficiently through the matrix B on all the components of the state x.

Inequalities of this kind play also a central role in *inverse problems*, where the goal is to reconstruct the properties of an unknown (or only partially known) medium or system by means of partial measurements. The observability inequality guarantees that the measurements $B^t\varphi$ are sufficient to detect all the components of the system.

The proof of Theorem 1.6.2 is quite simple. Actually, it suffices to write the solutions of (1.44) and (1.47) using the *variation of constants formula* and, then, to apply the *Cayley-Hamilton theorem*, that guarantees that any matrix is a root of its own characteristic polynomial.

Thus, to prove that (1.46) implies (1.48), it is sufficient to show that, when (1.46) is true, the mapping

$$\varphi^0 \mapsto \left(\int_0^T |B^t\varphi|^2\,dt \right)^{1/2}$$

is a norm in $\mathbf{R}^N$. To do that, it suffices to check that the following *uniqueness* or *unique continuation* result holds:

If $B^t\varphi = 0$ *for* $0 \le t \le T$ *then, necessarily,* $\varphi \equiv 0$.

It is in the proof of this result that the rank condition is needed.

Let us now see how, using (1.48), we can build controls such that the associated solutions to (1.44) satisfy (1.45). This will provide another idea of how controllability and optimal control problems are related.

Given initial and final states x^0 and x^1 and a control time $T > 0$, let us consider the quadratic functional I, with

$$I(\varphi^0) = \frac{1}{2}\int_0^T |B^t\varphi|^2\,dt - \langle x^1, \varphi^0\rangle + \langle x^0, \varphi(0)\rangle \quad \forall \varphi^0 \in \mathbf{R}^N, \tag{1.49}$$

where φ is the solution of the adjoint system (1.47) associated to the final state φ^0.

The function $\varphi^0 \mapsto I(\varphi^0)$ is strictly convex and continuous in $\mathbf{R}^N$. In view of (1.48), it is also coercive, that is,

$$\lim_{|\varphi^0|\to\infty} I(\varphi^0) = +\infty. \tag{1.50}$$

Therefore, I has a unique minimizer in $\mathbf{R}^N$, that we shall denote by $\hat{\varphi}^0$. Let us write the Euler equation associated to the minimization of the functional (1.49):

$$\int_0^T \langle B^t\hat{\varphi}, B^t\varphi\rangle\, dt - \langle x^1, \varphi^0\rangle + \langle x^0, \varphi(0)\rangle = 0 \quad \forall\varphi^0 \in \mathbf{R}^N, \quad \hat{\varphi}^0 \in \mathbf{R}^N. \quad (1.51)$$

Here, $\hat{\varphi}$ is the solution of the adjoint system (1.47) associated to the final state $\hat{\varphi}^0$.

From (1.51), we deduce that $\hat{u} = B^t\hat{\varphi}$ is a control for (1.44) that guarantees that (1.45) is satisfied. Indeed, if we denote by $\hat{x}$ the solution of (1.44) associated to $\hat{u}$, we have that

$$\int_0^T \langle B^t\hat{\varphi}, B^t\varphi\rangle\, dt = \langle \hat{x}(T), \varphi^0\rangle - \langle x^0, \varphi(0)\rangle \qquad \forall\varphi^0 \in \mathbf{R}^N. \quad (1.52)$$

Comparing (1.51) and (1.52), we see that the previous assertion is true.

It is interesting to observe that, from the rank condition, we can deduce several variants of the observability inequality (1.48). In particular,

$$|\varphi^0| \le C(T)\int_0^T |B^t\varphi|\, dt \quad (1.53)$$

This allows us to build controllers of different kinds.

Indeed, consider for instance the functional J_{bb}, given by

$$J_{bb}(\varphi^0) = \frac{1}{2}\left(\int_0^T |B^t\varphi|\, dt\right)^2 - \langle x^1, \varphi^0\rangle + \langle x^0, \varphi(0)\rangle \quad \forall\varphi^0 \in \mathbf{R}^N. \quad (1.54)$$

This is again strictly convex, continuous and coercive. Thus, it possesses exactly one minimizer $\hat{\varphi}^0_{bb}$. Let us denote by $\hat{\varphi}_{bb}$ the solution of the corresponding adjoint system. Arguing as above, it can be seen that the new control $\hat{u}_{bb}$, with

$$\hat{u}_{bb} = \left(\int_0^T |B^t\hat{\varphi}_{bb}|\, dt\right)\operatorname{sgn}(B^t\hat{\varphi}_{bb}), \quad (1.55)$$

makes the solution of (1.44) satisfy (1.45). This time, we have built a *bang-bang* control, whose components can only take two values:

$$\pm\int_0^T |B^t\hat{\varphi}_{bb}|\, dt.$$

The control $\hat{u}$ that we have obtained minimizing J is the one of minimal norm in $L^2(0, T; \mathbf{R}^M)$ among all controls guaranteeing (1.45). On the other hand, $\hat{u}_{bb}$ is the control of minimal L^∞ norm. The first one is smooth and the second one is piecewise constant and, therefore, discontinuous in general. However, the bang-bang control is easier to compute and apply since, as we saw explicitly in the case of the pendulum, we only need to determine its amplitude and the location of the switching points. Both controls $\hat{u}$ and $\hat{u}_{bb}$ are optimal with respect to some optimality criterium.

We have seen that, in the context of linear control systems, when controllability holds, the control may be computed by solving a minimization problem. This is also relevant from a computational viewpoint since it provides useful ideas to design efficient approximation methods.

1.7 Controllability of nonlinear finite dimensional systems

Let us now discuss the controllability of some nonlinear control systems. This is a very complex topic and it would be impossible to describe in a few pages all the significant results in this field. We will just recall some basic ideas.

When the goal is to produce small variations or deformations of the state, it might be sufficient to proceed using linearization arguments. More precisely, let us consider the system

$$\begin{cases} \dot{x} = f(x,u), \quad t>0, \\ x(0) = x^0, \end{cases} \tag{1.56}$$

where $f : \mathbf{R}^N \times \mathbf{R}^M \mapsto \mathbf{R}^N$ is smooth and $f(0,0)=0$. The linearized system at $u=0$, $x=0$ is the following:

$$\begin{cases} \dot{x} = \dfrac{\partial f}{\partial x}(0,0)x + \dfrac{\partial f}{\partial u}(0,0)u, \quad t>0, \\ x(0) = 0. \end{cases} \tag{1.57}$$

Obviously, (1.57) is of the form (1.44), with

$$A = \frac{\partial f}{\partial x}(0,0), \quad B = \frac{\partial f}{\partial u}(0,0), \quad x^0 = 0. \tag{1.58}$$

Therefore, the rank condition

$$\text{rank}\,[B|AB|\cdots|A^{N-1}B] = N \tag{1.59}$$

is the one that guarantees the controllability of (1.57).

Based on the *inverse function theorem*, it is not difficult to see that, if condition (1.59) is satisfied, then (1.56) is *locally controllable* in the following sense:

> *For every $T > 0$, there exists a neighborhood $\mathcal{B}_T$ of the origin in $\mathbf{R}^N$ such that, for any initial and final states $x_0, x_1 \in \mathcal{B}_T$, there exist controls u such that the associated solutions of the system* (1.56) *satisfy*
>
> $$x(T) = x^1 . \tag{1.60}$$

However, this analysis is not sufficient to obtain results of global nature.

A *natural* condition that can be imposed on the system (1.56) in order to guarantee global controllability is that, at each point $x^0 \in \mathbf{R}^N$, by choosing all admissible controls $u \in \mathcal{U}_{\mathrm{ad}}$, we can recover deformations of the state in all the directions of $\mathbf{R}^N$. But,

> Which are the directions in which the state x can be deformed starting from x^0 ?

Obviously, the state can be deformed in all directions $f(x_0, u)$ with $u \in \mathcal{U}_{\mathrm{ad}}$. But these are not all the directions of $\mathbf{R}^N$ when $M < N$. On the other hand, as we have seen in the linear case, there exist situations in which $M < N$ and, at the same time, controllability holds thanks to the rank condition (1.59).

In the nonlinear framework, the directions in which the state may be deformed around x^0 are actually those belonging to the *Lie algebra* generated by the vector fields $f(x^0, u)$, when u varies in the set of admissible controls $\mathcal{U}_{\mathrm{ad}}$. Recall that the Lie algebra $\mathcal{A}$ generated by a family $\mathcal{F}$ of regular vector fields is the set of *Lie brackets* $[f, g]$ with $f, g \in \mathcal{F}$, where

$$[f, g] = (\nabla g)f - (\nabla f)g$$

and all the fields that can be obtained iterating this process of computing Lie brackets.

The following result can be proved (see [206]):

Theorem 1.7.1 *Assume that, for each x^0, the Lie algebra generated by $f(x^0, u)$ with $u \in \mathcal{U}_{\mathrm{ad}}$ coincides with $\mathbf{R}^N$. Then* (1.56) *is controllable, i.e. it can be driven from any initial state to any final state in a sufficiently large time.*

The following simple model of driving a car provides a good example to apply these ideas.

Thus, let us consider a state with four components $x = (x_1, x_2, x_3, x_4)$ in which the first two, x_1 and x_2, provide the coordinates of the center of the axis $x_2 = 0$ of the vehicle, the third one, $x_3 = \varphi$, is the counterclockwise angle of the car with respect to the half axis $x_1 > 0$ and the fourth one, $x_4 = \theta$, is the angle of the front wheels with respect to the axis of the car. For simplicity, we will assume that the distance from the front to the rear wheels is $\ell = 1$.

The front wheels are then parallel to the vector $(\cos(\theta+\varphi), \sin(\theta+\varphi))$, so that the instantaneous velocity of the center of the front axis is parallel to this vector. Accordingly,

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = u_2(t)\begin{pmatrix} \cos(\theta+\varphi) \\ \sin(\theta+\varphi) \end{pmatrix}$$

for some scalar function $u_2 = u_2(t)$.

The center of the rear axis is the point $(x_1 - \cos\varphi,\ x_2 - \sin\varphi)$. The velocity of this point has to be parallel to the orientation of the rear wheels $(\cos\varphi,\ \sin\varphi)$, so that

$$(\sin\varphi)\frac{d}{dt}(x_1 - \cos\varphi) - (\cos\varphi)\frac{d}{dt}(x_2 - \sin\varphi) = 0.$$

In this way, we deduce that

$$\dot{\varphi} = u_2 \sin\theta.$$

On the other hand, we set

$$\dot{\theta} = u_1$$

and this reflects the fact that the velocity at which the angle of the wheels varies is the second variable that we can control. We obtain the following reversible system:

$$\dot{x} = u_1(t)\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} + u_2(t)\begin{pmatrix} \cos(\varphi+\theta) \\ \sin(\varphi+\theta) \\ \sin\theta \\ 0 \end{pmatrix}. \tag{1.61}$$

According to the previous analysis, in order to guarantee the controllability of (1.61), it is sufficient to check that the Lie algebra of the directions in which the control may be deformed coincides with $\mathbf{R}^4$ at each point.

With $(u_1, u_2) = (0, 1)$ and $(u_1, u_2) = (1, 0)$, we obtain the directions

$$\begin{pmatrix} \cos(\varphi+\theta) \\ \sin(\varphi+\theta) \\ \sin\theta \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \tag{1.62}$$

respectively. The corresponding Lie bracket provides the direction

$$\begin{pmatrix} -\sin(\varphi+\theta) \\ \cos(\varphi+\theta) \\ \cos\theta \\ 0 \end{pmatrix}, \tag{1.63}$$

whose Lie bracket with the first one in (1.62) provides the new direction

$$\begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \\ 0 \end{pmatrix}. \tag{1.64}$$

Taking into account that the determinant of the matrix formed by the four column vectors in (1.62), (1.63) and (1.64) is identically equal to 1, we deduce that, at each point, the set of directions in which the state may be deformed is the whole $\mathbf{R}^4$.

Thus, system (1.61) is controllable.

It is an interesting exercise to think on how one uses in practice the four vectors (1.62) – (1.64) to park a car. The reader interested in getting more deeply into this subject may consult the book by E. Sontag [206].

The analysis of the controllability of systems governed by partial differential equations has been the objective of a very intensive research the last decades. However, the subject is older than that.

In 1978, D.L. Russell [194] made a rather complete survey of the most relevant results that were available in the literature at that time. In that paper, the author described a number of different tools that were developed to address controllability problems, often inspired and related to other subjects concerning partial differential equations: multipliers, moment problems, nonharmonic Fourier series, etc. More recently, J.L. Lions introduced the so called *Hilbert Uniqueness Method* (H.U.M.; for instance, see [142, 143]) and this was the starting point of a fruitful period on the subject.

In this context, which is the usual for modelling problems from Continuum Mechanics, one needs to deal with infinite dimensional dynamical systems and this introduces a lot of nontrivial difficulties to the theory and raises many relevant and mathematically interesting questions. Furthermore, the solvability of the problem depends very much on the nature of the precise question under consideration and, in particular, the following features may play a crucial role: linearity or nonlinearity of the system, time reversibility, the structure of the set of admissible controls, etc.

For more details, the reader is referred to the books [126] and [130] and the survey papers [82], [237] and [235].

1.8 Control, complexity and numerical simulation

Real life systems are genuinely complex. *Internet*, the large quantity of components entering in the fabrication of a car or the decoding of human genoma are good examples of this fact.

The algebraic system (1.23) considered in Section 5 is of course academic but it suffices by itself to show that not all the components of the state are always sensitive to the chosen control. One can easily imagine how dramatic can the situation be when dealing with complex (industrial) systems. Indeed, determining whether a given controller allows to act on all the components of a system may be a very difficult task.

But complexity does not only arise for systems in Technology and Industry. It is also present in Nature. At this respect, it is worth recalling the following anecdote. In 1526, the Spanish King "Alfonso X El Sabio" got into the *Alcázar of Segovia* after a violent storm and exclaimed:

> "If God had consulted me when He was creating the world, I would have recommended a simpler system."

Recently we have learned about a great news, a historical achievement of Science: the complete decoding of human *genoma*. The genoma code is a good proof of the complexity which is intrinsic to life. And, however, one has not to forget that, although the decoding has been achieved, there will still be a lot to do before being able to use efficiently all this information for medical purposes.

Complexity is also closely related to numerical simulation. In practice, any efficient control strategy, in order to be implemented, has to go through numerical simulation. This requires discretizing the control system, which very often increases its already high complexity.

The recent advances produced in Informatics allow nowadays to use numerical simulation at any step of an industrial project: conception, development and qualification. This relative success of numerical methods in Engineering versus other traditional methods relies on the facts that the associated experimental costs are considerably lower and, also, that numerical simulation allows testing at the realistic scale, without the technical restrictions motivated by instrumentation.

This new scientific method, based on a combination of Mathematics and Informatics, is being seriously consolidated. Other Sciences are also closely involved in this melting, since many mathematical models stem from them: Mechanics, Physics, Chemistry, Biology, Economics, etc. Thus, we are now able to solve more sophisticated problems than before and the complexity of the systems we will be able to solve in the near future will keep increasing. Thanks in particular to parallelization techniques, the description and numerical simulation of complex systems in an acceptable time is more and more feasible.

However, this panorama leads to significant and challenging difficulties that we are now going to discuss.

The first one is that, in practice, the systems under consideration are in fact the coupling of several complex subsystems. Each of them has its own dynamics but the coupling may produce new and unexpected phenomena due to their interaction.

An example of this situation is found in the mathematical description of reactive fluids which are used, for instance, for the propulsion of spatial vehicles. For these systems, one has to perform a modular analysis, separating and simulating numerically each single element and, then, assembling the results. But this is a major task and much has still to be done[2].

There are many relevant examples of complex systems for which coupling can be the origin of important difficulties. In the context of *Aerospatial Technology*, besides the combustion of reactive fluids, we find fluid-structure interactions which are extremely important when driving the craft, because of the vibrations originated by combustion. Other significant examples are weather prediction and Climatology, where the interactions of atmosphere, ocean, earth, etc. play a crucial role. A more detailed description of the present situation of research and perspectives at this respect can be found in the paper [1], by J. Achache and A. Bensoussan.

In our context, the following must be taken into account:

- Only complex systems are actually relevant from the viewpoint of applications.
- Furthermore, in order to solve a relevant problem, we must first identify the various subsystems and the way they interact.

Let us now indicate some of the mathematical techniques that have been recently developed (and to some extent re-visited) to deal with complexity and perform the appropriate decomposition of large systems that we have mentioned as a need:

- **The solution of linear systems.**

  When the linear system we have to solve presents a block-sparse structure, it is convenient to apply methods combining appropriately the *local* solution of the subsystems corresponding to the individual blocks. This is a frequent situation when dealing with finite difference or finite element discretizations of a differential system.

  The most usual way to proceed is to introduce *preconditioners*, determined by the solutions to the subsystems, each of them being computed with one processor and, then, to perform the global solution with parallelized iterative methods.

[2] To understand the level of difficulty, it is sufficient to consider a hybrid parabolic-hyperbolic system and try to match the numerical methods obtained with a finite difference method in the hyperbolic component and a finite element method in the parabolic one.

- **Multigrid methods.**

  These are very popular today. Assume we are considering a linear system originated by the discretization of a differential equation. The main idea of a multigrid method is to "separate" the low and the high frequencies of the solution in the computation procedure. Thus we compute approximations of the solution at different levels, for instance working alternately with a coarse and a fine grid and incorporating adequate coupling mechanisms.

  The underlying reason is that any grid, even if it is very fine, is unable to capture sufficiently high frequency oscillations, just as an ordinary watch is unable to measure microseconds.

- **Domain decomposition methods.**

  Now, assume that (1.1) is a boundary value problem for a partial differential equation in the N-dimensional domain Ω. If Ω has a complex geometrical structure, it is very natural to decompose (1.1) in several similar systems written in simpler domains.

  This can be achieved with domain decomposition techniques. The main idea is to split $\overline{\Omega}$ in the form

  $$\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2} \cup \cdots \cup \overline{\Omega_m}\,, \tag{1.65}$$

  and to introduce then an iterative scheme based on computations on each Ω_i separately.

  Actually, this is not new. Some seminal ideas can be found in Volume II of the book [60] by R. Courant and D. Hilbert. Since then, there have been lots of works on domain decomposition methods applied to partial differential systems (see for instance [131]). However, the role of these methods in the solution of control problems has not still been analyzed completely.

- **Alternating direction methods.**

  Frequently, we have to consider models involving time-dependent partial differential equations in several space dimensions. After standard time discretization, one is led at each time step to a set of (stationary) partial differential problems whose solution, in many cases, is difficult to achieve.

  This is again connected to the need of decomposing complex systems in more simple subsystems. These ideas lead to the methods of alternating directions, of great use in this context. A complete analysis can be found in [223]. In the particular, but very illustrating context of the Navier-Stokes equations, these methods have been described for instance in [94] and [182].

However, from the viewpoint of Control Theory, alternating direction methods have not been, up to now, sufficiently explored.

The interaction of the various components of a complex system is also a difficulty of major importance in control problems. As we mentioned above, for real life control problems, we have first to choose an appropriate model and then we have also to make a choice of the control property. But necessarily one ends up introducing numerical discretization algorithms to make all this computable. Essentially, we will have to be able to compute an accurate approximation of the control and this will be made only if we solve numerically a *discretized control problem.*

At this point, let us observe that, as mentioned in [231], some models obtained after discretization (for instance via the finite element method) are not only relevant regarded as approximations of the underlying continuous models but also by themselves, as genuine models of the real physical world[3].

Let us consider a simple example in order to illustrate some extra, somehow unexpected, difficulties that the discretization may bring to the control process.

Consider again the state equation (1.1). To fix ideas, we will assume that our control problem is as follows

To find $u \in \mathcal{U}_{\rm ad}$ *such that*

$$\Phi(u, y(u)) \leq \Phi(v, y(v)) \quad \forall v \in \mathcal{U}_{\rm ad}\,, \tag{1.66}$$

where $\Phi = \Phi(v, y)$ *is a given function.*

Then, we are led to the following crucial question:

What is an appropriate discretized control problem ?

There are at least two reasonable possible answers:

- **First approximation method.**

  We first discretize $\mathcal{U}_{\rm ad}$ and (1.1) and obtain $\mathcal{U}_{{\rm ad},h}$ and the new (discrete) state equation

$$A_h(y_h) = f(v_h). \tag{1.67}$$

  Here, h stands for a small parameter that measures the *characteristic size* of the "numerical mesh". Later, we let $h \to 0$ to make the discrete problem converge to the continuous one. If $\mathcal{U}_{{\rm ad},h}$ and (1.67) are introduced

[3] The reader can find in [231] details on how the finite element method was born around 1960. In this article it is also explained that, since its origins, finite elements have been viewed as a tool to build legitimate discrete models for the mechanical systems arising in Nature and Engineering, as well as a method to approximate partial differential systems.

the right way, we can expect to obtain a "discrete state" $y_h(v_h)$ for each "discrete admissible" control $v_h \in \mathcal{U}_{\mathrm{ad},h}$.

Then, we search for an optimal control at the discrete level, i.e. a control $u_h \in \mathcal{U}_{\mathrm{ad},h}$ such that

$$\Phi(u_h, y_h(u_h)) \leq \Phi(v_h, y_h(v_h)) \quad \forall v_h \in \mathcal{U}_{\mathrm{ad},h}. \tag{1.68}$$

This corresponds to the following scheme:

$$\text{MODEL} \longrightarrow \text{DISCRETIZATION} \longrightarrow \text{CONTROL}.$$

Indeed, starting from the continuous control problem, we first discretize it and we then compute the control of the discretized model. This provides a first natural method for solving in practice the control problem.

- **Second approximation method.**

  However, we can also do as follows. We analyze the original control problem (1.1), (1.66) and we characterize the optimal solution and control in terms of an *optimality system*. We have already seen that, in practice, this is just to write the Euler or Euler-Lagrange equations associated to the minimization problem we are dealing with. We have already described how optimality systems can be found for some particular control problems.

  The optimality systems are of the form

$$A(y) = f(u), \quad B(y)p = g(u, y) \tag{1.69}$$

  (where $B(y)$ is a linear operator), together with an additional equation relating u, y and p. To simplify our exposition, let us assume that the latter can be written in the form

$$Q(u, y, p) = 0 \tag{1.70}$$

  for some mapping Q. The key point is that, if u, y and p solve the optimality system (1.69) – (1.70), then u is an optimal control and y is the associate state. Of course, p is the *adjoint state* associated to u and y.

  Then, we can discretize and solve numerically (1.69),(1.70). This corresponds to a different approach:

$$\text{MODEL} \longrightarrow \text{CONTROL} \longrightarrow \text{DISCRETIZATION}.$$

> Notice that, in this second approach, we have interchanged the control and discretization steps. Now, we first analyze the continuous control problem and, only later, we proceed to the numerical discretization.

It is not always true that these two methods provide the same results.

For example, it is shown in [113] that, with a finite element approximation, the first one may give erroneous results in vibration problems. This is connected to the lack of accuracy of finite elements in the computation of high frequency solutions to the wave equation, see [231][4].

On the other hand, it has been observed that, for the solution of a lot of *optimal design problems*, the first strategy is preferable; see for instance [167] and [181].

The commutativity of the DISCRETIZATION/CONTROL scheme is at present a subject that is not well understood and requires further investigation. We do not still have a significant set of results allowing to determine when these two approaches provide similar results and when they do not. Certainly, the answer depends heavily on the nature of the model under consideration. In this sense, control problems for elliptic and parabolic partial differential equations, because of their intrinsic dissipative feature, will be better behaved than hyperbolic systems. We refer the interested reader to [242] for a complete account of this fact. It is however expected that much progress will be made in this context in the near future.

1.9 Two challenging applications

In this Section, we will mention two control problems whose solution will probably play an important role in the context of applications in the near future.

1.9.1 Molecular control via laser technology

We have already said that there are many technological contexts where Control Theory plays a crucial role. One of them, which has had a very recent development and announces very promising perspectives, is the *laser control of chemical reactions*.

The basic principles used for the control of industrial processes in Chemistry have traditionally been the same for many years. Essentially, the strategies have been (a) to introduce changes in the temperature or pressure in the reactions and (b) to use *catalyzers*.

[4]Nevertheless, the disagreement of these two methods may be relevant not only as a purely numerical phenomenon but also at the level of modelling since, as we said above, in many engineering applications discrete models are often directly chosen.

Laser technology, developed in the last four decades, is now playing an increasingly important role in molecular design. Indeed, the basic principles in *Quantum Mechanics* rely on the wave nature of both light and matter. Accordingly, it is reasonable to believe that the use of laser will be an efficient mechanism for the control of chemical reactions.

The experimental results we have at our disposal at present allow us to expect that this approach will reach high levels of precision in the near future. However, there are still many important technical difficulties to overcome.

For instance, one of the greatest drawbacks is found when the molecules are "not very isolated". In this case, collisions make it difficult to define their phases and, as a consequence, it is very hard to choose an appropriate choice of the control. A second limitation, of a much more technological nature, is related to the design of lasers with well defined phases, not too sensitive to the instabilities of instruments.

For more details on the modelling and technological aspects, the reader is referred to the expository paper [25] by P. Brumer and M. Shapiro.

The goal of this subsection is to provide a brief introduction to the mathematical problems one finds when addressing the control of chemical reactions.

Laser control is a subject of high interest where Mathematics are not sufficiently developed. The models needed to describe these phenomena lead to complex (nonlinear) *Schrödinger equations* for which the results we are able to deduce are really poor at present. Thus,

- We do not dispose at this moment of a complete theory for the corresponding initial or initial/boundary value problems.

- Standard numerical methods are not sufficiently efficient and, accordingly, it is difficult to test the accuracy of the models that are by now available.

The control problems arising in this context are *bilinear*. This adds fundamental difficulties from a mathematical viewpoint and makes these problems extremely challenging. Indeed, we find here genuine nonlinear problems for which, apparently, the existing linear theory is insufficient to provide an answer in a first approach.

In fact, it suffices to analyze the most simple bilinear control problems where wave phenomena appear to understand the complexity of this topic. Thus, let us illustrate this situation with a model concerning the linear one-dimensional Schrödinger equation. It is clear that this is insufficient by itself to describe all the complex phenomena arising in molecular control via laser technology. But it suffices to present the main mathematical problem and difficulties arising in this context.

The system is the following:

$$\begin{cases} i\phi_t + \phi_{xx} + p(t)x\,\phi = 0 & 0 < x < 1, \quad 0 < t < T, \\ \phi(0,t) = \phi(1,t) = 0, & 0 < t < T, \\ \phi(x,0) = \phi^0(x), & 0 < x < 1. \end{cases} \tag{1.71}$$

In (1.71), $\phi = \phi(x,t)$ is the *state* and $p = p(t)$ is the control. Although ϕ is complex-valued, $p(t)$ is real for all t. The control p can be interpreted as the intensity of an applied electrical field and x is the (prescribed) direction of the laser.

The state $\phi = \phi(x,t)$ is the wave function of the molecular system. It can be regarded as a function that furnishes information on the location of an elementary particle: for arbitrary a and b with $0 \leq a < b \leq 1$, the quantity

$$P(a,b;t) = \int_a^b |\phi(x,t)|^2\,dx$$

can be viewed as the probability that the particle is located in (a,b) at time t.

The controllability problem for (1.71) is to find the set of attainable states $\phi(\cdot,T)$ at a final time T as p runs over the whole space $L^2(0,T)$.

It is worth mentioning that, contrarily to what happens to many other control problems, the set of attainable states at time T depends strongly on the initial data ϕ^0. In particular, when $\phi^0 = 0$ the unique solution of (1.71) is $\phi \equiv 0$ whatever p is and, therefore, the unique attainable state is $\phi(\cdot,T) \equiv 0$. It is thus clear that, if we want to consider a nontrivial situation, we must suppose that $\phi^0 \neq 0$.

We say that this is a *bilinear control problem*, since the unique nonlinearity in the model is the term $p(t)x\,\phi$, which is essentially the product of the control and the state. Although the nonlinearity might seem simple, this control problem becomes rather complex and out of the scope of the existing methods in the literature.

For an overview on the present state of the art of the control of systems governed by the Schrödinger equation, we refer to the survey article [243] and the references therein.

1.9.2 An environmental control problem

For those who live and work on the seaside or next to a river, the relevance of being able to predict drastic changes of weather or on the state of the sea is obvious. In particular, it is vital to predict whether flooding may arise, in order to be prepared in time.

Floodings are one of the most common environmental catastrophic events and cause regularly important damages in several regions of our planet. They are produced as the consequence of very complex interactions of tides, waves and storms. The varying wind and the fluctuations of the atmospherical pressure produced by a storm can be the origin of an elevation or descent of several meters of the sea level in a time period that can change from several hours to two or three days. The wind can cause waves of a period of 20 seconds and a wavelenght of 20 or 30 meters. The simultaneous combination of these two phenomena leads to a great risk of destruction and flooding.

The amplitude of the disaster depends frequently on the possible accumulation of factors or events with high tides. Indeed, when this exceptional elevation of water occurs during a high tide, the risk of flooding increases dangerously.

This problem is being considered increasingly as a priority by the authorities of many cities and countries. Indeed, the increase of temperature of the planet and the melting of polar ice are making these issues more and more relevant for an increasing population in all the continents.

For instance, it is well known that, since the Middle Age, regular floods in the Thames river cover important pieces of land in the city of London and cause tremendous damages to buildings and population.

When floods occur in the Thames river, the increase on the level of water can reach a height of 2 meters. On the other hand, the average level of water at the London bridge increases at a rate of about 75 centimeters per century due to melting of polar ice. Obviously, this makes the problem increasingly dangerous.

Before explaining how the British authorities have handled this problem, it is important to analyze the process that lead to these important floods.

It is rather complex. Indeed, low atmospheric pressures on the Canadian coast may produce an increase of about 30 centimeters in the average sea level in an area of about 1 600 square kilometers approximately. On the other hand, due to the north wind and ocean currents, this tremendous mass of water may move across the Atlantic Ocean at a velocity of about 80 to 90 kilometers per day to reach the coast of Great Britain. Occasionally, the north wind may even push this mass of water down along the coast of England to reach the Thames Estuary. Then, this mass of water is sent back along the Thames and the conditions for a disaster arise.

In 1953, a tremendous flooding happened killing over 300 people while 64 000 hectares of land were covered by water. After that, the British Government decided to create a Committee to analyze the problem and the possibilities of building defense mechanisms. There was consensus on the Committee about the need of some defense mechanism but not about which one should be implemented. Finally, in 1970 the decision of building a barrier, the *Thames Barrier*, was taken.

Obviously, the main goal of the barrier is to close the river when a dangerous increase of water level is detected. The barrier was built during 8 years and 4 000 workers participated on that gigantic engineering programme. The barrier was finally opened in 1984. It consists of 10 enormous steel gates built over the basement of reinforced concrete structures and endowed with sophisticated mechanisms that allow normal traffic on the river when the barrier is open but that allows closing and cutting the traffic and the flux of water when needed. Since its opening, the barrier has been closed three times up to now.

Obviously, as for other many control mechanisms, it is a priority to close the barrier a minimal number of times. Every time the barrier is closed, important economic losses are produced due to the suppression of river traffic. Furthermore, once the barrier is closed, it has to remain closed at least for 8 hours until the water level stabilizes at both sides. On the other hand, the process of closing the barrier takes two hours and, therefore, it is not possible to wait and see at place the flood arriving but, rather, one has to take the decision of closing on the basis of *predictions*. Consequently, extremely efficient methods of prediction are needed.

At present, the predictions are made by means of mathematical models that combine or match two different subsystems: the first one concerns the tides around the British Islands and the second one deals with weather prediction. In this way, every hour, predictions are made 30 hours ahead on several selected points of the coast.

The numerical simulation and solution of this model is performed on the supercomputer of the British Meteorological Office and the results are transferred to the computer of the Thames Barrier. The data are then introduced in another model, at a bigger scale, including the North Sea, the Thames Estuary and the low part of the river where the effect of tides is important. The models that are being used at present reduce to systems of partial differential equations and are solved by finite difference methods. The results obtained this way are compared to the average predictions and, in view of this analysis, the authorities have the responsibility of taking the decision of closing the barrier or keeping it opened.

The Thames Barrier provides, at present, a satisfactory solution to the problem of flooding in the London area. But this is not a long term solution since, as we said above, the average water level increases of approximately 75 centimeters per century and, consequently, in the future, this method of prevention will not suffice anymore.

We have mentioned here the main task that the Thames Barrier carries out: the prevention of flooding. But it also serves of course to prevent the water level to go down beyond some limits that put in danger the traffic along the river.

The Thames Barrier is surely one of the greatest achievements of Control Theory in the context of the environmental protection. Here, the combination of mathematical modelling, numerical simulation and Engineering has allowed to provide a satisfactory solution to an environmental problem of first magnitude.

The reader interested in learning more about the Thames Barrier is referred to [75].

1.10 The future

At present, there are many branches of Science and Technology in which Control Theory plays a central role and faces fascinating challenges. In some cases, one expects to solve the problems by means of technological developments that will make possible to implement more sophisticated control mechanisms. To some extent, this is the case for instance of the laser control of chemical reactions we have discussed above. But, in many other areas, important theoretical developments will also be required to solve the complex control problems that arise. In this Section, we will briefly mention some of the fields in which these challenges are present. The reader interested in learning more about these topics is referred to the SIAM Report [204].

• **Large space structures -** Quite frequently, we learn about the difficulties found while deploying an antenna by a satellite, or on getting the precise orientation of a telescope. In some cases, this may cause huge losses and damages and may even be a reason to render the whole space mission useless. The importance of space structures is increasing rapidly, both for communications and research within our planet and also in the space adventure. These structures are built coupling several components, rigid and flexible ones. The problem of stabilizing these structures so that they remain oriented in the right direction without too large deformations is therefore complex and relevant. Designing robust control mechanisms for these structures is a challenging problem that requires important cooperative developments in Control Theory, computational issues and Engineering.

• **Robotics -** This is a branch of Technology of primary importance, where the scientific challenges are diverse and numerous. These include, for instance, computer vision. Control Theory is also at the heart of this area and its development relies to a large extent on robust computational algorithms for controlling. It is not hard to imagine how difficult it is to get a robot "walking" along a stable dynamics or catching an objet with its "hands".

• **Information and energy networks -** The globalization of our planet is an irreversible process. This is valid in an increasing number of human activities as air traffic, generation and distribution of energy, informatic networks, etc. The dimensions and complexity of the networks one has to manage are so large that, very often, one has to take decisions locally, without having a complete global information, but taking into account that local decisions will have global effects. Therefore, there is a tremendous need of developing methods and techniques for the control of large interconnected systems.

• **Control of combustion -** This is an extremely important problem in Aerospatial and Aeronautical Industry. Indeed, the control of the instabilities that combustion produces is a great challenge. In the past, the emphasis has been put on design aspects, modifying the geometry of the system to interfere on the acoustic-combustion interaction or incorporating dissipative elements. The active control of combustion by means of thermal or acoustic mechanisms is also a subject in which almost everything is to be done.

• **Control of fluids -** The interaction between Control Theory and *Fluid Mechanics* is also very rich nowadays. This is an important topic in *Aeronautics*, for instance, since the structural dynamics of a plane in flight interacts with the flux of the neighboring air. In conventional planes, this fact can be ignored but, for the new generations, it will have to be taken into account, to avoid turbulent flow around the wings.

From a mathematical point of view, almost everything remains to be done in what concerns modelling, computational and control issues. A crucial contribution was made by J.L. Lions in [144], where the approximate controllability of the Navier-Stokes equations was conjectured. For an overview of the main existing results, see [81].

• **Solidification processes and steel industry -** The increasingly important development in *Material Sciences* has produced intensive research in solidification processes. The form and the stability of the liquid-solid interface are central aspects of this field, since an irregular interface may produce undesired products. The sources of instabilities can be of different nature: convection, surface tension, ... The *Free Boundary Problems* area has experienced important developments in the near past, but very little has been done from a control theoretical viewpoint. There are very interesting problems like, for instance, *building interfaces* by various indirect measurements, or its control by means of heating mechanisms, or applying electric or magnetic currents or rotations of the alloy in the furnace. Essentially, there is no mathematical theory to address these problems.

• **Control of plasma -** In order to solve the energetic needs of our planet, one of the main projects is the obtention of fusion reactions under control. At present, *Tokomak machines* provide one of the most promising approaches to this problem. Plasma is confined in a Tokomak machine by means of electromagnetic fields. The main problem consists then in keeping the plasma at high density and temperature on a desired configuration along long time intervals despite its instabilities. This may be done placing *sensors* that provide the information one needs to modify the currents rapidly to compensate the perturbations in the plasma. Still today there is a lot to be done in this area. There are also important identification problems arising due to the difficulties to get precise measurements. Therefore, this is a field that provides many challenging topics in the areas of Control Theory and *Inverse Problems Theory.*

• **Biomedical research -** The design of medical therapies depends very strongly on the understanding of the dynamics of Physiology. This is a very active topic nowadays in which almost everything is still to be done from a mathematical viewpoint. Control Theory will also play an important role in this field. As an example, we can mention the design of mechanisms for insulin supply endowed with control chips.

• **Hydrology -** The problem of governing water resources is extremely relevant nowadays. Sometimes this is because there are little resources, some others because they are polluted and, in general, because of the complexity of the network of supply to all consumers (domestic, agricultural, industrial, ...). The control problems arising in this context are also of different nature. For instance, the *parameter identification problem*, in which the goal is to determine the location of sensors that provide sufficient information for an efficient extraction and supply and, on the other hand, the design of efficient management strategies.

• **Recovery of natural resources -** Important efforts are being made on the modelling and theoretical and numerical analysis in the area of simulation of reservoirs of water, oil, minerals, etc. One of the main goals is to optimize the extraction strategies. Again, inverse problems arise and, also, issues related to the control of the interface between the injected and the extracted fluid.

• **Economics -** The increasingly important role that Mathematics are playing in the world of *Economics* and *Finances* is well known. Indeed, nowadays, it is very frequent to use Mathematics to predict the fluctuations in financial markets. The models are frequently stochastic and the existing *Stochastic Control Theory* may be of great help to design optimal strategies of investment and consumption.

• **Manufacturing systems -** Large automatic manufacturing systems are designed as flexible systems that allow rapid changes of the production planning as a function of demand. But this increasing flexibility is obtained at the price of an increasing complexity. In this context, Control Theory faces also the need of designing efficient computerized control systems.

• **Evaluation of efficiency on computerized systems -** The existing software packages to evaluate the efficiency of computer systems are based on its representation by means of the *Theory of Networks*. The development of parallel and synchronized computer systems makes them insufficient. Thus, it is necessary to develop new models and, at this level, the Stochastic Control Theory of *discrete systems* may play an important role.

• **Control of computer aided systems -** As we mentioned above, the complexity of the control problems we are facing nowadays is extremely high. Therefore, it is impossible to design efficient control strategies without the aid of computers and this has to be taken into account when designing these strategies. This is a multidisciplinary research field concerned with Control Theory, Computer Sciences, Numerical Analysis and Optimization, among other areas.

Appendix 1: Pontryagin's maximum principle

As we said in Section 3, one of the main contributions to Control Theory in the sixties was made by L. Pontryagin by means of *the maximum principle*. In this Appendix, we shall briefly recall the main underlying ideas.

In order to clarify the situation and show how powerful is this approach, we will consider a *minimal time control* problem. Thus, let us consider again the differential system

$$\left\{ \begin{array}{l} \dot{x} = f(x,u), \quad t > 0, \\ x(0) = x^0, \end{array} \right. \tag{1.72}$$

with state $x = (x_1(t), \ldots, x_N(t))$ and control $u = (u_1(t), \ldots, u_M(t))$.

For simplicity, we will assume that the function $f : \mathbf{R}^N \times \mathbf{R}^M \mapsto \mathbf{R}^N$ is well defined and smooth, although this is not strictly necessary (actually, this is one of the main contributions of Pontryagin's principle). We will also assume that a nonempty closed set $G \subset \mathbf{R}^M$ is given and that the family of admissible controls is

$$\mathcal{U}_{\rm ad} = \{\, u \in L^2(0, +\infty; \mathbf{R}^M) : u(t) \in G \ \text{ a.e.} \,\}. \tag{1.73}$$

Let us introduce a manifold $\mathcal{M}$ of $\mathbf{R}^N$, with

$$\mathcal{M} = \{\, x \in \mathbf{R}^N : \mu(x) = 0 \,\},$$

where $\mu : \mathbf{R}^N \mapsto \mathbf{R}^q$ is a regular map ($q \leq N$), so that the matrix $\nabla\mu(x)$ is of rank q at each point $x \in \mathcal{M}$ (thus, $\mathcal{M}$ is a smooth differential manifold of dimension $N - q$). Recall that the tangent space to $\mathcal{M}$ at a point $x \in \mathcal{M}$ is given by:

$$T_x\mathcal{M} = \{\, v \in \mathbf{R}^N : \nabla\mu(x) \cdot v = 0 \,\}.$$

Let us fix the initial state x^0 in $\mathbf{R}^N \setminus \mathcal{M}$. Then, to each control $u = u(t)$ we can associate a trajectory, defined by the solution of (1.72). Our minimal time control problem consists in finding a control in the admissible set $\mathcal{U}_{\rm ad}$ driving the corresponding trajectory to the manifold $\mathcal{M}$ in a time as short as possible.

In other words, we intend to minimize the quantity T subject to the following constraints:

- $T > 0$,
- For some $u \in \mathcal{U}_{\rm ad}$, the associated solution to (1.72) satisfies $x(T) \in \mathcal{M}$.

Obviously, the difficulty of the problem increases when the dimension of $\mathcal{M}$ decreases.

The following result holds (Pontryagin's maximum principle):

Theorem 1.10.1 *Assume that $\hat{T}$ is the minimal time and $\hat{u}$, defined for $t \in [0, \hat{T}]$, is an optimal control for this problem. Let $\hat{x}$ be the corresponding trajectory. Then there exists $\hat{p} = \hat{p}(t)$ such that the following identities hold almost everywhere in $[0, \hat{T}]$:*

$$\dot{\hat{x}} = f(\hat{x}, \hat{u}), \quad -\dot{\hat{p}} = \left(\frac{\partial f}{\partial x}(\hat{x}, \hat{u})\right)^t \cdot \hat{p} \tag{1.74}$$

and

$$H(\hat{x}(t), \hat{p}(t), \hat{u}) = \max_{v \in G} H(\hat{x}(t), \hat{p}(t), v), \tag{1.75}$$

where

$$H(x, p, v) = \langle f(x, v), p \rangle \quad \forall (x, p, v) \in \mathbf{R}^N \times \mathbf{R}^N \times G. \tag{1.76}$$

Furthermore, the quantity

$$H^*(\hat{x}, \hat{p}) = \max_{v \in G} H(\hat{x}, \hat{p}, v) \tag{1.77}$$

is constant and nonnegative (maximum condition) and we have

$$\hat{x}(\hat{T}) = x^0, \quad \hat{x}(\hat{T}) \in \mathcal{M} \tag{1.78}$$

and

$$\hat{p}(\hat{T}) \perp T_{\hat{x}(\hat{T})} \mathcal{M} \tag{1.79}$$

(transversality condition).

The function H is referred to as the *Hamiltonian* of (1.72) and the solutions $(\hat{x}, \hat{p}, \hat{u})$ of the equations (1.74)–(1.79) are called *extremal points*. Of course, $\hat{p}$ is the extremal *adjoint state*.

Very frequently in practice, in order to compute the minimal time $\hat{T}$ and the optimal control $\hat{u}$, system (1.74)–(1.79) is used as follows. First, assuming that $\hat{x}$ and $\hat{p}$ are known, we determine $\hat{u}(t)$ for each t from (1.75). Then, with $\hat{u}$ being determined in terms of $\hat{x}$ and $\hat{p}$, we solve (1.74) with the initial and final conditions (1.78) and (1.79).

Observe that this is a well posed boundary-value problem for the couple $(\hat{x}, \hat{p})$ in the time interval $(0, \hat{T})$.

From (1.74), the initial and final conditions and (1.75), provide the control in terms of the state. Consequently, the maximum principle can be viewed as a feedback law for determining a good control $\hat{u}$.

In order to clarify the statement in Theorem 1.10.1, we will now present a heuristic proof.

We introduce the Hilbert space $X \times \mathcal{U}$, where $\mathcal{U} = L^2(0, +\infty; \mathbf{R}^M)$ and X is the space of functions $x = x(t)$ satisfying $x \in L^2(0, +\infty; \mathbf{R}^N)$ and $\dot{x} \in L^2(0, +\infty; \mathbf{R}^N)$[5].

[5] This is the Sobolev space $H^1(0, +\infty; \mathbf{R}^N)$. More details can be found, for instance, in [24].

Let us consider the functional

$$F(T, x, u) = T \quad \forall (T, x, u) \in \mathbf{R} \times X \times \mathcal{U}.$$

Then, the problem under consideration is

$$\text{To minimize } F(T, x, u), \tag{1.80}$$

subject to the inequality constraint

$$T \geq 0, \tag{1.81}$$

the pointwise control constraints

$$u(t) \in G \quad \text{a.e. in } (0, T) \tag{1.82}$$

(that is to say $u \in \mathcal{U}_{\rm ad}$) and the equality constraints

$$\dot{x} - f(x, u) = 0 \quad \text{a.e. in } (0, T), \tag{1.83}$$

$$x(0) - x^0 = 0 \tag{1.84}$$

and

$$\mu(x(T)) = 0. \tag{1.85}$$

Let us assume that $(\hat{T}, \hat{x}, \hat{u})$ is a solution to this constrained extremal problem. One can then prove the existence of *Lagrange multipliers* $(\hat{p}, \hat{z}, \hat{w}) \in X \times \mathbf{R}^N \times \mathbf{R}^N$ such that $(\hat{T}, \hat{x}, \hat{u})$ is, together with $(\hat{p}, \hat{z}, \hat{w})$, a saddle point of the *Lagrangian*

$$\mathcal{L}(T, x, u; p, z, w) = T + \int_0^T \langle p, \dot{x} - f(x, u)\rangle\, dt + \langle z, x(0) - x^0\rangle + \langle w, \mu(x(T))\rangle$$

in $\mathbf{R}_+ \times X \times \mathcal{U}_{\rm ad} \times X \times \mathbf{R}^N \times \mathbf{R}^N$.

In other words, we have

$$\left\{ \begin{array}{l} \mathcal{L}(\hat{T}, \hat{x}, \hat{u}; p, z, w) \leq \mathcal{L}(\hat{T}, \hat{x}, \hat{u}; \hat{p}, \hat{z}, \hat{w}) \leq \mathcal{L}(T, x, u; \hat{p}, \hat{z}, \hat{w}) \\ \forall (T, x, u) \in \mathbf{R}_+ \times X \times \mathcal{U}_{\rm ad}\,, \quad \forall (p, z, w) \in X \times \mathbf{R}^N \times \mathbf{R}^N. \end{array} \right. \tag{1.86}$$

The first inequalities in (1.86) indicate that the equality constraints (1.83)–(1.85) are satisfied for $\hat{T}$, $\hat{x}$ and $\hat{u}$. Let us now see what is implied by the second inequalities in (1.86).

First, taking $T = \hat{T}$ and $x = \hat{x}$ and choosing u arbitrarily in $\mathcal{U}_{\rm ad}$, we find that

$$\int_0^{\hat{T}} \langle \hat{p}, f(\hat{x}, u)\rangle\, dt \leq \int_0^{\hat{T}} \langle \hat{p}, f(\hat{x}, \hat{u})\rangle\, dt \quad \forall u \in \mathcal{U}_{\rm ad}\,.$$

It is not difficult to see that this is equivalent to (1.75), in view of the definition of $\mathcal{U}_{\rm ad}$.

Secondly, taking $T = \hat{T}$ and $u = \hat{u}$, we see that

$$\int_0^{\hat{T}} \langle p, \dot{x} - f(x, \hat{u}) \rangle \, dt + \langle z, x(0) - x^0 \rangle + \langle w, \mu(x(\hat{T})) \rangle \geq 0 \quad \forall x \in X. \tag{1.87}$$

From (1.87) written for $x = \hat{x} \pm \varepsilon y$, taking into account that (1.83) – (1.85) are satisfied for $\hat{T}$, $\hat{x}$ and $\hat{u}$, after passing to the limit as $\varepsilon \to 0$, we easily find that

$$\int_0^{\hat{T}} \langle \hat{p}, \dot{y} - \frac{\partial f}{\partial x}(\hat{x}, \hat{u}) \cdot y \rangle \, dt + \langle \hat{z}, y(0) \rangle + \langle \hat{w}, \nabla \mu(\hat{x}(\hat{T})) \cdot y(\hat{T}) \rangle = 0 \quad \forall y \in X. \tag{1.88}$$

Taking $y \in X$ such that $y(0) = y(\hat{T}) = 0$, we can deduce at once the differential system satisfied by $\hat{p}$ in $(0, \hat{T})$. Indeed, after integration by parts, we have from (1.88) that

$$\int_0^{\hat{T}} \langle -\dot{\hat{p}} - \left(\frac{\partial f}{\partial x}(\hat{x}, \hat{u}) \right)^t \cdot \hat{p}, y \rangle \, dt = 0$$

for all such y. This leads to the second differential system in (1.74).

Finally, let us fix λ in $\mathbf{R}^N$ an let us take in (1.88) a function $y \in X$ such that $y(0) = 0$ and $y(\hat{T}) = \lambda$. Integrating again by parts, in view of (1.74), we find that

$$\langle \hat{p}(\hat{T}), \lambda \rangle + \langle \hat{w}, \nabla \mu(\hat{x}(\hat{T})) \cdot \lambda \rangle = 0$$

and, since λ is arbitrary, this implies

$$\hat{p}(\hat{T}) = -\left(\nabla \mu(\hat{x}(\hat{T})) \right)^t \hat{w}.$$

This yields the transversality condition (1.79).

We have presented here the maximum principle for a minimal time control problem, but there are many variants and generalizations.

For instance, let the final time $T > 0$ and a non-empty closed convex set $S \subset \mathbf{R}^N$ be fixed and let $\mathcal{U}_{\rm ad}$ be now the family of controls $u \in L^2(0, T; \mathbf{R}^M)$ with values in the closed set $G \subset \mathbf{R}^M$ such that the associated states $x = x(t)$ satisfy

$$x(0) = x^0, \quad x(T) \in S. \tag{1.89}$$

Let $f^0 : \mathbf{R}^N \times \mathbf{R}^M \mapsto \mathbf{R}$ be a smooth bounded function and let us put

$$F(u) = \int_0^T f^0(x(t), u(t)) \, dt \quad \forall u \in \mathcal{U}_{\rm ad}, \tag{1.90}$$

where x is the state associated to u through (1.72). In this case, the Hamiltonian H is given by

$$H(x,p,u) = \langle f(x,u),p\rangle + f^0(x,u) \quad \forall (x,p,u) \in \mathbf{R}^N \times \mathbf{R}^N \times G. \tag{1.91}$$

Then, if $\hat{u}$ minimizes F over $\mathcal{U}_{\rm ad}$ and $\hat{x}$ is the associated state, the maximum principle guarantees the existence of a function $\hat{p}$ such that the following system holds:

$$\dot{\hat{x}} = f(\hat{x},\hat{u}), \quad -\dot{\hat{p}} = \left(\frac{\partial f}{\partial x}(\hat{x},\hat{u})\right)^t \cdot \hat{p} + \frac{\partial f^0}{\partial x}(\hat{x},\hat{u}) \quad \text{a.e. in } (0,\mathrm{T}), \tag{1.92}$$

$$H(\hat{x}(t),\hat{p}(t),\hat{u}) = \max_{v\in G} H(\hat{x}(t),\hat{p}(t),v), \tag{1.93}$$

$$\hat{x}(0) = x^0, \quad \hat{x}(T) \in S \tag{1.94}$$

and

$$\langle \hat{p}(T), y - \hat{x}(T)\rangle \geq 0 \quad \forall y \in S. \tag{1.95}$$

For general nonlinear systems, the optimality conditions that the Pontryagin maximum principle provides may be difficult to analyze. In fact, in many cases, these conditions do not yield a complete information of the optimal trajectories. Very often, this requires appropriate geometrical tools, as the Lie brackets mentioned in Section 4. The interested reader is referred to H. Sussmann [209] for a more careful discussion of these issues.

In this context, the work by J.A. Reeds and L.A. Shepp [188] is worth mentioning. This paper is devoted to analyze a dynamical system for a vehicle, similar to the one considered at the end of Section 4, but allowing both backwards and forwards motion. As an example of the complexity of the dynamics of this system, it is interesting to point out that an optimal trajectory consists of, at most, five pieces. Each piece is either a segment or an arc of circumference, so that the whole set of possible optimal trajectories may be classified in 48 three-parameters families. More recently, an exhaustive analysis carried out in [210] by means of geometric tools allowed the authors to reduce the number of families actually to 46.

The extension of the maximum principle to control problems for partial differential equations has also been the objective of intensive research. As usual, when extending this principle, technical conditions are required to take into account the intrinsic difficulties of the infinite dimensional character of the system. The interested reader is referred to the books by H.O. Fattorini [77] and X. Li and J. Yong [140].

Appendix 2: Dynamical programming

We have already said in Section 3 that the *dynamical programming principle*, introduced by R. Bellman in the sixties, is another historical contribution to Control Theory.

The main goal of this principle is the same as of Pontryagin's main result: to characterize the optimal control by means of a system that may be viewed as a feedback law.

Bellman's central idea was to do it through the *value function* (also called the Bellman function) and, more precisely, to benefit from the fact that this function satisfies a *Hamilton-Jacobi equation.*

In order to give an introduction to this theory, let us consider for each $t \in [0,T]$ the following differential problem:

$$\left\{ \begin{array}{l} \dot{x}(s) = f(x(s),\, u(s)), \quad s \in [t,T], \\ x(t) = x^0\,. \end{array} \right. \tag{1.96}$$

Again, $x = x(s)$ plays the role of the state and takes values in $\mathbf{R}^N$ and $u = u(s)$ is the control and takes values in $\mathbf{R}^M$. The solution to (1.96) will be denoted by $x(\cdot;t,x^0)$.

We will assume that u can be any measurable function in $[0,T]$ with values in a compact set $G \subset \mathbf{R}^M$. The family of admissible controls will be denoted, as usual, by $\mathcal{U}_{\rm ad}$.

The final goal is to solve a control problem for the state equation in (1.96) in the whole interval $[0,T]$. But it will be also useful to consider (1.96) for each $t \in [0,T]$, with the "initial" data prescribed at time t.

Thus, for any $t \in [0,T]$, let us consider the problem of minimizing the cost $C(\cdot;t,x^0)$, with

$$C(u;t,x^0) = \int_t^T f^0(x(\tau;t,x^0),u(\tau))\,d\tau + f^1(T,x(T;t,x^0)) \quad \forall u \in \mathcal{U}_{\rm ad} \tag{1.97}$$

(the final goal is to minimize $C(\cdot;0,x^0)$ in the set of admissible controls $\mathcal{U}_{\rm ad}$). To simplify our exposition, we will assume that the functions f, f^0 and f^1 are regular and bounded with bounded derivatives.

The main idea in *dynamical programming* is to introduce and analyze the so called *value function* $V = V(x^0,t)$, where

$$V(x^0,t) = \inf_{u\in\mathcal{U}_{\rm ad}} C(u;t,x^0) \quad \forall x^0 \in \mathbf{R}^N, \quad \forall t \in [0,T]. \tag{1.98}$$

This function provides the minimal cost obtained when the system starts from x^0 at time t and evolves for $s \in [t,T]$. The main property of V is that it satisfies a *Hamilton-Jacobi* equation. This fact can be used to characterize and even compute the optimal control.

Before writing the Hamilton-Jacobi equation satisfied by V, it is convenient to state the following fundamental result:

Theorem 1.10.2 *The value function $V = V(x^0, t)$ satisfies the Bellman optimality principle, or dynamical programming principle. According to it, for any $x^0 \in \mathbf{R}^N$ and any $t \in [0, T]$, the following identity is satisfied:*

$$V(x^0,t) = \inf_{u \in \mathcal{U}_{\mathrm{ad}}} \left[V(x(s;t,x^0),s) + \int_t^s f^0(x(\tau;t,x^0),u(\tau))\,d\tau \right] \quad \forall s \in [t,T]. \tag{1.99}$$

In other words, the minimal cost that is produced starting from x^0 at time t coincides with the minimal cost generated starting from $x(s;t,x^0)$ at time s plus the "energy" lost during the time interval $[t,s]$. The underlying idea is that a control, to be optimal in the whole time interval $[0,T]$, has also to be optimal in every interval of the form $[t,T]$.

A consequence of (1.99) is the following:

Theorem 1.10.3 *The value function $V = V(x,t)$ is globally Lipschitz-continuous. Furthermore, it is the unique viscosity solution of the following Cauchy problem for the Hamilton-Jacobi-Bellman equation*

$$\begin{cases} V_t + \inf\limits_{v \in G} \left\{ \langle f(x,v), \nabla V \rangle + f^0(x,v) \right\} = 0, & (x,t) \in \mathbf{R}^N \times (0,T), \\ V(x,T) = f^1(T,x), & x \in \mathbf{R}^N. \end{cases} \tag{1.100}$$

The equation in (1.100) is, indeed, a Hamilton-Jacobi equation, i.e. an equation of the form

$$V_t + H(x, \nabla V) = 0,$$

with Hamiltonian

$$H(x,p) = \inf_{v \in G} \left\{ \langle f(x,v), p \rangle + f^0(x,v) \right\} \tag{1.101}$$

(recall (1.91)).

The notion of *viscosity solution* of a Hamilton-Jacobi equation was introduced to compensate the absence of existence and uniqueness of classical solution, two phenomena that can be easily observed using the method of characteristics. Let us briefly recall it.

Assume that $H = H(x,p)$ is a continuous function (defined for $(x,p) \in \mathbf{R}^N \times \mathbf{R}^N$) and $g = g(x)$ is a continuous bounded function in $\mathbf{R}^N$. Consider the following initial-value problem:

$$\begin{cases} y_t + H(x, \nabla y) = 0, & (x,t) \in \mathbf{R}^N \times (0,\infty), \\ y(x,0) = g(x), & x \in \mathbf{R}^N. \end{cases} \tag{1.102}$$

Let $y = y(x,t)$ be bounded and continuous. It will be said that y is a viscosity solution of (1.102) if the following holds:

- For each $v \in C^\infty(\mathbf{R}^N \times (0,\infty))$, one has

$$\left\{ \begin{array}{c} \text{If } y - v \text{ has a local maximum at } (x^0, t^0) \in \mathbf{R}^N \times (0,\infty), \text{ then} \\ v_t(x^0,t^0) + H(x^0, \nabla v(x^0,t^0)) \leq 0 \end{array} \right.$$

and

$$\left\{ \begin{array}{c} \text{If } y - v \text{ has a local minimum at } (x^0, t^0) \in \mathbf{R}^N \times (0,\infty), \text{ then} \\ v_t(x^0,t^0) + H(x^0, \nabla v(x^0,t^0)) \geq 0. \end{array} \right.$$

- $y(x,0) = g(x)$ for all $x \in \mathbf{R}^N$.

This definition is justified by the following fact. Assume that y is a classical solution to (1.102). It is then easy to see that, whenever $v \in C^\infty(\mathbf{R}^N \times (0,\infty))$ and $v_t(x^0,t^0)+H(x^0,\nabla v(x^0,t^0)) > 0$ (resp. < 0), the function $y-v$ cannot have a local maximum (resp. a local minimum) at (x^0,t^0). Consequently, a classical solution is a viscosity solution and the previous definition makes sense.

On the other hand, it can be checked that the solutions to (1.102) obtained by the *vanishing viscosity method* satisfy these conditions and, therefore, are viscosity solutions. The vanishing viscosity method consists in solving, for each $\varepsilon > 0$, the parabolic problem

$$\left\{ \begin{array}{ll} y_t + H(x, \nabla y) = \varepsilon \Delta y, & (x,t) \in \mathbf{R}^N \times (0,\infty), \\ y(x,0) = g(x), & x \in \mathbf{R}^N \end{array} \right. \tag{1.103}$$

and, then, passing to the limit as $\varepsilon \to 0^+$.

A very interesting feature of viscosity solutions is that the two properties entering in its definition suffice to prove uniqueness. The proof of this uniqueness result is inspired on the pioneering work by N. Kruzhkov [128] on entropy solutions for hyperbolic equations. The most relevant contributions to this subject are due to M. Crandall and P.L. Lions and L.C. Evans, see [61], [71].

But let us return to the dynamical programming principle (the fact that the value function V satisfies (1.99)) and let us see how can it be used.

One may proceed as follows. First, we solve (1.100) and obtain in this way the value function V. Then, we try to compute $\hat{u}(t)$ at each time t using the identities

$$f(\hat{x}(t), \hat{u}(t)) \cdot \nabla V(\hat{x}(t), t) + f^0(\hat{x}(t), \hat{u}(t)) = H(\hat{x}(t), \nabla V(\hat{x}(t), t)), \tag{1.104}$$

i.e. we look for the values $\hat{u}(t)$ such that the minimum of the Hamiltonian in (1.101) is achieved. In this way, we can expect to obtain a function $\hat{u} = \hat{u}(t)$ which is the optimal control.

Recall that, for each $\hat{u}$, the state $\hat{x}$ is obtained as the solution of

$$\left\{ \begin{array}{ll} \dot{\hat{x}}(s) = f\left(\hat{x}(s), \hat{u}(s)\right), & s \in [0, \hat{T}], \\ \hat{x}(0) = x^0. \end{array} \right. \tag{1.105}$$

Therefore, $\hat{x}$ is determined by $\hat{u}$ and (1.104) is an equation in which $\hat{u}(t)$ is in fact the sole unknown.

In this way, one gets indeed an optimal control $\hat{u}$ in feedback form that provides an optimal trajectory $\hat{x}$ (however, at the points where V is not smooth, important difficulties arise; for instance, see [80]).

If we compare the results obtained by means of the maximum principle and the dynamical programming principle, we see that, in both approaches, the main conclusions are the same. It could not be otherwise, since the objective was to characterize optimal controls and trajectories.

However, it is important to underline that the points of view are completely different. While Pontryagin's principle extends the notion of Lagrange multiplier, Bellman's principle provides a dynamical viewpoint in which the value function and its time evolution play a crucial role.

The reader interested in a simple but relatively complete introduction to Hamilton-Jacobi equations and dynamical programming can consult the book by L.C. Evans [72]. For a more complete analysis of these questions see for instance W. Fleming and M. Soner [80] and P.-L. Lions [150]. For an extension of these methods to partial differential equations, the reader is referred to the book by X. Li and J. Yong [140].

Chapter 2

An introduction to the controllability of linear PDE *(S. Micu and E. Zuazua)*

joint work with Sorin MICU, University of Craiova, Romania, to appear in Quelques questions de théorie du contrôle*. Sari, T., ed., Collection Travaux en Cours Hermann*

2.1 Introduction

These notes are a written abridged version of a course that both authors have delivered in the last five years in a number of schools and doctoral programs. Our main goal is to introduce some of the main results and tools of the modern theory of controllability of Partial Differential Equations (PDE). The notes are by no means complete. We focus on the most elementary material by making a particular choice of the problems under consideration.

Roughly speaking, the *controllability problem* may be formulated as follows. Consider an evolution system (either described in terms of Partial or Ordinary Differential Equations (PDE/ODE)). We are allowed to act on the trajectories of the system by means of a suitable control (the right hand side of the system, the boundary conditions, etc.). Then, given a time interval $t \in (0, T)$, and initial and final states we have to find a control such that the solution matches both the initial state at time $t = 0$ and the final one at time $t = T$.

This is a classical problem in Control Theory and there is a large literature on the topic. We refer for instance to the book by Lee and Marcus [136] for an introduction in the context of finite-dimensional systems. We also refer to the survey paper by Russell [194] and to the book of Lions [142, 143] for an introduction to the controllability of PDE, also referred to as Distributed Parameter Systems.

Research in this area has been very intensive in the last two decades and it would be impossible to report on the main progresses that have been made within these notes. For this reason we have chosen to collect some of the most relevant introductory material at the prize of not reaching the best results that are known today. The interested reader may learn more on this topic from the references above and those on the bibliography at the end of the article.

When dealing with controllability problems, to begin with, one has to distinguish between finite-dimensional systems modelled by ODE and infinite-dimensional distributed systems described by means of PDE. This modelling issue may be important in practice since finite-dimensional and infinite-dimensional systems may have quite different properties from a control theoretical point of view as we shall see in section 3.

Most of these notes deal with problems related to PDE. However, we start by an introductory section in which we present some of the basic problems and tools of control theory for finite-dimensional systems. The theory has evolved tremendously in the last decades to deal with nonlinearity and uncertainty but here we present the simplest results concerning the controllability of linear finite-dimensional systems and focus on developing tools that will later be useful to deal with PDE. As we shall see, in the finite-dimensional context *a system is controllable if and only if the algebraic Kalman rank condition is satisfied.* According to it, when a system is controllable for some time it is controllable for all time. But this is not longer true in the context of PDE. In particular, in the frame of the wave equation, a model in which propagation occurs with finite velocity, in order for controllability properties to be true the control time needs to be large enough so that the effect of the control may reach everywhere. In this first section we shall develop a variational approach to the control problem.

As we shall see, whenever a system is controllable, the control can be built by minimizing a suitable quadratic functional defined on the class of solutions of the adjoint system. Suitable variants of this functional allow building different types of controls: those of minimal L^2-norm turn out to be smooth while those of minimal L^∞-norm are of bang-bang form. The main difficulty when minimizing these functionals is to show that they are coercive. This turns out to be equivalent to the so called *observability property* of the adjoint equation, a property which is equivalent to the original control property of the state equation.

In sections 2 and 3 we introduce the problems of interior and boundary control of the linear constant coefficient wave equation. We describe the various variants, namely, approximate, exact and null controllability, and its mutual relations. Once again, the problem of exact controllability turns out to be equivalent to the observability of the adjoint system while approximate controllability is equivalent to a weaker uniqueness or unique continuation property. In section 4 we analyze the $1 - d$ case by means of Fourier series expansions and the classical Ingham's inequality which is a very useful tool to solve control problems for $1 - d$ wave-like and beam equations.

In sections 5 and 6 we discuss respectively the problems of interior and boundary control of the heat equation. We show that, as a consequence of Holmgren Uniqueness Theorem, the adjoint heat equation posesses the property of unique continuation in an arbitrarily small time. Accordingly the multi-dimensional heat equation is approximately controllable in an arbitrarily small time and with controls supported in any open subset of the domain where the equation holds. We also show that, in one space dimension, using Fourier series expansions, the null control problem, can be reduced to a problem of moments involving a sequence of real exponentials. We then build a biorthogonal family allowing to show that the system is null controllable in any time by means of a control acting on one extreme of the space interval where the heat equation holds.

As we said above these notes are not complete. The interested reader may learn more on this topic through the survey articles [237] and [241]. For the connections between controllability and the theory of homogenization we refer to section 4 below. We refer to [242] for a discussion of numerical approximation issues in controllability of PDE.

2.1.1 Controllability and stabilization of finite dimensional systems

This section is devoted to study some basic controllability and stabilization properties of finite dimensional systems.

The first two sections deal with the linear case. In Section 1 it is shown that the exact controllability property may be characterized by means of the *Kalman's algebraic rank condition*. In Section 2 a skew-adjoint system is considered. In the absence of control, the system is conservative and generates a group of isometries. It is shown that the system may be guaranteed to be uniformly exponentially stable if a well chosen feedback dissipative term is added to it. This is a particular case of the well known equivalence property between controllability and stabilizability of finite-dimensional systems ([226]).

2.1.2 Controllability of finite dimensional linear systems

Let $n, m \in \mathbb{N}^*$ and $T > 0$. We consider the following finite dimensional system:

$$\begin{cases} x'(t) = Ax(t) + Bu(t), & t \in (0,T), \\ x(0) = x^0. \end{cases} \tag{2.1}$$

In (2.1), A is a real $n \times n$ matrix, B is a real $n \times m$ matrix and x^0 a vector in $\mathbb{R}^n$. The function $x : [0,T] \longrightarrow \mathbb{R}^n$ represents the *state* and $u : [0,T] \longrightarrow \mathbb{R}^m$ the *control*. Both are vector functions of n and m components respectively depending exclusively on time t. Obviously, in practice $m \leq n$. The most desirable goal is, of course, controlling the system by means of a minimum number m of controls.

Given an initial datum $x^0 \in \mathbb{R}^n$ and a vector function $u \in L^2(0,T;\mathbb{R}^m)$, system (2.1) has a unique solution $x \in H^1(0,T;\mathbb{R}^n)$ characterized by the variation of constants formula:

$$x(t) = e^{At}x^0 + \int_0^t e^{A(t-s)}Bu(s)ds, \quad \forall t \in [0,T]. \tag{2.2}$$

Definition 2.1.1 *System* (2.1) *is* **exactly controllable** *in time* $T > 0$ *if given any initial and final one* $x^0, x^1 \in \mathbb{R}^n$ *there exists* $u \in L^2(0,T,\mathbb{R}^m)$ *such that the solution of* (2.1) *satisfies* $x(T) = x^1$.

According to this definition the aim of the control process consists in driving the solution x of (2.1) from the initial state x^0 to the final one x^1 in time T by acting on the system through the control u.

Remark that m is the number of controls entering in the system, while n stands for the number of components of the state to be controlled. As we mentioned before, in applications it is desirable to make the number of controls m to be as small as possible. But this, of course, may affect the control properties of the system. As we shall see later on, some systems with a large number of components n can be controlled with one control only (i. e. $m = 1$). But in order for this to be true, the control mechanism, i.e. the matrix (column vector when $m = 1$) B, needs to be chosen in a strategic way depending on the matrix A. Kalman's rank condition, that will be given in section 2.1.4, provides a simple characterization of controllability allowing to make an appropriate choice of the control matrix B.

Let us illustrate this with two examples. In the first one controllability does not hold because one of the components of the system is insensitive to the control. In the second one both components will be controlled by means of a scalar control.

Example 1. Consider the case

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{2.3}$$

Then the system

$$x' = Ax + Bu$$

can be written as

$$\begin{cases} x_1' = x_1 + u \\ x_2' = x_2, \end{cases}$$

or equivalently,

$$\begin{cases} x_1' = x_1 + u \\ x_2 = x_2^0 e^t, \end{cases}$$

where $x^0 = (x_1^0, x_2^0)$ are the initial data.

This system is not controllable since the control u does not act on the second component x_2 of the state which is completely determined by the initial data x_2^0. Hence, the system is not controllable. Nevertheless one can control the first component x_1 of the state. Consequently, the system is partially controllable.

■

Example 2. Not all systems with two components and a scalar control ($n = 2, m = 1$) behave so badly as in the previous example. This may be seen by analyzing the controlled harmonic oscillator

$$x'' + x = u, \tag{2.4}$$

which may be written as a system in the following way

$$\begin{cases} x' = y \\ y' = u - x. \end{cases}$$

The matrices A and B are now respectively

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Once again, we have at our disposal only one control u for both components x and y of the system. But, unlike in Example 1, now the control acts in the second equation where both components are present. Therefore, we cannot conclude immediately that the system is not controllable. In fact it is controllable. Indeed, given some arbitrary initial and final data, (x^0, y^0) and (x^1, y^1) respectively, it is easy to construct a regular function $z = z(t)$ such that

$$\begin{cases} z(0) = x^0, & z(T) = x^1, \\ z'(0) = y^0, & z'(T) = y^1. \end{cases} \tag{2.5}$$

In fact, there are infinitely many ways of constructing such functions. One can, for instance, choose a cubic polynomial function z. We can then define

$u = z'' + z$ as being the control since the solution x of equation (2.4) with this control and initial data (x^0, y^0) coincides with z, i.e. $x = z$, and therefore satisfies the control requirements (2.5).

This construction provides an example of system with two components ($n = 2$) which is controllable with one control only ($m = 1$). Moreover, this example shows that the control u is not unique. In fact there exist infinitely many controls and different controlled trajectories fulfilling the control requirements. In practice, choosing the control which is optimal (in some sense to be made precise) is an important issue that we shall also discuss. ∎

If we define the set of reachable states

$$R(T, x^0) = \{x(T) \in \mathbb{R}^n : x \text{ solution of (2.1) with } u \in (L^2(0,T))^m\}, \qquad (2.6)$$

the exact controllability property is equivalent to the fact that $R(T, x^0) = \mathbb{R}^n$ for any $x^0 \in \mathbb{R}^n$.

Remark 2.1.1 In the definition of exact controllability any initial datum x^0 is required to be driven to any final datum x^1. Nevertheless, in the view of the linearity of the system, without any loss of generality, we may suppose that $x^1 = 0$. Indeed, if $x^1 \neq 0$ we may solve

$$\begin{cases} y' = Ay, & t \in (0,T) \\ y(T) = x^1 \end{cases} \qquad (2.7)$$

backward in time and define the new state $z = x - y$ which verifies

$$\begin{cases} z' = Az + Bu \\ z(0) = x^0 - y(0). \end{cases} \qquad (2.8)$$

Remark that $x(T) = x^1$ if and only if $z(T) = 0$. Hence, driving the solution x of (2.1) from x^0 to x^1 is equivalent to leading the solution z of (2.8) from the initial data $z^0 = x^0 - y(0)$ to zero.

∎

The previous remark motivates the following definition:

Definition 2.1.2 *System* (2.1) *is said to be* **null-controllable** *in time* $T > 0$ *if given any initial data* $x^0 \in \mathbb{R}^n$ *there exists* $u \in L^2(0, T, \mathbb{R}^m)$ *such that* $x(T) = 0$.

Null-controllability holds if and only if $0 \in R(x^0, T)$ for any $x^0 \in \mathbb{R}^n$.

On the other hand, Remark 2.1.1 shows that *exact controllability and null controllability are equivalent properties in the case of finite dimensional linear systems.* But this is not necessarily the case for nonlinear systems, or, for strongly time irreversible infinite dimensional systems. For instance, the heat equation is a well known example of null-controllable system that is not exactly controllable.

2.1.3 Observability property

The exact controllability property is closely related to an inequality for the corresponding adjoint homogeneous system. This is the so called *observation or observability inequality*. In this section we introduce this notion and show its relation with the exact controllability property.

Let A^* be the adjoint matrix of A, i.e. the matrix with the property that $\langle Ax, y\rangle = \langle x, A^*y\rangle$ for all $x, y \in \mathbb{R}^n$. Consider the following homogeneous *adjoint system* of (2.1):

$$\begin{cases} -\varphi' = A^*\varphi, & t \in (0,T) \\ \varphi(T) = \varphi_T. \end{cases} \tag{2.9}$$

Remark that, for each $\varphi_T \in \mathbb{R}$, (2.9) may be solved backwards in time and it has a unique solution $\varphi \in C^\omega([0,T], \mathbb{R}^n)$ (the space of analytic functions defined in $[0,T]$ and with values in $\mathbb{R}^n$).

First of all we deduce an equivalent condition for the exact controllability property.

Lemma 2.1.1 *An initial datum $x^0 \in \mathbb{R}^n$ of* (2.1) *is driven to zero in time T by using a control $u \in L^2(0,T)$ if and only if*

$$\int_0^T \langle u, B^*\varphi\rangle dt + \langle x^0, \varphi(0)\rangle = 0 \tag{2.10}$$

for any $\varphi_T \in \mathbb{R}^n$, φ being the corresponding solution of (2.9).

Proof. Let φ_T be arbitrary in $\mathbb{R}^n$ and φ the corresponding solution of (2.9). By multiplying (2.1) by φ and (2.9) by x we deduce that

$$\langle x', \varphi\rangle = \langle Ax, \varphi\rangle + \langle Bu, \varphi\rangle; \quad -\langle x, \varphi'\rangle = \langle A^*\varphi, x\rangle.$$

Hence,

$$\frac{d}{dt}\langle x, \varphi\rangle = \langle Bu, \varphi\rangle$$

which, after integration in time, gives that

$$\langle x(T), \varphi_T\rangle - \langle x^0, \varphi(0)\rangle = \int_0^T \langle Bu, \varphi\rangle dt = \int_0^T \langle u, B^*\varphi\rangle dt. \tag{2.11}$$

We obtain that $x(T) = 0$ if and only if (2.10) is verified for any $\varphi_T \in \mathbb{R}^n$.

■

It is easy to see that (2.10) is in fact an optimality condition for the critical points of the quadratic functional $J : \mathbb{R}^n \to \mathbb{R}^n$,

$$J(\varphi_T) = \frac{1}{2}\int_0^T \mid B^*\varphi \mid^2 dt + \langle x^0, \varphi(0)\rangle$$

where φ is the solution of the adjoint system (2.9) with initial data φ_T at time $t = T$.

More precisely, we have the following result:

Lemma 2.1.2 *Suppose that J has a minimizer $\widehat{\varphi}_T \in \mathbb{R}^n$ and let $\widehat{\varphi}$ be the solution of the adjoint system* (2.9) *with initial data $\widehat{\varphi}_T$. Then*

$$u = B^*\widehat{\varphi} \tag{2.12}$$

is a control of system (2.1) *with initial data x^0.*

Proof. If $\widehat{\varphi}_T$ is a point where J achieves its minimum value, then

$$\lim_{h\to 0} \frac{J\left(\widehat{\varphi}_T + h\varphi_T\right) - J\left(\widehat{\varphi}_T\right)}{h} = 0, \quad \forall \varphi_T \in \mathbb{R}^n.$$

This is equivalent to

$$\int_0^T \langle B^*\widehat{\varphi}, B^*\varphi\rangle dt + \langle x^0, \varphi(0)\rangle = 0, \quad \forall \varphi_T \in \mathbb{R}^n,$$

which, in view of Lemma 2.1.1, implies that $u = B^*\widehat{\varphi}$ is a control for (2.1).

■

Remark 2.1.2 Lemma 2.1.2 gives a variational method to obtain the control as a minimum of the functional J. This is not the unique possible functional allowing to build the control. By modifying it conveniently, other types of controls (for instance *bang-bang* ones) can be obtained. We shall show this in section 2.1.5. Remark that the controls we found are of the form $B^*\varphi$, φ being a solution of the homogeneous adjoint problem (2.9). Therefore, they are analytic functions of time.

■

The following notion will play a fundamental role in solving the control problems.

Definition 2.1.3 *System* (2.9) *is said to be* **observable** *in time $T > 0$ if there exists $c > 0$ such that*

$$\int_0^T \mid B^*\varphi \mid^2 dt \geq c \mid \varphi(0) \mid^2, \tag{2.13}$$

for all $\varphi_T \in \mathbb{R}^n$, φ being the corresponding solution of (2.9).

In the sequel (2.13) will be called the **observation or observability inequality**. It guarantees that the solution of the adjoint problem at $t = 0$ is uniquely determined by the observed quantity $B^*\varphi(t)$ for $0 < t < T$. In other words, the information contained in this term completely characterizes the solution of (2.9).

Remark 2.1.3 The observation inequality (2.13) is equivalent to the following one: there exists $c > 0$ such that

$$\int_0^T \mid B^*\varphi \mid^2 dt \geq c \mid \varphi_T \mid^2, \tag{2.14}$$

for all $\varphi_T \in \mathbb{R}^n$, φ being the solution of (2.9).

Indeed, the equivalence follows from the fact that the map which associates to every $\varphi_T \in \mathbb{R}^n$ the vector $\varphi(0) \in \mathbb{R}^n$, is a bounded linear transformation in $\mathbb{R}^n$ with bounded inverse. We shall use the forms (2.13) or (2.14) of the observation inequality depending of the needs of each particular problem we shall deal with.

■

The following remark is very important in the context of finite dimensional spaces.

Proposition 2.1.1 *Inequality* (2.13) *is equivalent to the following unique continuation principle:*

$$B^*\varphi(t) = 0,\ \forall t \in [0, T] \Rightarrow \varphi_T = 0. \tag{2.15}$$

Proof. One of the implications follows immediately from (2.14). For the other one, let us define the semi-norm in $\mathbb{R}^n$

$$|\varphi_T|_* = \left[\int_0^T \mid B^*\varphi \mid^2 dt\right]^{1/2}.$$

Clearly, $| \cdot |_*$ is a norm in $\mathbb{R}^n$ if and only if (2.15) holds.

Since all the norms in $\mathbb{R}^n$ are equivalent, it follows that (2.15) is equivalent to (2.14). The proof ends by taking into account the previous Remark.

■

Remark 2.1.4 Let us remark that (2.13) and (2.15) will no longer be equivalent properties in infinite dimensional spaces. They will give rise to different notions of controllability (exact and approximate, respectively). This issue will be further developed in the following section. ■

The importance of the observation inequality relies on the fact that it implies exact controllability of (2.1). In this way the controllability property is reduced to the study of an inequality for the homogeneous system (2.9) which, at least conceptually, is a simpler problem. Let us analyze now the relation between the controllability and observability properties.

Theorem 2.1.1 *System* (2.1) *is exactly controllable in time* T *if and only if* (2.9) *is observable in time* T.

Proof. Let us prove first that observability implies controllability. According to Lemma 2.1.2, the exact controllability property in time T holds if for any $x^0 \in \mathbb{R}^n$, J has a minimum. Remark that J is continuous. Consequently, the existence of a minimum is ensured if J is coercive too, i.e.

$$\lim_{|\varphi_T| \to \infty} J(\varphi_T) = \infty. \tag{2.16}$$

The coercivity property (2.16) is a consequence of the observation property in time T. Indeed, from (2.13) we obtain that

$$J(\varphi_T) \geq \frac{c}{2} \mid \varphi_T \mid^2 - |\langle x^0, \varphi(0)\rangle|.$$

The right hand side tends to infinity when $|\varphi_T| \to \infty$ and J satisfies (2.16).

Reciprocally, suppose that system (2.1) is exactly controllable in time T. If (2.9) is not observable in time T, there exists a sequence $(\varphi_T^k)_{k\geq 1} \subset \mathbb{R}^n$ such that $|\varphi_T^k| = 1$ for all $k \geq 1$ and

$$\lim_{k\to\infty} \int_0^T |B^* \varphi^k|^2 dt = 0. \tag{2.17}$$

It follows that there exists a subsequence of $(\varphi_T^k)_{k\geq 1}$, denoted in the same way, which converges to $\varphi_T \in \mathbb{R}^n$ and $|\varphi_T| = 1$. Moreover, if φ is the solution of (2.9) with initial data φ_T, from (2.17) it follows that

$$\int_0^T |B^* \varphi|^2 dt = 0. \tag{2.18}$$

Since (2.1) is controllable, Lemma 2.1.1 gives that, for any initial data $x^0 \in \mathbb{R}^n$, there exists $u \in L^2(0,T)$ such that

$$\int_0^T \langle u, B^* \varphi_k \rangle dt = -\langle x^0, \varphi_k(0)\rangle, \quad \forall k \geq 1. \tag{2.19}$$

By passing to the limit in (2.19) and by taking into account (2.18), we obtain that $< x^0, \varphi(0) >= 0$. Since x^0 is arbitrary in $\mathbb{R}^n$, it follows that $\varphi(0) = 0$ and, consequently, $\varphi_T = 0$. This is in contradiction with the fact that $|\varphi_T| = 1$.

The proof of the theorem is now complete. ∎

Remark 2.1.5 The usefulness of Theorem 2.1.1 consists on the fact that it reduces the proof of the exact controllability to the study of the observation inequality.

■

2.1.4 Kalman's controllability condition

The following classical result is due to R. E. Kalman and gives a complete answer to the problem of exact controllability of finite dimensional linear systems. It shows, in particular, that the time of control is irrelevant.

Theorem 2.1.2 ([136]) *System* (2.1) *is exactly controllable in some time* T *if and only if*

$$\mathrm{rank}\,[B,\ AB, \cdots, A^{n-1}B] = n. \tag{2.20}$$

Consequently, if system (2.1) *is controllable in some time* $T > 0$ *it is controllable in any time.*

Remark 2.1.6 From now on we shall simply say that (A, B) is controllable if (2.20) holds. The matrix $[B, AB, \cdots, A^{n-1}B]$ will be called the *controllability matrix*.

■

Examples. In Example 1 from section 2.1.2 we had

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{2.21}$$

Therefore

$$[B,\ AB] = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.22}$$

which has rank 1. From Theorem 2.1.2 it follows that the system under consideration is not controllable. Nevertheless, in Example 2,

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.23}$$

and consequently

$$[B,\ AB] = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2.24}$$

which has rank 2 and the system is controllable as we have already observed.

■

Proof of Theorem 2.1.2 " $\Rightarrow$" Suppose that $\mathrm{rank}([B,\, AB, \cdots, A^{n-1}B]) < n$.

Then the rows of the controllability matrix $[B, AB, \cdots, A^{n-1}B]$ are linearly dependent and there exists a vector $v \in \mathbb{R}^n$, $v \neq 0$ such that

$$v^*[B,\, AB, \cdots, A^{n-1}B] = 0,$$

where the coefficients of the linear combination are the components of the vector v. Since $v^*[B,\, AB, \cdots, A^{n-1}B] = [v^*B,\, v^*AB, \cdots, v^*A^{n-1}B]$, $v^*B = v^*AB = \cdots = v^*\, A^{n-1}B = 0$. From Cayley-Hamilton Theorem we deduce that there exist constants $c_1, \cdots, c_n$ such that, $A^n = c_1A^{n-1} + \cdots + c_nI$ and therefore $v^*A^nB = 0$, too. In fact, it follows that $v^*A^kB = 0$ for all $k \in \mathbb{N}$ and consequently $v^*e^{At}B = 0$ for all t as well. But, from the variation of constants formula, the solution x of (2.1) satisfies

$$x(t) = e^{At}x^0 + \int_0^t e^{A(t-s)}Bu(s)ds. \tag{2.25}$$

Therefore

$$\langle v, x(T)\rangle = \langle v, e^{AT}x^0\rangle + \int_0^T \langle v, e^{A(T-s)}Bu(s)\rangle ds = \langle v, e^{AT}x^0\rangle,$$

where $\langle\, ,\, \rangle$ denotes the canonical inner product in $\mathbb{R}^n$. Hence, $\langle v, x(T)\rangle = \langle v, e^{AT}x^0\rangle$. This shows that the projection of the solution x at time T on the vector v is independent of the value of the control u. Hence, the system is not controllable.

■

Remark 2.1.7 The conservation property for the quantity $\langle v, x\rangle$ we have just proved holds for any vector v for which $v[B,\, AB, \cdots, A^{n-1}B] = 0$. Thus, if the rank of the matrix $[B,\, AB, \cdots, A^{n-1}B]$ is $n-k$, the reachable set that $x(T)$ runs is an affine subspace of $\mathbb{R}^n$ of dimension $n-k$.

■

" $\Leftarrow$" Suppose now that $\mathrm{rank}([B,\, AB, \cdots, A^{n-1}B]) = n$. According to Theorem 2.1.1 it is sufficient to show that system (2.9) is observable. By Proposition 2.1.1, (2.13) holds if and only if (2.15) is verified. Hence, the Theorem is proved if (2.15) holds. ¿From $B^*\varphi = 0$ and $\varphi(t) = e^{A^*(T-t)}\varphi_T$, it follows that $B^*e^{A^*(T-t)}\varphi_T \equiv 0$ for all $0 \leq t \leq T$. By computing the derivatives of this function in $t = T$ we obtain that

$$B^*[A^*]^k\varphi_T = 0 \quad \forall k \geq 0.$$

But since $\text{rank}(\big[B,\ AB, \cdots, A^{n-1}B\big]) = n$ we deduce that

$$\text{rank}(\big[B^*,\ B^*A^*, \cdots, B^*(A^*)^{n-1}\big]) = n$$

and therefore $\varphi_T = 0$. Hence, (2.15) is verified and the proof of Theorem 2.1.2 is now complete.

■

Remark 2.1.8 The set of controllable pairs (A, B) is open and dense. Indeed,

- If (A, B) is controllable there exists $\varepsilon > 0$ sufficiently small such that any (A^0, B^0) with $\mid A^0 - A \mid < \varepsilon$, $\mid B^0 - B \mid < \varepsilon$ is also controllable. This is a consequence of the fact that the determinant of a matrix depends continuously of its entries.
- On the other hand, if (A, B) is not controllable, for any $\varepsilon > 0$, there exists (A^0, B^0) with $\mid A - A^0 \mid < \varepsilon$ and $\mid B - B^0 \mid < \varepsilon$ such that (A^0, B^0) is controllable. This is a consequence of the fact that the determinant of a $n \times n$ matrix depends analytically of its entries and cannot vanish in a ball of $\mathbb{R}^{n\times n}$.

■

The following inequality shows that the norm of the control is proportional to the distance between $e^{AT}x^0$ (the state freely attained by the system in the absence of control, i. e. with $u = 0$) and the objective x^1.

Proposition 2.1.2 *Suppose that the pair (A, B) is controllable in time $T > 0$ and let u be the control obtained by minimizing the functional J. There exists a constant $C > 0$, depending on T, such that the following inequality holds*

$$\| u \|_{L^2(0,T)} \leq C|e^{AT}x^0 - x^1| \tag{2.26}$$

for any initial data x^0 and final objective x^1.

Proof. Let us first prove (2.26) for the particular case $x^1 = 0$.

Let u be the control for (2.1) obtained by minimizing the functional J. From (2.10) it follows that

$$||u||^2_{L^2(0,T)} = \int_0^T |B^*\widehat{\varphi}|^2 dt = - < x^0, \widehat{\varphi}(0) > .$$

If w is the solution of

$$\begin{cases} w'(t) = Aw(t), \quad t \in (0, T), \\ \quad w(0) = x^0 \end{cases} \tag{2.27}$$

then $w(t) = e^{At}x^0$ and

$$\frac{d}{dt} < w, \varphi >= 0$$

for all $\varphi_T \in \mathbb{R}^n$, φ being the corresponding solution of (2.9).

In particular, by taking $\varphi_T = \widehat{\varphi}_T$, the minimizer of J, it follows that

$$< x^0, \widehat{\varphi}(0) >=< w(0), \widehat{\varphi}(0) >=< w(T), \widehat{\varphi}_T >=< e^{AT}x^0, \widehat{\varphi}_T > .$$

We obtain that

$$||u||^2_{L^2(0,T)} = - < x^0, \widehat{\varphi}(0) >= - < e^{AT}x^0, \widehat{\varphi}_T >\leq |e^{AT}x^0|\,|\widehat{\varphi}_T|.$$

On the other hand, we have that

$$|\widehat{\varphi}_T| \leq c||B^*\widehat{\varphi}||_{L^2(0,T)} = c||u||_{L^2(0,T)}.$$

Thus, the control u verifies

$$||u||_{L^2(0,T)} \leq c|e^{AT}x^0|. \tag{2.28}$$

If $x^1 \neq 0$, Remark 2.1.1 implies that a control u driving the solution from x^0 to x^1 coincides with the one leading the solution from $x^0 - y(0)$ to zero, where y verifies (2.7). By using (2.28) we obtain that

$$||u||_{L^2(0,T)} \leq c|e^{TA}(x^0 - y(0))| = c|e^{TA}x^0 - x^1|$$

and (2.26) is proved.

$\blacksquare$

Remark 2.1.9 Linear scalar equations of any order provide examples of systems of arbitrarily large dimension that are controllable with only one control. Indeed, the system of order k

$$x^{(k)} + a_1x^{(k-1)} + \ldots + a_{k-1}x = u$$

is controllable. This can be easily obtained by observing that given k initial data and k final ones one can always find a trajectory z (in fact an infinite number of them) joining them in any time interval. This argument was already used in Example 2 for the case $k = 2$.

It is an interesting exercise to write down the matrices A and B in this case and to check that the rank condition in Theorem 2.1.2 is fulfilled.

$\blacksquare$

2.1.5 Bang-bang controls

Let us consider the particular case

$$B \in \mathcal{M}_{n\times 1}, \tag{2.29}$$

i. e. $m = 1$, in which only one control $u : [0, T] \to \mathbb{R}$ is available. In order to build bang-bang controls, it is convenient to consider the quadratic functional:

$$J_{bb}(\varphi^0) = \frac{1}{2}\left[\int_0^T \mid B^*\varphi \mid dt\right]^2 + \langle x^0, \varphi(0)\rangle \tag{2.30}$$

where φ is the solution of the adjoint system (2.9) with initial data φ_T.

Note that $B^* \in \mathcal{M}_{1\times n}$ and therefore $B^*\varphi(t) : [0, T] \to \mathbb{R}$ is a scalar function. It is also interesting to note that J_{bb} differs from J in the quadratic term. Indeed, in J we took the $L^2(0, T)$-norm of $B^*\varphi$ while here we consider the $L^1(0, T)$-norm.

The same argument used in the proof of Theorem 2.1.2 shows that J_{bb} is also continuous and coercive. It follows that J_{bb} attains a minimum in some point $\widehat{\varphi}_T \in \mathbb{R}^n$.

On the other hand, it is easy to see that

$$\begin{aligned}\lim_{h\to 0}\frac{1}{h}&\left[\left(\int_0^T \mid f + hg \mid dt\right)^2 - \left(\int_0^T \mid f \mid\right)^2\right] = \\ &= 2\int_0^T \mid f \mid dt \int_0^T \operatorname{sgn}(f(t))g(t)dt\end{aligned} \tag{2.31}$$

if the Lebesgue measure of the set $\{t \in (0, T) : f(t) = 0\}$ vanishes.

The sign function "sgn" is defined as a multi-valued function in the following way

$$\operatorname{sgn}(s) = \begin{cases} 1 & \text{when } s > 0 \\ -1 & \text{when } s < 0 \\ [-1, 1] & \text{when } s = 0 \end{cases}$$

Remark that in the previous limit there is no ambiguity in the definition of $\operatorname{sgn}(f(t))$ since the set of points $t \in [0, T]$ where $f = 0$ is assumed to be of zero Lebesgue measure and does not affect the value of the integral.

Identity (2.31) may be applied to the quadratic term of the functional J_{bb} since, taking into account that φ is the solution of the adjoint system (2.9), it is an analytic function and therefore, $B^*\varphi$ changes sign finitely many times in the interval $[0, T]$ except when $\widehat{\varphi}_T = 0$. In view of this, the Euler-Lagrange equation associated with the critical points of the functional J_{bb} is as follows:

$$\int_0^T \mid B^*\widehat{\varphi} \mid dt \int_0^T \operatorname{sgn}(B^*\widehat{\varphi})B^*\psi(t)dt + \langle x^0, \varphi(0)\rangle = 0$$

for all $\varphi_T \in \mathbb{R}$, where φ is the solution of the adjoint system (2.9) with initial data φ_T.

Consequently, the control we are looking for is $u = \int_0^T \mid B^*\widehat{\varphi} \mid dt \, \mathrm{sgn}(B^*\widehat{\varphi})$ where $\widehat{\varphi}$ is the solution of (2.9) with initial data $\widehat{\varphi}_T$.

Note that the control u is of bang-bang form. Indeed, u takes only two values $\pm \int_0^T \mid B^*\widehat{\varphi} \mid dt$. The control switches from one value to the other finitely many times when the function $B^*\widehat{\varphi}$ changes sign.

Remark 2.1.10 Other types of controls can be obtained by considering functionals of the form

$$J_p(\varphi^0) = \frac{1}{2}\left(\int_0^T \mid B^*\varphi \mid^p dt\right)^{2/p} + \langle x^0, \varphi^0 \rangle$$

with $1 < p < \infty$. The corresponding controls are

$$u = \left(\int_0^T \mid B^*\widehat{\varphi} \mid^p dt\right)^{(2-p)/p} |B^*\widehat{\varphi}|^{p-2} B^*\widehat{\varphi}$$

where $\widehat{\varphi}$ is the solution of (2.9) with initial datum $\widehat{\varphi}_T$, the minimizer of J_p.

It can be shown that, as expected, the controls obtained by minimizing this functionals give, in the limit when $p \to 1$, a bang-bang control.

■

The following property gives an important characterization of the controls we have studied.

Proposition 2.1.3 *The control $u_2 = B^*\widehat{\varphi}$ obtained by minimizing the functional J has minimal $L^2(0,T)$ norm among all possible controls. Analogously, the control $u_\infty = \int_0^T \mid B^*\widehat{\varphi} \mid dt \, \mathrm{sgn}(B^*\widehat{\varphi})$ obtained by minimizing the functional J_{bb} has minimal $L^\infty(0,T)$ norm among all possible controls.*

Proof. Let u be an arbitrary control for (2.1). Then (2.10) is verified both by u and u_2 for any φ_T. By taking $\varphi_T = \widehat{\varphi}_T$ (the minimizer of J) in (2.10) we obtain that

$$\int_0^T < u, B^*\widehat{\varphi} > dt = - < x^0, \widehat{\varphi}(0) >,$$

$$||u_2||^2_{L^2(0,T)} = \int_0^T < u_2, B^*\widehat{\varphi} > dt = - < x^0, \widehat{\varphi}(0) > .$$

Hence,

$$||u_2||^2_{L^2(0,T)} = \int_0^T < u, B^*\widehat{\varphi} > dt \leq ||u||_{L^2(0,T)}||B^*\widehat{\varphi}|| = ||u||_{L^2(0,T)}||u_2||_{L^2(0,T)}$$

and the first part of the proof is complete.

For the second part a similar argument may be used. Indeed, let again u be an arbitrary control for (2.1). Then (2.10) is verified by u and u_∞ for any φ_T. By taking $\varphi_T = \widehat{\varphi}_T$ (the minimizer of J_{bb}) in (2.10) we obtain that

$$\int_0^T B^*\widehat{\varphi} u dt = - < x^0, \widehat{\varphi}(0) >,$$

$$||u_\infty||^2_{L^\infty(0,T)} = \left(\int_0^T |B^*\widehat{\varphi}| dt \right)^2 = \int_0^T B^*\widehat{\varphi} u_\infty dt = - < x^0, \widehat{\varphi}(0) > .$$

Hence,

$$||u_\infty||^2_{L^\infty(0,T)} = \int_0^T B^*\widehat{\varphi}\, u dt \leq$$

$$\leq ||u||_{L^\infty(0,T)} \int_0^T |B^*\widehat{\varphi}| dt = ||u||_{L^\infty(0,T)}||u_\infty||_{L^\infty(0,T)}$$

and the proof finishes.

■

2.1.6 Stabilization of finite dimensional linear systems

In this section we assume that A is a skew-adjoint matrix, i. e. $A^* = -A$. In this case, $< Ax, x >= 0$.

Consider the system

$$\begin{cases} x' = Ax + Bu \\ x(0) = x^0. \end{cases} \tag{2.32}$$

Remark 2.1.11 The harmonic oscillator, $mx'' + kx = 0$, provides the simplest example of system with such properties. It will be studied with some detail at the end of the section.

■

When $u \equiv 0$, the energy of the solution of (2.32) is conserved. Indeed, by multiplying (2.32) by x, if $u \equiv 0$, one obtains

$$\frac{d}{dt}|x(t)|^2 = 0. \tag{2.33}$$

Hence,

$$|x(t)| = |x^0|, \quad \forall t \geq 0. \tag{2.34}$$

The problem of *stabilization* can be formulated in the following way. Suppose that the pair (A, B) is controllable. We then look for a matrix L such that the solution of system (2.32) with the *feedback* control

$$u(t) = Lx(t) \tag{2.35}$$

has **a uniform exponential decay**, i.e. there exist $c > 0$ and $\omega > 0$ such that

$$|x(t)| \leq ce^{-\omega t}|x^0| \tag{2.36}$$

for any solution.

Note that, according to the law (2.35), the control u is obtained in real time from the state x.

In other words, we are looking for matrices L such that the solution of the system

$$x' = (A + BL)x = Dx \tag{2.37}$$

has an uniform exponential decay rate.

Remark that we cannot expect more than (2.36). Indeed, the solutions of (2.37) may not satisfy $x(T) = 0$ in finite time T. Indeed, if it were the case, from the uniqueness of solutions of (2.37) with final state 0 in $t = T$, it would follow that $x^0 \equiv 0$. On the other hand, whatever L is, the matrix D has N eigenvalues λ_j with corresponding eigenvectors $e_j \in \mathbb{R}^n$. The solution $x(t) = e^{\lambda_j t}e_j$ of (2.37) shows that the decay of solutions can not be faster than exponential.

Theorem 2.1.3 *If A is skew-adjoint and the pair (A, B) is controllable then $L = -B^*$ stabilizes the system, i.e. the solution of*

$$\left\{ \begin{array}{c} x' = Ax - BB^*x \\ x(0) = x^0 \end{array} \right. \tag{2.38}$$

has an uniform exponential decay (2.36).

Proof. With $L = -B^*$ we obtain that

$$\frac{1}{2}\frac{d}{dt}|x(t)|^2 = - < BB^*x(t), x(t) >= - \mid B^*x(t) \mid^2 \leq 0.$$

Hence, the norm of the solution decreases in time.

Moreover,

$$|x(T)|^2 - |x(0)|^2 = -2\int_0^T \mid B^*x \mid^2 dt. \tag{2.39}$$

To prove the uniform exponential decay it is sufficient to show that there exist $T > 0$ and $c > 0$ such that

$$|x(0)|^2 \leq c \int_0^T | B^* x |^2 \, dt \tag{2.40}$$

for any solution x of (2.38). Indeed, from (2.39) and (2.40) we would obtain that

$$|x(T)|^2 - |x(0)|^2 \leq -\frac{2}{c}|x(0)|^2 \tag{2.41}$$

and consequently

$$|x(T)|^2 \leq \gamma |x(0)|^2 \tag{2.42}$$

with

$$\gamma = 1 - \frac{2}{c} < 1. \tag{2.43}$$

Hence,

$$|x(kT)|^2 \leq \gamma^k |x^0|^2 = e^{(\ln \gamma) k} |x^0|^2 \quad \forall k \in \mathbb{N}. \tag{2.44}$$

Now, given any $t > 0$ we write it in the form $t = kT + \delta$, with $\delta \in [0, T)$ and $k \in \mathbb{N}$ and we obtain that

$$|x(t)|^2 \leq |x(kT)|^2 \leq e^{-|\ln(\gamma)| k} |x^0|^2 =$$

$$= e^{-|\ln(\gamma)| \left(\frac{t}{T}\right)} e^{|\ln(\gamma)| \frac{\delta}{T}} |x^0|^2 \leq \frac{1}{\gamma} e^{-\frac{|\ln(\gamma)|}{T} t} |x^0|^2.$$

We have obtained the desired decay result (2.36) with

$$c = \frac{1}{\gamma}, \; \omega = \frac{| \ln(\gamma) |}{T}. \tag{2.45}$$

To prove (2.40) we decompose the solution x of (2.38) as $x = \varphi + y$ with φ and y solutions of the following systems:

$$\begin{cases} \varphi' = A\varphi \\ \varphi(0) = x^0, \end{cases} \tag{2.46}$$

and

$$\begin{cases} y' = Ay - BB^* x \\ y(0) = 0. \end{cases} \tag{2.47}$$

Remark that, since A is skew-adjoint, (2.46) is exactly the adjoint system (2.9) except for the fact that the initial data are taken at $t = 0$.

As we have seen in the proof of Theorem 2.1.2, the pair (A, B) being controllable, the following observability inequality holds for system (2.46):

$$|x^0|^2 \leq C \int_0^T | B^* \varphi |^2 \, dt. \tag{2.48}$$

Since $\varphi = x - y$ we deduce that

$$|x^0|^2 \leq 2C \left[\int_0^T \mid B^* x \mid^2 dt + \int_0^T \mid B^* y \mid^2 dt \right].$$

On the other hand, it is easy to show that the solution y of (2.47) satisfies:

$$\frac{1}{2}\frac{d}{dt} \mid y \mid^2 = -\langle B^* x, B^* y \rangle \leq |B^* x|\,|B^*|\,|y| \leq \frac{1}{2}\left(|y|^2 + |B^*|^2 |B^* x|^2 \right).$$

From Gronwall's inequality we deduce that

$$\mid y(t) \mid^2 \leq |B^*|^2 \int_0^t e^{t-s} \mid B^* x \mid^2 ds \leq |B^*|^2 e^T \int_0^T \mid B^* x \mid^2 dt \tag{2.49}$$

and consequently

$$\int_0^T \mid B^* y \mid^2 dt \leq |B|^2 \int_0^T \mid y \mid^2 dt \leq T|B|^4 e^T \int_0^T \mid B^* x \mid^2 dt.$$

Finally, we obtain that

$$\mid x^0 \mid^2 \leq 2C \int_0^T \mid B^* x \mid^2 dt + C|B^*|^4 e^T T \int_0^T \mid B^* x \mid^2 dt \leq C' \int_0^T \mid B^* x \mid^2 dt$$

and the proof of Theorem 2.1.3 is complete.

■

Example. Consider the damped harmonic oscillator:

$$mx'' + Rx + kx' = 0, \tag{2.50}$$

where m, k and R are positive constants.

Note that (2.50) may be written in the equivalent form

$$mx'' + Rx = -kx'$$

which indicates that an applied force, proportional to the velocity of the point-mass and of opposite sign, is acting on the oscillator.

It is easy to see that the solutions of this equation have an exponential decay property. Indeed, it is sufficient to remark that the two characteristic roots have negative real part. Indeed,

$$mr^2 + R + kr = 0 \Leftrightarrow r_{\pm} = \frac{-k \pm \sqrt{k^2 - 4mR}}{2m}$$

and therefore

$$\operatorname{Re} r_{\pm} = \begin{cases} -\frac{k}{2m} & \text{if } k^2 \leq 4mR \\ -\frac{k}{2m} \pm \sqrt{\frac{k^2}{4m} - \frac{R}{2m}} & \text{if } k^2 \geq 4mR. \end{cases}$$

Let us prove the exponential decay of the solutions of (2.50) by using Theorem 2.1.3. Firstly, we write (2.50) in the form (2.38). Setting

$$X = \begin{pmatrix} x \\ \sqrt{\frac{m}{R}} x' \end{pmatrix},$$

the conservative equation $mx'' + kx = 0$ corresponds to the system:

$$X' = AX, \quad \text{with } A = \begin{pmatrix} 0 & \sqrt{\frac{R}{m}} \\ -\sqrt{\frac{R}{m}} & 0 \end{pmatrix}.$$

Note that A is a skew-adjoint matrix. On the other hand, if we choose

$$B = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{k} \end{pmatrix}$$

we obtain that

$$BB^* = \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix}$$

and the system

$$X' = AX - BB^*X \tag{2.51}$$

is equivalent to (2.50).

Now, it is easy to see that the pair (A, B) is controllable since the rank of $[B, AB]$ is 2.

It follows that the solutions of (2.50) have the property of exponential decay as the explicit computation of the spectrum indicates.

■

If (A, B) is controllable, we have proved the uniform stability property of the system (2.32), under the hypothesis that A is skew-adjoint. However, this property holds even if A is an arbitrary matrix. More precisely, we have

Theorem 2.1.4 *If (A, B) is controllable then it is also stabilizable. Moreover, it is possible to prescribe any complex numbers λ_1, λ_2,...,λ_n as the eigenvalues of the closed loop matrix $A+BL$ by an appropriate choice of the feedback matrix L so that the decay rate may be made arbitrarily fast.*

In the statement of the Theorem we use the classical term *closed loop* system to refer to the system in which the control is given in feedback form.

The proof of Theorem 2.1.4 is obtained by reducing system (2.32) to the so called *control canonical form* (see [136] and [194]).

2.2 Interior controllability of the wave equation

In this section the problem of interior controllability of the wave equation is studied. The control is assumed to act on a subset of the domain where the solutions are defined. The problem of boundary controllability, which is also important in applications and has attracted a lot of attention, will be considered in the following section. In the later case the control acts on the boundary of the domain where the solutions are defined.

2.2.1 Introduction

Let Ω be a bounded open set of $\mathbb{R}^N$ with boundary of class C^2 and ω be an open nonempty subset of Ω. Given $T > 0$ consider the following non-homogeneous wave equation:

$$\begin{cases} u'' - \Delta u = f 1_\omega & \text{in } (0, T) \times \Omega \\ u = 0 & \text{on } (0, T) \times \partial\Omega \\ u(0, \cdot) = u^0,\ u'(0, \cdot) = u^1 & \text{in } \Omega. \end{cases} \tag{2.52}$$

By $'$ we denote the time derivative.

In (2.52) $u = u(t, x)$ is the state and $f = f(t, x)$ is the interior control function with support localized in ω. We aim at changing the dynamics of the system by acting on the subset ω of the domain Ω.

It is well known that the wave equation models many physical phenomena such as small vibrations of elastic bodies and the propagation of sound. For instance (2.52) provides a good approximation for the small amplitude vibrations of an elastic string or a flexible membrane occupying the region Ω at rest. The control f represents then a localized force acting on the vibrating structure.

The importance of the wave equation relies not only in the fact that it models a large class of vibrating phenomena but also because it is the most relevant hyperbolic partial differential equation. As we shall see latter on, the main properties of hyperbolic equations such as time-reversibility and the lack of regularizing effects, have some very important consequences in control problems too.

Therefore it is interesting to study the controllability of the wave equation as one of the fundamental models of continuum mechanics and, at the same time, as one of the most representative equations in the theory of partial differential equations.

2.2.2 Existence and uniqueness of solutions

The following theorem is a consequence of classical results of existence and uniqueness of solutions of nonhomogeneous evolution equations. All the details may be found, for instance, in [60].

Theorem 2.2.1 *For any $f \in L^2((0,T)\times\omega)$ and $(u^0,u^1)\in H_0^1(\Omega)\times L^2(\Omega)$ equation* (2.52) *has a unique weak solution*

$$(u,u') \in C([0,T], H_0^1(\Omega)\times L^2(\Omega))$$

given by the variation of constants formula

$$(u,u')(t) = S(t)(u^0,u^1) + \int_0^t S(t-s)(0, f(s)1_\omega)ds \tag{2.53}$$

where $(S(t))_{t\in\mathbb{R}}$ is the group of isometries generated by the wave operator in $H_0^1(\Omega)\times L^2(\Omega)$.

Moreover, if $f \in W^{1,1}((0,T);L^2(\omega))$ and $(u^0,u^1)\in[H^2(\Omega)\cap H_0^1(\Omega)]\times H_0^1(\Omega)$ equation (2.52) *has a strong solution*

$$(u,u') \in C^1([0,T], H_0^1(\Omega)\times L^2(\Omega)) \cap C([0,T],[H^2(\Omega)\cap H_0^1(\Omega)]\times H_0^1(\Omega))$$

and u verifies the wave equation (2.52) *in $L^2(\Omega)$ for all $t\geq 0$.*

Remark 2.2.1 The wave equation is reversible in time. Hence, we may solve it for $t\in(0,T)$ by considering initial data (u^0,u^1) in $t=0$ or final data (u_T^0,u_T^1) in $t=T$. In the former case the solution is given by (2.53) and in the later one by

$$(u,u')(t) = S(T-t)(u_T^0,-u_T^1) + \int_t^T S(s-t)(0,f(s)1_\omega)ds. \tag{2.54}$$

■

2.2.3 Controllability problems

Let $T>0$ and define, for any initial data $(u^0,u^1)\in H_0^1(\Omega)\times L^2(\Omega)$, the set of reachable states

$$R(T;(u^0,u^1)) = \{(u(T),u_t(T)) : u \text{ solution of (2.52) with } f\in L^2((0,T)\times\omega)\}.$$

Remark that, for any $(u^0,u^1)\in H_0^1(\Omega)\times L^2(\Omega)$, $R(T;(u^0,u^1))$ is an affine subspace of $H_0^1(\Omega)\times L^2(\Omega)$.

There are different notions of controllability that need to be distinguished.

Definition 2.2.1 *System* (2.52) *is* **approximately controllable in time** T *if, for every initial data $(u^0,u^1)\in H_0^1(\Omega)\times L^2(\Omega)$, the set of reachable states $R(T;(u^0,u^1))$ is dense in $H_0^1(\Omega)\times L^2(\Omega)$.*

Definition 2.2.2 *System* (2.52) *is* **exactly controllable in time** T *if, for every initial data $(u^0,u^1)\in H_0^1(\Omega)\times L^2(\Omega)$, the set of reachable states $R(T;(u^0,u^1))$ coincides with $H_0^1(\Omega)\times L^2(\Omega)$.*

Definition 2.2.3 *System* (2.52) *is* **null controllable in time** T *if, for every initial data* $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$*, the set of reachable states* $R(T; (u^0, u^1))$ *contains the element* $(0, 0)$.

Since the only dense and convex subset of $\mathbb{R}^n$ is $\mathbb{R}^n$, it follows that the approximate and exact controllability notions are equivalent in the finite-dimensional case. Nevertheless, for infinite dimensional systems as the wave equation, these two notions do not coincide.

Remark 2.2.2 In the notions of approximate and exact controllability it is sufficient to consider the case $(u^0, u^1) \equiv 0$ since $R(T; (u^0, u^1)) = R(T; (0, 0)) + S(T)(u^0, u^1)$.

■

In the view of the time-reversibility of the system we have:

Proposition 2.2.1 *System* (2.52) *is exactly controllable if and only if it is null controllable.*

Proof. Evidently, exact controllability implies null controllability.

Let us suppose now that $(0, 0) \in R(T; (u^0, u^1))$ for any $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$. Then any initial data in $H_0^1(\Omega) \times L^2(\Omega)$ can be driven to $(0, 0)$ in time T. From the reversibility of the wave equation we deduce that any state in $H_0^1(\Omega) \times L^2(\Omega)$ can be reached in time T by starting from $(0, 0)$. This means that $R(T, (0, 0)) = H_0^1(\Omega) \times L^2(\Omega)$ and the exact controllability property holds from Remark 2.2.2.

■

The previous Proposition guarantees that (2.52) is exactly controllable if and only if, for any $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$ there exists $f \in L^2((0, T) \times \omega)$ such that the corresponding solution (u, u') of (2.52) satisfies

$$u(T, \cdot\,) = u'(T, \cdot\,) = 0. \tag{2.55}$$

This is the most common form in which the exact controllability property for the wave equation is formulated.

Remark 2.2.3 The following facts indicate how the main distinguishing properties of wave equation affect its controllability properties:

- Since the wave equation is time-reversible and does not have any regularizing effect, one may not exclude the exact controllability to hold. Nevertheless, as we have said before, there are situations in which the exact controllability property is not verified but the approximate controllability holds. This depends on the geometric properties of Ω and ω.

- The wave equation is a prototype of equation with finite speed of propagation. Therefore, one cannot expect the previous controllability properties to hold unless the control time T is sufficiently large.

$\blacksquare$

2.2.4 Variational approach and observability

Let us first deduce a necessary and sufficient condition for the exact controllability property of (2.52) to hold. By $\langle \cdot, \cdot \rangle_{1,-1}$ we denote the duality product between $H_0^1(\Omega)$ and its dual, $H^{-1}(\Omega)$.

For $(\varphi_T^0, \varphi_T^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, consider the following backward homogeneous equation

$$\begin{cases} \varphi'' - \Delta\varphi = 0 & \text{in } (0,T) \times \Omega \\ \varphi = 0 & \text{on } (0,T) \times \partial\Omega \\ \varphi(T, \cdot) = \varphi_T^0,\ \varphi'(T, \cdot) = \varphi_T^1 & \text{in } \Omega. \end{cases} \tag{2.56}$$

Let $(\varphi, \varphi') \in C([0,T], L^2(\Omega) \times H^{-1}(\Omega))$ be the unique weak solution of (2.56).

Lemma 2.2.1 *The control $f \in L^2((0,T)\times\omega)$ drives the initial data $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$ of system* (2.52) *to zero in time T if and only if*

$$\int_0^T \int_\omega \varphi f dx dt = \left\langle \varphi'(0), u^0 \right\rangle_{1,-1} - \int_\Omega \varphi(0) u^1 dx, \tag{2.57}$$

for all $(\varphi_T^0, \varphi_T^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ where φ is the corresponding solution of (2.56).

Proof. Let us first suppose that $\left(u^0, u^1\right), \left(\varphi_T^0, \varphi_T^1\right) \in \mathcal{D}(\Omega) \times \mathcal{D}(\Omega)$, $f \in \mathcal{D}((0,T) \times \omega)$ and let u and φ be the (regular) solutions of (2.52) and (2.56) respectively.

We recall that $\mathcal{D}(M)$ denotes the set of $C^\infty(M)$ functions with compact support in M.

By multiplying the equation of u by φ and by integrating by parts one obtains

$$\int_0^T \int_\omega \varphi f dx dt = \int_0^T \int_\Omega \varphi \left(u'' - \Delta u\right) dx dt =$$
$$= \int_\Omega \left(\varphi u' - \varphi' u\right) dx \Big|_0^T + \int_0^T \int_\Omega u \left(\varphi'' - \Delta\varphi\right) dx dt =$$
$$= \int_\Omega \left[\varphi(T) u'(T) - \varphi'(T) u(T)\right] dx - \int_\Omega \left[\varphi(0) u'(0) - \varphi'(0) u(0)\right] dx.$$

Hence,

$$\int_0^T \int_\omega \varphi f dxdt = \int_\Omega \left[\varphi_T^0 u'(T) - \varphi_T^1 u(T)\right] dx - \int_\Omega \left[\varphi(0)u^1 - \varphi'(0)u^0\right] dx. \quad (2.58)$$

From a density argument we deduce, by passing to the limit in (2.58), that for any $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and $(\varphi_T^0, \varphi_T^1) \in L^2(\Omega) \times H^{-1}(\Omega)$,

$$\begin{gathered} \int_0^T \int_\omega \varphi f dxdt = \\ = -\left\langle \varphi_T^1, u(T)\right\rangle_{1,-1} + \int_\Omega \varphi_T^0 u'(T)dx + \left\langle \varphi'(0), u^0\right\rangle_{1,-1} - \int_\Omega \varphi(0)u^1 dx. \end{gathered} \quad (2.59)$$

Now, from (2.59), it follows immediately that (2.57) holds if and only if (u^0, u^1) is controllable to zero and f is the corresponding control. This completes the proof.

■

Let us define the duality product between $L^2(\Omega) \times H^{-1}(\Omega)$ and $H_0^1(\Omega) \times L^2(\Omega)$ by

$$\left\langle \left(\varphi^0, \varphi^1\right), \left(u^0, u^1\right)\right\rangle = \left\langle \varphi^1, u^0\right\rangle_{1,-1} - \int_\Omega \varphi^0 u^1 dx$$

for all $\left(\varphi^0, \varphi^1\right) \in L^2(\Omega) \times H^{-1}(\Omega)$ and $\left(u^0, u^1\right) \in H_0^1(\Omega) \times L^2(\Omega)$.

Remark that the map $\left(\varphi^0, \varphi^1\right) \to \left\langle \left(\varphi^0, \varphi^1\right), \left(u^0, u^1\right)\right\rangle$ is linear and continuous and its norm is equal to $||(u^0, u^1)||_{H_0^1 \times L^2}$.

For $(\varphi^0, \varphi^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, consider the following homogeneous equation

$$\begin{cases} \varphi'' - \Delta\varphi = 0 & \text{in } (0,T) \times \Omega \\ \varphi = 0 & \text{on } (0,T) \times \partial\Omega \\ \varphi(0, \cdot) = \varphi^0,\ \varphi'(0, \cdot) = \varphi^1 & \text{in } \Omega. \end{cases} \quad (2.60)$$

If $(\varphi, \varphi') \in C([0,T], L^2(\Omega) \times H^{-1}(\Omega))$ is the unique weak solution of (2.60), then

$$||\varphi||^2_{L^\infty(0,T;L^2(\Omega))} + ||\varphi'||^2_{L^\infty(0,T;H^{-1}(\Omega))} \leq ||(\varphi^0, \varphi^1)||^2_{L^2(\Omega)\times H^{-1}(\Omega)}. \quad (2.61)$$

Since the wave equation with homogeneous Dirichlet boundary conditions generates a group of isometries in $L^2(\Omega) \times H^{-1}(\Omega)$, Lemma 2.2.1 may be reformulated in the following way:

Lemma 2.2.2 *The initial data $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$ may be driven to zero in time T if and only if there exists $f \in L^2((0,T) \times \omega)$ such that*

$$\int_0^T \int_\omega \varphi f dxdt = \left\langle \left(\varphi^0, \varphi^1\right), \left(u^0, u^1\right)\right\rangle, \quad (2.62)$$

for all $(\varphi^0, \varphi^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ *where* φ *is the corresponding solution of* (2.60).

Relation (2.62) may be seen as an optimality condition for the critical points of the functional $\mathcal{J} : L^2(\Omega) \times H^{-1}(\Omega) \to \mathbb{R}$,

$$\mathcal{J}(\varphi^0, \varphi^1) = \frac{1}{2} \int_0^T \int_\omega |\varphi|^2 dxdt + \left\langle \left(\varphi^0, \varphi^1\right), \left(u^0, u^1\right) \right\rangle, \tag{2.63}$$

where φ is the solution of (2.60) with initial data $(\varphi^0, \varphi^1) \in L^2(\Omega) \times H^{-1}(\Omega)$.

We have:

Theorem 2.2.2 *Let* $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$ *and suppose that* $(\widehat{\varphi}^0, \widehat{\varphi}^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ *is a minimizer of* $\mathcal{J}$. *If* $\widehat{\varphi}$ *is the corresponding solution of* (2.60) *with initial data* $(\widehat{\varphi}^0, \widehat{\varphi}^1)$ *then*

$$f = \widehat{\varphi}_{|\omega} \tag{2.64}$$

is a control which leads (u^0, u^1) *to zero in time* T.

Proof. Since $\mathcal{J}$ achieves its minimum at $(\widehat{\varphi}^0, \widehat{\varphi}^1)$, the following relation holds

$$0 = \lim_{h \to 0} \frac{1}{h} \left(\mathcal{J}((\widehat{\varphi}^0, \widehat{\varphi}^1) + h(\varphi^0, \varphi^1)) - \mathcal{J}(\widehat{\varphi}^0, \widehat{\varphi}^1) \right) =$$

$$= \int_0^T \int_\omega \widehat{\varphi}\varphi dxdt + \int_\Omega u^1 \varphi^0 dx - < \varphi^1, u^0 >_{1,-1}$$

for any $(\varphi^0, \varphi^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ where φ is the solution of (2.60).

Lemma 2.2.2 shows that $f = \widehat{\varphi}_{|\omega}$ is a control which leads the initial data (u^0, u^1) to zero in time T.

$\blacksquare$

Let us now give sufficient conditions ensuring the existence of a minimizer for $\mathcal{J}$.

Definition 2.2.4 *Equation* (2.60) *is* **observable in time** T *if there exists a positive constant* $C_1 > 0$ *such that the following inequality is verified*

$$C_1 \parallel \left(\varphi^0, \varphi^1\right) \parallel^2_{L^2(\Omega) \times H^{-1}(\Omega)} \leq \int_0^T \int_\omega \mid \varphi \mid^2 dxdt, \tag{2.65}$$

for any $\left(\varphi^0, \varphi^1\right) \in L^2(\Omega) \times H^{-1}(\Omega)$ *where* φ *is the solution of* (2.60) *with initial data* (φ^0, φ^1).

Inequality (2.65) is called **observation or observability inequality**. It shows that the quantity $\int_0^T \int_\omega |\varphi|^2$ (the observed one) which depends only on the restriction of φ to the subset ω of Ω, uniquely determines the solution on (2.60).

Remark 2.2.4 The continuous dependence (2.61) of solutions of (2.60) with respect to its initial data guarantees that there exists a constant $C_2 > 0$ such that

$$\int_0^T \int_\omega |\varphi|^2 \, dxdt \leq C_2 \| \left(\varphi^0, \varphi^1\right) \|^2_{L^2(\Omega)\times H^{-1}(\Omega)} \tag{2.66}$$

for all $(\varphi^0, \varphi^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ and φ solution of (2.60).

■

Let us show that (2.65) is a sufficient condition for the exact controllability property to hold. First of all let us recall the following fundamental result in the Calculus of Variations which is a consequence of the so called Direct Method of the Calculus of Variations.

Theorem 2.2.3 *Let H be a reflexive Banach space, K a closed convex subset of H and $\varphi : K \to \mathbb{R}$ a function with the following properties:*

1. *φ is convex*
2. *φ is lower semi-continuous*
3. *If K is unbounded then φ is coercive, i. e.*

$$\lim_{||x||\to\infty} \varphi(x) = \infty. \tag{2.67}$$

Then φ attains its minimum in K, i. e. there exists $x_0 \in K$ such that

$$\varphi(x_0) = \min_{x\in K} \varphi(x). \tag{2.68}$$

For a proof of Theorem 2.2.3 see [24].

We have:

Theorem 2.2.4 *Let $(u^0, u^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and suppose that (2.60) is observable in time T. Then the functional $\mathcal{J}$ defined by (2.63) has an unique minimizer $(\widehat{\varphi}^0, \widehat{\varphi}^1) \in L^2(\Omega) \times H^{-1}(\Omega)$.*

Proof. It is easy to see that $\mathcal{J}$ is continuous and convex. Therefore, according to Theorem 2.2.3, the existence of a minimum is ensured if we prove that J is also coercive i.e.

$$\lim_{||(\varphi^0,\varphi^1)||_{L^2\times H^{-1}}\to\infty} \mathcal{J}(\varphi^0, \varphi^1) = \infty. \tag{2.69}$$

The coercivity of the functional $\mathcal{J}$ follows immediately from (2.65). Indeed,

$$\mathcal{J}(\varphi^0,\varphi^1) \geq \frac{1}{2}\left(\int_0^T\int_\omega |\varphi|^2 - ||(u^0,u^1)||_{H_0^1(\Omega)\times L^2(\Omega)}||(\varphi^0,\varphi^1)||_{L^2(\Omega)\times H^{-1}(\Omega)}\right)$$

$$\geq \frac{C_1}{2}||(\varphi^0,\varphi^1)||^2_{L^2(\Omega)\times H^{-1}(\Omega)} - \frac{1}{2}||(u^0,u^1)||_{H_0^1(\Omega)\times L^2(\Omega)}||(\varphi^0,\varphi^1)||_{L^2(\Omega)\times H^{-1}(\Omega)}.$$

It follows from Theorem 2.2.3 that $\mathcal{J}$ has a minimizer $(\widehat{\varphi}^0,\widehat{\varphi}^1)\in L^2(\Omega)\times H^{-1}(\Omega)$.

To prove the uniqueness of the minimizer it is sufficient to show that $\mathcal{J}$ is strictly convex. Let (φ^0,φ^1), $(\psi^0,\psi^1)\in L^2(\Omega)\times H^{-1}(\Omega)$ and $\lambda\in(0,1)$. We have that

$$\mathcal{J}(\lambda(\varphi^0,\varphi^1)+(1-\lambda)(\psi^0,\psi^1)) =$$

$$= \lambda\mathcal{J}(\varphi^0,\varphi^1)+(1-\lambda)\mathcal{J}(\psi^0,\psi^1) - \frac{\lambda(1-\lambda)}{2}\int_0^T\int_\omega |\varphi-\psi|^2 dxdt.$$

From (2.65) it follows that

$$\int_0^T\int_\omega |\varphi-\psi|^2 dxdt \geq C_1||(\varphi^0,\varphi^1)-(\psi^0,\psi^1)||_{L^2(\Omega)\times H^{-1}(\Omega)}.$$

Consequently, for any $(\varphi^0,\varphi^1)\neq(\psi^0,\psi^1)$,

$$\mathcal{J}(\lambda(\varphi^0,\varphi^1)+(1-\lambda)(\psi^0,\psi^1)) < \lambda\mathcal{J}(\varphi^0,\varphi^1)+(1-\lambda)\mathcal{J}(\psi^0,\psi^1)$$

and $\mathcal{J}$ is strictly convex.

■

Theorems 2.2.2 and 2.2.4 guarantee that, under hypothesis (2.65), system (2.52) is exactly controllable. Moreover, a control may be obtained as in (2.64) from the solution of the homogeneous equation (2.60) with the initial data minimizing the functional $\mathcal{J}$. Hence, the controllability problem is reduced to a minimization problem that may be solved by the Direct Method of the Calculus of Variations. This is very useful both from a theoretical and a numerical point of view.

The following proposition shows that the control obtained by this variational method is of minimal $L^2((0,T)\times\omega)$-norm.

Proposition 2.2.2 *Let $f=\widehat{\varphi}_{|\omega}$ be the control given by minimizing the functional $\mathcal{J}$. If $g\in L^2((0,T)\times\omega)$ is any other control driving to zero the initial data (u^0,u^1) then*

$$||f||_{L^2((0,T)\times\omega)} \leq ||g||_{L^2((0,T)\times\omega)}. \qquad (2.70)$$

Proof. Let $(\widehat{\varphi}^0, \widehat{\varphi}^1)$ the minimizer for the functional $\mathcal{J}$. Consider now relation (2.62) for the control $f = \widehat{\varphi}_{|\omega}$. By taking $(\widehat{\varphi}^0, \widehat{\varphi}^1)$ as test function we obtain that

$$||f||^2_{L^2((0,T)\times\omega)} = \int_0^T \int_\omega |\widehat{\varphi}|^2 dxdt = \left\langle \widehat{\varphi}^1, u^0 \right\rangle_{1,-1} - \int_\Omega \widehat{\varphi}^0 u^1 dx.$$

On the other hand, relation (2.62) for the control g and test function $(\widehat{\varphi}^0, \widehat{\varphi}^1)$ gives

$$\int_0^T \int_\omega g\widehat{\varphi} dxdt = \left\langle \widehat{\varphi}^1, u^0 \right\rangle_{1,-1} - \int_\Omega \widehat{\varphi}^0 u^1 dx.$$

We obtain that

$$||f||^2_{L^2((0,T)\times\omega)} = \left\langle \widehat{\varphi}^1, u^0 \right\rangle_{1,-1} - \int_\Omega \widehat{\varphi}^0 u^1 dx = \int_0^T \int_\omega g\widehat{\varphi} dxdt \leq$$

$$\leq ||g||_{L^2((0,T)\times\omega)} ||\widehat{\varphi}||_{L^2((0,T)\times\omega)} = ||g||_{L^2((0,T)\times\omega)} ||f||_{L^2((0,T)\times\omega)}$$

and (2.70) is proved.

■

2.2.5 Approximate controllability

Up to this point we have discussed only the exact controllability property of (2.52) which turns out to be equivalent to the observability property (2.65). Let us now address the approximate controllability one.

Let $\varepsilon > 0$ and $(u^0, u^1), (z^0, z^1) \in H_0^1(\Omega) \times L^2(\Omega)$. We are looking for a control function $f \in L^2((0,T)\times\omega)$ such that the corresponding solution (u, u') of (2.52) satisfies

$$||(u(T), u'(T)) - (z^0, z^1)||_{H_0^1(\Omega)\times L^2(\Omega)} \leq \varepsilon. \tag{2.71}$$

Recall that (2.52) is approximately controllable if, for any $\varepsilon > 0$ and (u^0, u^1), $(z^0, z^1) \in H_0^1(\Omega) \times L^2(\Omega)$, there exists $f \in L^2((0,T)\times\omega)$ such that (2.71) is verified.

By Remark 2.2.2, it is sufficient to study the case $(u^0, u^1) = (0, 0)$. From now on we assume that $(u^0, u^1) = (0, 0)$.

The variational approach considered in the previous sections may be also very useful for the study of the approximate controllability property. To see this, define the functional $\mathcal{J}_\varepsilon : L^2(\Omega) \times H^{-1}(\Omega) \to \mathbb{R}$,

$$\mathcal{J}_\varepsilon(\varphi^0, \varphi^1) =$$

$$= \frac{1}{2} \int_0^T \int_\omega |\varphi|^2 dxdt + \left\langle \left(\varphi^0, \varphi^1\right), \left(z^0, z^1\right) \right\rangle + \varepsilon ||(\varphi^0, \varphi^1)||_{L^2\times H^{-1}}, \tag{2.72}$$

where φ is the solution of (2.60) with initial data $(\varphi^0, \varphi^1) \in L^2(\Omega) \times H^{-1}(\Omega)$.

As in the exact controllability case, the existence of a minimum of the functional $\mathcal{J}_\varepsilon$ implies the existence of an approximate control.

Theorem 2.2.5 *Let $\varepsilon > 0$ and $(z^0, z^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and suppose that $(\widehat{\varphi}^0, \widehat{\varphi}^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ is a minimizer of $\mathcal{J}_\varepsilon$. If $\widehat{\varphi}$ is the corresponding solution of* (2.60) *with initial data $(\widehat{\varphi}^0, \widehat{\varphi}^1)$ then*

$$f = \widehat{\varphi}_{|\omega} \tag{2.73}$$

is an approximate control which leads the solution of (2.52) *from the zero initial data $(u^0, u^1) = (0, 0)$ to the state $(u(T), u'(T))$ such that* (2.71) *is verified.*

Proof. Let $(\widehat{\varphi}^0, \widehat{\varphi}^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ be a minimizer of $\mathcal{J}_\varepsilon$. It follows that, for any $h > 0$ and $(\varphi^0, \varphi^1) \in L^2(\Omega) \times H^{-1}(\Omega)$,

$$0 \leq \frac{1}{h}\left(\mathcal{J}_\varepsilon((\widehat{\varphi}^0, \widehat{\varphi}^1) + h(\varphi^0, \varphi^1)) - \mathcal{J}_\varepsilon(\widehat{\varphi}^0, \widehat{\varphi}^1)\right) \leq$$

$$\leq \int_0^T \int_\omega \widehat{\varphi}\varphi dxdt + \frac{h}{2}\int_0^T \int_\omega |\varphi|^2 dxdt + \left\langle \left(\varphi^0, \varphi^1\right), \left(z^0, z^1\right)\right\rangle + \varepsilon||(\varphi^0, \varphi^1)||_{L^2 \times H^{-1}}$$

being φ the solution of (2.60). By making $h \to 0$ we obtain that

$$-\varepsilon||(\varphi^0, \varphi^1)||_{L^2 \times H^{-1}} \leq \int_0^T \int_\omega \widehat{\varphi}\varphi dxdt + \left\langle \left(\varphi^0, \varphi^1\right), \left(z^0, z^1\right)\right\rangle.$$

A similar argument (with $h < 0$) leads to

$$\int_0^T \int_\omega \widehat{\varphi}\varphi dxdt + \left\langle \left(\varphi^0, \varphi^1\right), \left(z^0, z^1\right)\right\rangle \leq \varepsilon||(\varphi^0, \varphi^1)||_{L^2 \times H^{-1}}.$$

Hence, for any $(\varphi^0, \varphi^1) \in L^2(\Omega) \times H^{-1}(\Omega)$,

$$\left|\int_0^T \int_\omega \widehat{\varphi}\varphi dxdt + \left\langle \left(\varphi^0, \varphi^1\right), \left(z^0, z^1\right)\right\rangle\right| \leq \varepsilon||(\varphi^0, \varphi^1)||_{L^2 \times H^{-1}}. \tag{2.74}$$

Now, from (2.59) and (2.74) we obtain that

$$\left|\left\langle \left(\varphi^0, \varphi^1\right), \left[(z^0, z^1) - (u(T), u'(T))\right]\right\rangle\right| \leq \varepsilon||(\varphi^0, \varphi^1)||_{L^2 \times H^{-1}},$$

for any $(\varphi^0, \varphi^1) \in L^2(\Omega) \times H^{-1}(\Omega)$.

Consequently, (2.71) is verified and the proof finishes.

■

As we have seen in the previous section, the exact controllability property of (2.52) is equivalent to the observation property (2.65) of system (2.60). An unique continuation principle of the solutions of (2.60), which is a weaker version of the observability inequality (2.65), will play a similar role for the approximate controllability property and it will give a sufficient condition for the existence of a minimizer of $\mathcal{J}_\varepsilon$. More precisely, we have

Theorem 2.2.6 *The following properties are equivalent:*

1. *Equation* (2.52) *is approximately controllable.*
2. *The following unique continuation principle holds for the solutions of* (2.60)

$$\varphi_{|_{(0,T)\times\omega}} = 0 \Rightarrow (\varphi^0, \varphi^1) = (0, 0). \tag{2.75}$$

Proof. Let us first suppose that (2.52) is approximately controllable and let φ be a solution of (2.60) with initial data $(\varphi^0, \varphi^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ such that $\varphi_{|_{(0,T)\times\omega}} = 0$.

For any $\varepsilon > 0$ and $(z^0, z^1) \in H_0^1(\Omega) \times L^2(\Omega)$ there exists an approximate control function $f \in L^2((0, T) \times \omega)$ such that (2.71) is verified.

From (2.59) we deduce that $\big\langle (u(T), u'(T)), (\varphi^0, \varphi^1) \big\rangle = 0$. From the controllability property and the last relation we deduce that

$$\left| \big\langle (z^0, z^1), (\varphi^0, \varphi^1) \big\rangle \right| = \left| \big\langle [(z^0, z^1) - (u(T), u'(T))], (\varphi^0, \varphi^1) \big\rangle \right| \leq \varepsilon ||(\varphi^0, \varphi^1)||.$$

Since the last inequality is verified by any $(z^0, z^1) \in H_0^1(\Omega) \times L^2(\Omega)$ it follows that $(\varphi^0, \varphi^1) = (0, 0)$.

Hence the unique continuation principle (2.75) holds.

Reciprocally, suppose now that the unique continuation principle (2.75) is verified and let us show that (2.52) is approximately controllable.

In order to do that we use Theorem 2.2.5. Let $\varepsilon > 0$ and $(z^0, z^1) \in H_0^1(\Omega) \times L^2(\Omega)$ be given and consider the functional $\mathcal{J}_\varepsilon$. Theorem 2.2.5 ensures the approximate controllability property of (2.52) under the assumption that $\mathcal{J}_\varepsilon$ has a minimum. Let us show that this is true in our case.

The functional $\mathcal{J}_\varepsilon$ is convex and continuous in $L^2(\Omega) \times H^{-1}(\Omega)$. Thus, the existence of a minimum is ensured if $\mathcal{J}_\varepsilon$ is coercive, i. e.

$$\mathcal{J}_\varepsilon((\varphi^0, \varphi^1)) \to \infty \text{ when } ||(\varphi^0, \varphi^1)||_{L^2\times H^{-1}} \to \infty. \tag{2.76}$$

In fact we shall prove that

$$\underline{\lim}_{||(\varphi^0,\varphi^1)||_{L^2\times H^{-1}}\to\infty} \mathcal{J}_\varepsilon(\varphi^0, \varphi^1)/||(\varphi^0, \varphi^1)||_{L^2\times H^{-1}} \geq \varepsilon. \tag{2.77}$$

Evidently, (2.77) implies (2.76) and the proof of the theorem is complete.

In order to prove (2.77) let $\left((\varphi_j^0, \varphi_j^1)\right)_{j\geq 1} \subset L^2(\Omega)\times H^{-1}(\Omega)$ be a sequence of initial data for the adjoint system such that $\| (\varphi_j^0, \varphi_j^1) \|_{L^2\times H^{-1}} \to \infty$. We normalize them

$$(\widetilde{\varphi}_j^0, \widetilde{\varphi}_j^1) = (\varphi_j^0, \varphi_j^1)/ \| (\varphi_j^0, \varphi_j^1) \|_{L^2\times H^{-1}},$$

so that $\| (\widetilde{\varphi}_j^0, \widetilde{\varphi}_j^1) \|_{L^2\times H^{-1}} = 1$.

On the other hand, let $(\widetilde{\varphi}_j, \widetilde{\varphi}_j')$ be the solution of (2.60) with initial data $(\widetilde{\varphi}_j^0, \widetilde{\varphi}_j^1)$. Then

$$\frac{\mathcal{J}_\varepsilon((\varphi_j^0, \varphi_j^1))}{\| (\varphi_j^0, \varphi_j^1) \|} = \frac{1}{2} \| (\varphi_j^0, \varphi_j^1) \| \int_0^T \int_\omega | \widetilde{\varphi}_j |^2 \, dxdt + \left\langle (z^0, z^1), (\widetilde{\varphi}^0, \widetilde{\varphi}^1) \right\rangle + \varepsilon.$$

The following two cases may occur:

1) $\displaystyle\varliminf_{j\to\infty} \int_0^T \int_\omega | \widetilde{\varphi}_j |^2 > 0$. In this case we obtain immediately that

$$\frac{\mathcal{J}_\varepsilon((\varphi_j^0, \varphi_j^1))}{\| (\varphi_j^0, \varphi_j^1) \|} \to \infty.$$

2) $\displaystyle\varliminf_{j\to\infty} \int_0^T \int_\omega | \widetilde{\varphi}_j |^2 = 0$. In this case since $\left(\widetilde{\varphi}_j^0, \widetilde{\varphi}_j^1\right)_{j\geq 1}$ is bounded in $L^2 \times H^{-1}$, by extracting a subsequence we can guarantee that $(\widetilde{\varphi}_j^0, \widetilde{\varphi}_j^1)_{j\geq 1}$ converges weakly to (ψ^0, ψ^1) in $L^2(\Omega)\times H^{-1}(\Omega)$.

Moreover, if (ψ, ψ') is the solution of (2.60) with the initial data (ψ^0, ψ^1) at $t = T$, then $(\widetilde{\varphi}_j, \widetilde{\varphi}_j')_{j\geq 1}$ converges weakly to (ψ, ψ') in

$$L^2(0, T; L^2(\Omega) \times H^{-1}(\Omega)) \cap H^1(0, T; H^{-1}(\Omega) \times [H^2 \cap H_0^1(\Omega)]').$$

By lower semi-continuity,

$$\int_0^T \int_\omega \psi^2 dxdt \leq \varliminf_{j\to\infty} \int_0^T \int_\omega | \widetilde{\varphi}_j |^2 \, dxdt = 0$$

and therefore $\psi = 0$ in $\omega \times (0, T)$.

From the unique continuation principle we obtain that $(\psi^0, \psi^1) = (0, 0)$ and consequently,

$$(\widetilde{\varphi}_j^0, \widetilde{\varphi}_j^1) \rightharpoonup (0, 0) \text{ weakly in } L^2(\Omega)\times H^{-1}(\Omega).$$

Hence

$$\varliminf_{j\to\infty} \frac{\mathcal{J}_\varepsilon((\varphi_j^0, \varphi_j^1))}{\| (\varphi_j^0, \varphi_j^1) \|_{L^2\times H^{-1}}} \geq \varliminf_{j\to\infty} [\varepsilon + \left\langle (z^0, z^1), (\widetilde{\varphi}^0, \widetilde{\varphi}^1) \right\rangle] = \varepsilon,$$

and (2.77) follows. ■

When approximate controllability holds, then the following (apparently stronger) statement also holds:

Theorem 2.2.7 *Let E be a finite-dimensional subspace of $H_0^1(\Omega)\times L^2(\Omega)$ and let us denote by π_E the corresponding orthogonal projection. Then, if approximate controllability holds, for any $\left(u^0,u^1\right)$, $\left(z^0,z^1\right)\in H_0^1(\Omega)\times L^2(\Omega)$ and $\varepsilon>0$ there exists $f\in L^2((0,T)\times\omega)$ such that the solution of (2.52) satisfies*

$$\left|\left|\left(u(T)-z^0,u_t(T)-z^1\right)\right|\right|_{H_0^1(\Omega)\times L^2(\Omega)}\leq\varepsilon;\ \pi_E\left(u(T),u_t(T)\right)=\pi_E\left(z^0,z^1\right).$$

This property will be referred to as the **finite-approximate controllability property.** Its proof may be found in [236].

2.2.6 Comments

In this section we have presented some facts related with the exact and approximate controllability properties. The variational methods we have used allow to reduce them to an observation inequality and a unique continuation principle for the homogeneous adjoint equation respectively. The latter will be studied for some particular cases in section 2.4 by using nonharmonic Fourier analysis.

2.3 Boundary controllability of the wave equation

This section is devoted to study the boundary controllability problem for the wave equation. The control is assumed to act on a subset of the boundary of the domain where the solutions are defined.

2.3.1 Introduction

Let Ω be a bounded open set of $\mathbb{R}^N$ with boundary Γ of class C^2 and Γ_0 be an open nonempty subset of Γ. Given $T>0$ consider the following nonhomogeneous wave equation:

$$\left\{\begin{array}{ll}u''-\Delta u=0 & \text{in } (0,T)\times\Omega\\ u=f1_{\Gamma_0}(x) & \text{on } (0,T)\times\Gamma\\ u(0,\,\cdot\,)=u^0,\ u'(0,\,\cdot\,)=u^1 & \text{in } \Omega.\end{array}\right. \tag{2.78}$$

In (2.78) $u=u(t,x)$ is the state and $f=f(t,x)$ is a control function which acts on Γ_0. We aim at changing the dynamics of the system by acting on Γ_0.

2.3.2 Existence and uniqueness of solutions

The following theorem is a consequence of the classical results of existence and uniqueness of solutions of nonhomogeneous evolution equations. Full details may be found in [146] and [232].

Theorem 2.3.1 *For any* $f \in L^2((0,T)\times\Gamma_0)$ *and* $(u^0,u^1)\in L^2(\Omega)\times H^{-1}(\Omega)$ *equation* (2.78) *has a unique weak solution defined by transposition*

$$(u,u')\in C([0,T],L^2(\Omega)\times H^{-1}(\Omega)).$$

Moreover, the map $\{u^0,u^1,f\}\to\{u,u'\}$ *is linear and there exists* $C = C(T)>0$ *such that*

$$\begin{aligned}&||(u,u')||_{L^\infty(0,T;L^2(\Omega)\times H^{-1}(\Omega))}\leq\\ &\leq C\left(||(u^0,u^1)||_{L^2(\Omega)\times H^{-1}(\Omega)}+||f||_{L^2((0,T)\times\Gamma_0)}\right).\end{aligned}\tag{2.79}$$

Remark 2.3.1 The wave equation is reversible in time. Hence, we may solve (2.78) for $t\in(0,T)$ by considering final data at $t=T$ instead of initial data at $t=0$.

■

2.3.3 Controllability problems

Let $T>0$ and define, for any initial data $(u^0,u^1)\in L^2(\Omega)\times H^{-1}(\Omega)$, the set of reachable states

$$R(T;(u^0,u^1))=\{(u(T),u'(T)) : u \text{ solution of (2.78) with } f\in L^2((0,T)\times\Gamma_0)\}.$$

Remark that, for any $(u^0,u^1)\in L^2(\Omega)\times H^{-1}(\Omega)$, $R(T;(u^0,u^1))$ is a convex subset of $L^2(\Omega)\times H^{-1}(\Omega)$.

As in the previous section, several controllability problems may be addressed.

Definition 2.3.1 *System* (2.78) *is* **approximately controllable in time** T *if, for every initial data* $(u^0,u^1)\in L^2(\Omega)\times H^{-1}(\Omega)$*, the set of reachable states* $R(T;(u^0,u^1))$ *is dense in* $L^2(\Omega)\times H^{-1}(\Omega)$*.*

Definition 2.3.2 *System* (2.78) *is* **exactly controllable in time** T *if, for every initial data* $(u^0,u^1)\in L^2(\Omega)\times H^{-1}(\Omega)$*, the set of reachable states* $R(T;(u^0,u^1))$ *coincides with* $L^2(\Omega)\times H^{-1}(\Omega)$*.*

Definition 2.3.3 *System* (2.78) *is* **null controllable in time** T *if, for every initial data* $(u^0,u^1)\in L^2(\Omega)\times H^{-1}(\Omega)$*, the set of reachable states* $R(T;(u^0,u^1))$ *contains the element* $(0,0)$*.*

Remark 2.3.2 In the definitions of approximate and exact controllability it is sufficient to consider the case $(u^0, u^1) \equiv 0$ since

$$R(T;(u^0,u^1)) = R(T;(0,0)) + S(T)(u^0,u^1),$$

where $(S(t))_{t\in\mathbb{R}}$ is the group of isometries generated by the wave equation in $L^2(\Omega) \times H^{-1}(\Omega)$ with homogeneous Dirichlet boundary conditions.

■

Moreover, in view of the reversibility of the system we have

Proposition 2.3.1 *System* (2.78) *is exactly controllable if and only if it is null controllable.*

Proof. Evidently, exact controllability implies null controllability.

Let us suppose now that $(0,0) \in R(T;(u^0,u^1))$ for any $(u^0,u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$. It follows that any initial data in $L^2(\Omega) \times H^{-1}(\Omega)$ can be driven to $(0,0)$ in time T. From the reversibility of the wave equation we deduce that any state in $L^2(\Omega) \times H^{-1}(\Omega)$ can be reached in time T by starting from $(0,0)$. This means that $R(T,(0,0)) = L^2(\Omega) \times H^{-1}(\Omega)$ and the exact controllability property holds as a consequence of Remark 2.3.2.

■

The previous Proposition guarantees that (2.78) is exactly controllable if and only if, for any $(u^0,u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ there exists $f \in L^2((0,T)\times\Gamma_0)$ such that the corresponding solution (u,u') of (2.78) satisfies

$$u(T,\,\cdot\,) = u'(T,\,\cdot\,) = 0. \tag{2.80}$$

Remark 2.3.3 The following facts indicate the close connections between the controllability properties and some of the main features of hyperbolic equations:

- Since the wave equation is time-reversible and does not have any regularizing effect, the exact controllability property is very likely to hold. Nevertheless, as we have said before, the exact controllability property fails and the approximate controllability one holds in some situations. This is very closely related to the geometric properties of the subset Γ_0 of the boundary Γ where the control is applied.

- The wave equation is the prototype of partial differential equation with finite speed of propagation. Therefore, one cannot expect the previous controllability properties to hold unless the control time T is sufficiently large.

■

2.3.4 Variational approach

Let us first deduce a necessary and sufficient condition for the exact controllability of (2.78). As in the previous section, by $\langle \cdot\,,\,\cdot \rangle_{1,-1}$ we shall denote the duality product between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$.

For any $(\varphi_T^0, \varphi_T^1) \in H_0^1(\Omega) \times L^2(\Omega)$ let (φ, φ') be the solution of the following backward wave equation

$$\begin{cases} \varphi'' - \Delta\varphi = 0 & \text{in } (0,T)\times\Omega \\ \varphi\mid_{\partial\Omega} = 0 & \text{on } (0,T)\times\partial\Omega \\ \varphi(T,\,\cdot\,) = \varphi_T^0,\ \varphi'(T,\,\cdot\,) = \varphi_T^1. & \text{in } \Omega. \end{cases} \tag{2.81}$$

Lemma 2.3.1 *The initial data $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ is controllable to zero if and only if there exists $f \in L^2((0,T)\times\Gamma_0)$ such that*

$$\int_0^T \int_{\Gamma_0} \frac{\partial\varphi}{\partial n} f d\sigma dt + \int_\Omega u^0 \varphi'(0) dx - \left\langle u^1, \varphi(0) \right\rangle_{1,-1} = 0 \tag{2.82}$$

for all $(\varphi_T^0, \varphi_T^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and where (φ, φ') is the solution of the backward wave equation (2.81).

Proof. Let us first suppose that $\left(u^0, u^1\right), \left(\varphi_T^0, \varphi_T^1\right) \in \mathcal{D}(\Omega) \times \mathcal{D}(\Omega)$, $f \in \mathcal{D}((0,T)\times\Gamma_0)$ and let u and φ be the (regular) solutions of (2.78) and (2.81) respectively.

Multiplying the equation of u by φ and integrating by parts one obtains

$$\begin{aligned} 0 = \int_0^T \int_\Omega \varphi\,(u'' - \Delta u)\, dxdt = \int_\Omega (\varphi u' - \varphi' u)\, dx\Big|_0^T + \\ + \int_0^T \int_\Gamma \left(-\frac{\partial u}{\partial n}\varphi + \frac{\partial \varphi}{\partial n} u \right) d\sigma dt = \int_0^T \int_{\Gamma_0} \frac{\partial\varphi}{\partial n} u d\sigma dt + \\ + \int_\Omega [\varphi(T)u'(T) - \varphi'(T)u(T)]\, dx - \int_\Omega [\varphi(0)u'(0) - \varphi'(0)u(0)]\, dx \end{aligned}$$

Hence,

$$\int_0^T \int_{\Gamma_0} \frac{\partial\varphi}{\partial n} u d\sigma dt + \int_\Omega \left[\varphi_T^0 u'(T) - \varphi_T^1 u(T)\right] dx - \int_\Omega \left[\varphi(0)u^1 - \varphi'(0)u^0\right] dx = 0.$$

By a density argument we deduce that for any $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ and $(\varphi_T^0, \varphi_T^1) \in H_0^1(\Omega) \times L^2(\Omega)$,

$$\begin{aligned} \int_0^T \int_{\Gamma_0} \frac{\partial\varphi}{\partial n} u d\sigma dt = \\ = \int_\Omega u(T)\varphi_T^1 dx + \left\langle u'(T), \varphi_T^0 \right\rangle_{1,-1} + \int_\Omega u^0 \varphi'(0) dx - \left\langle u^1, \varphi(0) \right\rangle_{1,-1}. \end{aligned} \tag{2.83}$$

Now, from (2.83), it follows immediately that (2.82) holds if and only if (u^0, u^1) is controllable to zero. The proof finishes.

■

As in the previous section we introduce the duality product

$$\left\langle \left(\varphi^0, \varphi^1\right), \left(u^0, u^1\right)\right\rangle = \int_\Omega u^0 \varphi^1 dx - \left\langle u^1, \varphi^1 \right\rangle_{1,-1}$$

for all $\left(\varphi^0, \varphi^1\right) \in H_0^1(\Omega) \times L^2(\Omega)$ and $\left(u^0, u^1\right) \in L^2(\Omega) \times H^{-1}(\Omega)$.

For any $(\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega)$ let (φ, φ') be the finite energy solution of the following wave equation

$$\begin{cases} \varphi'' - \Delta\varphi = 0 & \text{in } (0,T) \times \Omega \\ \varphi\mid_{\partial\Omega} = 0 & \text{in } (0,T) \times \partial\Omega \\ \varphi(0,\,\cdot\,) = \varphi^0,\ \varphi'(0,\,\cdot\,) = \varphi^1 & \text{in } \Omega. \end{cases} \tag{2.84}$$

Since the wave equation generates a group of isometries in $H_0^1(\Omega) \times L^2(\Omega)$, Lemma 2.3.1 may be reformulated in the following way:

Lemma 2.3.2 *The initial data $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ is controllable to zero if and only if there exists $f \in L^2((0,T) \times \Gamma_0)$ such that*

$$\int_0^T \int_{\Gamma_0} \frac{\partial\varphi}{\partial n} f d\sigma dt + \left\langle \left(\varphi^0, \varphi^1\right), \left(u^0, u^1\right)\right\rangle = 0, \tag{2.85}$$

for all $(\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and where φ is the solution of (2.84).

Once again, (2.85) may be seen as an optimality condition for the critical points of the functional $\mathcal{J} : H_0^1(\Omega) \times L^2(\Omega) \longrightarrow \mathbb{R}$, defined by

$$\mathcal{J}(\varphi^0, \varphi^1) = \frac{1}{2}\int_0^T \int_{\Gamma_0} \left|\frac{\partial\varphi}{\partial n}\right|^2 d\sigma dt + \left\langle \left(\varphi^0, \varphi^1\right), \left(u^0, u^1\right)\right\rangle, \tag{2.86}$$

where φ is the solution of (2.84) with initial data $(\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega)$.

We have

Theorem 2.3.2 *Let $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$ and suppose that $(\widehat{\varphi}^0, \widehat{\varphi}^1) \in H_0^1(\Omega) \times L^2(\Omega)$ is a minimizer of $\mathcal{J}$. If $\widehat{\varphi}$ is the corresponding solution of* (2.84) *with initial data $(\widehat{\varphi}^0, \widehat{\varphi}^1)$ then $f = \frac{\partial\widehat{\varphi}}{\partial n}_{|\Gamma_0}$ is a control which leads (u^0, u^1) to zero in time T.*

Proof. Since, by assumption, $\mathcal{J}$ achieves its minimum at $(\widehat{\varphi}^0, \widehat{\varphi}^1)$, the following relation holds

$$0 = \lim_{h \to 0} \frac{1}{h} \left(\mathcal{J}((\widehat{\varphi}^0, \widehat{\varphi}^1) + h(\varphi^0, \varphi^1)) - \mathcal{J}(\widehat{\varphi}^0, \widehat{\varphi}^1) \right) =$$

$$= \int_0^T \int_{\Gamma_0} \frac{\partial \widehat{\varphi}}{\partial n} \frac{\partial \varphi}{\partial n} d\sigma dt + \int_\Omega u^0 \varphi^1 dx - < u^1, \varphi^0 >_{1,-1}$$

for any $(\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega)$ where φ is the solution of (2.84).

From Lemma 2.3.2 it follows that $f = \frac{\partial \widehat{\varphi}}{\partial n}_{|\Gamma_0}$ is a control which leads the initial data (u^0, u^1) to zero in time T.

■

Let us now give a general condition which ensures the existence of a minimizer for $\mathcal{J}$.

Definition 2.3.4 *Equation* (2.84) *is* **observable in time** T *if there exists a positive constant* $C_1 > 0$ *such that the following inequality is verified*

$$C_1 \parallel \left(\varphi^0, \varphi^1\right) \parallel^2_{H_0^1(\Omega) \times L^2(\Omega)} \leq \int_0^T \int_{\Gamma_0} \left| \frac{\partial \varphi}{\partial n} \right|^2 d\sigma dt, \tag{2.87}$$

for any $\left(\varphi^0, \varphi^1\right) \in H_0^1(\Omega) \times L^2(\Omega)$ *where* φ *is the solution of* (2.84) *with initial data* (φ^0, φ^1).

Inequality (2.87) is called **observation or observability inequality**. According to it, when it holds, the quantity $\int_0^T \int_{\Gamma_0} \left| \frac{\partial \varphi}{\partial n} \right|^2 d\sigma dt$ (the observed quantity) which depends only on the trace of $\frac{\partial \varphi}{\partial n}$ on $(0, T) \times \Gamma_0$, uniquely determines the solution of (2.84).

Remark 2.3.4 One may show that there exists a constant $C_2 > 0$ such that

$$\int_0^T \int_{\Gamma_0} \left| \frac{\partial \varphi}{\partial n} \right|^2 d\sigma dt \leq C_2 \parallel \left(\varphi^0, \varphi^1\right) \parallel^2_{H_0^1(\Omega) \times L^2(\Omega)} \tag{2.88}$$

for all $(\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and φ solution of (2.84).

Inequality (2.88) may be obtained by multiplier techniques (see [126] or [143]). Remark that, (2.88) says that $\frac{\partial \varphi}{\partial n}_{|\Gamma_0} \in L^2((0, T) \times \Gamma_0)$ which is a "hidden" regularity result, that may not be obtained by classical trace results.

■

Let us show that (2.87) is a sufficient condition for the exact controllability property to hold.

Theorem 2.3.3 *Suppose that* (2.84) *is observable in time T and let* $(u^0, u^1) \in L^2(\Omega) \times H^{-1}(\Omega)$. *The functional $\mathcal{J}$ defined by* (2.86) *has an unique minimizer* $(\widehat{\varphi}^0, \widehat{\varphi}^1) \in H_0^1(\Omega) \times L^2(\Omega)$.

Proof. It is easy to see that $\mathcal{J}$ is continuous and convex. The existence of a minimum is ensured if we prove that J is also coercive i.e.

$$\lim_{||(\varphi^0,\varphi^1)||_{H_0^1 \times L^2} \to \infty} \mathcal{J}(\varphi^0, \varphi^1) = \infty. \tag{2.89}$$

The coercivity of the functional $\mathcal{J}$ follows immediately from (2.87). Indeed,

$$\mathcal{J}(\varphi^0, \varphi^1) \geq$$

$$\geq \frac{1}{2}\left(\int_0^T \int_{\Gamma_0} \left|\frac{\partial \varphi}{\partial n}\right|^2 - ||(u^0, u^1)||_{H_0^1(\Omega)\times L^2(\Omega)} ||(\varphi^0, \varphi^1)||_{L^2(\Omega)\times H^{-1}(\Omega)}\right) \geq$$

$$\geq \frac{C_1}{2} ||(\varphi^0, \varphi^1)||^2_{L^2(\Omega)\times H^{-1}(\Omega)} - \frac{1}{2} ||(u^0, u^1)||_{H_0^1(\Omega)\times L^2(\Omega)} ||(\varphi^0, \varphi^1)||_{L^2(\Omega)\times H^{-1}(\Omega)}.$$

It follows from Theorem 2.2.3 that $\mathcal{J}$ has a minimizer $(\widehat{\varphi}^0, \widehat{\varphi}^1) \in H_0^1(\Omega) \times L^2(\Omega)$.

As in the proof of Theorem 2.2.4, it may be shown that $\mathcal{J}$ is strictly convex and therefore it achieves its minimum at a unique point.

■

Theorems 2.3.2 and 2.3.3 guarantee that, under the hypothesis (2.87), system (2.78) is exactly controllable. Moreover, a control may be obtained from the solution of the homogeneous system (2.81) with the initial data minimizing the functional $\mathcal{J}$. Hence, the controllability is reduced to a minimization problem. This is very useful both from the theoretical and numerical point of view.

As in Proposition 2.2.2 the control obtained by minimizing the functional $\mathcal{J}$ has minimal L^2-norm:

Proposition 2.3.2 *Let* $f = \frac{\partial \widehat{\varphi}}{\partial n}_{|\Gamma_0}$ *be the control given by minimizing the functional $\mathcal{J}$. If* $g \in L^2((0,T) \times \Gamma_0)$ *is any other control driving to zero the initial data* (u^0, u^1) *in time T, then*

$$||f||_{L^2((0,T)\times\Gamma_0)} \leq ||g||_{L^2((0,T)\times\Gamma_0)}. \tag{2.90}$$

Proof. It is similar to the proof of Property 2.2.2. We omit the details.

■

2.3.5 Approximate controllability

Let us now briefly present and discuss the approximate controllability property. Since many aspects are similar to the interior controllability case we only give the general ideas and let the details to the interested reader.

Let $\varepsilon > 0$ and $(u^0, u^1), (z^0, z^1) \in L^2(\Omega) \times H^{-1}(\Omega)$. We are looking for a control function $f \in L^2((0,T)\times\Gamma_0)$ such that the corresponding solution (u, u') of (2.78) satisfies

$$||(u(T), u'(T)) - (z^0, z^1)||_{L^2(\Omega)\times H^{-1}(\Omega)} \leq \varepsilon. \tag{2.91}$$

Recall that, (2.78) is approximately controllable if, for any $\varepsilon > 0$ and (u^0, u^1), $(z^0, z^1) \in L^2(\Omega) \times H^{-1}(\Omega)$, there exists $f \in L^2((0,T) \times \Gamma_0)$ such that (2.91) is verified.

By Remark 2.3.2, it is sufficient to study the case $(u^0, u^1) = (0, 0)$. Therefore, in this section we only address this case.

The variational approach may be also used for the study of the approximate controllability property.

To see this, define the functional $\mathcal{J}_\varepsilon : H_0^1(\Omega) \times L^2(\Omega) \to \mathbb{R}$,

$$\begin{aligned} &\mathcal{J}_\varepsilon(\varphi^0, \varphi^1) = \\ &= \frac{1}{2}\int_0^T \int_{\Gamma_0} \left|\frac{\partial\varphi}{\partial n}\right|^2 dxdt + \left\langle (\varphi^0, \varphi^1), (z^0, z^1) \right\rangle + \varepsilon||(\varphi^0, \varphi^1)||_{H_0^1\times L^2}, \end{aligned} \tag{2.92}$$

where φ is the solution of (2.81) with initial data $(\varphi^0, \varphi^1) \in H_0^1(\Omega) \times L^2(\Omega)$.

The following theorem shows how the functional $\mathcal{J}_\varepsilon$ may be used to study the approximate controllability property. In fact, as in the exact controllability case, the existence of a minimum of the functional $\mathcal{J}_\varepsilon$ implies the existence of an approximate control.

Theorem 2.3.4 *Let $\varepsilon > 0$, $(z^0, z^1) \in L^2(\Omega)\times H^{-1}(\Omega)$. Suppose that $(\widehat{\varphi}^0, \widehat{\varphi}^1) \in H_0^1(\Omega)\times L^2(\Omega)$ is a minimizer of $\mathcal{J}_\varepsilon$. If $\widehat{\varphi}$ is the corresponding solution of* (2.81) *with initial data $(\widehat{\varphi}^0, \widehat{\varphi}^1)$ then*

$$f = \frac{\partial\widehat{\varphi}}{\partial n}\Big|_{\Gamma_0} \tag{2.93}$$

is an approximate control which leads the solution of (2.78) *from the zero initial data $(u^0, u^1) = (0, 0)$ to the state $(u(T), u'(T))$ such that* (2.91) *is verified.*

Proof. It is similar to the proof of Theorem 2.3.4.

■

As we have seen, the exact controllability property of (2.78) is related to the observation property (2.65) of system (2.81). An unique continuation property of the solutions of (2.81) plays a similar role in the context of approximate controllability and guarantees the existence of a minimizer of $\mathcal{J}_\varepsilon$. More precisely, we have

Theorem 2.3.5 *The following properties are equivalent:*

1. *Equation* (2.78) *is approximately controllable.*
2. *The following unique continuation principle holds for the solutions of* (2.81)
$$\frac{\partial\varphi}{\partial n}\Big|_{(0,T)\times\Gamma_0} = 0 \Rightarrow (\varphi^0,\varphi^1) = (0,0). \tag{2.94}$$

Proof. The proof of the fact that the approximate controllability property implies the unique continuation principle (2.94) is similar to the corresponding one in Theorem 2.2.6 and we omit it.

Let us prove that, if the unique continuation principle (2.94) is verified, (2.78) is approximately controllable. By Theorem 2.3.4 it is sufficient to prove that $\mathcal{J}_\varepsilon$ defined by (2.92) has a minimum. The functional $\mathcal{J}_\varepsilon$ is convex and continuous in $H_0^1(\Omega)\times L^2(\Omega)$. Thus, the existence of a minimum is ensured if $\mathcal{J}_\varepsilon$ is coercive, i. e.

$$\mathcal{J}_\varepsilon((\varphi^0,\varphi^1)) \to \infty \text{ when } ||(\varphi^0,\varphi^1)||_{H_0^1\times L^2} \to \infty. \tag{2.95}$$

In fact we shall prove that

$$\underline{\lim}_{||(\varphi^0,\varphi^1)||_{H_0^1\times L^2}\to\infty} \mathcal{J}_\varepsilon(\varphi^0,\varphi^1)/||(\varphi^0,\varphi^1)||_{H_0^1\times L^2} \geq \varepsilon. \tag{2.96}$$

Evidently, (2.96) implies (2.95) and the proof of the theorem is complete.

In order to prove (2.96) let $\left((\varphi_j^0,\varphi_j^1)\right)_{j\geq 1} \subset H_0^1(\Omega)\times L^2(\Omega)$ be a sequence of initial data for the adjoint system with $\| (\varphi_j^0,\varphi_j^1) \|_{H_0^1\times L^2}\to\infty$. We normalize them

$$(\widetilde{\varphi}_j^0,\widetilde{\varphi}_j^1) = (\varphi_j^0,\varphi_j^1)/ \parallel (\varphi_j^0,\varphi_j^1) \parallel_{H_0^1\times L^2},$$

so that

$$\parallel (\widetilde{\varphi}_j^0,\widetilde{\varphi}_j^1) \parallel_{H_0^1\times L^2}= 1.$$

On the other hand, let $(\widetilde{\varphi}_j,\widetilde{\varphi}_j')$ be the solution of (2.81) with initial data $(\widetilde{\varphi}_j^0,\widetilde{\varphi}_j^1)$. Then

$$\frac{\mathcal{J}_\varepsilon((\varphi_j^0,\varphi_j^1))}{\parallel (\varphi_j^0,\varphi_j^1) \parallel} = \frac{1}{2} \parallel (\varphi_j^0,\varphi_j^1) \parallel \int_0^T \int_{\Gamma_0} \left|\frac{\partial\widetilde{\varphi}_j}{\partial n}\right|^2 d\sigma dt + \left\langle (z^0,z^1),(\widetilde{\varphi}^0,\widetilde{\varphi}^1)\right\rangle + \varepsilon.$$

The following two cases may occur:

1) $\underline{\lim}_{j\to\infty} \int_0^T \int_{\Gamma_0} \left| \frac{\partial \widetilde{\varphi}_j}{\partial n} \right|^2 > 0$. In this case we obtain immediately that

$$\frac{\mathcal{J}_\varepsilon((\varphi_j^0, \varphi_j^1))}{\| (\varphi_j^0, \varphi_j^1) \|} \to \infty.$$

2) $\underline{\lim}_{j\to\infty} \int_0^T \int_{\Gamma_0} \left| \frac{\partial \widetilde{\varphi}_j}{\partial n} \right|^2 = 0$. In this case, since $\left(\widetilde{\varphi}_j^0, \widetilde{\varphi}_j^1\right)_{j\geq 1}$ is bounded in $H_0^1 \times L^2$, by extracting a subsequence we can guarantee that $(\widetilde{\varphi}_j^0, \widetilde{\varphi}_j^1)_{j\geq 1}$ converges weakly to (ψ^0, ψ^1) in $H_0^1(\Omega) \times L^2(\Omega)$.

   Moreover, if (ψ, ψ') is the solution of (2.81) with initial data (ψ^0, ψ^1) at $t = T$, then $(\widetilde{\varphi}_j, \widetilde{\varphi}_j')_{j\geq 1}$ converges weakly to (ψ, ψ') in $L^2(0, T; H_0^1(\Omega) \times L^2(\Omega)) \cap H^1(0, T; L^2(\Omega) \times H^{-1}(\Omega))$.

   By lower semi-continuity,

$$\int_0^T \int_{\Gamma_0} \left| \frac{\partial \psi}{\partial n} \right|^2 d\sigma dt \leq \underline{\lim}_{j\to\infty} \int_0^T \int_{\Gamma_0} \left| \frac{\partial \widetilde{\varphi}_j}{\partial n} \right|^2 d\sigma dt = 0$$

   and therefore $\partial\psi/\partial n = 0$ on $\Gamma_0 \times (0, T)$.

   From the unique continuation principle we obtain that $(\psi^0, \psi^1) = (0, 0)$ and consequently,

$$(\widetilde{\varphi}_j^0, \widetilde{\varphi}_j^1) \rightharpoonup (0, 0) \text{ weakly in } H_0^1(\Omega) \times L^2(\Omega).$$

   Hence

$$\underline{\lim}_{j\to\infty} \frac{\mathcal{J}_\varepsilon((\varphi_j^0, \varphi_j^1))}{\| (\varphi_j^0, \varphi_j^1) \|} \geq \underline{\lim}_{j\to\infty} [\varepsilon + \langle (z^0, z^1), (\widetilde{\varphi}^0, \widetilde{\varphi}^1) \rangle] = \varepsilon,$$

   and (2.96) follows.

■

As mentioned in the previous section, when approximate controllability holds, the following (apparently stronger) statement also holds (see [236]):

Theorem 2.3.6 *Let E be a finite-dimensional subspace of $L^2(\Omega) \times H^{-1}(\Omega)$ and let us denote by π_E the corresponding orthogonal projection. Then, if approximate controllability holds, for any $\left(u^0, u^1\right)$, $\left(z^0, z^1\right) \in L^2(\Omega) \times H^{-1}(\Omega)$ and $\varepsilon > 0$ there exists $f \in L^2((0, T) \times \Gamma_0)$ such that the solution of (2.78) satisfies*

$$\left|\left|\left(u(T) - z^0, u_t(T) - z^1\right)\right|\right|_{L^2(\Omega)\times H^{-1}(\Omega)} \leq \varepsilon;\ \pi_E\left(u(T), u_t(T)\right) = \pi_E\left(z^0, z^1\right).$$

2.3.6 Comments

In the last two sections we have presented some results concerning the exact and approximate controllability of the wave equation. The variational methods we have used allow to reduce these properties to an observation inequality and a unique continuation principle for the adjoint homogeneous equation respectively.

Let us briefly make some remarks concerning the proof of the unique continuation principles (2.75) and (2.94).

Holmgren's Uniqueness Theorem (see [108]) may be used to show that (2.75) and (2.94) hold if T is large enough. We refer to chapter 1 in [142, 143], and [39] for a discussion of this problem. Consequently, approximate controllability holds if T is large enough.

The same results hold for wave equations with analytic coefficients too. However, the problem is not completely solved in the frame of the wave equation with lower order potentials $a \in L^\infty((0,T)\times\Omega)$ of the form

$$u_{tt} - \Delta u + a(x,t)u = f1_\omega \text{ in } (0,T)\times\Omega.$$

Once again the problem of approximate controllability of this system is equivalent to the unique continuation property of its adjoint. We refer to Alinhac [2], Tataru [213] and Robbiano-Zuilly [190] for deep results in this direction.

In the following section we shall prove the observability inequalities (2.65) and (2.87) in some simple one dimensional cases by using Fourier expansion of solutions. Other tools have been successfully used to prove these observability inequalities. Let us mention two of them.

1. **Multiplier techniques:** Ho in [107] proved that if one considers subsets of Γ of the form

   $$\Gamma_0 = \Gamma(x^0) = \left\{x \in \Gamma : (x - x^0)\cdot n(x) > 0\right\}$$

   for some $x^0 \in \mathbb{R}^N$ and if $T > 0$ is large enough, the boundary observability inequality (2.87), that is required to solve the boundary controllability problem, holds. The technique used consists of multiplying equation (2.84) by $q \cdot \nabla\varphi$ and integrating by parts in $(0,T)\times\Omega$. The multiplier q is an appropriate vector field defined in $\overline{\Omega}$. More precisely, $q(x) = x - x^0$ for any $x \in \overline{\Omega}$.

   Later on inequality (2.87) was proved in [143] for any

   $$T > T(x^0) = 2 \parallel x - x^0 \parallel_{L^\infty(\Omega)} .$$

   This is the optimal observability time that one may derive by means of multipliers. More recently Osses in [178] has introduced a new multiplier which is basically a rotation of the previous one and he has obtained a larger class of subsets of the boundary for which observability holds.

Concerning the interior controllability problem, one can easily prove that (2.87) implies (2.65) when ω is a neighborhood of $\Gamma(x^0)$ in Ω, i.e. $\omega = \Omega \cap \Theta$ where Θ is a neighborhood of $\Gamma(x^0)$ in $\mathbb{R}^n$, with $T > 2 \parallel x - x^0 \parallel_{L^\infty(\Omega\setminus\omega)}$ (see in [142, 143], vol. 1).

An extensive presentation and several applications of multiplier techniques are given in [126] and [142, 143].

2. **Microlocal analysis:** C. Bardos, G. Lebeau and J. Rauch [14] proved that, in the class of C^∞ domains, the observability inequality (2.65) holds if and only if (ω, T) satisfy the following *geometric control condition* in Ω*: Every ray of geometric optics that propagates in* Ω *and is reflected on its boundary* Γ *enters* ω *in time less than* T. This result was proved by means of microlocal analysis techniques. Recently the microlocal approach has been greatly simplified by N. Burq [26] by using the microlocal defect measures introduced by P. Gerard [91] in the context of the homogenization and the kinetic equations. In [26] the geometric control condition was shown to be sufficient for exact controllability for domains Ω of class C^3 and equations with C^2 coefficients.

Other methods have been developed to address the controllability problems such as moment problems, fundamental solutions, controllability via stabilization, etc. We will not present them here and we refer to the survey paper by D. L. Russell [194] for the interested reader.

2.4 Fourier techniques and the observability of the $1D$ wave equation

In sections 2.2 and 2.3 we have shown that the exact controllability problem may be reduced to the corresponding observability inequality. In this section we develop in detail some techniques based on Fourier analysis and more particularly on Ingham type inequalities allowing to obtain several observability results for linear 1-D wave equations. We refer to Avdonin and Ivanov [5] for a complete presentation of this approach.

2.4.1 Ingham's inequalities

In this section we present two inequalities which have been successfully used in the study of many 1-D control problems and, more precisely, to prove observation inequalities. They generalize the classical Parseval's equality for orthogonal sequences. Variants of these inequalities were studied in the works of

Paley and Wiener at the beginning of the past century (see [180]). The main inequality was proved by Ingham (see [114]) who gave a beautiful and elementary proof (see Theorems 2.4.1 and 2.4.2 below). Since then, many generalizations have been given (see, for instance, [10], [104], [8] and [117]).

Theorem 2.4.1 (Ingham [114]) *Let $(\lambda_n)_{n\in\mathbb{Z}}$ be a sequence of real numbers and $\gamma > 0$ be such that*

$$\lambda_{n+1} - \lambda_n \geq \gamma > 0, \quad \forall n \in \mathbb{Z}. \tag{2.97}$$

For any real T with

$$T > \pi/\gamma \tag{2.98}$$

there exists a positive constant $C_1 = C_1(T, \gamma) > 0$ such that, for any finite sequence $(a_n)_{n\in\mathbb{Z}}$,

$$C_1 \sum_{n\in\mathbb{Z}} | a_n |^2 \leq \int_{-T}^{T} \left| \sum_{n\in\mathbb{Z}} a_n e^{i\lambda_n t} \right|^2 dt. \tag{2.99}$$

Proof. We first reduce the problem to the case $T = \pi$ and $\gamma > 1$. Indeed, if T and γ are such that $T\gamma > \pi$, then

$$\int_{-T}^{T} \left| \sum_n a_n e^{i\lambda_n t} \right|^2 dt = \frac{T}{\pi} \int_{-\pi}^{\pi} \left| \sum_n a_n e^{i\frac{T\lambda_n}{\pi} s} \right|^2 ds = \frac{T}{\pi} \int_{-\pi}^{\pi} \left| \sum_n a_n e^{i\mu_n s} \right|^2 ds$$

where $\mu_n = T\lambda_n/\pi$. It follows that $\mu_{n+1} - \mu_n = T\left(\lambda_{n+1} - \lambda_n\right)/\pi \geq \gamma_1 := T\gamma/\pi > 1$.

We prove now that there exists $C_1' > 0$ such that

$$C_1' \sum_{n\in\mathbb{Z}} | a_n |^2 \leq \int_{-\pi}^{\pi} \left| \sum_{n\in\mathbb{Z}} a_n e^{i\mu_n t} \right|^2 dt.$$

Define the function

$$h : \mathbb{R} \to \mathbb{R}, \; h(t) = \begin{cases} \cos(t/2) & \text{if} \quad | t | \leq \pi \\ 0 & \text{if} \quad | t | > \pi \end{cases}$$

and let us compute its Fourier transform $K(\varphi)$,

$$K(\varphi) = \int_{-\pi}^{\pi} h(t) e^{it\varphi} dt = \int_{-\infty}^{\infty} h(t) e^{it\varphi} dt = \frac{4\cos \pi\varphi}{1 - 4\varphi^2}.$$

On the other hand, since $0 \leq h(t) \leq 1$ for any $t \in [-\pi, \pi]$, we have that

$$\int_{-\pi}^{\pi} \left| \sum_n a_n e^{i\mu_n t} \right|^2 dt \geq \int_{-\pi}^{\pi} h(t) \left| \sum_n a_n e^{i\mu_n t} \right|^2 dt = \sum_{n,m} a_n \bar{a}_m K(\mu_n - \mu_m) =$$

$$= K(0)\sum_n |\, a_n \,|^2 + \sum_{n\neq m} a_n \bar{a}_m K(\mu_n - \mu_m) \geq$$

$$\geq 4\sum_n |\, a_n \,|^2 - \frac{1}{2}\sum_{n\neq m}\left(|\, a_n \,|^2 + |\, a_m \,|^2\right) |\, K(\mu_n - \mu_m) \,| =$$

$$= 4\sum_n |\, a_n \,|^2 - \sum_n |\, a_n \,|^2 \sum_{m\neq n} |\, K(\mu_n - \mu_m) \,| \,.$$

Remark that

$$\sum_{m\neq n} |\, K(\mu_n - \mu_m) \,| \leq \sum_{m\neq n} \frac{4}{4\,|\, \mu_n - \mu_m \,|^2 - 1} \leq \sum_{m\neq n} \frac{4}{4\gamma_1^2 \,|\, n - m \,|^2 - 1} =$$

$$= 8\sum_{r\geq 1} \frac{1}{4\gamma_1^2 r^2 - 1} \leq \frac{8}{\gamma_1^2}\sum_{r\geq 1}\frac{1}{4r^2 - 1} = \frac{8}{\gamma_1^2}\frac{1}{2}\sum_{r\geq 1}\left(\frac{1}{2r-1} - \frac{1}{2r+1}\right) = \frac{4}{\gamma_1^2}.$$

Hence,

$$\int_{-\pi}^{\pi} \left| \sum_n a_n e^{i\mu_n t} \right|^2 dt \geq \left(4 - \frac{4}{\gamma_1^2}\right) \sum_n |\, a_n \,|^2 \,.$$

If we take

$$C_1 = \frac{T}{\pi}\left(4 - \frac{4}{\gamma_1^2}\right) = \frac{4\pi}{T}\left(T^2 - \frac{\pi^2}{\gamma^2}\right)$$

the proof is concluded.

■

Theorem 2.4.2 *Let $(\lambda_n)_{n\in\mathbb{Z}}$ be a sequence of real numbers and $\gamma > 0$ be such that*

$$\lambda_{n+1} - \lambda_n \geq \gamma > 0, \quad \forall n \in \mathbb{Z}. \tag{2.100}$$

For any $T > 0$ there exists a positive constant $C_2 = C_2(T, \gamma) > 0$ such that, for any finite sequence $(a_n)_{n\in\mathbb{Z}}$,

$$\int_{-T}^{T} \left| \sum_n a_n e^{i\lambda_n t} \right|^2 dt \leq C_2 \sum_n |\, a_n \,|^2 \,. \tag{2.101}$$

Proof. Let us first consider the case $T\gamma \geq \pi/2$. As in the proof of the previous theorem, we can reduce the problem to $T = \pi/2$ and $\gamma \geq 1$. Indeed,

$$\int_{-T}^{T} \left| \sum_n a_n e^{i\lambda_n t} \right|^2 dt \quad = \quad \frac{2T}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left| \sum_n a_n e^{i\mu_n s} \right|^2 ds$$

where $\mu_n = 2T\lambda_n/\pi$. It follows that $\mu_{n+1} - \mu_n = 2T\left(\lambda_{n+1} - \lambda_n\right)/\pi \geq \gamma_1 := 2T\gamma/\pi \geq 1$.

Let h be the function introduced in the proof of Theorem 2.4.1. Since $\sqrt{2}/2 \leq h(t) \leq 1$ for any $t \in [-\pi/2, \pi/2]$ we obtain that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left| \sum_n a_n e^{i\mu_n t} \right|^2 dt \leq 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} h(t) \left| \sum_n a_n e^{i\mu_n t} \right|^2 dt \leq$$

$$\leq 2 \int_{-\pi}^{\pi} h(t) \left| \sum_n a_n e^{i\mu_n t} \right|^2 dt = 2 \sum_{n,m} a_n \bar{a}_m K\left(\mu_n - \mu_m\right) =$$

$$= 8 \sum_n \mid a_n \mid^2 + 2 \sum_{n \neq m} a_n \bar{a}_m K\left(\mu_n - \mu_m\right) \leq$$

$$\leq 8 \sum_n \mid a_n \mid^2 + \sum_{n \neq m} \left(\mid a_n \mid^2 + \mid a_m \mid^2\right) \mid K\left(\mu_n - \mu_m\right) \mid .$$

As in the proof of Theorem 2.4.1 we obtain that

$$\sum_{m \neq n} \mid K(\mu_n - \mu_m) \mid \leq \frac{4}{\gamma_1^2}.$$

Hence,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left| \sum_n a_n e^{i\mu_n t} \right|^2 dt \leq 8 \sum_n \mid a_n \mid^2 + \frac{8}{\gamma_1^2} \sum_n \mid a_n \mid^2 \leq 8 \left(1 + \frac{1}{\gamma_1^2}\right) \sum_n \mid a_n \mid^2$$

and (2.101) follows with $C_2 = 8\left(4T^2/(\pi^2) + 1/\gamma^2\right)$.

When $T\gamma < \pi/2$ we have that

$$\int_{-T}^{T} \left| \sum a_n e^{i\lambda_n t} \right|^2 dt = \frac{1}{\gamma} \int_{-T\gamma}^{T\gamma} \left| \sum a_n e^{i\frac{\lambda_n}{\gamma} s} \right|^2 ds \leq \frac{1}{\gamma} \int_{-\pi/2}^{\pi/2} \left| \sum a_n e^{i\frac{\lambda_n}{\gamma} s} \right|^2 ds.$$

Since $\lambda_{n+1}/\gamma - \lambda_n/\gamma \geq 1$ from the analysis of the previous case we obtain that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left| \sum_n a_n e^{i\frac{\lambda_n}{\gamma} s} \right|^2 ds \leq 16 \sum_n \mid a_n \mid^2 .$$

Hence, (2.101) is proved with

$$C_2 = 8 \max\left\{ \left(\frac{4T^2}{\pi^2} + \frac{1}{\gamma^2}\right),\ \frac{2}{\gamma} \right\}$$

and the proof concludes. ■

Remark 2.4.1

- Inequality (2.101) holds for all $T > 0$. On the contrary, inequality (2.99) requires the length T of the time interval to be sufficiently large. Note that, when the distance between two consecutive exponents λ_n, the gap, becomes small the value of T must increase proportionally.

- In the first inequality (2.99) T depends on the minimum γ of the distances between every two consecutive exponents (gap). However, as we shall see in the next theorem, only the asymptotic distance as $n \to \infty$ between consecutive exponents really matters to determine the minimal control time T. Note also that the constant C_1 in (2.99) degenerates when T goes to π/γ.

- In the critical case $T = \pi/\gamma$ inequality (2.99) may hold or not, depending on the particular family of exponential functions. For instance, if $\lambda_n = n$ for all $n \in \mathbb{Z}$, (2.99) is verified for $T = \pi$. This may be seen immediately by using the orthogonality property of the complex exponentials in $(-\pi, \pi)$. Nevertheless, if $\lambda_n = n - 1/4$ and $\lambda_{-n} = -\lambda_n$ for all $n > 0$, (2.99) fails for $T = \pi$ (see, [114] or [225]).

■

As we have said before, the length $2T$ of the time interval in (2.99) does not depend on the smallest distance between two consecutive exponents but on the asymptotic gap defined by

$$\liminf_{|n| \to \infty} \mid \lambda_{n+1} - \lambda_n \mid = \gamma_\infty. \tag{2.102}$$

An induction argument due to A. Haraux (see [105]) allows to give a result similar to Theorem 2.4.1 above in which condition (2.97) for γ is replaced by a similar one for γ_∞.

Theorem 2.4.3 *Let $(\lambda_n)_{n\in\mathbb{Z}}$ be an increasing sequence of real numbers such that $\lambda_{n+1} - \lambda_n \geq \gamma > 0$ for any $n \in \mathbb{Z}$ and let $\gamma_\infty > 0$ be given by* (2.102)*. For any real T with*

$$T > \pi/\gamma_\infty \tag{2.103}$$

there exist two positive constants $C_1, C_2 > 0$ such that, for any finite sequence $(a_n)_{n\in\mathbb{Z}}$,

$$C_1 \sum_{n\in\mathbb{Z}} \mid a_n \mid^2 \leq \int_{-T}^{T} \left| \sum_{n\in\mathbb{Z}} a_n e^{i\lambda_n t} \right|^2 dt \leq C_2 \sum_{n\in\mathbb{Z}} \mid a_n \mid^2 . \tag{2.104}$$

Remark 2.4.2 When $\gamma_\infty = \gamma$, the sequence of Theorem 2.4.3 satisfies $\lambda_{n+1} - \lambda_n \geq \gamma_\infty > 0$ for all $n \in \mathbb{Z}$ and we can then apply Theorems 2.4.1 and 2.4.2. However, in general, $\gamma_\infty < \gamma$ and Theorem 2.4.3 gives a sharper bound on the minimal time T needed for (2.104) to hold.

Note that the existence of C_1 and C_2 in (2.104) is a consequence of Kahane's theorem (see [120]). However, if our purpose were to have an explicit estimate of C_1 or C_2 in terms of γ, γ_∞ then we would need to use the constructive argument below. It is important to note that these estimates depend strongly also on the number of eigenfrequencies λ that fail to fulfill the gap condition with the asymptotic gap γ_∞.

Proof of Theorem 2.4.3 The second inequality from (2.104) follows immediately by using Theorem 2.4.2. To prove the first inequality (2.104) we follow the induction argument due to Haraux [105].

Note that for any $\varepsilon_1 > 0$, there exists $N = N(\varepsilon_1) \in \mathbb{N}^*$ such that

$$|\lambda_{n+1} - \lambda_n| \geq \gamma_\infty - \varepsilon_1 \text{ for any } |n| > N. \tag{2.105}$$

We begin with the function $f_0(t) = \sum_{|n|>N} a_n e^{i\lambda_n t}$ and we add the missing exponentials one by one. From (2.105) we deduce that Theorems 2.4.1 and 2.4.2 may be applied to the family $\left(e^{i\lambda_n t}\right)_{|n|>N}$ for any $T > \pi/(\gamma_\infty - \varepsilon_1)$

$$C_1 \sum_{n>N} \mid a_n \mid^2 \leq \int_{-T}^{T} \mid f_0(t) \mid^2 dt \leq C_2 \sum_{n>N} \mid a_n \mid^2 . \tag{2.106}$$

Let now $f_1(t) = f_0 + a_N e^{i\lambda_N t} = \sum_{|n|>N} a_n e^{i\lambda_n t} + a_N e^{i\lambda_N t}$. Without loss of generality we may suppose that $\lambda_N = 0$ (since we can consider the function $f_1(t) e^{-i\lambda_N t}$ instead of $f_1(t)$).

Let $\varepsilon > 0$ be such that $T' = T - \varepsilon > \pi/\gamma_\infty$. We have

$$\int_0^\varepsilon (f_1(t+\eta) - f_1(t))\, d\eta = \sum_{n>N} a_n \left(\frac{e^{i\lambda_n \varepsilon} - 1}{i\lambda_n} - \varepsilon \right) e^{i\lambda_n t}, \quad \forall t \in [0, T'].$$

Applying now (2.106) to the function $h(t) = \displaystyle\int_0^\varepsilon (f_1(t+\eta) - f_1(t))\, d\eta$ we obtain that:

$$C_1 \sum_{n>N} \left| \frac{e^{i\lambda_n \varepsilon} - 1}{i\lambda_n} - \varepsilon \right|^2 |a_n|^2 \leq \int_{-T'}^{T'} \left| \int_0^\varepsilon (f_1(t+\eta) - f_1(t))\, d\eta \right|^2 dt. \tag{2.107}$$

Moreover,:

$$\left| e^{i\lambda_n \varepsilon} - 1 - i\lambda_n \varepsilon \right|^2 = |\cos(\lambda_n \varepsilon) - 1|^2 + |\sin(\lambda_n \varepsilon) - \lambda_n \varepsilon|^2 =$$

$$= 4\sin^4\left(\frac{\lambda_n\varepsilon}{2}\right) + (\sin(\lambda_n\varepsilon) - \lambda_n\varepsilon)^2 \geq \begin{cases} 4\left(\frac{\lambda_n\varepsilon}{\pi}\right)^4, & \text{if } |\lambda_n|\varepsilon \leq \pi \\ (\lambda_n\varepsilon)^2, & \text{if } |\lambda_n|\varepsilon > \pi. \end{cases}$$

Finally, taking into account that $|\lambda_n| \geq \gamma$, we obtain that,

$$\left|\frac{e^{i\lambda_n\varepsilon} - 1}{i\lambda_n} - \varepsilon\right|^2 \geq c\varepsilon^2.$$

We return now to (2.107) and we get that:

$$\varepsilon^2 C_1 \sum_{n>N} |a_n|^2 \leq \int_{-T'}^{T'} \left|\int_0^{\varepsilon} (f_1(t+\eta) - f_1(t))\, d\eta\right|^2 dt. \tag{2.108}$$

On the other hand

$$\int_{-T'}^{T'} \left|\int_0^{\varepsilon} (f_1(t+\eta) - f_1(t))\, d\eta\right|^2 dt \leq \int_{-T'}^{T'} \varepsilon \int_0^{\varepsilon} |f_1(t+\eta) - f_1(t)|^2\, d\eta dt \leq$$

$$\leq 2\varepsilon \int_{-T'}^{T'} \int_0^{\varepsilon} \left(|f_1(t+\eta)|^2 + |f_1(t)|^2\right) d\eta dt \leq 2\varepsilon^2 \int_{-T'}^{T} |f_1(t)|^2\, dt +$$

$$+2\varepsilon \int_0^{\varepsilon} \int_{-T'}^{T'} |f_1(t+\eta)|^2\, dt d\eta = 2\varepsilon^2 \int_{-T'}^{T} |f_1(t)|^2\, dt + 2\varepsilon \int_0^{\varepsilon} \int_{-T'+\eta}^{T'+\eta} |f_1(s)|^2\, ds d\eta$$

$$\leq 2\varepsilon^2 \int_{-T}^{T} |f_1(t)|^2\, dt + 2\varepsilon \int_0^{\varepsilon} \int_{-T}^{T} |f_1(s)|^2\, ds d\eta \leq 4\varepsilon^2 \int_{-T}^{T} |f_1(t)|^2\, dt.$$

From (2.108) it follows that

$$C_1 \sum_{n>N} |a_n|^2 \leq \int_{-T}^{T} |f_1(t)|^2\, dt. \tag{2.109}$$

On the other hand

$$|a_N|^2 = \left|f_1(t) - \sum_{n>N} a_n e^{i\lambda_n t}\right|^2 = \frac{1}{2T} \int_{-T}^{T} \left|f_1(t) - \sum_{n>N} a_n e^{i\lambda_n t}\right|^2 dt \leq$$

$$\leq \frac{1}{T} \left(\int_{-T}^{T} |f_1(t)|^2\, dt + \int_{-T}^{T} \left|\sum_{n>N} a_n e^{i\lambda_n t}\right|^2\right) dt \leq$$

$$\leq \frac{1}{T} \left(\int_{-T}^{T} |f_1(t)|^2\, dt + C_2^0 \sum_{n>N} |a_n|^2\right) \leq$$

$$\leq \frac{1}{T}\left(1+\frac{C_2}{C_1}\right)\int_{-T}^{T}|f_1(t)|^2\,dt.$$

From (2.109) we get that

$$C_1\sum_{n\geq N}|a_n|^2 \leq \int_{-T}^{T}|f_1(t)|^2\,dt.$$

Repeating this argument we may add all the terms $a_n e^{i\lambda_n t}$, $|n|\leq N$ and we obtain the desired inequalities.

■

2.4.2 Spectral analysis of the wave operator

The aim of this section is to give the Fourier expansion of solutions of the $1D$ linear wave equation

$$\begin{cases} \varphi'' - \varphi_{xx} + \alpha\varphi = 0, & x\in(0,1),\ t\in(0,T)\\ \varphi(t,0)=\varphi(t,1)=0, & t\in(0,T)\\ \varphi(0)=\varphi^0,\ \varphi'(0)=\varphi^1, & x\in(0,1)\end{cases} \tag{2.110}$$

where α is a real nonnegative number.

To do this let us first remark that (2.110) may be written as

$$\begin{cases} \varphi' = z\\ z' = \varphi_{xx}-\alpha\varphi\\ \varphi(t,0)=\varphi(t,1)=0\\ \varphi(0)=\varphi^0,\ z(0)=\varphi^1.\end{cases}$$

Nextly, denoting $\Phi=(\varphi,z)$, equation (2.110) is written in the following abstract Cauchy form:

$$\begin{cases} \Phi' + A\Phi = 0\\ \Phi(0)=\Phi^0.\end{cases} \tag{2.111}$$

The differential operator A from (2.111) is the unbounded operator in $H = L^2(0,1)\times H^{-1}(0,1)$, $A:\mathcal{D}(A)\subset H\to H$, defined by

$$\begin{aligned} &\mathcal{D}(A) = H_0^1(0,1)\times L^2(0,1)\\ &A(\varphi,z) = (-z, -\partial_x^2\varphi+\alpha\varphi) = \begin{pmatrix} 0 & -1\\ -\partial_x^2+\alpha & 0\end{pmatrix}\begin{pmatrix}\varphi\\ z\end{pmatrix}\end{aligned} \tag{2.112}$$

where the Laplace operator $-\partial_x^2$ is an unbounded operator defined in $H^{-1}(0,1)$ with domain $H_0^1(0,1)$:

$$-\partial_x^2 : H_0^1(0,1)\subset H^{-1}(0,1)\to H^{-1}(0,1),$$

$$\langle -\partial_x^2\varphi,\psi\rangle = \int_0^1 \varphi_x\psi_x dx,\quad \forall\varphi,\ \psi\in H_0^1(0,1).$$

Remark 2.4.3 The operator A is an isomorphism from $H_0^1(0,1) \times L^2(0,1)$ to $L^2(0,1) \times H^{-1}(0,1)$. We shall consider the space $H_0^1(0,1)$ with the inner product defined by

$$(u,v)_{H_0^1(0,1)} = \int_0^1 (u_x)(x)v_x(x)dx + \alpha \int_0^1 u(x)v(x)dx \tag{2.113}$$

which is equivalent to the usual one.

Lemma 2.4.1 *The eigenvalues of A are $\lambda_n = \operatorname{sgn}(n)\pi i\sqrt{n^2+\alpha}$, $n \in \mathbb{Z}^*$. The corresponding eigenfunctions are given by*

$$\Phi^n = \begin{pmatrix} \frac{1}{\lambda_n} \\ -1 \end{pmatrix} \sin(n\pi x),\ n \in \mathbb{Z}^*,$$

and form an orthonormal basis in $H_0^1(0,1) \times L^2(0,1)$.

Proof. Let us first determine the eigenvalues of A. If $\lambda \in \mathbb{C}$ and $\Phi = (\varphi, z) \in H_0^1(0,1) \times L^2(0,1)$ are such that $A\Phi = \lambda\Phi$ we obtain from the definition of A that

$$\begin{cases} -z = \lambda\varphi \\ -\partial_x^2\varphi + \alpha\varphi = \lambda z. \end{cases} \tag{2.114}$$

It is easy to see that

$$\partial_x^2\varphi - \alpha\varphi = \lambda^2\varphi; \quad \varphi(0) = \varphi(1) = 0; \quad \varphi \in C^2[0,1]. \tag{2.115}$$

The solutions of (2.115) are given by

$$\lambda_n = \operatorname{sgn}(n)\pi i\sqrt{n^2+\alpha}, \quad \varphi_n = c\sin(n\pi x), \quad n \in \mathbb{Z}^*$$

where c is an arbitrary complex constant.

Hence, the eigenvalues of A are $\lambda_n = \operatorname{sgn}(n)\pi i\sqrt{n^2+\alpha}$, $n \in \mathbb{Z}^*$ and the corresponding eigenfunctions are

$$\Phi^n = \begin{pmatrix} \frac{1}{\lambda_n} \\ -1 \end{pmatrix} \sin(n\pi x),\ n \in \mathbb{Z}^*.$$

It is easy to see that

- $$\| \Phi^n \|^2_{H_0^1 \times L^2} = \frac{1}{(n^2+\alpha)\pi^2}\left(\int_0^1 (n\pi\cos(n\pi x))^2\,dx + \alpha\int_0^1 \sin^2(n\pi x)dx\right) + \int_0^1 (\sin(n\pi x))^2 dx = 1$$

- $$(\Phi^n, \Phi^m) = \frac{1}{nm\pi^2} \int_0^1 (n\pi \cos(n\pi x) m\pi \cos(m\pi x))\, dx$$
$$+(\alpha + 1) \int_0^1 (\sin(n\pi x) \sin(m\pi x))\, dx = \delta_{nm}.$$

Hence, $(\Phi^n)_{n\in\mathbb{Z}^*}$ is an orthonormal sequence in $H_0^1(0,1) \times L^2(0,1)$.

The completeness of $(\Phi^n)_{n\in\mathbb{Z}^*}$ in $H_0^1(0,1) \times L^2(0,1)$ is a consequence of the fact that these are all the eigenfunctions of the compact skew-adjoint operator A^{-1}. It follows that $(\Phi^n)_{n\in\mathbb{Z}^*}$ is an orthonormal basis in $H_0^1(0,1) \times L^2(0,1)$.

■

Remark 2.4.4 Since $(\Phi^n)_{n\in\mathbb{Z}^*}$ is an orthonormal basis in $H_0^1(0,1) \times L^2(0,1)$ and A is an isomorphism from $H_0^1(0,1) \times L^2(0,1)$ to $L^2(0,1) \times H^{-1}(0,1)$ it follows immediately that $(A(\Phi^n))_{n\in\mathbb{Z}^*}$ is an orthonormal basis in $L^2(0,1) \times H^{-1}(0,1)$. Moreover $(\lambda_n \Phi^n)_{n\in\mathbb{Z}^*}$ is an orthonormal basis in $L^2(0,1)\times H^{-1}(0,1)$. We have that

- $\Phi = \sum_{n\in\mathbb{Z}^*} a_n\, \Phi^n \in H_0^1(0,1) \times L^2(0,1)$ if and only if $\sum_{n\in\mathbb{Z}^*} |a_n|^2 < \infty$.
- $\Phi = \sum_{n\in\mathbb{Z}^*} a_n\, \Phi^n \in L^2(0,1) \times H^{-1}(0,1)$ if and only if $\sum_{n\in\mathbb{Z}^*} \frac{|a_n|^2}{|\lambda_n|^2} < \infty$.

■

The Fourier expansion of the solution of (2.111) is given in the following Lemma.

Lemma 2.4.2 *The solution of* (2.111) *with the initial data*

$$W^0 = \sum_{n\in\mathbb{Z}^*} a_n\, \Phi^n \in L^2(0,1) \times H^{-1}(0,1) \tag{2.116}$$

is given by

$$W(t) = \sum_{n\in\mathbb{Z}^*} a_n\, e^{\lambda_n t} \Phi^n. \tag{2.117}$$

2.4.3 Observability for the interior controllability of the $1D$ wave equation

Consider an interval $J \subset [0,1]$ with $|\, J \,| > 0$ and a real time $T > 2$. We address the following control problem discussed in 2.2: given $(u^0, u^1) \in H_0^1(0,1) \times L^2(0,1)$ to find $f \in L^2((0,T) \times J)$ such that the solution u of the problem

$$\left\{ \begin{array}{ll} u'' - u_{xx} = f 1_J, & x \in (0,1),\ t \in (0,T) \\ u(t,0) = u(t,1) = 0, & t \in (0,T) \\ u(0) = u^0, u'(0) = u^1, & x \in (0,1) \end{array} \right. \tag{2.118}$$

satisfies

$$u(T,\cdot)=u'(T,\cdot)=0. \tag{2.119}$$

According to the developments of section 2.2, the control problem can be solved if the following inequalities hold for any $(\varphi^0,\varphi^1)\in L^2(0,1)\times H^{-1}(0,1)$

$$C_1 \parallel (\varphi^0,\varphi^1) \parallel^2_{L^2\times H^{-1}} \leq \int_0^T \int_J \mid \varphi(t,x) \mid^2 dxdt \leq C_2 \parallel (\varphi^0,\varphi^1) \parallel^2_{L^2\times H^{-1}} \tag{2.120}$$

where φ is the solution of the adjoint equation (2.110).

In this section we prove (2.120) by using the Fourier expansion of the solutions of (2.110). Similar results can be proved for more general potentials depending on x and t by multiplier methods and sidewiese energy estimates [233] and also using Carleman inequalities [227], [228].

Remark 2.4.5 In the sequel when (2.120) holds , for brevity, we will denote it as follows:

$$\parallel (\varphi^0,\varphi^1) \parallel^2_{L^2(0,1)\times H^{-1}(0,1)} \asymp \int_0^T \int_J \mid \varphi(t,x) \mid^2 dxdt. \tag{2.121}$$

Theorem 2.4.4 *Let $T\geq 2$. There exist two positive constants C_1 and C_2 such that* (2.120) *holds for any $(\varphi^0,\varphi^1)\in L^2(0,1)\times H^{-1}(0,1)$ and φ solution of* (2.110).

Proof. Firstly, we have that

$$\parallel (\varphi^0,\varphi^1) \parallel^2_{L^2\times H^{-1}} = \left|\left| A^{-1}\left(\sum_{n\in\mathbb{Z}^*} a_n \Phi^n\right)\right|\right|^2_{H_0^1\times L^2} =$$

$$= \left|\left| \sum_{n\in\mathbb{Z}^*} a_n \frac{1}{in\pi}\Phi^n \right|\right|^2_{H_0^1\times L^2} = \sum_{n\in\mathbb{Z}^*} \mid a_n \mid^2 \frac{1}{n^2\pi^2}.$$

Hence,

$$\parallel (\varphi^0,\varphi^1) \parallel^2_{L^2\times H^{-1}} = \sum_{n\in\mathbb{Z}^*} \mid a_n \mid^2 \frac{1}{n^2\pi^2}. \tag{2.122}$$

On the other hand, since $\varphi\in C([0,T],L^2(0,1))\subset L^2((0,T)\times(0,1))$, we obtain from Fubini's Theorem that

$$\int_0^T \int_J \mid w(t,x) \mid^2 dxdt = \int_J \int_0^T \left| \sum_{n\in\mathbb{Z}^*} a_n e^{in\pi t} \frac{1}{n\pi} \sin(n\pi x) \right|^2 dtdx.$$

Let first $T = 2$. From the orthogonality of the exponential functions in $L^2(0,2)$ we obtain that

$$\int_J \int_0^T \left| \sum_{n\in\mathbb{Z}^*} a_n e^{in\pi t} \frac{1}{n\pi} \sin(n\pi x) \right|^2 dtdx = \sum_{n\in\mathbb{Z}^*} \frac{|a_n|^2}{n^2\pi^2} \int_J \sin^2(n\pi x)dx.$$

If $T \geq 2$, it is immediate that

$$\int_J \int_0^T \left| \sum_{n\in\mathbb{Z}^*} \frac{a_n e^{in\pi t}}{n\pi} \sin(n\pi x) \right|^2 dtdx \geq \int_J \int_0^2 \left| \sum_{n\in\mathbb{Z}^*} \frac{a_n e^{in\pi t}}{n\pi} \sin(n\pi x) \right|^2 dtdx \geq$$

$$\geq \sum_{n\in\mathbb{Z}^*} \frac{|a_n|^2}{n^2\pi^2} \int_J \sin^2(n\pi x)dx.$$

On the other hand, by using the 2-periodicity in time of the exponentials and the fact that there exists $p > 0$ such that $2(p-1) \leq T < 2p$, it follows that

$$\int_J \int_0^T \left| \sum_{n\in\mathbb{Z}^*} \frac{a_n e^{in\pi t}}{n\pi} \sin(n\pi x) \right|^2 dtdx \leq p \int_J \int_0^2 \left| \sum_{n\in\mathbb{Z}^*} \frac{a_n e^{in\pi t}}{n\pi} \sin(n\pi x) \right|^2 dtdx$$

$$= p \sum_{n\in\mathbb{Z}^*} \frac{|a_n|^2}{n^2\pi^2} \int_J \sin^2(n\pi x)dx \leq \frac{T+2}{2} \sum_{n\in\mathbb{Z}^*} \frac{|a_n|^2}{n^2\pi^2} \int_J \sin^2(n\pi x)dx.$$

Hence, for any $T \geq 2$, we have that

$$\int_J \int_0^T \left| \sum_{n\in\mathbb{Z}^*} a_n e^{in\pi t} \frac{1}{n\pi} \sin(n\pi x) \right|^2 dxdt \asymp \sum_{n\in\mathbb{Z}^*} \frac{|a_n|^2}{n^2\pi^2} \int_J \sin^2(n\pi x)dx. \quad (2.123)$$

If we denote $b_n = \int_J \sin^2(n\pi x)dx$ then

$$B = \inf_{n\in\mathbb{Z}^*} b_n > 0. \quad (2.124)$$

Indeed,

$$b_n = \int_J \sin^2(n\pi x)dx = \frac{|\, J\,|}{2} - \int_J \frac{\cos(2n\pi x)}{2} \geq \frac{|\, J\,|}{2} - \frac{1}{2\,|\, n\,|\,\pi}.$$

Since $1/[2 \mid n \mid \pi]$ tends to zero when n tends to infinity, there exists $n_0 > 0$ such that

$$b_n \geq \frac{|\, J\,|}{2} - \frac{1}{2\,|\, n\,|\,\pi} > \frac{|\, J\,|}{4} > 0, \quad \forall \mid n \mid > n_0.$$

It follows that

$$B_1 = \inf_{|n|>n_0} b_n > 0 \quad (2.125)$$

and $B > 0$ since $b_n > 0$ for all n.

Moreover, since $b_n \leq |J|$ for any $n \in \mathbb{Z}^*$, it follows from (2.123) that

$$B \sum_{n \in \mathbb{Z}^*} \frac{|\, a_n \,|^2}{n^2\pi^2} \leq \int_0^T \int_J |\, \varphi(t,x) \,|^2 \, dxdt \leq |J| \sum_{n \in \mathbb{Z}^*} |\, a_n \,|^2 \, \frac{1}{n^2\pi^2}. \tag{2.126}$$

Finally, (2.120) follows immediately from (2.122) and (2.126).

■

As a direct consequence of Theorem 2.4.4 the following controllability result holds:

Theorem 2.4.5 *Let $J \subset [0,1]$ with $|\, J \,| > 0$ and a real $T \geq 2$. For any $(u^0, u^1) \in H_0^1(0,1) \times L^2(0,1)$ there exists $f \in L^2((0,T) \times J)$ such that the solution u of equation* (2.118) *satisfies* (2.119).

Remark 2.4.6 In order to obtain (2.123) for $T > 2$, Ingham's Theorem 2.4.1 could also be used. Indeed, the exponents are $\mu_n = n\pi$ and they satisfy the uniform gap condition $\gamma = \mu_{n+1} - \mu_n = \pi$, for all $n \in \mathbb{Z}^*$. It then follows from Ingham's Theorem 2.4.1 that, for any $T > 2\pi/\gamma = 2$, we have (2.123).

Note that the result may not be ontained in the critical case $T = 2$ by using Theorems 2.4.1 and 2.4.2. The critical time $T = 2$ is reached in this case because of the orthogonality properties of the trigonometric polynomials $e^{i\pi n t}$.

■

Consider now the equation

$$\begin{cases} u'' - u_{xx} + \alpha u = f 1_J, & x \in (0,1),\ t \in (0,T) \\ u(t,0) = u(t,1) = 0, & t \in (0,T) \\ u(0) = u^0, u'(0) = u^1, & x \in (0,1) \end{cases} \tag{2.127}$$

where α is a positive real number.

The controllability problem may be reduced once more to the proof of the following fact:

$$\int_J \int_0^T \left| \sum_{n \in \mathbb{Z}^*} a_n e^{\lambda_n t} \frac{1}{n\pi} \sin(n\pi x) \right|^2 dtdx \asymp \sum_{n \in \mathbb{Z}^*} \frac{|a_n|^2}{|\lambda_n|^2} \int_J \sin^2(n\pi x) dx \tag{2.128}$$

where $\lambda_n = \mathrm{sgn}(n)\pi i \sqrt{n^2 + \alpha}$ are the eigenvalues of problem (2.127).

Remark that,

$$\begin{array}{l} \gamma = \inf\{\lambda_{n+1} - \lambda_n\} = \inf \left\{ \dfrac{(2n+1)\pi}{\sqrt{(n+1)^2 + \alpha} + \sqrt{n^2 + \alpha}} \right\} > \dfrac{\pi}{2\sqrt{\alpha}}, \\ \gamma_\infty = \underline{\lim}_{n \to \infty} (\lambda_{n+1} - \lambda_n) = \pi. \end{array} \tag{2.129}$$

It follows from the generalized Ingham Theorem 2.4.3 that, for any $T > 2\pi/\gamma_\infty = 2$, (2.128) holds. Hence, the following controllability result is obtained:

Theorem 2.4.6 *Let $J \subset [0,1]$ with $\mid J \mid > 0$ and $T > 2$. For any $(u^0, u^1) \in H_0^1(0,1) \times L^2(0,1)$ there exists $f \in L^2((0,T) \times J)$ such that the solution u of equation* (2.127) *satisfies* (2.119).

Remark 2.4.7 Note that if we had applied Theorem 2.4.1 the controllability time would have been $T > 2\pi/\gamma \geq 4\sqrt{\alpha}$. But Theorem 2.4.3 gives a control time T independent of α.

Note that in this case the exponential functions $(e^{\lambda_n t})_n$ are not orthogonal in $L^2(0,T)$. Thus we can not use the same argument as in the proof on Theorem 2.4.4 and, accordingly, Ingham's Theorem is needed.

We have considered here the case where α is a positive constant. When α is negative the complex exponentials entering in the Fourier expansion of solutions may have eigenfrequencies λ_n which are not all purely real. In that case we can not apply directly Theorem 2.4.3. However, its method of proof allows also to deal with the situation where a finite number of eigenfrequencies are non real. Thus, the same result holds for all real α.

■

2.4.4 Boundary controllability of the $1D$ wave equation

In this section we study the following boundary controllability problem: given $T > 2$ and $(u^0, u^1) \in L^2(0,1) \times H^{-1}(0,1)$ to find a control $f \in L^2(0,T)$ such that the solution u of the problem:

$$\begin{cases} u'' - u_{xx} = 0 & x \in (0,1),\ t \in [0,T] \\ u(t,0) = 0 & t \in [0,T] \\ u(t,1) = f(t) & t \in [0,T] \\ u(0) = u^0, u'(0) = u^1 & x \in (0,1) \end{cases} \tag{2.130}$$

satisfies

$$u(T,\cdot) = u'(T,\cdot) = 0. \tag{2.131}$$

From the developments in section 2.3 it follows that the following inequalities are a necessary and sufficient condition for the controllability of (2.130)

$$C_1 \parallel (\varphi^0, \varphi^1) \parallel^2_{H_0^1 \times L^2} \leq \int_0^T |\varphi_x(t,1)|^2 \, dt \leq C_2 \parallel (\varphi^0, \varphi^1) \parallel^2_{H_0^1 \times L^2} \tag{2.132}$$

for any $(\varphi^0, \varphi^1) \in H_0^1(0,1) \times L^2(0,1)$ and φ solution of (2.110).

In order to prove (2.132) we use the Fourier decomposition of (2.110) given in the first section.

Theorem 2.4.7 *Let $T \geq 2$. There exist two positive constants C_1 and C_2 such that* (2.132) *holds for any* $(\varphi^0, \varphi^1) \in H_0^1(0,1) \times L^2(0,1)$ *and* φ *solution of* (2.110).

Proof. If $(\varphi^0, \varphi^1) = \sum_{n \in \mathbb{Z}^*} a_n \Phi_n$ we have that,

$$\| (\varphi^0, \varphi^1) \|^2_{H_0^1 \times L^2} = \left\| \sum_{n \in \mathbb{Z}^*} a_n \Phi_n \right\|^2_{H_0^1 \times L^2} = \sum_{n \in \mathbb{Z}^*} | a_n |^2 . \tag{2.133}$$

On the other hand

$$\int_0^T |\varphi_x(t,1)|^2 \, dt = \int_0^T \left| \sum_{n \in \mathbb{Z}^*} (-1)^n a_n e^{in\pi t} \right|^2 dt.$$

By using the orthogonality in $L^2(0,2)$ of the exponentials $(e^{in\pi t})_n$, we get that

$$\int_0^2 \left| \sum_{n \in \mathbb{Z}^*} (-1)^n a_n e^{in\pi t} \right|^2 dt = \sum_{n \in \mathbb{Z}^*} | a_n |^2 .$$

If $T > 2$, it is immediate that

$$\int_0^T \left| \sum_{n \in \mathbb{Z}^*} (-1)^n a_n e^{in\pi t} \right|^2 dt \geq \int_0^2 \left| \sum_{n \in \mathbb{Z}^*} (-1)^n a_n e^{in\pi t} \right|^2 dt = \sum_{n \in \mathbb{Z}^*} | a_n |^2 .$$

On the other hand, by using the 2-periodicity in time of the exponentials and the fact that there exists $p > 0$ such that $2(p-1) \leq T < 2p$, it follows that

$$\int_0^T \left| \sum_{n \in \mathbb{Z}^*} (-1)^n a_n e^{in\pi t} \right|^2 dt \geq p \int_0^2 \left| \sum_{n \in \mathbb{Z}^*} (-1)^n a_n e^{in\pi t} \right|^2 dt =$$

$$= p \sum_{n \in \mathbb{Z}^*} | a_n |^2 \leq \frac{T+2}{2} \sum_{n \in \mathbb{Z}^*} |a_n|^2.$$

Hence, for any $T \geq 2$, we have that

$$\int_0^T \left| \sum_{n \in \mathbb{Z}^*} (-1)^n a_n e^{in\pi t} \right|^2 dt \asymp \sum_{n \in \mathbb{Z}^*} | a_n |^2 . \tag{2.134}$$

Finally, from (2.133) and (2.134) we obtain that

$$\int_0^2 |\varphi_x(t,1)|^2 \, dt \asymp \| (\varphi^0, \varphi^1) \|^2_{H_0^1(0,1) \times L^2(0,1)}$$

and (2.132) is proved. ■

As a direct consequence of Theorems 2.4.7 the following controllability result holds:

Theorem 2.4.8 *Let $T \geq 2$. For any $(u^0, u^1) \in L^2(0,1) \times H^{-1}(0,1)$ there exists $f \in L^2(0,T)$ such that the solution u of equation* (2.130) *satisfies* (2.131).

As in the context of the interior controllability problem, one may address the following wave equation with potential

$$\begin{cases} u'' - u_{xx} + \alpha u = 0, & x \in (0,1),\ t \in (0,T) \\ u(t,0) = 0 & t \in [0,T] \\ u(t,1) = f(t) & t \in [0,T] \\ u(0) = u^0, u'(0) = u^1, & x \in (0,1) \end{cases} \tag{2.135}$$

where α is a positive real number.

The controllability problem is then reduced to the proof of the following inequality:

$$\int_0^T \left| \sum_{n \in \mathbb{Z}^*} (-1)^n \frac{n\pi}{\lambda_n} a_n e^{\lambda_n t} \right|^2 dt \asymp \sum_{n \in \mathbb{Z}^*} |a_n|^2 \tag{2.136}$$

where $\lambda_n = \mathrm{sgn}(n)\pi i \sqrt{n^2 + \alpha}$ are the eigenvalues of problem (2.135).

It follows from (2.129) and the generalized Ingham's Theorem 2.4.3 that, for any $T > 2\pi/\gamma_\infty = 2$, (2.136) holds. Hence, the following controllability result is obtained:

Theorem 2.4.9 *Let $T > 2$. For any $(u^0, u^1) \in L^2(0,1) \times H^{-1}(0,1)$ there exists $f \in L^2(0,T)$ such that the solution u of equation* (2.135) *satisfies* (2.131).

Remark 2.4.8 As we mentioned above, the classical Ingham inequality in (2.4.1) gives a suboptimal result in what concerns the time of control.

■

2.5 Interior controllability of the heat equation

In this section the interior controllability problem of the heat equation is studied. The control is assumed to act on a subset of the domain where the solutions are defined. The boundary controllability problem of the heat equation will be considered in the following section.

2.5.1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with boundary of class C^2 and ω a non empty open subset of Ω. Given $T > 0$ we consider the following non-homogeneous heat equation:

$$\begin{cases} u_t - \Delta u = f 1_\omega & \text{in } (0,T) \times \Omega \\ u = 0 & \text{on } (0,T) \times \partial\Omega \\ u(x,0) = u^0(x) & \text{in } \Omega. \end{cases} \tag{2.137}$$

In (2.137) $u = u(x,t)$ is the state and $f = f(x,t)$ is the control function with a support localized in ω. We aim at changing the dynamics of the system by acting on the subset ω of the domain Ω.

The heat equation is a model for many diffusion phenomena. For instance (2.137) provides a good description of the temperature distribution and evolution in a body occupying the region Ω. Then the control f represents a localized source of heat.

The interest on analyzing the heat equation above relies not only in the fact that it is a model for a large class of physical phenomena but also one of the most significant partial differential equations of parabolic type. As we shall see latter on, the main properties of parabolic equations such as time-irreversibility and regularizing effects have some very important consequences in control problems.

2.5.2 Existence and uniqueness of solutions

The following theorem is a consequence of classical results of existence and uniqueness of solutions of nonhomogeneous evolution equations. All the details may be found, for instance in [40].

Theorem 2.5.1 *For any $f \in L^2((0,T)\times\omega)$ and $u^0 \in L^2(\Omega)$ equation (2.137) has a unique weak solution $u \in C([0,T], L^2(\Omega))$ given by the variation of constants formula*

$$u(t) = S(t)u^0 + \int_0^t S(t-s)f(s)1_\omega ds \tag{2.138}$$

where $(S(t))_{t\in\mathbb{R}}$ is the semigroup of contractions generated by the heat operator in $L^2(\Omega)$.

Moreover, if $f \in W^{1,1}((0,T)\times L^2(\omega))$ and $u^0 \in H^2(\Omega)\cap H_0^1(\Omega)$, equation (2.137) has a classical solution $u \in C^1([0,T], L^2(\Omega))\cap C([0,T], H^2(\Omega)\cap H_1^0(\Omega))$ and (2.137) is verified in $L^2(\Omega)$ for all $t > 0$.

Let us recall the classical energy estimate for the heat equation. Multiplying in (2.137) by u and integrating in Ω we obtain that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\, u\,|^2\, dx + \int_\Omega |\, \nabla u\,|^2\, dx = \int_\Omega fudx \le \frac{1}{2}\int_\Omega |\, f\,|^2\, dx + \frac{1}{2}\int_\Omega |\, u\,|^2\, dx.$$

Hence, the scalar function $X = \int_\Omega |\, u\,|^2\, dx$ satisfies

$$X' \le X + \int_\Omega |\, f\,|^2\, dx$$

which, by Gronwall's inequality, gives

$$X(t) \le X(0)e^t + \int_0^t \int_\Omega |\, f\,|^2\, dxds \le X(0)e^t + \int_0^T \int_\Omega |\, f\,|^2\, dxdt.$$

On the other hand, integrating in (2.137) with respect to t, it follows that

$$\frac{1}{2}\int_\Omega u^2 dx\Big|_0^T + \int_0^T \int_\Omega |\nabla u|^2 \, dxdt \leq \frac{1}{2}\int_0^T \int_\Omega f^2 dxdt + \frac{1}{2}\int_0^T \int_\Omega u^2 dxdt$$

From the fact that $u \in L^\infty(0,T;L^2(\Omega))$ it follows that $u \in L^2(0,T;H_0^1(\Omega))$. Consequently, whenever $u_0 \in L^2(\Omega)$ and $f \in L^2(0,T;L^2(\omega))$ the solution u verifies

$$u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega)).$$

2.5.3 Controllability problems

Let $T > 0$ and define, for any initial data $u^0 \in L^2(\Omega)$, the set of reachable states

$$R(T;u^0) = \{u(T) : u \text{ solution of (2.137) with } f \in L^2((0,T)\times\omega)\}. \quad (2.139)$$

By definition, any state in $R(T;u^0)$ is reachable in time T by starting from u^0 at time $t = 0$ with the aid of a convenient control f.

As in the case of the wave equation several notions of controllability may be defined.

Definition 2.5.1 *System* (2.137) *is* **approximately controllable in time** T *if, for every initial data* $u^0 \in L^2(\Omega)$*, the set of reachable states* $R(T;u^0)$ *is dense in* $L^2(\Omega)$.

Definition 2.5.2 *System* (2.137) *is* **exactly controllable in time** T *if, for every initial data* $u^0 \in L^2(\Omega)$*, the set of reachable states* $R(T;u^0)$ *coincides with* $L^2(\Omega)$.

Definition 2.5.3 *System* (2.137) *is* **null controllable in time** T *if, for every initial data* $u^0 \in L^2(\Omega)$*, the set of reachable states* $R(T;u^0)$ *contains the element* 0.

Remark 2.5.1 Let us make the following remarks:

- One of the most important properties of the heat equation is its regularizing effect. When $\Omega\setminus\omega \neq \emptyset$, the solutions of (2.137) belong to $C^\infty(\Omega\setminus\omega)$ at time $t = T$. Hence, the restriction of the elements of $R(T,u^0)$ to $\Omega\setminus\omega$ are C^∞ functions. Then, the trivial case $\omega = \Omega$ (i. e. when the control acts on the entire domain Ω) being excepted, exact controllability may not hold. In this sense, the notion of exact controllability is not very relevant for the heat equation. This is due to its strong time irreversibility of the system under consideration.

- It is easy to see that if null controllability holds, then any initial data may be led to any final state of the form $S(T)v^0$ with $v^0 \in L^2(\Omega)$, i. e. to the range of the semigroup in time $t = T$.

  Indeed, let $u^0, v^0 \in L^2(\Omega)$ and remark that $R(T; u^0 - v^0) = R(T; u^0) - S(T)v^0$. Since $0 \in R(T; u^0 - v^0)$, it follows that $S(T)v^0 \in R(T; u^0)$.

  It is known that the null controllability holds for any time $T > 0$ and open set ω on which the control acts (see, for instance, [90]). The null controllability property holds in fact in a much more general setting of semilinear heat equations ([83] and [84]).

- Null controllability implies approximate controllability. Indeed, we have shown that, whenever null controllability holds, $S(T)[L^2(\Omega)] \subset R(T; u^0)$ for all $u^0 \in L^2(\Omega)$. Taking into account that all the eigenfunctions of the laplacian belong to $S(T)[L^2(\Omega)]$ we deduce that the set of reachable states is dense and, consequently, that approximate controllability holds.

- The problem of approximate controllability may be reduced to the case $u^0 \equiv 0$. Indeed, the linearity of the system we have considered implies that $R(T, u^0) = R(T, 0) + S(T)u^0$.

- Approximate controllability together with uniform estimates on the approximate controls as $\varepsilon \to 0$ may led to null controllability properties. More precisely, given u^1, we have that $u^1 \in R(T, u^0)$ if and only if there exists a sequence $(f_\varepsilon)_{\varepsilon>0}$ of controls such that $||u(T) - u^1||_{L^2(\Omega)} \leq \varepsilon$ and $(f_\varepsilon)_{\varepsilon>0}$ is bounded in $L^2(\omega \times (0, T))$. Indeed in this case any weak limit in $L^2(\omega \times (0, T))$ of the sequence $(f_\varepsilon)_{\varepsilon>0}$ of controls gives an exact control which makes that $u(T) = u_1$.

■

In this section we limit ourselves to study the approximate controllability problem. The main ingredients we shall develop are of variational nature. The problem will be reduced to prove unique continuation properties. Null-controllability will be addressed in the following section.

2.5.4 Approximate controllability of the heat equation

Given any $T > 0$ and any nonempty open subset ω of Ω we analyze in this section the approximate controllability problem for system (2.137).

Theorem 2.5.2 *Let ω be an open nonempty subset of Ω and $T > 0$. Then* (2.137) *is approximately controllable in time T.*

Remark 2.5.2 The fact that the heat equation is approximately controllable in arbitrary time T and with control in any subset of Ω is due to the infinite velocity of propagation which characterizes the heat equation.

Nevertheless, the infinite velocity of propagation by itself does not allow to deduce quantitative estimates for the norm of the controls. Indeed, as it was proved in [164], the heat equation in an infinite domain $(0,\infty)$ of $\mathbb{R}$ is approximately controllable but, in spite of the infinite velocity of propagation, it is not null-controllable.

■

Remark 2.5.3 There are several possible proofs for the approximate controllability property. We shall present here two of them. The first one is presented below and uses Hahn-Banach Theorem. The second one is constructive and uses a variational technique similar to the one we have used for the wave equation. We give it in the following section.

■

Proof of the Theorem 2.5.2 As we have said before, it is sufficient to consider only the case $u^0 = 0$. Thus we assume that $u^0 = 0$.

From Hahn-Banach Theorem, $R(T, u^0)$ is dense in $L^2(\Omega)$ if the following property holds: There is no $\varphi_T \in L^2(\Omega)$, $\varphi_T \neq 0$ such that $\displaystyle\int_\Omega u(T)\varphi_T dx = 0$ for all u solution of (2.137) with $f \in L^2(\omega \times (0,T))$.

Accordingly, the proof can be reduced to showing that, if $\varphi_T \in L^2(\Omega)$ is such that $\int_\Omega u(T)\varphi_T dx = 0$, for all solution u of (2.137) then $\varphi_T = 0$.

To do this we consider the adjoint equation:

$$\begin{cases} \varphi_t + \Delta\varphi = 0 & \text{in } (0,T)\times\Omega \\ \varphi\mid_{\partial\Omega} = 0 & \text{on } (0,T)\times\partial\Omega \\ \varphi(T) = \varphi_T & \text{in } \Omega. \end{cases} \tag{2.140}$$

We multiply the equation satisfied by φ by u and then the equation of u by φ. Integrating by parts and taking into account that $u^0 \equiv 0$ the following identity is obtained

$$\int_0^T\int_\omega f\varphi dxdt = \int_{\Omega\times(0,T)} (u_t - \Delta u)\varphi dxdt = -\int_{\Omega\times(0,T)} (\varphi_t + \Delta\varphi) u dxdt +$$

$$+\int_\Omega u\varphi dx\Big|_0^T + \int_0^T\int_{\partial\Omega}\left(-\frac{\partial u}{\partial n}\varphi + u\frac{\partial\varphi}{\partial n}\right) d\sigma dt = \int_\Omega u(T)\varphi_T dx.$$

Hence, $\int_\Omega u(T)\varphi_T dx = 0$ if and only if $\displaystyle\int_0^T\int_\omega f\varphi dxdt = 0$. If the later relation holds for any $f \in L^2(\omega\times(0,T))$, we deduce that $\varphi \equiv 0$ in $\omega \times (0,T)$.

Let us now recall the following result whose proof may be found in [108]:

Holmgren Uniqueness Theorem. *Let P be a differential operator with constant coefficients in $\mathbb{R}^n$. Let u be a solution of $Pu = 0$ in Q_1 where Q_1 is an open set of $\mathbb{R}^n$. Suppose that $u = 0$ in Q_2 where Q_2 is an open nonempty subset of Q_1.*

Then $u = 0$ in Q_3, where Q_3 is the open subset of Q_1 which contains Q_2 and such that any characteristic hyperplane of the operator P which intersects Q_3 also intersects Q_1.

In our particular case $P = \partial_t + \Delta_x$ is a differential operator in $\mathbb{R}^{n+1}$ and its principal part is $P_p = \Delta_x$. A hyperplane of $\mathbb{R}^{n+1}$ is characteristic if its normal vector $(\xi, \zeta) \in \mathbb{R}^{n+1}$ is a zero of P_p, i. e. of $P_p(\xi, \zeta) = |\xi|^2$. Hence, normal vectors are of the form $(0, \pm 1)$ and the characteristic hyperplanes are horizontal, parallel to the hyperplane $t = 0$.

Consequently, for the adjoint heat equation under consideration (2.140), we can apply Holmgren's Uniqueness Theorem with $Q_1 = (0,T) \times \Omega$, $Q_2 = (0,T) \times \omega$ and $Q_3 = (0,T) \times \Omega$. Then the fact that $\varphi = 0$ in $(0,T) \times \omega$ implies $\varphi = 0$ in $(0,T) \times \Omega$. Consequently $\varphi_T \equiv 0$ and the proof is complete.

■

2.5.5 Variational approach to approximate controllability

In this section we give a new proof of the approximate controllability result in Theorem 2.5.2. This proof has the advantage of being constructive and it allows to compute explicitly approximate controls.

Let us fix the control time $T > 0$ and the initial datum $u^0 = 0$. Let $u^1 \in L^2(\Omega)$ be the final target and $\varepsilon > 0$ be given. Recall that we are looking for a control f such that the solution of (2.137) satisfies

$$||u(T) - u_1||_{L^2(\Omega)} \leq \varepsilon. \tag{2.141}$$

We define the following functional:

$$J_\varepsilon : L^2(\Omega) \to \mathbb{R} \tag{2.142}$$

$$J_\varepsilon(\varphi_T) = \frac{1}{2} \int_0^T \int_\omega \varphi^2 dx dt + \varepsilon \parallel \varphi_T \parallel_{L^2(\Omega)} - \int_\Omega u_1 \varphi_T dx \tag{2.143}$$

where φ is the solution of the adjoint equation (2.140) with initial data φ_T.

The following Lemma ensures that the minimum of J_ε gives a control for our problem.

Lemma 2.5.1 *If $\widehat{\varphi}_T$ is a minimum point of J_ε in $L^2(\Omega)$ and $\widehat{\varphi}$ is the solution of* (2.140) *with initial data $\widehat{\varphi}_T$, then $f = \widehat{\varphi}_{|\omega}$ is a control for* (2.137)*, i. e.* (2.141) *is satisfied.*

Proof. In the sequel we simply denote J_ε by J.

Suppose that J attains its minimum value at $\widehat{\varphi}_T \in L^2(\Omega)$. Then for any $\psi_0 \in L^2(\Omega)$ and $h \in \mathbb{R}$ we have $J(\widehat{\varphi}_T) \leq J\left(\widehat{\varphi}_T + h\psi_0\right)$. On the other hand,

$$J\left(\widehat{\varphi}_T + h\psi_0\right) =$$
$$= \frac{1}{2}\int_0^T\int_\omega \mid \widehat{\varphi} + h\psi \mid^2 dxdt + \varepsilon \parallel \widehat{\varphi}_T + h\psi_0 \parallel_{L^2(\Omega)} - \int_\Omega u_1(\widehat{\varphi}_T + h\psi_0)dx$$
$$= \frac{1}{2}\int_0^T\int_\omega \mid \widehat{\varphi} \mid^2 dxdt + \frac{h^2}{2}\int_0^T\int_\omega \mid \psi \mid^2 dxdt + h\int_0^T\int_\omega \widehat{\varphi}\psi dxdt +$$
$$+\varepsilon \parallel \widehat{\varphi}_T + h\psi_0 \parallel_{L^2(\Omega)} - \int_\Omega u_1(\widehat{\varphi}_T + h\psi_0)dx.$$

Thus

$$\begin{aligned} 0 \leq \quad & \varepsilon\left[\parallel \widehat{\varphi}_T + h\psi_0 \parallel_{L^2(\Omega)} - \parallel \widehat{\varphi}_T \parallel_{L^2(\Omega)}\right] + \frac{h^2}{2}\int_{(0,T)\times\omega}\psi^2 dxdt \\ & +h\left[\int_0^T\int_\omega \widehat{\varphi}\psi dxdt - \int_\Omega u_1\psi_0 dx\right]. \end{aligned}$$

Since

$$\parallel \widehat{\varphi}_T + h\psi_0 \parallel_{L^2(\Omega)} - \parallel \widehat{\varphi}_T \parallel_{L^2(\Omega)} \leq |h| \parallel \psi_0 \parallel_{L^2(\Omega)}$$

we obtain

$$0 \leq \varepsilon\ |h| \parallel \psi_0 \parallel_{L^2(\Omega)} + \frac{h^2}{2}\int_0^T\int_\omega \psi^2 dxdt + h\int_0^T\int_\omega \widehat{\varphi}\psi dxdt - h\int_\Omega u_1\psi_0 dx$$

for all $h \in \mathbb{R}$ and $\psi_0 \in L^2(\Omega)$.

Dividing by $h > 0$ and by passing to the limit $h \to 0$ we obtain

$$0 \leq \varepsilon \parallel \psi_0 \parallel_{L^2(\Omega)} + \int_0^T\int_\omega \widehat{\varphi}\psi dxdt - \int_\Omega u_1\psi_0 dx. \tag{2.144}$$

The same calculations with $h < 0$ gives that

$$\left|\int_0^T\int_\omega \widehat{\varphi}\psi dxdt - \int_\Omega u_1\psi_0 dx\right| \leq \varepsilon \parallel \psi_0 \parallel \quad \forall \psi_0 \in L^2(\Omega). \tag{2.145}$$

On the other hand, if we take the control $f = \widehat{\varphi}$ in (2.137), by multiplying in (2.137) by ψ solution of (2.140) and by integrating by parts we get that

$$\int_0^T\int_\omega \widehat{\varphi}\psi dxdt = \int_\Omega u(T)\psi_0 dx. \tag{2.146}$$

From the last two relations it follows that

$$\Big| \int_\Omega (u(T) - u_1)\psi_0 dx \Big| \leq \varepsilon ||\psi_0||_{L^2(\Omega)}, \quad \forall \psi_0 \in L^2(\Omega) \tag{2.147}$$

which is equivalent to

$$||u(T) - u_1||_{L^2(\Omega)} \leq \varepsilon.$$

The proof of the Lemma is now complete.

■

Let us now show that J attains its minimum in $L^2(\Omega)$.

Lemma 2.5.2 *There exists $\widehat{\varphi}_T \in L^2(\Omega)$ such that*

$$J(\widehat{\varphi}_T) = \min_{\varphi_T \in L^2(\Omega)} J(\varphi_T). \tag{2.148}$$

Proof. It is easy to see that J is convex and continuous in $L^2(\Omega)$. By Theorem 2.2.3, the existence of a minimum is ensured if J is coercive, i. e.

$$J(\varphi_T) \to \infty \text{ when } ||\varphi_T||_{L^2(\Omega)} \to \infty. \tag{2.149}$$

In fact we shall prove that

$$\underline{\lim}_{||\varphi_T||_{L^2(\Omega)} \to \infty} J(\varphi_T)/||\varphi_T||_{L^2(\Omega)} \geq \varepsilon. \tag{2.150}$$

Evidently, (2.150) implies (2.149) and the proof of the Lemma is complete.

In order to prove (2.150) let $(\varphi_{T,j}) \subset L^2(\Omega)$ be a sequence of initial data for the adjoint system with $\| \varphi_{T,j} \|_{L^2(\Omega)} \to \infty$. We normalize them

$$\widetilde{\varphi}_{T,j} = \varphi_{T,j} / \| \varphi_{T,j} \|_{L^2(\Omega)},$$

so that $\| \widetilde{\varphi}_{T,j} \|_{L^2(\Omega)} = 1$.

On the other hand, let $\widetilde{\varphi}_j$ be the solution of (2.140) with initial data $\widetilde{\varphi}_{T,j}$. Then

$$J(\varphi_{T,j}) / \| \varphi_{T,j} \|_{L^2(\Omega)} = \frac{1}{2} \| \varphi_{T,j} \|_{L^2(\Omega)} \int_0^T \int_\omega | \widetilde{\varphi}_j |^2 \, dxdt + \varepsilon - \int_\Omega u_1 \widetilde{\varphi}_{T,j} dx.$$

The following two cases may occur:

1) $\underline{\lim}_{j \to \infty} \int_0^T \int_\omega | \widetilde{\varphi}_j |^2 > 0$. In this case we obtain immediately that

$$J(\varphi_{T,j}) / \| \varphi_{T,j} \|_{L^2(\Omega)} \to \infty.$$

2) $\underline{\lim}_{j\to\infty} \int_0^T \int_\omega \mid \widetilde{\varphi}_j \mid^2 = 0$. In this case since $\widetilde{\varphi}_{T,j}$ is bounded in $L^2(\Omega)$, by extracting a subsequence we can guarantee that $\widetilde{\varphi}_{T,j} \rightharpoonup \psi_0$ weakly in $L^2(\Omega)$ and $\widetilde{\varphi}_j \rightharpoonup \psi$ weakly in $L^2(0,T;H_0^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega))$, where ψ is the solution of (2.140) with initial data ψ_0 at $t = T$. Moreover, by lower semi-continuity,

$$\int_0^T \int_\omega \psi^2 dxdt \leq \underline{\lim}_{j\to\infty} \int_0^T \int_\omega \mid \widetilde{\varphi}_j \mid^2 dxdt = 0$$

and therefore $\psi = 0$ en $\omega \times (0,T)$.

Holmgren Uniqueness Theorem implies that $\psi \equiv 0$ in $\Omega \times (0,T)$ and consequently $\psi_0 = 0$.

Therefore, $\widetilde{\varphi}_{T,j} \rightharpoonup 0$ weakly in $L^2(\Omega)$ and consequently $\int_\Omega u_1 \widetilde{\varphi}_{T,j} dx$ tends to 0 as well.

Hence

$$\underline{\lim}_{j\to\infty} \frac{J(\varphi_{T,j})}{\| \varphi_{T,j} \|} \geq \underline{\lim}_{j\to\infty} [\varepsilon - \int_\Omega u_1 \widetilde{\varphi}_{T,j} dx] = \varepsilon,$$

and (2.150) follows.

■

Remark 2.5.4 Lemmas 2.5.1 and 2.5.2 give a second proof of Theorem 2.5.2. This approach does not only guarantee the existence of a control but also provides a method to obtain the control by minimizing a convex, continuous and coercive functional in $L^2(\Omega)$.

In the proof of the coercivity, the relevance of the term $\varepsilon||\varphi_T||_{L^2(\Omega)}$ is clear. Indeed, the coercivity of J depends heavily on this term. This is not only for technical reasons. The existence of a minimum of J with $\varepsilon = 0$ implies the existence of a control which makes $u(T) = u^1$. But this is not true unless u^1 is very regular in $\Omega \setminus \omega$. Therefore, for general $u^1 \in L^2(\Omega)$, the term $\varepsilon||\varphi_T||_{L^2(\Omega)}$ is needed.

Note that both proofs are based on the unique continuation property which guarantees that if φ is a solution of the adjoint system such that $\varphi = 0$ in $\omega \times (0,T)$, then $\varphi \equiv 0$. As we have seen, this property is a consequence of Holmgren Uniqueness Theorem.

■

The second proof, based on the minimization of J, with some changes on the definition of the functional as indicated in section 2.1.1, allows proving approximate controllability by means of other controls, for instance, of bang-bang form. We address these variants in the following sections.

2.5.6 Finite-approximate control

Let E be a subspace of $L^2(\Omega)$ of finite dimension and Π_E be the orthogonal projection over E. As a consequence of the approximate controllability property in Theorem 2.5.2 the following stronger result may be proved: *given u^0 and u^1 in $L^2(\Omega)$ and $\varepsilon > 0$ there exists a control f such that the solution of (2.137) satisfies simultaneously*

$$\Pi_E(u(T)) = \Pi_E(u_1), \quad \| u(T) - u^1 \|_{L^2(\Omega)} \leq \varepsilon. \tag{2.151}$$

This property not only says that the distance between $u(T)$ and the target u^1 is less that ε but also that the projection of $u(T)$ and u^1 over E coincide.

This property, introduced in [236], will be called **finite-approximate controllability**. It may be proved easily by taking into account the following property of Hilbert spaces: *If $L : E \to F$ is linear and continuous between the Hilbert spaces E and F and the range of L is dense in F, then, for any finite set $f_1, f_2, \dots, f_N \in F$, the set $\{Le : (Le, f_j)_F = 0 \quad \forall j = 1, 2, \dots, N\}$ is dense in the orthogonal complement of* $\mathrm{Span}\{f_1, f_2, \dots, f_N\}$.

Nevertheless, as we have said before, this result may also be proved directly, by considering a slightly modified form of the functional J used in the second proof of Theorem 2.5.2. We introduce

$$J_E(\varphi_T) = \frac{1}{2}\int_0^T \int_\omega \varphi^2 dxdt + \varepsilon \| (I - \Pi_E)\varphi_T \|_{L^2(\Omega)} - \int_\Omega u_1 \varphi_T dx.$$

The functional J_E is again convex and continuous in $L^2(\Omega)$. Moreover, it is coercive. The proof of the coercivity of J_E is similar to that of J. It is sufficient to note that if $\widehat{\varphi}_{T,j}$ tends weakly to zero in $L^2(\Omega)$, then $\Pi_E(\widehat{\varphi}_{T,j})$ converges (strongly) to zero in $L^2(\Omega)$.

Therefore $\| (I - \Pi_E)\widehat{\varphi}_{T,j} \|_{L^2(\Omega)} / \| \widehat{\varphi}_{T,j} \|_{L^2(\Omega)}$ tends to 1. According to this, the new functional J_E satisfies the coercivity property (2.150).

It is also easy to see that the minimum of J_E gives the finite-approximate control we were looking for.

2.5.7 Bang-bang control

In the study of finite dimensional systems we have seen that one may find "bang-bang" controls which take only two values $\pm\lambda$ for some $\lambda > 0$.

In the case of the heat equation it is also easy to construct controls of this type. In fact a convenient change in the functional J will ensure the existence of "bang-bang" controls. We consider:

$$J_{bb}(\varphi_T) = \frac{1}{2}\left(\int_0^T \int_\omega | \varphi | \, dxdt\right)^2 + \varepsilon \| \varphi_T \|_{L^2(\Omega)} - \int_\Omega u_1 \varphi_T dx.$$

Remark that the only change in the definition of J_{bb} is in the first term in which the norm of φ in $L^2(\omega \times (0,T))$ has been replaced by its norm in $L^1((0,T) \times \omega)$.

Once again we are dealing with a convex and continuous functional in $L^2(\Omega)$. The proof of the coercivity of J_{bb} is the same as in the case of the functional J. We obtain that:

$$\liminf_{\|\varphi_T\|_{L^2(\Omega)} \to \infty} \frac{J_{bb}(\varphi_T)}{\| \varphi_T \|_{L^2(\Omega)}} \geq \varepsilon.$$

Hence, J_{bb} attains a minimum in some $\widehat{\varphi}_T$ of $L^2(\Omega)$. It is easy to see that, if $\widehat{\varphi}$ is the corresponding solution of the adjoint system with $\widehat{\varphi}_T$ as initial data, then there exists $f \in \int_\omega \int_0^T |\widehat{\varphi}| dx \, \mathrm{sgn}(\widehat{\varphi})$ such that the solution of (2.137) with this control satisfies $||u(T) - u_1|| \leq \varepsilon$.

On the other hand, since $\widehat{\varphi}$ is a solution of the adjoint heat equation, it is real analytic in $\Omega \times (0,T)$. Hence, the set $\{t : \widehat{\varphi} = 0\}$ is of zero measure in $\Omega \times (0,T)$. Hence, we may consider

$$f = \int_\omega \int_0^T |\widehat{\varphi}| dxdt \, \mathrm{sgn}(\widehat{\varphi}) \tag{2.152}$$

which represents a bang-bang control. Remark that the sign of the control changes when the sign of $\widehat{\varphi}$ changes. Consequently, the geometry of the sets where the control has a given sign can be quite complex.

Note also that the amplitude of the bang-bang control is $\int_\omega \int_0^T |\widehat{\varphi}| dxdt$ which, evidently depends of the distance from the final target u^1 to the uncontrolled final state $S(T)u^0$ and of the control time T.

Remark 2.5.5 As it was shown in [73], the bang-bang control obtained by minimizing the functional J_{bb} is the one of minimal norm in $L^\infty((0,T) \times \omega)$ among all the admissible ones. The control obtained by minimizing the functional J has the minimal norm in $L^2((0,T) \times \omega)$.

■

Remark 2.5.6 The problem of finding bang-bang controls guaranteeing the finite-approximate property may also be considered. It is sufficient to take the following combination of the functionals J_E and J_{bb}:

$$J_{bb,E}(\varphi_T) = \frac{1}{2} \left(\int_0^T \int_\omega | \varphi | \, dxdt \right)^2 + \varepsilon \parallel (I - \Pi_E)\varphi_T \parallel_{L^2(\Omega)} - \int_\Omega u_1 \varphi_T dx.$$

■

2.5.8 Comments

The null controllability problem for system (2.137) is equivalent to the following observability inequality for the adjoint system (2.140):

$$\| \varphi(0) \|_{L^2(\Omega)}^2 \leq C \int_0^T \int_\omega \varphi^2 dxdt, \quad \forall \varphi^0 \in L^2(\Omega). \tag{2.153}$$

Once (2.153) is known to hold one can obtain the control with minimal L^2-norm among the admissible ones. To do that it is sufficient to minimize the functional

$$J(\varphi^0) = \frac{1}{2} \int_0^T \int_\omega \varphi^2 dxdt + \int_\Omega \varphi(0) u^0 dx \tag{2.154}$$

over the Hilbert space

$$H = \{\varphi^0 : \text{ the solution } \varphi \text{ of (2.140) satisfies } \int_0^T \int_\omega \varphi^2 dxdt < \infty\}.$$

To be more precise, H is the completion of $L^2(\Omega)$ with respect to the norm $[\int_0^T \int_\omega \varphi^2 dxdt]^{1/2}$. In fact, H is much larger than $L^2(\Omega)$. We refer to [83] for precise estimates on the nature of this space.

Observe that J is convex and continuous in H. On the other hand (2.153) guarantees the coercivity of J and the existence of its minimizer.

Due to the irreversibility of the system, (2.153) is not easy to prove. For instance, multiplier methods do not apply. Let us mention two different approaches used for the proof of (2.153).

1. **Results based on the observation of the wave or elliptic equations:** In [195] it was shown that if the wave equation is exactly controllable for some $T > 0$ with controls supported in ω, then the heat equation (2.137) is null controllable for all $T > 0$ with controls supported in ω. As a consequence of this result and in view of the controllability results for the wave equation, it follows that the heat equation (2.137) is null controllable for all $T > 0$ provided ω satisfies the geometric control condition. However, the geometric control condition does not seem to be natural at all in the context of the heat equation.

   Later on, Lebeau and Robbiano [134] proved that the heat equation (2.137) is null controllable for every open, non-empty subset ω of Ω and $T > 0$. This result shows, as expected, that the geometric control condition is unnecessary in the context of the heat equation. A simplified proof of it was given in [135] where the linear system of thermoelasticity was addressed. The main ingredient in the proof is the following observability estimate for the eigenfunctions $\{\psi_j\}$ of the Laplace operator

$$\int_\omega \left| \sum_{\lambda_j \leq \mu} a_j \psi_j(x) \right|^2 dx \geq C_1 e^{-C_2\sqrt{\mu}} \sum_{\lambda_j \leq \mu} | a_j |^2 \tag{2.155}$$

which holds for any $\{a_j\} \in \ell^2$ and for all $\mu > 0$ and where $C_1, C_2 > 0$ are two positive constants.

This result was implicitly used in [134] and it was proved in [135] by means of Carleman's inequalities for elliptic equations.

2. **Carleman inequalities for parabolic equations:** The null controllability of the heat equation with variable coefficients and lower order time-dependent terms has been studied by Fursikov and Imanuvilov (see for instance [44], [87], [88], [89], [110] and [111]). Their approach is based on the use of the Carleman inequalities for parabolic equations and is different to the one we have presented above. In [90], Carleman estimates are systematically applied to solve observability problem for linearized parabolic equations.

In [78] the boundary null controllability of the heat equation was proved in one space dimension using moment problems and classical results on the linear independence in $L^2(0,T)$ of families of real exponentials. We shall describe this method in the next section.

2.6 Boundary controllability of the $1D$ heat equation

In this section the boundary null-controllability problem of the heat equation is studied. We do it by reducing the control problem to an equivalent problem of moments. The latter is solved with the aid of a biorthogonal sequence to a family of real exponential functions. This technique was used in the study of several control problems (the heat equation being one of the most relevant examples of application) in the late 60's and early 70's by R. D. Russell and H. O. Fattorini (see, for instance, [78] and [79]).

2.6.1 Introduction

Given $T>0$ arbitrary, $u^0 \in L^2(0,1)$ and $f \in L^2(0,T)$ we consider the following non-homogeneous 1-D problem:

$$\begin{cases} u_t - u_{xx} = 0 & x \in (0,1),\ t \in (0,T) \\ u(t,0) = 0, \quad u(t,1) = f(t) & t \in (0,T) \\ u(0,x) = u^0(x) & x \in (0,1). \end{cases} \tag{2.156}$$

In (2.156) $u = u(x,t)$ is the state and $f = f(t)$ is the control function which acts on the extreme $x = 1$. We aim at changing the dynamics of the system by acting on the boundary of the domain $(0,1)$.

2.6.2 Existence and uniqueness of solutions

The following theorem is a consequence of classical results of existence and uniqueness of solutions of nonhomogeneous evolution equations. All the details may be found, for instance in [156].

Theorem 2.6.1 *For any $f \in L^2(0,T)$ and $u^0 \in L^2(\Omega)$ equation* (2.156) *has a unique weak solution $u \in C([0,T], H^{-1}(\Omega))$.*

Moreover, the map $\{u^0, f\} \to \{u\}$ is linear and there exists $C = C(T) > 0$ such that

$$||u||_{L^\infty(0,T;H^{-1}(\Omega))} \le C\left(||u^0||_{L^2(\Omega)} + ||f||_{L^2(0,T)}\right). \tag{2.157}$$

2.6.3 Controllability and the problem of moments

In this section we introduce several notions of controllability.

Let $T > 0$ and define, for any initial data $u^0 \in L^2(\Omega)$, the set of reachable states

$$R(T; u^0) = \{u(T) : u \text{ solution of (2.156) with } f \in L^2(0,T)\}. \tag{2.158}$$

An element of $R(T, u^0)$ is a state of (2.156) reachable in time T by starting from u^0 with the aid of a control f.

As in the previous section, several notions of controllability may be defined.

Definition 2.6.1 *System* (2.156) *is* **approximately controllable in time** *T if, for every initial data $u^0 \in L^2(\Omega)$, the set of reachable states $R(T; u^0)$ is dense in $L^2(\Omega)$.*

Definition 2.6.2 *System* (2.156) *is* **exactly controllable in time** *T if, for every initial data $u^0 \in L^2(\Omega)$, the set of reachable states $R(T; u^0)$ coincides with $L^2(\Omega)$.*

Definition 2.6.3 *System* (2.156) *is* **null controllable in time** T *if, for every initial data* $u^0 \in L^2(\Omega)$*, the set of reachable states* $R(T; u^0)$ *contains the element* 0.

Remark 2.6.1 Note that the regularity of solutions stated above does not guarantee that $u(T)$ belongs to $L^2(\Omega)$. In view of this it could seem that the definitions above do not make sense. Note however that, due to the regularizing effect of the heat equation, if the control f vanishes in an arbitrarily small neighborhood of $t = T$ then $u(T)$ is in C^∞ and in particular in $L^2(\Omega)$. According to this, the above definitions make sense by introducing this minor restrictions on the controls under consideration.

■

Remark 2.6.2 Let us make the following remarks, which are very close to those we did in the context of interior control:

- The linearity of the system under consideration implies that $R(T, u^0) = R(T, 0) + S(T)u^0$ and, consequently, without loss of generality one may assume that $u^0 = 0$.

- Due to the regularizing effect the solutions of (2.156) are in C^∞ far away from the boundary at time $t = T$. Hence, the elements of $R(T, u^0)$ are C^∞ functions in $[0, 1)$. Then, exact controllability may not hold.

- It is easy to see that if null controllability holds, then any initial data may be led to any final state of the form $S(T)v^0$ with $v^0 \in L^2(\Omega)$.

  Indeed, let $u^0, v^0 \in L^2(\Omega)$ and remark that $R(T; u^0 - v^0) = R(T; u^0) - S(T)v^0$. Since $0 \in R(T; u^0 - v^0)$, it follows that $S(T)v^0 \in R(T; u^0)$.

- Null controllability implies approximate controllability. Indeed we have that $S(T)[L^2(\Omega)] \subset R(T; u^0)$ and $S(T)[L^2(\Omega)]$ is dense in $L^2(\Omega)$.

- Note that $u^1 \in R(T, u^0)$ if and only if there exists a sequence $(f_\varepsilon)_{\varepsilon>0}$ of controls such that $||u(T) - u^1||_{L^2(\Omega)} \leq \varepsilon$ and $(f_\varepsilon)_{\varepsilon>0}$ is bounded in $L^2(0, T)$. Indeed, in this case, any weak limit in $L^2(0, T)$ of the sequence $(f_\varepsilon)_{\varepsilon>0}$ gives an exact control which makes that $u(T) = u_1$.

■

Remark 2.6.3 As we shall see, null controllability of the heat equation holds in an arbitrarily small time. This is due to the infinity speed of propagation. It is important to underline, however, that, despite of the infinite speed of propagation, the null controllability of the heat equation does not hold in an infinite domain. We refer to [164] for a further discussion of this issue.

The techniques we shall develop in this section do not apply in unbounded domains. Although, as shown in [164], using the similarity variables, one can find a spectral decomposition of solutions of the heat equation on the whole or half line, the spectrum is too dense and biorthogonal families do not exist.

■

In this section the null-controllability problem will be considered. Let us first give the following characterization of the null-controllability property of (2.156).

Lemma 2.6.1 *Equation* (2.156) *is null-controllable in time* $T>0$ *if and only if, for any* $u^0 \in L^2(0,1)$ *there exists* $f \in L^2(0,T)$ *such that the following relation holds*

$$\int_0^T f(t)\varphi_x(t,1)dt = \int_0^1 u^0(x)\varphi(0,x)dx, \tag{2.159}$$

for any $\varphi_T \in L^2(0,1)$, *where* $\varphi(t,x)$ *is the solution of the backward adjoint problem*

$$\begin{cases} \varphi_t + \varphi_{xx} = 0 & x \in (0,1),\ t \in (0,T) \\ \varphi(t,0) = \varphi(t,1) = 0 & t \in (0,T) \\ \varphi(T,x) = \varphi_T(x) & x \in (0,1). \end{cases} \tag{2.160}$$

Proof. Let $f \in L^2(0,T)$ be arbitrary and u the solution of (2.156). If $\varphi_T \in L^2(0,1)$ and φ is the solution of (2.160) then, by multiplying (2.156) by φ and by integrating by parts we obtain that

$$0 = \int_0^T \int_0^1 (u_t - u_{xx})\varphi dxdt = \int_0^1 u\varphi dx\Big|_0^T + \int_0^T (-u_x\varphi + u\varphi_x)dt\Big|_0^1 +$$

$$+ \int_0^T \int_0^1 u(-\varphi_t - \varphi_{xx})dxdt = \int_0^1 u\varphi dx\Big|_0^T + \int_0^T f(t)\varphi_x(t,1)dt.$$

Consequently

$$\int_0^T f(t)\varphi_x(t,1)dt = \int_0^1 u^0(x)\varphi(0,x)dx - \int_0^1 u(T,x)\varphi_T(x)dx. \tag{2.161}$$

Now, if (2.159) is verified, it follows that $\int_0^1 u(T,x)\varphi_T(x)dx = 0$, for all $\varphi^1 \in L^2(0,1)$ and $u(T)=0$.

Hence, the solution is controllable to zero and f is a control for (2.156).

Reciprocally, if f is a control for (2.156), we have that $u(T)=0$. From (2.161) it follows that (2.159) holds and the proof finishes.

■

From the previous Lemma we deduce the following result:

Proposition 2.6.1 *Equation* (2.156) *is null-controllable in time* $T > 0$ *if and only if for any* $u^0 \in L^2(0,1)$, *with Fourier expansion*

$$u^0(x) = \sum_{n \geq 1} a_n \sin(\pi n x),$$

there exists a function $w \in L^2(0,T)$ *such that,*

$$\int_0^T w(t) e^{-n^2\pi^2 t} dt = (-1)^n \frac{a_n}{2n\pi} e^{-n^2\pi^2 T}, \quad n = 1, 2, ... \tag{2.162}$$

Remark 2.6.4 Problem (2.162) is usually refered to as **problem of moments.**

Proof. From the previous Lemma we know that $f \in L^2(0,T)$ is a control for (2.156) if and only if it satisfies (2.159). But, since $(\sin(n\pi x))_{n\geq 1}$ forms an orthogonal basis in $L^2(0,1)$, (2.159) is verified if and only if it is verified by $\varphi_n^1 = \sin(n\pi x)$, $n = 1, 2, ...$

If $\varphi_n^1 = \sin(n\pi x)$ then the corresponding solution of (2.160) is $\varphi(t,x) = e^{-n^2\pi^2(T-t)} \sin(n\pi x)$ and from (2.159) we obtain that

$$\int_0^T f(t)(-1)^n n\pi e^{-n^2\pi^2(T-t)} = \frac{a_n}{2} e^{-n^2\pi^2 T}.$$

The proof ends by taking $w(t) = f(T-t)$.

∎

The control property has been reduced to the problem of moments (2.162). The latter will be solved by using biorthogonal techniques. The main ideas are due to R.D. Russell and H.O. Fattorini (see, for instance, [78] and [79]).

The eigenvalues of the heat equation are $\lambda_n = n^2\pi^2$, $n \geq 1$. Let $\Lambda = \left(e^{-\lambda_n t}\right)_{n\geq 1}$ be the family of the corresponding real exponential functions.

Definition 2.6.4 $(\theta_m)_{m\geq 1}$ *is a* **biorthogonal sequence** *to* Λ *in* $L^2(0,T)$ *if and only if*

$$\int_0^T e^{-\lambda_n t} \theta_m(t) dt = \delta_{nm}, \quad \forall n, m = 1, 2, ...$$

If there exists a biorthogonal sequence $(\theta_m)_{m\geq 1}$, the problem of moments (2.162) may be solved immediately by setting

$$w(t) = \sum_{m\geq 1} (-1)^m \frac{a_m}{2m\pi} e^{-m^2\pi^2 T} \theta_m(t). \tag{2.163}$$

As soon as the series converges in $L^2(0,T)$, this provides the solution to (2.162).

We have the following controllability result:

Theorem 2.6.2 *Given $T > 0$, suppose that there exists a biorthogonal sequence $(\theta_m)_{m\geq 1}$ to Λ in $L^2(0,T)$ such that*

$$||\theta_m||_{L^2(0,T)} \leq M e^{\omega m}, \quad \forall m \geq 1 \tag{2.164}$$

where M and ω are two positive constants.

Then (2.156) *is null-controllable in time T.*

Proof. From Proposition 2.6.1 it follows that it is sufficient to show that for any $u^0 \in L^2(0,1)$ with Fourier expansion

$$u^0 = \sum_{n\geq 1} a_n \sin(n\pi x),$$

there exists a function $w \in L^2(0,T)$ which verifies (2.162).

Consider

$$w(t) = \sum_{m\geq 1} (-1)^m \frac{a_m}{2m\pi} e^{-m^2\pi^2 T} \theta_m(t). \tag{2.165}$$

Note that the series which defines w is convergent in $L^2(0,T)$. Indeed,

$$\sum_{m\geq 1} \left|\left|(-1)^m \frac{a_m}{2m\pi} e^{-m^2\pi^2 T}\theta_m\right|\right|_{L^2(0,T)} = \sum_{m\geq 1} \frac{|a_m|}{2m\pi} e^{-m^2\pi^2 T} ||\theta_m||_{L^2(0,T)} \leq$$

$$\leq M \sum_{m\geq 1} \frac{|a_m|}{2m\pi} e^{-m^2\pi^2 T + \omega m} < \infty$$

where we have used the estimates (2.164) of the norm of the biorthogonal sequence (θ_m).

On the other hand, (2.165) implies that w satisfies (2.162) and the proof finishes.

■

Theorem 2.6.2 shows that, the null-controllability problem (2.156) is solved if we prove the existence of a biorthogonal sequence $(\theta_m)_{m\geq 1}$ to Λ in $L^2(0,T)$ which verifies (2.164). The following sections are devoted to accomplish this task.

2.6.4 Existence of a biorthogonal sequence

The existence of a biorthogonal sequence to the family Λ is a consequence of the following Theorem (see, for instance, [203]).

Theorem 2.6.3 *(Münz) Let $0 < \mu_1 \leq \mu_2 \leq ... \leq \mu_n \leq ...$ be a sequence of real numbers. The family of exponential functions $(e^{-\mu_n t})_{n\geq 1}$ is complete in $L^2(0,T)$ if and only if*

$$\sum_{n\geq 1} \frac{1}{\mu_n} = \infty. \tag{2.166}$$

Given any $T > 0$, from Münz's Theorem we obtain that the space generated by the family Λ is a proper space of $L^2(0,T)$ since

$$\sum_{n\geq 1} \frac{1}{\lambda_n} = \sum_{n\geq 1} \frac{1}{n^2\pi^2} < \infty.$$

Let $E(\Lambda, T)$ be the space generated by Λ in $L^2(0,T)$ and $E(m, \Lambda, T)$ be the subspace generated by $\left(e^{-\lambda_n t}\right)_{\substack{n\geq 1\\ n\neq m}}$ in $L^2(0,T)$.

We also introduce the notation $p_n(t) = e^{-\lambda_n t}$.

Theorem 2.6.4 *Given any $T > 0$, there exists a unique sequence $(\theta_m(T,\,\cdot\,))_{m\geq 1}$, biorthogonal to the family Λ, such that*

$$(\theta_m(T,\,\cdot\,))_{m\geq 1} \subset E(\Lambda, T).$$

Moreover, this biorthogonal sequence has minimal $L^2(0,T)$-norm.

Proof. Since Λ is not complete in $L^2(0,T)$, it is also minimal. Thus, $p_m \notin E(m, \Lambda, T)$, for each $m \in I$.

Let r_m be the orthogonal projection p_m over the space $E(m, \Lambda, T)$ and define

$$\theta_m(T,\,\cdot\,) = \frac{p_m - r_m}{||p_m - r_m||^2_{L^2(0,T)}}. \tag{2.167}$$

From the projection properties, it follows that

1. $r_m \in E(m, \Lambda, T)$ verifies $||p_m(t) - r_m(t)||_{L^2(0,T)} = \min_{r\in E(m,\Lambda,T)} ||p_m - r||_{L^2(0,T)}$
2. $(p_m - r_m) \perp E(m, \Lambda, T)$
3. $(p_m - r_m) \perp p_n \in E(m, \Lambda, T)$, $\forall n \neq m$
4. $(p_m - r_m) \perp r_m \in E(m, \Lambda, T)$.

From the previous properties and (2.167) we deduce that

1. $\int_0^T \theta_m(T,t)p_n(t)dt = \delta_{m,n}$
2. $\theta_m(T,\cdot) = \frac{p_m - r_m}{||p_m - r_m||^2} \in E(\Lambda, T)$.

Thus, (2.167) gives a biorthogonal sequence $(\theta_m(T,\cdot))_{m\geq 1} \subset E(\Lambda,T)$ to the family Λ.

The uniqueness of the biorthogonal sequence is obtained immediately. Indeed, if $(\theta'_m)_{m\geq 1} \subset E(\Lambda,T)$ is another biorthogonal sequence to the family Λ, then

$$\left.\begin{array}{l} (\theta_m - \theta'_m) \in E(\Lambda,T) \\ p_n \perp (\theta_m - \theta'_m), \ \forall n \geq 1 \end{array}\right\} \Rightarrow \theta_m - \theta'_m = 0$$

where we have taken into account that $(p_m)_{m\geq 1}$ is complete in $E(\Lambda,T)$.

To prove the minimality of the norm of $(\theta_m(T,\cdot))_{m\geq 1}$, let us consider any other biorthogonal sequence $(\theta'_m)_{m\geq 1} \subset L^2(0,T)$.

$E(\Lambda,T)$ being closed in $L^2(0,T)$, its orthogonal complement, $E(\Lambda,T)^\perp$, is well defined. Thus, for any $m \geq 1$, there exists a unique $q_m \in E(\Lambda,T)^\perp$ such that $\theta'_m = \theta_m + q_m$.

Finally,

$$||\theta'_m||^2 = ||\theta_m + q_m||^2 = ||\theta_m||^2 + ||q_m||^2 \geq ||\theta_m||^2$$

and the proof ends.

■

Remark 2.6.5 The previous Theorem gives a biorthogonal sequence of minimal norm. This property is important since the convergence of the series of (2.163) depends directly of these norms.

■

The existence of a biorthogonal sequence $(\theta_m)_{m\geq 1}$ to the family Λ being proved, the next step is to evaluate its $L^2(0,T)$-norm. This will be done in two steps. First for the case $T=\infty$ and next for $T<\infty$.

2.6.5 Estimate of the norm of the biorthogonal sequence: $T = \infty$

Theorem 2.6.5 *There exist two positive constants M and ω such that the biorthogonal of minimal norm $(\theta_m(\infty,\cdot)_{m\geq 1}$ given by Theorem 2.6.4 satisfies the following estimate*

$$||\theta_m(\infty,\cdot)||_{L^2(0,\infty)} \leq M\pi e^{\omega m}, \quad \forall m \geq 1. \tag{2.168}$$

Proof. Let us introduce the following notations: $E^n := E^n(\Lambda, \infty)$ is the subspace generated by $\Lambda^n := \left(e^{-\lambda_k t}\right)_{1\leq k\leq n}$ in $L^2(0,T)$ and $E^n_m := E^2(m, \Lambda, \infty)$ is the subspace generated by $\left(e^{-\lambda_k t}\right)_{\substack{1\leq k\leq n\\ k\neq m}}$ in $L^2(0,T)$.

Remark that E^n and E^n_m are finite dimensional spaces and

$$E(\Lambda,\infty) = \cup_{n\geq 1} E^n, \quad E(m,\Lambda,\infty) = \cup_{n\geq 1} E^n_m.$$

We have that, for each $n \geq 1$, there exists a unique biorthogonal family $(\theta^n_m)_{1\leq m\leq n} \subset E^n$, to the family of exponentials $\left(e^{-\lambda_k t}\right)_{1\leq k\leq n}$. More precisely,

$$\theta^n_m = \frac{p_m - r^n_m}{||p_m - r^n_m||^2_{L^2(0,\infty)}}, \tag{2.169}$$

where r^n_m is the orthogonal projection of p_m over E^n_m.

If

$$\theta^n_m = \sum_{k=1}^n c^m_k p_k \tag{2.170}$$

then, by multiplying (2.170) by p_l and by integrating in $(0,\infty)$, it follows that

$$\delta_{m,l} = \sum_{k\geq 1} c^m_k \int_0^T p_l(t) p_k(t) dt, \quad 1 \leq m, l \leq n. \tag{2.171}$$

Moreover, by multiplying in (2.170) by θ^n_m and by integrating in $(0,\infty)$, we obtain that

$$||\theta^n_m||^2_{L^2(0,\infty)} = c^m_m. \tag{2.172}$$

If G denotes the Gramm matrix of the family Λ, i. e. the matrix of elements

$$g^l_k = \int_0^\infty p_k(t) p_l(t) dt, \quad 1 \leq k, l \leq n$$

we deduce from (2.171) that c^m_k are the elements of the inverse of G. Cramer's rule implies that

$$c^m_m = \frac{|G_m|}{|G|} \tag{2.173}$$

where $|G|$ is the determinant of matrix G and $|G_m|$ is the determinant of the matrix G_m obtained by changing the $m-$th column of G by the $m-$th vector of the canonical basis.

It follows that

$$||\theta^n_m||_{L^2(0,\infty)} = \sqrt{\frac{|G_m|}{|G|}}. \tag{2.174}$$

The elements of G may be computed explicitly

$$g^n_k = \int_0^\infty p_k(t) p_n(t) = \int_0^\infty e^{-(n^2+k^2)\pi^2 t} dt = \frac{1}{n^2\pi^2 + k^2\pi^2}.$$

Remark 2.6.6 A formula, similar to (2.174), may be obtained for any $T > 0$. Nevertheless, the determinants may be estimated only in the case $T = \infty$.

■

To compute the determinats $|G|$ and $|G_m|$ we use the following lemma (see [47]):

Lemma 2.6.2 *If $C = (c_{ij})_{1 \le i,j \le n}$ is a matrix of coefficients $c_{ij} = 1/(a_i + b_j)$ then*

$$|C| = \frac{\prod_{1 \le i < j \le n}(a_i - a_j)(b_i - b_j)}{\prod_{1 \le i,j \le n}(a_i + b_j)}. \tag{2.175}$$

It follows that

$$|G| = \frac{\prod_{1 \le i < j \le n}(i^2\pi^2 - j^2\pi^2)^2}{\prod_{1 \le i,j \le n}(i^2\pi^2 + j^2\pi^2)}, \quad |G_m| = \frac{\prod'_{1 \le i < j \le n}(i^2\pi^2 - j^2\pi^2)^2}{\prod'_{1 \le i,j \le n}(i^2\pi^2 + j^2\pi^2)}$$

where $'$ means that the index m has been skipped in the product.

Hence,

$$\frac{|G_m|}{|G|} = 2m^2\pi^2 \prod_{k=1}^{n}{}' \frac{(m^2 + k^2)^2}{(m^2 - k^2)^2}. \tag{2.176}$$

From (2.174) and (2.176) we deduce that

$$||\theta_m^n||_{L^2(0,\infty)} = \sqrt{2}m\pi \prod_{k=1}^{n}{}' \frac{m^2 + k^2}{|m^2 - k^2|}. \tag{2.177}$$

Lemma 2.6.3 *The norm of the biorthogonal sequence $(\theta_m(\infty, \cdot))_{m \ge 1}$ to the family Λ in $L^2(0, \infty)$ given by Theorem 2.6.4, verifies*

$$||\theta_m(\infty, \cdot)||_{L^2(0,\infty)} = \sqrt{2}m\pi \prod_{k=1}^{\infty}{}' \frac{m^2 + k^2}{|m^2 - k^2|}. \tag{2.178}$$

Proof. It consists in passing to the limit in (2.177) as $n \to \infty$. Remark first that, for each $m \ge 1$, the product

$$\prod_{k=1}^{\infty}{}' \frac{m^2 + k^2}{|m^2 - k^2|}$$

is convergent since

$$1 \le \prod_{k=1}^{\infty}{}' \frac{m^2 + k^2}{|m^2 - k^2|} = \exp\left(\sum_{k=1}^{\infty}{}' \ln\left(\frac{m^2 + k^2}{|m^2 - k^2|} \right) \right) \le$$

$$\leq \exp\left(\sum_{k=1}^{\infty}{}' \ln\left(1+\frac{2m^2}{|m^2-k^2|}\right)\right) \leq \exp\left(2m^2\sum_{k=1}^{\infty}{}' \frac{1}{|m^2-k^2|}\right) < \infty.$$

Consequently, the limit $\lim_{n\to\infty} ||\theta_m^n||_{L^2(0,\infty)} = L \geq 1$ exists. The proof ends if we prove that

$$\lim_{n\to\infty} ||\theta_m^n||_{L^2(0,\infty)} = ||\theta_m||_{L^2(0,\infty)}. \tag{2.179}$$

Identity (2.169) implies that $\lim_{n\to\infty} ||p_m - r_m^n||_{L^2(0,\infty)} = 1/L$ and (2.179) is equivalent to

$$\lim_{n\to\infty} ||p_m - r_m^n||_{L^2(0,\infty)} = ||p_m - r_m||_{L^2(0,\infty)}. \tag{2.180}$$

Let now $\varepsilon > 0$ be arbitrary. Since $r_m \in E(m,\Lambda,\infty)$ it follows that there exist $n(\varepsilon) \in \mathbb{N}^*$ and $r_m^\varepsilon \in E_m^{n(\varepsilon)}$ with

$$||r_m - r_m^\varepsilon||_{L^2(0,\infty)} < \varepsilon.$$

For any $n \geq n(\varepsilon)$ we have that

$$||p_m - r_m|| = \min_{r\in E(m,\Lambda,\infty)} ||p_m - r|| \leq ||p_m - r_m^n|| = \min_{r\in E_m^n} ||p_m - r|| \leq$$

$$\leq ||p_m - r_m^\varepsilon|| \leq ||p_m - r_m|| + ||r_m - r_m^\varepsilon|| < ||p_m - r_m|| + \varepsilon.$$

Thus, (2.180) holds and Lemma 2.6.3 is proved.

■

Finally, to evaluate $\theta_m(\infty,\cdot)$ we use the following estimate

Lemma 2.6.4 *There exist two positive constants M and ω such that for any $m \geq 1$,*

$$\prod_{k=1}^{\infty}{}' \frac{m^2+k^2}{|m^2-k^2|} \leq Me^{\omega m}. \tag{2.181}$$

Proof. Remark that

$$\prod_k{}' \frac{m^2+k^2}{|m^2-k^2|} = \exp\left[\sum_k{}' \ln\left(\frac{m^2+k^2}{|m^2-k^2|}\right)\right] \leq \exp\left[\sum_k{}' \ln\left(1+\frac{2m^2}{|m^2-k^2|}\right)\right].$$

Now

$$\sum_k{}' \ln\left(1+\frac{2m^2}{|m^2-k^2|}\right) \leq \int_1^m \ln\left(1+\frac{2m^2}{m^2-x^2}\right)dx+$$

$$+\int_m^{2m} \ln\left(1+\frac{2m^2}{x^2-m^2}\right)dx+\int_{2m}^{\infty} \ln\left(1+\frac{2m^2}{x^2-m^2}\right)dx=$$

$$=m\left[\int_0^1 \ln\left(1+\frac{2}{1-x^2}\right)dx+\int_1^2 \ln\left(1+\frac{2}{x^2-1}\right)dx+\right.$$

$$\left.+\int_2^{\infty} \ln\left(1+\frac{2}{x^2-1}\right)dx\right]=m\left(I_1+I_2+I_3\right).$$

We evaluate now each one of these integrals.

$$I_1=\int_0^1 \ln\left(1+\frac{2}{1-x^2}\right)dx=\int_0^1 \ln\left(1+\frac{2}{(1-x)(1+x)}\right)dx\leq$$

$$\int_0^1 \ln\left(1+\frac{2}{1-x}\right)dx=-\int_0^1 (1-x)'\ln\left(1+\frac{2}{1-x}\right)dx=$$

$$=-\left.(1-x)\ln\left(1+\frac{2}{1-x}\right)\right|_0^1+\int_0^1 \frac{2}{3-x}dx=c_1<\infty,$$

$$I_2=\int_1^2 \ln\left(1+\frac{2}{x^2-1}\right)dx\leq\int_1^2 \ln\left(1+\frac{2}{(x-1)^2}\right)dx=$$

$$=\int_1^2 (x-1)'\ln\left(1+\frac{2}{(x-1)^2}\right)dx=$$

$$=-\left.(x-1)\ln\left(1+\frac{2}{(x-1)^2}\right)\right|_0^1+\int_1^2 \frac{2}{2+(x-1)^2}dx=c_2<\infty.$$

$$I_3=\int_2^{\infty} \ln\left(1+\frac{2}{x^2-1}\right)dx\leq\int_2^{\infty} \ln\left(1+\frac{2}{(x-1)^2}\right)dx\leq$$

$$\leq\int_2^{\infty} \frac{2}{(x-1)^2}dx=c_3<\infty.$$

The proof finishes by taking $\omega=c_1+c_2+c_3$.

$\blacksquare$

The proof of Theorem 2.6.5 ends by taking into account relation (2.178) and Lemma 2.6.4.

$\blacksquare$

2.6.6 Estimate of the norm of the biorthogonal sequence: $T < \infty$

We consider now $T < \infty$. To evaluate the norm of the biorthogonal sequence $(\theta_m(T, \cdot))_{m\geq 1}$ in $L^2(0,T)$ the following result is necessary. The first version of this result may be found in [203] (see also [78] and [104]).

Theorem 2.6.6 *Let Λ be the family of exponential functions $\left(e^{-\lambda_n t}\right)_{n\geq 1}$ and let T be arbitrary in $(0,\infty)$. The restriction operator*

$$R_T : E(\Lambda, \infty) \to E(\Lambda, T), \quad R_T(v) = v_{|_{[0,T]}}$$

is invertible and there exists a constant $C > 0$, which only depends on T, such that

$$||R_T^{-1}|| \leq C. \tag{2.182}$$

Proof. Suppose that, on the contrary, for some $T > 0$ there exists a sequence of exponential polynomials

$$P_k(t) = \sum_{n=1}^{N(k)} a_{kn} e^{-\lambda_n t} \subset E(\Lambda, T)$$

such that

$$\lim_{k\to\infty} ||P_k||_{L^2(0,T)} = 0 \tag{2.183}$$

and

$$||P_k||_{L^2(0,\infty)} = 1, \quad \forall k \geq 1. \tag{2.184}$$

By using the estimates from Theorem 2.6.5 we obtain that

$$|a_{mn}| = \left| \int_0^\infty P_k(t)\theta_m(\infty, t)dt \right| \leq ||P_k||_{L^2(0,\infty)} ||\theta_m(\infty, \cdot)||_{L^2(0,\infty)} \leq M e^{\omega m}.$$

Thus

$$|P_k(z)| \leq \sum_{n=1}^{N(k)} |a_{kn}| \left|e^{-\lambda_n z}\right| \leq M \sum_{n=1}^{\infty} e^{\omega n - n^2\pi^2 \mathcal{R}e(z)}. \tag{2.185}$$

If $r > 0$ is given let $\Delta_r = \{z \in \mathbb{C} : \mathcal{R}e(z) > r\}$. For all $z \in \Delta_r$, we have that

$$|P_k(z)| \leq M \sum_{n=1}^{\infty} e^{\omega n - n^2\pi^2 r} \leq M(\omega, r). \tag{2.186}$$

Hence, the family $(P_k)_{k\geq 1}$ consists of uniformly bounded entire functions. From Montel's Theorem (see [43]) it follows that there exists a subsequence, denoted in the same way, which converges uniformly on compact sets of Δ_r to an analytic function P.

Choose $r < T$. From (2.183) it follows that $\lim_{k\to\infty} ||P_k||_{L^2(r,T)} = 0$ and therefore $P(t) = 0$ for all $t \in (r, T)$. Since P is analytic in Δ_r, P must be identically zero in Δ_r.

Hence, $(P_k)_{k\geq 1}$ converges uniformly to zero on compact sets of Δ_r.

Let us now return to (2.185). There exists $r_0 > 0$ such that

$$|P_k(z)| \leq Me^{-\mathcal{R}e(z)}, \quad \forall z \in \Delta_{r_0}. \tag{2.187}$$

Indeed, there exists $r_0 > 0$ such that

$$\omega n - n^2\pi^2\mathcal{R}e(z) \leq -\mathcal{R}e(z) - n, \quad \forall z \in \Delta_{r_0}$$

and therefore, for any $z \in \Delta_{r_0}$,

$$|P_k(z)| \leq M\sum_{n\geq 1} e^{\omega n - n^2\pi^2\mathcal{R}e(z)} \leq Me^{-\mathcal{R}e(z)}\sum_{n\geq 1} e^{-n} = \frac{M}{e-1}e^{-\mathcal{R}e(z)}.$$

Lebesgue's Theorem implies that

$$\lim_{k\to\infty} ||P_k||_{L^2(r,\infty)} = 0$$

and consequently

$$\lim_{k\to\infty} ||P_k||_{L^2(0,r)} = 1.$$

If we take $r < T$ the last relation contradicts (2.184) and the proof ends.

■

We can now evaluate the norm of the biorthogonal sequence.

Theorem 2.6.7 *There exist two positive constants M and ω with the property that*

$$||\theta_m(T,\cdot)||_{L^2(0,T)} \leq Me^{\omega m}, \quad \forall m \geq 1 \tag{2.188}$$

where $(\theta_m(T,\cdot))_{m\geq 1}$ is the biorthogonal sequence to the family Λ in $L^2(0,T)$ which belongs to $E(\Lambda, T)$ and it is given in Theorem 2.6.4.

Proof. Let $(R_T^{-1})^* : E(\Lambda,\infty) \to E(\Lambda, T)$ be the adjoint of the bounded operator R_T^{-1}. We have that

$$\delta_{kj} = \int_0^\infty p_k(t)\theta_j(\infty,t)dt = \int_0^\infty (R_T^{-1}R_T)(p_k(t))\theta_j(\infty,t)dt =$$

$$= \int_0^T R_T(p_k(t))(R_T^{-1})^*(\theta_j(\infty,t))dt.$$

Since $(R_T^{-1})^*(\theta_j(\infty,\cdot)) \in E(\Lambda,T)$, from the uniqueness of the biorthogonal sequence in $E(\Lambda,T)$, we finally obtain that

$$(R_T^{-1})^*(\theta_j(\infty,\cdot)) = \theta_j(T,\cdot), \quad \forall j \geq 1.$$

Hence

$$||\theta_j(T,\cdot)||_{L^2(0,T)} = \left|\left|(R_T^{-1})^*(\theta_j(\infty,\cdot))\right|\right|_{L^2(0,T)} \leq ||R_T^{-1}||\,||\theta_j(\infty,\cdot)||_{L^2(0,\infty)},$$

since $||(R_T^{-1})^*|| = ||R_T^{-1}||$.

The proof finishes by taking into account the estimates from Theorem 2.6.5.

■

Remark 2.6.7 From the proof of Theorem 2.6.7 it follows that the constant ω does not depend of T.

■

Chapter 3

Propagation, Observation, Control and Finite-Difference Numerical Approximation of Waves

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3.1 Introduction

In recent years important progress has been made on problems of observation and control of wave phenomena. There is now a well established theory for wave equations with sufficiently smooth coefficients for a number of systems and variants: Lamé and Maxwell systems, Schrödinger and plate equations, etc. However, when waves propagate in highly heterogeneous or discrete media much less is known.

These problems of observability and controllability can be stated as follows:

- *Observability.* Assuming that waves propagate according to a given wave equation and with suitable boundary conditions, can one guarantee that their whole energy can be estimated (independently of the solution) in terms of the energy concentrated on a given subregion of the domain (or its boundary) where propagation occurs in a given time interval ?

- *Controllability.* Can solutions be driven to a given state at a given final time by means of a control acting on the system on that subregion?

It is well known that the two problems are equivalent provided one chooses an appropriate functional setting, which depends on the equation (see for, instance, [142, 143],[241]). It is also well known that in order for the observation and/or control property to hold, a *Geometric Control Condition (GCC)* should be satisfied [14]. According to the GCC all rays of Geometric Optics should intersect the observation/control region in the time given to observe/control.

In this work we shall mainly focus on the issue of how these two properties behave under numerical approximation schemes. More precisely, we shall discuss the problem of whether, when the continuous wave model under consideration has an observation and/or control property, it is preserved for numerical approximations, and whether this holds uniformly with respect to the mesh size so that, in the limit as the mesh size tends to zero, one recovers the observation/control property of the continuous model.

But, before getting into the matter, let us briefly indicate some of the industrial and/or applied contexts in which this kind of problems arises. The interested reader on learning more on this matter is refered to the SIAM Report [204], or, for more historical and engineering oriented applications, to [138]. The problem of controllability is classical in the field of Mathematical Control Theory and Control Engineering. We refer to the books by Lee and Marcus [136] and Sontag [206] for a classical and more modern, respectively, account of the theory for finite-dimensional systems with applications. The book by Fattorini [76] provides an updated account of theory in the context of semigroups which is therefore more adapted to the models governed by partial differential equations (PDE) and provides also some interesting examples of applications.

The problems of controllability and/or observability are in fact only some of those arising in the applications of control theory nowadays. In fact, an important part of the modelling effort needs to be devoted to defining the appropriate control problem to be addressed. But, whatever the control question we address is, the deep mathematical issues that arise when facing these problems of observability and controllability end up entering in one way or another. Indeed, understanding the properties of observation and controllability for a given system requires, first, analyzing the fine dynamical properties of the system and, second, the effect of the controllers on its dynamics.

In the context of control for PDE one needs to distinguish, necessarily, the elliptic, parabolic and hyperbolic ones since their distinguished qualitative properties make them to behave also very differently from a control point of view. The issue of controllability being typically of dynamic nature (although it also makes senses for elliptic or stationary problems) it is natural to address parabolic and hyperbolic equations, and, in particular, the heat and the wave equation.

Most of this article is devoted to the wave equation (although we shall also discuss briefly the beam equation, the Schrödinger equation and the heat equation). The wave equation is a simplified hyperbolic problem arising in many areas of Mechanics, Engineering and Technology. It is indeed, a model for describing the vibrations of structures, the propagation of acoustic or seismic waves, etc. Therefore, the control of the wave equation enters in a way or another in problems related with control mechanisms for structures, buildings in the presence of earthquakes, for noise reduction in cavities and vehicles, etc. We refer to [11], and [192] for insteresting practical applications in these areas. But the wave equation, as we said, is also a prototype of infinite-dimensional, purely conservative dynamical system. As we shall see, most of the theory can be adapted to deal also with Schrödinger equation which opens the frame of applications to the challenging areas of Quantum computing and control (see [25]). It is well known that the interaction of waves with a numerical mesh produces dispersion phenomena and spurious[1] high frequency oscillations ([224], [217]). In particular, because of this nonphysical interaction of waves with the discrete medium, the velocity of propagation of numerical waves and, more precisely, the so called *group velocity*[2] may converge to zero when the wavelength of solutions is of the order of the size of the mesh and the latter tends to zero. As a consequence of this fact, the time needed to uniformly (with respect to the mesh size) observe (or control) the numerical waves from the boundary or from a subset of the medium in which they propagate may tend to infinity as the mesh becomes finer. Thus, the observation and control properties of the discrete model may eventually disappear.

This effect is compatible and not in contradiction (as one's first intuition might suggest) with the convergence of the numerical scheme in the classical sense and with the fact that the observation and control properties of the continuous model do hold. Indeed, convergent numerical schemes may have an unstable behavior in what concerns observability. In fact, we shall only discuss classical and well known convergent semi-discrete and fully discrete approximations of the wave equation, but we shall see that, despite the schemes under consideration are convergent, the failure of convergence occurs at the level of observation and control properties. As we said above, this is due to the fact that most numerical schemes exhibit dispersion diagrams (we shall

[1]The adjective spurious will be used to designate any component of the numerical solution that does not correspond to a solution of the underlying PDE. In the context of the wave equation, this happens at the high frequencies and, consequently, these spurious solutions weakly converge to zero as the mesh size tends to zero. Consequently, the existence of these spurious oscillations is compatible with the convergence (in the classical sense) of the numerical scheme, which does indeed hold for fixed initial data.

[2]At the numerical level it is important to distinguish phase velocity and group velocity. Phase velocity refers to the velocity of propagation of individual monocromatic waves, while group velocity corresponds to the velocity of propagation of wave packets, that may significantly differ from the phase velocity when waves of similar frequencies are combined. See, for instance, [217].

give a few examples below) showing that the group velocity of high frequency numerical solutions tends to zero.

The main objectives of this paper are:

- To explain how numerical dispersion and spurious high frequency oscillations occur;
- To describe their consequences for observation/control problems;
- To describe possible remedies for these pathologies;
- To consider to what extent these phenomena occur for other models like plate or heat-like equations.

The previous discussion can be summarized by saying that discretization and observation or control do not commute:

$$Continuous\ Model + Observation/Control + Numerics$$

$$\neq$$

$$Continuous\ Model + Numerics + Observation/Control.$$

Indeed, here are mainly two alternative approaches to follow. The PDE approach consists on approximating the control of the underlying PDE through its corresponding optimality system or Euler-Lagrange equations. This provides convergent algorithms that produce good numerical approximations of the true control of the continuous PDE but, certainly one needs to go through PDE theory to develop it. But we may also first discretize the continuous model, then compute the control of the discrete system and use it as a numerical approximation of the continuous one. One of the main goals of this article is to explain that this second procedure, which is often used in the literature without comment, may diverge. We shall describe how this divergence may occur and develop some numerical remedies. In other words, the topic of the manuscript may also be viewed as a particular instance of *black-box* versus problem specific control. In the black box approach, when willing to control a PDE we make a finite-dimensional model approximating the PDE and control it. The other approach is to develop the theory of control for the PDE and discretize the control obtained that way. It is often considered that the black-box method is more robust. In this article we show that it may fail dramatically for wave-like problems. [3]

Summarizing, *controlling a discrete version of a continuous wave model is often a bad way of controlling the continuous wave model itself*: stable solvers

[3] There are however some other situations in which it works. We refer to E. Casas [29] for the analysis of finite-element approximations of elliptic optimal control problems and to [46] for an optimal shape design problem for the Laplace operator.

for solving the initial-boundary value problem for the wave equation do not need to yield stable control solvers.

It is also worth underlying that the instabilities we shall describe have a rather catastrophic nature. Indeed the divergence rate of controls is not polynomial on the number of nodes of the mesh but rather exponential. This shows that the stability can not restablished by simply changing norms on the observed quantities or relaxing the regularity of controls by a finite number of derivatives.

Up to now, we have discussed control problems in quite a vague way. In fact, rigorously speaking, the discussion above concerns the problem of *exact controllability* in which the goal is to drive the solution of an evolution problem to a given final state exactly in a given time. It is in this setting where the pathological numerical high frequency waves may lead to lack of convergence. But this lack of convergence does not occur if the control problem is relaxed to an approximate or optimal control problem. In this paper we shall illustrate this fact by showing that, although controls may diverge when we impose an exact control condition requiring the whole solution to vanish at the final time, when relaxing this condition (by simply requiring the solution to become smaller in norm than a given arbitrarily small number ε (approximate control) or to minimize the norm of the solution within a class of bounded controls (optimal control)) then the controls are bounded and converge as $h \to 0$ to the corresponding control of the continuous wave equation.

However, even if one is interested on those weakened versions of the control problem, taking into account that the exact controllability one can be obtained as a suitable limit of them, *the previous discussion indicates the instability and extreme sensitivity of all control problems for waves under numerical discretizations.*

As a consequence of this, computing efficiently the control of the continuous wave model may be a difficult task, which has has been undertaken in a number of works by Glowinski, Li, and Lions [98], Glowinski [95], and Asch and Lebeau [4], among others. The effort that has been carried out in these papers is closely related to the existing work on developing numerical schemes with suitable dispersion properties ([224], [217]), based on the classical notion of *group velocity*. But a full understanding of these connections in the context of control and observation of numerical waves requires an additional effort to which this paper is devoted.

In this paper, avoiding unnecessary technical difficulties, we shall present the main advances in this field, explaining the problems under consideration, the existing results and methods and also some open problems that, in our opinion, are particularly important. We shall describe some possible alternatives for avoiding these high frequency spurious oscillations, including Tychonoff regularization, multigrid methods, mixed finite elements, numerical viscosity, and filtering of high frequencies. All these methods, although they may look very different one from another in a first view, are in fact different ways of taking care of the spurious high frequency oscillations that common numerical approximation methods introduce. Despite the fact that the proofs of convergence may be lengthy and technically difficult (and often constitute open problems), once the high frequency numerical pathologies under consideration are well understood, it is easy to believe that they are indeed appropriate methods for computing the controls.

Our analysis is mainly based on the Fourier decomposition of solutions and classical results on the theory of non-harmonic Fourier series. In recent works by F. Macià [157], [158] tools of discrete Wigner measures (in the spirit of Gérard [91] and Lions and Paul [152]) have been developed to show that, as in the continuous wave equation, in the absence of boundary effects, one can characterize the observability property in terms of geometric properties related to the propagation of bicharacteristic rays. In this respect it is important to observe that the bicharacteristic rays associated with the discrete model do not obey the same Hamiltonian system as the continuous ones but have their own dynamics (as was pointed out before in [217]). As a consequence, numerical solutions develop quite different dynamical properties at high frequencies since both velocity and direction of propagation of energy may differ from those of the continuous model. Ray analysis allows one to be very precise when filtering the high frequencies and to do this filtering microlocally [4]. In this article we shall briefly comment on this discrete ray theory but shall mainly focus on the Fourier point of view, which is sufficient to understand the main issues under consideration. This ray theory provides a rigorous justification of a fact that can be formally analyzed and understood through the notion of group velocity of propagation of numerical waves [217].

All we have said up to now concerning the wave equation can be applied with minor changes to several other models that are purely conservative. However, many models from physics and mechanics have some damping mechanism built in. When the damping term is "mild" the qualitative properties are the same as those we discussed above. That is for instance the case for the dissipative wave equation

$$u_{tt} - \Delta u + k u_t = 0$$

that, under the change of variables $v = e^{-kt/2} u$, can be transformed into the

[4]Microlocal analysis deals, roughly speaking, with the possibility of localizing functions and its singularities not only in the physical space but also in the frequency domain. In particular, one can localize in the frequency domain not only according to the size of frequencies but also to sectors in the euclidean space in which they belong to. This allows introducing the notion of microlocal regularity, see for instance ([109])

wave equation plus potential

$$v_{tt} - \Delta v - \frac{k^2}{4} v = 0.$$

In the latter the presence of the zero order potential introduces a compact perturbation of the d'Alembertian and does not change the dynamics of the system in what concerns the problems of observability and controllability under consideration. Therefore, the presence of the damping term in the equation for u introduces, roughly, a decay rate[5] in time of the order of $e^{-kt/2}$ but does not change the properties of the system in what concerns control/observation.

However, some other dissipative mechanisms may have much stronger effects. This is for instance the case for the thermal effects arising in the heat equation itself but also in some other more sophisticated systems, like the system of thermoelasticity. Thus, we shall also analyze the 1D heat equation and see that, because of its intrinsic and strong dissipativity properties, the controls of the simplest numerical approximation schemes remain bounded and converge as the mesh size tends to zero to the control of the continuous heat equation, in contrast with the pathological behavior for wave equations. This fact can be easily understood: the dissipative effect of the 1D heat equation acts as a filtering mechanism by itself and is strong enough to exclude high frequency spurious oscillations. However, the situation is more complex in several space dimensions, where the thermal effects are not enough to guarantee the uniform boundedness of the controls. We shall discuss this interesting open problem that, in any case, indicates that viscosity helps to reestablish uniform observation and control properties in numerical approximation schemes. We shall also see that plate and Schrödinger equations behave better than the wave equation. This is due to the fact that the dispersive nature of the continuous model also helps at the discrete level and since it allows the group velocity of high frequency numerical waves not to vanish in some cases.

Most of the analysis we shall present here has been also developed in the context of a more difficult problem, related to the behavior of the observation/control properties of the wave equation in the context of homogenization. There, the coefficients of the wave equation oscillate rapidly on a scale ε that tends to zero, so that the equation homogenizes to a constant coefficient one. In that framework it was pointed out that the interaction of high frequency waves with the microstructure produce localized waves at high frequency. These localized waves are an impediment for the uniform observation/control properties to hold. This suggests the need for filtering of high frequencies. It has been proved in a number of situations that this filtering technique suffices to reestablish uniform observation and control properties ([33] and [133]).

[5]This is true for low frequency solutions. But, the decay rate may be lower for low frequency ones when k is large enough. This can be easily seen by means of Fourier decomposition. This is the so-called *overdamping* phenomenon.

The analogies between both problems (homogenization and numerical approximation) are clear: the mesh size h in numerical approximation schemes plays the same role as the ε parameter in homogenization (see [238] and [38] for a discussion of the connection between these problems). Although the analysis of the numerical problem is much easier from a technical point of view, it was only developed after the problem of homogenization was understood. This is due in part to the fact that, from a control theoretical point of view, there was a conceptual difficulty to match the existing finite-dimensional and infinite-dimensional theories. In this article we illustrate how to do this in the context of the wave equation, a model of purely conservative dynamics in infinite dimension.

The rest of this paper is organized as follows. In section 3.2 we recall the basic ingredients of the finite-diemnsional theory we will need along the paper. In particular we shall introduce the Kalman rank condition. Section 3.3 is devoted to presenting and discussing the problems of observability and controllability for the constant coefficient wave equation. In section 3.4 we briefly discuss some aspects related to the multi-dimensional wave equation such as the concentration and the lack of propagation of waves. In section 3.5 we discuss the finite-difference space semi-discretization of the 1D wave equation and present the main results on the lack of controllability and observability. We also comment on how filtering of high frequencies can be used to get uniform controllability results and on the impact of all this on other relaxed versions of the control problem (approximate controllability and optimal control). Section 3.6 is devoted to analyzing semi-discretizations for the 2D wave equation in a square. In Section 3.7 we discuss some other methods for curing the high frequency pathologies: viscous numerical damping, mixed finite elements, etc. As we shall see, in this case, numerical approximations affect not only the velocity of propagation of energy but also its direction, and further filtering is needed. More precisely, one has to restrict the wavelength of solutions in all space directions to get uniform observability and control properties. Finally, in Section 3.8 we discuss the finite difference space semi-discretizations for the heat and beam equations, showing that both viscous and dispersive properties of the original continuous models may help in the numerical approximation procedure. We close this paper with some further comments and a list of open problems.

The interested reader is referred to the survey articles [237] and [241] for a more complete discussion of the state of the art in the controllability of partial differential equations.

3.2 Preliminaries on finite-dimensional systems

Most of this article is devoted to analyze the wave equation and its numerical approximations. Numerical approximation schemes and more precisely those that are semi-discrete (discrete in space and continuous in time) yield finite-dimensional systems of ODE's. There is by now an extensive literature on the control of finite-dimensional systems and the problem is completely understood for linear systems ([136], [206]). As we have mentioned above, the problem of convergence of controls as the mesh-size in the numerical approximation tends to zero is very closely related to passing to the limit as the dimension of finite-dimensional systems tends to infinity. The later topic is widely open and this article may be considered as a contribution in this direction.

In this section we briefly summarize the most basic material on finite-dimensional systems that will be used along this article (we refer to [166] for more details).

Consider the finite-dimensional system of dimenion N:

$$x' + Ax = Bv, \quad 0 \leq t \leq T; \quad x(0) = x_0, \tag{3.1}$$

where x is the N-dimensional state and v is the M-dimensional control, with $M \leq N$.

Here A is an $N \times N$ matrix with constant real coefficients and B is an $M \times N$ matrix. The matrix A determines the dynamics of the system and the matrix B models the way controls act on the system.

Obviously, in practice, it would be desirable to control the N components of the system with a low number of controls and the best would be to do it by means of a scalar control, in which case $M = 1$.

System (3.1) is said to be controllable in time T when every initial datum $x_0 \in \mathbf{R}^N$ can be driven to any final datum x_1 in $\mathbf{R}^N$ in time T.

It turns out that for finite-dimensional systems there is a necessary and sufficient condition for controllability which is of purely algebraic nature. It is the so called *Kalman condition*: System (3.1) is controllable in some time $T > 0$ iff

$$\operatorname{rank}[B, AB, ..., A^{N-1}B] = N. \tag{3.2}$$

According to this, in particular, system (3.1) is controllable in some time T if and only if it is controllable for all time.

There is a direct proof of this result which uses the representation of solutions of (3.1) by means of the variations of constants formula. However, the methods we shall develop along this article rely more on the dual (but completely equivalent!) problem of observability of the adjoint system.

Consider the *adjoint system*

$$-\varphi' + A^*\varphi, \quad 0 \leq t \leq T; \quad \varphi(T) = \varphi_0. \tag{3.3}$$

It is not difficult to see that system (3.1) is controllable in time T if and only if the adjoint system (3.3) is *observable* in time T, i. e. if there exists a constant $C > 0$ such that, for all solution φ of (3.3),

$$|\varphi_0|^2 \leq C \int_0^T |B^* \varphi|^2 dt. \tag{3.4}$$

Before analyzing (3.4) in more detail let us see that this observability inequality does indeed imply the controllability of the state equation.

Assume the observability inequality (3.4) holds and consider the following quadratic functional $J : \mathbf{R}^N \to \mathbf{R}$:

$$J(\varphi_0) = \frac{1}{2} \int_0^T |B^* \varphi(t)|^2 dt - < x_1, \varphi_0 > + < x_0, \varphi(0) > . \tag{3.5}$$

It is easy to see that, if $\tilde{\varphi}_0$ is a minimizer for J then the control $v = B^* \tilde{\varphi}$, where $\tilde{\varphi}$ is the solution of the adjoint system with that datum at time $t = T$, is such that the solution x of the state equation satisfies the control requirement $x(T) = x_1$. Indeed, it is sufficient to write down explicitly the fact that the differential of J at the minimizer vanishes.

Thus, the controllability problem is reduced to minimizing the functional J. This can be done easily applying the Direct Method of the Calculus of Variations if the coercivity of J is proved since we are in finite dimensions, and the functional J is continuous and convex.

Therefore, the key point is whether the functional J is coercive or not. In fact, coercivity requires the Kalman condition to be satisfied. Indeed, when (3.4) holds the following variant holds as well, with possibly a different constant $C > 0$:

$$|\varphi_0|^2 + |\varphi(0)|^2 \leq C \int_0^T |B^* \varphi|^2 dt. \tag{3.6}$$

In view of this property the coercity of J is easy to prove.

This property turns out to be equivalent to the adjoint rank condition

$$\operatorname{rank}[B^*, B^* A^*, ..., B^* [A^*]^{N-1}] = N \tag{3.7}$$

which is obviously equivalent to the previous one (3.2).

To see the equivalence between (3.6) and (3.7) let us note that, since we are in finite-dimension, using the fact that all norms are equivalent, the observability inequality (3.6) is in fact equivalent to a uniqueness property: *Does the fact that $B^* \varphi$ vanish for all $0 \leq t \leq T$ imply that $\varphi \equiv 0$?* And, as we shall see, this uniqueness property is precisely equivalent to the adjoint Kalman condition (3.7).

Before proving this we note that $B^* \varphi$ is only an M-dimensional projection of the solution φ who has N components. Therefore, in order for this property to be true the operator B^* has to be chosen in a strategic way, depending of the state matrix A. The Kalman condition is the right test to check whether the choice of B^* (or B) is appropriate.

Let us finally prove that the uniqueness property holds when the adjoint rank condition (3.7) is fulfilled. In fact, taking into account that solutions φ are analytic in time, the fact that $B^*\varphi$ vanishes is equivalent to the fact that all the derivatives of $B^*\varphi$ of any order at time $t = T$ vanish.

But the solution φ admits the representation $\varphi(t) = e^{A^*(t-T)}\varphi_0$ and therefore all the derivaties of $B^*\varphi$ at time $t = T$ vanish if and only if $B^*\left[A^*\right]^k\varphi_0 \equiv 0$ for all $k \geq 0$. But, according to the Cayley-Hamilton's Theorem this is equivalent to the fac that $B^*[A^*]^k\varphi_0 \equiv 0$ for all $k = 0, ..., N-1$. Finally, the latter is equivalent to the fact that $\varphi_0 \equiv 0$ (i.e. $\varphi \equiv 0$) if and only if the adjoint Kalman rank condition (3.7) is fulfilled.

It is important to note that in this finite-dimensional context, the time T plays no role. In particular, whether a system is controllable (or its adjoint observable) is independent of the time T of control. In particular one of the main features of the wave equation, namely, that the system is controllable/observable only if $T \geq 2$ does not arise in finite-dimensional systems.

The main task to be undertaken in order to pass to the limit in numerical approximations of control problems for wave equations as the mesh-size tends to zero is to explain why, even though at the finite-dimensional level the value of the control time T is irrelevant, it may play a key role for the controllability/observability of the continuous PDE.

3.3 The constant coefficient wave equation

3.3.1 Problem formulation

In order to motivate the problems we have in mind let us first consider the constant coefficient 1D wave equation:

$$\begin{cases} u_{tt} - u_{xx} = 0, & 0 < x < 1,\ 0 < t < T \\ u(0,t) = u(1,t) = 0, & 0 < t < T \\ u(x,0) = u^0(x),\ u_t(x,0) = u^1(x), & 0 < x < 1. \end{cases} \tag{3.8}$$

In (3.8) $u = u(x,t)$ describes the displacement of a vibrating string occupying the interval $(0,1)$.

The energy of solutions of (3.8) is conserved in time, i.e.

$$E(t) = \frac{1}{2}\int_0^1 \left[|u_x(x,t)|^2 + |u_t(x,t)|^2\right] dx = E(0), \ \ \forall 0 \leq t \leq T. \tag{3.9}$$

The problem of boundary observability of (3.8) can be formulated, roughly, as follows: *To give sufficient conditions on the length of the time interval T*

such that there exists a constant $C(T) > 0$ *so that the following inequality holds for all solutions of (3.8):*

$$E(0) \leq C(T) \int_0^T |u_x(1,t)|^2 \, dt. \tag{3.10}$$

Inequality (3.10), when it holds, guarantees that the total energy of solutions can be "observed" or estimated from the energy concentrated or measured on the extreme $x = 1$ of the string during the time interval $(0, T)$.

Here and in the sequel, the best constant $C(T)$ in inequality (3.10) will be referred to as the *observability constant*.

Of course, (3.10) is just an example of a variety of similar observability problems. Among its possible variants, the following are worth mentioning: (a) one could observe the energy concentrated on the extreme $x = 0$ or in the two extremes $x = 0$ and 1 simultaneously; (b) the $L^2(0,T)$-norm of $u_x(1,t)$ could be replaced by some other norm, (c) one could also observe the energy concentrated in a subinterval (α, β) of the space interval $(0,1)$ occupied by the string, etc.

The observability problem above is equivalent to a boundary controllability problem. More precisely, the observability inequality (3.10) holds, if and only if, for any $(y^0, y^1) \in L^2(0,1) \times H^{-1}(0,1)$ there exists $v \in L^2(0,T)$ such that the solution of the controlled wave equation

$$\begin{cases} y_{tt} - y_{xx} = 0, & 0 < x < 1, \, 0 < t < T \\ y(0,t) = 0; \, y(1,t) = v(t), & 0 < t < T \\ y(x,0) = y^0(x), \, y_t(x,0) = y^1(x), & 0 < x < 1 \end{cases} \tag{3.11}$$

satisfies

$$y(x,T) = y_t(x,T) = 0, \quad 0 < x < 1. \tag{3.12}$$

Needless to say, in this control problem the goal is to drive solutions to equilibrium at the time $t = T$. Once the configuration is reached at time $t = T$, the solution remains at rest for all $t \geq T$, by taking null control for $t \geq T$, i. e. $v \equiv 0$ for $t \geq T$.

The exact controllability property of the controlled state equation (3.11) *is completely equivalent* [6] *to the observability inequality for the adjoint system* (3.8).

At this respect it is convenient to note that (3.8) is not, strictly speaking, the adjoint of (3.11). The initial data for the adjoint system should be given at time $t = T$. But, in view of the time-irreversibility of the wave equations under consideration this is irrelevant. The same holds for the time discretizations we shall consider. Obviously, one has to be more careful about this when dealing

[6] We refer to J.L. Lions [142, 143] for a systematic analysis of the equivalence between controllability and observability through the so called Hilbert Uniqueness Method (HUM).

with time irreversible systems as the heat equation in section 3.8.1. We claim that system (3.8) is controllable if and only if (3.11) is controllable. Let us check first that observability implies controllability. The proof is of a constructive nature and allows to build the control of minimal norm ($L^2(0,T)$-norm in the present situation) by minimizing a convex, continuous and coercive functional in a Hilbert space. In the present case, given $\left(y^0, y^1\right) \in L^2(0,1) \times H^{-1}(0,1)$ the control $v \in L^2(0,T)$ of minimal norm for which (3.12) holds is of the form

$$v(t) = u_x^*(1,t), \tag{3.13}$$

where u^* is the solution of the adjoint system (3.8) corresponding to initial data $(u^{0,*}, u^{1,*}) \in H_0^1(0,1) \times L^2(0,1)$ minimizing the functional[7],

$$J((u^0, u^1)) = \frac{1}{2} \int_0^T |u_x(1,t)|^2 dt + \int_0^1 y^0 u^1 dx - \int_0^1 y^1 u^0 dx, \tag{3.14}$$

in the space $H_0^1(0,1) \times L^2(0,1)$.

Note that J is convex. The continuity of J in $H_0^1(0,1) \times L^2(0,1)$ is guaranteed by the fact that the solutions of (3.8) satisfy the extra regularity property that $u_x(1,t) \in L^2(0,T)$ (a fact that holds also for the Dirichlet problem for the wave equation in several space dimensions, see [126], [142, 143]). More, precisely, for all $T > 0$ there exists a constant $C_*(T) > 0$ such that

$$\int_0^T \left[|u_x(0,t)|^2 + |u_x(1,t)|^2 \right] dt \le C_*(T) E(0), \tag{3.15}$$

for all solution of (3.8).

Thus, in order to guarantee that the functional J achieves its minimum, it is sufficient to prove that it is coercive. This is guaranteed by the observability inequality (3.10).

Once coercivity is known to hold the Direct Method of the Calculus of Variations (DMCV) allows showing that the minimum of J over $H_0^1(0,1) \times L^2(0,1)$ is achieved. By the strict convexity of J the minimum is unique and we denote it, as above, by $(u^{0,*}, u^{1,*}) \in H_0^1(0,1) \times L^2(0,1)$, the corresponding solution of the adjoint system (3.8) being u^*.

The functional J is of class C^1. Consequently, the gradient of J at the minimizer vanishes. This yields the following Euler-Lagrange equations:

$$\begin{aligned} < DJ((u^{0,*}, u^{1,*})), (w^0, w^1) >= \int_0^T u_x^*(1,t) w_x(1,t) dt \\ + \int_0^1 y^0 w^1 dx - < y^1, w^0 >_{H^{-1} \times H_0^1} = 0, \end{aligned} \tag{3.16}$$

[7] The integral $\int_0^1 y^1 u^0 dx$ represents the duality $< y^1, u^0 >_{H^{-1} \times H_0^1}$.

for all $(w^0, w^1) \in H_0^1(0,1) \times L^2(0,1)$, where w stands for the solution of the adjoint equation with initial data (w^0, w^1). By choosing the control as in (3.13) this identity yields:

$$\int_0^T v(t) w_x(1,t)dt + \int_0^1 y^0 w^1 dx - < y^1, w^0 >_{H^{-1}\times H_0^1} = 0. \tag{3.17}$$

On the other hand, multiplying in (3.11) by w and integrating by parts we get:

$$\begin{aligned}\int_0^T v(t) w_x(1,t)dt + \int_0^1 y^0 w^1 dx - < y^1, w^0 >_{H^{-1}\times H_0^1} \\ - \int_0^1 y(T) w_t(T) dx + < y_t(T), w(T) >_{H^{-1}\times H_0^1} = 0.\end{aligned} \tag{3.18}$$

Combining these two identities we get:

$$\int_0^1 y(T) w_t(T) dx - < y_t(T), w(T) >_{H^{-1}\times H_0^1} = 0, \tag{3.19}$$

for all $(w^0, w^1) \in H_0^1(0,1) \times L^2(0,1)$, which is equivalent to the exact controllability condition $y(T) \equiv y_t(T) \equiv 0$.

This argument shows that *observability implies controllability* [8]. The reverse is also true. If controllability holds, then the linear map that to all initial data $(y^0, y^1) \in L^2(0,1) \times H^{-1}(0,1)$ of the state equation (3.11) associates the control v of minimal $L^2(0,T)$-norm is bounded. Multiplying in the state equation (3.11) with that control by u, solution of the adjoint system, and using that $y(T) \equiv y_t(T) \equiv 0$ we obtain the identity:

$$\int_0^T v(t) u_x(1,t)dt + \int_0^1 y^0 u^1 dx - < y^1, u^0 >_{H^{-1}\times H_0^1} = 0. \tag{3.20}$$

Consequently,

$$\begin{aligned}\left|\int_0^1 [y^0 u^1 - y^1 u^0] dx\right| = \left|\int_0^T v(t) u_x(1,t) dt\right| \le ||v||_{L^2(0,T)} ||u_x(1,t)||_{L^2(0,T)} \\ \le C ||(y^0, y^1)||_{L^2(0,1)\times H^{-1}(0,1)} ||u_x(1,t)||_{L^2(0,T)},\end{aligned} \tag{3.21}$$

for all $(y^0, y^1) \in L^2(0,1) \times H^{-1}(0,1)$, which implies the observability inequality (3.10).

[8] In the particular case under consideration one can even get a feedback law for the control. Indeed, using the decomposition of the d' Alembert operator $\partial_t^2 - \partial_x^2$ into $\partial_t^2 - \partial_x^2 = (\partial_t + \partial_x)(\partial_t - \partial_x)$ it is easy to see that the solution of the wave equation with Dirichlet boundary condition $y = 0$ at $x = 0$ and the dissipative boundary condition $y_x + y_t = 0$ at $x = 1$ vanishes in time $T = 2$. This gives a feedback control mechanism. This is however a very specific property of the linear constant coefficient $1D$ wave equation.

Throughout this paper we shall mainly focus on the problem of observability. However, in view of the equivalence above, all the results we shall present have immediate consequences for controllability. The most important ones will also be stated. Note however that controllability is not the only application of the observability inequalities, which are also of systematic use in the context of Inverse Problems (Isakov, [116]). We shall discuss this issue briefly in Open Problem #10 in Section 3.9.2.

3.3.2 Observability

The first easy fact to be mentioned is that system (3.8) is observable if $T \geq 2$. More precisely, the following can be proved:

Proposition 3.3.1 *For any $T \geq 2$, system* (3.8) *is observable. In other words, for any $T \geq 2$ there exists $C(T) > 0$ such that* (3.10) *holds for any solution of* (3.8). *Conversely, if $T < 2$,* (3.8) *is not observable, or, equivalently,*

$$\sup_{u \text{ solution of } (3.8)} \left[\frac{E(0)}{\int_0^T \mid u_x(1,t) \mid^2 dt} \right] = \infty. \tag{3.22}$$

The proof of observability for $T \geq 2$ can be carried out in several ways, including Fourier series, multipliers (Komornik, [126]; Lions, [142, 143]), Carleman inequalities (Zhang, [227]), and microlocal tools (Bardos et al., [14]; Burq and Gérard, [27]). Let us explain how it can be proved using Fourier series. Solutions of (3.8) can be written in the form

$$u = \sum_{k \geq 1} \left(a_k \cos(k\pi t) + \frac{b_k}{k\pi} \sin(k\pi t) \right) \sin(k\pi x) \tag{3.23}$$

where a_k, b_k are such that

$$u^0(x) = \sum_{k \geq 1} a_k \sin(k\pi x), \quad u^1(x) = \sum_{k \geq 1} b_k \sin(k\pi x).$$

It follows that

$$E(0) = \frac{1}{4} \sum_{k \geq 1} \left[a_k^2 k^2 \pi^2 + b_k^2 \right].$$

On the other hand,

$$u_x(1,t) = \sum_{k \geq 1} (-1)^k \left[k\pi a_k \sin(k\pi t) + b_k \cos(k\pi t) \right].$$

Using the orthogonality properties of $\sin(k\pi t)$ and $\cos(k\pi t)$ in $L^2(0,2)$, it follows that

$$\int_0^2 |u_x(1,t)|^2\, dt = \sum_{k\geq 1} \left(\pi^2 k^2 a_k^2 + b_k^2\right).$$

The two identities above show that the observability inequality holds when $T=2$ and therefore for any $T>2$ as well. In fact, in this particular case, we actually have the identity

$$E(0) = \frac{1}{4}\int_0^2 |u_x(1,t)|^2\, dt. \tag{3.24}$$

On the other hand, for $T<2$ the observability inequality does not hold. Indeed, suppose that $T \leq 2-2\delta$ with $\delta>0$. We solve the wave equation

$$\begin{cases} u_{tt} - u_{xx} = 0, & 0<x<1,\ 0<t<T \\ u(0,t) = u(1,t) = 0, & 0<t<T \end{cases} \tag{3.25}$$

with data at time $t=T/2$ with support in the subinterval $(0,\delta)$. Note that, in view of the time reversibility of the wave equation, the solution is determined uniquely for $t \geq T/2$ and $t \leq T/2$.This solution is such that $u_x(1,t)=0$ for $\delta<t<T-\delta$. This can be seen using the classical fact that the time segment $x=1$, $t\in(\delta, T-\delta)$ remains outside the domain of influence of the space segment $t=T/2$, $x\in(0,\delta)$ (see Figure 1 below). This is a consequence of the fact that the velocity of propagation in this system is one and shows that the observability inequality fails for any time interval of length less[9]than 2.

Proposition 3.3.1 states that a necessary and sufficient condition for the observability to hold is that $T\geq 2$. We have just seen that the necessity is a consequence of the finite speed of propagation. The sufficiency, which was proved using Fourier series, is also related to the finite speed of propagation. Indeed, when developing solutions of (3.8) in Fourier series the solution is decomposed into the particular solutions $u_k = \sin(k\pi t)\sin(k\pi x)$, $\quad \tilde{u}_k = \cos(k\pi t)\sin(k\pi x)$. Observe that both u_k and $\tilde{u}_k$ can be written in the form

$$u_k = \frac{\cos(k\pi(t-x)) - \cos(k\pi(t+x))}{2}, \quad \tilde{u}_k = \frac{\sin(k\pi(x+t)) - \sin(k\pi(t-x))}{2}$$

and therefore they are linear combinations of functions of the form $f(x+t)$ and $g(x-t)$ for suitable profiles f and g. This shows that, regardless of the frequency of oscillation of the initial data of the equation, solutions propagate with velocity 1 in space and therefore can be observed at the end $x=1$ of the string, at the latest, at time $T=2$. Note that the observability time is twice

[9]This simple construction provides a $1D$ motivation of the Geometric Control Condition (GCC) mentioned in the introduction which is essentially necessary and sufficient for observability to hold in several space dimensions too.

the length of the string. This is due to the fact that an initial disturbance concentrated near $x = 1$ may propagate to the left (in the space variable) as t increases and only reach the extreme $x = 1$ of the interval after bouncing at the left extreme $x = 0$ (as described in Figure 1). A simple computation shows that this requires the time interval to be $T \geq 2$ for observability to hold.

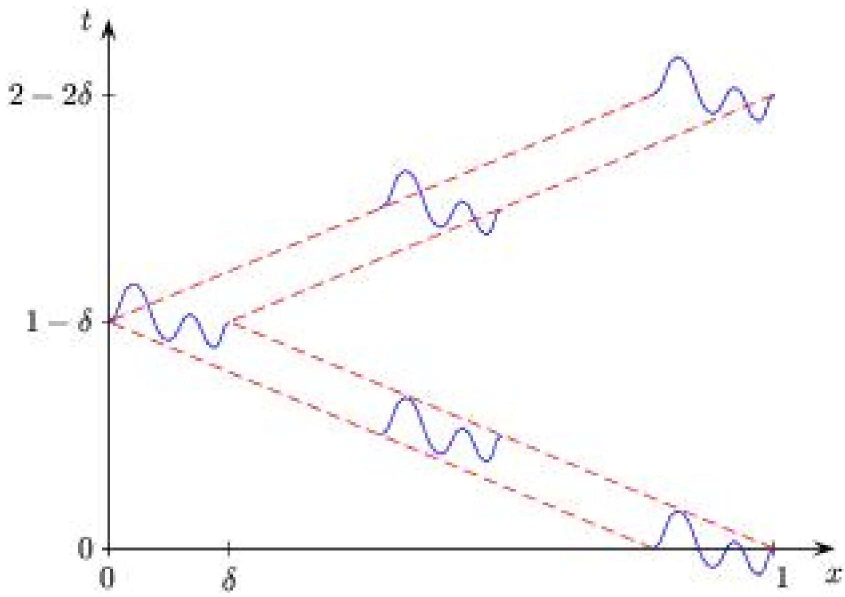

Figure 3.1: Wave localized at $t = 0$ near the endpoint $x = 1$ that propagates with velocity 1 to the left, bounces at $x = 0$ and reaches $x = 1$ again in a time of the order of 2.

As we have seen, in 1D and with constant coefficients, the observability inequality is easy to understand. The same results are true for sufficiently smooth coefficients (BV-regularity suffices). However, when the coefficients are simply Hölder continuous, these properties may fail, thereby contradicting a first intuition (see ([36]).

3.4 The multi-dimensional wave equation

In several space dimensions the observability problem for the wave equation is much more complex and can not be solved using Fourier series. The velocity of propagation is still one for all solutions but energy propagates along bicharacteristic rays.

But, before going further let us give the precise definition of bicharacteristic ray.

Consider the wave equation with a scalar, positive and smooth variable coefficient $a = a(x)$:

$$u_{tt} - \operatorname{div}(a(x)\nabla u) = 0. \tag{3.26}$$

Bicharacteristic rays solve the Hamiltonian system

$$\begin{cases} x'(s) = -a(x)\xi \\ t'(s) = \ \tau \\ \xi'(s) = \nabla a(x)|\xi|^2 \\ \tau'(s) = 0. \end{cases} \tag{3.27}$$

Rays allow describing microlocally how the energy of solutions propagates. The projections of the bicharacteristic rays in the (x, t) variables are the rays of Geometric Optics that play a fundamental role in the analysis of the observation and control properties through the Geometric Control Condition (GCC). As time evolves the rays move in the physical space according to the solutions of (3.27). Moreover, the direction in the Fourier space (ξ, τ) in which the energy of solutions is concentrated as they propagate is given precisely by the projection of the bicharacteristic ray in the (ξ, τ) variables. When the coefficient $a = a(x)$ is constant the ray is a straight line and carries the energy outward, which is always concentrated in the same direction in the Fourier space, as expected. But for variable coefficients the dynamics is more complex. This Hamiltonian system describes the dynamics of rays in the interior of the domain where the equation is satisfied. When rays reach the boundary they are reflected according to the laws of Geometric Optics.[10]

When the coefficient $a = a(x)$ varies in space, the dynamics of this system may be quite complex and can lead to some unexpected behaviour. An example wil be given later.

Let us now formulate the control problem precisely and discuss it in some more detail. We shall address here only the case of smooth domains.[11]

Let Ω be a bounded domain of $\mathbf{R}^n, n \geq 1$, with boundary Γ of class C^2. Let ω be an open and non-empty subset of Ω and $T > 0$.

Consider the linear controlled wave equation in the cylinder $Q = \Omega \times (0, T)$:

$$\begin{cases} y_{tt} - \Delta y = f 1_\omega & \text{in} \quad Q \\ y = 0 & \text{on} \quad \Sigma \\ y(x, 0) = y^0(x), y_t(x, 0) = y^1(x) & \text{in} \quad \Omega. \end{cases} \tag{3.28}$$

[10]Note however that tangent rays may be diffractive or even enter the boundary. We refer to [14] for a deeper discussion of these issues.

[11]We refer to Grisvard [100] for a discussion of these problems in the context of non-smooth domains.

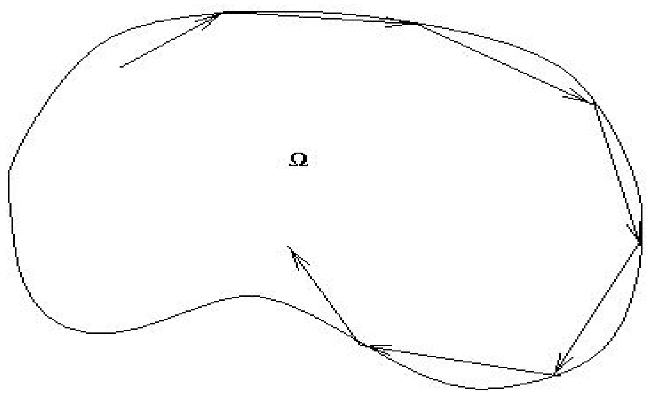

Figure 3.2: Ray that propagates inside the domain Ω following straight lines that are reflected on the boundary according to the laws of geometrical optics.

In (3.26) Σ represents the lateral boundary of the cylinder Q, i.e. $\Sigma = \Gamma \times (0,T)$, 1_ω is the characteristic function of the set ω, $y = y(x,t)$ is the state and $f = f(x,t)$ is the control variable. Since f is multiplied by 1_ω the action of the control is localized in ω.

When $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and $f \in L^2(Q)$ the system (3.26) has a unique solution $y \in C\left([0,T]; H_0^1(\Omega)\right) \cap C^1\left([0,T]; L^2(\Omega)\right)$.

The problem of *controllability*, generally speaking, is as follows: *Given* $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$, *find* $f \in L^2(Q)$ *such that the solution of system* (3.26) *satisfies*

$$y(T) \equiv y_t(T) \equiv 0. \tag{3.29}$$

In order to guarantee that the observability inequality holds, it is necessary that all rays reach the observation subset of the boundary in a uniform time and therefore this observation subset has to be selected in an appropriate way and has to be, in general, large enough. As we mentioned above it has to fulfill the Geometric Control Condition (see for instance Bardos et al. [14] and Burq and Gérard, [27]). For instance, when the domain is a ball, the subset of the boundary where the control is being applied needs to contain a point of each diameter. Otherwise, if a diameter skips the control region, it may support solutions that are not observed (see Ralston [194]).

Several remarks are in order.

Remark 3.4.1

(a) Since we are dealing with solutions of the wave equation, for any of these properties to hold, the control time T has to be sufficiently large due to the finite speed of propagation, the trivial case $\omega = \Omega$ being excepted. But, as we said above, the time being large enough does not suffice, since the control subdomain ω needs to satisfy the GCC. Figure 3 provides an example of this fact.

(b) The controllability problem may also be formulated in other function spaces in which the wave equation is well posed.

(c) Most of the literature on the controllability of the wave equation has been written in the framework of the *boundary control* problem discussed in the previous section. The control problems formulated above for (3.26) are usually referred to as *internal controllability* problems since the control acts on the subset ω of Ω. The latter is easier to deal with since it avoids considering non-homogeneous boundary conditions, in which case solutions have to be defined in the sense of transposition [142, 143].

■

Using HUM [12] and arguing as in section 3.3, the exact controllability property can be shown to be equivalent to the following *observability inequality*:

$$\left|\left|\left(u^0, u^1\right)\right|\right|^2_{L^2(\Omega)\times H^{-1}(\Omega)} \leq C \int_0^T \int_\omega u^2 dxdt \tag{3.30}$$

for every solution of the adjoint uncontrolled system

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in} \quad Q \\ u = 0 & \text{on} \quad \Sigma \\ u(x,0) = u^0(x), u_t(x,0) = u^1(x) & \text{in} \quad \Omega. \end{cases} \tag{3.31}$$

This inequality makes it possible to estimate the total energy of the solution of (3.4) by means of measurements on the control region $\omega \times (0,T)$.

When (3.30) holds one can minimize the functional

$$J\left(u^0, u^1\right) = \frac{1}{2}\int_0^T \int_\omega u^2 dxdt + \left\langle \left(u^0, u^1\right), \left(y^1, -y^0\right)\right\rangle \tag{3.32}$$

in the space $L^2(\Omega) \times H^{-1}(\Omega)$. Indeed, the following is easy to prove arguing as in the $1D$ case: *When the observability inequality* (3.30) *holds, the functional* J *has a unique minimizer* $\left(\widehat{u}^0, \widehat{u}^1\right)$ *in* $L^2(\Omega) \times H^{-1}(\Omega)$ *for all* $\left(u^0, u^1\right) \in H^1_0(\Omega) \times$

[12] HUM (Hilbert Uniqueness Method) was introduced by J. L. Lions (see [142, 143] as a systematic method to address controllability problems for Partial Differential Equations.

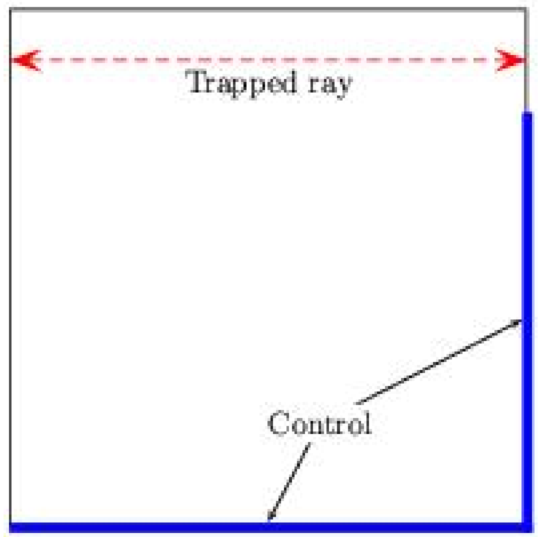

Figure 3.3: A geometric configuration in which the Geometric Control Condition is not satisfied, whatever $T > 0$ is. The domain where waves evolve is a square. The control is located on a subset of two adjacent sides of the boundary, leaving a small vertical subsegment uncontrolled. There is an horizontal line that constitutes a ray that bounces back and forth for all time perpendicularly on two points of the vertical boundaries where the control does not act.

$L^2(\Omega)$. *The control* $f = \widehat{u}$ *with* $\widehat{u}$ *a solution of* (3.31) *corresponding to* $\left(\widehat{u}^0, \widehat{u}^1\right)$ *is such that the solution of* (3.26) *satisfies* (4.4).

Consequently, in this way, the exact controllability problem is reduced to the analysis of the observability inequality (3.30) which is the key ingredient to prove the coercivity of J .

Let us now discuss what is known about (3.30):

(a) Using multiplier techniques Ho in [107] proved that if one considers subsets of Γ of the form $\Gamma(x^0) = \left\{x \in \Gamma : (x - x^0) \cdot n(x) > 0\right\}$ for some $x^0 \in \mathbf{R}^n$ (we denote by $n(x)$ the outward unit normal to Ω in $x \in \Gamma$ and by $\cdot$ the scalar product in $\mathbf{R}^n$) and if $T > 0$ is large enough, the following boundary observability inequality holds:

$$\left|\left|\left(u^0, u^1\right)\right|\right|^2_{H_0^1(\Omega)\times L^2(\Omega)} \leq C \int_0^T \int_{\Gamma(x^0)} \left|\frac{\partial u}{\partial n}\right|^2 d\sigma dt \qquad (3.33)$$

for all $\left(u^0, u^1\right) \in H_0^1(\Omega) \times L^2(\Omega)$.

This is the observability inequality that is required to solve the boundary controllability problem.

Later inequality (3.33) was proved in [142, 143], for any $T > T(x^0) = 2 \parallel x - x^0 \parallel_{L^\infty(\Omega)}$. This is the optimal observability time that one may derive by means of multipliers. More recently Osses in [178] introduced a new multiplier which is basically a rotation of the previous one, making it possible to obtain a larger class of subsets of the boundary for which observability holds.

Proceeding as in [142, 143], one can easily prove that (3.33) implies (3.30) when ω is a neighborhood of $\Gamma(x^0)$ in Ω, i.e. $\omega = \Omega \cap \Theta$ where Θ is a neighborhood of $\Gamma(x^0)$ in $\mathbf{R}^n$, with $T > 2 \parallel x - x^0 \parallel_{L^\infty(\Omega \setminus \omega)}$. In particular, exact controllability holds when ω is a neighborhood of the boundary of Ω.

(b) C. Bardos, G. Lebeau and J. Rauch [14] proved that, in the class of C^∞ domains, the observability inequality (3.30) holds if and only if the pair (ω, T) representing the control subdomain and time satisfies the following *geometric control condition (GCC)* in Ω*: Every ray of geometrical optics that propagates in* Ω *and is reflected on its boundary* Γ *intersects* ω *in time less than* T.

This result was proved by means of microlocal analysis. Recently the microlocal approach has been greatly simplified by N. Burq [26] by using the microlocal defect measures introduced by P. Gérard [91] in the context of homogenization and kinetic equations. In [26] the GCC was shown to be sufficient for exact controllability for domains Ω of class C^3 and equations with C^2 coefficients. The result for variable coefficients is the same: The observability inequality and, thus, the exact controllability property holds if and only if all rays of geometrical optics intersect the control region before the control time. However, it is important to note that, although in the constant coefficient equation all rays are straight lines, in the variable coefficient case this is no longer the case. In particular, there are smooth coefficients for which there are periodic rays that follow closed trajectories. This may happen in the interior of the domain Ω. Then, for instance, there are variable coefficient wave equations that are not exactly controllable when ω is a neighborhood of the boundary. Note that this is the typical geometrical situation in which the constant coefficient wave equation is exactly controllable. Indeed, in this case, the rays being straight lines, controllability holds when the control time exceeds the diameter of $\Omega \setminus \omega$ since it guarantees that all rays reach the subdomain ω. But, as we said, this property may fail for variable coefficient wave equations

3.5 1D Finite-Difference Semi-Discretizations

3.5.1 Orientation

In section 3.3 we showed how the observability/controllability problem for the constant coefficient wave equation can be solved by Fourier series expansions. We now address the problem of the continuous dependence of the observability constant $C(T)$ in (3.10) with respect to finite difference space semi-discretizations as the parameter h of the discretization tends to zero. This problem arises naturally in the numerical implementation of the controllability and observability properties of the continuous wave equation but is of independent interest in the analysis of discrete models for vibrations.

There are several important facts and results that deserve to be underlined and that we shall discuss below:

- The observability constant for the semi-discrete model tends to infinity for any T as $h \to 0$. This is related to the fact that the velocity of propagation of solutions tends to zero as $h \to 0$ and the wavelength of solutions is of the same order as the size of the mesh.
- As a consequence of this fact and of Banach-Steinhaus Theorem, there are initial data for the wave equation for which the controls of the semi-discrete models diverge. This proves that one can not simply rely on the classical convergence (consistency + stability) analysis of the underlying numerical schemes to design algorithms for computing the controls.
- The observability constant may be uniform if the high frequencies are filtered in an appropriate manner.

Let us now formulate these problems and state the corresponding results in a more precise way.

3.5.2 Finite-difference approximations

Given $N \in \mathbf{N}$ we define $h = 1/(N+1) > 0$. We consider the mesh

$$x_0 = 0;\ x_j = jh,\ j = 1, \dots, N;\ x_{N+1} = 1, \tag{3.34}$$

which divides $[0,1]$ into $N+1$ subintervals $I_j = [x_j, x_{j+1}]$, $j = 0, ..., N$.

Consider the following finite difference approximation of the wave equation (3.8):

$$\begin{cases} u_j'' - \frac{1}{h^2}\left[u_{j+1} + u_{j-1} - 2u_j\right] = 0, & 0 < t < T,\ j = 1, \dots, N \\ u_j(t) = 0, & j = 0,\ N+1,\ 0 < t < T \\ u_j(0) = u_j^0,\ u_j'(0) = u_j^1, & j = 1, \dots, N. \end{cases} \tag{3.35}$$

Observe that (3.35) is a coupled system of N linear differential equations of second order. The function $u_j(t)$ provides an approximation of $u(x_j, t)$ for all $j = 1, \ldots, N$, u being the solution of the continuous wave equation (3.8). The conditions $u_0 = u_{N+1} = 0$ take account of the homogeneous Dirichlet boundary conditions, and the second order differentiation with respect to x has been replaced by the three-point finite difference $[u_{j+1} + u_{j-1} - 2u_j]/h^2$.

We shall use a vector notation to simplify the expressions. In particular, the column vector

$$\overrightarrow{u}(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_N(t) \end{pmatrix} \tag{3.36}$$

will represent the whole set of unknowns of the system. Introducing the matrix

$$A_h = \frac{1}{h^2} \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \tag{3.37}$$

the system (3.35) reads as follows

$$\begin{cases} \vec{u}''(t) + A_h \vec{u}(t) = 0, & 0 < t < T \\ \vec{u}(0) = \vec{u}^0, \ \vec{u}'(0) = \vec{u}^1. \end{cases} \tag{3.38}$$

Obviously the solution $\overrightarrow{u}$ of (3.38) depends also on h. But, for the sake of simplicity, we shall only use the index h when strictly needed.

The energy of the solutions of (3.35) is as follows:

$$E_h(t) = \frac{h}{2} \sum_{j=0}^{N} \left[| u_j' |^2 + \left| \frac{u_{j+1} - u_j}{h} \right|^2 \right], \tag{3.39}$$

and it is constant in time. It is also a natural discretization of the continuous energy (3.9).

The problem of observability of system (3.35) can be formulated as follows: *To find $T > 0$ and $C_h(T) > 0$ such that*

$$E_h(0) \leq C_h(T) \int_0^T \left| \frac{u_N(t)}{h} \right|^2 dt \tag{3.40}$$

holds for all solutions of (3.35).

Observe that $| u_N/h |^2$ is a natural approximation [13] of $| u_x(1,t) |^2$ for the solution of the continuous system (3.8). Indeed $u_x(1,t) \sim [u_{N+1}(t) - u_N(t)]/h$ and, taking into account that $u_{N+1} = 0$, it follows that $u_x(1,t) \sim -u_N(t)/h$.

[13] Here and in the sequel u_N refers to the N-th component of the solution $\vec{u}$ of the semidiscrete system, which obvioulsy depends also on h. This dependence is not made explicit in the notation not to make it more complex.

System (3.35) is finite-dimensional. Therefore, if observability holds for some $T > 0$, then it holds for all $T > 0$ as we have have seen in section 3.3.

We are interested mainly in the uniformity of the constant $C_h(T)$ as $h \to 0$. If $C_h(T)$ remains bounded as $h \to 0$ we say that system (3.35) is uniformly (with respect to h) observable as $h \to 0$. Taking into account that the observability of the limit system (3.8) only holds for $T \geq 2$, it would be natural to expect $T \geq 2$ to be a necessary condition for the uniform observability of (3.35). This is indeed the case but, as we shall see, the condition $T \geq 2$ is far from being sufficient. In fact, *uniform observability fails for all* $T > 0$. In order to explain this fact it is convenient to analyze the spectrum of (3.35).

Let us consider the eigenvalue problem

$$-\left[w_{j+1} + w_{j-1} - 2w_j\right]/h^2 = \lambda w_j, \; j = 1, \ldots, N; \quad w_0 = w_{N+1} = 0. \tag{3.41}$$

The spectrum can be computed explicitly in this case (Isaacson and Keller [115]). The eigenvalues $0 < \lambda_1(h) < \lambda_2(h) < \ldots < \lambda_N(h)$ are

$$\lambda_k^h = \frac{4}{h^2}\sin^2\left(\frac{k\pi h}{2}\right) \tag{3.42}$$

and the corresponding eigenvectors are

$$\vec{w}_k^h = (w_{k,1}, \ldots, w_{k,N})^T : w_{k,j} = \sin(k\pi j h), \; k, j = 1, \ldots, N, \tag{3.43}$$

Obviously,

$$\lambda_k^h \to \lambda_k = k^2\pi^2, \text{ as } h \to 0 \tag{3.44}$$

for each $k \geq 1$, $\lambda_k = k^2\pi^2$ being the $k-$th eigenvalue of the continuous wave equation (3.8). On the other hand we see that the eigenvectors $\vec{w}_k^h$ of the discrete system (3.41) coincide with the restriction to the mesh-points of the eigenfunctions $w_k(x) = \sin(k\pi x)$ of the continuous wave equation (3.8). [14]

According to (3.42) we have

$$\sqrt{\lambda_k^h} = \frac{2}{h}\sin\left(\frac{k\pi h}{2}\right),$$

and therefore, in a first approximation, we have

$$\left|\sqrt{\lambda_k^h} - k\pi\right| \sim \frac{k^3\pi^3h^2}{24}. \tag{3.45}$$

This indicates that the convergence in (3.44) is only uniform in the range $k \ll h^{-2/3}$. Thus, one can not expect to solve completely the problem of uniform observability for the semi-discrete system (3.35) as a consequence of the observability property of the continuous wave equation and a perturbation argument with respect to h. A more careful analysis of the behavior of the eigenvalues and eigenvectors at high frequencies is needed.

[14]This is a non-generic fact that only occurs for the constant coefficient 1D problem with uniform meshes.

3.5.3 Non uniform observability

The following identity holds ([112], [113]):

Lemma 3.5.1 *For any $h > 0$ and any eigenvector of* (3.41) *associated with the eigenvalue λ,*

$$h \sum_{j=0}^{N} \left| \frac{w_{j+1} - w_j}{h} \right|^2 = \frac{2}{4 - \lambda h^2} \left| \frac{w_N}{h} \right|^2 . \tag{3.46}$$

We now observe that the largest eigenvalue λ_N^h of (3.41) is such that

$$\lambda_N^h h^2 \to 4 \text{ as } h \to 0. \tag{3.47}$$

Indeed

$$\lambda_N^h h^2 = 4 \sin^2 \left(\frac{\pi N h}{2} \right) = 4 \sin^2 \left(\frac{\pi(1-h)}{2} \right) = 4 \cos^2(\pi h/2) \to 4 \text{ as } h \to 0.$$

Combining (3.46) and (3.47) we get the following result on non-uniform observability:

Theorem 3.5.1 *For any $T > 0$ it follows that, as $h \to 0$,*

$$\sup_{u \text{ solution of } (3.35)} \left[\frac{E_h(0)}{\int_0^T \mid u_N/h \mid^2 dt} \right] \to \infty. \tag{3.48}$$

Proof of Theorem 3.5.1 We consider solutions of (3.35) of the form $\vec{u}^h = \cos\left(\sqrt{\lambda_N^h} t\right) \vec{w}_N^h$, where λ_N^h and $\vec{w}_N^h$ are the $N-$th eigenvalue and eigenvector of (3.41) respectively. We have

$$E_h(0) = \frac{h}{2} \sum_{j=0}^{N} \left| \frac{w_{N,j+1}^h - w_{N,j}^h}{h} \right|^2 \tag{3.49}$$

and

$$\int_0^T \left| \frac{u_N^h}{h} \right|^2 dt = \left| \frac{w_{N,N}^h}{h} \right|^2 \int_0^T \cos^2 \left(\sqrt{\lambda_N^h} t \right) dt. \tag{3.50}$$

Taking into account that $\lambda_N^h \to \infty$ as $h \to 0$ it follows that

$$\int_0^T \cos^2 \left(\sqrt{\lambda_N^h} t \right) dt \to T/2 \quad \text{as} \quad h \to 0. \tag{3.51}$$

By combining (3.46), (3.49), (3.50) and (3.51), (3.48) follows immediately.

Remark 3.5.1 Note that the construction above applies to any sequence of eigenvalues $\lambda^h_{j(h)}$.

■

It is important to note that the solution we have used in the proof of this theorem is not the only impediment for the uniform observability inequality to hold.

Indeed, let us consider the following solution of the semi-discrete system (3.35), constituted by the last two eigenvectors:

$$\vec{u} = \frac{1}{\sqrt{\lambda_N}}\left[\exp(i\sqrt{\lambda_N}t)\vec{w}_N - \exp(i\sqrt{\lambda_{N-1}}t)\vec{w}_{N-1}\right]. \tag{3.52}$$

This solution is a simple superposition of two monochromatic semi-discrete waves corresponding to the last two eigenfrequencies of the system. The total energy of this solution is of the order 1 (because each of both components has been normalized in the energy norm and the eigenvectors are orthogonal one to each other). However, the trace of its discrete normal derivative is of the order of h in $L^2(0,T)$. This is due to two facts.

- First, the trace of the discrete normal derivative of each eigenvector is very small compared to its total energy.
- Second and more important, the gap between $\sqrt{\lambda_N}$ and $\sqrt{\lambda_{N-1}}$ is of the order of h, as it is shown in Figure 4. This wave packet has then a group velocity of the order of h.

  Thus, by Taylor expansion, the difference between the two time-dependent complex exponentials $\exp(i\sqrt{\lambda_N}t)$ and $\exp(i\sqrt{\lambda_{N-1}}t)$ is of the order Th. Thus, in order for it to be of the order of 1 in $L^2(0,T)$, we need a time T of the order of $1/h$. In fact, by drawing the graph of the function in (3.52) above one can immediately see that it corresponds to a wave propagating at a velocity of the order of h (Figure 5).

This construction makes it possible to show that, whatever the time T is, the observability constant $C_h(T)$ in the semi-discrete system is at least of order $1/h$. In fact, this idea can be used to show that the observability constant has to blow-up at infinite order. To do this it is sufficient to proceed as above but combining an increasing number of eigenfrequencies. This argument allows one to show that the observability constant has to blow-up as an arbitrary negative power of h. Actually, S. Micu in [163] proved that the constant $C_h(T)$ blows up exponentially[15] by means of a careful analysis of the biorthogonal sequences to the family of exponentials $\{\exp(i\sqrt{\lambda_j}t)\}_{j=1,\dots,N}$ as $h \to 0$.

[15]According to [163] we know that the observability constant $C_h(T)$ necessarily blows-up exponentially as $h \to 0$. On the other hand, it is known that the observability inequality

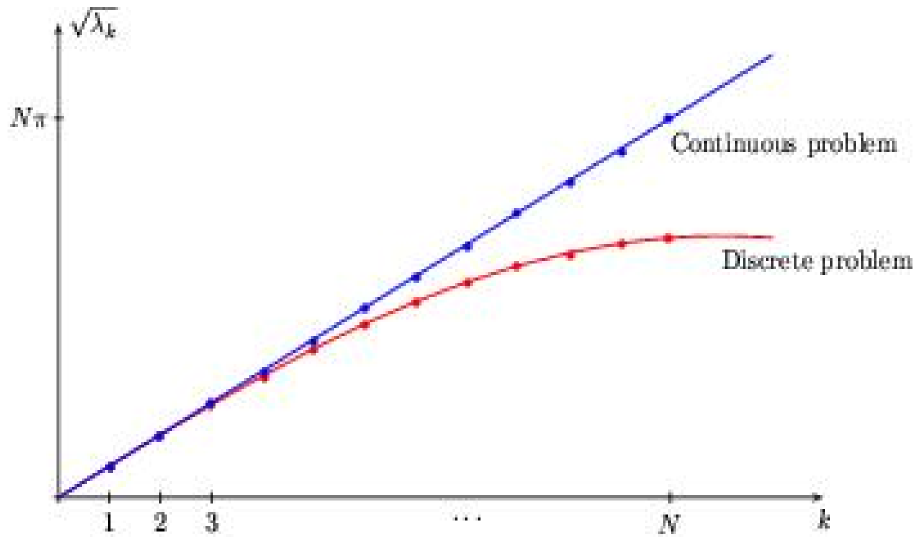

Figure 3.4: Square roots of the eigenvalues in the continuous and discrete case. The gaps between these numbers are clearly independent of k in the continuous case and of order h for large k in the discrete one.

All these high frequency pathologies are in fact very closely related with the notion of group velocity. We refer to [224], [217] for an in depth analysis of this notion that we discuss briefly in the context of this example.

According to the fact that the eigenvector $\vec{w}_j$ is a sinusoidal function (see (3.43)) we see that these functions can also be written as linear combinations

---

is true if $C_h(T)$ is large enough. To see this we can apply in this semi-discrete system the classical method of sidewise energy estimates for $1D$ wave equations (see [233]). Recall that solutions of the semi-discrete system vanish at the boundary point $x = 1$, i.e., $u_{N+1} \equiv 0$. On the other hand, the right hand side of the observability inequality provides an estimate of u_N in $L^2(0,T)$. The semi-discrete equation at the node $j = N$ reads as follows:

$$u_{N-1} = h^2 u_N'' + 2u_N, \quad 0 < t < T. \tag{3.53}$$

This provides an estimate of u_{N-1} in $L^2(0,T)$. Indeed, in principle, in view of (3.53), one should lose two time derivatives when doing this. However, this can be compensated by the fact that we are dealing with a finite-dimensional model in which two time derivatives of u are of the order of $A_h u$, A_h being the matrix in (3.37), which is of norm $4/h^2$. Iterating this argument we can end up getting an estimate in $L^2(0,T)$ for all u_j with $j = 1, ..., N$. But, taking into account that $N \sim 1/h$, the constant in the bound will necessarily be exponential in $1/h$. The problem of obtaining sharp asymptotic (as $h \to 0$) estimates on the observability constant is open.

of complex exponentials (in space-time):

$$\exp\left[\pm ij\pi\left[\frac{\sqrt{\lambda_j}}{j\pi}t - x\right]\right].$$

In view of this, we see that each monochromatic wave propagates at a speed

$$\frac{\sqrt{\lambda_j}}{j\pi} = \frac{2\sin(j\pi h/2)}{j\pi h} = \frac{\omega_h(\xi)}{\xi}\bigg|_{\{\xi=j\pi h\}} = c_h(\xi)\bigg|_{\{\xi=j\pi h\}}, \qquad (3.54)$$

with

$$\omega_h(\xi) = 2\sin(\xi/2).$$

This is the so called *phase velocity*. The velocity of propagation of monochromatic semi-discrete waves (3.54) turns out to be bounded above and below by positive constants, independently of h, i. e.

$$0 < \alpha \leq c_h(\xi) \leq \beta < \infty, \quad \forall h > 0, \forall \xi \in [0, \pi].$$

Note that $[0, \pi]$ is the relevant range of frequencies. Indeed, $\xi = j\pi h$ and $j = 1, ..., N$ and $Nh = 1 - h$.

However, it is well known that, even though the velocity of propagation of each eigenmode is bounded above and below, wave packets may travel at a different speed because of the cancellation phenomena we have exhibited above (see (3.52)). The corresponding speed for those semi-discrete wave packets accumulating is given by the derivative of $\omega_h(\cdot)$ (see [217]). At the high frequencies ($j \sim N$) the derivative of $\omega_h(\xi)$ at $\xi = N\pi h = \pi(1-h)$, is of the order of h and therefore the wave packet (3.52) propagates with velocity of the order of h.

Note that the fact that this group velocity is of the order of h is equivalent to the fact that the gap between $\sqrt{\lambda_{N-1}}$ and $\sqrt{\lambda_N}$ is of order h.

Indeed, the definition of group velocity as the derivative of ω_h is a natural consequence of the classical properties of the superposition of linear harmonic oscillators with close but not identical phases (see [56]). The group velocity is thus, simply, the derivative of the curve in the dispersion diagram of Figure 4 describing the velocity of propagation of monocromatic waves, as a function of frequency. Taking into account that this curve is constituted by the square roots of the eigenvalues of the numerical scheme, we see that there is a one-to-one correspondence between group velocity and spectral gap. In particular, the group velocity decreases when the gap between consecutive eigenvalues does it.

According to this analysis, the fact that group velocity is bounded below is a necessary condition for the uniform observability inequality to hold. Moreover, this is equivalent to an uniform spectral gap condition.

The convergence property of the numerical scheme only guarantees that the group velocity is correct for low frequency wave packets.[16] The negative results we have mentioned above are a new reading of well-known pathologies of finite differences scheme for the wave equation.

The careful analysis of this negative example is extremely useful when designing possible remedies, i.e., to determine how one could modify the numerical scheme in order to reestablish the uniform observability inequality, since we have only found two obstacles and both happen at high frequencies. The first remedy is very natural: To cut off the high frequencies or, in other words, to ignore the high frequency components of the numerical solutions. This method is analyzed in the following section.

3.5.4 Fourier Filtering

Filtering works as soon as we deal with solutions where the only Fourier components are those corresponding to the eigenvalues $\lambda \leq \gamma h^{-2}$ with $0 < \gamma < 4$ or with indices $0 < j < \delta h^{-1}$ with $0 < \delta < 1$, the observability inequality becomes uniform. Note that these classes of solutions correspond to taking projections of the complete solutions by cutting off all frequencies with $\gamma h^{-2} < \lambda < 4h^{-2}$.

All this might seem surprising in a first approach to the problem but it is in fact very natural. The numerical scheme, which converges in the classical sense, reproduces, at low frequencies, as $h \to 0$, the whole dynamics of the continuous wave equation. But, it also introduces a lot of high frequency spurious solutions. The scheme then becomes more accurate if we ignore that part of the solutions and, at the same time, makes the observability inequality uniform provided the time is taken to be large enough.[17]

It is also important to observe that the high frequency pathologies we have described can not be avoided by simply taking, for instance, a different approximation of the discrete normal derivative. Indeed, the fact that high frequency wave packets propagate at velocity h is due to the scheme itself and, therefore, can not be compensated by suitable boundary measurements. More precisely, even if we have had the right uniform observability inequality for each individual eigenvector, the uniform observability inequality would still be false for the semi-discrete wave equation and the observability constant will blow up exponentially.

To prove the main uniform observability result for system (3.35), in addition

---

[16] Note that in Figure 4 the semidiscrete and continuous curves are tangent. This is in agreement with the convergence property of the numerical algorithm under consideration and with the fact that low frequency wave packets travel essentially with the velocity of the continuous model.

[17] As we will see below, computing the optimal time for the observability inequality to hold requires taking again into account the notion of group velocity. The minimal time for uniform observability turns out to depend on the cutoff parameter γ.

to the sharp spectral results of the previous section we shall use a classical result due to Ingham in the theory of non-harmonic Fourier series (see Ingham [114] and Young [225]).

Ingham's Theorem. *Let $\{\mu_k\}_{k\in\mathbf{Z}}$ be a sequence of real numbers such that*

$$\mu_{k+1} - \mu_k \geq \gamma > 0, \forall k \in \mathbf{Z}. \tag{3.55}$$

Then, for any $T > 2\pi/\gamma$ there exists a positive constant $C(T,\gamma) > 0$ such that

$$\frac{1}{C(T,\gamma)} \sum_{k\in\mathbf{Z}} | a_k |^2 \leq \int_0^T \left| \sum_{k\in\mathbf{Z}} a_k e^{i\mu_k t} \right|^2 dt \leq C(T,\gamma) \sum_{k\in\mathbf{Z}} | a_k |^2 \tag{3.56}$$

for all sequences of complex numbers $\{a_k\} \in \ell^2$.

Remark 3.5.2 Ingham's inequality can be viewed as a generalization of the orthogonality property of trigonometric functions we used to prove the observability of the $1D$ wave equation in Section 3.3. Indeed, assume that $\mu_k = k\gamma$, $k \in \mathbf{Z}$ for some $\gamma > 0$. Then (3.55) holds with equality for all k. We set $T = 2\pi/\gamma$. Then

$$\int_0^{2\pi/\gamma} \left| \sum_{k\in\mathbf{Z}} a_k e^{i\gamma k t} \right|^2 dt = \frac{2\pi}{\gamma} \sum_{k\in\mathbf{Z}} | a_k |^2 . \tag{3.57}$$

Note that under the weaker gap condition (3.55) we obtain upper and lower bounds instead of identity (3.57). Observe also that Ingham's inequality does not apply at the minimal time $2\pi/\gamma$. This fact is also sharp [225].

■

In the previous section we have seen that, in the absence of spectral gap (or, when the group velocity vanishes) the uniform observability inequality fails. Ingham's inequality provides the positive counterpart, showing that, as soon as the gap condition is satisfied, there is uniform observability provided the time is large enough. Note that the observability time ($T > 2\pi/\gamma$) is inversely proportional to the gap, and this is once more in agreement with the interpretation of the previous section.

All these facts confirm that a suitable cutoff or filtering of the spurious numerical high frequencies may be a cure for these pathologies.

Let us now describe the basic *Fourier filtering mechanism.* We recall that solutions of (3.35) can be developed in Fourier series as follows:

$$\vec{u} = \sum_{k=1}^{N} \left(a_k \cos\left(\sqrt{\lambda_k^h} t\right) + \frac{b_k}{\sqrt{\lambda_k^h}} \sin\left(\sqrt{\lambda_k^h} t\right) \right) \vec{w}_k^h \tag{3.58}$$

where a_k, b_k are the Fourier coefficients of the initial data, i.e.,

$$\vec{u}^0 = \sum_{k=1}^{N} a_k \vec{w}_k^h, \quad \vec{u}^1 = \sum_{k=1}^{N} b_k \vec{w}_k^h.$$

Given $0 < \delta < 1$, we introduce the following classes of solutions of (3.35):

$$\mathcal{C}_\delta(h) = \left\{ \vec{u} \text{ sol. of (3.35) s.t. } \vec{u} = \sum_{k=1}^{[\delta/h]} \left(a_k \cos\left(\sqrt{\lambda_k^h}\, t\right) + \frac{b_k}{\sqrt{\lambda_k^h}} \sin\left(\sqrt{\lambda_k^h}\, t\right)\right) \vec{w}_k^h \right\}. \tag{3.59}$$

Note that in the class $\mathcal{C}_\delta(h)$ the high frequencies corresponding to the indices $j > [\delta(N+1)]$ have been cut off. We have the following result:

Theorem 3.5.2 *(*[112], [113]*) For any $\delta > 0$ there exists $T(\delta) > 0$ such that for all $T > T(\delta)$ there exists $C = C(T, \delta) > 0$ such that*

$$\frac{1}{C} E_h(0) \le \int_0^T \left| \frac{u_N(t)}{h} \right|^2 dt \le C E_h(0) \tag{3.60}$$

for every solution u of (3.35) *in the class $\mathcal{C}_\delta(h)$, and for all $h > 0$. Moreover, the minimal time $T(\delta)$ for which* (3.60) *holds is such that $T(\delta) \to 2$ as $\delta \to 0$ and $T(\delta) \to \infty$ as $\delta \to 1$.*

Remark 3.5.3 Theorem 3.5.2 guarantees the uniform observability in each class $\mathcal{C}_\delta(h)$, for all $0 < \delta < 1$, provided the time T is larger than $T(\delta)$.

The last statement in the Theorem shows that when the filtering parameter δ tends to zero, i.e. when the solutions under consideration contain fewer and fewer frequencies, the time for uniform observability converges to $T = 2$, which is the corresponding one for the continuous equation. This is in agreement with the observation that the group velocity of the low frequency semi-discrete waves coincides with the velocity of propagation in the continuous model.

By contrast, when the filtering parameter increases, i.e. when the solutions under consideration contain more and more frequencies, the time of uniform control tends to infinity. This is in agreement and explains further the negative showing that, if the absence of filtering, there is no finite time T for which the uniform observability inequality holds.

The proof of Theorem 3.5.2 below provides an explicit estimate on the minimal observability time in the class $\mathcal{C}_\delta(h)$: $T(\delta) = 2/\cos(\pi\delta/2)$.

■

Remark 3.5.4 In the context of the numerical computation of the boundary control for the wave equation the need of an appropriate filtering of the high frequencies was observed by R. Glowinski [95]. This issue was further investigated numerically by M. Asch and G. Lebeau in [4]. There, finite difference schemes were used to test the Geometric Control Condition in various geometrical situations and to analyze the cost of the control as a function of time.

■

Proof of Theorem 3.5.2 The statement in Theorem 3.5.2 is a consequence of Ingham's inequality and the gap properties of the semi-discrete spectra. Let us analyze the gap between consecutive eigenvalues. We have

$$\sqrt{\lambda_k^h} - \sqrt{\lambda_{k-1}^h} = \pi \cos\left(\frac{\pi(k-1+\eta)h}{2}\right)$$

for some $0 < \eta < 1$. Observe that $\cos\left((\pi(k-1+\eta)h)/2\right) \geq \cos\left(\pi k h/2\right).$ Therefore $\sqrt{\lambda_k^h} - \sqrt{\lambda_{k-1}^h} \geq \pi \cos\left(\pi k h/2\right).$ It follows that

$$\sqrt{\lambda_k^h} - \sqrt{\lambda_{k-1}^h} \geq \pi \cos\left(\pi\delta/2\right), \text{ for } k \leq \delta h^{-1}. \tag{3.61}$$

We are now in the conditions for applying Ingham's Theorem. We rewrite the solution $\vec{u} \in \mathcal{C}_\delta(h)$ of (3.35) as

$$\vec{u} = \sum_{\substack{|k| \leq \delta/h \\ k \neq 0}} c_k e^{i\mu_k^h t} \vec{w}_k^h \tag{3.62}$$

where

$$\mu_{-k}^h = -\mu_k^h;\ \mu_k^h = \sqrt{\lambda_k^h};\ \vec{w}_{-k} = \vec{w}_k;\ c_k = \frac{a_k - ib_k/\mu_k^h}{2}; c_{-k} = \overline{c_k}.$$

Then,

$$u_N(t) = \sum_{\substack{|k| \leq \delta/h \\ k \neq 0}} c_k e^{i\mu_k^h t} w_{k,N}.$$

Therefore

$$\int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt = \int_0^T \left|\sum_{\substack{|k| \leq \delta/h \\ k \neq 0}} c_k e^{i\mu_k^h t} \frac{w_{k,N}}{h}\right|^2 dt. \tag{3.63}$$

In view of the gap property (3.61) and, according to Ingham's inequality, it follows that if $T > T(\delta)$ with

$$T(\delta) = 2/\cos(\pi\delta/2) \tag{3.64}$$

there exists a constant $C = C(T,\delta) > 0$ such that

$$\frac{1}{C}\sum_{\substack{|k|\le\delta/h\\ k\neq 0}} |\ c_k\ |^2 \left|\frac{w_{k,N}}{h}\right|^2 \le \int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt \le C \sum_{\substack{|k|\le\delta/h\\ k\neq 0}} |\ c_k\ |^2 \left|\frac{w_{k,N}}{h}\right|^2 \quad (3.65)$$

for every solution of (3.35) in the class $\mathcal{C}_\delta(h)$. On the other hand, we observe that

$$\lambda_k^h h^2 = 4\sin^2\left(\frac{\pi k h}{2}\right) \le 4\sin^2\left(\frac{\pi\delta}{2}\right) \quad (3.66)$$

for all $k \le \delta/h$. Therefore, according to Lemma 3.5.1, it follows that

$$\frac{1}{2}\left|\frac{w_N}{h}\right|^2 \le h\sum_{j=0}^{N}\left|\frac{w_{j+1}-w_j}{h}\right|^2 \le \frac{1}{2\cos^2(\pi\delta/2)}\left|\frac{w_N}{h}\right|^2 \quad (3.67)$$

for all eigenvalues with index $k \le \delta/h$.

Combining (3.65) and (3.67) we deduce that for all $T > T(\delta)$ there exists $C > 0$ such that

$$\frac{1}{C}\sum_{\substack{|k|\le\delta/h\\ k\neq 0}} |\ c_k\ |^2 \le \int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt \le C \sum_{\substack{|k|\le\delta/h\\ k\neq 0}} |\ c_k\ |^2 . \quad (3.68)$$

Finally we observe that

$$\sum_{\substack{|k|\le\delta/h\\ k\neq 0}} |\ c_k\ |^2 \sim E_h(0).$$

This concludes the proof of Theorem 3.5.2.

3.5.5 Conclusion and controllability results

We have shown that the uniform observability property of the finite difference approximations (3.35) fails for any $T > 0$. On the other hand, we have proved that by filtering the high frequencies or, in other words, considering solutions in the classes $\mathcal{C}_\delta(h)$ with $0 < \delta < 1$, the uniform observability holds in a minimal time $T(\delta)$ that satisfies

- $T(\delta) \to \infty$ as $\delta \to 1$;
- $T(\delta) \to 2$ as $\delta \to 0$.

Observe that, as $\delta \to 0$, we recover the minimal observability time $T = 2$ of the continuous wave equation (3.8). This allows us to obtain, for all

$T > 2$, the observability property of the continuous wave equation (3.8) as the limit $h \to 0$ of uniform observability inequalities for the semi-discrete systems (3.35). Indeed, given any $T > 2$ there exists $\delta > 0$ such that $T > T(\delta)$ and, consequently, by filtering the high frequencies corresponding to the indices $k > \delta N$, the uniform observability in time T is guaranteed. As we mentioned above, this confirms that the semi-discrete scheme provides a better approximation of the wave equation when the high frequencies are filtered.

On the other hand, we have seen that when $\lambda \sim 4/h^2$ or, equivalently, $k \sim 1/h$, the gap between consecutive eigenvalues vanishes. This allows the construction of high-frequency wave-packets that travel at a group velocity of the order of h and it forces the uniform observability time $T(\delta)$ in the classes $\mathcal{C}_\delta(h)$ to tend to infinity as $\delta \to 1$.

Here we have used Inghman's inequality and spectral analysis. These results may also be obtained using discrete multiplier techniques ([112] and [113]).

In this sub-section we explain the consequences of these results in the context of controllability. Before doing this, it is important to distinguish two notions of the controllability of any evolution system, regardless of whether it is finite or infinite-dimensional:

- To control exactly to zero the whole solution for initial data in a given subspace.
- To control the projection of the solution over a given subspace for all initial data.

Before discussing these issues in detail it is necessary to write down the control problem we are analyzing. The state equation is as follows:

$$\begin{cases} y_j'' - \frac{1}{h^2}\left[y_{j+1} + y_{j-1} - 2y_j\right] = 0, & 0 < t < T,\ j = 1, \dots, N \\ y_0(0,t) = 0;\ y_{N+1}(1,t) = v(t), & 0 < t < T \\ y_j(0) = y_j^0,\ y_j'(0) = y_j^1, & j = 1, \dots, N, \end{cases} \tag{3.69}$$

and the question we consider is whether, for a given $T > 0$ and given initial data $(\vec{y}^0, \vec{y}^1)$, there exists a control $v_h \in L^2(0,T)$ such that

$$\vec{y}(T) = \vec{y}'(T) = 0. \tag{3.70}$$

As we shall see below, the answer to this question is positive. However, this does not mean that the controls will be bounded as h tends to zero. In fact they are not, even if $T \geq 2$ as one could predict from the results concerning the wave equation.

We have the following main results:

- Taking into account that for all $h > 0$ the Kalman rank condition is satisfied, for all $T > 0$ and all $h > 0$ the semi-discrete system (3.69) is controllable. In other words, for all $T > 0$, $h > 0$ and initial data $(\vec{y}^0, \vec{y}^1)$, there exists $v \in L^2(0,T)$ such that the solution $\vec{y}$ of (3.69) satisfies (3.70). Moreover, the control v of minimal $L^2(0,T)$-norm can be built as in section 3.3. It suffices to minimize the functional

$$J_h((\vec{u}^0, \vec{u}^1)) = \frac{1}{2}\int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt + h\sum_{j=1}^N y_j^0 u_j^1 - h\sum_{j=1}^N y_j^1 u_j^0 \tag{3.71}$$

over the space of all initial data $(\vec{u}^0, \vec{u}^1)$ for the adjoint semi-discrete system (3.35).

Of course, this strictly convex and continuous functional is coercive and, consequently, has a unique minimizer. The coercivity of the functional is a consequence of the observability inequality (3.40) that does indeed hold for all $T > 0$ and $h > 0$. Once we know that the minimum of J_h is achieved, the control is easy to compute. It suffices to take

$$v_h(t) = u_N^*(t)/h,\ 0 < t < T, \tag{3.72}$$

where $\vec{u}^*$ is the solution of the semi-discrete adjoint system (3.35), corresponding to the initial data $(\vec{u}^{0,*}, \vec{u}^{1,*})$ that minimize the functional J_h, as control to guarantee that (3.70) holds.

The control we obtain in this way is optimal in the sense that it is the one of minimal $L^2(0,T)$-norm. We can also get an upper bound on its size. Indeed, using the fact that $J_h \leq 0$ at the minimum (which is a trivial fact since $J_h((0,0)) \leq 0$), and the observability inequality (3.40), we deduce that

$$||v_h||_{L^2(0,T)} \leq 4C_h(T)||(y^0, y^1)||_{*,h}, \tag{3.73}$$

where $||\cdot||_{*,h}$ denotes the norm

$$||(y^0, y^1)||_{*,h} = \sup_{(u_j^0, u_j^1)_{j=1,\dots,N}} \left[\left|h\sum_{j=1}^N y_j^0 u_j^1 - h\sum_{j=1}^N y_j^1 u_j^0\right| \Big/ E_h^{1/2}(u^0, u^1)\right] \tag{3.74}$$

It is easy to see that this norm converges as $h \to 0$ to the norm in $L^2(0,1) \times H^{-1}(0,1)$. This norm can also be written in terms of the Fourier coefficients. It becomes a weighted euclidean norm whose weights are uniformly (with respect to h) equivalent to those of the continuous $L^2 \times H^{-1}$-norm.

Remark 3.5.5 In [112] and [113], in one space dimension, similar results were proved for the finite element space semi-discretization of the wave equation (3.8) as well. In Figure 7 below we plot the dispersion diagram for the piecewise linear finite element space semi-discretization. This time the discrete spectrum and, consequently, the dispersion diagram lies above the one corresponding to the continuous wave equation. But, the group velocity of high frequency numerical solutions vanishes again. This is easily seen on the slope of the discrete dispersion curve.

■

- The estimate (3.73) is sharp. On the other hand, for all $T > 0$ the constant $C_h(T)$ diverges as $h \to 0$. This shows that there are initial data for the wave equation in $L^2(0,1) \times H^{-1}(0,1)$ such that the controls of the semi-discrete systems $v_h = v_h(t)$ diverge as $h \to 0$. There are different ways of making this result precise. For instance, given initial data $(y^0, y^1) \in L^2(0,1) \times H^{-1}(0,1)$ for the continuous system, we can consider in the semi-discrete control system (3.69) the initial data that take the same Fourier coefficients as (y^0, y^1) for the indices $j = 1, \ldots, N$. It then follows by the Banach-Steinhaus Theorem that, because of the divergence of the observability constant $C_h(T)$, there is necessarily some initial data $(y^0, y^1) \in L^2(0,1) \times H^{-1}(0,1)$ for the continuous system such that the corresponding controls v_h for the semi-discrete system diverge in $L^2(0,T)$ as $h \to 0$. Indeed, assume that for any initial data $(y^0, y^1) \in L^2(0,1) \times H^{-1}(0,1)$, the controls v_h remain uniformly bounded in $L^2(0,T)$ as $h \to 0$. Then, according to the uniform boundedness principle, we would deduce that the maps that associate the controls v_h to the initial data are also uniformly bounded. But this implies the uniform boundedness of the observability constant.

  This lack of convergence is in fact easy to understand. As we have shown above, the semi-discrete system generates a lot of spurious high frequency oscillations. The control of the semi-discrete system has to take these into account. When doing this it gets further and further away from the true control of the continuous wave equation.

  At this respect it is important to note that the following alternative holds:

  a) Either the controls v_h of the numerical approximation schemes are not bounded in $L^2(0,T)$ as $h \to 0$;

  or

  b) If the controls are bounded, they converge in $L^2(0,T)$ to the control v of the continuous wave equation. Indeed, once the controls are bounded a classical argument of weak convergence and passing to the limit on the semi-discrete controlled systems shows that the limit of the controls is a control for the limit system. A Γ-convergence argument allows showing that it is the control of minimal $L^2(0;T)$ obtained by minimizing the functional J in (3.14). Finally, the convergence of the norms of the controls together with their weak convergence yields the strong convergence result (see [137] for details of the proofs in the case of beam equations.)

- The observability inequality is uniform in the class of filtered solutions $\mathcal{C}_\delta(h)$, for $T > T(\delta)$. As a consequence of this, one can control uniformly the projection of the solutions of the semi-discretized systems over subspaces in which the high frequencies have been filtered. More precisely, if the control requirement (3.70) is weakened to

$$\pi_\delta \vec{y}(T) = \pi_\delta \vec{y}\,'(T) = 0, \tag{3.75}$$

  where π_δ denotes the projection of the solution of the semi-discrete system (3.69) over the subspace of the eigenfrequencies involved in the filtered space $\mathcal{C}_\delta(h)$, then the corresponding control remains uniformly bounded as $h \to 0$ provided $T > T(\delta)$. The control that produces (3.75) can be obtained by minimizing the functional J_h in (3.71) over the subspace $\mathcal{C}_\delta(h)$. Note that the uniform (with respect to h) coercivity of this functional and, consequently, the uniform bound on the controls holds as a consequence of the uniform observability inequality.

  One may recover the controllability property of the continuous wave equation as a limit of this partial controllability results since, as $h \to 0$, the projections π_δ end up covering the whole range of frequencies.

  It is important to underline that the time of control depends on the filtering parameter δ in the projections π_δ. But, as we mentioned above, for any $T > 2$ there is a $\delta \in (0,1)$ for which $T > T(\delta)$ and so that the uniform (with respect to h) results apply.

However, although the divergence of the controls occurs for some data, it is hard to observe in numerical simulations. This fact has been recently explained by S. Micu in [163]. According to [163], if the initial data of the wave equation has only a finite number of non vanishing Fourier components, the controls of the semi-discrete models are bounded as $h \to 0$ and converge to the control of the continuous wave equation.[18] The proof consists in writing the control problem as a moment problem and then getting estimates on its solutions by means of sharp and quite technical estimates of the family of biorthogonals to the family of complex exponentials $\{\exp(\pm i\sqrt{\lambda_j}t)\}_{j=1,\dots,N}$. The interested reader will find an introduction to these techniques, based on moment problem theory, in the survey paper by D. Russell [194].

[18]In fact, the result in [163] is much more precise since it indicates that, as $h \to 0$, one can control uniformly a space of initial data in which the number of Fourier component increases and tends to infinity.

The result in [163], in terms of the observability inequality, is equivalent to proving that for $T > 0$ large enough, and for any finite $M > 0$, there exists a constant $C_M > 0$, independent of h, such that

$$E_h(\pi_M(u^0, u^1)) \leq C_M \int_0^T \left| \frac{u_N(t)}{h} \right|^2 dt \tag{3.76}$$

for any solution of the semi-discrete adjoint system. Here π_M denotes the projection over the subspace generated by the first M eigenvectors. The inequality (3.76) provides an estimate of $\pi_M(u^0, u^1)$ for any solution, regardless of the number of Fourier components it involves, rather than an estimate for the solutions involving only those M components. We do not know if this type of estimate can be obtained by multiplier methods or Ingham type inequalities. Very likely the celebrated Beurling-Malliavin Theorem can be of some use when doing this, but this issue remains to be clarified.[19]

Nevertheless, even though the method converges for some initial data it is of unstable nature and therefore not of practical use. Similar results hold for full discretizations. Once more, except for the very particular case of the $1D$ wave equation with centered finite differences an equal time and space steps, filtering of high frequencies is needed (see [172], [173]).

3.5.6 Numerical experiments

In this section we briefly illustrate by some simple but convincing numerical experiments the theory provided along this section. These experiments have been developed by J. Rasmussen [187] using MatLab.

We consider the wave equation in the space interval $(0, 1)$ with control time $T = 4$.

We address the case of continuous and piecewise constant initial data y^0 of the form in Figure 8 below together with $y^1 \equiv 0$. In this simple situation and when the control time $T = 4$ the Dirichlet control can be computed at the end point $x = 1$ explicitly. This can be done using Fourier series and the time periodicity of solutions of the adjoint system (with time period $= 2$) which guarantees complete time orthogonality of the different Fourier components when $T = 4$.

Obviously the time $T = 4$ is sufficient for exact controllability to hold, the minimal control time being $T = 2$.

We see that the exact explicit control in Figure 9 looks very much like the initial datum to be controlled itslef. This can be easily understood by applying the d'Alembert formula for the explicit representation of solutions of the wave equation in $1D$.

[19]See [106] and [64] for applications of the Beurling-Malliavin Theorem in the control of plates and networks of vibrating strings.

We now consider the finite-difference semi-discrete approximation of the wave equation by finite-differences. First of all, we ignore all the discussion of the present section about the need of filtering. Thus, we merely compute the exact control of the semi-discrete system (3.69). This is done as follows. As described in section 3.5.5, the control is characterized through the minimization of the functional J_h in (3.71) over the space of all solutions of the adjoint equation (3.8). This allows writing the control v_h as the solution of an equation of the form $\Lambda_h(v_h) = \{-y^1, y^0\}$, where $\{y^0, y^1\}$ is the initial datum to be controlled.

The operator Λ_h can be computed by first solving the adjoint system and then the state equation with the normal derivative of the adjoint state as boundary datum and starting from equilibrium at time $t = T$ (see [142, 143]). Of course, in practice, we do not deal with the continuous adjoint equation but rather with a fully discrete approximation. We simply take the centered discretization in time with time-step $\Delta t = 0.99\,\Delta x$ ($\Delta x = h$), which, of course, guarantees the convergence of the scheme and the fact that our computations yield results which are very close to the semi-discrete case. Applying this procedure to the initial datum under consideration we get the exact control.

In Figure 10 below we draw the evolution on the error of the control as the number of mesh-points N increases. The solid line describes the evolution of the error when simply controlling the finite difference semi-discretization. This solid line diverges very fast as N increasing as a clear evidence of the lack of convergence of the control of the discrete system towards the control of the continuous one as $h \to 0$. In the dotted line of Figure 10 we describe the evolution of the error when the filtering parameter is taken to be $\gamma = 0.6$. This filtering parameter has been chosen in order to guarantee the uniform observability of the filtered solutions of the adjoint semi-discrete and fully discrete (with $\Delta t = 0.99\Delta x$) schemes in time $T = 4$ and, concequently, the convergence of controls as $h \to 0$ as well. The decay of the error as the number of mesh-points N increases, or, equivalently, when $h \to 0$, is obvious in the figure.

The control for the filtered problem is obtained by restricting and inverting the operator Λ_h above to the solutions of the adjoint system that only involve the Fourier components that remain after filtering.

In figures 11 we show the evolution of the control of the discrete problem as the number of mesh-points N icreases, or, equivalently, when the mesh-size h tends to zero. We see that ,when $N = 20$, a low number of mesh-points, the control captures essentially the form of the continuous control in Figure 9 but with some extra unwanted oscillations. The situation is very similar when $N = 40$. But when $N = 100$ we see that these oscillations become wild and for $N = 160$ the dynamics of the control is completely chaotic. This is a good example of lack of convergence in the absence of filtering and confirms the predictions of the theory.

We do the same experiment but now with filtering parameter = 0.6. Theory predicts convergence of controls in $L^2(0,T)$. The numerical experiments we draw in Figure 12 confirm this fact. These figures exhibit a Gibbs phenomenon which is compatible with L^2-convergence.

3.5.7 Robustness of the optimal and approximate control problems

In the previous sections we have shown that the exact controllability property behaves badly under most classical finite difference approximations. It is natural to analyze to what extent the high frequency spurious pathologies do affect other control problems and properties. The following two are worth considering:

- *Approximate controllability.*

  Approximate controllability is a relaxed version of the exact controllability property. The goal this time is to drive the solution of the controlled wave equation (3.11) not exactly to the equilibrium as in (3.12) but rather to an ε-state such that

  $$||y(T)||_{L^2(0,1)} + ||y_t(T)||_{H^{-1}(0,1)} \leq \varepsilon. \tag{3.77}$$

  When for all initial data (y^0, y^1) in $L^2(0,1) \times H^{-1}(0,1)$ and for all ε there is a control v such that (3.77) holds, we say that the system (3.12) is approximately controllable. Obviously, approximate controllability is a weaker notion than exact controllability and whenever the wave equation is exactly controllable, it is approximately controllable too.

  Approximate controllability does not require the GCC to hold. In fact, approximate controllability holds for controls acting on any open subset of the domain where the equation holds (or from its boundary) if the time is large enough.

  To be more precise, in $1D$, although exact controllability requires an observability inequality of the form of (3.10) to hold, for approximate controllability one only requires the following uniqueness property: $u \equiv 0$ *whenever the solution u of* (3.8) *is such that* $u_x(1,t) \equiv 0$*, in* $(0,T)$. This uniqueness property holds for $T \geq 2$ as well and can be easily proved using Fourier series or d'Alembert's formula. Its multidimensional version holds as well, as an immediate consequence of Holmgren's Uniqueness Theorem (see [142, 143]) for general wave equations with analytic coefficients and without geometric conditions, other than the time being large enough. In 1D, because of the trivial geometry, both the uniqueness property and the observability inequality hold simultaneously for $T \geq 2$.

Of course, the approximate controllability property by itself, as stated, does not provide any information of what the cost of controlling to an ε-state as in (3.77), i.e. on what is the norm of the control v_ε needed to achieve the approximate control condition (3.77). Roughly speaking, when exact controllability does not hold (for instance, in several space dimensions, when the GCC is not fulfilled), the cost of controlling blows up exponentially as ε tends to zero (see [189]).[20] But this issue will not be addressed here.

Thus, let us fix some $\varepsilon > 0$ and continue our discussion in the $1D$ case. Once ε is fixed, we know that when $T \geq 2$, for all initial data (y^0, y^1) in $L^2(0,1) \times H^{-1}(0,1)$, there exists a control $v_\varepsilon \in L^2(0,T)$ such that (3.77) holds.

The question we are interested in is the behavior of this property under numerical discretization.

Thus, let us consider the semi-discrete controlled version of the wave equation (3.69). We also fix the initial data in (3.69) "independently of h" (roughly, by taking a projection over the discrete mesh of fixed initial data (y^0, y^1) or by truncating its Fourier series).

Of course, (3.69) is also approximately controllable.[21] The question we address is as follows : *Given initial data which are "independent of h", with ε fixed, and given also the control time $T \geq 2$, is the control v_h of the semi-discrete system* (3.69) *(such that the discrete version of* (3.77) *holds) uniformly bounded as $h \to 0$?*

In the previous sections we have shown that the answer to this question in the context of the exact controllability (which corresponds to taking $\varepsilon = 0$) is negative. However, here, *in the context of approximate controllability, the controls v_h do remain uniformly bounded as $h \to 0$. Moreover, they can be chosen such that they converge to a limit control v for which* (3.77) *is realized for the continuous wave equation.*

This positive result on the uniformity of the approximate controllability property under numerical approximation when $\varepsilon > 0$ does not contradict the fact that the controls blow up for exact controllability. These are in fact two complementary and compatible facts. For approximate controllability, one is allowed to concentrate an ε amount of energy on the solution at the final time $t = T$. For the semi-discrete problem this is

[20] This type of result has been also proved in the context of the heat equation in [83]. But there the difficulty does not come from the geometry but rather from the regularizing effect of the heat equation.

[21] In fact, in finite dimensions, exact and approximate controllability are equivalent notions and, as we have seen, the Kalman condition is satisfied for system (3.69).

done precisely in the high frequency components that are badly controllable as $h \to 0$, and this makes it possible to keep the control fulfilling (3.77), bounded as $h \to 0$.

The approximate control of the semi-discrete system can be obtained by minimizing the functional

$$J_h^*(\vec{u}^0, \vec{u}^1) = \frac{1}{2}\int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt \varepsilon ||(\vec{u}^0, \vec{u}^1)||_{\mathcal{H}^1 \times \ell^2} + h\sum_{j=1}^N y_j^0 u_j^1 - h\sum_{j=1}^N y_j^1 u_j^0 \tag{3.78}$$

over the space of all initial data $(\vec{u}^0, \vec{u}^1)$ for the adjoint semi-discrete system (3.35). In J_h^*, $||\cdot||_{\mathcal{H}^1 \times \ell^2}$ stands for the discrete energy norm, i.e. $||\cdot|| = \sqrt{2E_h}$. Note that there is an extra term $\varepsilon||(\vec{u}^0, \vec{u}^1)||_{\mathcal{H}^1 \times \ell^2}$ in this new functional compared with the one we used to obtain the exact control (see (3.71)). Thanks to this term, the functional J_h^* satisfies an extra coercivity property that can be proved to be uniform as $h \to 0$. More precisely, it follows that

$$\lim_{||(\vec{u}^0, \vec{u}^1)||_{\mathcal{H}^1 \times \ell^2} \to \infty} \frac{J_h^*(\vec{u}^0, \vec{u}^1)}{||(\vec{u}^0, \vec{u}^1)||_{\mathcal{H}^1 \times \ell^2}} \geq \varepsilon, \tag{3.79}$$

uniformly in h, provided $T \geq 2$.

Note that, at this level, the fact that $T \geq 2$ is essential. Indeed, in order to show that the coercivity property above is uniform in $0 < h < 1$ we have to argue by contradiction as in [234]. In particular, we have to consider the case where $h \to 0$ and solutions of the adjoint semi-discrete system (3.35) converge to a solution of the continuous adjoint wave equation (3.25) such that $u_x(1,t) \equiv 0$ in $(0,T)$. Of course, if this happens with $T \geq 2$ we can immediately deduce that $u \equiv 0$, which yields the desired contradiction.

Once the uniform coercivity of the functional is proved, their minimizers are uniformly bounded and in particular, the controls, which are once again given by the formula (3.25), turn out to be uniformly bounded in $L^2(0,T)$. Once this is known it is not hard to prove by a Γ-convergence argument (see [65], [234]) that these controls converge in $L^2(0,T)$ to the control $v \in L^2(0,T)$ for the continuous wave equation that one gets by minimizing the functional

$$\begin{aligned} J^*(u^0, u^1) &= \frac{1}{2}\int_0^T |u_x(1,t)|^2 dt + \varepsilon||(u^0, u^1)||_{H^1(0,1) \times L^2(0,1))} + \\ &\quad + \int_0^1 [y^0 u^1 dx - y^1 u^0] dx \end{aligned} \tag{3.80}$$

in the space $H_0^1(0,1) \times L^2(0,1)$ for the solutions of the continuous adjoint wave equation (3.25). This control v is once again obtained as in (3.12) where u^* is the solution of (3.12) with the initial data minimizing the functional J^* and it turns out to be the function of minimal $L^2(0,T)$-norm among all admissible controls satisfying (3.77).

This shows that the approximate controllability property is well-behaved under the semi-discrete finite-difference discretization of the wave equation. But the argument is in fact much more general and can be applied in several space dimensions too, and for other numerical approximation schemes.

The result above guarantees the convergence of the approximate controls, with $\varepsilon > 0$ fixed, *but only for fixed initial data.* It is important to note the bounds are not uniform on all possible initial data, even if they are normalized to have unit norm. Indeed, a more careful analysis based on the construction of biorthogonal families in [163] shows that, even when $\varepsilon > 0$ is kept fixed, the bound on the control as $h \to 0$ blows up as the frequency of the initial datum to be controlled increases.

A careful analysis of the proof above shows that the classical convergence property of numerical schemes suffices for convergence at the level of approximate controllability too, in contrast to the unstability phenomena observed in the context of exact controllability.

■

- *Optimal control.*

  Finite horizon optimal control problems can also be viewed as relaxed versions of the exact controllability one.

  Let us consider the following example in which the goal is to drive the solution at time $t = T$ as closely as possible to the desired equilibrium state but penalizing the use of the control. In the continuous context the problem can be simply formulated as that of minimizing the functional

$$L^k(v) = \frac{k}{2} ||(y(T), y_t(T))||^2_{L^2(0,1)\times H^{-1}(0,1)} + \frac{1}{2} ||v||^2_{L^2(0,T)} \tag{3.81}$$

  over $v \in L^2(0,T)$. This functional is continuous, convex and coercive in the Hilbert space $L^2(0,T)$. Thus it admits a unique minimizer that we denote by v_k. The corresponding optimal state is denoted by y_k. The penalization parameter establishes a balance between reaching the distance to the target and the use of the control. As k increases, the need of getting close to the target (the $(0,0)$ state) is emphasized and the penalization on the use of control is relaxed.

When exact controllability holds, i.e. when $T \geq 2$, it is not hard to see that the control one obtains by minimizing L^k converges, as $k \to \infty$, to an exact control for the wave equation.

Of course, once $k > 0$ is fixed, the optimal control v_k does not guarantee that the target is achieved in an exact way. One can then measure the rate of convergence of the optimal solution $(y_k(T), y_{k,t}(T))$ towards $(0,0)$ as $k \to \infty$. When approximate controllability holds but exact controllability does not (a typical situation in several space dimensions when the GCC is not satisfied), the convergence of $(y_k(T), y_{k,t}(T))$ to $(0,0)$ in $L^2(0,1) \times H^{-1}(0,1)$ as $k \to \infty$ is very slow.[22]

But here, once again, we fix any $k > 0$ and we discuss the behavior of the optimal control problem for the semi-discrete equation as $h \to 0$.

It is easy to write the semi-discrete version of the problem of minimizing the functional L^k. Indeed, it suffices to introduce the corresponding semi-discrete functional L_h^k replacing the $L^2 \times H^{-1}$-norm in the definition of L^k by the discrete norm introduced in (3.74). It is also easy to prove by the arguments we have developed in the context of approximate controllability, that, *as $h \to 0$, the control v_h^k that minimizes L_h^k in $L^2(0,T)$ converges to the minimizer of the functional L^k and the optimal solutions y_h^k of the semi-discrete system converge to the optimal solution y^k of the continuous wave equation in the appropriate topology*[23] *as $h \to 0$ too.*

This shows that the optimal control problem is also well-behaved with respect to numerical approximation schemes, like the approximate control problem.

The reason for this is basically the same: In the optimal control problem the target is not required to be achieved exactly and, therefore, the pathological high frequency spurious numerical components are not required to be controlled.

■

In view of this discussion it becomes clear that the source of divergence in the limit process as $h \to 0$ in the exact controllability problem is the requirement of driving the high frequency components of the numerical solution exactly to zero. As we mentioned in the introduction, taking into account that optimal

[22] We should mention the works of Lebeau [132] and L. Robbiano [189] where the cost of approximate controllability is analyzed for the wave equation when the GCC fails. On the other hand, in [83], due to the regularizing effect of the heat equation, it was proved that convergence holds at a logarithmic rate.

[23] Roughly, in $C([0,T]; L^2(0,1)) \cap C^1[0,T]; H^{-1}(0,1))$.

and approximate controllability problems are relaxed versions of the exact controllability one, this negative result should be considered as a warning about the limit process as $h \to 0$ in general control problems.

3.6 Space discretizations of the 2D wave equations

In this section we briefly discuss the results in [235] on the space finite difference semi-discretizations of the 2D wave equation in the square $\Omega = (0, \pi) \times (0, \pi)$ of $\mathbf{R}^2$:

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in} \quad Q = \Omega \times (0, T) \\ u = 0 & \text{on} \quad \Sigma = \partial\Omega \times (0, T) \\ u(x, 0) = u^0(x),\ u_t(x, 0) = u^1(x) & \text{in} \quad \Omega. \end{cases} \tag{3.82}$$

Obviously, the fact that classical finite differences provide divergent results for $1D$ problems in what concerns observability and controllability indicate that the same should be true in $2D$ as well. This is indeed the case. However the multidimensional case exhibits some new features and deserves additional analysis, in particular in what concerns filtering techniques. Given $\left(u^0, u^1\right) \in H_0^1(\Omega) \times L^2(\Omega)$, system (3.82) admits a unique solution $u \in C\left([0, T]; H_0^1(\Omega)\right) \cap C^1\left([0, T]; L^2(\Omega)\right)$. Moreover, the energy

$$E(t) = \frac{1}{2} \int_\Omega \left[\mid u_t(x, t) \mid^2 + \mid \nabla u(x, t) \mid^2\right] dx \tag{3.83}$$

remains constant, i.e.

$$E(t) = E(0), \quad \forall 0 < t < T. \tag{3.84}$$

Let Γ_0 denote a subset of the boundary of Ω constituted by two consecutive sides, for instance,

$$\Gamma_0 = \{(x_1, \pi) : x_1 \in (0, \pi)\} \cup \{(\pi, x_2) : x_2 \in (0, \pi)\}. \tag{3.85}$$

It is well known (see [142, 143]) that for $T > 2\sqrt{2}\pi$ there exists $C(T) > 0$ such that

$$E(0) \leq C(T) \int_0^T \int_{\Gamma_0} \left|\frac{\partial u}{\partial n}\right|^2 d\sigma dt \tag{3.86}$$

holds for every finite-energy solution of (3.82). In (3.86), n denotes the outward unit normal to Ω, $\partial \cdot / \partial n$ the normal derivative and $d\sigma$ the surface measure.

Remark 3.6.1 The lower bound $2\sqrt{2}\pi$ on the minimal observability time is sharp.

On the other hand inequality (3.86) fails if in the right-hand side, instead of Γ_0, we only consider the energy concentrated on a strict subset of Γ_0, a situation which is drawn in Figure 3.

■

Let us now introduce the standard 5-point finite difference space semi-discretization scheme for the 2D wave equation. Given $N \in \mathbf{N}$ we set

$$h = \frac{\pi}{N+1}. \tag{3.87}$$

We denote by $u_{j,k}(t)$ the approximation of the solution u of (3.82) at the point $x_{j,k} = (jh, kh)$. The finite difference semi-discretization of (3.82) is as follows:

$$\begin{cases} u''_{jk} - \frac{1}{h^2}\left[u_{j+1,k} + u_{j-1,k} - 4u_{j,k} + u_{j,k+1} + u_{j,k-1}\right] = 0, & \\ \qquad\qquad 0 < t < T, j,\ k = 1, \dots, N & \\ u_{j,k} = 0, & 0 < t < T,\ j = 0, N+1; k = 0, N+1 \\ u_{j,k}(0) = u^0_{j,k}, u'_{j,k}(0) = u^1_{j,k}, & j,\ k = 1, \dots, N. \end{cases} \tag{3.88}$$

This is a coupled system of N^2 linear differential equations of second order.

It is well known that this semi-discrete scheme provides a convergent numerical scheme for the approximation of the wave equation. Let us now introduce the *discrete energy* associated with (3.88):

$$E_h(t) = \frac{h^2}{2} \sum_{j=0}^{N} \sum_{k=0}^{N} \left[\mid u'_{jk}(t) \mid^2 + \left| \frac{u_{j+1,k}(t) - u_{j,k}(t)}{h} \right|^2 + \left| \frac{u_{j,k+1}(t) - u_{j,k}(t)}{h} \right|^2 \right], \tag{3.89}$$

that remains constant in time, i.e.,

$$E_h(t) = E_h(0), \quad \forall 0 < t < T \tag{3.90}$$

for every solution of (3.88).

Note that the discrete version of the energy observed on the boundary is given by

$$\int_0^T \int_{\Gamma_0} \left| \frac{\partial u}{\partial n} \right|^2 d\sigma dt \sim \int_0^T \left[h \sum_{j=1}^{N} \left| \frac{u_{j,N}(t)}{h} \right|^2 + h \sum_{k=1}^{N} \left| \frac{u_{N,k}(t)}{h} \right|^2 \right] dt. \tag{3.91}$$

The discrete version of (3.86) is then an inequality of the form

$$E_h(0) \leq C_h(T) \int_0^T \left[h \sum_{j=1}^{N} \left| \frac{u_{j,N}(t)}{h} \right|^2 + h \sum_{k=1}^{N} \left| \frac{u_{N,k}(t)}{h} \right|^2 \right] dt. \tag{3.92}$$

This inequality holds for any $T > 0$ and $h > 0$ as in (3.40), for a suitable constant $C_h(T) > 0$. [24]

The problem we discuss here is the 2D version of the 1D one we analyzed in section 3.5 and can be formulated as follows: *Assuming* $T > 2\sqrt{2}\pi$, *is the constant* $C_h(T)$ *in* (3.92) *uniformly bounded as* $h \to 0$? *In other words, can we recover the observability inequality* (3.86) *as the limit as* $h \to 0$ *of the inequalities* (3.92) *for the semi-discrete systems* (3.88)?

As in the 1D case the constants $C_h(T)$ in (3.92) necessarily blow up as $h \to 0$, for all $T > 0$.

Theorem 3.6.1 ([235]) *For any* $T > 0$ *we have*

$$\sup_{u \text{ solution of } (3.88)} \left[\frac{E_h(0)}{\int_0^T \left[h \sum_{j=1}^N \left| \frac{u_{j,N}(t)}{h} \right|^2 + h \sum_{k=1}^N \left| \frac{u_{N,k}}{h} \right|^2 \right] dt} \right] \to \infty \, as \ h \to 0. \tag{3.93}$$

The proof of this negative result can be carried out as in the 1D case, analyzing the solutions in separated variables corresponding to the eigenvector with largest eigenvalue.

In order to prove the positive counterpart we have to filter the high frequencies. To do this we consider the eigenvalue problem associated with (3.88):

$$\begin{cases} -\dfrac{1}{h^2}\left[\varphi_{j+1,k} + \varphi_{j-1,k} - 4\varphi_{j,k} + \varphi_{j,k+1} + \varphi_{j,k-1}\right] = \lambda\varphi_{j,k}, \ j,k = 1,\dots,N \\ \varphi_{j,k} = 0, \qquad j = 0, N+1; k = 0, \ N+1. \end{cases} \tag{3.94}$$

This system admits N^2 eigenvalues that can be computed explicitly (see [115], p. 459):

$$\lambda^{p,q}(h) = 4\left[\frac{1}{h^2}\sin^2\left(\frac{ph}{2}\right) + \frac{1}{h^2}\sin^2\left(\frac{qh}{2}\right)\right], \quad p,q = 1,\dots,N \tag{3.95}$$

with corresponding eigenvectors

$$\varphi^{p,q} = \left(\varphi^{p,q}_{j,k}\right)_{1\le j,k\le N}, \quad \varphi^{p,q}_{j,k} = \sin(jph)\sin(kqh). \tag{3.96}$$

[24] As in the 1D case this can be proved in two different ways: a) Using the characterization of observability/controllability by means of Kalman's rank condition [136]; b) Using a propagation argument. Indeed, using the information that the right side of (3.92) provides and the semi-discrete system, this information can be propagated to all the nodes $j,k = 1,\dots,N$ and an inequality of the form (3.92) can be obtained. This argument was developed in section 3.5 in 1D. Of course, in 2D one has to be much more careful, since one has to take into account also the direction of propagation on the numerical mesh. However, as proved in [45], this argument can be successfully applied if one uses the information on the nodes in the neighborhood of one of the sides of the square to get a global estimate of the energy.

The following is a sharp upper bound for the eigenvalues of (3.94):

$$\lambda \leq 4\left[\frac{1}{h^2}+\frac{1}{h^2}\right]=8\left[\frac{1}{h^2}\right]. \tag{3.97}$$

Let us also recall what the spectrum of the continuous system is. The eigenvalue problem associated with (3.82) is

$$-\Delta\varphi=\lambda\varphi \text{ in } \Omega;\ \varphi=0 \text{ on } \partial\Omega, \tag{3.98}$$

and its eigenvalues are

$$\lambda^{p,q}=p^2+q^2, \quad p,q\geq 1 \tag{3.99}$$

with corresponding eigenfunctions

$$\varphi^{p,q}(x_1,x_2)=\sin\left(px_1\right)\sin\left(qx_2\right). \tag{3.100}$$

Once again the discrete eigenvalues and eigenvectors converge as $h\to 0$ to those of the continuous Laplacian. Moreover, in the particular case under consideration, the discrete eigenvectors are actually the restriction to the discrete mesh of the eigenfunctions of the continuous laplacian which is a non-generic fact.

Solutions of (3.88) can be developed in Fourier series and one can introduce classes of solutions of the form of the form $\mathcal{C}_\gamma(h)$ consisting on the solutions that only involve the eigenvalues λ such that $\lambda h^2\leq\gamma$.

According to the upper bound (3.97), when $\gamma=8, \mathcal{C}_\gamma\left(h\right)=\mathcal{C}_8\left(h\right)$ coincides with the space of all solutions of (3.88). However, when $0<\gamma<8$, solutions in the class $\mathcal{C}_\gamma\left(h\right)$ do not contain the contribution of the high frequencies $\lambda>\gamma\ h^{-2}$ that have been truncated or filtered.

The distinguishing property in this $2D$ is that, contrarily to the $1D$ one, it is not sufficient to filter by any $\gamma<8$. In fact, the observability inequality is not uniform as $h\to 0$ in the classes $\mathcal{C}_\gamma\left(h\right)$ when $\gamma\geq 4$. This is due to the fact that, in those classes, there exist solutions corresponding to high frequency oscillations in one direction and very slow oscillations in the other one. These are the separated variable solutions corresponding to the eigenvectors $\varphi^{(p,q)}$ with indices $(p,q)=(N,1)$ and $(p,q)=(1,N)$, for instance. Thus, further filtering is needed in order to guarantee the uniform observability inequality. Roughly speaking, one needs to filter efficiently in both space directions, and this requires taking $\gamma<4$ (see [235]).

In order to better understand the necessity of filtering and getting sharp observability times it is convenient to adopt the approach of [157], [158] based on the use of discrete Wigner measures. The symbol of the semi-discrete system (3.88) for solutions of wavelength h is

$$\tau^2-4\left(\sin^2(\xi_1/2)+\sin^2(\xi_2/2)\right) \tag{3.101}$$

and can be easily obtained as in the von Neumann analysis of the stability of numerical schemes by taking the Fourier transform of the semi-discrete equation: the continuous one in time and the discrete one in space. [25]

Note that, in the symbol in (3.101) the parameter h disappears. This is due to the fact that we are analyzing the propagation of waves of wavelength of the order of h.

The bicharacteristic rays are then defined as follows

$$\left\{\begin{array}{l} x_j'(s) = -2\sin(\xi_j/2)\cos(\xi_j/2) = -\sin(\xi_j), \quad j=1,2 \\ t'(s) = \tau \\ \xi_j'(s) = 0, \quad j=1,2 \\ \tau'(s) = 0. \end{array}\right. \tag{3.102}$$

It is interesting to note that the rays are straight lines, as for the constant coefficient wave equation, as a consequence of the fact that the coefficients of the equation and the numerical discretization are both constant. We see however that in (3.102) both the direction and the velocity of propagation change with respect to those of the continuous wave equation.

Let us now consider initial data for this Hamiltonian system with the following particular structure: x_0 is any point in the domain Ω, the initial time $t_0 = 0$ and the initial microlocal direction (τ^*, ξ^*) is such that

$$(\tau^*)^2 = 4\left(\sin^2(\xi_1^*/2) + \sin^2(\xi_2^*/2)\right). \tag{3.103}$$

Note that the last condition is compatible with the choice $\xi_1^* = 0$ and $\xi_2^* = \pi$ together with $\tau^* = 2$. Thus, let us consider the initial microlocal direction $\xi_2^* = \pi$ and $\tau^* = 2$. In this case the ray remains constant in time, $x(t) = x_0$, since, according to the first equation in (3.102), x_j' vanishes both for $j = 1$ and $j = 2$. Thus, the projection of the ray over the space x does not move as time evolves. This ray never reaches the exterior boundary $\partial\Omega$ where the equation evolves and excludes the possibility of having a uniform boundary observability property. More precisely, this construction allows one to show that, as $h \to 0$, there exists a sequence of solutions of the semi-discrete problem whose energy is concentrated in any finite time interval $0 \le t \le T$, as much as one wishes in a neighborhood of the point x_0.

Note that this example corresponds to the case of very slow oscillations in the space variable x_1 and very rapid ones in the x_2-direction and it can be ruled out, precisely, by taking the filtering parameter $\gamma < 4$. In view of the structure of the Hamiltonian system, it is clear that one can be more precise when choosing the space of filtered solutions. Indeed, it is sufficient to exclude

[25]This argument can be easily adapted to the case where the numerical approximation scheme is discrete in both space and time by taking discrete Fourier transforms in both variables.

by filtering the rays that do not propagate at all to guarantee the existence of a minimal velocity of propagation (see Figure 13 above).[26]

All the results we have presented in this section have their counterpart in the context of controllability which are close analogues of those developed previously in the $!D$ case. As far as we know, the 2D counterpart of the 1D positive result in [163], showing that initial data involving a finite number of Fourier components are uniformly controllable as $h \to 0$, has not been proved (see open problem # 7).

3.7 Other remedies for high frequency pathologies

In the previous sections we have described the high frequency spurious oscillations that arise in finite difference space semi-discretizations of the wave equation and how they produce divergence of the controls as the mesh size tends to zero. We have also shown that there is a remedy for this, which consists in filtering the high frequencies by truncating the Fourier series. However, this method, which is natural from a theoretical point of view, can be hard to implement in numerical simulations. Indeed, solving the semi-discrete system provides the nodal values of the solution. One then needs to compute its Fourier coefficients and, once this is done, to recalculate the nodal values of the filtered/truncated solution. Therefore, it is convenient to explore other ways of avoiding these high frequency pathologies that do not require going back and forth from the physical space to the frequency one. Here we shall briefly discuss other cures that have been proposed in the literature.

3.7.1 Tychonoff regularization

Glowinski et al. in [98] proposed a Tychonoff regularization technique that allows one to recover the uniform (with respect to the mesh size) coercivity of the functional that one needs to minimize to get the controls in the HUM approach. The method was tested to be efficient in numerical experiments. The convergence of the argument has not been proved so far, as far as we

[26] Roughly speaking, this suffices for the observability inequality to hold uniformly in h for a sufficiently large time [157], [158]. This ray approach makes it possible to obtain the optimal uniform observability time depending on the class of filtered solutions under consideration. The optimal time is simply that needed by all characteristic rays entering in the class of filtered solutions to reach the controlled region. It is in fact the discrete version of the Geometric Control Condition (GCC) for the continuous wave equation. Moreover, if the filtering is done so that the wavelength of the solutions under consideration is of an order strictly less than h, then one recovers the classical observability result for the constant coefficient continuous wave equation with the optimal observability time.

know. Here we give a sketch of proof of convergence in the particular case under consideration.

Let us recall that the lack of uniform observability makes the functionals (3.71) not uniformly coercive, as we mentioned in section 3.5.5. As a consequence of this, for some initial data, the controls v_h diverge as $h \to 0$. In order to avoid this lack of uniform coercivity, the functional J_h can be reinforced by means of a Tychonoff regularization procedure.[27] Consider the new functional

$$J_h^*((u_j^0, u_j^1)_{j=1,\dots,N}) = \frac{1}{2}\int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt + h^3 \sum_{j=0}^{N} \int_0^T (\frac{u'_{j+1} - u'_j}{h})^2 dt + + h\sum_{j=1}^{N} y_j^0 u_j^1 - h\sum_{j=1}^{N} y_j^1 u_j^0. \tag{3.104}$$

This functional is coercive when $T > 2$ and, more importantly, its coercivity is uniform in h. This is a consequence of the following observability inequality (see [214]):

$$E_h(0) \leq C(T)\Big[\int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt + h^3 \sum_{j=0}^{N} \int_0^T \left(\frac{u'_{j+1} - u'_j}{h}\right)^2 dt\Big]. \tag{3.105}$$

This inequality holds for all $T > 2$ for a suitable $C(T) > 0$ which is independent of h and of the solution of the semi-discrete problem (3.35) under consideration.

Note that in (3.105) we have the extra term

$$h^3 \sum_{j=0}^{N} \int_0^T \left(\frac{u'_{j+1} - u'_j}{h}\right)^2 dt, \tag{3.106}$$

which has also been used in the regularization of the functional J_h^* in (3.104). By inspection of the solutions of (3.35) in separated variables it is easy to understand why this added term is a suitable one to reestablish the uniform observability property. Indeed, consider the solution of the semi-discrete system $u = \exp(\pm i\sqrt{\lambda_j}t)w_j$. The extra term we have added is of the order of $h^2\lambda_j E_h(0)$. Obviously this term is negligible as $h \to 0$ for the low frequency solutions (for j fixed), but becomes relevant for the high frequency ones when $\lambda_j \sim 1/h^2$. Accordingly, when inequality (3.40) fails, i.e. for the high frequency solutions, the extra term in (3.105) reestablishes the uniform character of the

[27] This functional is a variant of the one proposed in [98] where the added term was $h^2||(\vec{u}^0, \vec{u}^1)||^2_{H^2\times H^1}$ instead of $h^3 \sum_{j=0}^{N} \int_0^T \left(\frac{u'_{j+1}-u'_j}{h}\right)^2 dt$. Both terms have the same scales, so that both are negligible at low frequencies but are of the order of the energy for the high ones. The one introduced in (3.104) arises naturally in view of (3.105).

estimate with respect to h. It is important to underline that both terms are needed for (3.105) to hold. Indeed, (3.106) by itself does not suffice since its contribution vanishes as $h \to 0$ for the low frequency solutions.

As we said above, this uniform observability inequality guarantees the uniform boundedness of the minima of J_h^* and the corresponding controls. But there is an important price to pay. The control that J_h^* yields is not only at the boundary but also distributed everywhere in the interior of the domain. The corresponding control system reads as follows:

$$\begin{cases} y_j'' - \frac{1}{h^2}\left[y_{j+1} + y_{j-1} - 2y_j\right] = h^2 g_{h,j}', & 0 < t < T,\ j = 1, \dots, N \\ y_0(0,t) = 0;\ y_{N+1}(1,t) = v_h(t), & 0 < t < T \\ y_j(0) = y_j^0,\ y_j'(0) = y_j^1, & j = 1, \dots, N. \end{cases} \tag{3.107}$$

And the controlled state satisfies $\vec{y}_h(T) \equiv \vec{y}'_h(T) \equiv 0$. In this case, roughly speaking, when the initial data are fixed independently of h (for instance we consider initial data in $L^2(0,1) \times H^{-1}(0,1)$ and we choose those in (3.107) as the corresponding Fourier truncation) then there exist controls $v_h \in L^2(0,T)$ and g_h such that the solution of (3.107) reaches equilibrium at time T with the following uniform bounds:

$$v_h \ \text{ is uniformly bounded in } \ L^2(0,T), \tag{3.108}$$

$$||\overrightarrow{(A_h)^{-1/2} g_h}||_h \ \text{ is uniformly bounded in } \ L^2(0,T) \tag{3.109}$$

where A_h is the matrix in (3.37), and $||\cdot||_h$ stands for the standard euclidean norm

$$||\vec{f_h}||_h = \left[h \sum_{j=1}^{N} |f_{h,j}|^2 \right]^{1/2}. \tag{3.110}$$

These bounds on the controls can be obtained directly form the coercivity property of the functional J_h^* we minimize which is a consequence of the uniform observability inequality (3.105). The role that the two controls play is of different nature: The internal control $h^2 g_h'$ takes care of the high frequency spurious oscillations, and the boundary control deals with the low frequency components. In fact, it can be shown that, as $h \to 0$, the boundary control v_h converges to the control v of (3.11) in $L^2(0,T)$. In this sense, the limit of the control system (3.107) is the boundary control problem for the wave equation. To better understand this fact it is important to observe that, due to the h^2 multiplicative factor on the internal control, its effect vanishes in the limit. Indeed, in view of the uniform bound (3.109), roughly speaking,[28] the internal control is of the order of h^2 in the space $H^{-1}(0,T;H^{-1}(0,1))$ and therefore, tends to zero in the distributional sense. The fact that the natural

[28]To make this more precise we should introduce Sobolev spaces of negative order at the discrete level as in (3.74). This can be done using Fourier series representations or extension operators from the discrete grid to the continuous space variable.

space for the internal control is $H^{-1}(0,T;H^{-1}(0,1))$ comes from the nature of the regularizing term introduced in the functional J_h^*. Indeed, its continuous counterpart is

$$\int_0^T \int_0^1 |\nabla u_t|^2 dxdt$$

and it can be seen that, by duality, it produces controls of the form $\partial_t \partial_x(f)$ with $f \in L^2((0,1)\times(0,T))$. The discrete internal control reproduces this structure.

It is also easy to see that the control $h^2 g'_{h,j}$ is bounded in L^2 with respect to both space and time. This is due to two facts: a) the norm of the operator $(A_h)^{1/2}$ is of order $1/h$, and b) taking one time derivative produces multiplicative factors of order $\sqrt{\lambda}$ for the solutions in separated variables. Since the maximum of the square roots of the eigenvalues at the discrete level is of order $1/h$, this yields a contribution of order $1/h$ too. These two contributions are balanced by the multiplicative factor h^2. Now recall that the natural space for the controlled trajectories is $L^\infty(0,T;L^2(0,1)) \cap W^{1,\infty}(0,T;H^{-1}(0,1))$ at the continuous level, with the corresponding counterpart for the discrete one. However, the right-hand side terms in L^2 for the wave equation produces finite energy solutions in $L^\infty(0,T;H^1(0,1)) \cap W^{1,\infty}(0,T;L^2(0,1))$. Thus, the added internal control only produces a compact correction on the solution at the level of the space $L^\infty(0,T;L^2(0,1)) \cap W^{1,\infty}(0,T;H^{-1}(0,1))$. As a consequence of this one can show, for instance, that, using only boundary controls, one can reach states at time T that weakly (resp. strongly) converge to zero as $h \to 0$ in $H^1(0,1) \times L^2(0,1)$ (resp. $L^2(0,1) \times H^{-1}(0,1)$).

Summarizing, we may say that a Tychonoff regularization procedure may allow controlling uniformly the semi-discrete system at the price of adding an extra internal control but in such a way that the boundary component of the controls converge to the boundary control for the continuous wave equation. Consequently, in practice, one can ignore the internal control this procedure gives and only keep the boundary one that, even though it does not exactly control the numerical approximation scheme it does converge to the right control of the wave equation. Thus, the method is efficient for computing approximations of the boundary control for the wave equation.

3.7.2 A two-grid algorithm

Glowinski and Li in [97] introduced a two-grid algorithm that also makes it possible to compute efficiently the control of the continuous model. The method was further developed by Glowinski in [95].

The relevance and impact of using two grids can be easily understood in view of the analysis above of the 1D semi-discrete model. In section 3.5 we have seen that that all the eigenvalues of the semi-discrete system satisfy $\lambda \leq 4/h^2$. We have also seen that the observability inequality becomes uniform when one considers solutions involving eigenvectors corresponding to eigenvalues $\lambda \leq 4\gamma/h^2$, with $\gamma < 1$. Glowinski's algorithm is based on the idea of using two grids: one with step size h and a coarser one of size $2h$. In the coarser mesh the eigenvalues obey the sharp bound $\lambda \leq 1/h^2$. Thus, the oscillations in the coarse mesh that correspond to the largest eigenvalues $\lambda \sim 1/h^2$, in the finer mesh are associated to eigenvalues in the class of filtered solutions with parameter $\gamma = 1/2$. Then this corresponds to a situation where the observability inequality is uniform for $T > 2/\cos(\pi/8)$. Note however that, once again, the time needed for this procedure to work is greater than the minimal control time for the wave equation.

This explains the efficiency of the two-grid algorithm for computing the control of the continuous wave equation.

This method was introduced by Glowinski [95] in the context of the full finite difference and finite element discretizations in 2D. It was then further developed in the framework of finite differences by M. Asch and G. Lebeau in [4], where the GCC for the wave equation in different geometries was tested numerically.

The convergence of this method has recently been proved rigorously in [174] for finite difference and finite element semi-discrete approximation in one space dimension.

In practice, the 2-grid algorithm works as follows: The time T needs to be larger than 4, twice the control time for the wave equation. This may be predicted by the analysis of the corresponding dispersion diagram. One then minimizes J_h over the subspace of data obtained by interpolation over the coarse mesh. This gives a sequence of bounded (as h tends to zero) controls. The controls, for h fixed, only give a result of partial controllability in the sense that only a projection of solutions of the controlled system over the coarse grid vanishes. But the limit of these controls as h tends to zero is an exact control for the wave equation. Consequently, the 2-grid algorithm is a good method for getting numerical approximations of the control of the wave equation.

The key point in the proof of this result is an uniform (with respect to h) observability inequality for the adjoint system over the subspace of data interpolated from the coarse grid.

3.7.3 Mixed finite elements

Let us now discuss a different approach that is somewhat simpler than the previous ones. It consists in using mixed finite element methods rather than finite differences or standard finite elements, which require some filtering, Tychonoff regularization or multigrid techniques, as we have shown.

First of all, it is important to underline that the analysis we have developed in section 3.5 for the finite difference space semi-discretization of the 1D wave equation can be carried out with minor changes for finite element semi-discretizations as well. In particular, due to the high frequency spurious oscillations, uniform observability does not hold [113]. It is thus natural to consider mixed finite element (m.f.e.) methods. This idea was introduced by Banks et al. [12] in the context of boundary stabilization of the wave equation. Here we adapt that approach to the analysis of controllability and observability. A variant of this method was introduced in [96].

The starting point is writing the adjoint wave equation (3.8) in the system form

$$u_t = v, \quad v_t = u_{xx}.$$

We now use two different Galerkin basis for the approximation of u and v. Since u lies in H_0^1, we use classical piecewise linear finite elements, and for v piecewise constant ones.

In these bases, and after some work which is needed to handle the fact that the left- and right-hand side terms of the equations in this system do not have the same regularity, one is led to the following semi-discrete system:

$$\begin{cases} \frac{1}{4}\left[u''_{j+1} + u''_{j-1} + 2u''_j\right] = \frac{1}{h^2}\left[u_{j+1} + u_{j-1} - 2u_j\right], & 0 < t < T,\ j = 1, \ldots, N \\ u_j(t) = 0, & j = 0,\ N+1 \\ u_j(0) = u_j^0,\ u'_j(0) = u_j^1, & j = 1, \ldots, N. \end{cases} \tag{3.111}$$

This system is a good approximation of the wave equation and converges in classical terms. Moreover, the spectrum of the mass and stiffness matrices involved in this scheme can be computed explicitly and the eigenvectors are those of (3.43), i.e. the restriction of the sinusoidal eigenfunctions of the laplacian to the mesh points. The eigenvalues are now

$$\lambda_k = \frac{4}{h^2}\tan^2(k\pi h/2), \quad k = 1, ..., N. \tag{3.112}$$

For this spectrum the gap between the square roots of consecutive eigenvalues is uniformly bounded from below, and in fact tends to infinity for the highest frequencies as $h \to 0$ (Figure 14). According to this and applying Ingham's inequality, the uniform observability property is proved (see [32]). Note however that one can not expect the inequality (3.40) to hold since, it is not even uniform for the eigenvectors. One gets instead that, for all $T > 2$, there exists $C(T) > 0$ such that

$$E_h(0) \le C_h(T) \int_0^T \left[\left|\frac{u_N(t)}{h}\right|^2 + h^2\left|\frac{u'_N(t)}{h^2}\right|^2\right] dt \tag{3.113}$$

for every solution of (3.111) and for all $h > 0$. As a consequence, the corresponding systems are also uniformly controllable and the controls converge as $h \to 0$. In [32] similar results have also been proved for a suitable $2D$ mixed finite element scheme.

One of the drawbacks of this method is that the CFL stability condition that is required when dealing with fully discrete approximations based on this method is much stronger than for classical finite difference or finite element methods because of the sparsity of the spectrum. In this case, for instance, when considering centered time discretization, one requires

$$\Delta t \leq c(\Delta x)^2, \tag{3.114}$$

in opposition to the classical stability condition $\Delta t \leq c\Delta x$ one gets for classical schemes. Thus, applying this method in numerical simulations requires the use of implicit time-discretization schemes and this makes the method to be computationally expensive.

Recently, A. Munch in [171] has introduced a variant of this scheme for which the dispersion diagram behaves better in the sense that there is less dispersion for the highest frequencies. Accordingly, the stability condition is significantly improved with respect to (3.114). This idea of correcting the dispersion diagram by means of adding higher order terms in the approximation of the scheme has also be used before by S. Krenk [127], for instance.

3.8 Other models

In this article we have seen that most numerical schemes for the wave equation produce high frequency pathologies that make the boundary observability inequalities to be nonuniform and produce divergence of the controls of the semi-discrete or discrete systems as the mesh size tends to zero. We have also seen some possible remedies.

However, other equations behave much better due to diffusive or dispersive effects. As we shall see in the present section, these high frequency pathologies do not arise when dealing with the 1D heat and beam equation.

3.8.1 Finite difference space semi-discretizations of the heat equation

The convergence of numerical schemes for control problems associated with parabolic equations has been extensively studied in the literature ([125], [193], [218],...). But this has been done mainly in the context of optimal control and very little is known about the controllability issues that we address now.

Let us consider the following 1D heat equation with control acting at the boundary point $x = L$:

$$\begin{cases} y_t - y_{xx} = 0, & 0 < x < L,\, 0 < t < T \\ y(0,t) = 0,\, y(L,t) = v(t), & 0 < t < T \\ y(x,0) = y^0(x), & 0 < x < L. \end{cases} \tag{3.115}$$

This is the so called boundary control problem. It is by now well known that (3.115) is null controllable in any time $T > 0$ (see for instance D.L. Russell [194], [195]). To be more precise, the following holds: *For any $T > 0$, and $y^0 \in L^2(0,L)$ there exists a control $v \in L^2(0,T)$ such that the solution y of* (3.115) *satisfies*

$$y(x,T) \equiv 0 \text{ in } (0,L). \tag{3.116}$$

This null controllability result is equivalent to a suitable observability inequality for the adjoint system:

$$\begin{cases} u_t + u_{xx} = 0, & 0 < x < L,\, 0 < t < T, \\ u(0,t) = u(L,t) = 0, & 0 < t < T \\ u(x,T) = u^0(x), & 0 < x < L. \end{cases} \tag{3.117}$$

Note that, in this case, due to the time irreversibility of the state equation and its adjoint, in order to guarantee that the latter is well-posed, we take the initial conditions at the final time $t = T$. The corresponding observability inequality is as follows: *For any $T > 0$ there exists $C(T) > 0$ such that*

$$\int_0^L u^2(x,0)dx \le C \int_0^T |u_x(L,t)|^2 \, dt \tag{3.118}$$

holds for every solution of (3.117).[29]

Let us consider now semi-discrete versions of (3.115) and (3.117):

$$\begin{cases} y_j' - \frac{1}{h^2}\left[y_{j+1} + y_{j-1} - 2y_j\right] = 0, & 0 < t < T, \quad j = 1, \dots, N \\ y_0 = 0,\, y_{N+1} = v, & 0 < t < T \\ y_j(0) = y_j^0, & j = 1, \dots, N; \end{cases} \tag{3.119}$$

$$\begin{cases} u_j' + \frac{1}{h^2}\left[u_{j+1} + u_{j-1} - 2u_j\right] = 0, & 0 < t < T,\, j = 1, \dots, N \\ u_0 = u_{N+1} = 0, & 0 < t < T \\ u_j(T) = u_j^0, & j = 1, \dots, N. \end{cases} \tag{3.120}$$

According to the Kalman criterion for controllability in section 2.20, for any $h > 0$ and for all time $T > 0$ system (3.119) is controllable and (3.120) observable. In fact, in this case, in contrast with the results we have described for the wave equation, these properties hold uniformly as $h \to 0$. More precisely, the following results hold:

[29] This inequality has been greatly generalized to heat equations with potentials in several space dimensions, with explicit observability constants depending on the potentials, etc. (see for instance [90], [83])

Theorem 3.8.1 [155] *For any $T>0$ there exists a positive constant $C(T)>0$ such that*

$$h\sum_{j=1}^{N}|u_j(0)|^2 \le C\int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt \tag{3.121}$$

holds for any solution of (3.120) *and any $h>0$.*

Theorem 3.8.2 [155] *For any $T>0$ and $\{y_1^0,\ldots,y_N^0\}$ there exists a control $v\in L^2(0,T)$ such that the solution of* (3.119) *satisfies*

$$y_j(T)=0,\ j=1,\ldots,N. \tag{3.122}$$

Moreover, there exists a constant $C(T)>0$, independent of $h>0$, such that

$$\| v \|^2_{L^2(0,T)} \le Ch\sum_{j=1}^{N}\left|y_j^0\right|^2. \tag{3.123}$$

These results were proved in [155] using Fourier series and a classical result on the sums of real exponentials (see for instance Fattorini-Russell [75]) that plays the role of Ingham's inequality in the context of parabolic equations.

Let us recall briefly: Given $\xi>0$ and a decreasing function $M:(0,\infty)\to\mathbf{N}$ such that $M(\delta)\to\infty$ as $\delta\to 0$, we introduce the class $\mathcal{L}(\xi,M)$ of increasing sequences of positive real numbers $\{\mu_j\}_{j\ge 1}$ such that

$$\mu_{j+1}-\mu_j\ge\xi>0, \quad \forall j\ge 1, \qquad \sum_{k\ge M(\delta)}\mu_k^{-1}\le\delta, \qquad \forall\delta>0.$$

The following holds:

Proposition 3.8.1 *Given a class of sequences $\mathcal{L}(\xi,M)$ and $T>0$ there exists a constant $C>0$ (which depends on ξ, M and T) such that*

$$\int_0^T\left|\sum_{k=1}^{\infty}a_k e^{-\mu_k t}\right|^2 dt \ge \frac{C}{\left(\sum_{k\ge 1}\mu_k^{-1}\right)}\sum_{k\ge 1}\frac{|\, a_k\,|^2\, e^{-2\mu_k T}}{\mu_k} \tag{3.124}$$

for all $\{\mu_j\}\in\mathcal{L}(\xi,N)$ and all sequence $\{a_k\}\in\ell^2$.

It is easy to see[30] that the eigenvalues of the semi-discrete laplacian $\{\lambda_j^h\}_{j=1,\ldots,N}$ in (3.42) belong to one of these uniform classes $\mathcal{L}(\xi,M)$. Consequently, applying the uniform inequality (3.124) together with the Fourier

[30] Indeed, in view of the explicit form of these eigenvalues there exists $c>0$ such that $\lambda_j^h\ge cj^2$ for all $h>0$ and $j=1,...,N$. On the other hand, the uniform gap condition is also satisfied. Recall that, in the context of the wave equation, the lack of gap for the square roots of these eigenvalues was observed for the high frequencies. In particular it was found that $\sqrt{\lambda_N^h}-\sqrt{\lambda_{N-1}^h}\sim h$. But then, $\lambda_N^h-\lambda_{N-1}^h\sim(\sqrt{\lambda_N^h}-\sqrt{\lambda_{N-1}^h})(\sqrt{\lambda_N^h}+\sqrt{\lambda_{N-1}^h})\sim 1$, since $\sqrt{\lambda_N^h}+\sqrt{\lambda_{N-1}^h}\sim 1/h$. This fact describes clearly why the gap condition is fulfilled in this case.

representation of solutions of (3.120), one gets, for all $T > 0$, a uniform observability inequality of the form (3.121) for the solutions of the semi-discrete systems (3.120). There is one slight difficulty when doing this. The boundary observability property is not uniform for the high frequency eigenvectors. However, this is compensated in this case by the strong dissipative effect of the heat equation. Indeed, note that solutions of (3.120) can be written in Fourier series as

$$\vec{u}(t) = \sum_{j=1}^{N} a_j e^{-\lambda_j^h (T-t)} \vec{w}_j^h, \tag{3.125}$$

where $\{a_j\}_{j=1,\ldots,N}$ are the Fourier coefficients of the initial data of (3.120) at $t = T$. The solution at the "final"[31] time can be represented as follows:

$$\vec{u}(0) = \sum_{j=1}^{N} a_j e^{-\lambda_j^h T} \vec{w}_j^h. \tag{3.126}$$

We see in this formula that the high frequencies are damped out by an exponentially small factor that compensates for the lack of uniform boundary observability of the high frequency eigenvectors.

Once the uniform observability inequality of Theorem 3.8.1 is proved, the controls for the semi-discrete heat equation (3.119) can be easily constructed by means of the minimization method described in section 3.5.5. The fact that the observability inequality is uniform implies the uniform bound (3.123) on the controls. The null controls for the semi-discrete equation (3.120) one obtains in this way are such that, as $h \to 0$, they tend to the null control for the continuous heat equation (3.115) (see [155]).

3.8.2 The beam equation

In a recent work by L. León and the author [137] the problem of boundary controllability of finite difference space semi-discretizations of the beam equation $y_{tt} + y_{xxxx} = 0$ was addressed. This model has important differences from the wave equation even in the continuous case. First of all, at the continuous level, it turns out that the gap between consecutive eigenfrequencies tends to infinity. For instance, with the boundary conditions $y = y_{xx} = 0, \quad x = 0, \pi$, the solution admits the Fourier representation formula

$$y(x,t) = \sum_{k \in \mathbf{Z}} a_k e^{i\lambda_k t} \sin(kx),$$

[31] Note that $t = 0$ is the final time for the adjoint equation (3.120), which is solved backwards from $t = T$ to $t = 0$.

where $\lambda_k = \text{sgn}(k)k^2$. Obviously, the gap between consecutive eigenvalues is uniformly bounded from below. More precisely,

$$\lambda_{k+1} - \lambda_k = 2k + 1 \to \infty, \quad \text{as } k \to \infty.$$

This allows use to apply a variant of Ingham's inequality for an arbitrarily small control time $T > 0$ (see [165]).[32] As a consequence, boundary exact controllability holds for any $T > 0$ too.

When considering finite difference space semi-discretizations things are better than for the wave equation too. Indeed, as it is proved in [137], roughly speaking, the asymptotic gap[33] also tends to infinity as $k \to \infty$, uniformly on the parameter h. This allows proving the uniform observability and controllability (as $h \to 0$) of the finite difference semi-discretizations. However, as we mentioned in section 3.5, due to the bad approximation that finite differences provide at the level of observing the high frequency eigenfunctions, the control has to be split in two parts. The main part that strongly converges to the control of the continuous equation in the sharp $L^2(0,T)$ space and the oscillatory one that converges to zero in a weaker space $H^{-1}(0,T)$. Thus, in the context of the beam equation, with the most classical finite difference semi-discretization, we get what we got for the wave equation with mixed finite elements. This fact was further explained by means of tools related with discrete Wigner measures in [157] , [158].

The same results apply for the Schrödinger equation.

Note however that, as we shall see in open problem # 3 below, the situation is more complex in several space dimensions in which the dissipative and dispersive effects added by the heat and Schrödinger equations do not suffice.

3.9 Further comments and open problems

3.9.1 Further comments

a) We have considered finite difference space semi-discretizations of the wave equation. We have addressed the problem of boundary observability and, more

[32] Although in the classical Ingham inequality the gap between consecutive eigenfrequencies is assumed to be uniformly bounded from below for all indices k, in fact, in order for Ingham inequality to be true, it is sufficient to assume that all eigenfrequencies are distinct and that there is an asymptotic gap as $k \to \infty$. We refer to [165] for a precise statement where explicit estimates of the constants arising in the inequalities are given.

[33] In fact one needs to be more careful since, for $h > 0$ fixed, the gap between consecutive eigenfrequencies is not increasing. Indeed, in order to guarantee that the gap is asymptotically larger than any constant $L > 0$ one has to filter not only a finite number of low frequencies but also the highest ones. However, the methods and results in [165] apply in this context too (see [137]).

precisely, the problem of whether the observability estimates are uniform when the mesh size tends to zero.

We have proved the following results:

- Uniform observability does not hold for any time T.
- Uniform observability holds if the time T is large enough provided we filter appropriately the high frequencies.

We have also mentioned the main consequences concerning controllability. In this article we have collected the existing work in this subject, to the best of our knowledge.

b) The problem of controllability has been addressed. Nevertheless, similar developments could be carried out, with the same conclusions, in the context of stabilization. The connections between controllability and stabilization are well known (see for instance [194], [226]).

In the context of the wave equation, it is well known that the GCC suffices for stabilization and more precisely to guarantee the uniform exponential decay of solutions when a damping term, supported in the control region, is added to the system. More precisely, when the subdomain ω satisfies the GCC the solutions of the damped wave equation

$$y_{tt} - \Delta y + 1_\omega y_t = 0$$

with homogeneous Dirichlet boundary conditions are known to decay exponentially in the energy space. In other words, there exist constants $C > 0$ and $\gamma > 0$ such that

$$E(t) \leq Ce^{-\gamma t} E(0)$$

holds for every finite energy solution of the Dirichlet problem for this damped wave equation.

It is then natural to analyze whether the decay rate is uniform with respect to the mesh size for numerical discretizations. The answer is in general negative. Indeed, due to spurious high frequency oscillations, the decay rate fails to be uniform, for instance, for the classical finite difference semi-discrete approximation of the wave equation. This was established rigorously by F. Macià [157], [158] using Wigner measures. This negative result also has important consequences in many other issues related with control theory like infinite horizon control problems, Riccati equations for the optimal stabilizing feedback ([184]), etc. But these issues have not been studied so far (see open problem 8 below).

We shall simply mention here that, even if the most natural semi-discretization fails to be uniformly exponentially stable, the uniformity of the exponential decay rate can be reestablished if we add an internal viscous damping term to the equation (see [214], [170]). This is closely related to the enhanced observability inequality (3.105) in which the extra internal viscous term added in the observed quantity guarantees the observability constant to be uniform. We shall return to this issue in open problem # 5 below.

c) According to the analysis above it could seem that most control problems behave badly with respect to numerical approximations. However, this is no longer true for classical optimal control problems (LQR, finite-time-horizon optimal control,...) or even for approximate controllability problems in which the objective is to drive the solution to any state of size less than a given ε, as shown in section 3.5.7. This is even true for more sophisticated problems arising in the context of homogenization (see, for instance [234] and [38]). The ε parameter arising in homogenization theory to denote the small period of the oscillating coefficient and the h parameter describing the mesh size in numerical approximation problems, play a similar role. The problem is more difficult to handle in the context of homogenization because computations are less explicit than in numerical problems but the behavior is basically the same: high frequency solutions have to be filtered in order to avoid zero group velocity. However, surprisingly, the numerical approximation problems only became understood once the main features of the behavior of controllability problems in homogenization had been discovered [33].

3.9.2 Open problems

1. *Semilinear equations.* The questions we have addressed in this article are completely open in the case of the semilinear heat and wave equations with globally Lipschitz nonlinearities. In the context of continuous models there are a number of fixed point techniques that allow one to extend the results of controllability of linear waves and heat equations to semilinear equations with moderate nonlinearities (globally Lipschitz ones, for instance [241]). These techniques need to be combined with Carleman or multiplier inequalities ([90], [227]) allowing one to estimate the dependence of the observability constants on the potential of the linearized equation. However, the analysis we have pursued in this article relies very much on the Fourier decomposition of solutions, which does not suffice to obtain explicit estimates on the observability constants in terms of the potential of the equation. Thus, extending the positive results of uniform controllability presented in this paper (by means of filtering, mixed finite elements, multi-grid techniques, etc.) to the numerical approximation schemes of semilinear PDE is a completely open subject of research.

2. *Wavelets and spectral methods.* In the previous sections we have described how filtering of high frequencies can be used to get uniform observ-

ability and controllability results. It would be interesting to develop the same analysis in the context of numerical schemes based on wavelet analysis in which the filtering of high frequencies can be easy to implement in practice.

Matache, Negreanu and Schwab in [161] have developed a wavelet based algorithm which is inspired in the multi-grid ideas described in section 3.7.2. Their method is extremely efficient in numerical experiments giving in practice the optimal control time. But a further theoretical study of this method is still to be done.

Spectral methods are also very natural to be considered in this setting. Obviously, if the method is based on the exact eigenfunctions of the wave equation (which, in practice, are only available for constant coefficient $1D$-problems) then the convergence is guaranteed (see [13]). On the other hand, in a recent paper [22] it has been shown that the superconvergence properties that the spectral methods provide may help at the level of controlling the wave equation in the sense that less filtering of high frequencies is required. A complete investigation of the use of spectral methods for the observation and control of the wave equation remains to be carried out.

3. *Discrete unique-continuation.* In the context of the continuous wave equation we have seen that the observability inequality and, consequently, exact controllability holds if and only if the domain where the control is being applied satisfies the GCC. However, very often in practice, it is natural to consider controls that are supported in a small subdomain. In those cases, when the control time is large enough, one obtains approximate controllability results as discussed in section 3.5.7. Approximate controllability is equivalent to a uniqueness or unique-continuation property for the adjoint system[34]: *If the solution u of* (3.31) *vanishes in $\omega \times (0,T)$, then it vanishes everywhere.* We emphasize that this property holds whatever the open subset ω of Ω may be, provided T is large enough, by Holmgren's Uniqueness Theorem.

One could expect the same result to hold also for semi-discrete and discrete equations. But the corresponding theory has not been developed. The following example due to O. Kavian [122] shows that, at the discrete level, new phenomena arise. It concerns the eigenvalue problem (3.94) for the 5-point finite difference scheme for the laplacian in the square. A grid function taking alternating values ± 1 along a diagonal and vanishing everywhere else is an eigenvector of (3.94) with eigenvalue $\lambda = 4/h^2$. According to this example, even at the level of the elliptic equation, the domain ω where the solution vanishes has to be assumed to be large enough to guarantee the unique continuation property. In [45] it was proved that when ω is a "neighborhood of one side of the boundary", then unique continuation holds for the discrete Dirichlet problem in any discrete domain. Here by a "neighborhood of one side of the

[34] This has been proved in detail in section 2.20 in the context of finite-simensional systems but is also true in the context of PDE ([142, 143]).

boundary" we refer to the nodes of the mesh that are located immediately to one side of the boundary nodal points (left, right, top or bottom). Indeed, if one knows that the solution vanishes at the nodes immediately to one side of the boundary, taking into account that they vanish in the boundary too, the 5-point numerical scheme allows propagating the information and showing that the solution vanishes at all nodal points of the whole domain.

Getting optimal geometric conditions on the set ω depending on the domain Ω where the equation holds, the discrete equation itself, the boundary conditions and, possibly, the frequency of oscillation of the solution for the unique continuation property to hold at the discrete level is an interesting and widely open subject of research.

One of the main tools for dealing with unique continuation properties of PDE are the so called *Carleman inequalities*. It would be interesting to develop the corresponding discrete theory.

4. *Hybrid hyperbolic-parabolic equations.* We have discussed discretizations of the wave equation and have seen that, for most schemes, there are high frequency spurious oscillations that need to be filtered to guarantee uniform observability and controllability. However, we have seen that the situation is much better for the 1D heat equation. Nevertheless, it should also be taken into account that, according to the counterexample above showing that unique continuation may fail for the 2D eigenvalue problem, one can not expect the uniform observability property to hold uniformly for the semi-discretized 2D heat equation for any control sub-domain. Understanding the need of filtering of high frequencies in parabolic equations is also an interesting open problem, very closely related to the unique continuation problem above.

It would also be interesting to analyze mixed models involving wave and heat components. There are two examples of such systems: a) Systems of thermoelasticity and b) Models for fluid-structure interaction (see [135] for the system of thermoelasticity and [229], [230] and [235] for the analysis of a system coupling the wave and the heat equation along an interface.) In particular, it would be interesting to analyze to which extent the presence of the parabolic component makes unnecessary the filtering of the high frequencies for the uniform observability property to hold for space or space-time discretizations.

5. *Viscous numerical damping.* In [214] we analyzed finite difference semi-discretizations of the damped wave equation

$$u_{tt} - u_{xx} + \chi_\omega u_t = 0, \tag{3.127}$$

where χ_ω denotes the characteristic function of the set ω where the damping term is effective. In particular we analyzed the following semi-discrete approximation in which an extra numerical viscous damping term is present:

$$\begin{cases} u_j'' - \frac{1}{h^2}\left[u_{j+1} + u_{j-1} - 2u_j\right] - & \left[u_{j+1}' + u_{j-1}' - 2u_j'\right] - u_j'\chi_\omega = 0, \\ & 0 < t < T,\ j = 1, \ldots, N \\ u_j(t) = 0, & 0 < t < T,\ j = 0,\ N+1 \\ u_j(0) = u_j^0, \quad u_j^1(0) = u_j^1, & j = 1, \ldots, N. \end{cases} \tag{3.128}$$

It was proved that this type of scheme preserves the uniform stabilization properties of the wave equation (3.127). To be more precise we recall that solutions of the 1D wave equation (3.127) in a bounded interval with Dirichlet boundary conditions decay exponentially uniformly as $t \to \infty$ when a damping term as above is added, ω being an open non-empty subinterval (see [184]). Using the numerical scheme above, this exponential decay property is kept with a uniform rate as h tends to zero. The extra numerical damping that this scheme introduces adding the term $\left[u_{j+1}' + u_{j-1}' - 2u_j'\right]$ damps out the high frequency spurious oscillations that the classical finite difference discretization scheme introduces and that produce a lack of uniform exponential decay in the presence of damping.

The problem of whether this numerical scheme is uniformly observable or controllable as h tends to zero is an interesting open problem.

Note that the system above, in the absence of the damping term lozalized in ω, can be written in the vector form

$$\vec{u}'' + A_h\vec{u} + h^2 A_h\vec{u}' = 0. \tag{3.129}$$

Here $\vec{u}$ stands, as usual, for the vector unknown $(u_1, ..., u_N)^T$ and A_h for the tridiagonal matrix associated with the finite difference approximation of the laplacian (3.37). In this form it is clear that the scheme above corresponds to a viscous approximation of the wave equation. Indeed, taking into account that A_h provides an approximation of $-\partial_x^2$, the presence of the extra multiplicative factor h^2 in the numerical damping term guarantees that it vanishes asymptotically as h tends to zero. This is true for the classical convergence theory but it remains to be proved for observability and controllability.

6. *Multigrid methods.* In section 3.7.2 we presented the two-grid algorithm introduced by R. Glowinski [95] and we explained heuristically why it is a remedy for high frequency spurious oscillations. In [95] the efficiency of the method was exhibited in several numerical examples and the convergence proved in [173] in $1D$. The problem of convergence is open in sevral space dimensions.

7. *Uniform control of the low frequencies.* As we mentioned in the end of section 3.5.5, the 2D counterpart of the 1D positive result in [163] showing that the initial data involving a finite number of Fourier components are uniformly

controllable as $h \to 0$ has not been proved in the literature. Such a result is very likely to hold for quite general approximation schemes and domains. But, up to now, it has only been proved in 1D for finite difference semi-discretizations. The methods involving Wigner measures developed in [157], [158] do not seem sufficient to address this issue. On the other hand, moment problems techniques, which require quite technical developments in [163] to deal with the 1D case, also seem to be hard to adapt to a more general setting. This is an interesting (and, very likely, difficult) open problem.

8. *Other control theoretical issues.* As we have mentioned above, the topics we have discussed make sense in other contexts of Control Theory. In particular, similar questions arise concerning problems of stabilization, the infinite horizon optimal control problem, the Riccati equations for optimal feedback operators or the reciprocal systems discussed by R. Curtain in [63]. The phenomena we have discussed in this paper, related to high frequency spurious oscillations, certainly affect the results we can expect in these other problems too. But the corresponding analysis has not been done.

9. *Extending the Wigner measure theory.* As we mentioned above, F. Macià in [157], [158] has developed a discrete Wigner measure theory to describe the propagation of semi-discrete and discrete waves at high frequency. However, this was done for regular grids and without taking into account boundary effects. Therefore, a lot has to be done in order to fully develop the Wigner measure tools. The notion of polarization developed in [28] remains also to be analyzed in the discrete setting.

10. *Theory of inverse problems and Optimal Design.* This paper has been devoted mainly to the property of observability and its consequences for controllability. But, as we mentioned from the beginning, most of the results we have developed have consequences in other fields. This is the case for instance for the theory of inverse problems, where one of the most classical problems is the one of reconstructing the coefficients of a given PDE in terms of boundary measurements (see [116]). Assuming that one has a positive answer to this problem in an appropriate functional setting it is natural to consider the problem of numerical approximation. Then, the following question arises: *Is solving the discrete version of the inverse problem for a discretized model an efficient way of getting a numerical approximation of the solution of the continuous inverse problem?*. Thus, as in the context of control we are analyzing whether the procedure of numerical approximation and that of solving the inverse problem commute.

According to the analysis above we can immediately say that, in general, the answer to this problem is negative. Consider for instance the wave equation

$$\begin{cases} \rho u_{tt} - u_{xx} = 0, & 0 < x < 1,\ 0 < t < T \\ u(0,t) = u(1,t) = 0, & 0 < t < T, \end{cases} \tag{3.130}$$

with a constant but unknown density $\rho > 0$. Solutions of this equation are time-periodic of period $2\sqrt{\rho}$ and this can be immediately observed on the trace of normal derivatives of solutions at either of the two boundary points $x = 0$ or $x = 1$, by inspection of the Fourier series representation of solutions of (3.130). Thus, roughly speaking, we can assert that the value of ρ can be determined by means of boundary measurements.

Let us now consider the semi-discrete version of (3.130). In this case, according to the analysis above, the solutions do not have any well-defined time-periodicity property. On the contrary, for any given values of ρ and h, (3.130) admits a whole range of solutions that travel at different group velocities, ranging from $h/\sqrt{\rho}$ (for the high frequencies) to $1/\sqrt{\rho}$ (for the low frequency ones). In particular, the high frequency numerical solutions do behave more like a solution of the wave equation with an effective density ρ/h^2. This argument shows that the mapping that allows determining the value of the constant density from boundary measurements is unstable under numerical discretization.

Of course, most of the remedies that have been introduced in this paper to avoid the failure of uniform controllability and/or observability can also be used in this context of inverse problems. But developing these ideas remains to be done.

The same can be said about optimal design problems. Indeed, in this context very little is known about the convergence of the optimal designs for the numerical discretized models towards the optimal design of the continuous models and, to a large extent, the difficulties one has to face in this context is very similar that those we addressed all along this paper. We refer to [46] for an analysis of this problem in optimal shape design for the Dirichlet laplacian.

11. *Finite versus infinite-dimensional nonlinear control.* Most of this work has been devoted to analyzing linear problems. There is still a lot to be done to understand the connections between finite-dimensional and infinite-dimensional control theory, and, in particular, concerning numerical approximations and their behavior with respect to the control property. According to the analysis above, the problem is quite complex even in the linear case. Needless to say, one expects a much higher degree of complexity in the nonlinear frame.

There are a number of examples in which the finite-dimensional versions of important nonlinear PDE have been solved from the point of view of controllability. Among them the following are worth mentioning:

a) The Galerkin approximations of the bilinear control problem for the Schrödinger equation arising in Quantum Chemistry ([185] and [220]).

b) The control of the Galerkin approximations of the Navier-Stokes equations [151].

In both cases nothing is known about the possible convergence of the controls of the finite-dimensional system to the control of a PDE as the dimension of the Galerkin subspace tends to infinity. This problem seems to be very complex. However, the degree of difficulty may be different in both cases. Indeed, in the case of the continuous Navier-Stokes and Euler equations for incompressible fluids there are a number of results in the literature indicating that they are indeed controllable ([90], [58], [59]). However, for the bilinear control of the Schrödinger equations, it is known that the reachable set is very small in general, which indicates that one can only expect very weak controllability properties. This weakness of the controllability property at the continuous level makes it even harder to address the problem of passing to the limit on the finite-dimensional Galerkin approximations as the dimension tends to infinity.

This problem is certainly one of the most relevant ones in the frame of controllability of PDE and its numerical approximations.

12. *Wave equations with irregular coefficients.* The methods we have developed do not suffice to deal with wave equations with non-smooth coefficients. However, at the continuous level, in one space dimension, observability and exact controllability hold for the wave equation with piecewise constant coefficients with a finite number of jump discontinuities (or, more generally, for BV coefficients). It would be interesting to see if the main results presented in this paper hold in this setting too. This seems to be a completely open problem. We refer to the book by G. Cohen [56] for the analysis of reflection and transmission indices for numerical schemes for wave equations with interfaces.

13. *Convergence rates.* In this article we have described several numerical methods that do provide convergence of controls. The problem of the rate of convergence has not been addressed so far. Recently important progresses have done at this respect in the context of optimal control problems for semilinear elliptic equations ([29], [30]).

Figure 3.5: Time evolution of solution (3.52) for $h = 1/61$ ($N = 60$) and $0 \leq t \leq 120$. It is clear that, according to the figure, the solution seems to exhibit a time-periodicity property with period τ of the order of $\tau \sim 50$. Note however that all solutions of the wave equation are time-periodic of period 2. In the figure it is also clear that fronts propagate in space at velocity of the order of $1/50$. This is in agreement with the prediction of the theory in the sense that high frequency wave packets travel at a group velocity of the order of h.

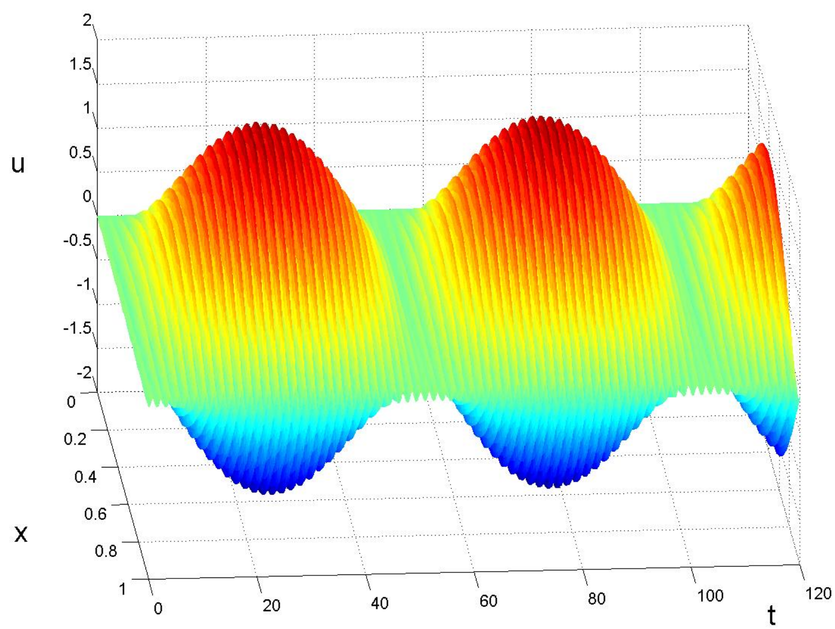

Figure 3.6: 3D view of the solution of Figure 3.5.3.

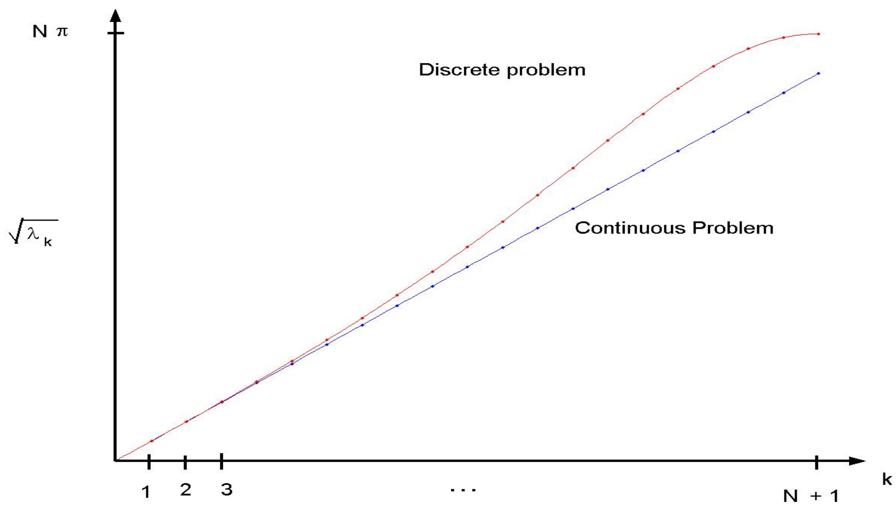

Figure 3.7: Dispersion diagram for the piecewise linear finite element space semi-discretization versus the continuous wave equation.

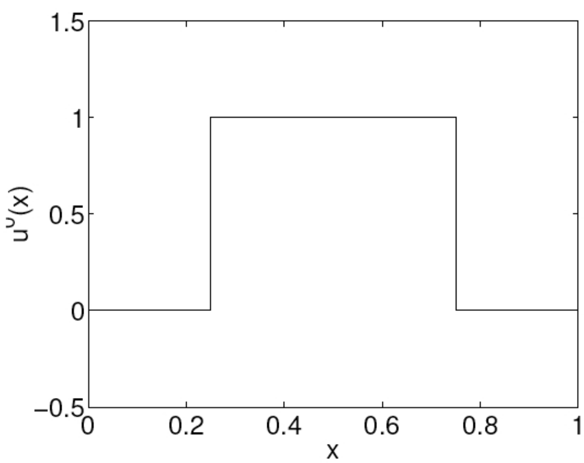

Figure 3.8: Plot of the initial datum to be controlled for the string occupying the space interval $0 < x < 1$.

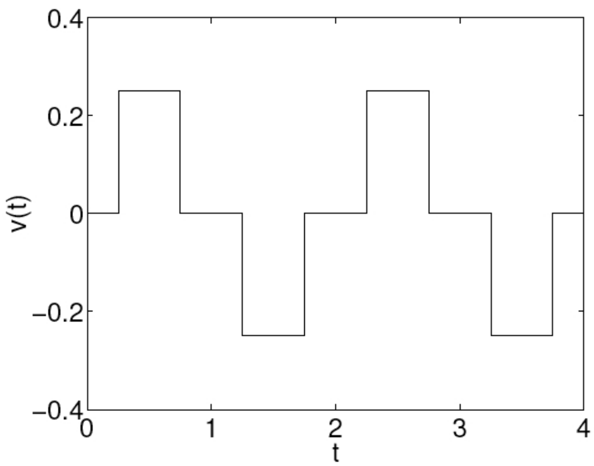

Figure 3.9: Plot of the time evolution of the exact control for the wave equation in time $T = 4$ with initial data as in Figure 8 above.

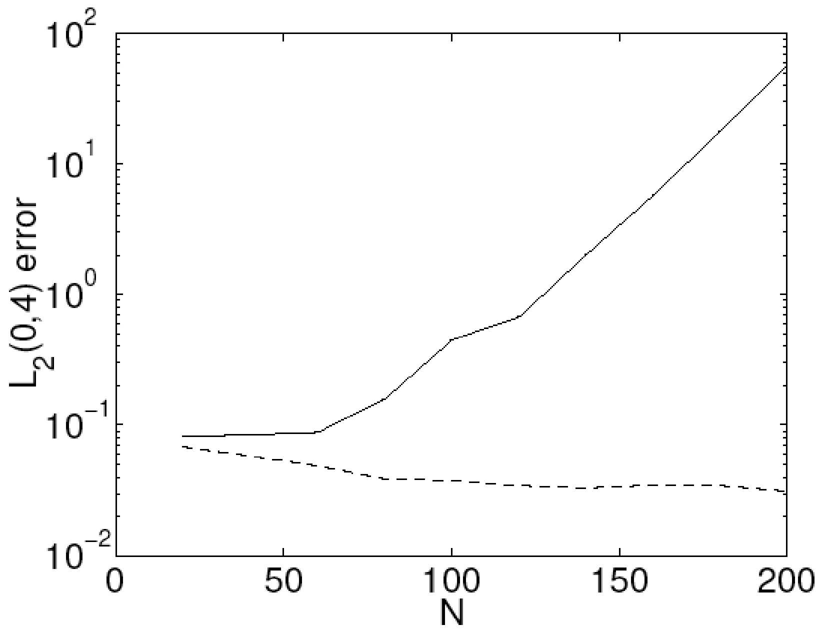

Figure 3.10: Plot, in solid line the error in the computation of the control without filtering, versus the error (in dotted line) for the discrete case with filtering parameter 0.6.

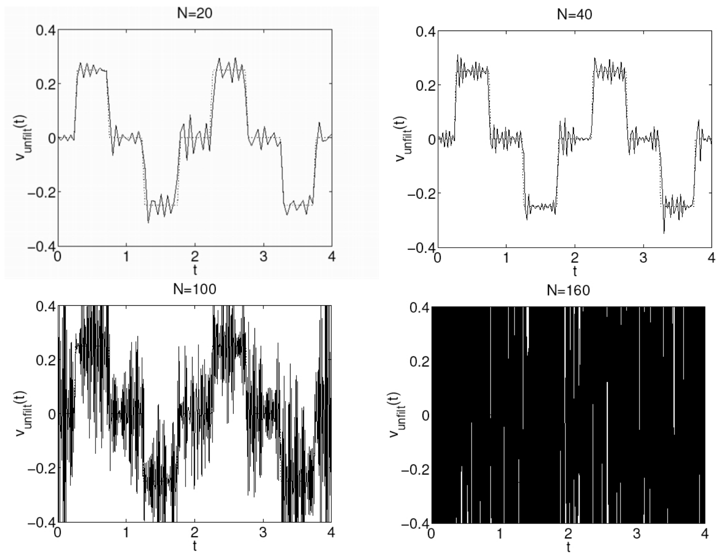

Figure 3.11: Divergent evolution of the control, in the absence of filtering, when the number N of mesh-points increases.

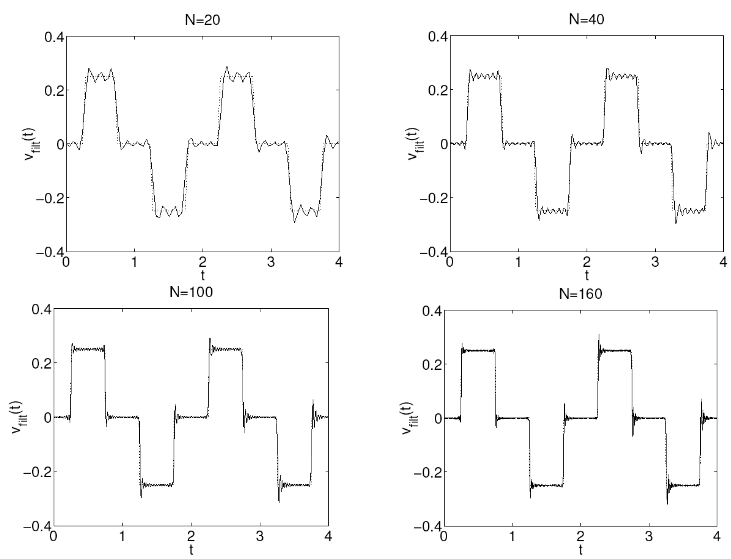

Figure 3.12: Convergent evolution of the control, with filtering parameter $= 0.6$, when the number N of mesh-points increases.

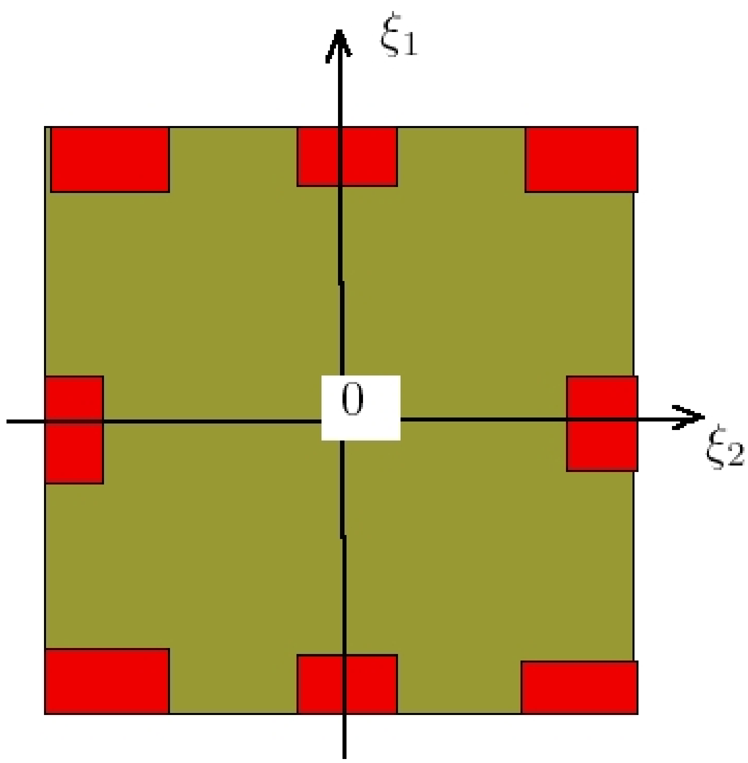

Figure 3.13: This figure represents the zones in the frequency space that need to be filtered out in order to guarantee a uniform minimal velocity of propagation of rays as $h \to 0$. When the filtering excludes the areas within the eight small neighborhoods of the distinguished points on the boundary of the frequency cell, the velocity of propagation of rays is uniform. Obviously the minimal velocity depends on the size of these patches that have been removed by filtering and, consequently, so does the observation/control time.

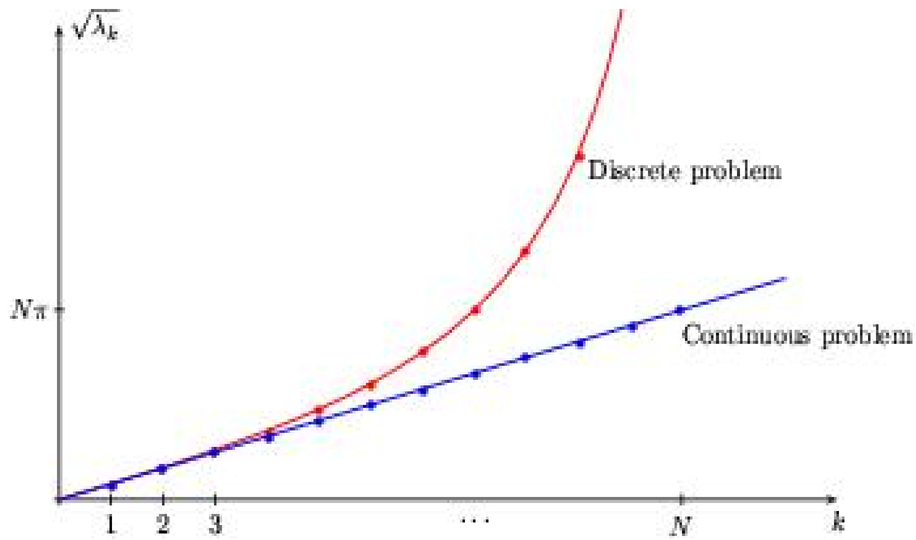

Figure 3.14: Square roots of the eigenvalues in the continuous and discrete cases with mixed finite elements (compare with Figure 5).

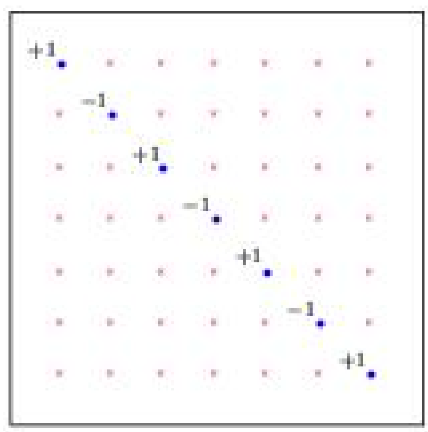

Figure 3.15: The eigenvector for the 5-point finite difference scheme for the laplacian in the square, with eigenvalue $\lambda = 4/h^2$, taking alternating values ± 1 along a diagonal and vanishing everywhere else in the domain.

Chapter 4

Some Topics on the Control and Homogenization of Parabolic Partial Differential Equations *(C. Castro and E. Zuazua)*

joint work with Carlos Castro, Universidad Politécnica de Madrid, Spain, published in Homogenization 2001. Proceedings of the First HMS2000 International School and Conference on Homogenization, *L. Carbone and R. De Arcangelis Eds., GAKUTO Internat. Ser. Math. Sci. Appl.* **18**, *Gakkotosho, Tokyo, Naples, 45-94.*

4.1 Introduction

These Notes have been conceived as a complementary material to the series of lectures we have delivered in the School held in Napoli in June 2001, in the frame of the European TMR Network "Homogenization & Multiple Scales" supported by the EU. Our goal is to address some topics related to the controllability of partial differential equations and homogenization.

Before describing the content of these Notes in detail it is convenient to remind some basic notions in controllability and homogenization.

The problem of controllability may be formulated as follows. Consider an evolution system (either described in terms of Partial or Ordinary Differential Equations). We are allowed to act on the trajectories of the system by means of a suitable choice of the control (the right hand side of the system, the boundary conditions, etc.). Then, given a time interval $t \in (0, T)$, and initial and final states, the problem consists in finding a control such that the solution matches both the initial state at time $t = 0$ and the final one at time $t = T$.

This is a classical problem in Control Theory and there is a large literature on the topic. We refer for instance to the classical book by Lee and Marcus [136] for an introduction in the context of finite-dimensional systems described in terms of Ordinary Differential Equations (ODE). We also refer to the survey paper by Russell [194] and to the book by Lions [143] for an introduction to the controllability of systems modeled by means of PDE also referred to as Distributed Parameter Systems.

In the PDE context the most classical models are those of the wave and the heat equation. They are relevant not only because they (or their variants) arise in most physical applications but also because they constitute prototypes of evolution PDE that are time-reversible and strongly irreversible, respectively, a fact that is determinant when analyzing controllability problems.

In recent years there has been considerable progress in the understanding of the controllability property of these systems. For instance, it is by now well known that the wave equation is controllable in the energy space, roughly speaking, if and only if a Geometric Control Condition (GCC) is satisfied. This condition asserts that every ray of Geometric Optics reaches the control set in a time which is less than the control time (see Bardos, Lebeau and Rauch [14]). On the other hand, it is also by now well known that the heat equation is null-controllable with controls supported in arbitrarily small open sets and in any time (see Fursikov and Imanuvilov [90]). Here null-controllability means that every initial state may be driven to the zero solution and this turns out to be the natural notion of controllability because of the strong time irreversibility of the heat equation. Many other systems including that of thermoelasticity, plate models, Schrödinger and KdV equations, etc. have been also addressed recently. But, describing the state of the art in the field is out of the scope of these Notes. The reader interested in an updated presentation of some of the most relevant progresses in the field is referred to the survey articles [237] and [241] by the second author and the references therein.

On the other hand, the subject of Homogenization has also undergone spectacular progresses in the last decades. There is also an extensive literature in this area. The classical book by Bensousan, Lions and Papanicolau [18] and the more recent one by Cioranescu and Donato [48] contain many of the existing results and mathematical techniques in this area.

The goal of the theory of Homogenization is to derive macroscopic (simplified) models for physical phenomena in which microscopic heterogeneities arise. From a mathematical point of view the most classical problem is that of describing the limiting behavior of the solutions of an elliptic boundary value problem with variable, periodic coefficients, in which the period tends to zero. The same problem can be considered when the coefficients are constant but the domain is perforated or when both heterogeneities arise together. Some of the most fundamental contributions in the field of Homogenization have been done in these apparently simple (but sophisticated enough to require important analytical developments) problems. See, for instance, Spagnolo [207], Tartar [211], and Cioranescu and Murat [52].

Of course, it is also natural to address the problem of Homogenization in the context of controllability or viceversa. For instance, consider a wave or heat equation with rapidly oscillating coefficients at the scale ε. Under rather natural conditions, these systems are controllable (in a sense to be made precise in each situation) for every value of ε. We then fix the initial and the final data. The following questions arise then naturally. *Does the control remain bounded as the size of the microstructure ε tends to zero? Does it converge? Does the limit of the controls provide a good control for the limiting macroscopic model?*

Obviously, these questions make sense not only in the context of Homogenization but for many other singular perturbation problems like, for instance, thin domains, change of type of operators, etc. We refer to volume 2 of Lions' book [143] for a systematic analysis of these questions.

In these Notes we shall focus on this type of problems in the context of one of the most classical issues in Homogenization: rapidly oscillating coefficients. Here, we shall consider only the linear heat equation. Of course, the same questions arise for many other systems including wave equations, the system of elasticity and thermoelasticity, the nonlinear versions of the models addressed here, etc. But these issues will not be considered here. We will also duscuss the problem of controllability of a fixed heat equation but when the control is located at a single point that oscillates rapidly in time.

In the context of the controllability of heat equations or, more generally speaking, linear parabolic equations, there are two fundamental notions of controllability that make sense. The first one is the property of *approximate controllability* in which one is interested in whether solutions at the final time, cover a dense subspace of the natural energy space when the control varies in the space of admissible controls (L^2 in most cases). In the linear setting, this question reduces to an unique continuation property of solutions of the adjoint uncontrolled system. This unique continuation property turns out to be often a consequence of Holmgren Uniqueness Theorem when the coefficients are analytic (see for instance John [118]) or Carleman type inequalities, as in Fursikov and Imanuvilov [90], for equations with non-smooth coefficients. Once the unique continuation property is known, the approximate control may be computed by minimizing a convex, coercive functional in a suitable Hilbert space (usually the space of L^2 functions in the domain where the equation holds). The coercivity of this functional turns out to be a consequence of the unique continuation property and its proof does not require sophisticated estimates. When dealing with homogenization problems, one is lead to analyze the limiting behavior of the minimizers as ε tends to zero. This analysis may be carried out using classical tools in Γ-convergence theory since, the uniform coercivity of the functionals is easy to get.

The situation is much more delicate when dealing with the problem of *null-controllability* for parabolic equations. Recall that we are then interested in driving the solution exactly to zero at the final time. Due to the backward uniqueness property of parabolic equations, approximate controllability turns out to be a consequence of null controllability. When the coefficients are sufficiently smooth, the null controllability property is by now well known for linear parabolic equations in arbitrarily small time and with controls supported in any non-empty open subset of the domain where the equation holds ([87]) But this property requires precise observability estimates for the adjoint system providing an estimate of the global energy of the solution in terms of the energy localized in the subdomain where the control is to be supported. The existing tools for deriving such observability estimates (mainly Carleman inequalities except for $1-d$ problems where the inequalities may be also derived by means of Fourier series developments) do not provide uniform estimates as the ε parameter tends to zero. The problem has been solved so far only in the case of rapidly oscillating coefficients in $1-d$ (see López and Zuazua [154], [156]). It is important to observe that, eventually, under suitable regularity assumptions on the coefficients, the property of null controllability turns out to be uniform and that the controls of the ε problem end up converging to the control of the homogenized equation as ε tends to zero, but this is a consequence of a fine analysis in which different techniques are applied for the control of the low and the high frequencies. It is also worth mentioning that the strong dissipativity of the parabolic equation plays a crucial role in the proof of this result since it allows to compensate the very large cost of controlling the high frequencies as ε tends to zero.

Up to now we have discussed controllability problems in the context of homogenization. Recently it has been observed that the same questions arise when pursuing the numerical analysis of the controllability problem, even for equations with constant coefficients. Indeed, in numerical approximation problems the parameter h denoting the mesh-size (that, consequently, is devoted to tend to zero) plays the same role as the ε parameter, describing the size of the microstructure, arising in homogenization problems. We refer to [233] for a discussion of this analogy in the context of the controllability of the $1-d$ wave equation.

But before discussing further the question we have in mind let us formulate it in a precise way. Given an evolution controllable (in a sense to be made precise in each particular problem under consideration) PDE we consider its numerical discretization (it may be a semi-discretization in space or a complete discretization in space-time). Let us denote by h the characteristic size of the numerical mesh. The following two questions then arise as in the context of homogenization. *Is the h-problem controllable? If yes, do the controls of the h-problem tend to a control of the evolution PDE?*. As in homogenization problems, the answer to these questions depends both in the controllability problem under consideration and on the type of PDE we are dealing with. Roughly speaking, it can be said that, in the context of approximate controllability, the answer to the two questions above is positive, regardless of the type of PDE under consideration. By the contrary, when dealing with null controllability problems, although the h-problem is typically controllable, the controls do not necessarily converge as h goes to zero because of the high frequency spurious numerical solutions. This is for instance the case for the wave equation. However, in the context of the heat equation, due to its strong dissipative effect on high frequencies (even at the numerical h-level), the controls do converge, at least in one space dimension. In these Notes we shall briefly recall the result by A. López and the second author [155] on the uniform null controllability of the finite difference space semi-discretization of the 1-d heat equation. We refer to [245] for a detailed discussion of these issues.

These Notes are organized as follows: In section 2 we present the problem of approximate controllability for the heat equation and show how it can be proved to be uniform in the context of homogenization as ε tends to zero. In section 3 we address the problem of the null controllability for the $1-d$ heat equation with rapidly oscillating coefficients and describe the results of López and the second author [154], [156] showing that this property is also uniform as ε tends to zero. In section 4 we introduce a rapidly oscillating (in time) pointwise control problem and we discuss the limit of the controls as the oscillation parameter ε tends to zero. The results of this section are new and have not been published before. Finally, in section 5, following [155] we briefly discuss the problem of the uniform controllability of space semi-discretizations of the heat equation in 1-d as h tends to zero. We end up with section 6 in which we include a list of open problems and a selected list of bibliographical references.

These Notes are dedicated to the memory of Jeannine Saint-Jean Paulin and Jacques-Louis Lions. Jeannine did fundamental contributions in this subject considering both optimal control and controllability problems in the context of homogenization and singular perturbations. Some of her works are listed in the bibliography at the end of this paper. The influence of the thinking and methods of Jacques-Louis Lions is obvious all along these Notes. Most of the material we present here is a consequence of work motivated by the many discussions we had with him.

4.2 Approximate controllability of the linear heat equation

4.2.1 The constant coefficient heat equation

Let Ω be a bounded domain of $\mathbf{R}^n, n \geq 1$, with boundary Γ of class C^2. Let ω be an open and non-empty subset of Ω and $T > 0$.

Consider the linear controlled heat equation in the cylinder $Q = \Omega \times (0, T)$:

$$\begin{cases} u_t - \Delta u = f 1_\omega & \text{in} \quad Q \\ u = 0 & \text{on} \quad \Sigma \\ u(x, 0) = u^0(x) & \text{in} \quad \Omega. \end{cases} \tag{4.1}$$

In (4.1) Σ represents the lateral boundary of the cylinder Q, i.e. $\Sigma = \Gamma \times (0, T), 1_\omega$ is the characteristic function of the set $\omega, u = u(x, t)$ is the state and $f = f(x, t)$ is the control variable. Since f is multiplied by 1_ω the action of the control is localized in ω.

We assume that $u^0 \in L^2(\Omega)$ and $f \in L^2(Q)$ so that (4.1) admits an unique solution $u \in C\left([0, T]\,; L^2(\Omega)\right) \cap L^2\left(0, T; H_0^1(\Omega)\right)$.

The problem of *controllability* consists roughly on *describing the set of reachable final states*

$$R(T; u^0) = \left\{u(T) : f \in L^2(Q)\right\}.$$

One may distinguish the following degrees of controllability:

(a) System (4.1) is said to be *approximately controllable* if $R(T; u^0)$ is dense in $L^2(\Omega)$ for all $u^0 \in L^2(\Omega)$.

(b) System (4.1) is *exactly controllable* if $R(T; u^0) = L^2(\Omega)$ for all $u^0 \in L^2(\Omega)$.

(c) System (4.1) is *null controllable* if $0 \in R(T; u^0)$ for all $u^0 \in L^2(\Omega)$.

Remark 4.2.1

(a) Approximate controllability holds for every open non-empty subset ω of Ω and for every $T > 0$.

(b) It is easy to see that exact controllability may not hold except in the case in which $\omega = \Omega$. Indeed, due to the regularizing effect of the heat equation, solutions of (4.1) at time $t = T$ are smooth in $\Omega \backslash \overline{\omega}$. Therefore $R(T; u^0)$ is strictly contained in $L^2(\Omega)$ for all $u^0 \in L^2(\Omega)$.

(c) Null controllability implies that all the range of the semigroup generated by the heat equation is reachable too. More precisely, let us denote by $S(t)$ the semigroup generated by (4.1) with $f = 0$. Then, as a consequence of the null-controllability property, for any $u^0 \in L^2(\Omega)$ and $u^1 \in S(T)\left[L^2(\Omega)\right]$ there exists $f \in L^2(\omega \times (0,T))$ such that the solution $u = u(x,t)$ satisfies $u(T) = u^1$.

(d) Null controllability implies approximate controllability. This is so because of remark (c) above and the fact that $S(T)[L^2(\Omega)]$ is dense in $L^2(\Omega)$. In the case of the linear heat equation this can be seen easily developing solutions in Fourier series. However, if the equation contains time dependent coefficients the density of the range of the semigroup, by duality, may be reduced to a backward uniqueness property in the spirit of Lions and Malgrange [147] (see also Ghidaglia [93]).

■

In this section we focus on the *approximate controllability problem.*

System (4.1) is approximately controllable for any open, non-empty subset ω of Ω and $T > 0$. To see this one can apply Hahn-Banach's Theorem or use the variational approach developed in [145]. In both cases the approximate controllability is reduced to a unique continuation property of the adjoint system

$$\begin{cases} -\varphi_t - \Delta\varphi = 0 & \text{in} \quad Q \\ \varphi = 0 & \text{on} \quad \Sigma \\ \varphi(x,T) = \varphi^0(x) & \text{in} \quad \Omega. \end{cases} \tag{4.2}$$

More precisely, approximate controllability holds if and only if the following uniqueness property is true: *If* φ *solves* (4.2) *and* $\varphi = 0$ *in* $\omega \times (0,T)$ *then, necessarily*, $\varphi \equiv 0$, i.e. $\varphi^0 \equiv 0$.

This uniqueness property holds for every open non-empty subset ω of Ω and $T > 0$ by Holmgren's Uniqueness Theorem.

Following the variational approach of [145] the control can be constructed as follows. First of all we observe that it is sufficient to consider the particular case $u^0 \equiv 0$. Then, for any u^1 in $L^2(\Omega), \delta > 0$ we introduce the functional

$$J_\delta(\varphi^0) = \frac{1}{2}\int_0^T \int_\omega \varphi^2 dxdt + \delta \left|\left|\varphi^0\right|\right|_{L^2(\Omega)} - \int_\Omega \varphi^0 u^1 dx. \tag{4.3}$$

The functional J_δ is continuous and convex in $L^2(\Omega)$. On the other hand, in view of the unique continuation property above, one can prove that

$$\lim_{\|\varphi^0\|_{L^2(\Omega)}\to\infty} \frac{J_\delta(\varphi^0)}{\| \varphi^0 \|_{L^2(\Omega)}} \geq \delta \tag{4.4}$$

(we refer to [73] for the details of the proof). Thus, J_δ has a minimizer in $L^2(\Omega)$. Let us denote it by $\bar{\varphi}^0$. Let $\bar{\varphi}$ be the solution of (4.2) with the minimizer $\bar{\varphi}^0$ as initial datum at $t = T$. Then, the control $f = \bar{\varphi}$ is such that the solution u of (4.1) satisfies

$$\left| u(T) - u^1 \right|_{L^2(\Omega)} \leq \delta. \tag{4.5}$$

Obviously, (4.5) for any initial and final data $u^0, u^1 \in L^2(\Omega)$ and for any $\delta > 0$ is equivalent to the approximate controllability property.

Consequently, the following holds:

Theorem 4.2.1 ([73]) *Let ω be any open non-empty subset of Ω and $T > 0$ be any positive control time. Then, for any $u^0, u^1 \in L^2(\Omega)$, $\delta > 0$ there exists a control $f \in L^2(Q)$ such that the solution u of* (4.1) *satisfies* (4.5).

4.2.2 The heat equation with rapidly oscillating coefficients

In this section we consider the approximate controllability of the heat equation with periodic coefficients of small period $\varepsilon \to 0$. More precisely, we introduce a small parameter in the equations and we study how this small parameter affects both the controls and the solutions.

Consider the following system:

$$\left\{ \begin{array}{lll} \rho\left(\frac{x}{\varepsilon}\right) u_t - \operatorname{div}\left(a\left(\frac{x}{\varepsilon}\right)\nabla u\right) = f 1_\omega & \text{in} & Q \\ u = 0 & \text{on} & \Sigma \\ u(x,0) = u^0(x) & \text{in} & \Omega, \end{array} \right. \tag{4.6}$$

where $\varepsilon > 0$, $\rho \in L^\infty(\mathbf{R}^n)$ and $a \in C^1(\mathbf{R}^n)$ are such that

$$\left\{ \begin{array}{l} 0 < \rho_m \leq \rho(x) \leq \rho_M \text{ a.e. in } \mathbf{R}^n \\ 0 < a_m \leq a(x) \leq a_M \text{ a.e. in } \mathbf{R}^n \\ \rho, a \text{ are periodic of period 1 in each variable } x_i,\ i = 1, \ldots, n. \end{array} \right. \tag{4.7}$$

We assume that $u^0 \in L^2(\Omega)$ and $f \in L^2(Q)$ so that (4.6) admits an unique solution $u_\varepsilon \in C\left([0,T]; L^2(\Omega)\right) \cap L^2\left(0,T; H_0^1(\Omega)\right)$.

When $\varepsilon \to 0$ the solutions of (4.6) converge to the solutions of the following limit system where we have replaced the oscillating coefficients $\rho(x/\varepsilon)$ and $a(x/\varepsilon)$ by the average $\bar{\rho} = \int_{[0,1]^n} \rho(x)dx$ and the homogenized constant matrix A respectively :

$$\left\{ \begin{array}{lll} \bar{\rho} u_t - \operatorname{div}(A\nabla u) = f 1_\omega & \text{in} & Q \\ u = 0 & \text{on} & \Sigma \\ u(x,0) = u^0(x) & \text{in} & \Omega. \end{array} \right. \tag{4.8}$$

More precisely, the following holds

Theorem 4.2.2 ([23]) *Let us consider in* (4.6) *a sequence of initial data* $u_\varepsilon^0 \in L^2(\Omega)$ *and a sequence of right hand sides* $f_\varepsilon \in L^2(\omega \times (0,T))$. *Then,*

i) If u_ε^0 *(resp.* f_ε*) weakly converges in* $L^2(\Omega)$ *(resp.* $L^2(\omega \times (0,T))$*) to* u^0 *(resp.* f*) as* $\varepsilon \to 0$, *the solutions* u_ε *of* (4.6) *satisfy*

$$u_\varepsilon \to u \text{ weakly-* in } L^\infty(0,T;L^2(\Omega))$$

as $\varepsilon \to 0$, *where* u *is the solution of the limit system* (4.8).

ii) If u_ε^0 *(resp.* f_ε*) strongly converges in* $L^2(\Omega)$ *(resp.* $L^2(\omega \times (0,T))$*) to* u^0 *(resp.* f*) as* $\varepsilon \to 0$, *the solutions* u_ε *of* (4.6) *satisfy*

$$u_\varepsilon \to u \text{ strongly in } C([0,T];L^2(\Omega))$$

as $\varepsilon \to 0$, *where* u *is the solution of the limit system* (4.8).

We consider the following approximate controllability problem for system (4.6): Given u^0, u^1 in $L^2(\Omega)$ and $\alpha > 0$, to find a control $f_\varepsilon \in L^2(\omega \times (0,T))$ such that the solution $u_\varepsilon = u_\varepsilon(x,t)$ of (4.6) satisfies

$$\left| u_\varepsilon(T) - u^1 \right|_{L^2(\Omega)} \leq \alpha. \tag{4.9}$$

Obviously, the control f_ε also depends on α but we do not make this dependence explicit in the notation for simplicity.

We also study the uniform boundedness of the control f_ε in $L^2(\omega \times (0,T))$ and its possible convergence to a control and a solution of the limit heat equation (4.8) as $\varepsilon \to 0$.

For ε fixed, the approximate controllability of system (4.6) is a direct consequence of the unique continuation of solutions of the homogeneous adjoint equation:

$$\begin{cases} -\rho\left(\frac{x}{\varepsilon}\right)\varphi_t - \operatorname{div}\left(a\left(\frac{x}{\varepsilon}\right)\nabla\varphi\right) = 0, & \text{in } \Omega \times (0,T) \\ \varphi = 0, & \text{on } \Gamma \times (0,T) \\ \varphi(x,T) = \varphi^0(x) & \text{in } \Omega. \end{cases} \tag{4.10}$$

More precisely, for ε fixed, since $\varphi = 0$ in $\omega \times (0,T)$ implies $\varphi^0 = 0$ (see, for instance, Saut and Scheurer [202]), we can derive the approximate controllability of system (4.6) by Hahn-Banach's Theorem or by the variational approach in [145], as in the previous section. Note however that the approach based in the Hahn Banach Theorem does not provide any information on the dependence of the control on the initial and final data and on the parameter ε. Therefore we follow the second method, i.e. the variational approach in [145], presented in the previous section.

Let us recall that when $u^0 = 0$ the control f_ε is of the form $f_\varepsilon = \overline{\varphi}_\varepsilon$ where $\overline{\varphi}_\varepsilon$ solves (4.10) with initial data $\overline{\varphi}^0_\varepsilon$, the minimizer of the functional

$$J_\varepsilon(\varphi^0) = \frac{1}{2}\int_0^T \int_\omega |\varphi|^2 dxdt + \alpha \left|\left|\rho\left(\frac{x}{\varepsilon}\right)\varphi^0\right|\right|_{L^2(\Omega)} - \int_\Omega \rho\left(\frac{x}{\varepsilon}\right) u^1 \varphi^0 dx \quad (4.11)$$

over $L^2(\Omega)$. This control is such that (4.9) holds. Indeed, the minimizer $\overline{\varphi}^0_\varepsilon$ satisfies the Euler equation

$$\int_0^T \int_\omega \overline{\varphi}_\varepsilon \varphi dxdt + \alpha \frac{\int_\Omega \left[\rho\left(\frac{x}{\varepsilon}\right)\right]^2 \overline{\varphi}^0_\varepsilon \varphi^0}{\left|\left|\rho\left(\frac{x}{\varepsilon}\right)\overline{\varphi}^0_\varepsilon\right|\right|_{L^2(\Omega)}} - \int_\Omega \rho\left(\frac{x}{\varepsilon}\right) u^1 \varphi^0 dx = 0 \quad (4.12)$$

for all $\overline{\varphi}^0 \in L^2(\Omega)$ where φ is the solution of (4.10). On the other hand, multiplying the first equation in system (4.6), with $u^0 = 0$ and $f = \overline{\varphi}_\varepsilon$ as a control, by φ and integrating by parts we obtain

$$\int_\Omega \rho\left(\frac{x}{\varepsilon}\right) u(T)\varphi^0 dx = \int_0^T \int_\omega \overline{\varphi}_\varepsilon \varphi dxdt, \quad (4.13)$$

for all $\varphi^0 \in L^2(\Omega)$. Combining now (4.12) and (4.13) we easily deduce that

$$u(T) - u^1 = -\alpha \frac{\rho\left(\frac{x}{\varepsilon}\right)\overline{\varphi}^0_\varepsilon}{\left|\left|\rho\left(\frac{x}{\varepsilon}\right)\overline{\varphi}^0_\varepsilon\right|\right|_{L^2(\Omega)}} \quad (4.14)$$

and (4.9) holds.

The adjoint system associated to the limit system (4.8) is given by

$$\begin{cases} -\bar{\rho}\varphi_t - \operatorname{div}(A\nabla\varphi) = 0, & \text{in } \Omega \times (0,T) \\ \varphi = 0, & \text{on } \Gamma \times (0,T) \\ \varphi(x,T) = \varphi^0(x) & \text{in } \Omega, \end{cases} \quad (4.15)$$

and the corresponding functional associated to (4.8) and (4.15) is given by

$$J(\varphi^0) = \frac{1}{2}\int_0^T \int_\omega |\varphi|^2 dxdt + \alpha \left|\left|\bar{\rho}\varphi^0\right|\right|_{L^2(\Omega)} - \int_\Omega \bar{\rho} u^1 \varphi^0 dx, \quad (4.16)$$

where φ is the solution of (4.15) with final data φ^0.

For simplicity we consider first the case where $u^0 = 0$ and $\left|\left|u^1\right|\right|_{L^2(\Omega)} \geq \alpha$. The main result is as follows:

Theorem 4.2.3 ([235]) *If $u^0 = 0$ and $\alpha > 0$ the approximate controls f_ε obtained by minimizing J_ε over $L^2(\Omega)$ are uniformly bounded in $C([0,T]; L^2(\Omega))$. Moreover, they strongly converge in $C([0,T]; L^2(\Omega))$ as $\varepsilon \to 0$ to the control f associated to the minimizer of the limit functional J, which is an approximate control for the limit system* (4.8).

On the other hand, the solutions u_ε of (4.6) *converge strongly in $C([0,T]; L^2(\Omega))$ as $\varepsilon \to 0$ to the solution u of the limit problem* (4.8).

Let us now consider the case where u^0 is non-zero. We set $v_\varepsilon^1 = v_\varepsilon(T)$ where v_ε is the solution of (4.6) with $f = 0$. It is easy to check that v_ε^1 is uniformly bounded in $H_0^1(\Omega)$. Indeed, multiplying the equation satisfied by v_ε by $\frac{\partial}{\partial t}v_\varepsilon$ and integrating we obtain

$$\frac{d}{dt}\int_\Omega a(\frac{x}{\varepsilon})|\nabla v_\varepsilon|^2 dx = -2\int_\Omega \rho(\frac{x}{\varepsilon})|v_{\varepsilon,t}|^2 \leq 0$$

and therefore

$$\left|v_\varepsilon^1\right|^2_{H_0^1} \leq \int_\Omega \frac{a(\frac{x}{\varepsilon})}{a_m}|\nabla v_\varepsilon(T)|^2 dx \leq \int_\Omega \frac{a(\frac{x}{\varepsilon})}{a_m}|\nabla u^0|^2 dx \leq \frac{a_M}{a_m}\left|u^0\right|^2_{H_0^1}.$$

Then, v_ε^1 weakly converges to $v^1 = v(T)$ where v is the solution of (4.8) with $f = 0$. Now observe that the solution u of (4.6) can be written as $u = v_\varepsilon + w_\varepsilon$ where w_ε is the solution of (4.6) with zero initial data that satisfies $w_\varepsilon(T) = u(T) - v_\varepsilon^1$. In this way, the controllability problem for u can be reduced to a controllability problem for w with zero initial data $w^0 = 0$ but, instead of having a fixed target u^1, we have a sequence of targets $w_\varepsilon^1 = u^1 - v_\varepsilon^1$ that converge weakly in $H_0^1(\Omega)$. In this case, in the definition of the functional J_ε we have to replace u^1 by u_ε^1.

We have the following result:

Theorem 4.2.4 *Assume that $u^0 = 0$, $\alpha > 0$ and consider a sequence of final data u_ε^1 in $L^2(\Omega)$ such that, as $\varepsilon \to 0$, they converge in $L^2(\Omega)$ to $u^1 \in L^2(\Omega)$. Then, the conclusions of Theorem* 4.2.3 *hold.*

Consequently, the conclusions of Theorem 4.2.3 *on the convergence of the controls f_ε and the solutions u_ε hold also for any u^0, $u^1 \in L^2(\Omega)$ and $\alpha > 0$.*

Theorem 4.2.3 is a particular case of Theorem 4.2.4. Thus we will focus in the proof of Theorem 4.2.4.

Proof of Theorem 4.2.4 Let us recall that, in the setting of Theorem 4.2.4, the functional J_ε is given by:

$$J_\varepsilon(\varphi^0) = \frac{1}{2}\int_0^T\int_\omega |\varphi|^2 dxdt + \alpha\left|\left|\rho\left(\frac{x}{\varepsilon}\right)\varphi^0\right|\right|_{L^2(\Omega)} - \int_\Omega \rho\left(\frac{x}{\varepsilon}\right)u_\varepsilon^1\varphi^0 dx \quad (4.17)$$

We set

$$M_\varepsilon = \inf_{\varphi^0 \in L^2(\Omega)} J_\varepsilon(\varphi^0). \quad (4.18)$$

For each $\varepsilon > 0$ the functional J_ε is continuous, convex and coercive. Therefore it attains its minimum M_ε in $L^2(\Omega)$. Moreover, if $f = \overline{\varphi}_\varepsilon$ where $\overline{\varphi}_\varepsilon$ solves (4.10) with data $\overline{\varphi}_\varepsilon^0$, the solution of (4.6) satisfies (4.9) (see [73] and [74]).

The following lemma establishes the uniform bound of the minimizers:

Lemma 4.2.1 *We have*

$$\varliminf_{\substack{||\varphi^0||_{L^2(\Omega)}\to\infty\\ \varepsilon\to 0}} \frac{J_\varepsilon(\varphi^0)}{||\varphi^0||_{L^2(\Omega)}} \geq \alpha. \tag{4.19}$$

Furthermore, the minimizers $\{\overline{\varphi}^0_\varepsilon\}_{\varepsilon\geq 0}$ *are uniformly bounded in* $L^2(\Omega)$.

Proof of Lemma 4.2.1 Let us consider sequences $\varepsilon_j \to 0$ and $\varphi^0_{\varepsilon_j} \in L^2(\Omega)$ such that $\left|\left|\varphi^0_{\varepsilon_j}\right|\right|_{L^2(\Omega)} \to \infty$ as $j \to \infty$. Note that, obviously, this implies that $\left|\left|\rho(x/\varepsilon_j)\varphi^0_{\varepsilon_j}\right|\right|_{L^2(\Omega)} \to \infty$.

Let us introduce the normalized data

$$\psi^0_{\varepsilon_j} = \frac{\varphi^0_{\varepsilon_j}}{\left|\left|\varphi^0_{\varepsilon_j}\right|\right|_{L^2(\Omega)}}$$

and the corresponding solutions of (4.10):

$$\psi_{\varepsilon_j} = \frac{\varphi_{\varepsilon_j}}{\left|\left|\varphi^0_{\varepsilon_j}\right|\right|_{L^2(\Omega)}}.$$

We have

$$\begin{aligned} I_j \quad &= \quad \frac{J_{\varepsilon_j}(\varphi^0_{\varepsilon_j})}{\left|\left|\rho\left(\frac{x}{\varepsilon_j}\right)\varphi^0_{\varepsilon_j}\right|\right|_{L^2(\Omega)}} = \frac{1}{2}\left|\left|\rho\left(\frac{x}{\varepsilon_j}\right)\varphi^0_{\varepsilon_j}\right|\right|_{L^2(\Omega)} \int_0^T\int_\omega |\psi_{\varepsilon_j}|^2 dxdt + \\ &+ \quad \alpha - \int_\Omega \rho\left(\frac{x}{\varepsilon_j}\right) u^1_{\varepsilon_j}\psi^0_{\varepsilon_j}. \end{aligned}$$

We distinguish the following two cases:

Case 1. $\varliminf_{j\to\infty}\int_0^T\int_\omega|\psi_{\varepsilon_j}|^2dxdt > 0$. In this case, we have clearly $\varliminf_{j\to\infty} I_j = \infty$.

Case 2. $\varliminf_{j\to\infty}\int_0^T\int_\omega|\psi_{\varepsilon_j}|^2dxdt = 0$. In this case we argue by contradiction. Assume that there exists a subsequence, still denoted by the index j, such that

$$\int_0^T\int_\omega|\psi_{\varepsilon_j}|^2dxdt \to 0 \tag{4.20}$$

and

$$\varliminf_{j\to\infty} I_j < \alpha. \tag{4.21}$$

By extracting a subsequence, still denoted by the index j, we have

$$\rho\left(\frac{x}{\varepsilon_j}\right)\psi^0_{\varepsilon_j} \rightharpoonup \bar{\rho}\psi^0 \text{ weakly in } L^2(\Omega).$$

By Theorem 4.2.3 we have

$$\psi_{\varepsilon_j} \rightharpoonup \psi \text{ weakly-* in } L^\infty(0,T;L^2(\Omega))$$

where ψ is the solution of the homogenized problem (4.15) with initial data ψ^0. In view of (4.20) we have

$$\psi = 0 \text{ in } \omega \times (0,T)$$

and by Holmgren's Uniqueness Theorem (see, for example [118]) this implies that $\psi^0 = 0$. Thus

$$\rho\left(\frac{x}{\varepsilon_j}\right)\psi^0_{\varepsilon_j} \rightharpoonup 0 \text{ weakly in } L^2(\Omega)$$

and therefore

$$\varliminf_{j\to\infty} I_j \geq \varliminf_{j\to\infty} (\alpha - \int_\Omega \rho\left(\frac{x}{\varepsilon_j}\right) u^1_{\varepsilon_j}\psi^0_{\varepsilon_j}) = \alpha$$

since $u^1_{\varepsilon_j}$ converges strongly in $L^2(\Omega)$. This is in contradiction with (4.21) and concludes the proof of (4.19).

On the other hand, it is obvious that $M_\varepsilon \leq 0$ for all $\varepsilon > 0$, since $J_\varepsilon(0) = 0$. Thus, (4.19) implies the uniform boundedness of the minimizers in $L^2(\Omega)$.

Concerning the convergence of the minimizers we have the following lemma:

Lemma 4.2.2 *The sequence $\rho\left(\frac{x}{\varepsilon}\right)\overline{\varphi}^0_\varepsilon$, where $\overline{\varphi}^0_\varepsilon$ are the minimizers of J_ε, converges strongly in $L^2(\Omega)$ as $\varepsilon \to 0$ to $\bar{\rho}\,\overline{\varphi}^0$ where $\overline{\varphi}^0$ is the minimizer of J and M_ε converges to*

$$M = \inf_{\overline{\varphi}^0 \in L^2(\Omega)} J(\overline{\varphi}^0). \tag{4.22}$$

Moreover, the corresponding solutions $\overline{\varphi}_\varepsilon$ of (4.10) converge in $C([0,T];L^2(\Omega))$ to the solution $\overline{\varphi}$ of (4.15) as $\varepsilon \to 0$.

Proof of Lemma 4.2.2 In view of the uniform bound of the minimizers provided by Lemma 4.2.1, by extracting a subsequence, that we still denote by ε, we have

$$\rho\left(\frac{x}{\varepsilon}\right)\overline{\varphi}^0_\varepsilon \rightharpoonup \bar{\rho}\psi^0 \text{ weakly in } L^2(\Omega)$$

as $\varepsilon \to 0$. It is sufficient to check that $\overline{\varphi}^0 = \psi^0$ or, equivalently,

$$J(\psi^0) \leq J(\varphi^0) \text{ for all } \varphi^0 \in L^2(\Omega). \tag{4.23}$$

From Theorem 4.2.2 we know that

$$\overline{\varphi}_\varepsilon \rightharpoonup \psi \text{ weakly-* in } L^\infty(0,T;L^2(\Omega))$$

where ψ is the solution of (4.15) with initial data ψ^0. By the lower semicontinuity of the first term in J and taking into account that u^1_ε converges strongly to u^1 in $L^2(\Omega)$ we deduce that

$$J(\psi^0) \le \varliminf_{\varepsilon\to 0} J_\varepsilon(\overline{\varphi}^0_\varepsilon). \tag{4.24}$$

On the other hand, for each $\varphi^0 \in L^2(\Omega)$ we have

$$\varliminf_{\varepsilon\to 0} J_\varepsilon(\overline{\varphi}^0_\varepsilon) \le \varliminf_{\varepsilon\to 0} J_\varepsilon(\frac{\bar\rho}{\rho\,(x/\varepsilon)}\varphi^0). \tag{4.25}$$

Observe also that for $\varphi^0 \in L^2(\Omega)$ fixed,

$$\lim_{\varepsilon\to 0} J_\varepsilon(\frac{\bar\rho}{\rho\left(\frac{x}{\varepsilon}\right)}\varphi^0) = J(\varphi^0). \tag{4.26}$$

Indeed,

$$J_\varepsilon(\frac{\bar\rho}{\rho\left(\frac{x}{\varepsilon}\right)}\varphi^0) - J(\varphi^0) = \frac{1}{2}\int_0^T\int_\omega |\varphi_\varepsilon|^2 - \frac{1}{2}\int_0^T\int_\omega |\varphi|^2, \tag{4.27}$$

where φ_ε is the solution of the adjoint system (4.15) with initial data $\frac{\bar\rho}{\rho\left(\frac{x}{\varepsilon}\right)}\varphi^0$. By Theorem 4.2.2, the solutions φ_ε of (4.10) converge strongly to the solution φ of (4.15) in $L^2(\Omega\times(0,T))$ and (4.27) converges to zero as $\varepsilon\to 0$.

From (4.24)-(4.26) we obtain (4.23).

This concludes the proof of the weak convergence of the minimizers and it also shows that

$$M \le \varliminf_{\varepsilon\to 0} M_\varepsilon. \tag{4.28}$$

On the other hand, in view of (4.26) we have

$$M = J(\overline{\varphi}^0) = \limsup_{\varepsilon\to 0} J_\varepsilon(\frac{\bar\rho}{\rho\left(\frac{x}{\varepsilon}\right)}\overline{\varphi}^0) \ge \limsup_{\varepsilon\to 0} J_\varepsilon(\overline{\varphi}^0_\varepsilon) = \limsup_{\varepsilon\to 0} M_\varepsilon. \tag{4.29}$$

From (4.28) and (4.29) we deduce the convergence of the minima, i.e. $M_\varepsilon \to M$.

Observe that (4.22) combined with the weak convergence of $\rho\,(x/\varepsilon)\,\overline{\varphi}^0_\varepsilon$ in $L^2(\Omega)$ and the strong convergence of u^1_ε in $L^2(\Omega)$, implies that

$$\begin{aligned}\lim\nolimits_{\varepsilon\to 0}&\left(\frac{1}{2}\int_0^T\int_\omega |\overline{\varphi}_\varepsilon|^2 dxdt + \alpha\left|\left|\rho\left(\frac{x}{\varepsilon}\right)\overline{\varphi}^0_\varepsilon\right|\right|_{L^2(\Omega)}\right) = \\ &= \frac{1}{2}\int_0^T\int_\omega |\overline{\varphi}|^2 dxdt + \alpha\left|\left|\overline{\rho\varphi^0}\right|\right|_{L^2(\Omega)}.\end{aligned}$$

This identity, combined with the weak convergence of $\rho\,(x/\varepsilon)\,\overline{\varphi}_\varepsilon^0$ to $\overline{\rho\varphi}^0$ in $L^2(\Omega)$ and the weak convergence of $\overline{\varphi}_\varepsilon$ to $\overline{\varphi}$ in $L^2(\omega \times (0,T))$ implies that

$$\rho\left(\frac{x}{\varepsilon}\right)\overline{\varphi}_\varepsilon^0 \to \overline{\rho\varphi}^0 \text{ strongly in } L^2(\Omega). \tag{4.30}$$

Therefore, by Theorem 4.2.2 we have

$$\overline{\varphi}_\varepsilon \to \overline{\varphi} \text{ strongly in } C([0,T];L^2(\Omega)).$$

This concludes the proof of Lemma 4.2.2.

In view of (4.30) the strong convergence in $C([0,T];L^2(\Omega))$ of u_ε is a consequence of Theorem 4.2.2.

Remark 4.2.2 All along this section we have assumed that the coefficient a in the equation must be C^1 while the regularity required for ρ is only L^∞. This is to guarantee the unique continuation of solutions of (4.10) and more precisely the fact that the following property holds: 'If φ solves (4.10) and $\varphi = 0$ in $\omega \times (0,T)$, then $\varphi \equiv 0$'. When $a \in C^1(\Omega)$ this condition is well-known and may be obtained by means of Carleman Inequalities ([111]).

Note however that the homogenized adjoint system has constant coefficients because of the periodicity assumption on a. Thus, unique continuation for this system is a consequence of Holmgren's Uniqueness Theorem.

Remark 4.2.3 This result was proved in [235] in the particular case where the density ρ is constant. The results we have presented here are new.

4.3 Null controllability of the heat equation

In this section we analyze the null controllability of the heat equation. We divide this section in two parts: first we consider the constant coefficients case and afterwards the case of periodic rapidly oscillating coefficients.

4.3.1 The constant coefficient heat equation

Let us consider again the controlled linear heat equation (4.1):

$$\begin{cases} u_t - \Delta u = f1_\omega & \text{in} \quad Q \\ u = 0 & \text{on} \quad \Sigma \\ u(x,0) = u^0(x) & \text{in} \quad \Omega. \end{cases} \tag{4.1}$$

As in the approximate controllability case, the null controllability can be reduced to an observability property for the homogeneous adjoint system

$$\begin{cases} -\varphi_t - \Delta\varphi = 0 & \text{in} \quad Q \\ \varphi = 0 & \text{on} \quad \Sigma \\ \varphi(x,T) = \varphi^0(x) & \text{in} \quad \Omega. \end{cases} \tag{4.2}$$

More precisely, the null controllability problem for system (4.1) is equivalent to the following observability inequality for the adjoint system (4.2):

$$\| \varphi(0) \|^2_{L^2(\Omega)} \leq C \int_0^T \int_\omega \varphi^2 dx dt, \quad \forall \varphi^0 \in L^2(\Omega). \tag{4.3}$$

Due to the irreversibility of the system, (4.3) is not easy to prove. For instance, classical multiplier methods as in [143] do not apply.

In [194] the boundary null controllability of the heat equation was proved in one space dimension using moment problems and classical results on the linear independence in $L^2(0,T)$ of families of real exponentials. Later on, in [195], it was shown that, *if the wave equation is exactly controllable for some* $T > 0$ *with controls supported in* ω, *then the heat equation* (4.1) *is null controllable for all* $T > 0$ *with controls supported in* ω. As a consequence of this result and in view of the controllability results above, it follows that the heat equation (4.1) is null controllable for all $T > 0$ provided ω satisfies the geometric control condition.

However, the geometric control condition is not natural at all in the context of the control of the heat equation.

More recently Lebeau and Robbiano [134] have proved that *the heat equation* (4.1) *is null controllable for every open, non-empty subset* ω *of* Ω *and* $T > 0$. This result shows, as expected, that the geometric control condition is unnecessary in the context of the heat equation.

A slightly simplified proof of this result from [134] was given in [135] where the linear system of thermoelasticity was addressed. Let us describe briefly this proof. The main ingredient of it is an observability estimate for the eigenfunctions of the Laplace operator:

$$\begin{cases} -\Delta w_j = \lambda_j w_j & \text{in} \quad \Omega \\ w_j = 0 & \text{on} \quad \partial\Omega. \end{cases} \tag{4.4}$$

Recall that the eigenvalues $\{\lambda_j\}$ form an increasing sequence of positive numbers such that $\lambda_j \to \infty$ as $j \to \infty$ and that the eigenfunctions $\{w_j\}$ may be chosen such that they form an orthonormal basis of $L^2(\Omega)$.

The following holds:

Theorem 4.3.1 ([134], [135]) *Let* Ω *be a bounded domain of class* C^∞. *For any non-empty open subset* ω *of* Ω *there exist positive constants* $C_1, C_2 > 0$ *such that*

$$\int_\omega \left| \sum_{\lambda_j \leq \mu} a_j \psi_j(x) \right|^2 dx \geq C_1 e^{-C_2\sqrt{\mu}} \sum_{\lambda_j \leq \mu} | a_j |^2 \tag{4.5}$$

for all $\{a_j\} \in \ell^2$ *and for all* $\mu > 0$.

This result was implicitly used in [134] and it was proved in [135] by means of Carleman's inequalities.

As a consequence of (4.5) one can prove that the observability inequality (4.3) holds for solutions of (4.2) with initial data in $E_\mu = \text{span}\,\{w_j\}_{\lambda_j \le \mu}$, the constant being of the order of $\exp\left(C\sqrt{\mu}\right)$. This shows that the projection of solutions over E_μ can be controlled to zero with a control of size $\exp\left(C\sqrt{\mu}\right)$. Thus, when controlling the frequencies $\lambda_j \le \mu$ one increases the $L^2(\Omega)$-norm of the high frequencies $\lambda_j > \mu$ by a multiplicative factor of the order of $\exp\left(C\sqrt{\mu}\right)$. However, as it was observed in [134], solutions of the heat equation (4.1) without control ($f = 0$) and such that the projection of the initial data over E_μ vanishes, decay in $L^2(\Omega)$ at a rate of the order of $\exp(-\mu t)$. Thus, if we divide the time interval $[0, T]$ in two parts $[0, T/2]$ and $[T/2, T]$, we control to zero the frequencies $\lambda_j \le \mu$ in the interval $[0, T/2]$ and then allow the equation to evolve without control in the interval $[T/2, T]$, it follows that, at time $t = T$, the projection of the solution u over E_μ vanishes and the norm of the high frequencies does not exceed the norm of the initial data u^0.

This argument allows to control to zero the projection over E_μ for any $\mu > 0$ but not the whole solution. To do that an iterative argument is needed. We refer to [134] and [135] for the proof.

Remark 4.3.1

(a) Once (4.3) is known to hold one can obtain the control with minimal $L^2(\omega \times (0, T))$-norm among the admissible ones. To do that it is sufficient to minimize the functional

$$J(\varphi^0) = \frac{1}{2}\int_0^T \int_\omega \varphi^2 dx dt + \int_\Omega \varphi(0) u^0 dx \tag{4.6}$$

over the Hilbert space

$$H = \{\varphi^0 : \text{ the solution } \varphi \text{ of (4.2) satisfies } \int_0^T \int_\omega \varphi^2 dx dt < \infty\}.$$

Observe that J is continuous and convex in H. On the other hand (4.3) guarantees the coercivity of J and the existence of its minimizer. The space H is difficult to characterize in terms of the usual energy spaces associated to the heat equation. In this sense, a more natural approach may be to consider the modified functional

$$J_\alpha(\varphi^0) = \frac{1}{2}\int_0^T \int_\omega \varphi^2 dx dt + \int_\Omega \varphi(0) u^0 dx + \alpha \left|\varphi^0\right|_{L^2(\Omega)}, \tag{4.7}$$

over the space $L^2(\Omega)$. The minimizers $\varphi^0_\alpha \in L^2(\Omega)$ allow us to construct a sequence of approximate controls in such a way that the solutions of the heat equation (4.1) u_α, with control $\varphi_\alpha\big|_\omega$, satisfy $\Big|u_\alpha(T)\Big|_{L^2(\Omega)} \leq \alpha$. When $\alpha \to 0$ the minimizers φ_α^0 converge to the minimizer of J in H, and the controls $\varphi_\alpha\big|_\omega$ converge to the null control $\varphi\big|_\omega$ in $L^2(\omega \times (0,T))$.

(b) As a consequence of the internal null controllability of the heat equation one can deduce easily the null boundary controllability with controls in an arbitrarily small open subset of the boundary.

(c) The method of proof of the null controllability we have described is based on the possibility of developing solutions in Fourier series. Thus it can be applied in a more general class of heat equations with variable time-independent coefficients. The same can be said about the methods of [195].

■

The null controllability of the heat equation with lower order time-dependent terms of the form

$$\begin{cases} u_t - \Delta u + a(x,t)u = f1_\omega & \text{in} \quad Q \\ u = 0 & \text{on} \quad \Sigma \\ u(x,0) = u^0(x) & \text{in} \quad \Omega \end{cases} \tag{4.8}$$

has been studied by Fursikov and Imanuvilov (see for instance [44], [87], [90]). Their approach, based on the use of Carleman inequalities, is different to the one we have presented here. As a consequence of their null controllability results it follows that an observability inequality of the form (4.3) holds for the solutions of the adjoint system

$$\begin{cases} -\varphi_t - \Delta\varphi + a(x,t)\varphi = 0 & \text{in} \quad Q \\ \varphi = 0 & \text{on} \quad \Sigma \\ \varphi(x,T) = \varphi^0(x) & \text{in} \quad \Omega \end{cases} \tag{4.9}$$

when ω is any open subset of Ω.

The same approach provides also the null controllability of the variable coefficients heat equation when the control acts in the boundary, i.e.

$$\begin{cases} \rho(x)u_t - \Delta u = 0 & \text{in} \quad Q \\ u = v & \text{on} \quad \Gamma \times (0,T), \qquad u = 0 \text{ on } \Sigma\backslash\Gamma \times (0,T) \\ u(x,0) = u^0(x) & \text{in} \quad \Omega \end{cases} \tag{4.10}$$

where v is the control which acts in one part of the boundary Γ. In this case the corresponding observability inequality for the adjoint system requires the coefficient ρ to be C^1. However, in the one-dimensional case it is possible to go further. Indeed, this is a consequence of the controllability of the one-dimensional wave equation when coefficients are in BV, and the well-known argument by Russell (see [194]) which establishes the equivalence between the control of the heat and wave equation (see [86]) for the details).

4.3.2 The heat equation with rapidly oscillating coefficients in 1-d

In this section we discuss the null controllability of the heat equation with rapidly oscillating coefficients in one space dimension. The complete analysis of this problem and the results of this section were obtained by López and Zuazua in [154], [156].

Once again we introduce a small parameter ε, we assume the periodicity of the coefficients, and we study the behavior of both the controls and the solutions when $\varepsilon \to 0$. We restrict ourselves to the particular case of the boundary controllability problem in the one-dimensional heat equation with oscillating density. In higher dimensions the questions addressed here are basically open.

Let $\rho \in C^2(\mathbf{R})$ be a periodic function satisfying

$$0 < \rho_m \leq \rho(x) \leq \rho_M < +\infty, \ \forall x \in \mathbf{R}. \tag{4.11}$$

Without loss of generality we may assume that ρ is periodic of period 1. We denote by $\bar{\rho}$ its average

$$\bar{\rho} = \int_0^1 \rho(x) dx. \tag{4.12}$$

Given $\varepsilon > 0$ we consider the heat equation with oscillatory density:

$$\begin{cases} \rho(x/\varepsilon) u_t - u_{xx} = 0, \ 0 < x < 1, \ 0 < t < T, \\ u(0,t) = 0; \ u(1,t) = v(t), \ 0 < t < T, \\ u(x,0) = u_0(x). \end{cases} \tag{4.13}$$

In (4.13) $v = v(t)$ denotes a control acting on the system through the extreme $x = 1$ of the interval.

Following the methods of [75] and [90] one can show that system (4.13) is null-controllable for any $T > 0$ and any $0 < \varepsilon < 1$. In other words, for any $T > 0$ and $0 < \varepsilon < 1$ and any $u_0 \in L^2(0,1)$ there exists a control $v \in L^2(0,T)$ such that

$$u(x,T) = 0, \quad 0 < x < 1. \tag{4.14}$$

Moreover, there exists a positive constant $C(\varepsilon, T)$ such that

$$\| v \|_{L^2(0,T)} \leq C(\varepsilon, T) \| u_0 \|_{L^2(0,1)}, \ \forall u_0 \in L^2(0,1). \tag{4.15}$$

We show that $C(\varepsilon, T)$ remains bounded as $\varepsilon \to 0$.

Observe that, passing to the limit in (4.13), formally, we obtain the averaged system

$$\begin{cases} \bar{\rho} u_t - u_{xx} = 0,\ 0 < x < 1,\ 0 < t < T \\ u(0,t) = 0,\ u(1,t) = v(t), 0 < t < T \\ u(x,0) = u_0(x),\ 0 < x < 1. \end{cases} \tag{4.16}$$

The limit system (4.16) is also null-controllable. Thus, the problem of the uniform null-control for system (4.13) as $\varepsilon \to 0$ makes sense.

The main result is as follows:

Theorem 4.3.2 *Assume that $\rho \in W^{2,\infty}(\mathbf{R})$ is a periodic function satisfying* (4.11)*. Let $T > 0$. Then, for any $u^0 \in L^2(0,1)$ and $0 < \varepsilon < 1$ there exists a control $v_\varepsilon \in L^2(0,T)$ such that the solution u_ε of* (4.13) *satisfies*

$$u_\varepsilon(x,T) = 0, \quad x \in (0,1). \tag{4.17}$$

Moreover, there exists a constant $C > 0$ independent of $0 < \varepsilon < 1$ such that

$$\| v_\varepsilon \|_{L^2(0,T)} \leq C \| u^0 \|_{L^2(0,1)},\ \forall u^0 \in L^2(0,1),\ \forall 0 < \varepsilon < 1. \tag{4.18}$$

Finally, for $u^0 \in L^2(0,1)$ fixed, the control v_ε may be built so that

$$v_\varepsilon \to v \text{ in } L^2(0,T) \text{ as } \varepsilon \to 0, \tag{4.19}$$

v being a control for the limit problem (4.16)*, so that the solution u of* (4.16) *satisfies* (4.14)*.*

Remark 4.3.2 It is interesting to compare this result with those obtained in the context of the wave equation with rapidly oscillatory coefficients,

$$\rho(x/\varepsilon) u_{tt} - u_{xx} = 0. \tag{4.20}$$

As it was shown in [6], [36], in the context of (4.20), the control may blow-up as $\varepsilon \to 0$. Then, to obtain a uniform controllability result it was necessary to relax the null-controllability condition to controlling only the projection of solutions over a suitable subspace containing only the "low frequencies" of the system. However, in the context of the heat equation with rapidly oscillating coefficients the null controllability of the whole solution holds, uniformly with respect to $\varepsilon \to 0$.

■

The uniform controllability result of Theorem 4.3.2 is equivalent to a uniform boundary observability property for the adjoint system

$$\begin{cases} \rho(x/\varepsilon)\varphi_t + \varphi_{xx} = 0,\ 0 < x < 1,\ 0 < t < T \\ \varphi(0,t) = \varphi(1,t) = 0,\ 0 < t < T, \\ \varphi(x,T) = \varphi^0(x),\ 0 < x < 1. \end{cases} \tag{4.21}$$

As an immediate corollary of Theorem 4.3.2 the following holds:

Theorem 4.3.3 *Under the assumptions of Theorem* 4.3.2, *there exists a constant* $C > 0$ *which is independent of* $0 < \varepsilon < 1$, *such that*

$$\| \varphi(x,0) \|^2_{L^2(0,1)} \leq C \int_0^T |\varphi_x(1,t)|^2 \, dt \tag{4.22}$$

holds for every $\varphi^0 \in L^2(0,1)$ *and* $0 < \varepsilon < 1$.

Remark 4.3.3 The analogue of (4.22) for the wave equation

$$\left\{ \begin{array}{l} \rho(x/\varepsilon)\varphi_{tt} - \varphi_{xx} = 0, \; 0 < x < 1, \; 0 < t < T \\ \varphi(0,t) = \varphi(1,t) = 0, \; 0 < t < T \end{array} \right. \tag{4.23}$$

is false. This is due to the fact that there exist eigenvalues $\lambda_\varepsilon \sim c/\varepsilon^2$ such that the corresponding eigenfunction $w_\varepsilon(x)$ of the problem

$$-w_{xx} = \lambda_\varepsilon \rho(x/\varepsilon) w, \; 0 < x < 1, \; w(0) = w(1) = 0 \tag{4.24}$$

satisfies

$$\int_0^1 | \, w_x \, |^2 \, dx \Big/ \, | \, w_x(1) \, |^2 \geq C_1 \exp(C_2/\varepsilon). \tag{4.25}$$

As a consequence of (4.25) it is easy to see that the solutions $\varphi_\varepsilon = \cos\left(\sqrt{\lambda_\varepsilon} t\right) w_\varepsilon(x)$ of (4.23) are such that

$$\int_0^1 | \, \varphi_{\varepsilon,x}(x,0) \, |^2 \, dx \Big/ \int_0^T | \, \varphi_{\varepsilon,x}(1,t) \, |^2 \, dt \to \infty, \text{ as } \varepsilon \to 0. \tag{4.26}$$

whatever $T > 0$ is.

Note however that these eigenfunctions are not an obstacle for (4.22) to hold. Indeed, the corresponding solution to (4.21) is $\varphi_\varepsilon = e^{-\lambda_\varepsilon (T-t)} w_\varepsilon(x)$ and then

$$\int_0^1 | \, \varphi_\varepsilon(x,0) \, |^2 \, dx \Big/ \int_0^T | \, \varphi_{\varepsilon,x}(1,t) \, |^2 \, dt \sim e^{-cT/\varepsilon^2} e^{c/\varepsilon} \to 0 \text{ as } \varepsilon \to 0. \tag{4.27}$$

Thus, the strong dissipativity of the parabolic equation compensates the concentration of energy that the high frequency eigenfunctions may present.

■

Note that we obtain the uniform observability inequality (4.22) as a consequence of the uniform controllability result of Theorem 4.3.2. This is contrast with the most classical approach in which the controllability is obtained as a consequence of a suitable observability inequality (see [143]).

The proof of Theorem 4.3.2 is based on a control strategy in three steps that is inspired in [134]. First, using the theory of non-harmonic Fourier series and sharp results on the spectrum of (4.24) the uniform controllability of a suitable projection of the solution over the low frequencies is proved. Then letting the equation to evolve freely without control during a time interval, the control of the low frequencies is kept while the size of the state decreases. Finally, using a global Carleman inequality as in [90] the whole solution may be controlled to zero. The control of the last step can be guaranteed to be uniformly bounded (in fact it tends to zero), since the norm of the solution after the two previous steps is small enough.

The rest of this section is divided in three parts in which we sketch the proof of each step. The first one is devoted to prove the uniform controllability of the low frequencies. The second one is devoted to obtain the global Carleman inequality. In the last one the controls are built and its asymptotic behavior is analyzed.

4.3.2.1 Uniform controllability of the low frequencies

Let us denote by $\{\lambda_{j,\varepsilon}\}_{j\geq 1}$ the eigenvalues of system (4.24), i.e.,

$$0 < \lambda_{1,\varepsilon} < \lambda_{2,\varepsilon} < \cdots < \lambda_{k,3} < \cdots \to \infty. \tag{4.28}$$

Let us denote by $\{w_{j\varepsilon}\}$ the corresponding eigenfunctions so that they constitute an orthonormal basis of $L^2(0,1)$ for each $0 < \varepsilon < 1$.

Let us recall the following sharp spectral result from [34]:

Lemma 4.3.1 *There exist $c, \gamma > 0$ such that*

$$\min_{\lambda \leq c\varepsilon^{-2}} \left| \sqrt{\lambda_{j+1,\varepsilon}} - \sqrt{\lambda_{j,\varepsilon}} \right| \geq \gamma > 0 \tag{4.29}$$

for all $0 < \varepsilon < 1$. Moreover, there exists $C > 0$ such that

$$C \left| w_{j,\varepsilon}(1) \right|^2 \geq \lambda_{j,\varepsilon} \tag{4.30}$$

for all $0 < \varepsilon < 1$ and all eigenvalues in the range $\lambda \leq c\varepsilon^{-2}$.

Note that the periodicity of ρ is required in Lemma 4.3.1.

We now need a result on series of real exponentials. Given $\xi > 0$ and a decreasing function $N : (0, \infty) \to \mathbb{N}$ such that $N(\delta) \to \infty$ as $\delta \to 0$, we introduce the class $\mathcal{L}(\xi, N)$ of increasing sequences of positive real numbers $\{\mu_j\}_{j\geq 1}$ such that:

$$\mu_{j+1} - \mu_j \geq \xi > 0, \, \forall j \geq 1 \tag{4.31}$$

$$\sum_{k \geq N(\delta)} \mu_k^{-1} \leq \delta, \, \forall \delta > 0. \tag{4.32}$$

Using the techniques and results in [75] the following can be proved:

Lemma 4.3.2 *Given a class of sequences $\mathcal{L}(\xi, N)$ and $T > 0$ there exists a constant $C > 0$ (which depends on ξ, N and T) such that*

$$\int_0^T \left| \sum_{k=1}^{\infty} a_k e^{-\mu_k t} \right|^2 dt \geq \frac{C}{\left(\sum_{k \geq 1} \mu_k^{-1} \right)} \sum_{k \geq 1} \frac{| a_k |^2}{\mu_k} e^{-2\mu_k T}, \tag{4.33}$$

for all $\{\mu_k\} \in \mathcal{L}(\xi, N)$ and all sequence $\{a_k\}$ of real numbers.

We now develop solutions of the adjoint system (4.21) in Fourier series

$$\varphi_\varepsilon(x, t) = \sum_{j \geq 1} a_{j,\varepsilon} e^{-\lambda_{j,\varepsilon} t} w_{j,\varepsilon}(x), \tag{4.34}$$

where $\{a_{j,\varepsilon}\}$ are the Fourier coefficients of the datum φ^0 in the basis $\{w_{j,\varepsilon}\}$, i.e.

$$a_{j,\varepsilon} = \int_0^1 \varphi^0(x) w_{j,\varepsilon}(x) dx. \tag{4.35}$$

Let us now denote by E_ε the subspace of $L^2(0,1)$ generated by the low frequency eigenfunctions corresponding to $\lambda \leq c\varepsilon^{-2}$, $c > 0$ being as in Lemma 4.3.1:

$$E_\varepsilon = \operatorname*{span}_{\lambda_{j,\varepsilon} \leq c\varepsilon^{-2}} \{w_{j,\varepsilon}\}. \tag{4.36}$$

As an immediate consequence of Lemmas 4.3.1 and 4.3.2 the following holds [154], [156]:

Proposition 4.3.1 *For any $T > 0$ there exists $C(T) > 0$ such that*

$$\| \varphi_\varepsilon(x, 0) \|^2_{L^2(0,1)} \leq C \int_0^T |\varphi_{\varepsilon,x}(1, t)|^2 dt \tag{4.37}$$

for every solution of (4.21) with $\varphi^0 \in E_\varepsilon$ and $0 < \varepsilon < 1$.

Let us denote by π_ε the orthogonal projection from $L^2(0,1)$ into E_ε. The following uniform, partial controllability result holds [154], [156]:

Proposition 4.3.2 *For any $T > 0$, $0 < \varepsilon < 1$ and $u^0 \in L^2(0,1)$ there exists a control $v_\varepsilon \in L^2(0,T)$ such that the solution of (4.13) satisfies*

$$\pi_\varepsilon (u_\varepsilon(T)) = 0. \tag{4.38}$$

Moreover, there exists a constant $C = C(T) > 0$ independent of $\varepsilon > 0$ such that

$$\| v_\varepsilon \|_{L^2(0,T)} \leq C(T) \| u^0 \|_{L^2(0,1)} \tag{4.39}$$

for all $u^0 \in L^2(0,1)$ and $0 < \varepsilon < 1$.

Proposition 4.3.2 follows immediately from Proposition 4.3.1 applying HUM (see [143]).

4.3.2.2 Global non-uniform controllability

Let us consider the variable coefficient adjoint heat equation

$$\begin{cases} a(x)\theta_t + \theta_{xx} = 0, & 0 < x <, \quad 0 < t < T, \\ \theta(0,t) = \theta(1,t) = 0, & 0 < t < T \\ \theta(x,T) = \theta^0(x), & 0 < x < 1 \end{cases} \tag{4.40}$$

with $a \in W^{2,\infty}(0,1)$ such that

$$0 < a_0 \leq a(x) \leq a_1 < \infty, \, \forall x \in (0,1). \tag{4.41}$$

The following holds [154, 156]:

Lemma 4.3.3 *For any $T > 0$ there exist constants $C_1 > 0$ and $C_2(T) > 0$ such that*

$$\| \theta(x,0) \|^2_{L^2(0,1)} \leq C_1 \exp\left(C_1 \| a \|_{W^{1,\infty}} + C_2 \| a \|^{2/3}_{W^{2,\infty}} \right) \int_0^T |\theta_x(1,t)|^2 \, dt \tag{4.42}$$

for every solution of (4.40) *and every $a \in W^{2,\infty}(0,1)$.*

Sketch of the proof of Lemma 4.3.3

By a classical change of variables (4.40) may be reduced to an equation of the form

$$\psi_t + \psi_{xx} + b(x)\psi = 0$$

where b depends on a and its derivatives up to the second order. When $a \in W^{2,\infty}(0,1)$ the potential b turns out to belong to $L^\infty(0,1)$. Applying the global Carleman inequalities as in [90], and going back to the original variables, (4.42) is obtained.

■

Applying (4.42) to solutions of (4.21) and taking into account that $\left|\rho_\varepsilon\right|_{W^{1,\infty}} \sim 1/\varepsilon$ and $\left|\rho_\varepsilon\right|_{W^{2,\infty}} \sim 1/\varepsilon^2$ as $\varepsilon \to 0$, the following holds [154, 156]:

Proposition 4.3.3 *For any $T > 0$ there exist constants $C_1, C_2 > 0$ such that*

$$\| \varphi_\varepsilon(x,0) \|^2_{L^2(0,1)} \leq C_1 \exp\left(C_1 \varepsilon^{-1} + C_2 \varepsilon^{-4/3} \right) \int_0^T |\varphi_{\varepsilon,x}(1,t)|^2 \, dt \tag{4.43}$$

for any solution of (4.21) *and any $0 < \varepsilon < 1$.*

As an immediate consequence of the observability inequality (4.43) the following null-controllability result holds [154, 156]:

Proposition 4.3.4 *For any $T > 0$, $u^0 \in L^2(0,1)$ and $0 < \varepsilon < 1$, there exists a control $v_\varepsilon \in L^2(0,T)$ such that the solution u_ε of* (4.13) *satisfies* (4.17). *Moreover,*

$$\| v_\varepsilon \|_{L^2(0,T)} \leq C_1 \exp\left(C_1\varepsilon^{-1} + C_2\varepsilon^{-4/3}\right) \| u^0 \|_{L^2(0,1)} \tag{4.44}$$

for all $u^0 \in L^2(0,1)$ and $0 < \varepsilon < 1$.

Note that none of the estimates (4.43) and (4.44) are uniform as $\varepsilon \to 0$. However, (4.43) provides an observability inequality for all solutions of (4.21), and (4.44) an estimate of the control driving the whole solution to rest. In this sense these results are stronger than Propositions 4.3.1 and 4.3.2.

4.3.2.3 Control strategy and passage to the limit

Given $T > 0$, in order to control system (4.13) uniformly to zero we divide the control interval in three parts: $[0,T] = I_1 \cup I_2 \cup I_3$, where $I_j = [(j-1)T/3, jT/3], j = 1,2,3$.

We fix $u^0 \in L^2(0,1)$. Then, in the first interval I_1 we apply Proposition 4.3.2. We obtain controls $v_\varepsilon^1 \in L^2(I_1)$ such that

$$\| v_\varepsilon^1 \|_{L^2(I_1)} \leq C \| u^0 \|_{L^2(0,1)}, \tag{4.45}$$

and

$$\begin{aligned} &\| u_\varepsilon(t) \|_{L^2(0,1)} \leq C \| u^0 \|_{L^2(0,1)}, \qquad \text{for all } t > T/3; \\ &\pi_\varepsilon(u_\varepsilon(T/3)) = 0, \end{aligned} \tag{4.46}$$

for all $0 < \varepsilon < 1$, with $C > 0$ independent of $0 < \varepsilon < 1$.

In the second interval I_2 we let the solution of (4.13) to evolve freely without control (i.e. $v \equiv 0$). In view of (4.46), using the decay of solutions of the heat equation and the invariance of the subspace E_ε under the flow we deduce that

$$\| u_\varepsilon(2T/3) \|_{L^2(0,1)} \leq C \exp(-cT/3\varepsilon^2) \| u^0 \|_{L^2(0,1)} \tag{4.47}$$

$$\pi_\varepsilon(u_\varepsilon(2T/3)) = 0. \tag{4.48}$$

In the last interval I_3 we apply Proposition 4.3.4 so that the solution u_ε of (4.13) achieves the rest at $t = T$. This provides controls $v_\varepsilon^2 \in L^2(I_3)$ such that

$$\| v_\varepsilon^2 \|_{L^2(I_3)} \leq C_1 \exp\left(C_1\varepsilon^{-1} + C_2(T)\varepsilon^{-4/3}\right) \| u_\varepsilon(2T/3) \|_{L^2(0,1)} . \tag{4.49}$$

Note however that, according to (4.47)-(4.49), we have

$$\| v_\varepsilon^2 \|_{L^2(I_3)} \to 0 \text{ exponentially as } \varepsilon \to 0. \tag{4.50}$$

The control $v_\varepsilon \in L^2(0,T)$ for system (4.13) we were looking for is:

$$v_\varepsilon(t) = \begin{cases} v_\varepsilon^1(t), & \text{if} \quad 0 \le t \le T/3 \\ 0, & \text{if} \quad T/3 \le t \le 2T/3 \\ v_\varepsilon^2(t), & \text{if} \quad 2T/3 \le t \le T. \end{cases} \tag{4.51}$$

It is clear that the uniform bound of the controls holds. In fact, it can be seen that

$$v_\varepsilon^1 \to v^1 \text{ in } L^2(0,T/3), \qquad \text{as } \varepsilon \to 0 \tag{4.52}$$

where v^1 is a null control for the limit system (4.16) in the interval $[0,T/3]$. Combining (4.50) and (4.52) we deduce that

$$v_\varepsilon \to v \text{ in } L^2(0,T) \tag{4.53}$$

where

$$v = \begin{cases} v_1, & \text{if} \quad 0 \le t \le T/3, \\ 0, & \text{if} \quad T/3 \le t \le T. \end{cases} \tag{4.54}$$

Moreover the solution u of the limit system (4.16) satisfies

$$u(t) = 0, \forall T/3 \le t \le T. \tag{4.55}$$

4.4 Rapidly oscillating controllers

Let Ω be a bounded smooth domain of $\mathbf{R}^n$, $n = 1,2,3$ and consider the system:

$$\begin{cases} u_t - \Delta u = f(t)\delta(x - a_\varepsilon(t)) & \text{in} \quad Q \\ u = 0 & \text{on} \quad \Sigma \\ u(x,0) = u^0(x) & \text{in} \quad \Omega, \end{cases} \tag{4.1}$$

where $\delta(x)$ represents the Dirac measure centered at $x = 0$, $a_\epsilon(t) = a(t/\epsilon)$, $\epsilon > 0$ is a small parameter and $a : I\!\!R \to \Omega$ is a periodic and real analytic function. We assume without loss of generality that $a(t)$ is periodic of period 2π.

Note that here the control $f(t)$ acts on a periodic oscillating point $x = a_\varepsilon(t)$. Typically a_ε oscillates around a point $x_0 \in \Omega$ with a small amplitude. For instance $a_\varepsilon(t) = x_0 + v\cos(t/\varepsilon)$ with $v \in R^n$.

System (4.1) is well defined in different Sobolev spaces, depending on the dimension n. We introduce the spaces:

$$H_0 = \begin{cases} L^2(\Omega) \text{ if } n = 1, \\ H_0^1(\Omega) \text{ if } n = 2,3, \end{cases} \quad \text{and} \quad H_1 = \begin{cases} H_0^1(\Omega) \text{ if } n = 1, \\ H^2 \cap H_0^1(\Omega) \text{ if } n = 2,3, \end{cases} \tag{4.2}$$

and we denote by H_0' and H_1' their duals.

We assume that $u^0 \in H_0'$ and $f \in L^2(0,T)$. The Dirac measure $\delta(x - a_\varepsilon(t))$ in the right hand side of (4.1) satisfies

$$\delta(x - a_\varepsilon(t)) \in H_1', \qquad \text{for any } t \in [0,T],$$

and system (4.1) admits an unique solution in the class (see [141])

$$u_\varepsilon \in C\left([0,T]\,;H_0'\right) \text{ where } H_0' = \begin{cases} L^2(\Omega) \text{ if } n = 1, \\ H^{-1}(\Omega) \text{ if } n = 2,3. \end{cases}$$

We consider the following approximate controllability problem for system (4.1): *Given u^0, $u^1 \in H_0'$ and $\alpha > 0$, to find a control $f_\varepsilon \in L^2(0,T)$ such that the solution $u_\varepsilon = u_\varepsilon(x,t)$ of (4.1) satisfies*

$$\|u_\varepsilon(T) - u^1\|_{H_0'} \leq \alpha. \tag{4.3}$$

As in the previous sections we also study the boundedness of the control as $\varepsilon \to 0$ and its convergence. We prove that, indeed, the controls remain bounded as $\varepsilon \to 0$. Moreover, we prove that, in the limit, the control no longer acts in a single point for each t but in an interior space-curve with a suitable density. More precisely, the limit control problem is of the form

$$\begin{cases} u_t - \Delta u = f(x,t) m_a(x) 1_\gamma & \text{in} \quad Q \\ u = 0 & \text{on} \quad \Sigma \\ u(x,0) = u^0(x) & \text{in} \quad \Omega, \end{cases} \tag{4.4}$$

where $\gamma \subset \Omega$ is an interior curve and $m_a(x)$ is a limit density, which only depends on a and that will be given explicitly below. This fact was illustrated by Berggren in the one dimensional case by means of a formal argument and some numerical experiments (see [19]).

The rest of this section is divided in three subsections. First we consider the pointwise control of system (4.1) in a general framework, i.e. with controls supported over a general curve $b(t)$. As a particular case we obtain the controllability of (4.1) for any $\varepsilon > 0$ under certain hypothesis on $a_\varepsilon(t)$. In the second subsection we prove a convergence result for the solutions of (4.1) with $f(t) = 0$ that we use in the third subsection to prove the convergence of the controls and the controlled solutions as $\varepsilon \to 0$ towards (4.4), in a suitable sense.

4.4.1 Pointwise control of the heat equation

When $\varepsilon > 0$ is fixed the approximate controllability of system (4.1) is a consequence of the following unique continuation property for the adjoint system: *If φ solves*

$$\begin{cases} -\varphi_t - \Delta\varphi = 0 & \text{in} \quad Q \\ \varphi = 0 & \text{on} \quad \Sigma \\ \varphi(x,T) = \varphi^0(x) & \text{in} \quad \Omega, \end{cases} \tag{4.5}$$

can we guarantee that

$$\varphi(a_\varepsilon(t), t) = 0, \quad \forall t \in [0, T] \quad \Rightarrow \quad \varphi \equiv 0? \tag{4.6}$$

This section is devoted to analyze this uniqueness problem. Taking into account that $\varepsilon > 0$ is fixed and in order to simplify the notation we denote the curve where the control is supported by $x = b(t)$ instead of $x = a_\varepsilon(t)$.

The following lemma reduces the unique continuation problem (4.6) to a certain unique continuation property for the eigenfunctions of system (4.5):

$$\begin{cases} -\Delta w(x) = \lambda w(x), & x \in \Omega \\ w(x) = 0, & x \in \partial\Omega. \end{cases} \tag{4.7}$$

Lemma 4.4.1 *Assume that $b : [0, T] \to \Omega$ satisfies the hypothesis:*

$$b(t) \textit{ can be extended to a real analytic function } \bar{b} : (-\infty, T] \to \overline{\Omega}. \tag{4.8}$$

Let us consider the set of accumulation points

$$P = \left\{ x \in \overline{\Omega}, \ s.t. \ \exists t_n \to -\infty \ with \ \bar{b}(t_n) \to x \right\}, \tag{4.9}$$

and for each $x \in P$, the set of 'accumulation directions'

$$D_x = \left\{ v \in R^n, \ s.t. \ \exists t_n \to -\infty \ with \ \bar{b}(t_n) \to x \ and \ \frac{\bar{b}(t_n) - x}{\left|\left|\bar{b}(t_n) - x\right|\right|} \to v \right\}. \tag{4.10}$$

Assume that the following unique continuation property holds for the eigenfunctions of (4.5)*:*

$$\left.\begin{array}{lll} w \ eigenfunction \ of \ (4.7) & & \\ w(x) = 0, & \forall x \in P, \ and & \\ \nabla w(x) \cdot v = 0, & \forall v \in D_x, & \forall x \in P \end{array}\right\} \Rightarrow w \equiv 0. \tag{4.11}$$

Then we have the following unique continuation property for the solutions of the adjoint problem (4.5)*:*

$$\varphi(b(t), t) = 0 \quad \forall t \in [0, T] \Rightarrow \varphi \equiv 0. \tag{4.12}$$

Proof. Let $\varphi \in C([0, T]; H_1)$ be a solution of (4.5) with $\varphi(b(t), t) = 0$ for all $t \in [0, T]$. Obviously, this solution can be extended naturally to all $t \leq T$. As the Laplace operator generates an analytic semigroup, the solution of system (4.5) $\varphi : \Omega \times (-\infty, T) \to \mathbb{R}$ is analytic. On the other hand, $\bar{b}(t)$ is also analytic and the composition $\varphi(\bar{b}(t), t)$ is still analytic. Then the fact that $\varphi(\bar{b}(t), t)$ vanishes for $t \in [0, T]$ implies that

$$\varphi(\bar{b}(t), t) = 0 \quad \forall t \in (-\infty, T].$$

Let us introduce the Fourier representation of φ

$$\varphi(x,t) = \sum_{j=1}^{\infty} e^{-\lambda_j (T-t)} \sum_{k=1}^{l(j)} c_{j,k} w_{j,k}(x)$$

where

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_j < \cdots$$

are the eigenvalues of (4.7) and $\{w_{j,k}(x)\}_{k=1,\dots,k(l)}$ is a system of linear independent eigenfunctions associated to λ_j. We assume that $\{w_{j,k}(x)\}_{j,k\geq 1}$ is chosen to be orthonormal in H_1 (recall that $H_1 = H_0^1(\Omega)$ if $n = 1$ and $H_1 = H^2 \cap H_0^1(\Omega)$ if $n = 2, 3$). Taking into account that $\varphi(T) = \varphi^0 \in H_1$ we deduce that

$$\sum_{j,k} |c_{j,k}|^2 < \infty.$$

Then

$$0 = \varphi(\bar{b}(t), t) = \sum_{j=1}^{\infty} e^{-\lambda_j (T-t)} \sum_{k=1}^{l(j)} w_{j,k}(\bar{b}(t)), \quad \forall t \in (-\infty, T]. \tag{4.13}$$

This implies that

$$\sum_{k=1}^{l(j)} c_{j,k} w_{j,k}(x_0) = \sum_{k=1}^{l(j)} c_{j,k} \nabla w_{j,k}(x_0) \cdot v = 0, \\ \text{for all } j \text{ and } \forall x_0 \in P, \, \forall v \in D_{x_0}. \tag{4.14}$$

Assuming for the moment that (4.14) holds, taking into account the fact that $\sum_{k=1}^{l(j)} c_{j,k} w_{j,k}$ is an eigenfunction and by the unique continuation hypothesis for the eigenfunctions (4.11) we obtain

$$\sum_{k=1}^{l(j)} c_{j,k} w_{j,k} \equiv 0, \quad \text{for all } j \geq 1,$$

Therefore $c_{j,k} = 0$ for all $k = 1, \dots l(j)$ because of the linear independence of $w_{j,k}$. This concludes the proof of the lemma.

Finally, let us prove (4.14). Multiplying the series in (4.13) by $e^{\lambda_1 (T-t)}$ and taking into account that λ_1 is simple we obtain

$$c_{1,1} w_{1,1}(\bar{b}(t)) + \sum_{j=2}^{\infty} e^{(\lambda_1 - \lambda_j)(T-t)} \sum_{k=1}^{l(j)} c_{j,k} w_{j,k}(\bar{b}(t)) = 0 \quad \forall t \in (-\infty, T]. \tag{4.15}$$

The second term on the left hand side converges to zero as $t \to -\infty$. Indeed,

$$\left|\sum_{j=2}^{\infty} e^{(\lambda_1-\lambda_j)(T-t)} \sum_{k=1}^{l(k)} c_{j,k} w_{j,k}(\bar{b}(t))\right|^2 \leq \left\|\sum_{j=2}^{\infty} e^{(\lambda_1-\lambda_j)(T-t)} \sum_{k=1}^{l(k)} c_{j,k} w_{j,k}\right\|^2_{L^\infty(\Omega)}$$

$$\leq \left\|\sum_{j=2}^{\infty} e^{(\lambda_1-\lambda_j)(T-t)} \sum_{k=1}^{l(k)} c_{j,k} w_{j,k}\right\|^2_{H_1} = \sum_{j=2}^{\infty} e^{2(\lambda_1-\lambda_j)(T-t)} \sum_{k=1}^{l(k)} |c_{j,k}|^2$$

which converges to zero as $t \to -\infty$.

Let $x_0 \in P$ and $t_n \to -\infty$ such that $\bar{b}(t_n) \to x_0$ as $t_n \to -\infty$. Passing to the limit as $t_n \to \infty$ in (4.15) we obtain

$$c_{1,1} w_{1,1}(x_0) = 0.$$

Analogously, multiplying (4.15) by $e^{-(\lambda_1-\lambda_2)(T-t)}$ we obtain

$$\sum_{k=1}^{l(2)} c_{2,k} w_{2,k}(\bar{b}(t)) + \sum_{j=3}^{\infty} e^{(\lambda_2-\lambda_j)(T-t)} \sum_{k=1}^{l(j)} c_{j,k} w_{j,k}(\bar{b}(t)) = 0 \quad \forall t \in (-\infty, T]. \tag{4.16}$$

Once again, the second term on the left hand side converges to zero as $t \to -\infty$. Then, passing to the limit as $t_n \to \infty$ in (4.16) we obtain

$$\sum_{k=1}^{l(2)} c_{1,k} w_{1,k}(x_0) = 0.$$

Following an induction argument we easily obtain

$$\sum_{k=1}^{l(j)} c_{j,k} w_{j,k}(x_0) = 0, \quad \text{for all } j \geq 1 \text{ and } \forall x_0 \in P. \tag{4.17}$$

To finish the proof of (4.14) we have to check that

$$\sum_{k=1}^{l(j)} c_{j,k} \nabla w_{j,k}(x_0) \cdot v = 0, \quad \text{for all } j \text{ and } \forall x_0 \in P,\ \forall v \in D_{x_0}. \tag{4.18}$$

Following the same argument as above, if $x_0 \in P$, $v \in D_{x_0}$ and $t_n \to -\infty$ as in (4.10) we have that

$$c_{1,1} \nabla w_{1,1}(x_0) \cdot v = c_{1,1} \lim_{t_n \to -\infty} \frac{w_{1,1}(\bar{b}(t_n)) - w_{1,1}(x_0)}{\|\bar{b}(t_n) - x_0\|} = 0 \tag{4.19}$$

since from (4.15) and (4.17),

$$
\begin{aligned}
& \left| c_{1,1} \frac{w_{1,1}(\bar{b}(t_n)) - w_{1,1}(x_0)}{\bar{b}(t_n) - x_0} \right| = \left| c_{1,1} \frac{w_{1,1}(\bar{b}(t_n))}{\bar{b}(t_n) - x_0} \right| \\
= & \left| \sum_{j=2}^{\infty} e^{(\lambda_1 - \lambda_j)(T - t_n)} \left[\sum_{k=1}^{l(j)} c_{j,k} \frac{w_{j,k}(\bar{b}(t_n))}{\bar{b}(t_n) - x_0} - \frac{1}{\bar{b}(t_n) - x_0} \sum_{k=1}^{l(j)} c_{j,k} w_{j,k}(x_0) \right] \right| \\
= & \left| \sum_{j=2}^{\infty} e^{(\lambda_1 - \lambda_j)(T - t_n)} \left[\sum_{k=1}^{l(j)} c_{j,k} \frac{w_{j,k}(\bar{b}(t_n)) - w_{j,k}(x_0)}{\bar{b}(t_n) - x_0} \right] \right| \\
\leq & \sum_{j=2}^{\infty} e^{(\lambda_1 - \lambda_j)(T - t_n)} \sum_{k=1}^{l(j)} |c_{j,k}| \left| \frac{w_{j,k}(\bar{b}(t_n)) - w_{j,k}(x_0)}{\bar{b}(t_n) - x_0} \right| \\
\leq & C \sum_{j=2}^{\infty} e^{(\lambda_1 - \lambda_j)(T - t_n)} \sum_{k=1}^{l(j)} |c_{j,k}| \; ||\nabla w_{j,k}||_{L^\infty} \\
\leq & C \sum_{j=2}^{\infty} e^{(\lambda_1 - \lambda_j)(T - t_n)} \sum_{k=1}^{l(j)} |c_{j,k}| \; ||\nabla w_{j,k}||_{H_1} \leq C \sum_{j=2}^{\infty} |c_{j,k}| \, \lambda_j e^{(\lambda_1 - \lambda_j)(T - t_n)},
\end{aligned}
$$

The last term in this expression converges to zero as $t_n \to -\infty$ and then $c_{1,1} \nabla w_{1,1}(x_0) \cdot v = 0$ for any $v \in D_x$.

Once again, following an induction argument we easily obtain

$$
\sum_{k=1}^{l(j)} c_{j,k} \nabla w_{j,k}(x_0) \cdot v = 0, \quad \text{for all } j \text{ and } \forall x_0 \in P, \, \forall v \in D_{x_0}. \tag{4.20}
$$

This concludes the proof of (4.18) and therefore the proof of the Lemma.

■

Examples: The assumptions of the Lemma hold in the following particular cases:

1. **Static control:** Assume that $b(t) = x_0$ is constant. Then $P = \{x_0\}$ and D_{x_0} is empty. Let (λ_j, w_j) be the eigenpairs of (4.7). According to Lemma 4.4.1, if the spectrum of the laplacian in Ω is simple (which is generically true, with respect to the geometry of the domain) and

$$
w_j(x_0) \neq 0, \quad \forall j, \tag{4.21}
$$

then (4.12) holds. The set of points $x \in \Omega$ which satisfy (4.21) are usually referred to as *strategic* and they are dense in Ω.

2. **Oscillating control:** Assume that $b(t)$ is periodic and analytic. Then P coincides with the range of $b(t)$ i.e.

$$P = \{x \in \Omega \text{ s.t. } \exists t \in \mathbf{R} \text{ with } b(t) = x\},$$

   while for each $x \in P$, D_x is the set of tangent vectors to P at x. We say that $b(t)$ is *strategic* if (4.11) holds. Note that nonstrategic curves are those for which P is included in a nodal curve.

   In the one-dimensional case nodal curves are reduced to points and therefore (4.11) holds as long as $a(t)$ is non-constant. This is the case addressed by Berggrem in [19].

3. **Quasi-static control**: Assume that $b(t) = x_0 + a(t)$, a being a nonconstant analytic function satisfying $\lim_{t\to-\infty} a(t) = 0$. Then $P = \{x_0\}$. Let also D_{x_0} be the set of accumulation directions.

   Obviously, if the spectrum of the Laplacian is simple and x_0 is *strategic*, the unique continuation property (4.11) holds as indicated in the first example above.

   Let us consider now the particular 1-d case. Then the spectrum of the Laplacian is simple and x_0 being strategic can be understood as an irrationality condition. Even if x_0 is non-strategic, if D_{x_0} is non empty, the unique continuation property (4.11) holds. Indeed, in 1-d, the eigenfunctions solve a second order ODE and $w(x_0) = 0$ together with $w'(x_0) = 0$ implies that $w \equiv 0$.

   This example shows the interest of the extra information that the proof of Lemma 4.4.1 provides about the gradient of the eigenfunctions on the D_{x_0}-directions. In the particular case under consideration it proves that uniqueness does hold for $x(t) = x_0 + \exp(t)$, even when x_0 is non-strategic.

Remark 4.4.1

(a) The argument in the proof of Lemma 4.4.1 can be iterated to get further information on the derivative of the eigenfunctions of higher order. A careful analysis of this fact will be developed elsewhere.

(b) When the curve $x = b(t)$ is periodic, P coincides with the range of b in Ω and D_{x_0} is constituted by the tangent vectors to P at x_0. In this case, the fact that $\nabla w(x_0) \cdot v$ vanishes for all $x_0 \in P$ and $v \in D_{x_0}$ does not add anything new with respect to the fact that $w = 0$ on P.

    The fact that $\nabla w(x_0) \cdot v = 0$ is only of interest when the curve $b(t)$ is non-periodic.

4.4.2 A convergence result

In this section we prove the following lemma:

Lemma 4.4.2 *Let $a(s) : \mathbb{R} \to \Omega$ be an analytic $2\pi-$periodic curve. Consider a sequence $u_\varepsilon^0 \rightharpoonup u^0$ that weakly converges in H_0 ($H_0 = L^2(\Omega)$ if $n = 1$ and $H_0 = H_0^1(\Omega)$ if $n = 2, 3$). Let u_ε, u be the solutions of the homogeneous system (4.1) with $f = 0$, and initial data u_ε^0, u^0 respectively. Let $a_\varepsilon(t) = a(t/\varepsilon)$. Then*

$$\int_0^T |u_\varepsilon(a_\varepsilon(t), t)|^2 dt \to \int_0^T \int_\gamma |u(x,t)|^2 m_a(x) d\gamma dt, \tag{4.22}$$

where γ is the range of $a(t)$ and $m_a(x)$ is defined as follows: Let $\{I_h\}_{h=1}^H \subset (0, 2\pi))$ be the set of closed time intervals where $a(t) : (0, 2\pi) \to \mathbf{R}$ is one-to-one and $\gamma_h = a(I_h) \subset \Omega$. Note that the number H of subintervals $\gamma_h \subset [0, 2\pi]$, must be finite since the analyticity of $a(t)$. Then

$$m_a(x) = \frac{1}{2\pi} \sum_{h=1}^H \frac{1}{|a'(a^{-1}(x))|}, \forall x \in \gamma \tag{4.23}$$

where $a^{-1}(x)$ is the inverse function of a. Note that m_a is defined over the whole curve γ since

$$\bigcup_h \gamma_h = \gamma.$$

Moreover, if $\varphi \in C_0^\infty((0,1) \times (0,T))$ then

$$\int_0^T u_\varepsilon(a_\varepsilon(t), t)\varphi(a_\varepsilon(t), t) \ dt \to \int_0^T \int_\gamma u(x,t)\varphi(x,t) m_a(x) \ d\gamma dt. \tag{4.24}$$

Remark 4.4.2 The function $m_a(x)$ may be singular at the extremes of the intervals I_h if $a'(s) = 0$ for some point s. For example, in the one dimensional case studied in [19], $\Omega = (0, 1)$, $a(t) = x_0 + \delta \cos(t)$ and

$$m_a(x) = \begin{cases} \frac{1}{\pi\sqrt{\delta^2 - (x - x_0)^2}} & \text{if } |x - x_0| < \delta, \\ 0 \text{ otherwise,} \end{cases}$$

which is singular at $x = x_0 \pm \delta$. Observe however that $m_a(x) \in L^1(\Omega)$ and the integral in (4.24) is well-defined.

In fact the singular integral in (4.24) is always well-defined since, as we will see below in (4.31), we have

$$\int_\gamma u(x,t)\varphi(x,t) m_a(x) \ d\gamma = \frac{1}{2\pi} \int_0^{2\pi} u(a(s), t)\varphi(a(s), t) \ ds$$

which is obviously finite.

Proof. The sequence $u_\varepsilon(x,t)$ of solutions of the homogeneous system (4.1) with $f=0$ and initial data u_ε^0 can be written in the Fourier representation

$$u_\varepsilon(x,t)=\sum_{j=1}^{\infty}e^{-\lambda_j t}\sum_{k=1}^{l(j)}c_{j,k}^{\varepsilon}w_{j,k}(x).$$

We assume that $(w_{j,k})_{j,k\geq 1}$ constitute an orthonormal basis in H_0. Analogously, the solution $u(x,t)$ of the homogeneous system (4.1) with $f=0$ and initial data u^0, is

$$u(x,t)=\sum_{j=1}^{\infty}e^{-\lambda_j t}\sum_{k=1}^{l(j)}c_{j,k}w_{j,k}(x).$$

Due to the weak convergence of the initial data $u_\varepsilon^0 \rightharpoonup u^0$ in $L^2(\Omega)$ we have

$$\sum_{j,k\geq 1}|c_{j,k}^{\varepsilon}|^2\leq C, \qquad \sum_{j,k\geq 1}|c_{j,k}|^2\leq C, \tag{4.25}$$

with C independent of ε. Moreover,

$$c_{j,k}^{\varepsilon}\to c_{j,k}, \qquad \text{as } \varepsilon\to 0, \qquad \forall j,k\geq 1.$$

Let us prove the convergence stated in (4.22). To avoid the singularity of the solution u_ε at $t=0$ we divide the left hand side in two parts

$$\int_0^T|u_\varepsilon(a_\varepsilon(t),t)|^2dt=\int_0^\delta|u_\varepsilon(a_\varepsilon(t),t)|^2dt+\int_\delta^T|u_\varepsilon(a_\varepsilon(t),t)|^2dt \tag{4.26}$$

with $\delta>0$ to be chosen later. By classical estimates of the heat kernel (see [40], p. 44) we know that

$$||u_\varepsilon(\cdot,t)||_{L^\infty(\Omega)}\leq Ct^{-\frac{n}{2q}}\left|\left|u_\varepsilon^0\right|\right|_{L^q(\Omega)}$$

and therefore the first integral in (4.26) can be estimated by

$$\int_0^\delta|u_\varepsilon(a_\varepsilon(t),t)|^2dt\leq\begin{cases} C\delta^{1/2}\left|\left|u_\varepsilon^0\right|\right|_{L^2(\Omega)}^2 & \text{if } n=1,\\ \frac{(4\pi\delta)^{1-n/4}}{4\pi(4-n)}\left|\left|u_\varepsilon^0\right|\right|_{L^4(\Omega)}\leq C\delta^{1-n/4}\left|\left|u_\varepsilon^0\right|\right|_{H_0^1(\Omega)} & \text{if } n=2,3.\end{cases}$$

Taking the bound on the initial data into account, we see that the first integral in (4.26) converges to zero as $\delta\to 0$ uniformly in ε.

Thus it suffices to show that the second integral in (4.26), for $\delta>0$ fixed, tends to

$$\int_\delta^T\int_\gamma|u(x,t)|^2m_a(x)d\gamma dt$$

as $\varepsilon \to 0$. We have

$$\begin{aligned}\int_\delta^T |u_\varepsilon(a_\varepsilon(t),t)|^2 dt &= \int_\delta^T \sum_{j,i=1}^{\infty}\sum_{k=1}^{l(j)}\sum_{m=1}^{l(i)} e^{-(\lambda_i+\lambda_j)t} c^\varepsilon_{j,k} c^\varepsilon_{i,m} w_{j,k}(a_\varepsilon(t)) w_{i,m}(a_\varepsilon(t))\ dt \\ &= \sum_{j,i=1}^{\infty}\sum_{k=1}^{l(j)}\sum_{m=1}^{l(i)} \int_\delta^T e^{-(\lambda_i+\lambda_j)t} c^\varepsilon_{j,k} c^\varepsilon_{i,m} w_{j,k}(a_\varepsilon(t)) w_{i,m}(a_\varepsilon(t))\ dt.\end{aligned}$$

Now we take the limit as $\varepsilon \to 0$,

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_\delta^T |u_\varepsilon(a_\varepsilon(t),t)|^2 dt \\ &= \lim_{\varepsilon\to 0}\sum_{j,i=1}^{\infty}\sum_{k=1}^{l(j)}\sum_{m=1}^{l(i)} \int_\delta^T e^{-(\lambda_i+\lambda_j)t} c^\varepsilon_{j,k} c^\varepsilon_{i,m} w_{j,k}(a_\varepsilon(t)) w_{i,m}(a_\varepsilon(t))\ dt \\ &= \sum_{j,i=1}^{\infty}\sum_{k=1}^{l(j)}\sum_{m=1}^{l(i)} c_{j,k} c_{i,m} \lim_{\varepsilon\to 0}\int_\delta^T e^{-(\lambda_i+\lambda_j)t} w_{j,k}(a_\varepsilon) w_{i,m}(a_\varepsilon)\ dt. \quad (4.27)\end{aligned}$$

Interchanging the sum and the limit is justified because of the dominated convergence theorem. Indeed, each term of the series can be bounded above as follows

$$\begin{aligned}&\left| c^\varepsilon_{j,k} c^\varepsilon_{i,m} \int_\delta^T e^{-(\lambda_i+\lambda_j)t} w_{j,k}(a_\varepsilon(t)) w_{i,m}(a_\varepsilon(t))\ dt \right| \\ \leq\ & \left(\sum_{i=1}^{\infty}\sum_{m=1}^{l(i)} |c^\varepsilon_{i,m}|^2\right) ||w_{j,k}||_{L^\infty(\Omega)}\, ||w_{i,m}||_{L^\infty(\Omega)} \int_\delta^T e^{-(\lambda_i+\lambda_j)t} dt \\ \leq\ & \left(\sum_{i=1}^{\infty}\sum_{m=1}^{l(i)} |c^\varepsilon_{i,m}|^2\right) \sqrt{\lambda_j\lambda_i}\frac{e^{-(\lambda_i+\lambda_j)\delta} - e^{-(\lambda_i+\lambda_j)T}}{\lambda_i+\lambda_j} \qquad (4.28)\end{aligned}$$

where we have used the normalization of the eigenfunctions and the fact that

$$||w_{j,k}||_{L^\infty(\Omega)} \leq ||w_{j,k}||_{H_1} = \sqrt{\lambda_j}\, ||w_{j,k}||_{H_0} = \sqrt{\lambda_j}.$$

Note that the series on the right hand side of (4.28) is bounded uniformly in $\varepsilon \to 0$ by (4.25), while the other one satisfies

$$\sqrt{\lambda_j\lambda_i}\frac{e^{-(\lambda_i+\lambda_j)\delta} - e^{-(\lambda_i+\lambda_j)T}}{\lambda_i+\lambda_j} \leq e^{-(\lambda_i+\lambda_j)\delta}, \qquad (4.29)$$

and the sum in i and j of all these numbers is finite due to the well-known asymptotic behavior of the eigenvalues of the Laplace operator. Indeed,

$$\sum_{i,j\geq 1} e^{-(\lambda_i+\lambda_j)\delta} = \left(\sum_{j\geq 1} e^{-\lambda_j\delta}\right)^2$$

and this sum can be estimated above taking into account the asymptotic behavior of the eigenvalues of the Laplace operator. Recall that the number of eigenvalues less than a constant λ is asymptotically equal to $\lambda|\Omega|/4\pi$ if $n=2$, and $\lambda^{3/2}|\Omega|/6\pi^2$ if $n=3$ (see [66], p. 442). Indeed, for example, in the case $n=3$ we have

$$\sum_{j\geq 1} e^{-\lambda_j\delta} = \sum_{k=1}^{\infty} \sum_{k-1\leq\lambda_j\leq k} e^{-\lambda_j\delta} \leq C\sum_{k=1}^{\infty} k^{3/2}e^{-(k-1)\delta} < \infty.$$

Once we have checked (4.27), we observe that

$$w_{j,k}(a_\varepsilon(t))w_{i,m}(a_\varepsilon(t)) = w_{j,k}(a(t/\varepsilon))w_{i,m}(a(t/\varepsilon))$$

where $w_{j,k}(a(s))w_{i,m}(a(s))$ is 2π-periodic. Therefore, as $\varepsilon \to 0$, the function $w_{j,k}(a_\varepsilon(t))w_{i,m}(a_\varepsilon(t))$ converges weakly to its average in L^2_{loc}, i.e.

$$\lim_{\varepsilon\to 0}\int_\delta^T e^{-(\lambda_i+\lambda_j)t} w_{j,k}(a_\varepsilon(t))w_{i,m}(a_\varepsilon(t))\ dt$$

$$= \frac{1}{2\pi}\int_\delta^T e^{-(\lambda_i+\lambda_j)t}\int_0^{2\pi} w_{j,k}(a(s))w_{i,m}(a(s))\ dsdt. \tag{4.30}$$

This last integral can be simplified studying separately the intervals where $a(s)$ is one-to-one $\{I_h\}_{h=1}^H$. Note that the whole interval $[0,2\pi]$ is divided in the subintervals I_h. Indeed, if there is a subinterval $I \subset (0,2\pi)$ such that I is not included in $\bigcup_{h=1}^H I_h$ then $a(s)$ must be constant on I and then constant everywhere because of the analyticity of a. Then,

$$\int_0^{2\pi} w_{j,k}(a(s))w_{i,m}(a(s))\ ds = \sum_{h=1}^H \int_{I_h} w_{j,k}(a(s))w_{i,m}(a(s))\ \frac{1}{|a'(s)|}\,|a'(s)|\,ds$$

$$= \sum_{h=1}^H \int_{\gamma_h} w_{j,k}\big|_{\gamma_h}\ w_{i,m}\big|_{\gamma_h}\ \frac{1}{|a'(a^{-1}(x))|}d\gamma_h. \tag{4.31}$$

Substituting (4.30) and (4.31) in (4.27) we obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_\delta^T |u_\varepsilon(a_\varepsilon(t),t)|^2 dt \\
= &\sum_{j,i=1}^{\infty}\sum_{k=1}^{l(j)}\sum_{m=1}^{l(i)} c_{j,k}c_{i,m}\frac{1}{2\pi}\int_\delta^T e^{-(\lambda_i+\lambda_j)t}\sum_{h=1}^{H}\int_{\gamma_h} w_{j,k}|_{\gamma_h}\; w_{i,m}|_{\gamma_h}\frac{1}{|a'(a^{-1}(x))|}d\gamma_h \\
= &\int_\delta^T\int_\gamma |u(x,t)|^2 m_a(x)d\gamma,
\end{aligned}$$

for all $\delta > 0$.

The proof of (4.23) is similar. We only have to take into account that $C_0^\infty(\Omega)\times C_0^\infty(0,T)$ is sequentially dense in $C_0^\infty(\Omega\times(0,T))$ and then it suffices to check (4.23) for test functions in separated variables. This concludes the proof of the lemma.

4.4.3 Oscillating pointwise control of the heat equation

In this section we finally consider the control problem (4.1). We prove the following:

Theorem 4.4.1 *Let us assume that the curve $a(t):\mathbf{R}\to\Omega$ is a 2π-periodic real analytic function, $\varepsilon > 0$ is a small parameter and $a_\varepsilon(t) = a(t/\varepsilon)$. Let us assume also that $a(t)$ is a strategic curve, i.e. the range of a is not included in a nodal curve (see the example 2 after the proof of Lemma* 4.4.1*). Under these hypothesis, system* (4.1) *is approximately controllable for all $\varepsilon > 0$.*

Given $u^0, u^1 \in H_0'$ and $\alpha > 0$ there exists a sequence of approximate controls $f_\varepsilon \in L^2(0,T)$ of system (4.1) *which is uniformly bounded in $L^2(0,T)$ such that the solutions u_ε of* (4.1) *satisfy* (4.3)*. Moreover, the controls can be chosen such that they strongly converge in the following sense:*

$$f_\varepsilon(t)\delta(x-a_\varepsilon(t)) \to f(x,t)m_a(x)1_\gamma \ \text{ in } L^2(0,T;H_1') \text{ as } \varepsilon\to 0, \tag{4.32}$$

where f is an approximate control for the limit system (4.4)*.*

On the other hand, with the above controls the solutions u_ε of (4.1) *converges strongly in $C([0,T];H_0')$ as $\varepsilon\to 0$ to the solution u of the limit problem* (4.4)*.*

Remark 4.4.3

(a) Note that $a(t)$ is *strategic* in the sense of the statement above (see also the example 2) if and only if $a(t/\varepsilon)$ is strategic for all $\varepsilon > 0$.

(b) As a consequence of the statement in Theorem 4.4.1, system (4.4) is approximately controllable. In fact, Theorem 4.4.1 guarantees that the control of (4.4) may be achieved as limit when $\varepsilon \to 0$ of the controls of (4.1) in the sense of (4.32).

Note however that one could prove directly the approximate controllability of (4.4). Indeed, the fact that $a(t)$ is periodic and non-strategic and that P (defined as in Lemma 4.4.1) coincides with γ guarantees that the unique continuation property below holds:

$$\text{If } \varphi \text{ solves (4.5) and } \varphi = 0 \text{ on } \gamma \times (0,T) \text{ then } \varphi \equiv 0. \tag{4.33}$$

This unique continuation property turns out to be equivalent to the approximate controllability of (4.4).

Proof. We first restrict ourselves to the case where $u^0 = 0$ and $\left|u^1\right|_{H_0} \geq \alpha$.

Given $\varepsilon > 0$, system (4.1) is approximate controllable. Indeed according to Lemma 4.4.1) and, in view of the assumptions on the curve $a(t)$, the unique continuation property (4.12) holds with $b(t) = a(t/\varepsilon)$ for all $\varepsilon > 0$. Then the control that makes (4.3) to hold is given by $f_\varepsilon = \overline{\varphi}_\varepsilon(a_\varepsilon(t), t)$, where $\overline{\varphi}_\varepsilon$ solves (4.5) with the initial data $\overline{\varphi}_\varepsilon^0$ being the minimizer of the functional

$$J_\varepsilon(\varphi^0) = \frac{1}{2}\int_0^T |\varphi(a_\varepsilon(t),t)|^2 dt + \alpha \left|\left|\varphi^0\right|\right|_{H_0} - < u^1, \varphi^0 >_{H_0', H_0} \tag{4.34}$$

over H_0. Note, in particular, that the coercivity of this functional is guaranteed by the unique continuation property (4.12).

The adjoint system associated to the limit system (4.4) is also given by (4.5) and the corresponding functional associated to (4.4) is given by

$$J(\varphi^0) = \frac{1}{2}\int_0^T \int_\gamma |\varphi(x,t)|^2 m_a(x) d\gamma dt + \alpha \left|\left|\varphi^0\right|\right|_{H_0} - < u^1, \varphi^0 >_{H_0', H_0}, \tag{4.35}$$

where φ is the solution of (4.5) with final data φ^0.

We set

$$M_\varepsilon = \inf_{\varphi^0 \in H_0} J_\varepsilon(\varphi^0). \tag{4.36}$$

For each $\varepsilon > 0$ the functional J_ε attains its minimum M_ε in H_0. This is a consequence of the unique continuation property (4.6) which allows us to prove the coercivity of J_ε for each $\varepsilon > 0$. This unique continuation property is obtained applying the result of Lemma 4.4.1 to the curve $b(t) = a_\varepsilon(t)$, which satisfies the hypothesis of Lemma 4.4.1.

Lemma 4.4.3 below establishes that the coerciveness of J_ε is indeed uniform in ε. Moreover, if $f(t) = \overline{\varphi}_\varepsilon(a_\varepsilon(t), t)$ where $\overline{\varphi}_\varepsilon$ solve (4.5) with data $\overline{\varphi}_\varepsilon^0$, the solution of (4.1) satisfies (4.3).

Lemma 4.4.3 *We have*

$$\underline{\lim}_{||\varphi^0||_{H_0}\to\infty\,\varepsilon\to 0} \frac{J_\varepsilon(\varphi^0)}{||\varphi^0||_{H_0}} \geq \alpha. \tag{4.37}$$

Furthermore, the minimizers $\{\overline{\varphi_\varepsilon^0}\}_{\varepsilon\geq 0}$ are uniformly bounded in H_0.

Proof of Lemma 4.4.3 Let us consider sequences $\varepsilon_j \to 0$ and $\varphi^0_{\varepsilon_j} \in H_0$ such that $\left|\left|\varphi^0_{\varepsilon_j}\right|\right|_{H_0} \to \infty$ as $j \to \infty$.

Let us introduce the normalized data

$$\psi^0_{\varepsilon_j} = \frac{\varphi^0_{\varepsilon_j}}{\left|\left|\varphi^0_{\varepsilon_j}\right|\right|_{H_0}}$$

and the corresponding solutions of (4.5):

$$\psi_{\varepsilon_j} = \frac{\varphi_{\varepsilon_j}}{\left|\left|\varphi^0_{\varepsilon_j}\right|\right|_{H_0}}.$$

We have

$$\begin{aligned} I_j &= \frac{J_{\varepsilon_j}(\varphi^0_{\varepsilon_j})}{\left|\left|\varphi^0_{\varepsilon_j}\right|\right|_{H_0}} = \frac{1}{2}\left|\left|\varphi^0_{\varepsilon_j}\right|\right|_{H_0}\int_0^T |\psi_{\varepsilon_j}(a_{\varepsilon_j}(t),t)|^2 dt + \\ &+ \alpha - < u^1, \psi^0_{\varepsilon_j} >_{H_0', H_0}. \end{aligned}$$

We distinguish the following two cases:

Case 1. $\underline{\lim}_{j\to\infty}\int_0^T |\psi_{\varepsilon_j}(a_{\varepsilon_j}(t),t)|^2 dt > 0$. In this case, we have clearly $\underline{\lim}_{j\to\infty} I_j = \infty$.

Case 2. $\underline{\lim}_{j\to\infty}\int_0^T |\psi_{\varepsilon_j}(a_{\varepsilon_j}(t),t)|^2 dt = 0$. In this case we argue by contradiction. Assume that there exists a subsequence, still denoted by the index j, such that

$$\int_0^T |\psi_{\varepsilon_j}(a_{\varepsilon_j}(t),t)|^2 dt \to 0 \tag{4.38}$$

and

$$\underline{\lim}_{j\to\infty} I_j < \alpha. \tag{4.39}$$

By extracting a subsequence, still denoted by the index j, we have

$$\psi^0_{\varepsilon_j} \rightharpoonup \psi^0 \text{ weakly in } H_0,$$

and therefore

$$\psi_{\varepsilon_j} \rightharpoonup \psi \text{ weakly-* in } L^\infty(0,T;H_0)$$

where ψ is the solution of (4.5) with initial data ψ^0. By Lemma 4.4.2 we have

$$\psi = 0 \text{ in } \gamma \times (0,T).$$

Now, recall that by hypothesis a_ε is a strategic curve and then Lemma 4.4.1 establishes that $\psi^0 = 0$. Thus

$$\psi^0_{\varepsilon_j} \rightharpoonup 0 \text{ weakly in } H_0$$

and therefore

$$\underline{\lim}_{j\to\infty} I_j \geq \lim \inf_{j\to\infty} (\alpha - < u^1, \psi^0_{\varepsilon_j} >_{H_0', H_0}) = \alpha$$

since u^1_j converges strongly in H_0. This is in contradiction with (4.39) and concludes the proof of (4.37).

On the other hand, it is obvious that $I_\varepsilon \leq 0$ for all $\varepsilon > 0$. Thus, (4.37) implies the uniform boundedness of the minimizers in H_0.

Concerning the convergence of the minimizers we have the following lemma:

Lemma 4.4.4 *The minimizers $\overline{\varphi}^0_\varepsilon$ of J_ε converge strongly in H_0 as $\varepsilon \to 0$ to the minimizer $\overline{\varphi}^0$ of J in* (4.35) *and M_ε converges to*

$$M = \inf_{\overline{\varphi}^0 \in H_0} J(\overline{\varphi}^0). \tag{4.40}$$

Moreover, the corresponding solutions $\overline{\varphi}_\varepsilon$ of (4.5) *converge in $C([0,T];H_0)$ to the solution $\overline{\varphi}$ as $\varepsilon \to 0$.*

Proof of Lemma 4.4.4 By extracting a subsequence, that we still denote by ε, we have

$$\overline{\varphi}^0_\varepsilon \rightharpoonup \psi^0 \text{ weakly in } H_0$$

as $\varepsilon \to 0$. It is sufficient to check that $\overline{\varphi}^0 = \psi^0$ or, equivalently,

$$J(\psi^0) \leq J(\varphi^0) \text{ for all } \varphi^0 \in H_0. \tag{4.41}$$

We know that

$$\overline{\varphi}_\varepsilon \rightharpoonup \psi \text{ weakly-* in } L^\infty(0,T;H_0)$$

where ψ is the solution of (4.5) with initial data ψ^0. By Lemma 4.4.2 we deduce that

$$J(\psi^0) = \lim_{\varepsilon\to 0} J_\varepsilon(\overline{\varphi}^0_\varepsilon). \tag{4.42}$$

On the other hand, for each $\varphi^0 \in H_0$ we have

$$\lim_{\varepsilon \to 0} J_\varepsilon(\overline{\varphi}^0_\varepsilon) \leq \lim_{\varepsilon \to 0} J_\varepsilon(\varphi^0). \tag{4.43}$$

Observe also that for $\varphi^0 \in H_0$ fixed, Lemma 4.4.2 ensures that

$$\lim_{\varepsilon \to 0} J_\varepsilon(\varphi^0) = J(\varphi^0). \tag{4.44}$$

Combining (4.42)-(4.44) it is easy to see that and (4.41) holds.

This concludes the proof of the weak convergence of the minimizers and it also shows that

$$\begin{aligned} \lim_{\varepsilon \to 0} M_\varepsilon \quad &\geq \quad M = J(\bar{\varphi}^0) = \limsup_{\varepsilon \to 0} J_\varepsilon(\overline{\varphi}^0) \\ &\geq \quad \limsup_{\varepsilon \to 0} J_\varepsilon(\overline{\varphi}^0_\varepsilon) = \limsup_{\varepsilon \to 0} M_\varepsilon. \end{aligned} \tag{4.45}$$

Therefore we deduce the convergence $M_\varepsilon \to M$.

Observe that (4.40) combined with the weak convergence of $\overline{\varphi}^0_\varepsilon$ in H_0, implies that

$$\lim_{\varepsilon \to 0} \left(\frac{1}{2} \int_0^T |\overline{\varphi}_\varepsilon(a_\varepsilon(t), t)|^2 dt + \alpha \left|\left|\overline{\varphi}^0_\varepsilon\right|\right|_{H_0} \right) =$$

$$= \frac{1}{2} \int_0^T \int_\gamma |\overline{\varphi}|^2 m_a(x) d\gamma dt + \alpha \left|\left|\overline{\varphi}^0\right|\right|_{H_0},$$

since the last term in $J_\varepsilon(\bar{\varphi}^0_\varepsilon)$, which is linear in $\bar{\varphi}^0_\varepsilon$ passes trivially to the limit.

This identity, combined with the weak convergence of $\overline{\varphi}^0_\varepsilon$ to $\overline{\varphi}^0$ in H_0 and Lemma 4.4.2 implies that

$$\overline{\varphi}^0_\varepsilon \to \overline{\varphi}^0 \text{ strongly in } H_0. \tag{4.46}$$

Therefore, we have

$$\overline{\varphi}_\varepsilon \to \overline{\varphi} \text{ strongly in } C([0,T]; H_0).$$

This concludes the proof of Theorem 4.4.1 when $u^0 = 0$ and $\left|\left|u^1\right|\right|_{L^2(\Omega)} \geq \alpha$.

Let us consider now the case where u^0 is non-zero. We set $v^1 = v(T)$ where v is the solution of (4.1) with $f = 0$. Now observe that the solution u of (4.1) can be written as $u = v + w$ where w is the solution of (4.1) with zero initial data that satisfies $w(T) = u(T) - v^1$. In this way, the controllability problem for u can be reduced to a controllability problem for w with zero initial data $w^0 = 0$. This is the problem we solved. The proof is now complete.

■

Remark 4.4.4 The proof guarantees that the coercivity property (4.37) is also true for the limit functional J. This fact could also be proved arguing directly on J together with the unique continuation property (4.33).

4.5 Finite-difference space semi-discretizations of the heat equation

Let us consider now the following $1-d$ heat equation with control acting on the extreme $x = L$:

$$\begin{cases} u_t - u_{xx} = 0, \ 0 < x < L, \ 0 < t < T \\ u(0,t) = 0, \ u(L,t) = v(t), \ 0 < t < T \\ u(x,0) = u_0(x), \ 0 < x < L. \end{cases} \tag{4.1}$$

As we have seen in section 3, it is well known that system (4.1) is null controllable. To be more precise, the following holds: *For any $T > 0$, and $u_0 \in L^2(0,L)$ there exists a control $v \in L^2(0,T)$ such that the solution u of* (4.1) *satisfies*

$$u(x,T) \equiv 0 \ in \ (0,L). \tag{4.2}$$

This null controllability result is equivalent to a suitable observability inequality for the adjoint system:

$$\begin{cases} \varphi_t + \varphi_{xx} = 0, \ 0 < x < L, \ 0 < t < T, \\ \varphi(0,t) = \varphi(L,t) = 0, \ 0 < t < T \\ \varphi(x,T) = \varphi_0(x), \ 0 < x < L. \end{cases} \tag{4.3}$$

The corresponding observability inequality is as follows: *For any $T > 0$ there exists $C(T) > 0$ such that*

$$\int_0^L \varphi^2(x,0)dx \le C \int_0^T |\varphi_x(L,t)|^2 \, dt \tag{4.4}$$

holds for every solution of (4.3).

Let us consider now the semi-discrete versions of systems (4.1) and (4.3):

$$\begin{cases} u_j' - [u_{j+1} + u_{j-1} - 2u_j] / \, h^2 = 0, \ 0 < t < T, \ j = 1, \cdots, N \\ u_0 = 0, \ u_{N+1} = v, \ 0 < t < T \\ u_j(0) = u_{0,j}, \ j = 1, \cdots, N; \end{cases} \tag{4.5}$$

$$\begin{cases} \varphi_j' + [\varphi_{j+1} + \varphi_{j-1} - 2\varphi_j] / \, h^2 = 0, \ 0 < t < T, \ j = 1, \cdots, N \\ \varphi_0 = \varphi_{N+1} = 0, \ 0 < t < T \\ \varphi_j(T) = \varphi_{0,j}, \ j = 1, \cdots, N. \end{cases} \tag{4.6}$$

Here and in the sequel $h = L/(N+1)$ with $N \in \mathbf{N}$. The parameter h measuring the size of the numerical mesh is devoted to tend to zero.

In this case, in contrast with the results we have described on the wave equation, systems (4.5) and (4.6) are uniformly controllable and observable respectively as $h \to 0$.

More precisely, the following results hold:

Theorem 4.5.1 ([155]) *For any $T > 0$ there exists a positive constant $C(T) > 0$ such that*

$$h\sum_{j=1}^{N}|\varphi_j(0)|^2 \leq C\int_0^T \left|\frac{\varphi_N(t)}{h}\right|^2 dt \tag{4.7}$$

holds for any solution of (4.6) *and any $h > 0$.*

Theorem 4.5.2 ([155]) *For any $T > 0$ and $\{u_{0,1}, \cdots, u_{0,N}\}$ there exists a control $v \in L^2(0,T)$ such that the solution of* (4.5) *satisfies*

$$u_j(T) = 0,\ j = 1, \cdots, N. \tag{4.8}$$

Moreover, there exists a constant $C(T) > 0$ independent of $h > 0$ such that

$$\| v \|^2_{L^2(0,T)} \leq Ch\sum_{j=1}^{N}|u_{0,j}|^2. \tag{4.9}$$

These results were proved in [155] using Fourier series and Lemma 4.3.2.

One can even prove that the null controls for the semi-discrete equation (4.5) can be built so that, as $h \to 0$, they tend to the null control for the continuous heat equation (4.1). According to this result the control of the heat equation is the limit of the controls of the semi-discrete systems (4.5) and this is relevant in the context of the Numerical Analysis (see chapter 3).

In this problem the parameter h plays the role of the parameter ε in the homogenization problem discussed in section 3. But things are much simpler here since the spectrum of the finite-difference scheme can be computed explicitly ([113]). Moreover, in this case, the three-steps control method described in section 3 is not required since the high frequency components do not arise in the semi-discrete setting.

As we shall see below the extension of these results to the multi-dimensional setting is a widely open subject of research.

4.6 Open problems

There is an important number of relevant open problems in this field. Here we mention some of the most significant ones:

1. **Heat equation in perforated domains:** Let us consider the heat equation in a perforated domain Ω_ε of $\mathbf{R}^n$, $n \geq 2$. Does null controllability hold uniformly as the size of the holes tends to zero? Is this true when the size of the holes is sufficiently small?

   At this respect it is important to note that, according to the results by Donato and Nabil [67], the property of approximate controllability is indeed uniform. But, as we have shown along these Notes, there is a big gap between approximate and null controllability.

2. **Heat equation with rapidly oscillating coefficients:** Do the results of section 3 on the uniform null controllability of the heat equation with rapidly oscillating coefficients hold in the multi-dimensional case? Note in particular that one may expect this result to be true without geometric conditions in the control subdomain.

   On the other hand, even in one space dimension, do the results of section 3 on the uniform null controllability hold for general bounded measurable coefficients without further regularity assumptions?

3. **Heat equation with irregular coefficients.** As far as we know there is no example in the literature of heat equation, with bounded, measurable and coercive coefficients for which the null controllability does not hold. The problem of finding counterexamples or relaxing the additional regularity assumptions on the coefficients we have used along these Notes is open. On the other hand, the existing results that are based in the use of Carleman inequalities require some regularity assumptions on the coefficients. Roughly speaking, null controllability is known to hold when the coefficients are of class C^1 ([90]). In the one-dimensional case, in [86], it was proved that the BV regularity of coefficients suffices.

   At this respect it is important to note that, in the context of the 1-d wave equation, the Hölder continuity of the coefficients is not enough to guarantee the null controllability (see [37]). Indeed, in that case, the minimum regularity for the coefficients required to obtain controllability is BV. The counterexample in [37] for Hölder continuous coefficients is based in a construction of a sequence of high-frequency eigenfunctions which is mainly concentrated around a fixed point. In the context of the heat equation these high-frequency eigenfunctions dissipate too fast and do not produce any counterexample to the null controllability problem.

4. **Nonlinear problems.** The extension of the results of these Notes to nonlinear problems is a wide open subject. In [83, 84] the problem of null and approximate controllabilty was treated for semi-linear heat equations and, in particular, it was proved that null controllability may hold for some nonlinearities for which, in the absence of control, blow-up phenomena arise. Similar problems were addressed in [7] for nonlinearities involving gradient terms. However, nothing is known in the context of homogenization.

5. **Numerical approximations.** We have presented here some results showing the analogy of the behavior of the homogenization and numerical

problems with respect to controllability. However, the examples considered so far are quite simple. There is much to be done to develop a complete theory and, in particular, to address problems in several space dimensions.

6. **Rapidly oscillating pointwise controllers.** In section 4.4 we have addressed the problem of the approximate controllability of the constant coefficients heat equation with pointwise controllers that are localized in a point that oscillates rapidly in time. We have shown that the approximate controllability property is uniform as the oscillation parameter tends to zero and we have shown that, in the limit, one recovers the approximate controllability property with a control distributed along an interior curve. Do the same results hold in the context of null controllability?

7. **Uniqueness in the context of pointwise control.** In section 4 (Lemma) we have proved an uniqueness result for the solutions of the heat equation vanishing on the curve $x = b(t)$, $0 \leq t \leq T$. This proof requires of the time-analyticity of the solutions and their decomposition in Fourier series. It would be interesting to develop other tools (based, for instance, in Carleman inequalities) allowing to extend this uniqueness result to more general situations like, for instance, heat equations with potentials depending both on space and time.

Chapter 5

Null control of a $1-d$ model of mixed hyperbolic-parabolic type

in "Optimal Control and Partial Differential Equations", J. L. Menaldi et al., eds., IOS Press, 2001, pp. 198–210.

5.1 Introduction and main result

In this article we consider the problem of null controllability for the following mixed system of hyperbolic-parabolic type:

$$\begin{cases} y_{tt} - y_{xx} = 0, & -1 < x < 0, \quad t > 0 \\ z_t - z_{xx} = 0, & 0 < x < 1, \quad t > 0 \\ y = z, \, y_x = z_x, & x = 0, \quad t > 0 \\ y(-1,t) = v(t), & t > 0 \\ z(1,t) = 0, & t > 0 \\ y(x,0) = y_0(x), \, y_t(x,0) = y_1(x), & -1 < x < 0 \\ z(x,0) = z_0(x), & 0 < x < 1. \end{cases} \tag{5.1}$$

This system represents the coupling between the wave equation arising on the interval $(-1,0)$ with state y and the heat equation that holds on the interval $(0,1)$ with state z. At the interface, the point $x=0$, we impose the continuity of (y,z) and (y_x,z_x). The system is complemented with boundary conditions at the free ends $x=\pm 1$ and initial conditions at time $t=0$. The control $v=v(t)$ acts on the system through the extreme point $x=-1$.

This system might be viewed as a "toy model" of fluid-structure interaction. We refer to [149] and [177] for an analysis of the approximate controllability property for other, more complete, models in this context.

A lot of progress has been done in what concerns the controllability of heat and wave equations. In both cases, following J.L. Lions' HUM method (see [142, 143]), the problem may be reduced to the obtention of suitable observability inequalities for the underlying uncontrolled adjoint systems. However, the techniques that have been developed to obtain such estimates differ very much from one case to the other one. In the context of the wave equation one may use multipliers (see for instance [142, 143]) or microlocal analysis ([14]) while, in the context of parabolic equations, one uses Carleman inequalities (see for instance [90], [134], [83]). Carleman inequalities have also been used to obtain observability estimates for wave equations ([212]), but, up to now, as far as we know, there is no theory describing how the Carleman inequalities for the parabolic equation may be obtained continuously from the Carleman inequalities for hyperbolic equations. This problem was addressed in [153] by viewing the heat equation $u_t - \Delta u = 0$ as limit of wave equations of the form $\varepsilon u_{tt} - \Delta u + u_t = 0$. But in [153], the Carleman inequalities were not uniform as $\varepsilon \to 0$ and therefore, Carleman inequalities were combined with a careful spectral analysis.

Summarizing, one may say that the techniques that have been developed to prove observability inequalities for wave and heat equations are difficult to combine and therefore there is, to some extent, a lack of tools to address controllability problems for systems in which both hyperbolic and parabolic components are present.

However, some examples have been addressed with succes. For instance, in [135] and [235] we considered the system of three-dimensional elasticity. There, using decoupling techniques, we were able to overcome these difficulties. However, in doing that, the fact that the hyperbolic and parabolic component of the solution of the system of thermoelasticity occupy the same domain played a crucial role.

The model we discuss here has the added difficulty that the two equations hold in two different domains and that they are only coupled through an interface where we impose transmission conditions guaranteeing the well-posedness of the initial-boundary value problem. On the contrary, our analysis is by now restricted to the $1-d$ case.

In the absence of control, i.e. when $v \equiv 0$, the energy

$$E(t) = \frac{1}{2}\int_{-1}^{0}\left[\mid y_x(x,t)\mid^2 + \mid y_t(x,t)\mid^2\right]dx + \frac{1}{2}\int_0^1 |z_x(x,t)|^2 dx \tag{5.2}$$

is decreasing. More precisely,

$$\frac{dE}{dt}(t) = -\int_0^1 |z_{xx}(x,t)|^2 dx. \tag{5.3}$$

Therefore, when $v \equiv 0$, for initial data

$$(y_0, y_1) \in H^1(-1,0) \times L^2(-1,0),\ z_0 \in H^1(0,1) \tag{5.4}$$

with

$$y_0(-1) = 0, y_0(0) = z_0(0), z_0(1) = 0, \tag{5.5}$$

system (5.1) admits an unique solution

$$\begin{cases} y \in C\left([0,\infty);\ H^1(-1,0)\right) \cap C^1\left([0,\infty);\ L^2(-1,0)\right) \\ z \in C\left([0,\infty);\ H^1(0,1)\right) \cap L^2(0,T; H^2(0,1)). \end{cases} \tag{5.6}$$

Note that, when (5.4) hold, y_0 and z_0 are simply the restriction of a function of $H_0^1(-1,1)$ to the left and right intervals $(-1,0)$ and $(0,1)$, respectively. Thus, as a consequence of (5.6), and abusing of notation, we may write that

$$(y,z) \in C\left([0,\infty);\ H_0^1(-1,1)\right). \tag{5.7}$$

The same existence and uniqueness result holds when $v \not\equiv 0$ but it is smooth enough.

Here we are interested on the problem of null-controllability. More precisely, given $T > 0$ and initial data $\{(y_0,y_1),\ z_0\}$ as above, we look for a control $v = v(t)$ (say, in $L^2(0,T)$), such that the solution of (5.1) is at rest at time $t = T$.

Here, being at rest at time $t = T$ means fulfilling the conditions

$$y(x,T) \equiv y_t(x,T) \equiv 0,\ -1 < x < 0;\ z(x,T) \equiv 0,\ 0 < x < 1. \tag{5.8}$$

As we mentioned above, there is a large literature in the subject in what concerns wave and heat equations, but much less is known when both components are coupled. We refer to the survey articles [237] and [241] for a description of the state of the art in this field.

If we relax the controllabillity condition (5.8) to a weaker one requiring the distance of the solution at time T to the target to be less than an arbitrarily small ε, i. e. to the so called approximate controllability property, the main difficulties disappear. Indeed, as a consequence of Holmgren's Uniqueness Theorem, this property turns out to hold even in several space dimensions. But, as we shall see, when doing this, the main difficulty arising when analyzing the null-control problem, i. e. the so called observability inequality, is avoided.

5.2 Observability of the adjoint system

As usual, when studying controllability problems, the key point is the obtention of suitable observability estimates for the adjoint system. Once this is done the null control may be easily obtained minimizing a suitable quadratic functional on a Hilbert space.

Let us therefore consider the adjoint system

$$\begin{cases} \varphi_{tt} - \varphi_{xx} = f & \text{in} \quad (-1,0) \times (0,T) \\ -\psi_t - \psi_{xx} = g & \text{in} \quad (0,1) \times (0,T) \\ \varphi(0,t) = \psi(0,t) & \text{for} \quad t \in (0,T) \\ \varphi_x(0,t) = \psi_x(0,t) & \text{for} \quad t \in (0,T) \\ \varphi(-1,t) = \psi(1,t) = 0 & \text{for} \quad t \in (0,T) \\ \varphi(x,T) = \varphi_0(x), \ \varphi_t(x,T) = \varphi_1(x) & \text{in} \quad (-1,0) \\ \psi(x,T) = \psi_0(x) & \text{in} \quad (0,1). \end{cases} \tag{5.9}$$

Multiplying in (5.9) formally by (y,z) and integrating by parts it follows that

$$\int_{-1}^{0}\int_{0}^{T} fydxdt + \int_{0}^{1}\int_{0}^{T} gzdxdt \tag{5.10}$$
$$= \int_{0}^{T} \varphi_x(-1,t)v(t)dt - \int_{0}^{1} \left[\psi_0(x)z(x,T) - \psi(x,0)z_0(x)\right] dx$$
$$+ \int_{-1}^{0} \left[\varphi_1(x)y(x,T) - \varphi_0(x)y_t(x,T) - \varphi_t(x,0)y_0(x) + \varphi(x,0)y_1(x)\right] dx.$$

Obviously, in the obtention of (5.10) the transmission conditions in (5.1) and (5.9) have played a crucial role to cancel the terms appearing at the interface $x = 0$ when integrating by parts.

Using classical energy estimates it can be shown that, when

$$f \in L^1\left(0,T; L^2(-1,0)\right), \ g \in L^2\left(0,T; L^2(0,1)\right), \ (\varphi_0, \psi_0) \in H_0^1(-1,1)$$

and $\varphi_1 \in L^2(-1,0)$, system (5.9) admits an unique solution

$$\begin{cases} (\varphi,\psi) \in C\left([0,T];\, H_0^1(-1,1)\right); \ \varphi_t \in C^1\left([0,T];\, L^2(-1,0)\right) \\ \psi \in L^2\left(0,T;\, H^2(0,1)\right). \end{cases} \tag{5.11}$$

It is then easy to see using the classical results on the "hidden regularity" of solutions of the wave equation that

$$\varphi_x(-1,t) \in L^2(0,T) \tag{5.12}$$

as well, since this property holds locally around the boundary for finite energy solutions of the wave equation (see [142, 143], Tome 1). Thus, in the present case, the presence of the heat component to the right of $x = 0$ is not an obstacle for this property of regularity of the trace of the normal derivative of the wave component to hold.

By transposition, we deduce that, whenever $v \in L^2(0,T)$, $y_0 \in L^2(-1,0)$ and $(y_1, z_0) \in H^{-1}(-1,1)$, system (5.1) admits an unique solution

$$y \in C\left([0,T];\, L^2(-1,0)\right),\ (y_t, z) \in C\left([0,T];\, H^{-1}(-1,1)\right). \tag{5.13}$$

Our goal is to prove the null-controllability of system (5.1) in this functional setting.

For this we need the following observability property for the solutions of the adjoint system:

Proposition 5.2.1 *Assume that $f \equiv g \equiv 0$.*
Let $T > 2$. Then, there exists $C > 0$ such that

$$\| \left(\varphi(x,0),\, \psi(x,0)\right) \|^2_{H^1_0(-1,1)} + \| \varphi_t(x,0) \|^2_{L^2(-1,0)} \leq C \, \| \varphi_x(-1,t) \|^2_{L^2(0,T)} \tag{5.14}$$

for every solution of (5.9) *with $f \equiv g \equiv 0$.*

Proof. We proceed in three steps.

Step 1. Sidewise energy estimates for the wave equation.

Arguing as in [235] and using the fact that φ satisfies the homogeneous wave equation on the left space interval $x \in (-1,0)$ (since $f \equiv 0$) we deduce that

$$\int_{1+x}^{T-(1+x)} \left[|\varphi_t(x,t)|^2 + |\varphi_x(x,t)|^2\right] dt \leq \int_0^T |\varphi_x(-1,t)|^2\, dt, \quad \forall x \in [-1,0]. \tag{5.15}$$

In particular, integrating with respect to $x \in (-1,0)$:

$$\int_{-1}^0 \int_{1+x}^{T-(1+x)} \left(\varphi_t^2 + \varphi_x^2\right) dxdt \leq \int_0^T |\varphi_x(-1,t)|^2\, dt \tag{5.16}$$

and, at $x = 0$,

$$\int_1^{T-1} \left[|\varphi_t(0,t)|^2 + |\varphi_x(0,t)|^2\right] dt \leq \int_0^T |\varphi_x(-1,t)|^2\, dt. \tag{5.17}$$

Using the fact that $\varphi = 0$ at $x = -1$ and Poincaré inequality we also deduce that

$$\int_1^{T-1} |\varphi(0,t)|^2\, dt \leq \int_{-1}^0 \int_{1+x}^{T-(1+x)} \left(\varphi_t^2 + \varphi_x^2\right) dxdt. \tag{5.18}$$

This inequality, combined with (5.16) yields

$$\int_1^{T-1} |\varphi(0,t)|^2\, dt \leq C \int_0^T |\varphi_x(-1,t)|^2\, dt \tag{5.19}$$

for some $C > 0$, independent of φ.

Step 2. Estimates for the heat equation.

In view of (5.17)-(5.18) and using the transmission conditions at $x = 0$ we deduce that

$$\int_1^{T-1} \left[| \psi(0,t) |^2 + | \psi_t(0,t) |^2 + | \psi_x(0,t) |^2 \right] dt \leq C \int_0^T |\varphi_x(-1,t)|^2 \, dt. \tag{5.20}$$

Our goal in this second step is to determine how much of the energy of ψ we can estimate in terms of the left hand side of (5.20). Note that (5.20) provides estimates on the Cauchy data of ψ at $x = 0$ in the time interval $(1, T-1)$, which is non empty because of the assumption $T > 2$. In order to simplify the notation, in this step we translate the interval $(1, T-1)$ into $(0, T')$ with $T' = T - 2$. This can be done because the system under consideration is time independent. On the other hand, taking into account that the inequalities for the heat equation we shall use hold in any interval of time, we can replace T' by T to simplify the notation.

We have to use the fact that ψ satisfies

$$\begin{cases} \psi_t + \psi_{xx} = 0, & \text{in} \quad (0,1) \times (0,T) \\ \psi(1,t) = 0, & \text{for} \quad t \in (0,T). \end{cases} \tag{5.21}$$

Note that the boundary condition of ψ at $x = 0$ is unknown, although, according to (5.20), we have an estimate on its $H^1(0,T)$ norm.

We decompose ψ as follows:

$$\psi = \theta + \eta \tag{5.22}$$

with θ solution of

$$\begin{cases} \theta_t + \theta_{xx} = 0 & \text{in} \quad (0,1) \times (0,T) \\ \theta(x,T) = 0 & \text{in} \quad (0,1) \\ \theta(0,t) = \psi(0,t) & \text{for} \quad t \in (0,T) \\ \theta(1,t) = 0 & \text{for} \quad t \in (0,T), \end{cases} \tag{5.23}$$

and η solving

$$\begin{cases} \eta_t + \eta_{xx} = 0 & \text{in} \quad (0,1) \times (0,T) \\ \eta(x,T) = \psi(x,T) & \text{in} \quad (0,1) \\ \eta(0,t) = \eta(1,t) = 0 & \text{for} \quad t \in (0,T). \end{cases} \tag{5.24}$$

Analyzing the regularity of solutions of (5.23) one can deduce that

$$\| \theta \|_{L^2\left(0,T;\, H^{5/2-\delta}(0,1)\right)} + \| \theta_t \|_{L^2\left(0,T;\, H^{1/2-\delta}(0,1)\right)} \leq C_\delta \, \| \psi(0,t) \|_{H^1(0,1)} \tag{5.25}$$

for all $\delta > 0$.

In particular

$$\| \theta_x(0,t) \|_{L^2(0,T)} \leq C \| \psi(0,t) \|_{H^1(0,T)} . \tag{5.26}$$

Combining (5.20) and (5.26) we deduce that

$$\begin{aligned} \| \eta_x(0,t) \|^2_{L^2(0,T)} &\leq C \left[\| \psi(0,t) \|^2_{H^1(0,T)} + \| \psi_x(0,t) \|^2_{L^2(0,T)} \right] \\ &\leq C \| \varphi_x(-1,t) \|^2_{L^2(0,T)} . \end{aligned} \tag{5.27}$$

Now, using the classical observability estimates (see [155] and [183]) for the solutions η of the heat equation (5.24) with homogeneous Dirichlet boundary conditions we deduce that

$$\| \eta \|_{L^2(0,T-s;\, H^\sigma(0,1))} \leq C_{s,\sigma} \| \eta_x(0,t) \|_{L^2(0,T)} \tag{5.28}$$

for all $s \in (0,T)$ and for all $\sigma > 0$, with $C_{s,\sigma}$ independent of η, which, combined with (5.27), yields

$$\| \eta \|_{L^2(0,T-s;\, H^\sigma(0,1))} \leq C_{s,\sigma} \| \varphi_x(-1,t) \|_{L^2(0,T)} \tag{5.29}$$

Combining (5.25) and (5.29) and going back to the time interval $(1, T-1)$ we deduce that

$$\| \psi \|_{L^2(1,T-1-\delta,\, H^1(0,1))} \leq C_\delta \| \varphi_x(-1,t) \|_{L^2(0,T)} \tag{5.30}$$

for all $\delta \in (0, T-2)$.

Step 3. Conclusion.

Combining (5.16) and (5.30) we have that

$$\begin{aligned} &\int_1^{T-1-\delta} \int_{-1}^0 \left[|\varphi_t(x,t)|^2 + |\varphi_x(x,t)|^2 \right] dxdt \\ &\leq + \int_1^{T-1-\delta} \int_0^1 |\psi_x(x,t)|^2 \, dxdt C_\delta \int_0^T |\varphi_x(-1,t)|^2 \, dt, \end{aligned} \tag{5.31}$$

for all $\delta > 0$ with $T - 2 - \delta > 0$.

Taking into account that the energy

$$E(t) = \frac{1}{2} \int_{-1}^0 \left[|\varphi_t(x,t)|^2 + |\varphi_x(x,t)|^2 \right] dx + \frac{1}{2} \int_0^1 |\psi_x(x,t)|^2 \, dx$$

is a non decreasing function of time when (φ, ψ) solve (5.9) with $f \equiv g \equiv 0$, inequality (5.14) holds.

■

5.3 Null-controllability

As a consequence of Proposition 5.2.1 the following null-controllability property of system (5.1) may be deduced:

Theorem 5.3.1 *Assume that* $T > 2$. *Then, for every* $y_0 \in L^2(-1, 0)$, $(y_1, z_0) \in H^{-1}(-1, 1)$ *there exists a control* $v \in L^2(0, T)$ *such that the solution* (y, z) *of* (5.1) *satisfies* (5.8).

The proof of Theorem 5.3.1 may be done as in [83]. Using the variational approach to approximate controllability (see [73]), for any $\varepsilon > 0$, one can easily find a control v_ε such that

$$\| y(T) \|_{L^2(-1,0)} + \| (y_t(T), z(T)) \|_{H^{-1}(-1,1)} \leq \varepsilon.$$

Moreover, according to (5.14) one can show that v_ε remains bounded in $L^2(0, T)$ as $\varepsilon \to 0$. Passing to the limit as $\varepsilon \to 0$ one gets the desired null-control.

5.4 Further comments

- The tools we have developed can be easily extended to treat similar systems with variable coefficients. One can also handle the case in which the space interval is divided in three pieces so that the heat equation arises in the middle one and the wave equation holds in the other two. Controlling on both extremes of the interval through the two wave equations allows to control to zero the whole process.

- The same techniques allow to treat other boundary and transmission conditions. For instance, in the context of fluid-structure interaction it is more natural to consider transmission conditions of the form:

$$y_t = z; \quad y_x = z_x \quad \text{at } x = 0, \text{ for all } t > 0. \tag{5.32}$$

  When doing this, y_t represents the velocity in the displacement of the structure and z the velocity of the fluid and the energy of the system is then:

$$E(t) = \frac{1}{2} \int_{-1}^{0} \left[| y_x(x, t) |^2 + | y_t(x, t) |^2 \right] dx + \frac{1}{2} \int_{0}^{1} |z(x, t)|^2 dx \tag{5.33}$$

  The method of proof of the observability inequality developed in section 2 applies in this case too.

However, many interesting questions are completely open. For instance:

- A similar result is true when the control acts on the right extreme point $x = 1$ through the parabolic component?

  In what concerns observability, this problem is equivalent to replacing $\| \varphi_x(-1,t) \|_{L^2(0,T)}$ by $\| \psi_x(1,t) \|_{L^2(0,T)}$ in (5.14). The proof given above does not apply readily in this case because of the lack of sidewise energy estimates for the heat equation.

  The same question arises for the boundary conditions (4.1).

- Extending the result of this paper to the case of several space dimensions is also a challenging open problem. Given a domain Ω and an open subset $\omega \subset\subset \Omega$ we consider the wave equation in the outer region $\Omega \backslash \bar{\omega}$ and the heat equation in the inner one ω, coupled by suitable transmission conditions in the interface $\partial\omega$ as in (5.1) or (5.33). Can we control the whole process acting on the outer boundary $\partial\Omega$ on the wave component during a large enough time?

  The techniques developed in the literature up to now to deal with multi-dimensional controllability problems seem to be insufficient to address this question.

Chapter 6

Control, observation and polynomial decay for a coupled heat-wave system *(X. Zhang and E. Zuazua)*

joint work with X. Zhang, in C. R. Acad. Sci. Paris, I, **336**, *823–828.*

6.1 Introduction

In this chapter, we consider first the null controllability problem of the following $1-d$ linearized model for fluid-structure interaction with boundary control either through the hyperbolic component:

$$\begin{cases} u_t - u_{xx} = 0 & \text{in } (0,T)\times(0,1), \\ v_{tt} - v_{xx} = 0 & \text{in } (0,T)\times(-1,0), \\ u(t,1) = 0, \quad v(t,-1) = g_1(t) & t\in(0,T), \\ u(t,0) = v_t(t,0), \quad u_x(t,0) = v_x(t,0) & t\in(0,T), \\ u(0) = u_0 & \text{in } (0,1), \\ v(0) = v_0, \quad v_t(0) = v_1 & \text{in } (-1,0), \end{cases} \tag{6.1}$$

or through the parabolic one:

$$\begin{cases} u_t - u_{xx} = 0 & \text{in } (0,T)\times(0,1), \\ v_{tt} - v_{xx} = 0 & \text{in } (0,T)\times(-1,0), \\ u(t,1) = g_2(t), \quad v(t,-1) = 0 & t\in(0,T), \\ u(t,0) = v_t(t,0), \quad u_x(t,0) = v_x(t,0) & t\in(0,T), \\ u(0) = u_0 & \text{in } (0,1), \\ v(0) = v_0, \quad v_t(0) = v_1 & \text{in } (-1,0). \end{cases} \tag{6.2}$$

Here $T > 0$ is a finite control time, which will be needed to be large enough for the control problems to have a positive answer. Similar null controllability problems for systems (6.1) and (6.2) with the transmission condition $u(t,0) = v_t(t,0)$ replaced by $u(t,0) = v(t,0)$ were considered in [240] and [229]. Note however that, the transmission condition considered in this paper is more natural from the modelling point of view: u may be viewed as the velocity of the linearized $1-d$ fluid; while v_t represents the velocity of the deformation of the structure.

In (6.1), $g_1(t) \in H_0^1(0,T)$ is the control acting on the system through the wave extreme $x = -1$; while the state space is the Hilbert space $\mathcal{H} \equiv L^2(0,1) \times H^1(-1,0) \times L^2(-1,0)$ with the canonical norm.

Put $H = \{(\phi,\psi,\eta) \mid \phi \in L^2(0,1), \psi \in H^1(-1,0) \text{ with } \psi(-1) = 0, \eta \in L^2(-1,0)\}$. Obviously, H is a Hilbert space with the norm $|(\phi,\psi,\eta)|_H = \left[|\phi|^2_{L^2(0,1)} + |\psi_x|^2_{L^2(-1,0)} + |\eta|^2_{L^2(-1,0)}\right]^{1/2}$. By means of the transposition method, it is easy to show that, for any $(u_0,v_0,v_1) \in H (\subsetneq \mathcal{H})$ and $g_1 \in H_0^1(0,T)$, system (6.1) admits a unique solution (u,v,v_t) in the class $C([0,T];\mathcal{H})$ with $(u(T),v(T),v_t(T)) \in H$. Note that, of course, the trajectories of (6.1) are not in H unless $g_1 \equiv 0$ (since the second component of the element in H vanishes at $x = -1$).

In (6.2), $g_2(t) \in H_0^1(0,T)$ is the control acting on the system through the heat extreme $x = 1$; while the state space is H. Using again the transposition method, it is easy to show that, for any $(u_0,v_0,v_1) \in H$ and $g_2 \in H_0^1(0,T)$, system (6.2) admits a unique solution (u,v,v_t) in the class $C([0,T];H)$.

Our first goal is to select a control g_1 (*resp.* g_2) such that the solution of (6.1) (*resp.* (6.2)) vanishes at time $t = T$. By a classical duality argument ([139]), this may be reduced to the obtention of boundary observability estimates for the following system through the wave and heat components, respectively.

$$\begin{cases} y_t - y_{xx} = 0 & \text{in } (0,\infty)\times(0,1), \\ z_{tt} - z_{xx} = 0 & \text{in } (0,\infty)\times(-1,0), \\ y(t,1) = z(t,-1) = 0, y(t,0) = z_t(t,0), y_x(t,0) = z_x(t,0) & t\in(0,\infty), \\ y(0) = y_0 & \text{in } (0,1), \\ z(0) = z_0, z_t(0) = z_1 & \text{in } (-1,0). \end{cases} \tag{6.3}$$

System (6.3) is well-posed in H. Moreover, the energy of system (6.3),

$$E(t) \triangleq \frac{1}{2}\left[\int_{-1}^{0}\left(|z_x(t,x)|^2+|z_t(t,x)|^2\right)dx+\int_0^1|y(t,x)|^2dx\right],$$

decreases along trajectories. More precisely,

$$\frac{d}{dt}E(t)=-\frac{1}{2}\int_0^1|y_x|^2dx.$$

This formula shows that the only dissipation mechanism of system (6.3) comes from the heat component. The decay rate of $E(t)$ will also be addressed in this Note. As we shall see, unlike the pure heat equation or the $1-d$ wave equation dissipated on a subinterval, this dissipation mechanism is not strong enough to produce an exponential decay of the energy.

In order to show the boundary observability of (6.3) in H through the wave component, we proceed as in [240] by combining the sidewise energy estimate for the wave equation and the Carleman inequalities for the heat equation. However, due to the new transmission condition $y(t,0)=z_t(t,0)$ on the interface, some undesired lower order term occurs in the observability inequality. Hence, we will need to use the classical Compactness-Uniqueness Argument ([244]) to absorb it (Note that this argument is not necessary in [240] and [229]). On the other hand, the functional setting of the observability inequality differs from that in [240].

As for the boundary observability estimates for (6.3) through the heat component, similar to [229], we need to develop first a careful spectral analysis for the underlying semigroup of (6.3). Our spectral analysis yields:

a) Lack of observability of system (6.3) in H from the heat extreme $x=1$ with a defect of infinite order;

b) A new Ingham-type inequality for mixed parabolic and hyperbolic spectra;

c) The observability of system (6.3) in a Hilbert space with, roughly speaking, exponentially small weight for the Fourier coefficients of the hyperbolic eigenvectors;

d) And then the null controllability of system (6.2) in a Hilbert space with, roughly speaking, exponentially large weight for the Fourier coefficients of the hyperbolic eigenvectors.

6.2 Boundary control and observation through the wave component

We begin with the following observability estimate:

Theorem 6.2.1 *Let $T > 2$. Then there is a constant $C > 0$ such that every solution of equation* (6.3) *satisfies*

$$|(y(T), z(T), z_t(T))|_H^2 \leq C|z_x(\cdot, -1)|^2_{L^2(0,T)}, \qquad \forall\ (y_0, z_0, z_1) \in H. \tag{6.4}$$

By means of the duality argument, Theorem 6.2.1 yields the null controllability of (6.1) but with the trajectories in a Hilbert space larger than $\mathcal{H}$. In order to obtain the null controllability of (6.1) in $\mathcal{H}$, we need to derive another observability inequality, which reads:

Theorem 6.2.2 *Let $T > 2$. Then there is a constant $C > 0$ such that every solution of equation* (6.3) *satisfies*

$$\begin{aligned} |(y(T), z(T), z_t(T))|_H^2 \quad &\leq \quad C\left| z_x(\cdot, -1) - \frac{1}{T}\int_0^T z_x(t, -1)dt \right|^2_{L^2(0,T)}, \\ &\forall\ (y_0, z_0, z_1) \in H. \end{aligned} \tag{6.5}$$

Note that Theorem 6.2.1 will play a key role in Section 4 when deducing the Ingham-type inequality. Theorem 6.2.2 states that the observability is still true by making weaker, zero average, boundary measurements. As far as we know, the fact this inequality holds is also new in the case of a simple wave equation.

As we mentioned before, similar to [240], the proof of Theorems 6.2.1 and 6.2.2 is based on the sidewise energy estimate for the wave equation and the Carleman inequalities for the heat equation. However, some elementary but key technique of lifting the underlying Hilbert space and the classical Compactness-Uniqueness Argument (*see* [244]) are also necessary in the proof. Note that one does need the later two techniques in [240] and [229].

Theorem 6.2.2 implies the null controllability of system (6.1) through the wave component:

Theorem 6.2.3 *Let $T > 2$. Then for every $(u_0, v_0, v_1) \in H$, there exists a control $g_1 \in H_0^1(0, T)$ such that the solution (u, v, v_t) of system* (6.1) *satisfies $u(T) = 0$ in $(0, 1)$ and $v(T) = v_t(T) = 0$ in $(-1, 0)$.*

6.3 Spectral analysis

System (6.3) can be written in an abstract form $Y_t = \mathcal{A}Y$ with $Y(0) = Y_0$. Here $\mathcal{A} :\ D(\mathcal{A}) \subset H \to H$ is an unbounded operator defined as follows: $\mathcal{A}Y = (f_{xx}, h, g_{xx})$, where $Y = (f, g, h) \in D(\mathcal{A})$, and $D(\mathcal{A}) \equiv \{(f, g, h) \mid f \in H^2(0, 1),\ g \in H^2(-1, 0),\ h \in H^1(-1, 0),\ f(1) = g(-1) = h(-1) = 0,\ f(0) =$

$h(0),\ f_x(0) = g_x(0)\}$. It is easy to show that $\mathcal{A}$ generates a contractive C_0-semigroup in H with compact resolvent. Hence $\mathcal{A}$ has a sequence of eigenvalues (in $\mathbb{C}$) tending to ∞.

The main result in this section reads:

Theorem 6.3.1 *The large eigenvalues of $\mathcal{A}$ can be divided into two classes $\{\lambda_\ell^p\}_{\ell=\ell_1}^\infty$ and $\{\lambda_k^h\}_{|k|=k_1}^\infty$, where ℓ_1 and k_1 are suitable positive integers, which satisfy the following asymptotic estimates as ℓ and k tend to ∞ respectively:*

$$\lambda_\ell^p = -(1/2+\ell)^2\pi^2+2+O(\ell^{-1}), \quad \lambda_k^h = -\frac{1}{\sqrt{2|k|\pi}}+k\pi i+\frac{\operatorname{sgn}(k)}{\sqrt{2|k|\pi}}i+O(|k|^{-1}). \tag{6.6}$$

Furthermore there exist integers $n_0 > 0$, $\tilde{\ell}_1 \geq \ell_1$ and $\tilde{k}_1 \geq k_1$ such that $\{u_{j,0}, \cdots, u_{j,m_j-1}\}_{j=1}^{n_0} \cup \{u_\ell^p\}_{\ell=\tilde{\ell}_1}^\infty \cup \{u_k^h\}_{|k|=\tilde{k}_1}^\infty$ form a Riesz basis of H, where $u_{j,0}$ is an eigenvector of $\mathcal{A}$ with respect to some eigenvalue μ_j with algebraic multiplicity m_j, and $\{u_{j,1}, \cdots, u_{j,m_j-1}\}$ is the associated Jordan chain, and u_ℓ^p and u_k^h are eigenvectors of $\mathcal{A}$ with respect to eigenvalues λ_ℓ^p and λ_k^h, respectively.

Here and in the sequel the superindex p stands for "parabolic" while h for "hyperbolic". This theorem indeed shows that there are two distinguished branches of the spectrum at high frequencies. The parabolic eigenvalues are indeed close to those of a heat equation while the hyperbolic ones behaves like those of the wave equation with a weak damping term. It can be shown that the first order approximation of the parabolic component of the parabolic eigenvalues are eigenfunctions of the heat equation in the interval $(0, 1)$ with Dirichlet boundary condition at $x = 1$ and Neumann boundary condition at the transmission point $x = 0$; while the first order approximation of the hyperbolic ones are eigenfunctions of the wave equation in the interval $(-1, 0)$ with Dirichlet boundary conditions. The leading terms of the parabolic and hyperbolic eigenvalues in (6.6) correspond to the same boundary conditions. Note that the first order approximation of eigenvectors for the system discussed in [229] have a different behavior since the boundary conditions for parabolic and hyperbolic eigenvectors are reversed in that case.

6.4 Ingham-type inequality for mixed parabolic-hyperbolic spectra

By means of our spectral decomposition result the observability estimate (6.4) can be written as an Ingham-type inequality (Recall Theorem 6.3.1 for n_0, m_j, $\tilde{\ell}_1$, $\tilde{k}_1$ and μ_j, λ_ℓ^p and λ_k^h):

Lemma 6.4.1 *Let $T > 2$. Then there is a constant $C = C(T) > 0$ such that*

$$\begin{aligned} &\sum_{j=1}^{n_0} \sum_{k=0}^{m_j-1} |a_{j,k}|^2 + \sum_{\ell=\tilde{\ell}_1}^{\infty} |a_\ell|^2 e^{2(T-1)\mathrm{Re}\,\lambda_\ell^p} + \sum_{|k|=\tilde{k}_1}^{\infty} |b_k|^2 \\ &\leq C \int_0^T \left| \sum_{j=1}^{n_0} e^{\mu_j t} \sum_{k=0}^{m_j-1} a_{j,k} t^k + \sum_{\ell=\tilde{\ell}_1}^{\infty} a_\ell e^{\lambda_\ell^p t} + \sum_{|k|=\tilde{k}_1}^{\infty} b_k e^{\lambda_k^h t} \right|^2 dt \end{aligned} \tag{6.7}$$

holds for all complex numbers $a_{j,k}$ ($k = 0, 1, \cdots, m_j - 1$; $j = 1, 2, \cdots, n_0$), and all square-summable sequences $\{a_\ell\}_{\ell=\tilde{\ell}_1}^{\infty}$ and $\{b_k\}_{|k|=\tilde{k}_1}^{\infty}$ in $\mathbb{C}$.

The Ingham-type inequality (6.7) is similar to the one in [229] but for different sequences $\{\lambda_\ell^p\}_{\ell=\tilde{\ell}_1}^{\infty}$ and $\{\lambda_k^h\}_{|k|=\tilde{k}_1}^{\infty}$. At this point we would like to underline that, as far as we know, there is no a direct proof of inequalities of the form (6.7) in the literature devoted to this issue. It is in fact a consequence of estimate (6.4) obtained by PDE techniques and the spectral analysis above.

6.5 Boundary control and observation through the heat component

We begin with the following negative result on the observability for system (6.3) in H, which implies the lack of boundary observability in H from the heat component with a defect of infinite order.

Theorem 6.5.1 *Let $T > 0$ and $s \geq 0$. Then*

$$\sup_{(y_0,z_0,z_1)\in H\setminus\{0\}} \frac{|(y(T), z(T), z_t(T))|_H}{|y_x(\cdot, 1)|_{H^s(0,T)}} = +\infty,$$

where (y, z, z_t) is the solution of system (6.3) with initial data (y_0, z_0, z_1).

Theorem 6.5.1 is a consequence of Theorem 6.3.1. Indeed, from Theorem 6.3.1, one may deduce that the parabolic component of solutions of system (6.3) decays rapidly while its hyperbolic component is "almost" conservative. Moreover, the hyperbolic eigenvectors are mostly concentrated on the wave interval. This makes the observability inequality from the heat extreme to fail in any Sobolev space.

By means of the well-known duality relationship between controllability and observability, from Theorem 6.5.1, one concludes that system (6.2) is not null controllable in H with $L^2(0, T)$-controls at $x = 1$ neither, with controls in any negative index Sobolev space of the form $H^{-s}(0, T)$.

However, the Ingham-type inequality (6.7), combined with Theorem 6.3.1 and a sharp description of the asymptotic form of eigenvectors, allows to get an observability inequality from the parabolic extreme in space with suitable exponential weights in the Fourier coefficients. This is precisely what we shall do in the sequel.

Put (Recall Theorem 6.3.1 for n_0, m_j, $u_{j,k}$, $\tilde{\ell}_1$, $\tilde{k}_1$, u_ℓ^p and u_k^h)

$$V = \left\{ \sum_{j=1}^{n_0} \sum_{k=0}^{m_j-1} a_{j,k} u_{j,k} + \sum_{\ell=\tilde{\ell}_1}^{\infty} a_\ell u_\ell^p + \sum_{|k|=\tilde{k}_1}^{\infty} b_k u_k^h \;\middle|\; a_{j,k}, a_\ell, b_k \in \mathbb{C}, \quad \sum_{\ell=\tilde{\ell}_1}^{\infty} |a_\ell|^2 + \sum_{|k|=\tilde{k}_1}^{\infty} |k| e^{\sqrt{2|k|\pi}} |b_k|^2 < \infty \right\},$$

$$V' = \left\{ \sum_{j=1}^{n_0} \sum_{k=0}^{m_j-1} a_{j,k} u_{j,k} + \sum_{\ell=\tilde{\ell}_1}^{\infty} a_\ell u_\ell^p + \sum_{|k|=\tilde{k}_1}^{\infty} b_k u_k^h \;\middle|\; a_{j,k}, a_\ell, b_k \in \mathbb{C}, \quad \sum_{\ell=\tilde{\ell}_1}^{\infty} |a_\ell|^2 + \sum_{|k|=\tilde{k}_1}^{\infty} \frac{|b_k|^2}{|k| e^{\sqrt{2|k|\pi}}} < \infty \right\}.$$

V and V', endowed with their canonical norms, are mutually dual Hilbert spaces.

We have the following null controllability result on system (6.2):

Theorem 6.5.2 *Let $T > 2$. Then for every $(u_0, v_0, v_1) \in V$, there exists a control $g_2 \in H_0^1(0,T)$ such that the solution (u, v, v_t) of system* (6.2) *satisfies $u(T) = 0$ in $(0,1)$ and $v(T) = v_t(T) = 0$ in $(-1,0)$.*

In order to prove Theorem 6.5.2, we need to derive the following key observability estimate:

Theorem 6.5.3 *For any $T > 2$, there is a constant $C > 0$ such that every solution of* (6.3) *satisfies*

$$|(y(T), z(T), z_t(T))|_{V'}^2 \le C |y_x(\cdot,1)|_{L^2(0,T)}^2, \qquad \forall\, (y_0, z_0, z_1) \in V'. \tag{6.8}$$

Inequality (6.8) follows from Lemma 6.4.1 together with Theorem 6.3.1.

6.6 Polynomial decay rate

According to the asymptotic form of the hyperbolic eigenvalues in (6.6) it is clear that the decay rate of the energy is not uniform. Indeed, as (6.6) shows, $\operatorname{Re}\lambda_k^h \sim -c/\sqrt{|k|}$ for a positive constant $c > 0$. In this situation, the best we can expect is a polynomial decay rate for sufficiently smooth solutions. The following result is a consequence of Theorem 6.3.1, which provides a sharp polynomial decay rate.

Theorem 6.6.1 *There is a constant $C > 0$ such that for any $(y_0, z_0, z_1) \in D(\mathcal{A})$, the solution of* (6.3) *satisfies*

$$|(y(t), z(t), z_t(t))|_H \leq Ct^{-2}|(y_0, z_0, z_1)|_{D(\mathcal{A})}, \qquad \forall\, t > 0.$$

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