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# HAIMANS WORK ON THE $N!$ THEOREM, AND BEYOND

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INSTITUT  
FOURIER

**2008 SUMMER SCHOOL**

**Geometric Methods in  
Representation Theory**

**HAIMANS WORK ON THE  $N!$  THEOREM, AND BEYOND**

Iain GORDON

## 2008 SUMMER SCHOOL PROCEEDINGS :

### Lecture notes

- José BERTIN: *The punctal Hilbert scheme: an introduction*
- Michel BRION: *Representations of quivers*
- Victor GINZBURG: *Lectures on Nakajima's quiver varieties*
- Iain GORDON: *Haiman's work on the  $n!$  theorem, and beyond*
- Jens Carsten JANTZEN: *Moment graphs and representations*
- Bernard LECLERC: *Fock space representations*
- Olivier SCHIFFMANN: *Lectures on Hall algebras*
- Olivier SCHIFFMANN: *Lectures on canonical and crystal bases of Hall algebras*

### Articles

- Karin BAUR: *Cluster categories,  $m$ -cluster categories and diagonals in polygons*
- Ada BORALEVI: *On simplicity and stability of tangent bundles of rational homogeneous varieties*
- Laurent EVAIN: *Intersection theory on punctual Hilbert schemes*
- Daniel JUTEAU, Carl MAUTNER and Geordie WILLIAMSON: *Perverse sheaves and modular representation theory*
- Manfred LEHN and Christoph SORGER: *A symplectic resolution for the binary tetrahedral group*
- Ivan LOSEV: *On the structure of the category  $O$  for  $W$ -algebras*
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- Olivier SERMAN: *Orthogonal and symplectic bundles on curves and quiver representations*
- Dmitri SHMELKIN: *Some remarks on Nakajima's quiver varieties of type A*
- Francesco VACCARINO: *Moduli of representations, symmetric products and the non commutative Hilbert scheme*

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IAIN GORDON

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## INTRODUCTION

In the late 1980's Macdonald introduced some remarkable symmetric functions which now bear his name. They depend on two parameters,  $t$  and  $q$  and under various specialisations recover well-known symmetric functions that we have grown to love, including Hall-Littlewood functions, Jack functions, monomial symmetric functions, Schur functions. Based on empirical evidence, Macdonald conjectured several fundamental and non-obvious properties, including that when expressed in the Schur basis, the transition functions for his symmetric functions actually belong to  $\mathbb{N}[q^{\pm 1}, t^{\pm 1}]$ . This is called the *Macdonald positivity conjecture*. Such a result has predecessors for some of the above symmetric functions in fewer parameters, and is of interest because it suggests something is being counted, and even being counted with respect to a bigrading (to account for the  $t$  and  $q$ ).

It is now known what is being counted (or better to say, we know one thing that is being counted by the Macdonald functions): the Macdonald functions count some bigraded copies of the regular representation of the symmetric group. But where do such representations come from? The symmetric group  $S_n$  acts naturally on a set of commuting variables  $x_1, \dots, x_n$ , but such an action will only produce a grading (and indeed had been used in the study of Hall-Littlewood functions). To get the bigrading Garsia and Haiman introduced a second set of variables  $y_1, \dots, y_n$  and then proceeded to seek candidates for associated spaces that might produce the regular representation. They found some very natural spaces that, in low degree, did exactly what was required; they conjectured that in general these would produce the required realisation of Macdonald polynomials. Since this conjecture predicted that a space of polynomials (in  $2n$  variables) carried the regular representation of  $S_n$ , it was known as the  *$n!$  conjecture*. This conjecture became rather famous: it was easy to state, and attractive since it generalised many celebrated results from symmetric function theory, representation theory and geometry. On the other hand, having two sets of variables seemed to make things much more difficult. However, what made the conjecture **really** interesting was that thanks to Haiman and Procesi, it introduced a new object to the field, namely  $\text{Hilb}^n \mathbb{C}^2$ , and consequently many new structures.

After a long battle, Haiman succeeded to confirm the  $n!$  conjecture. He showed that bigraded  $S_n$ -equivariant components of special fibres of an exotic bundle on  $\text{Hilb}^n \mathbb{C}^2$  – called the Procesi bundle – are being counted by Macdonald's polynomials. His work is a mixture of combinatorics, representation theory, algebraic geometry and homological algebra. The conjecture has inspired and fed into many other recent developments in algebra, combinatorics and geometry. These include the discovery of symplectic reflection algebras

by Etingof-Ginzburg, the homological symplectic McKay correspondence of Bezrukavnikov-Kaledin, new combinatorial statistics for partitions attached to Dyck paths introduced by Haglund, Haiman, Loehr, Warrington and others.

In these lectures we will outline the whole story, but at a rather general level. There are already several excellent expository articles written on this topic by Haiman and available on his homepage. They contain varying levels of detail, but serve as wonderful guides to his two main papers on these topics, [12] and [13].

The search for further understanding of the spaces described by Macdonald polynomials goes on; exciting progress is mentioned towards the end of these lectures.

## LECTURE 1

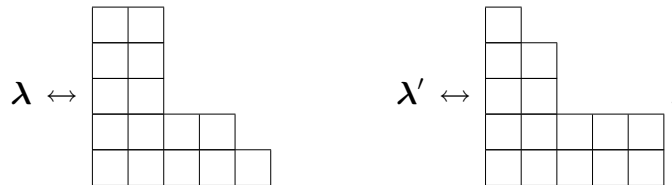
The best reference for much of the content in this lecture is *the* book [17].

**1.1. Symmetric functions and the Frobenius map.** Recall that a sequence of integers  $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r > 0)$  is a partition of  $|\lambda| = \sum \lambda_i$ , written  $\lambda \vdash |\lambda|$ . We write  $\ell(\lambda) = r$  and set  $n(\lambda) = \sum_i (i-1)\lambda_i$ . We let  $\lambda'$  denote the transpose of  $\lambda$ . The dominance ordering on partitions is defined by

$$\lambda \leq \mu \text{ if and only if } \lambda_1 + \dots + \lambda_i \leq \mu_1 + \dots + \mu_i \text{ for each } i > 0.$$

Note that  $\lambda \leq \mu$  if and only if  $\lambda' \geq \mu'$ .

We identify a partition  $\lambda$  with its Young diagram  $\lambda = \{(p, q) \in \mathbb{N} \times \mathbb{N} : p < \lambda_{q+1}\}$ . For example  $\lambda = (5, 4, 2, 2, 2)$  gives the following partitions of 15



Here  $\lambda' > \lambda$ .

Let  $\Lambda$  be the ring of symmetric functions, i.e.

$$\Lambda = \bigoplus_{k \geq 0} \varprojlim_{\leftarrow} \mathbb{Q}[z_1, \dots, z_n]_k^{S_n}.$$

These are functions of bounded degree, but in infinitely many variables  $z = (z_1, z_2, \dots)$ . Later, we will extend scalars in  $\Lambda$  from  $\mathbb{Q}$  to either  $\mathbb{Q}(t)$  or  $\mathbb{Q}(q, t)$ . We will write  $\Lambda_t$  or  $\Lambda_{q,t}$  respectively.

There are several natural bases for  $\Lambda$ , all indexed by partitions.

- **Power**  $p_\lambda = p_{\lambda_1} p_{\lambda_2} \dots p_{\lambda_r}$  where  $p_t = \sum_{i \geq 1} z_i^t$ .

- **Monomial**  $m_\lambda = \sum_{\alpha \text{ permutation of } \lambda} z^\alpha$  where if  $\alpha = (\alpha_i)_{i \geq 1}$  then  $z^\alpha = \prod_{i \geq 1} z_i^{\alpha_i}$ .
- **Complete**  $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_r}$  where  $h_t = \sum_{|\mu|=t} m_\mu$ .
- **Schur**  $s_\lambda = \det(z_i^{\lambda_j + n - j}) / \det(z_i^{n - j})$ .

The last three bases are actually *integral*, meaning that they form bases for the ring of symmetric functions over  $\mathbb{Z}$ .

There is an inner product  $\langle -, - \rangle$  on  $\Lambda$ , preserving degree. It is characterised by any of the following:

$$\begin{aligned} \langle s_\lambda, s_\mu \rangle &= \delta_{\lambda, \mu} \\ \langle p_\lambda, p_\mu \rangle &= \delta_{\lambda, \mu} z_\lambda \\ \langle h_\lambda, m_\mu \rangle &= \delta_{\lambda, \mu}. \end{aligned}$$

Here  $z_\lambda = \prod_{i \geq 1} i^{m_i} m_i!$  where  $\lambda = (1^{m_1}, 2^{m_2}, \dots)$ .

A good reason to care about symmetric functions is the following isometry of algebras,  $F$ , called the *Frobenius map*. To define it let  $\text{Rep}(S_n)$  denote the Grothendieck group of complex representations of  $S_n$ . Then  $F : \bigoplus_{n \geq 0} \text{Rep}(S_n) \otimes_{\mathbb{Z}} \mathbb{Q} \longrightarrow \Lambda$  where  $[A] \in \text{Rep}(S_n)$  is sent to

$$F_A(z) := \frac{1}{n!} \sum_{w \in S_n} \chi^A(w) p_{\tau(w)}(z)$$

where  $\tau(w)$  is the partition describing the cycle type of  $w \in S_n$ . On the left-hand-side we have an inner product given by the inner product on characters; multiplication is induced from

$$[A] \star [B] = [\text{Ind}_{S_n \times S_m}^{S_{n+m}} (A \boxtimes B)]$$

for  $[A] \in \text{Rep}(S_n), [B] \in \text{Rep}(S_m)$ . Here  $\text{Ind}$  denotes the induction functor, which for  $H \leq G$  is defined on representations of  $H$  by  $\text{Ind}_H^G(M) = \mathbb{C}G \otimes_{\mathbb{C}H} M$ .

As an exercise you should show that  $F_{\text{triv}_n}(z) = h_n(z) = s_{(n)}(z)$  and that more generally  $L_\lambda$ , the irreducible representation of  $S_n$  associated to  $\lambda$ , is sent to  $s_\lambda(z)$ , i.e. that  $\chi^\lambda(w) = \langle s_\lambda, p_{\tau(w)} \rangle$  (we write  $\chi^\lambda$  for  $\chi^{L_\lambda}$ ).

One consequence of this definition and these observations is that

$$F_{\text{Ind}_{S_\mu}^{S_n}(\text{triv}_\mu)}(z) = \prod_{i=1}^r F_{\text{triv}_{\mu_i}}(z) = \prod_{i=1}^r h_{\mu_i} = h_\mu$$

where  $S_\mu = S_{\mu_1} \times S_{\mu_2} \times \cdots \times S_{\mu_r}$  is a Young subgroup of  $S_n$ . Thus

$$\langle s_\lambda, h_\mu \rangle = \langle \chi^\lambda, \text{Ind}_{S_\mu}^{S_n}(\text{triv}_\mu) \rangle = K_{\lambda, \mu},$$

a Kostka number (and in particular non-negative). So  $h_\mu = \sum_\lambda K_{\lambda,\mu} s_\lambda$ . Similarly, we see that  $s_\lambda = \sum_\mu K_{\lambda,\mu} m_\mu$ . Thus the Frobenius map gives positivity results (and interpretation) for transition matrices in symmetric function theory.

The Frobenius map generalises to a map on multi-graded  $S_n$ -representations. This means for instance that if  $A = \bigoplus_{r,s \in \mathbb{Z}} A_{r,s}$  is a direct sum decomposition of  $S_n$ -representations labelled by pairs of integers, then we can set

$$F_A(z; q, t) = \sum_{r,s \in \mathbb{Z}} F_{A_{r,s}}(z) q^s t^r \in \Lambda_{q,t}.$$

So there is a relationship between bigraded representations of  $S_n$  and  $(q, t)$ -symmetric function theory.

**1.2. Plethysm.** For any  $A \in \Lambda_{q,t}$  we introduce the following  $\mathbb{Q}(q, t)$ -linear operation on  $\Lambda_{q,t}$ . For each  $k > 0$  set  $p_k[A] = A|_{q \rightarrow q^k, t \rightarrow t^k, z_i \rightarrow z_i^k}$ . Since the  $p_k$  freely generate  $\Lambda_{q,t}$  as a  $\mathbb{Q}(q, t)$ -algebra this leads to an endomorphism  $ev_A : \Lambda_{q,t} \rightarrow \Lambda_{q,t}$  sending  $p_{i_1} \cdots p_{i_t}$  to  $p_{i_1}[A] \cdots p_{i_t}[A]$ . This defines the *plethystic substitution*  $f[A] = ev_A(f)$ .

For instance, if we set  $Z = z_1 + z_2 + \cdots = p_1 = h_1 = m_{(1)} = s_{(1)}$  then we see

$$p_k[Z] = p_k(z); \quad p_k[-Z] = -p_k(z); \quad \text{and so } p_k[Z(1-t)] = p_k(z)(1-t^k).$$

Similarly, using  $Z/(1-t) = Z + tZ + t^2Z + \cdots$ , we find

$$p_k[Z/(1-t)] = \sum_{i \geq 0} p_k(z) t^{ik} = \frac{1}{1-t^k} p_k(z).$$

In other words the substitutions involving  $Z(1-t)$  and  $Z/(1-t)$  are inverse to one another.

**Proposition.** *Let  $[A] \in \text{Rep}(S_n)$  be considered as a graded representation concentrated in degree 0, and let  $V = \mathbb{C}^n$ , the natural permutation representation of  $S_n$ , be concentrated in degree 1. Then*

- $F_A[Z(1-t)] = \sum_{k \geq 0} (-t)^k F_{\wedge^k V \otimes A}(z; t)$ , and
- $F_A[Z/(1-t)] = \sum_{k \geq 0} t^k F_{S^k V \otimes A}(z; t)$ .

It is an exercise to check this for  $A = \text{triv}_n$  (where  $F_A(z) = h_n(z)$ ) and then to deduce the general case from that.

Now it is not hard to check (using Cauchy's formula) that for any non-zero  $A \in \mathbb{Q}(q, t)$   $\langle f[ZA], g \rangle = \langle f, g[ZA] \rangle$ . This allows us to define new inner products on  $\Lambda_{q,t}$ . The ones we will be interested in are

$$\langle f, g \rangle_t := \langle f, g[Z/(1-t)] \rangle \quad \text{and} \quad \langle f, g \rangle_{q,t} := \langle f, g[Z \frac{1-q}{1-t}] \rangle. \quad (1)$$

Observe that there is a chain between all of our inner products – set first  $q = 0$ , then  $t = 0$ .



1.3. **Macdonald polynomials.** In [18, TO COME] we saw Hall-Littlewood polynomials  $P_{\mu}(z; t)$ . They have a dual basis  $Q_{\mu}(z; t)$  with respect to the form  $\langle -, - \rangle_t$  above. If we set

$$\tilde{H}_{\mu}(z; t) := t^{n(\mu)} Q_{\mu}[Z/(1-t)]$$

then we can characterise the  $\tilde{H}_{\mu}$ 's (called *transformed Hall-Littlewood polynomials*) by

- (1)  $\tilde{H}_{\mu}(z; t) \in \mathbb{Q}(t)\{s_{\lambda} : \lambda \geq \mu\}$
- (2)  $\tilde{H}_{\mu}[Z(1-t); t] \in \mathbb{Q}(t)\{s_{\lambda} : \lambda \geq \mu'\}$
- (3)  $\langle s_{(n)}, \tilde{H}_{\mu}(z; t) \rangle = 1$ .

Set  $\tilde{H}_{\mu}(z; t) = \sum_{\lambda} K_{\lambda, \mu}(t) s_{\lambda}(z)$ . It can be checked that we have a specialisation  $\tilde{H}_{\mu}(z; 1) = h_{\mu}(z)$ , so we see that  $K_{\lambda, \mu}(t)$  are  $t$ -versions of the Kostka numbers. They are called Kostka-Foulkes polynomials (they belong to  $\mathbb{N}[t^{\pm 1}]$  – we will explain a geometric origin of this later).

Macdonald proved the existence of the following set of polynomials  $\tilde{H}_{\mu}(z; q, t)$  which generalise the Hall-Littlewood polynomials to the  $(q, t)$ -case.

- (1)  $\tilde{H}_{\mu}[Z(1-q); q, t] \in \mathbb{Q}(q, t)\{s_{\lambda} : \lambda \geq \mu\}$
- (2)  $\tilde{H}_{\mu}[Z(1-t); q, t] \in \mathbb{Q}(q, t)\{s_{\lambda} : \lambda \geq \mu'\}$
- (3)  $\langle s_{(n)}, \tilde{H}_{\mu}(z; q, t) \rangle = 1$ .

The difficulty with proving the existence of such symmetric functions is that the ordering we take on partitions is not total, but only the dominance ordering. Using Gram-Schmidt it is easy to find such bases in terms of total orderings, but the equations above then specify some extra vanishing that is not at all clear. Once we have established existence, uniqueness of such polynomials is straightforward. Indeed refine the dominance order to a total order. Then (1) shows that the transition matrix between any two sets of polynomials is upper triangular, while (2) shows it is lower triangular. Condition (3) then fixes the the resulting diagonal matrix.

Macdonald found the  $\tilde{H}_{\mu}(z; q, t)$ 's by constructing them as eigenfunctions of the following operator on  $\Lambda_{q, t}$ :

$$D := \frac{1}{(1-q)(1-t)}(1 - D_0)$$

where  $D_0(f) = u^0$  term of  $f[Z + (1-q)(1-t)u^{-1}] \prod_{i \geq 1} (1 - uz_i)$ . Their eigenvalues are

$$B_{\mu}(q, t) = \sum_{(i, j) \in \mu} q^i t^j, \tag{2}$$

and since the indices specify  $\mu$ , the eigenspaces are all one-dimensional.

These are not the original Macdonald polynomials, but, up to a constant factor, elementary transformations of them:

$$M_{\mu}(z; q, t) = t^{n(\mu)} \tilde{H}_{\mu}[Z(1-t^{-1}); q, t^{-1}].$$

Setting  $q = 0$  produces the old  $\tilde{H}_\mu(z; t)$  and there is also an obvious  $(q, t)$ -symmetry

$$\tilde{H}_\mu(z; q, t) = \tilde{H}_{\mu'}(z; t, q).$$

We are led to

**Macdonald positivity conjecture:** If we expand in the Schur basis  $\tilde{H}_\mu(z; q, t) = \sum_{\lambda} K_{\lambda, \mu}(q, t) s_\lambda(z)$ , then  $K_{\lambda, \mu}(q, t) \in \mathbb{N}[q^{\pm 1}, t^{\pm 1}]$ .

The  $(q, t)$ -elements  $K_{\lambda, \mu}(q, t)$  are *Kostka-Macdonald* polynomials.

We have some obvious and some not-so-obvious properties of  $K_{\lambda, \mu}(q, t)$ :

- a  $(q, t)$ -symmetry  $K_{\lambda, \mu}(q, t) = K_{\lambda, \mu'}(t, q)$
- setting  $q = 0$  produces the Kostka-Foulkes polynomials
- $\tilde{H}_\mu(z; 1, 1) = F_{\mathbb{C}S_n}(z)$  so that  $K_{\lambda, \mu}(1, 1) = \chi^\lambda(1)$ , independent of  $\mu$ .

Let's discuss this last property. We'll calculate what's happening at  $q = 1$  first. The operator  $D$  above has a limit as  $q$  tends to 1 (you should check!) and is a derivation, which we shall call  $D_{q=1}$ . Thus  $\tilde{H}_\mu(z; 1, t)$  is an eigenfunction for  $D_{q=1}$  with eigenvalue  $B_\mu(1, t) = \sum_{i \geq 1} (1 + t + \dots + t^{\mu_i - 1}) =: [\mu'_i]_t$ . But thanks to the derivation property the product  $\prod_{i \geq 1} \tilde{H}_{(1^{\mu'_i})}(z; 1, t)$  is also an eigenfunction with the same eigenvalue. Thus they are multiples of one another because we had one dimensional eigenspaces.

Now condition (2) shows that up to constant factor we have

$$\tilde{H}_{(1^{\mu'_i})}[Z(1-t); q, t] = s_{(\mu'_i)}(z) = h_{(\mu'_i)}(z).$$

On using condition (3) to calculate the constant we find that

$$\tilde{H}_\mu(z; 1, t) = (1-t)^{|\mu|} \left( \prod_i [\mu'_i]_t! \right) h_\mu[Z/(1-t)].$$

There's still a little algebra to do, but basically setting  $t = 1$  and applying the hook length formula shows that the right-hand-side here becomes  $\sum_{\lambda} s_\lambda(z) \chi^\lambda(1)$ , as required.

The upshot is that Macdonald positivity predicts that for each partition of  $n$ , there exists a bigrading on the regular representation  $\mathbb{C}S_n$ .

**1.4. The Garsia-Haiman model.** We'll explain some more motivation for the following construction in the next lecture. At the moment we want to find candidates for bigraded regular representations of  $S_n$ .

*Definition 1.* Let  $\underline{x} = (x_1, \dots, x_n)$ ,  $\underline{y} = (y_1, \dots, y_n)$  be pairwise commuting sets of indeterminates. For  $\lambda \vdash n$  set

$$\Delta_\lambda = \det(x_i^{q_j} y_i^{p_j})_{1 \leq i, j \leq n} \quad \text{for } (p_j, q_j) \in \lambda.$$

These elements are defined only up to sign;  $\Delta_{(1^n)}$ ,  $\Delta_{(n)}$  are Vandermonde determinants in the  $x$ 's and  $y$ 's respectively.

*Definition 2.*  $V(\boldsymbol{\lambda}) := \mathbb{C}[\partial_{\underline{x}}, \partial_{\underline{y}}](\Delta_{\boldsymbol{\lambda}}) \subset \mathbb{C}[\underline{x}, \underline{y}]$ , a finite dimensional vector space. ( $\mathbb{C}[\partial_{\underline{x}}, \partial_{\underline{y}}]$  denotes the polynomial ring of constant coefficient differential operators  $\mathbb{C}[\partial/\partial x_1, \dots, \partial/\partial y_n]$ .)

Now  $V(\boldsymbol{\lambda})$  has an action of  $S_n$  induced from  $w x_i = x_{w(i)}, w y_j = y_{w(j)}$ . To check this for yourself, note that  $w \Delta_{\boldsymbol{\lambda}} = (-1)^{\text{sign}(w)} \Delta_{\boldsymbol{\lambda}}$ . There is a bigrading on  $V(\boldsymbol{\lambda})$  induced from  $\deg x_i = (1, 0)$  and  $\deg y_j = (0, 1)$ . Thus  $V(\boldsymbol{\lambda})$  is a bigraded  $S_n$ -representation and under the Frobenius map it produces a  $(q, t)$ -symmetric function. Finally, switching the rôles of  $\underline{x}$  and  $\underline{y}$  swaps  $V_{q,t}(\boldsymbol{\lambda})$  and  $V_{t,q}(\boldsymbol{\lambda}')$  where we've included the  $q, t$  subscript to indicate how the bigrading gets shifted.

**The Garsia-Haiman  $n!$  conjecture.**  $F_{V(\boldsymbol{\mu})}(z; q, t) = \tilde{H}_{\boldsymbol{\mu}}(z; q, t)$ .

Of course, it is called the  $n!$  conjecture because it predicts that  $V(\boldsymbol{\mu})$  carries the regular representation of  $S_n$  and hence has dimension  $n!$ . It also immediately implies Macdonald positivity. The conjecture is true for  $\boldsymbol{\mu} = (n)$  and for  $\boldsymbol{\mu} = (1^n)$  by classical considerations on Vandermonde determinants.

There is a uniform approach to this conjecture. We can swap between subspaces and quotients of  $\mathbb{C}[\underline{x}, \underline{y}]$  using the non-degenerate  $S_n$ -equivariant bidegree preserving form

$$(f, g) = f(\partial_{\underline{x}}, \partial_{\underline{y}}) \cdot g(\underline{x}, \underline{y})|_{\underline{x}=\underline{y}=0} : \mathbb{C}[\underline{x}, \underline{y}] \otimes \mathbb{C}[\underline{x}, \underline{y}] \longrightarrow \mathbb{C}.$$

Using this form we see that  $V(\boldsymbol{\lambda})$  is isomorphic as a bigraded  $S_n$ -representation to the quotient  $R_{\boldsymbol{\lambda}} = \mathbb{C}[\underline{x}, \underline{y}]/J_{\boldsymbol{\lambda}}$  where  $J_{\boldsymbol{\lambda}} = V(\boldsymbol{\lambda})^\perp$ . Moreover,  $J_{\boldsymbol{\lambda}}$  is an ideal since  $V(\boldsymbol{\lambda})$  is closed under application of  $\partial_{\underline{x}}$  and  $\partial_{\underline{y}}$  and so  $R_{\boldsymbol{\lambda}}$  is an algebra. Now if the conjecture were true then each  $V(\boldsymbol{\lambda})$  would contain only one copy of the trivial module. It would follow that each  $R_{\boldsymbol{\lambda}}$  was actually a quotient of the ring of *diagonal coinvariants*

$$R_n := \frac{\mathbb{C}[\underline{x}, \underline{y}]}{\langle \mathbb{C}[\underline{x}, \underline{y}]_+^{S_n} \rangle}.$$

(Under the form above this corresponds to the space of *diagonal harmonics*, i.e. the  $f \in \mathbb{C}[\underline{x}, \underline{y}]$  such that  $i(\partial_{\underline{x}}, \partial_{\underline{y}})f = 0$  for all  $i \in \mathbb{C}[\underline{x}, \underline{y}]_+^{S_n}$ .)

Garsia and Haiman also studied  $R_n$  and came up with a remarkable conjecture to describe its structure. To state this we need to introduce a few more combinatorial polynomials in addition to  $B_{\boldsymbol{\mu}}(q, t)$  defined in (2). We will do this by example.

Let  $x \in \boldsymbol{\mu} = (5, 4, 2, 2, 2)$ . Then we define the arm, leg, co-arm and co-leg of  $x$  to be the number of  $a$ 's,  $l$ 's,  $a'$ 's and  $l'$ 's below; they will be written  $a(x), l(x), a'(x)$  and  $l'(x)$  respectively.

	$l$				
	$l$				
	$l$				
$a'$	$x$	$a$	$a$		
	$l'$				

Now set

$$\Pi_{\boldsymbol{\mu}}(q, t) = \prod_{x \in \boldsymbol{\mu} \setminus (0,0)} (1 - q^{a'(x)} t^{l'(x)}), \quad (3)$$

$$h_{\boldsymbol{\mu}}(q, t) = \prod_{x \in \boldsymbol{\mu}} (1 - t^{l(x)+1} q^{-a(x)}), \quad (4)$$

$$h'_{\boldsymbol{\mu}}(q, t) = \prod_{x \in \boldsymbol{\mu}} (1 - q^{a(x)+1} t^{-l(x)}). \quad (5)$$

**Garsia-Haiman  $(n+1)^{n-1}$  Conjecture.** Keep the above notation. Then

$$F_{R_n}(z; q, t) = \sum_{\boldsymbol{\mu} \vdash n} \frac{(1-t)(1-q) \Pi_{\boldsymbol{\mu}}(q, t) B_{\boldsymbol{\mu}}(q, t) \tilde{H}_{\boldsymbol{\mu}}(z; q, t)}{h_{\boldsymbol{\mu}}(q, t) h'_{\boldsymbol{\mu}}(q, t)}.$$

On specialising  $q = t = 1$  (and working a bit) one sees that this predicts  $\dim R_n = (n+1)^{n-1}$  and that for the sign isotypic component  $\dim R_n^{\text{sign}} = C_n$  where  $C_n = \frac{1}{n+1} \binom{2n}{n}$  is the  $n$ th Catalan number. In particular, the Hilbert series of  $R_n^{\text{sign}}$  is predicted to produce a  $(q, t)$ -Catalan number.

## LECTURE 2

**2.1. The one variable case.** Recall the transformed Hall-Littlewood polynomial  $\tilde{H}_{\boldsymbol{\mu}}(z; t)$  and the corresponding Kostka-Foulkes polynomials. They have the following geometric description.

Let  $\mathcal{B} = \{F^\bullet : \{0\} \subset F^1 \subset F^2 \subset \dots \subset F^{n-1} \subset \mathbb{C}^n, \dim F^i = i\}$  denote the flag manifold. It is a homogeneous  $GL_n(\mathbb{C})$ -space and its cotangent bundle has the description

$$T^*\mathcal{B} = \{(X, F^\bullet) : X \in Mat_n(\mathbb{C}), F^\bullet \in \mathcal{B}, X(F^i) \subseteq F^{i-1}\} \subset Mat_n(\mathbb{C}) \times \mathcal{B},$$

and so comes equipped with  $GL_n(\mathbb{C})$ -equivariant projections  $\pi$  onto  $\mathcal{N} \subset Mat_n(\mathbb{C})$ , the variety of nilpotent matrices, and  $\rho$  onto  $\mathcal{B}$ . Fix  $X \in \mathcal{N}$  a nilpotent matrix with Jordan

blocks of size  $\boldsymbol{\mu} = (\mu_1 \geq \mu_2 \geq \dots \geq \mu_r)$ . Set  $\mathcal{B}_{\boldsymbol{\mu}} = \pi^{-1}(X)$ , a *Springer fibre*. (By  $GL_n(\mathbb{C})$ -equivariance, this is independent up to isomorphism of the choice of representative  $X$ , and so the notation is well-defined.) Note that  $\mathcal{B}_{(1^n)} = \mathcal{B}$  and  $\mathcal{B}_{(n)} = \{\text{pt}\}$ .

**Theorem.** *There is an action of  $S_n$  on  $R_{\boldsymbol{\mu}}(\underline{x}) := H^\bullet(\mathcal{B}_{\boldsymbol{\mu}}, \mathbb{C})$  making it isomorphic to  $\text{Ind}_{S_{\boldsymbol{\mu}}}^{S_n}(\text{triv}_n)$ . Moreover, after halving cohomological degree, we have*

$$F_{R_{\boldsymbol{\mu}}(\underline{x})}(z; t) = \tilde{H}_{\boldsymbol{\mu}}(z; t).$$

This generalises the classical theorem of Borel which states that

$$R_{(1^n)}(\underline{x}) = H^\bullet(\mathcal{B}) \cong \frac{\mathbb{C}[\underline{x}]}{\langle \mathbb{C}[\underline{x}]_+^{S_n} \rangle}$$

carries the regular representation of  $S_n$ . It is the work of many authors; a good reference for an overview is [19].

We will use the following facts without proof.

- $\dim \mathcal{B}_{\boldsymbol{\mu}} = n(\boldsymbol{\mu})$ .
- The inclusion  $\iota : \mathcal{B}_{\boldsymbol{\mu}} \longrightarrow \mathcal{B}$  induces a surjective homomorphism  $H^\bullet(\mathcal{B}, \mathbb{C}) \longrightarrow H^\bullet(\mathcal{B}_{\boldsymbol{\mu}}, \mathbb{C})$  (this is particular to the  $GL_n(\mathbb{C})$ -case).
- The irreducible  $S_n$ -representation  $L_{\boldsymbol{\mu}}$  appears once in  $\mathbb{C}[\underline{x}]_{n(\boldsymbol{\mu})}$  and in no lower degree. (This copy gives rise to the copy of  $L_{\boldsymbol{\mu}}$  in  $H^{2n(\boldsymbol{\mu})}(\mathcal{B}_{\boldsymbol{\mu}}, \mathbb{C})$  that produces the Springer correspondence for  $GL_n(\mathbb{C})$ .)

It is straightforward to describe the elements of  $\mathbb{C}[\underline{x}]_{n(\boldsymbol{\mu})}$  belonging to  $L_{\boldsymbol{\mu}}$ . They are described by the *Garnir polynomials*. These are the  $S_n$ -orbit of the product of the Vandermondes

$$G_{\boldsymbol{\mu}}(\underline{x}) = \Delta_{(1^{\mu_1})}(x_1, \dots, x_{\mu_1}) \Delta_{(1^{\mu_2})}(x_{\mu_1+1}, \dots, x_{\mu_1+\mu_2}) \cdots .$$

We can characterise  $R_{\boldsymbol{\mu}}(\underline{x}) = \frac{\mathbb{C}[\underline{x}]}{J_{\boldsymbol{\mu}}(\underline{x})}$  by

$J_{\boldsymbol{\mu}}(\underline{x}) \subset \mathbb{C}[\underline{x}]$  is the largest homogeneous  $S_n$ -stable ideal having zero intersection with the copy of  $L_{\boldsymbol{\mu}}$  in degree  $n(\boldsymbol{\mu})$ .

Garsia and Procesi give another, more elementary construction of  $J_{\boldsymbol{\mu}}(\underline{x})$  which will be useful for us. Let  $\mathbf{a} \in \mathbb{C}^n$  be a point with  $\mu_1$  co-ordinates equal to  $a_1$ ,  $\mu_2$  co-ordinates equal to  $a_2$ , and so on, with  $a_1, \dots, a_r$  all pairwise distinct. Let  $J_{\mathbf{a}}$  be the ideal vanishing on the set  $S_n \cdot \mathbf{a} \subset \mathbb{C}^n$ . Then  $\dim \mathbb{C}[\underline{x}]/J_{\mathbf{a}} = |S_n|/|S_{\boldsymbol{\mu}}| = \dim R_{\boldsymbol{\mu}}(\underline{x})$ . They show that

$$\text{gr } J_{\mathbf{a}} = J_{\boldsymbol{\mu}}(\underline{x}).$$

In other words  $R_{\boldsymbol{\mu}}(\underline{x})$  is the limit as  $u \rightarrow 0$  of the set of points  $S_n \cdot (u\mathbf{a})$ .

**2.2. The two variable case.** Recall  $J_\mu$  from 1.4 and the quotient  $R_\mu = \mathbb{C}[\underline{x}, \underline{y}]/J_\mu$  which conjecturally carries the regular representation of  $S_n$ . Where did  $J_\mu$  come from? Well, we observed that  $\tilde{H}_\mu(z; 0, t) = \tilde{H}_\mu(z; t)$  and similarly that  $\tilde{H}_\mu(z; q, 0) = \tilde{H}_{\mu'}(z; q)$ . This suggests that  $R_\mu$  should in fact be a quotient of  $R_\mu(\underline{x}) \otimes R_{\mu'}(\underline{y})$ . Moreover, the quotient is supposed to carry the regular representation. Now take a partition  $\mu$  and to each row associate a distinct element  $a_i \in \mathbb{C}$  and to each column a distinct element  $b_j \in \mathbb{C}$ . In this way each box  $(p, q) \in \mu$  gives rise to  $(a_p, b_q)$  and these points are pairwise distinct. Thus we have a generic point  $\mathbf{p} \in (\mathbb{C}^2)^n$ . Let  $J'_\mathbf{p}$  be the ideal of  $\mathbb{C}[\underline{x}, \underline{y}]$  associated to the (free) orbit  $S_n \cdot \mathbf{p}$  with  $n!$  points. Then taking the highest degree terms produces

$$J_\mathbf{p} := \text{gr } J'_\mathbf{p}$$

and we see that  $\mathbb{C}[\underline{x}, \underline{y}]/J_\mathbf{p}$  carries the regular representation of  $S_n$ . Moreover, killing the  $y$ 's produces  $\mathbb{C}[\underline{x}]/J_\mu(\underline{x})$  and killing the  $x$ 's produces  $\mathbb{C}[\underline{y}]/J_{\mu'}(\underline{y})$  as we hoped. **But** it is not clear that  $J_\mathbf{p}$  is bigraded! And how does it depend on the choice of  $\mathbf{p}$ ?

To deal with this we need a way to control the degenerations of  $J'_\mathbf{p}$  and the corresponding quotient of  $\mathbb{C}[\underline{x}, \underline{y}]$ .

We begin to do this by characterising  $J_\mu$  in the spirit of the one variable characterisation in 2.1.

**Lemma.** *There exists a unique  $S_n$ -stable doubly homogeneous ideal  $I \subseteq R_\mu(\underline{x}) \otimes R_{\mu'}(\underline{y})$  such that*

- (1)  $\mathbb{C}[\underline{x}, \underline{y}]/I$  contains a copy of the sign representation
- (2)  $\mathbb{C}[\underline{x}, \underline{y}]/I$  is Gorenstein (i.e. its highest degree term is the entire socle and it is one dimensional).

*This ideal  $I$  is  $J_\mu$ .*

*Proof.* Set

$$I = \{f : (f) \subset R_\mu(\underline{x}) \otimes R_{\mu'}(\underline{y}) \text{ intersects the unique copy of } \mathbf{sign} \text{ in zero}\}.$$

Note that if  $L_\tau$  is a summand of  $R_\mu(\underline{x})$  (respectively  $R_{\mu'}(\underline{y})$ ) then  $\tau \geq \mu$  (respectively  $\tau \geq \mu'$ ). But  $\mathbf{sign}$  is a summand of  $L_\tau \otimes L_\nu$  if and only if  $\nu = \tau'$ . Thus there is only one copy of  $\mathbf{sign}$  in the tensor product  $R_\mu(\underline{x}) \otimes R_{\mu'}(\underline{y})$  and so the above definition is sensible.

We claim that  $I$  is the ideal of the lemma. (1) is obvious, and so is (2). So we need only check uniqueness.

Take any  $I'$  satisfying (1) and (2). Then if  $f \in I'$  then (1) ensures that  $(f) \subseteq I$  and so  $I' \subseteq I$ . On the other hand if (2) holds then the socle is one dimensional and since  $\mathbf{sign}$  appears in the top degree  $n(\mu) + n(\mu')$  it must be that this is the socle. But the quotient

$I/I'$  must intersect the socle if non-zero and this would imply that  $I$  contains **sign**, which is nonsense. Thus  $I = I'$ .

If we see why  $J_\mu$  belongs to  $R_\mu(\underline{x}) \otimes R_{\mu'}(\underline{y})$  then the last claim is obvious because (1) is true for  $J_\mu$  since  $\Delta_\mu$  belongs to  $R_\mu$  and (2) follows immediately from the non-degenerate pairing  $R_\mu \otimes V(\mu) \rightarrow \mathbb{C}$ . On dualising we know that  $J_\mu(x)^\perp = \mathbb{C}[\partial_x](S_n G_\mu(\underline{x}))$ , and so we must show that all partial derivatives coming from  $J_\mu \cap \mathbb{C}[\underline{x}]$  annihilate  $S_n G_\mu(\underline{x})$ . But  $G_\mu(\underline{x})$  belongs to  $\mathbb{C}[\partial_x, \partial_y]\Delta_\mu$  since it can be obtained by applying as many  $\partial_y$ 's as possible. Thus all partial derivatives from  $J_\mu$  kill  $S_n G_\mu(\underline{x})$ , as required.  $\square$

Let  $f \in \mathbb{C}[\underline{x}, \underline{y}]$  be a homogeneous element belonging to  $J_{\mathbf{p}}$ . This means that  $f|_{S_n \cdot \mathbf{p}} = g|_{S_n \cdot \mathbf{p}}$  for a polynomial  $g$  of degree lower than  $\deg(f)$ . Now  $\mathbb{C}[S_n \cdot \mathbf{p}]$ , functions on  $S_n \cdot \mathbf{p}$ , produce the regular representation and so, up to scalar, there is a unique function transforming via **sign**. Moreover in the quotient  $\mathbb{C}[\underline{x}, \underline{y}] \rightarrow \mathbb{C}[S_n \cdot \mathbf{p}]$  no polynomial of degree less than  $n(\mu) + n(\mu')$  realises this function since that is true on the associated graded space. Now suppose that  $(f)$  intersects **sign** on  $S_n \cdot \mathbf{p}$  in degree  $n(\mu) + n(\mu')$ . Then we find  $h$  such that  $(hf)|_{S_n \cdot \mathbf{p}}$  produces **sign** and hence so does  $(hg)|_{S_n \cdot \mathbf{p}}$ . But this is a contradiction because this polynomial has degree less than  $n(\mu) + n(\mu')$ . Therefore  $J_{\mathbf{p}} \subseteq J_\mu$ .

We **deduce** that  $\dim R_\mu \leq n!$ . To get equality we need that  $J_{\mathbf{p}} = J_\mu$ , in which case  $J_{\mathbf{p}}$  is indeed independent of  $\mathbf{p}$  and bigraded.

**2.3. Here comes the Hilbert scheme.** Recall from [3, TO COME] that  $J'_{\mathbf{p}}$  above is a (generic) point of  $\text{Hilb}^{S_n}(\mathbb{C}^{2n})$ . We don't yet know much about this variety, but we do know that it has a universal family: a bundle carrying the regular representation, denoted for the moment by  $\mathcal{X}_n$ . Similarly  $H_n = \text{Hilb}^n \mathbb{C}^2$  has a universal property too. Now we have a morphism

$$\phi : \text{Hilb}^{S_n}(\mathbb{C}^{2n}) \longrightarrow H_n$$

induced from the bundle  $\mathcal{X}_n^{S_{n-1}}$  where  $S_{n-1} = S_{\{1,2,\dots,n-1\}}$  and so this is a bundle of rank  $n$  of  $\mathbb{C}[x_n, y_n]$ -algebras. Moreover this map commutes with the Hilbert-Chow morphisms. In particular, it is an isomorphism generically. If this were an isomorphism then there would be a vector bundle of rank  $n!$  on  $H_n$  (which we already understand generically). A candidate is

$$\begin{array}{ccc} \mathbb{C}^{2n} \times_{\mathbb{C}^{2n}/S_n} H_n & \longrightarrow & \mathbb{C}^{2n} \\ \downarrow & & \downarrow \text{orbit} \\ H_n & \xrightarrow{\pi} & \mathbb{C}^{2n}/S_n. \end{array}$$

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However the fibre product is not reduced (exercise for  $n = 2$ ) and so we should take the unique reduced structure, which we denote  $X_n := (\mathbb{C}^{2n} \times_{\mathbb{C}^{2n}/S_n} H_n)_{\text{red}}$

$$\begin{array}{ccc} X_n & \longrightarrow & \mathbb{C}^{2n} \\ \rho_n \downarrow & & \downarrow \text{orbit} \\ H_n & \xrightarrow{\pi} & \mathbb{C}^{2n}/S_n. \end{array}$$

We call  $X_n$  the *isospectral Hilbert scheme*: its points have the form  $(I, P_1, \dots, P_n) \in H_n \times (\mathbb{C}^2)^n$  where  $\sum P_i$  is the support of  $I$  counted with multiplicity. Now the generic fibre of  $\rho_n$  has rank  $n!$  and carries the regular representation and an action of  $\mathbb{C}[\underline{x}, \underline{y}]$  and so if  $\rho_n$  were flat then we'd find a morphism

$$\eta : H_n \longrightarrow \text{Hilb}^{S_n}(\mathbb{C}^{2n})$$

arising from  $X_n$  and again commuting with the Hilbert-Chow morphisms. But then  $\eta$  and  $\phi$  would be generically inverse and hence everywhere inverse!

**Theorem** (Haiman, [12]).  *$X_n$  is Cohen-Macaulay and Gorenstein.*

These are two technical homological conditions. Recall the definition of a system of parameters.

- Cohen-Macaulay says that  $\rho_n$  is flat (i.e. finite over smooth is flat).
- Gorenstein says that  $X_n$  is first Cohen-Macaulay and then that if  $J$  is an ideal of  $\mathcal{O}_{X_n}$  generated locally by a system of parameters, then for  $q \in V(J)$  we have  $\mathcal{O}_{X_n, q}/J$  is artinian Gorenstein (i.e. has a duality, generalising the graded case mentioned before).

This gets us half-way to the proof of the  $n!$  conjecture (and it's the hard half) as follows.

**Corollary.**  $\dim R_{\mu} = n!$ .

*Proof.* The Cohen-Macaulayness of  $X_n$  implies that  $\rho_n$  is flat which, by the argument above, in turn implies that  $\phi : \text{Hilb}^{S_n}(\mathbb{C}^{2n}) \longrightarrow H_n$  is an isomorphism and that  $\mathcal{P} := (\rho_n)_* \mathcal{O}_{X_n}$  is the vector bundle on  $H_n$  corresponding to the tautological bundle of  $\text{Hilb}^{S_n}(\mathbb{C}^{2n})$ .

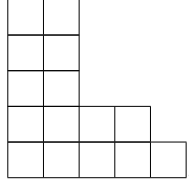
Now let  $\mathbf{p} \in \mathbb{C}^{2n}$  be a point associated to  $\mu$  as in the first paragraph of 2.2. Recall that  $J'_{\mathbf{p}} \in \text{Hilb}^{S_n}(\mathbb{C}^{2n})$  and that  $J_{\mathbf{p}} = \lim_{u \rightarrow 0} J'_{u\mathbf{p}}$ .

Under the isomorphism  $\phi$  we see that  $\phi(J_{\mathbf{p}}) = I(S)$  where  $S$  is the set of points  $\{(a_q, b_p) : (p, q) \in \mu\}$ . Now  $\phi$  is equivariant for rescaling and hence commutes with limits with respect to  $u$ . Thus we find that

$$\phi(J_{\mathbf{p}}) = \lim_{u \rightarrow 0} \phi(J'_{u\mathbf{p}}) = \lim_{u \rightarrow 0} I(uS)$$



and we claim that this final limit is just  $I_\mu$ , the monomial ideal belonging to  $H_n$  described in [3, TO COME]. To see this, we'll just do an example which illustrates the general principle. Look at the box (4, 1) which is just outside the Young diagram for this  $\mu$



Then  $(x - a_1)(y - b_1)(y - b_2)(y - b_3)(y - b_4) \in I(S)$  (since the  $(x - a_1)$  factor ensures anything involving the first row disappears, and the  $y$  factors clear out the first four columns). But then  $xy^4 \in \lim_{u \rightarrow 0} I(uS)$ . Thus we see that the limit contains the generators for  $I_\mu$ . But since the limit and  $I_\mu$  both have colength  $n$  we have the desired equality.

We deduce that  $\mathcal{P}(I_\mu) = \mathbb{C}[\underline{x}, \underline{y}] / J_\mu$ . But this is Gorenstein by the theorem (since it is  $\mathcal{O}_{X_n, Q_\mu} / I_\mu$  where  $Q_\mu = (I_\mu, 0, \dots, 0) \in X_n$ ) and it certainly contains a copy of the sign representation. Hence  $J_\mu = J_\mu$  by our characterisation lemma and we are finished.  $\square$

*Definition 3.* In the above corollary we saw the appearance of an unusual vector bundle of rank  $n!$  over  $H_n$ . We will call  $\mathcal{P} = (\rho_n)_* \mathcal{O}_{X_n}$  the *Procesi bundle*.

**2.4. Outline of proof of Haiman's big theorem.** For any irreducible scheme  $X$  we have the dualising complex  $\omega_X \in D(\text{coh } X)$ , the derived category of coherent sheaves on  $X$ . There is a nice characterisation of Cohen-Macaulayness and Gorensteinness in terms of the dualising complex.

- $X$  is Cohen-Macaulay if and only if  $\omega_X$  is concentrated in homological degree  $-\dim X$ .
- $X$  is Gorenstein if and only if  $\omega_X$  is concentrated in homological degree  $-\dim X$  and in that degree it is a line bundle, isomorphic to the canonical bundle  $\Omega_X^{\dim X}$  on  $X$ .

So the proof of the big theorem involves confirming the second equivalent notion for  $X_n$ . This is done by induction on  $n$  with the following claim:

$$\boxed{X_n \text{ is Gorenstein with canonical bundle } \rho_n^* \mathcal{O}(-1)}$$

Here  $\mathcal{O}(-1) = (\wedge^n B)^\vee$  where  $B$  denotes the tautological bundle on  $H_n$ .

To deal with the induction step Haiman uses the nested Hilbert scheme (which, by the way, also appears in Nakajima and Grojnowski's work on Heisenberg actions on cohomology of Hilbert schemes). It is

$$H_{n-1, n} = \{(I_{n-1}, I_n) : I_n \subset I_{n-1}\} \subset H_{n-1} \times H_n,$$

a closed subvariety with projections onto both  $H_{n-1}$  and  $H_n$ . A theorem of Cheah proves that it is smooth (very unusual!). Observe that for any point of  $H_{n-1, n}$  there exists a unique

$P \in \mathbb{C}^2$  such that  $\dim(\mathbb{C}[x, y]/I_n)_P = \dim(\mathbb{C}[x, y]/I_{n-1})_P + 1$  which in turn produces a morphism from  $H_{n-1, n}$  to  $\mathbb{C}^{2(n-1)}/S_{n-1} \times \mathbb{C}^2$ . Define the reduced variety

$$X_{n-1, n} = (H_{n-1, n} \times_{\mathbb{C}^{2(n-1)}/S_{n-1} \times \mathbb{C}^2} \mathbb{C}^{2n})_{\text{red}}.$$

We find a commutative diagram with obvious morphisms

$$\begin{array}{ccc} X_{n-1, n} & \xrightarrow{g} & X_n \\ \downarrow f & \searrow & \downarrow \rho_n \\ & H_{n-1, n} & \longrightarrow H_n \\ & \downarrow & \\ X_{n-1} & \xrightarrow{\rho_{n-1}} & H_{n-1} \end{array}$$

Now induction gives  $\omega_{X_{n-1}} = \rho_{n-1}^* \mathcal{O}(-1)$ . We can pull this back to describe the relative dualising complex of  $X_{n-1, n}$  over  $H_{n-1, n}$  as  $\mathcal{O}(-1, 0)$  (the notation just denotes the pullback from  $X_{n-1}$ ). Haiman calculates explicitly that  $\omega_{H_{n-1, n}} = \mathcal{O}(1, -1)$  (where the notation here denotes the tensor of the pullbacks of  $\mathcal{O}_{H_{n-1}}(1)$  and  $\mathcal{O}_{H_n}(-1)$  respectively) from which it follows that full dualising complex on  $X_{n-1, n}$  is  $g^* \mathcal{O}(-1)$ . Now we push this forward along  $g$  and use the projection formula to find  $\omega_{X_n} = Rg_* \mathcal{O}_{X_{n-1, n}} \otimes \mathcal{O}(-1)$ . Thus the crucial point to prove is that  $Rg_* \mathcal{O}_{X_{n-1, n}} \cong \mathcal{O}_{X_n}$ .

To prove this consider first the locus where  $(I, P_1, \dots, P_n) \in X_n$  has points  $P_i$  not all equal. Then it turns out that  $X_n$  splits locally into  $X_k \times X_l$  for appropriate  $k$  and  $l$  and then the isomorphism is already proved locally by induction. So we need to deal with glueing these different pieces together and also with the locus of equal  $P_i$ 's. The glueing is rather straightforward and attaching the locus of equal  $P_i$ 's – whose points are contained in  $V(y_1 - y_2, y_2 - y_3, \dots, y_{n-1} - y_n) \subset X_n$  – boils down, by local cohomology, to the fact that  $(y_1 - y_2, y_2 - y_3, \dots, y_{n-1} - y_n)$  form a regular sequence in  $X_n$ . This follows from the result that the composed projections

$$X_n \longrightarrow \mathbb{C}^{2n} \longrightarrow \mathbb{C}^n$$

onto  $\underline{y}$ -coordinates is flat. This in turn is a consequence of the relatively straightforward observation generalising the Proj construction of  $H_n$  given in [3, TO COME] that

$$X_n = \text{Proj } \mathbb{C}[\underline{x}, \underline{y}][tJ] = \text{Proj } \bigoplus_{d \geq 0} J^d$$

(where  $J = \mathbb{C}[\underline{x}, \underline{y}] \cdot \mathbb{C}[\underline{x}, \underline{y}]_+^{\text{sign}}$ ), and the *heart of the matter* which is that  $J^d$  is free over  $\mathbb{C}[\underline{y}]$  for all  $d \geq 0$ .

This is a consequence of the following result.

**Theorem** (Polygraph Theorem). *Let*

$$Z(n, \ell) = \{(P_1, \dots, P_n, Q_1, \dots, Q_\ell) : Q_i \in \{P_1, \dots, P_n\} \text{ for all } i\} \subset \mathbb{C}^{2n+2\ell}.$$

*Set*  $R(n, \ell) = \mathbb{C}[Z(n, \ell)]$ . *Then*  $R(n, \ell)$  *is free over*  $\mathbb{C}[y]$ .

Haiman shows that  $J^d = R(n, dn)^{\text{sign}}$ , hence implying the regular sequence property required. The proof of the above theorem is rather tricky, and is a drawback of the current proof of the  $n!$  conjecture. I will leave it to you to imagine why Haiman christened it the ‘‘Polygraph Theorem’’.

### LECTURE 3

**3.1. Identifying  $F_{R_\mu}(z; q, t)$  and  $\tilde{H}_\mu(z; q, t)$ .** Recall there were three parts to the definition of  $\tilde{H}_\mu(z; q, t)$ :

- (1)  $\tilde{H}_\mu[Z(1 - q); q, t] \in \mathbb{Q}(q, t)\{s_\lambda : \lambda \geq \mu\}$
- (2)  $\tilde{H}_\mu[Z(1 - t); q, t] \in \mathbb{Q}(q, t)\{s_\lambda : \lambda \geq \mu'\}$
- (3)  $\langle s_{(n)}, \tilde{H}_\mu(z; t) \rangle = 1$ .

It is obvious that (3) holds for  $F_{R_\mu}(z; q, t)$ : it simply expresses the fact that the trivial representation appears exactly once in  $R_\mu$ , and that it is in degree  $(0, 0)$ . (It comes from the constants in  $R_\mu$ .)

The proofs of (1) and (2) are similar, so we’ll just deal with (1). We begin by introducing a slightly more geometric version of the Frobenius map (the most natural geometric one is yet to come).

Set  $T = (\mathbb{C}^\times)^2$ , the two-dimensional complex torus. It acts naturally on  $\mathbb{C}^2$  via

$$\begin{pmatrix} t^{-1} & \\ & q^{-1} \end{pmatrix}$$

and so by functoriality on everything in sight, for instance  $\mathbb{C}[\underline{x}, \underline{y}], H_n, \mathcal{P}$ . The  $T$ -fixed points on  $H_n$  have already been identified in [3, TO COME]. They are given by the monomial ideals  $I_\mu$ , one for each partition  $\mu$  of  $n$ . Let

$$S = \mathcal{O}_{H_n, I_\mu},$$

a local ring on which  $T$  acts since  $I_\mu$  was a  $T$ -fixed point. Let  $\mathfrak{m}$  be the maximal ideal of  $S$ .

We first want to calculate the cotangent space of  $H_n$  at  $I_\mu$  as a  $T$ -representation, i.e. to describe the bigrading on the  $2n$ -dimensional vector space  $T_{I_\mu}^* H_n$ . This was done implicitly in [3] as follows. Recall that  $I_\mu \in U_\mu \subset H_n$  where  $U_\mu$  is one of our standard open sets in  $H_n$ . The functions on  $U_\mu$  have the form  $\Delta_M / \Delta_\mu$  for all  $M \subset \mathbb{N}^2$  subsets of cardinality of  $n$ . Now [3, Proposition 3.11 TO COME] shows that a basis for  $\mathfrak{m}/\mathfrak{m}^2$  – the cotangent space at  $I_\mu$  –



has degree  $(0, 1)$  and taking the corresponding Koszul resolution for  $A/(y_1, \dots, y_n)A$  we see, exactly as in the proof of the proposition of 1.2, that

$$F_{A/(y_1, \dots, y_n)A}(z; q, t) = F_A[Z(1 - q); q, t].$$

But  $A$  is a free  $S$ -module with  $T \times S_n$ -homogeneous basis provided by  $R_\mu$ . Thus

$$F_{R_\mu}[Z(1 - q); q, t] = p_S(q, t)^{-1} F_A[Z(1 - q); q, t] \in \mathbb{Q}(q, t) F_{A/(y_1, \dots, y_n)A}(z; q, t),$$

where  $p_S(q, t)$  denotes the bigraded Hilbert series of  $S$ .

Now  $A/(y_1, \dots, y_n)A$  has a generating set given by  $A/((y_1, \dots, y_n) + \mathfrak{m})A = R_\mu/(y_1, \dots, y_n)R_\mu$ . As we have already observed,  $R_\mu/(y_1, \dots, y_n)R_\mu$  is generated over  $\mathbb{C}[\partial_x]$  by the Garnir elements and so, by the theorem of 2.1, only features  $L_\lambda$  with  $\lambda \geq \mu$ .

Finally if we decompose  $A/(y_1, \dots, y_n)A$  into its different  $S_n$ -isotypic components and use Nakayama's lemma we see that this then implies that  $F_{A/(y_1, \dots, y_n)A}(z; q, t) \in \mathbb{Q}(q, t)\{s_\lambda : \lambda \geq \mu\}$ , as required.  $\square$

So the  $n!$  conjecture is now proved, and with it Macdonald positivity. (But don't forget that we've said nothing about diagonal coinvariants yet.)

**3.2. Homological consequences of  $H_n = \text{Hilb}^{S_n}(\mathbb{C}^{2n})$ .** In order to get better character formulae (for objects not necessarily supported only at one point like  $I_\mu \in H_n$ ) we need a stronger link between combinatorics and geometry. This is given by the following localisation theorem which is a special case of a general result called the Atiyah-Bott-Lefschetz formula.

**Theorem.** *Let  $\mathcal{A}$  be a  $T \times S_n$ -equivariant coherent sheaf on  $H_n$ . Let  $H^i(H_n, \mathcal{A})$  denote the  $i$ th sheaf cohomology of  $\mathcal{A}$ , a finitely generated  $\mathcal{O}(H_n) = \mathbb{C}[\underline{x}, \underline{y}]^{S_n}$ -module. Set  $F_{\mathcal{A}}^i(z; q, t) = F_{H^i(H_n, \mathcal{A})}(z; q, t)$ . Then*

$$\sum_{i \geq 0} (-1)^i F_{\mathcal{A}}^i(z; q, t) = \sum_{\mu \vdash n} \left( \frac{\sum (-1)^i F_{\text{Tor}_i(\mathbb{C}_\mu, \mathcal{A})}(q, t)}{\det_{T_{I_\mu}^*} (1 - (q, t))} \right)$$

where  $\mathbb{C}_\mu = \mathcal{O}_{H_n, I_\mu}/\mathfrak{m}$  denotes the irreducible skyscraper sheaf supported at the point  $I_\mu \in H_n$  on which  $T$  acts trivially.

Each term on the right hand side is a contribution from a  $T$ -fixed point and corresponds to what we looked at in 3.1 (where we worked only in a neighbourhood of  $I_\mu$  and so also higher cohomology vanished, making the left hand side simpler too).

So in order to get good combinatorial formulas associated to global sections  $H^0(H_n, \mathcal{A})$  we see from the left hand side of the formula that we need cohomology vanishing. To get good cohomology vanishing we need control on the derived category of coherent sheaves on  $H_n$ .

**Theorem** (Bridgeland-King-Reid, [6]). *Let  $G$  be a finite group and let  $X$  be a  $G$ -scheme over  $\mathbb{C}$ . Assume that the stabiliser subgroup of any point  $x \in X$  acts on the tangent space  $T_x X$  as a subgroup of  $SL(T_x X)$ . As in [3, TO COME] let  $Y = G - H_X^r$ , the (reduced)  $G$ -Hilbert scheme, and*

$$\pi : Y \longrightarrow X/G$$

*the corresponding Hilbert-Chow morphism. Assume that  $\pi$  is semismall: i.e. that for all irreducible subvarieties  $Z \subseteq Y$  we have  $2\text{codim}_Y Z \geq \text{codim}_{X/G} \pi(Z)$ . Then*

- (1)  *$(Y, \pi)$  is a crepant resolution of  $X/G$*
- (2) *There is an equivalence of triangulated categories  $\Phi : D^b(\text{coh } Y) \longrightarrow D^b(G - \text{coh } X)$  given by  $\Phi = Rf_* \circ \rho^*$  where  $\rho$  and  $f$  are the restrictions to the universal subscheme  $Z \subset Y \times X$  (attached to the  $G$ -Hilbert scheme) of the projections to  $Y$  and  $X$  respectively.*

We've already seen that thanks to Haiman's theorem of 2.3,  $\text{Hilb}^{S_n}(\mathbb{C}^{2n}) = S_n - H_{\mathbb{C}^{2n}}^r \cong H_n$ . Thus by [3, Theorem 3.6] the assumption on the semismallness of the Hilbert-Chow morphism  $\pi$  holds in this case, and so we get an equivalence of derived categories

$$\Phi : D^b(\text{coh } H_n) \longrightarrow D^b(S_n - \text{coh } \mathbb{C}^{2n}) = D^b(\mathbb{C}[\underline{x}, \underline{y}] \rtimes S_n - \text{mod}).$$

We can be a little bit more explicit about this. We've also seen in 2.3 that  $Z$ , the universal subscheme, is identified with  $X_n$  and hence it follows that

$$\Phi(-) = R\Gamma(\mathcal{P} \otimes -)$$

where  $\mathcal{P} = (\rho_n)_* \mathcal{O}_{X_n}$  is the Procesi bundle. Moreover, [6] gives an explicit inverse too:

$$\Psi(-) = ((\rho_n)_*(\omega_{X_n} \otimes^L Lf^*(-)))^G.$$

We saw in the second lecture that  $\omega_{X_n} = (\rho_n)^* \mathcal{O}(-1)$  where  $\mathcal{O}(-1)$  is the line bundle  $(\wedge^n B)^\vee$  on  $H_n$ . This carries the **sign** representation of  $S_n$  and so it follows that

$$\Psi(-) = \mathcal{O}(-1) \otimes^L ((\rho_n)_* Lf^*(-))^{\text{sign}}.$$

**3.3. Polygraphs revisited.** The tautological bundle  $B$  is described as a variety finite over  $H_n$  by  $F = \{(I, P) : P \in V(I)\} \subset H_n \times \mathbb{C}^2$ . Now consider the product

$$X_n \times F^\ell / H_n = \{(I, P_1, \dots, P_n, Q_1, \dots, Q_\ell) : Q_i \in V(I) \text{ for all } i\} \subset H_n \times \mathbb{C}^{2n+2\ell}.$$

Let  $\theta$  denote the projection to  $\mathbb{C}^{2n+2\ell}$ . It is clear that

$$\text{im } \theta = \{(P_1, \dots, P_n, Q_1, \dots, Q_\ell) : Q_i \in \{P_1, \dots, P_n\} \text{ for all } i\} = Z(n, \ell),$$

where  $Z(n, \ell)$  is as defined in 2.4. In other words there exists an injective algebra homomorphism

$$\theta^* : R(n, \ell) = \mathbb{C}[Z(n, \ell)] \longrightarrow \mathcal{O}(X_n \times F^\ell / H_n) = \Gamma(H_n, \mathcal{P} \otimes B^{\otimes \ell}).$$

In the notation of the previous section we can write this a homomorphism

$$\theta^* : R(n, \ell) \longrightarrow \Phi(B^{\otimes \ell})$$

in  $D^b(S_n - \text{coh } \mathbb{C}^{2n})$ .

**Theorem** (Haiman, [13]).  *$\theta^*$  is an isomorphism. In particular  $H^i(H_n, \mathcal{P} \otimes B^{\otimes \ell}) = 0$  for all  $i > 0$ .*

*Proof.* We need to show that inverse to  $\theta^*$ ,  $\Psi(R(n, \ell)) \longrightarrow B^{\otimes \ell}$ , is an isomorphism. It is a general fact of derived categories that this morphism fits into a cone

$$\Psi(R(n, \ell)) \longrightarrow B^{\otimes \ell} \longrightarrow C \longrightarrow \Psi(R(n, \ell))[1]$$

and thus we need to show that  $C = 0$  in  $D^b(\text{coh } H_n)$ .

Recall that by the Polygraph Theorem of 2.4  $R(n, \ell)$  is free over  $\mathbb{C}[\underline{y}]$ . Observe that the quotient  $R(n, \ell)/(y_1, \dots, y_n)R(n, \ell)$  is stable under the automorphisms induced from translation in the  $x$ -co-ordinates. This in turn all means that  $R(n, \ell)$  is free over  $\mathbb{C}[\underline{x}_1, \underline{y}]$  (to check this just take the union of the modules generated by  $(x_2 - x_1)^{i_2} \dots (x_n - x_1)^{i_n} (a_1 - x_1)^{j_1} \dots (a_\ell - x_1)^{j_\ell}$  where the  $a$ 's are generating the  $x$ -variables for  $\mathbb{C}^{2\ell}$ ). It follows that  $R(n, \ell)$  has a  $\mathbb{C}[\underline{x}, \underline{y}]$ -free  $S_n$ -resolution of length  $n - 1$ .

Applying this to our cone we see that  $C$  has a locally free resolution of length  $n$ . Moreover,  $\theta^*$  is an isomorphism over generic points and by a standard induction we see that this passes to the set where not all points  $(P_1, \dots, P_n)$  coincide. Hence  $C$  vanishes over such points.

So we have reached the stage where  $C$  is a complex on  $H_n$  with a locally free resolution of length  $n$  and which vanishes on an open set whose complement  $Z = \mathbb{C}^2 \times \pi^{-1}(0)$  has codimension  $2n - (2 + (n - 1)) = n - 1$ . The *new intersection theorem* of Peskine, Szpiro and Roberts (which is also a key tool in [6]) says that if

$$C_\bullet : 0 \longrightarrow C_n \longrightarrow \dots \longrightarrow C_1 \longrightarrow C_0 \longrightarrow 0$$

is a complex of locally free coherent sheaves on  $X$ , then every component of  $\text{Supp}(\mathcal{H}(C_\bullet))$  has codimension at most  $n$  in  $X$ . Put in a more suggestive way for us, if the codimension of the support is greater than  $n$ , then  $C_\bullet$  must be exact.

Haiman shows by calculation that  $C$ , the third term in our cone, is exact on  $U := (H_n \setminus Z) \cup U_{(n)} \cup U_{(1^n)}$ . It's not hard to show that the complement of  $U$  in  $H_n$  has codimension  $n + 1$ , and so  $C$  is exact, and hence  $\theta^*$  is an isomorphism as required.  $\square$

3.4. **The zero fibre**  $Z_n \subset H_n$ . Our final goal in this lecture is a homological description of  $Z_n := \pi^{-1}(0)$  where, as always,  $\pi : H_n \longrightarrow \mathbb{C}^{2n}/S_n$  is the Hilbert-Chow morphism.

Define a homomorphism  $\text{tr} : B \longrightarrow \mathcal{O}_{H_n}$  as follows. On a sufficiently small open set  $U$ ,  $B(U) \cong \mathcal{O}_{H_n}(U)^n$ . Multiplication by an element  $f$  of the algebra  $B(U)$  then produces an  $n$ -by- $n$  matrix over  $\mathcal{O}_{H_n}(U)$ . We set  $\text{tr}(f)$  to be the trace of this matrix.

Let  $U$  be the open set of  $H_n$  where all points  $P_1, \dots, P_n$  in the support of the ideal are distinct. Call these points  $(x_1, y_1), \dots, (x_n, y_n)$ . For any  $I \in U$  we can diagonalise the action of  $x$  and  $y$  on the fibre  $B(I)$  (and so on an neighbourhood of  $I$ ) by

$$x \mapsto \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}, \quad y \mapsto \begin{pmatrix} y_1 & & \\ & \ddots & \\ & & y_n \end{pmatrix}.$$

This shows that  $\text{tr}(x^r y^s) = p_{r,s}(\underline{x}, \underline{y}) := \sum_{i=1}^n x_i^r y_i^s$  and thus, since  $p_{r,s}(\underline{x}, \underline{y})$  is lifted from  $\mathcal{O}_{H_n}(H_n) = \mathbb{C}[\underline{x}, \underline{y}]^{S_n}$  and therefore regular, we have equality everywhere

$$\text{tr}(x^r y^s) = p_{r,s}(\underline{x}, \underline{y}).$$

There is an inclusion  $\mathcal{O}_{H_n} \hookrightarrow B$  to the identity element on  $B$  and, by the above calculation, this is split by  $\frac{1}{n} \text{tr}$ . Hence if we set  $B' = \ker(\text{tr})$  then we have

$$B = \mathcal{O}_{H_n} \oplus B'$$

and  $B'$  has global sections spanned over  $\mathcal{O}_{H_n}(H_n)$  by  $x^r y^s - \frac{1}{n} p_{r,s}(\underline{x}, \underline{y})$  with  $r + s > 0$ .

Let  $\mathcal{J}$  be the coherent subsheaf of  $B$  generated over  $B$  by  $x, y, B'$ ; let  $\mathcal{O}_t$  (respectively  $\mathcal{O}_q$ ) denote the trivial bundle on  $H_n$  with  $T$ -equivariant structure twisted by  $t$  (respectively  $q$ ). Multiplication on  $B$  will be denoted by  $m$  and the inclusion of  $\mathcal{J}$  into  $B$  by  $\iota$ . Then the natural homomorphism

$$B \otimes (B' \oplus \mathcal{O}_t \oplus \mathcal{O}_q) \xrightarrow{\iota \circ m} B$$

has image  $\mathcal{J}$ .

We claim that

$$B/\mathcal{J} \cong \mathcal{O}_{Z_n} \tag{6}$$

(the left hand side will be a sheaf of algebras supported on  $Z_n \subseteq H_n$ ).

This granted, we see that  $\mathcal{J}$  is locally generated by  $(n-1) + 2 = n+1$  equations and that  $\text{codim}_F(V(\mathcal{J})) = 2n - (n-1) = n+1$ , giving a complete intersection. This in turn yields a Koszul resolution

$$\cdots \longrightarrow B \otimes \wedge^k(B' \oplus \mathcal{O}_t \oplus \mathcal{O}_q) \longrightarrow \cdots \longrightarrow B \otimes (B' \oplus \mathcal{O}_t \oplus \mathcal{O}_q) \longrightarrow B \longrightarrow \mathcal{O}_{Z_n} \longrightarrow 0.$$



Label this locally free resolution of  $\mathcal{O}_{Z_n}$  by  $V_\bullet$  (we do not include  $\mathcal{O}_{Z_n}$ , so that  $V_\bullet \cong \mathcal{O}_{Z_n}$  in  $D^b(\text{coh } H_n)$ ). Now each term in  $V_\bullet$  is a direct summand of a tensor power of  $B$  and so we can use our cohomology vanishing results of 3.3 to deduce that

$$H^i(H_n, \mathcal{P} \otimes B^{\otimes \ell} \otimes V_k) = 0$$

for all  $i > 0$  and  $k \geq 0$ . Thus we have

$$H^i(Z_n, \mathcal{P} \otimes B^{\otimes \ell}) = H^i(H_n, \mathcal{P} \otimes B^{\otimes \ell} \otimes \mathcal{O}_{Z_n}) = H^i(H_n, \mathcal{P} \otimes B^{\otimes \ell} \otimes V_\bullet)$$

and this vanishes if  $i > 0$  and produces a surjection

$$H^0(H_n, \mathcal{P} \otimes B^{\otimes \ell} \otimes B) \longrightarrow H^0(Z_n, \mathcal{P} \otimes B^{\otimes \ell}).$$

The kernel of this homomorphism from  $R(n, \ell + 1)$  is generated by  $x, y$  (in the  $(\ell + 1)$ -st place) and by  $\mathfrak{m} = \mathbb{C}[\underline{x}, \underline{y}]_+^{S_n}$ . It follows, after a little work, that we find an isomorphism

$$\frac{R(n, \ell)}{\mathfrak{m}R(n, \ell)} \cong H^0(Z_n, \mathcal{P} \otimes B^{\otimes \ell}). \quad (7)$$

We still have to justify the claim (6). Recall  $F \subset H_n \times \mathbb{C}^2$  with projection  $\eta$  to  $H_n$ . Let  $\eta^{-1}(Z_n)_{\text{red}}$  be the reduced pullback of  $Z_n$ . Then  $\eta^{-1}(Z_n)_{\text{red}} \subset H_n \times \{0\}$  and so  $\eta$  provides an isomorphism

$$\eta^{-1}(Z_n)_{\text{red}} \xrightarrow{\sim} (Z_n)_{\text{red}}.$$

Now  $x, y$  and  $p_{r,s}(\underline{x}, \underline{y})$  with  $r + s > 0$  all vanish on  $\eta^{-1}(Z_n)_{\text{red}}$  and are locally given by  $(n + 1)$ -equations (defining  $\mathcal{J}$ ). But  $\eta^{-1}(Z_n)_{\text{red}}$  has codimension  $n + 1$  in  $F$  and so since  $F$  is Cohen-Macaulay (it is flat over  $H_n$ , a smooth space) we find that the equations cut out a complete intersection that define a subvariety isomorphic to  $Z_n$  (because  $p_{r,s}(\underline{x}, \underline{y})$  with  $r + s > 0$  generate the ideal of  $0 \in \mathbb{C}^{2n}/S_n$ ). (With a little extra work –  $Z_n$  is Cohen-Macaulay and generically reduced (on the intersection with the chart  $U_{(1^n)}$  for instance) – this argument extends to prove the reducedness of  $Z_n$  too.)

## LECTURE 4

**4.1. Diagonal coinvariants.** Recall we have the ring of diagonal coinvariants

$$R_n = \mathbb{C}[\underline{x}, \underline{y}] / \mathfrak{m}\mathbb{C}[\underline{x}, \underline{y}]$$

where  $\mathfrak{m} = \mathbb{C}[\underline{x}, \underline{y}]_+^{S_n}$ . From (7) we know that  $H^0(Z_n, \mathcal{P}) = R_n$  and also that  $H^i(Z_n, \mathcal{P}) = 0$  for all  $i > 0$ . Thus

$$F_{R_n}(z; q, t) = F_{\mathcal{P} \otimes \mathcal{O}_{Z_n}}(z; q, t)$$

Now recall the locally free resolution of  $\mathcal{O}_{Z_n}$  whose  $k$ th term is  $V_k = B \otimes \wedge^k(B' \oplus \mathcal{O}_t \oplus \mathcal{O}_q)$ . Since we also have  $H^i(H_n, \mathcal{P} \otimes V_k) = 0$  for all  $i > 0$  we see that by the Atiyah-Bott-Lefschetz localisation theorem

$$F_{\mathcal{P} \otimes \mathcal{O}_{Z_n}}(z; q, t) = \sum_{k=0}^{n+1} (-1)^k F_{\mathcal{P} \otimes V_k}(z; q, t) = \sum_{\mu \vdash n} \frac{\sum_{k=0}^{n+1} (-1)^k F_{\mathcal{P} \otimes V_k(I_\mu)}(z; q, t)}{\det_{T_{I_\mu}^*}(1 - (q, t))}.$$

Now, by the  $n!$  theorem  $F_{\mathcal{P}(I_\mu)}(z; q, t) = \tilde{H}_\mu(z; q, t)$ , whilst  $V_k$  has no  $S_n$ -action at all and so only produces terms from  $\mathbb{Q}(q, t)$ . These are easy to describe. As  $B(I_\mu)$  has basis  $x^p y^q$  for  $(p, q) \in \mu$  we see that it has graded series  $\sum_{x \in \mu} q^{a'(x)} t^{l'(x)} = B_\mu(q, t)$ . Similarly  $B' \oplus \mathcal{O}_t \oplus \mathcal{O}_q$  has terms  $q^{a'(x)} t^{l'(x)}$  (where  $x \in \mu \setminus \{(0, 0)\}$ ) and  $t$  and  $q$ . Thus, using the notation of (3), the exterior algebra  $V_\bullet$  has graded series

$$(1 - q)(1 - t) \prod_{x \in \mu \setminus \{(0, 0)\}} (1 - q^{a'(x)} t^{l'(x)}) = (1 - q)(1 - t) \Pi_\mu(q, t).$$

Putting this together with the formula we already know for  $\det_{T_{I_\mu}^*}(1 - (q, t))$  shows that

$$F_{R_n}(z; q, t) = \sum_{\mu \vdash n} \frac{(1 - t)(1 - q) \Pi_\mu(q, t) B_\mu(q, t) \tilde{H}_\mu(z; q, t)}{h_\mu(q, t) h'_\mu(q, t)}.$$

This confirms the Garsia-Haiman  $(n + 1)^{n-1}$  conjecture.

**4.2. New proofs.** We have now outlined the proofs of Macdonald positivity and of the Garsia-Haiman  $(n + 1)^{n-1}$  conjecture, both via the Hilbert scheme.

However, there are two new proofs of Macdonald positivity now available. The both have the same first building block.

**Block 1:** Building on work of Haglund, Haglund-Haiman-Loehr, [10], give a decomposition of the  $\tilde{H}_\mu(z; q, t)$ 's into LLT polynomials (Lascoux-Leclerc-Thibon polynomials) with coefficients given by a combinatorial statistic.

The LLT polynomials are more combinatorial than the Macdonald polynomials and so combinatorists like this approach. However, to get Macdonald positivity it is still necessary to show that LLT polynomials (or at least the subclass appearing in the above description of Macdonald polynomials) are Schur positive. This has now been proved twice.

**Block 2:** (A) Assaf, [1], gives a combinatorial proof of the Schur positivity of LLT polynomials using “dual equivalence graphs”. This is combinatorial.

(B) Grojnowski-Haiman, [9], give a proof of Schur positivity of LLT polynomials (and their generalisations to other groups) using geometric representation theory.

However, there is still a thousand dollar question (literally): to produce an explicit basis for each of the spaces  $V(\mu)$ .

**Diagonal harmonics.** Garsia and Haiman long ago showed that the conjectural formula for  $R_n$  could be described simply as  $\nabla_{S_{(1^n)}}$ . Here  $\nabla$  is the operator on  $\Lambda_{q,t}$  which is defined by the following rule

$$\nabla \tilde{H}_\mu(z; q, t) = t^{n(\mu)} q^{n(\mu')} \tilde{H}_\mu(z; q, t).$$

Despite its elementary construction,  $\nabla$  is a rather mysterious operator which is very interesting to combinatorists. In particular it features in a series of conjectures by Loehr and Warrington, [16], who give a conjectural combinatorial description of  $\nabla_{S_\mu}$  in terms of the monomial basis  $m_\lambda$  and which in particular generalises a conjectural combinatorial description of  $\nabla_{S_{(1^n)}}$  of Haglund-Haiman-Loehr-Remmel-Ulyanov, [11].

**4.3. Beyond  $S_n$ : coinvariants.** Let's just make explicit the strategy we had for dealing with diagonal coinvariants.

$$\begin{array}{ccc} X_n & \longrightarrow & \mathbb{C}^{2n} \\ \downarrow & & \downarrow \\ H_n & \longrightarrow & \mathbb{C}^{2n}/S_n \end{array}$$

Here  $X_n$  is the reduced fibre product. On the right hand side, the scheme-theoretic zero fibre produces the ring of diagonal coinvariants. Thus, under pullback, it should come from functions on  $X_n$  supported on  $Z_n$ , i.e.  $H^0(Z_n, \mathcal{P})$ . Well, we saw that it was and then we studied  $H^0(Z_n, \mathcal{P})$  using two important facts:

- smoothness of  $H_n$ , producing a crepant resolution of  $\mathbb{C}^{2n}/S_n$ ;
- homological precision via the McKay correspondence to understand the space of sections combinatorially.

Now a natural generalisation is to replace  $(\mathbb{C}^{2n}, S_n)$  with  $(\mathfrak{h} \times \mathfrak{h}^*, G)$  where  $G$  is a complex reflection group and  $\mathfrak{h}$  its reflection representation. There is, however, a serious problem.

**Theorem** (Ginzburg-Kaledin [7], Bellamy [2]). *The singular Gorenstein variety  $\mathfrak{h} \times \mathfrak{h}^*/G$  has a crepant resolution if and only if  $G = (\mathbb{Z}/r\mathbb{Z})^n \rtimes S_n$  or  $G =$  binary tetrahedral group  $G_4$  (this is the group  $E_6$  in the description of finite subgroups of  $SU(2)$ , but with a different non-defining two dimensional representation producing the reflection representation).*

The non-existence is proved using a mixture of Poisson deformation theory, algebraic geometry and representation of symplectic reflection algebras. The existence part is mostly straightforward. Indeed we can think of  $(\mathfrak{h} \times \mathfrak{h}^*)/(\mathbb{Z}/r\mathbb{Z})^n \rtimes S_n$  as  $(\mathbb{C}^2/\mathbb{Z}/r\mathbb{Z})^n/S_n$  then we make the resolution in two steps. At the first step we use the minimal resolution  $Y \longrightarrow \mathbb{C}^2/(\mathbb{Z}/r\mathbb{Z})$  of the kleinian singularity. Then we use the Hilbert scheme:

$$\text{Hilb}^n Y \longrightarrow Y^n/S_n \longrightarrow (\mathbb{C}^2/\mathbb{Z}/r\mathbb{Z})^n/S_n.$$

The resolution for  $G_4$  was discussed in Sorger's lecture, see [15].

So the geometric tactic above is not available for studying diagonal coinvariants for general complex reflection groups. But there is still hope:

**Theorem** (Gordon [8]). *Let  $W$  be a finite Coxeter group. Then there exists a natural  $W$ -stable bigraded quotient  $R_n(W)$  of  $\mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]/\langle \mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]_+^W \rangle$  which has analogous combinatorial and  $W$ -representation theoretic properties to  $R_n = R_n(S_n)$ .*

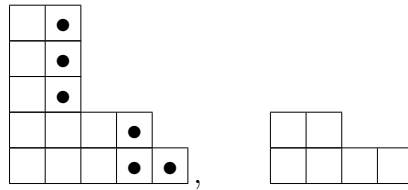
This is proved using the representation theory of symplectic reflection algebras. Let us remark that it was already known to Haiman that the whole coinvariant ring,  $\mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]/\langle \mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]_+^W \rangle$ , was too large to have the correct properties. There is a conjecture due to Haiman:

**Conjecture:**(Haiman, [14]) Let  $K = \ker(\mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]/\langle \mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]_+^W \rangle \longrightarrow R_n(W))$ . Then  $K$  is the largest ideal of  $\mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]/\langle \mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]_+^W \rangle$  that does not contain a copy of the representation  $\text{sign}$ .

4.4. **Beyond  $S_n$ : symmetric functions.** Fix  $r \geq 2, n \geq 2$ . (The  $r = 1$  case will be what we covered before concerning  $H_n$ ; the  $n = 1$  case corresponding to the geometry associated to kleinian singularities discussed in [4].) Set  $\Gamma = \mathbb{Z}/r\mathbb{Z}$ ,  $\Gamma_n = (\mathbb{Z}/r\mathbb{Z})^n \rtimes S_n =: (\mathbb{Z}/r\mathbb{Z}) \wr S_n$ .

*Definition 4.* A *ribbon* in a partition is a skew subdiagram containing no  $2 \times 2$  square. An  *$r$ -core* is a partition containing no ribbon of size  $r$ . The  *$r$ -core* of a partition is the  $r$ -core that remains when all ribbons of size  $r$  are (repeatedly) removed from the given partition.

In the following diagram of  $\mu$  we mark out two distinct ribbons of size 3. We also draw a 3-core (the 3-core of  $\mu$  is the empty partition).



Let  $\nu_0$  be a core. There is a combinatorial procedure from the classical era that identifies partitions of size  $nr + |\nu_0|$  with  $r$ -core  $\nu_0$  with the  $r$ -multipartitions of  $n$ :

$$\mu \mapsto \text{Quot}_r(\mu).$$

Basic group representation theory shows that the complex irreducible representations of  $\Gamma_n$  are labelled by  $r$ -multipartitions of  $n$ , i.e.  $(\lambda^1, \lambda^2, \dots, \lambda^r)$  with  $\sum |\lambda^i| = n$ .

Recall the definition of the Macdonald polynomials from 1.3 as well as the lemma in 1.2 which gives the representation theoretic interpretation of the plethystic substitution  $Z \mapsto Z(1 - q)$ .

**Conjecture:**(Haiman, [14]) There exists a basis  $\{H_{\mu}(q, t)\}$  of  $\mathbb{Q}(q, t) \otimes \text{Rep}(\Gamma_n)$  indexed by partitions of size of  $|\nu_0| + nr$  and core  $\nu_0$  such that

- (1)  $H_{\mu}(q, t) \otimes \sum_i (-q)^i \chi^{\wedge^{i\mathfrak{h}}} \in \mathbb{Q}(q, t) \{\chi^{\text{Quot}_r(\lambda)} : \lambda \geq \mu, \text{Core}_r(\lambda) = \nu_0\}$
- (2)  $H_{\mu}(q, t) \otimes \sum_i (-t)^{-i} \chi^{\wedge^{i\mathfrak{h}}} \in \mathbb{Q}(q, t) \{\chi^{\text{Quot}_r(\lambda)} : \lambda \leq \mu, \text{Core}_r(\lambda) = \nu_0\}$
- (3)  $\langle \chi^{\text{triv}}, H_{\mu}(z; t) \rangle = 1$ .

If such elements exist, they will be called *wreath Macdonald polynomials*. It is relatively straightforward (with only a little work) to see that this generalises the definition of the Macdonald polynomials when  $r = 1$ . The difficulty, as usual, in showing the existence in this level of generality is that we are using a partial and *not* total order on partitions; no analogue of the operator  $D$  of Macdonald is known.

**Conjecture:**(Haiman, [14]) The coefficients of the wreath Macdonald polynomials  $H_{\mu}(q, t)$  in their Schur expansion all belong to  $\mathbb{N}[q^{\pm 1}, t^{\pm 1}]$ .

At the heart of the wreath Macdonald polynomials is the ordering on  $r$ -multipartitions induced from the core  $\nu_0$ . This is constructed by identifying the  $r$ -multipartitions of  $n$  with the partitions of  $nr + |\nu_0|$  whose core is  $\nu_0$  and then ordering them by the dominance ordering on the partitions. This ordering varies as  $\nu_0$  varies, but since  $r$ -multipartitions of  $n$  is a finite set there are only a finite number of orderings and therefore a finite number of possible wreath Macdonald polynomials of degree  $n$ . For instance if  $\nu_0$  is large compared to  $n$  then there is only one ordering (conjecturally this is related to the work of Yvonne which featured in Leclerc's lectures, and Rouquier's generalisation of it to the context of symplectic reflection algebras).

4.5. **Beyond  $S_n$ : geometry.**  $\langle \sigma \rangle = \Gamma$  acts on  $\mathbb{C}^2$  via

$$\sigma \mapsto \begin{pmatrix} \eta & \\ & \eta^{-1} \end{pmatrix}$$

where  $\eta = \exp(2\pi\sqrt{-1}/r)$ . Now let  $K = |\nu_0| + nr$ . Then  $\Gamma$  acts on  $\text{Hilb}^K \mathbb{C}^2$  and the fixed point subvariety  $(\text{Hilb}^K \mathbb{C}^2)^{\Gamma}$  is smooth, but has several components.

Consider  $\mathbb{C}[x, y]/I_{\nu_0}$ , which is supported only at  $0 \in \mathbb{C}^2$ , and  $n$  distinct generic  $\Gamma$ -orbits in  $\mathbb{C}^2$ . Glueing these together produces a point in  $(\text{Hilb}^K \mathbb{C}^2)^{\Gamma}$ ; varying the  $n$ -generic orbits

then gives us a space of dimension  $2n$ . The closure of this set in  $\text{Hilb}^K \mathbb{C}^2$  determines a connected component of  $(\text{Hilb}^K \mathbb{C}^2)^\Gamma$ , which we will label by  $H_n(\nu_0)$ .

Restricting the Chow morphism produces a projective morphism

$$\pi : H_n(\nu_0) \longrightarrow \mathbb{C}^{2K}/S_K$$

whose image is  $|\nu_0| \cdot [0] + \sum_{i=1}^n (\sum_{g \in \Gamma} gP_i)$ . After subtracting  $|\nu_0| \cdot [0]$ , this is just isomorphic to  $(\mathbb{C}^2/\Gamma)^n/S_n = \mathbb{C}^{2n}/\Gamma_n$ . In fact,

$$\pi : H_n(\nu_0) \longrightarrow \mathbb{C}^{2n}/\Gamma_n$$

is a crepant resolution of singularities. (If we take  $\nu_0$  to be large enough, then this is the resolution  $\text{Hilb}^n(Y)$  where  $Y$  is the minimal resolution of the kleinian singularity of type  $A_{r-1}$ . In general, it is not isomorphic to this.) The  $T$ -fixed points of  $H_n(\nu_0)$  are, by construction, the partitions of  $K$  with core  $\nu_0$ , a set in natural bijection with  $r$ -multipartitions of  $n$  and so we might hope that there is a connection with wreath Macdonald polynomials.

Now  $H_n(\nu_0)$  carries a tautological bundle of rank  $nr$ , which we will label by  $B_n(\nu_0)$ . It is obtained from the tautological bundle  $B_K$  of  $\text{Hilb}^K \mathbb{C}^2$  of rank  $K$  by first restricting to  $H_n(\nu_0)$  and then observing that at any fibre  $B_K(I) = \mathbb{C}[x, y]/I$  there is a submodule  $I_{\nu_0}/I$ . Factoring out this sub-bundle gives  $B_n(\nu_0)$ . Clearly this is  $\Gamma$ -equivariant and, thanks to the description above of a generic point in  $H_n(\nu_0)$  we see that each fibre of  $B_n(\nu_0)$  carries  $n$  copies of the regular representation of  $\Gamma$ .

*Remark.* In fact,  $H_n(\nu_0)$  also has a description as a Nakajima quiver variety on the cyclic quiver with  $r$  vertices, dimension vector  $n\delta = n(1, 1, \dots, 1)$ , and framing vector  $\epsilon = \omega_i$  for some (any)  $i$ . The bundle just described above is simply the tautological bundle of this quiver variety.

**Conjecture:**(Haiman, [14]) The variety  $H_n(\nu_0)$  has a vector bundle  $\mathcal{P}_n(\nu_0)$  that is  $T$ -equivariant and that has an action of  $\mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*] \rtimes \Gamma_n$  on each fibre and that satisfies

- (1) each fibre carries the regular  $\Gamma_n$ -representation;
- (2)  $\mathcal{P}_n(\nu_0)^{\Gamma_{n-1}} \cong B_n(\nu_0)$  as bigraded bundles of  $\mathbb{C}[x, y] \rtimes \Gamma$ -modules;
- (3) for  $\mu \vdash K$  with  $r$ -core  $\nu_0$  the fibre  $\mathcal{P}_n(\nu_0)(I_\mu)$  has bigraded character equal to  $H_\mu(q, t)$ .

It is expected that  $\mathcal{P}_n(\nu_0)$  is the tautological bundle for an alternative realisation of  $H_n(\nu_0)$  as a moduli space of  $\Gamma_n$ -constellations. A  $\Gamma_n$ -constellation is a  $\mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*] \rtimes \Gamma_n$ -module which is just the regular representation as a  $\Gamma_n$ -module. (This moduli space would be constructed using King's construction via some stability parameter  $\theta \in K(\mathbb{C}\Gamma_n)^\vee$  which satisfies  $\theta(\mathbb{C}\Gamma_n) = 0$ ; the case of a  $G$ -cluster is  $\theta(\chi) < 0$  for all irreducible  $\chi \neq \chi^{\text{triv.}}$ .) With such a realisation we would be able to apply [6] to get an equivalence

$$D^b(\text{coh } H_n(\nu_0)) \longrightarrow D^b(\mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*] \rtimes \Gamma_n - \text{mod}).$$

Such an equivalence is already known thanks to a remarkable theorem of Bezrukavnikov-Kaledin, [5].

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