# Moment graphs and representations

Jens Carsten Jantzen

# 2008 SUMMER SCHOOL

# Geometric Methods in Representation Theory

**Moment graphs and representations**

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**2008** SUMMER SCHOOL PROCEEDINGS :

## Lecture notes

-**José BERTIN**: *The punctal Hilbert scheme: an introductio*n

- **Michel BRION**: *Representations of quivers*
- **Victor GINZBURG**: *Lectures on Nakajima's quiver varieties*
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# Moment graphs and representations

Jens Carsten Jantzen*

In a 1979 paper Kazhdan and Lusztig introduced certain polynomials that nowadays are called Kazhdan-Lusztig polynomials. They conjectured that these polynomials determine the characters of infinite dimensional simple highest weight modules for complex semi-simple Lie algebras. Soon afterwards Lusztig made an analogous conjecture for the characters of irreducible representations of semi-simple algebraic groups in prime characteristics.

The characteristic 0 conjecture was proved within a few years. Concerning prime characteristics the best result known says that the conjecture holds in all characteristics $p$ greater than an unknown bound depending on the type of the group.

In both cases the proofs rely on the fact (proved by Kazhdan and Lusztig) that the Kazhdan-Lusztig polynomials describe the intersection cohomology of Schubert varieties. It was then quite complicated to link the representation theory to the intersection cohomology. In the characteristic 0 case this involved $\mathcal{D}$–modules and the Riemann-Hilbert correspondence. The proof of the weaker result in prime characteristics went via quantum groups and Kac-Moody Lie algebras.

In these notes I want to report on a more direct link between representations and cohomology. Most of this is due to Peter Fiebig. An essential tool is an alternative description of the intersection cohomology found by Tom Braden and Robert MacPherson. A crucial point is that on one hand one has to replace the usual intersection cohomology by *equivariant* intersection cohomology, while on the other hand one has to work with *deformations* of representations, i.e., with lifts of the modules to a suitable local ring that has our original ground field as its residue field.

Braden and MacPherson looked at varieties with an action of an (algebraic) torus; under certain assumptions (satisfied by Schubert varieties) they showed that the equivariant intersection cohomology is given by a combinatorially defined sheaf on a graph, the *moment graph* of the variety with the torus action.

Fiebig then constructed a functor from deformed representations to sheaves on a moment graph. This functor takes projective indecomposable modules to the sheaves defined by Braden and MacPherson. This is then the basis for a comparison between character formulae and intersection cohomology.

In Section 4 of these notes I describe Fiebig's construction in the characteristic 0 case. While Fiebig actually works with general (symmetrisable) Kac-Moody algebras, I have restricted myself here to the less complicated case of finite dimensional semi-simple Lie algebras. The prime characteristic case is then discussed in Section 5, but with crucial proofs replaced by references to Fiebig's papers.

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The two middle sections 2 and 3 discuss moment graphs and sheaves on them. I describe the Braden-MacPherson construction and follow Fiebig's approach to a localisation functor and its properties.

The first section looks at some cohomological background. A proof of the fact that the Braden-MacPherson sheaf describes the equivariant intersection cohomology was beyond the reach of these notes. Instead I go through the central definitions in equivariant cohomology and try to make it plausible that moment graphs have something to do with equivariant cohomology.

For advice on Section 1 I would like to thank Michel Brion and Jørgen Tornehave.

## 1 Cohomology

For general background in algebraic topology one may consult [Ha]. For more information on fibre bundles, see [Hm]. (I actually looked at the first edition published by McGraw-Hill.)

**1.1.** (**A simple calculation**) Consider the polynomial ring $S = k[x_1, x_2, x_3]$ in three indeterminates over a field $k$. Set $\alpha = x_1 - x_2$ and $\beta = x_2 - x_3$. Let us determine the following $S$–subalgebra of $S^3 = S \times S \times S$:

$$Z = \{\,(a,b,c) \in S^3 \mid a \equiv b \,(\operatorname{mod} S\alpha),\ b \equiv c \,(\operatorname{mod} S\beta),\ a \equiv c \,(\operatorname{mod} S(\alpha+\beta))\,\}. \tag{1}$$

We have clearly $(c,c,c) \in Z$ for all $c \in S$; it follows that $Z = S\,(1,1,1) \oplus Z'$ with

$$Z' = \{\,(a,b,0) \in S^3 \mid a \equiv b \,(\operatorname{mod} S\alpha),\ b \in S\beta,\ a \in S(\alpha+\beta)\,\}.$$

Any triple $(b\,(\alpha+\beta), b\beta, 0)$ with $b \in S$ belongs to $Z'$. This yields $Z' = S\,(\alpha+\beta,\beta,0) \oplus Z''$ where $Z''$ consists of all $(a,0,0)$ with $a \in S\,\alpha \cap S\,(\alpha+\beta)$. Since $\alpha$ and $\alpha+\beta$ are non-associated prime elements in the unique factorisation domain $S$, the last condition is equivalent to $a \in S\,\alpha\,(\alpha+\beta)$. So we get finally

$$Z = S\,(1,1,1) \oplus S\,(\alpha+\beta,\beta,0) \oplus S\,(\alpha\,(\alpha+\beta),0,0). \tag{2}$$

So $Z$ is a free $S$–module of rank 3.

Consider $S$ as a graded ring with the usual grading doubled; so each $x_i$ is homogeneous of degree 2. Then also $S^3$ and $Z$ are naturally graded. Now (2) says that we have an isomorphism of graded $S$–modules

$$Z \simeq S \oplus S\langle 2\rangle \oplus S\langle 4\rangle \tag{3}$$

where quite generally $\langle n\rangle$ indicates a shift in the grading moving the homogeneous part of degree $m$ into degree $n+m$.

The point about all this is that we have above calculated (in case $k = \mathbf{C}$) the equivariant cohomology $H_T^\bullet(\mathbf{P}^2(\mathbf{C});\mathbf{C})$ where $T$ is the algebraic torus $T = \mathbf{C}^\times \times \mathbf{C}^\times \times \mathbf{C}^\times$ acting on $\mathbf{P}^2(\mathbf{C})$ via $(t_1,t_2,t_3)\cdot[x:y:z] = [t_1x : t_2y : t_3z]$ in homogeneous coordinates. Actually we have also calculated the ordinary cohomology $H^\bullet(\mathbf{P}^2(\mathbf{C});\mathbf{C})$ that we get (in this case) as $Z/\mathfrak{m}Z$ where $\mathfrak{m}$ is the maximal ideal of $S$ generated by the $x_i$, $1 \le i \le 3$. So we regain the well-known fact that $H^{2r}(\mathbf{P}^2(\mathbf{C});\mathbf{C}) \simeq \mathbf{C}$ for $0 \le r \le 2$ while all remaining cohomology groups are 0.

**1.2.** (**Principal bundles**) Let $G$ be a topological group. Recall that a *$G$–space* is a topological space $X$ with a continuous action $G \times X \to X$ of $G$ on $X$. If $X$ is a $G$–space, then we denote by $X/G$ the space of all orbits $Gx$ with $x \in X$ endowed with the quotient topology: If $\pi: X \to X/G$ takes any $x \in X$ to its orbit $Gx$, then $U \subset X/G$ is open if and only if $\pi^{-1}(U)$ is open in $X$. It then follows that $\pi$ is open since $\pi^{-1}(\pi(V)) = \bigcup_{g \in G} gV$ for any $V \subset X$.

A (numerable) *principal $G$–bundle* is a triple $(E, p, B)$ where $E$ is a $G$–space, $B$ a topological space and $p: E \to B$ a continuous map such that there exists a numerable covering of $B$ by open subsets $U$ such that there exists a homeomorphism

$$\varphi_U: U \times G \to p^{-1}(U) \qquad \text{with } p \circ \varphi_U(u, g) = u \text{ and } \varphi_U(u, gh) = g\,\varphi_U(u, h) \tag{1}$$

for all $u \in U$ and $g, h \in G$. (The numerability condition is automatically satisfied if $B$ is a paracompact Hausdorff space. We assume in the following all bundles to be numerable.)

Note that these conditions imply that the fibres of $p$ are exactly the $G$–orbits on $E$, that each fibre $p^{-1}(b)$ with $b \in B$ is homeomorphic to $G$, and that $G$ acts freely on $E$: If $g \in G$ and $x \in E$ with $g\,x = x$, then $g = 1$. It also follows that $Gx \mapsto p(x)$ is a homeomorphism from $E/G$ onto $B$ and that $p$ is open.

For example the canonical map $p: \mathbf{C}^{n+1} \setminus \{0\} \to \mathbf{P}^n(\mathbf{C})$ is a principal bundle for the multiplicative group $\mathbf{C}^\times$. If we restrict $p$ to the vectors of length 1, then we get a principal bundle $S^{2n+1} \to \mathbf{P}^n(\mathbf{C})$ for the group $S^1$ of complex numbers of length 1.

If $G$ is a Lie group and $H$ a closed Lie subgroup of $G$, then the canonical map $G \to G/H$ is a principal bundle for $H$ acting on $G$ by right multiplication. This is a fundamental result in Lie group theory.

If $(E, p, B)$ is a principal bundle for a Lie group $G$ and if $H$ is a closed Lie subgroup of $G$, then $(E, \overline{p}, E/H)$ is a principal bundle for $H$ where $\overline{p}: E \to E/H$ maps any $v \in E$ to its $H$-orbit $Hv$.

**1.3.** (**Universal principal bundles**) Let $(E, p, B)$ be a principal bundle for a topological group $G$ and let $f: B' \to B$ be a continuous map of topological spaces. Then one constructs an *induced* principal bundle $f^*(E, p, B) = (E', p', B')$: One takes $E'$ as the fibre product

$$E' = B' \times_B E = \{\, (v, x) \in B' \times E \mid f(v) = p(x) \,\}$$

and one defines $p'$ as the projection $p'(v, x) = v$. The action of $G$ on $E'$ is given by $g\,(v, x) = (v, gx)$; this makes sense as $p(gx) = p(x) = f(v)$. Consider an open subset $U$ in $B$ such that there exists a homeomorphism $\varphi_U$ as in 1.2(1). Then $V := f^{-1}(U)$ is open in $B'$, we have $(p')^{-1}(V) \subset V \times p^{-1}(U)$ and $\mathrm{id}_V \times \varphi_U$ induces a homeomorphism

$$\{\, (v, u, g) \in V \times U \times G \mid f(v) = u \,\} \longrightarrow (p')^{-1}(V),$$

hence using $(v, g) \mapsto (v, f(v), g)$ a homeomorphism $\psi_V : V \times G \to (p')^{-1}(V)$ satisfying $p' \circ \psi_V(v, g) = v$ and $g\,\psi_V(v, h) = \psi_V(v, gh)$ for all $v \in V$ and $g, h \in G$.

One can show: If $f_1: B' \to B$ and $f_2: B' \to B$ are homotopic continuous maps, then the induced principal bundles $f_1^*(E, p, B)$ and $f_2^*(E, p, B)$ are isomorphic over $B'$. Here two principal $G$–bundles $(E_1, p_1, B)$ and $(E_2, p_2, B)$ are called *isomorphic over* $B$ if there exists a homeomorphism $\varphi\colon E_1 \to E_2$ with $p_2 \circ \varphi = p_1$ and $\varphi(g\,x) = g\,\varphi(x)$ for all $x \in E_1$.

A principal bundle $(E_G, p_G, B_G)$ for a topological group $G$ is called a *universal principal bundle* for $G$ if for every principal $G$–bundle $(E, p, B)$ there exists a continuous map $f\colon B \to B_G$ such that $(E, p, B)$ is isomorphic to $f^*(E_G, p_G, B_G)$ over $B$ and if $f$ is uniquely determined up to homotopy by this property.

Milnor has given a general construction that associates to any topological group a universal principal bundle. A theorem of Dold (in *Ann. of Math.* **78** (1963), 223–255) says that a principal $G$–bundle $(E, p, B)$ is universal if and only if $E$ is contractible.

In case $G = S^1$ Milnor's construction leads to the following: Consider for any positive integer $n$ the principal $G$–bundle $p_n\colon E_G^n = S^{2n+1} \to B_G^n = \mathbf{P}^n(\mathbf{C})$ as in 1.2. We have natural embeddings $E_G^n \to E_G^{n+1}$ and $B_G^n \to B_G^{n+1}$ induced by the embedding $\mathbf{C}^n \to \mathbf{C}^{n+1}$ mapping any $(x_1, x_2, \dots, x_n)$ to $(x_1, x_2, \dots, x_n, 0)$. These embeddings are compatible with the action of $G$ and with the maps $p_n$ and $p_{n+1}$. Take now the limits

$$E_G = \varinjlim E_G^n = S^\infty \qquad \text{and} \qquad B_G = \varinjlim B_G^n = \mathbf{P}^\infty(\mathbf{C})$$

with the inductive topology. We get a map $p_G\colon E_G \to B_G$ inducing all $p_n$, and $(E_G, p_G, B_G)$ is then a universal principal bundle for $G = S^1$.

For the multiplicative group $G = \mathbf{C}^\times$ one can get a universal principal bundle by a similar procedure: One sets now $E_G^n = \mathbf{C}^{n+1} \setminus \{0\}$ and $B_G^n = \mathbf{P}^n(\mathbf{C})$ with the canonical map $p_n$ and takes the limit as above. So one gets now $E_G = \mathbf{C}^\infty \setminus \{0\}$ and $B_G = \mathbf{P}^\infty(\mathbf{C})$. Since $S^\infty$ is a deformation retract of $\mathbf{C}^\infty \setminus \{0\}$ and since $S^\infty$ is contractible (e.g., by the preceding example and Dold's theorem*), also $\mathbf{C}^\infty \setminus \{0\}$ is contractible. Therefore $(E_G, p_G, B_G)$ is a universal principal bundle for $G = \mathbf{C}^\times$.

*Remarks*: 1) Let $(E_G, p, B_G)$ be a universal principal bundle for a topological group $G$. Dold's theorem implies that $B_G$ is pathwise connected. Furthermore one gets from the long exact homotopy sequence of this fibration: If $G$ is connected, then $B_G$ is simply connected.

2) Let $G$ be a Lie group and $H$ a closed Lie subgroup of $G$. If $(E_G, p, B_G)$ is a universal principal $G$–bundle, then $(E_G, \overline{p}, E_G/H)$ with $\overline{p}(x) = Hx$ for all $x \in E_G$ is a universal principal $H$–bundle: We noted at the end of 1.2 that we get here a principal $H$–bundle; it is universal by Dold's theorem.

**1.4.** (**Equivariant cohomology**) Let $G$ be a topological group and $X$ a $G$–space. We can associate to each principal $G$–bundle $(E, p, B)$ a fibre bundle $(X_E, q, B)$: We let $G$ act on $X \times E$ diagonally, i.e., via $g\,(x, y) = (gx, gy)$, and set $X_E$ equal to the orbit space $(X \times E)/G$. We define $q$ by $q(G\,(x, y)) = p(y)$. Let $\pi\colon X \times E \to X_E$ denote the map sending each element to its orbit under $G$.

Consider an open subset $U$ in $B$ with a homeomorphism $\varphi_U\colon U \times G \to p^{-1}(U)$ as in 1.2(1). We have $q^{-1}(U) = (X \times p^{-1}(U))/G$ and $\pi^{-1}(q^{-1}(U)) = X \times p^{-1}(U)$. Now

$$\widehat{\psi}_U\colon X \times U \times G \longrightarrow \pi^{-1}(q^{-1}(U)), \qquad (x, u, g) \mapsto (gx, \varphi_U(u, g))$$

* At `en.wikipedia.org/wiki/Contractibility_of_unit_sphere_in_Hilbert_space` you can find the standard proofs.

is a homeomorphism (the composition of $(x,u,g) \mapsto (gx,u,g)$ with $\mathrm{id}_X \times \varphi_U$) that is $G$–equivariant if we let $G$ act on $X \times U \times G$ via $h(x,u,g) = (x,u,hg)$. For this action $X \times U$ is homeomorphic to $(X \times U \times G)/G$, mapping $(x,u)$ to $G(x,u,1)$. It follows that we get a homeomorphism

$$\psi_U : X \times U \longrightarrow q^{-1}(U) = \pi^{-1}(q^{-1}(U))/G, \qquad (x,u) \mapsto G\,(x, \varphi_U(u,1)).$$

This shows that $(X_E, q, B)$ is a locally trivial fibration with all fibres homeomorphic to $X$. We get also that $(X \times E, \pi, X_E)$ is a principal $G$–bundle: For any open subset $U$ as above $\widehat{\psi}_U \circ (\psi_U^{-1} \times \mathrm{id}_G)$ is a homeomorphism from $q^{-1}(U) \times G$ onto $\pi^{-1}(q^{-1}(U))$ satisfying the conditions in 1.2(1).

Apply this construction to a universal principal $G$–bundle $(E_G, p, B_G)$. In this case we use the notation $X_G = (X \times E_G)/G$ and get thus a fibre bundle $(X_G, q, B_G)$. We then define the *equivariant cohomology* $H^\bullet_G(X; \mathbf{C})$ of the $G$–space $X$ as the ordinary cohomology of $X_G$:

$$H^\bullet_G(X; \mathbf{C}) = H^\bullet(X_G; \mathbf{C}). \tag{1}$$

(We could of course also use other coefficients than $\mathbf{C}$.)

At first sight it looks as if this definition depends on the choice of the universal principal $G$–bundle $(E_G, p, B_G)$. Let us show that this choice does not matter. Suppose that $(E'_G, p', B'_G)$ is another universal principal $G$–bundle. Consider $X \times E_G \times E'_G$ as a $G$–space with $G$ acting on all three factors. We get natural maps

$$q_1 : (X \times E_G \times E'_G)/G \longrightarrow (X \times E_G)/G \quad \text{and} \quad q_2 : (X \times E_G \times E'_G)/G \longrightarrow (X \times E'_G)/G$$

These are locally trivial fibrations with fibres homeomorphic to $E'_G$ (for $q_1$) or to $E_G$ (for $q_2$). For example the construction above of a fibre bundle yields $q_1$ if we start with the $G$–space $E'_G$ and the principal $G$–bundle given by the orbit map $X \times E_G \to (X \times E_G)/G$.

Since the fibre $E'_G$ of the locally trivial fibration $q_1$ is contractible, the long exact homotopy sequence of the fibration shows that $q_1$ induces isomorphisms of all homotopy groups. Now a result of Whitehead implies that $q_1^*$ is an isomorphism of cohomology algebras $H^\bullet((X \times E_G)/G; \mathbf{C}) \xrightarrow{\sim} H^\bullet((X \times E_G \times E'_G)/G; \mathbf{C})$. The same argument applies to $q_2$ and get thus an isomorphism

$$(q_2^*)^{-1} \circ q_1^* : H^\bullet((X \times E_G)/G; \mathbf{C}) \xrightarrow{\sim} H^\bullet((X \times E'_G)/G; \mathbf{C}).$$

Denote this isomorphism for the moment by $\alpha(E'_G, E_G)$. If now $(E''_G, p'', B''_G)$ is a third universal principal $G$–bundle, then one checks that $\alpha(E''_G, E_G) = \alpha(E''_G, E'_G) \circ \alpha(E'_G, E_G)$. We can now formally define $H^\bullet_G(X; \mathbf{C})$ as the limit of the family of all $H^\bullet((X \times E_G)/G; \mathbf{C})$ and of all $\alpha(E'_G, E_G)$ over all universal principal $G$–bundles $(E_G, p, B_G)$.

**1.5.** (**Elementary properties**) Let again $G$ be a topological group and $(E_G, p, B_G)$ a universal principal $G$–bundle. If $f\colon X \to Y$ is a morphism of $G$–spaces (i.e., a continuous $G$–equivariant map), then $f \times \mathrm{id}$ is a morphism $X \times E_G \to Y \times E_G$ of $G$–spaces and induces a continuous map $\overline{f}\colon X_E \to Y_E$, $G(x,z) \mapsto G(f(x),z)$ of the orbit spaces. We get thus a homomorphism

$$\overline{f}^*\colon H_G^\bullet(Y;\mathbf{C}) \to H_G^\bullet(X;\mathbf{C}) \tag{1}$$

in the equivariant cohomology.

If $X$ is a point, then $(X \times E_G)/G = (\{\mathrm{pt}\} \times E_G)/G$ identifies with $E_G/G \simeq B_G$. We get thus

$$H_G^\bullet(\mathrm{pt};\mathbf{C}) \simeq H^\bullet(B_G;\mathbf{C}), \tag{2}$$

an isomorphism of algebras.

More generally, if $G$ acts trivially on a topological space $X$, then $(X \times E_G)/G$ is homeomorphic to $X \times (E_G/G)$ under $G(x,y) \mapsto (x, Gy)$. So the Künneth formula yields an isomorphism $H_G^\bullet(X;\mathbf{C}) \simeq H^\bullet(X;\mathbf{C}) \otimes H^\bullet(B_G;\mathbf{C})$.

For an arbitrary $G$–space $X$ the map sending all of $X$ to a point is a morphism of $G$–spaces. Therefore we get a homomorphism $H^\bullet(B_G;\mathbf{C}) \to H_G^\bullet(X;\mathbf{C})$ that makes $H_G^\bullet(X;\mathbf{C})$ into an $H^\bullet(B_G;\mathbf{C})$–algebra. Any $\overline{f}^*$ as in (1) is then a homomorphism of $H^\bullet(B_G;\mathbf{C})$–algebras.

If $H$ is a closed subgroup of $G$, then we can regard $G/H$ as a $G$–space. We get then a homeomorphism

$$(G/H \times E_G)/G \longrightarrow E_G/H, \qquad G(gH, x) \mapsto Hg^{-1}x.$$

The inverse map takes any $H$–orbit $Hx$ to $G(1H, x)$. If $G$ is a Lie group and if $H$ is a closed Lie subgroup of $G$, then $E_G \to E_G/H$ is a universal principal $H$–bundle, as noted in 1.3. So in this case we can take $B_H = E_G/H$ and get an isomorphism

$$H_G^\bullet(G/H;\mathbf{C}) \xrightarrow{\sim} H^\bullet(B_H;\mathbf{C}). \tag{3}$$

The structure as an $H_G^\bullet(\mathrm{pt};\mathbf{C})$–algebra on $H_G^\bullet(G/H;\mathbf{C})$ is induced by the homomorphism $q^*$ with $q\colon E_G/H \to B_G$, $q(Hx) = p(x)$.

**1.6.** (**Tori**) If $G$ and $G'$ are topological groups, if $(E, p, B)$ is a (universal) principal $G$–bundle and if $(E', p', B')$ is a (universal) principal $G'$–bundle, then $(E \times E', p \times p', B \times B')$ is a (universal) principal $(G \times G')$–bundle.

Consider an (algebraic) torus $T = \mathbf{C}^\times \times \mathbf{C}^\times \times \cdots \times \mathbf{C}^\times$ ($d$ factors). Then we get principal $T$–bundles $(E_T^n, p_n, B_T^n)$ setting $E_T^n = (\mathbf{C}^{n+1} \setminus \{0\})^d$ and $B_T^n = \mathbf{P}^n(\mathbf{C})^d$, cf. 1.3, and we get a universal principal $T$–bundle $(E_T, p_T, B_T)$ with $E_T = (\mathbf{C}^\infty \setminus \{0\})^d$ and $B_T = \mathbf{P}^\infty(\mathbf{C})^d$. As in 1.3 we can identify $E_T$ and $B_T$ as inductive limits of all $E_T^n$ and all $B_T^n$ respectively.

If $X$ is a $T$–space, then $X_T = (X \times E_T)/T$ is the inductive limit of all $X_T^n = (X \times E_T^n)/T$. Since we are looking at cohomology with coefficients in a field, we get therefore

$$H_T^i(X;\mathbf{C}) = \varprojlim H^i((X \times E_T^n)/T;\mathbf{C}) \qquad \text{for all } i \in \mathbf{N}, \tag{1}$$

cf. [Ha], Thm. 3F.5.

We get in particular that each $H^i(B_T;\mathbf{C})$ is the inverse limit of the $H^i(\mathbf{P}^n(\mathbf{C})^d;\mathbf{C})$, $n\in\mathbf{N}$. There is an isomorphism of graded rings

$$\mathbf{C}[x_1,x_2,\dots,x_d]/(x_1^{n+1},x_2^{n+1},\dots,x_d^{n+1}) \xrightarrow{\sim} H^\bullet(\mathbf{P}^n(\mathbf{C})^d;\mathbf{C}). \tag{2}$$

Here $\mathbf{C}[x_1,x_2,\dots,x_d]$ is the polynomial ring over $\mathbf{C}$ in $d$ indeterminates, graded such that each $x_i$ has degree 2. So for $n\geq 1$ the map in (2) sends $\sum_{i=1}^d \mathbf{C}x_i$ bijectively to $H^2(\mathbf{P}^n(\mathbf{C})^d;\mathbf{C})$. Furthermore the inclusion $\iota_n\colon \mathbf{P}^n(\mathbf{C})^d\to\mathbf{P}^{n+1}(\mathbf{C})^d$ induces for $n\geq 1$ an isomorphism $\iota_n^*\colon H^2(\mathbf{P}^{n+1}(\mathbf{C})^d;\mathbf{C}) \xrightarrow{\sim} H^2(\mathbf{P}^n(\mathbf{C})^d;\mathbf{C})$. (Recall that one gets $\mathbf{P}^{n+1}(\mathbf{C})$ from $\mathbf{P}^n(\mathbf{C})$ by adjoining a $(2n+2)$-cell and use the cell decomposition to compute the cohomology.) It follows that we get an isomorphism of graded rings

$$\mathbf{C}[x_1,x_2,\dots,x_d] \xrightarrow{\sim} H^\bullet(B_T;\mathbf{C}). \tag{3}$$

Consider the maximal compact subgroup $K=S^1\times S^1\times\cdots\times S^1$ ($d$ factors) of $T$. Setting $E_K^n=(S^{2n+1})^d\subset E_T^n$ and $B_K^n=B_T^n$ we get principal $K$–bundles $(E_K^n,p'_n,B_K^n)$ with $p'_n$ the restriction of $p_n$. Similarly we get a universal principal $K$–bundle $(E_K,p_K,B_K)$ with $E_K=(S^\infty)^d\subset E_T$ and $B_K=B_T$. As in the case of $T$ we get for any $K$–space $X$ that $H_K^\bullet(X;\mathbf{C})$ is the inverse limit of all $H^\bullet((X\times E_K^n)/K;\mathbf{C})$.

We can regard any $T$–space $X$ as a $K$–space by restricting the action of $T$ to $K$. The inclusion of $E_K^n=(S^{2n+1})^d$ into $E_T^n=(\mathbf{C}^{n+1}\setminus\{0\})^d$ induces a continuous map of orbit spaces

$$\psi\colon (X\times E_K^n)/K\to (X\times E_T^n)/T.$$

We can cover $B_T^n$ by open subsets $U$ such that there exists a homeomorphism $\varphi_U\colon U\times T\to p_n^{-1}(U)$ as in 1.2(1) and such that $\varphi_U$ restricts to a similar homeomorphism $\varphi'_U\colon U\times K\to (p'_n)^{-1}(U)$. Then the inverse images of $U$ both in $(X\times E_K^n)/K$ and in $(X\times E_T^n)/T$ identify with $X\times U$, cf. 1.4. Under this identification $\psi$ corresponds to the identity map. Therefore $\psi$ is a homeomorphism and induces an isomorphism of cohomology groups. Taking inverse limits (or working directly with $E_K$ and $E_T$) we get thus an isomorphism

$$H_T^\bullet(X;\mathbf{C}) \xrightarrow{\sim} H_K^\bullet(X;\mathbf{C}). \tag{4}$$

**1.7.** (**Line bundles**) We need a more canonical description of the isomorphism 1.6(3). This involves (complex) line bundles and their Chern classes.

Let $G$ be a topological group and $(E,p,B)$ a principal $G$–bundle. We associate to any continuous group homomorphism $\lambda\colon G\to\mathbf{C}^\times$ a line bundle $\mathcal{L}(\lambda)=\mathcal{L}(\lambda;E)$ on $B$ as follows: Denote by $\mathbf{C}_\lambda$ the $G$–space equal to $\mathbf{C}$ as a topological space such that $g\,a=\lambda(g)\,a$ for all $g\in G$ and $a\in\mathbf{C}$. Set $\mathcal{L}(\lambda)=(\mathbf{C}_\lambda\times E)/G$ and define $q_\lambda\colon\mathcal{L}(\lambda)\to B$ by $q_\lambda(G(a,x))=p(x)$. Then $(\mathcal{L}(\lambda),q_\lambda,B)$ is a fibre bundle as described in 1.4. It is in fact a line bundle: Each fibre $q_\lambda^{-1}(p(x))$ with $x\in E$ gets a vector space structure such that $\mathbf{C}\to q_\lambda^{-1}(p(x))$, $a\mapsto G(a,x)$ is an isomorphism of vector spaces. This structure is independent of the choice of $x$ in $p^{-1}(p(x))=Gx$ since $G$ acts linearly on $\mathbf{C}_\lambda$. The homeomorphisms $\psi_U$ as in 1.4 are compatible with this structure.

If also $\mu$ is a continuous group homomorphism $G \to \mathbf{C}^\times$, then one gets an isomorphism of line bundles

$$\mathcal{L}(\lambda) \otimes \mathcal{L}(\mu) \xrightarrow{\sim} \mathcal{L}(\lambda + \mu) \tag{1}$$

where we use an additive notation for the group of continuous group homomorphisms from $G$ to $\mathbf{C}^\times$; so we have $(\lambda + \mu)\,(g) = \lambda(g)\,\mu(g)$.

Let also $G'$ be a topological group with a principal $G'$–bundle $(E', p', B')$. Suppose that we have continuous maps $\varphi \colon B' \to B$ and $\psi \colon E' \to E$ and a continuous group homomorphism $\alpha \colon G' \to G$ such that $p \circ \psi = \varphi \circ p'$ and $\psi(hy) = \alpha(h)\,\psi(y)$ for all $h \in G'$ and $y \in E'$. Then the pull-back $\varphi^*\mathcal{L}(\lambda)$ of the line bundle $\mathcal{L}(\lambda)$ under $\varphi$ is isomorphic to the line bundle $\mathcal{L}(\lambda \circ \alpha)$:

$$\mathcal{L}(\lambda \circ \alpha) \xrightarrow{\sim} \varphi^*\mathcal{L}(\lambda). \tag{2}$$

Recall that the pull-back $\varphi^*\mathcal{L}(\lambda)$ is the fibre product of $B'$ and $\mathcal{L}(\lambda)$ over $B$, hence consists of all pairs $(b', G(a, x))$ with $b' \in B'$, $a \in \mathbf{C}$, and $x \in E$. The isomorphism in (2) sends any orbit $G'(a, y)$ with $a \in \mathbf{C}$ and $y \in E'$ to $(p(y), G(a, \psi(y)))$.

This result implies in particular for any inner automorphism $\mathrm{Int}\,(g) \colon h \mapsto ghg^{-1}$ of $G$ that

$$\mathcal{L}(\lambda \circ \mathrm{Int}\,(g)) \xrightarrow{\sim} \mathcal{L}(\lambda). \tag{3}$$

Take above $G' = G$ and $(E', p', B') = (E, p, B)$. Then the assumptions are satisfied by $\varphi = \mathrm{id}_B$ and $\alpha = \mathrm{Int}\,(g)$ if we set $\psi(x) = gx$ for all $x \in E$.

Consider for example $G = \mathbf{C}^\times$ and the principal bundle $(\mathbf{C}^{n+1} \setminus \{0\}, \pi, \mathbf{P}^n(\mathbf{C}))$ with the canonical map $\pi$. Choose $\lambda \colon \mathbf{C}^\times \to \mathbf{C}^\times$ as the map $g \mapsto g^{-1}$. Then the map

$$\mathbf{C}_\lambda \times (\mathbf{C}^{n+1} \setminus \{0\}) \to \mathbf{C}^{n+1} \times \mathbf{P}^n(\mathbf{C}), \qquad (a, v) \mapsto (av, \mathbf{C}v)$$

is constant on the orbits of $G$ and induces an isomorphism of line bundles

$$\mathcal{L}(\lambda) \xrightarrow{\sim} \{\, (w, \mathbf{C}v) \in \mathbf{C}^{n+1} \times \mathbf{P}^n(\mathbf{C}) \mid w \in \mathbf{C}v \,\}. \tag{4}$$

Here the right hand is usually known as the *tautological* line bundle on $\mathbf{P}^n(\mathbf{C})$. In algebraic geometry this bundle is usually denoted by $\mathcal{O}(-1)$.

**1.8.** (**Chern classes**) If $q \colon \mathcal{L} \to B$ is a (complex) line bundle on a topological space $B$, then the Chern class of $\mathcal{L}$ is an element $c_1(\mathcal{L}) \in H^2(B; \mathbf{C})$. (Actually it is an element in $H^2(B; \mathbf{Z})$ that we here replace by its image in $H^2(B; \mathbf{C})$.) Isomorphic line bundles have the same Chern class. If $\mathcal{L}'$ is another line bundle on $B$, then we have $c_1(\mathcal{L} \otimes \mathcal{L}') = c_1(\mathcal{L}) + c_1(\mathcal{L}')$. If $f \colon B' \to B$ is a continuous map, then one gets $f^*(c_1(\mathcal{L})) = c_1(f^*\mathcal{L})$ in $H^2(B'; \mathbf{C})$.

Let $G$ be a topological group and $(E, p, B)$ a principal $G$–bundle. Given a continuous group homomorphism $\lambda \colon G \to \mathbf{C}^\times$ we get a line bundle $\mathcal{L}(\lambda)$ on $B$ as in 1.7, hence a Chern class $c_1(\lambda) = c_1(\mathcal{L}(\lambda))$ in $H^2(B; \mathbf{C})$. If also $\mu \colon G \to \mathbf{C}^\times$ is a continuous group homomorphism, then we get

$$c_1(\lambda + \mu) = c_1(\lambda) + c_1(\mu) \tag{1}$$

from 1.7(1).

We can apply this construction to a universal principal $G$–bundle and get thus via 1.5(2) a class $c_1(\lambda)$ in $H^2_G(\mathrm{pt};\mathbf{C})$. This class is independent of the choice of the universal principal $G$–bundle used in 1.5(2): If $(E_G, p, B_G)$ and $(E'_G, p', B'_G)$ are two such bundles, then we identify $H^\bullet(B_G;\mathbf{C})$ with $H^\bullet(B'_G;\mathbf{C})$ by $(q_2^*)^{-1}\circ q_1^*$ where $q_1\colon (E_G\times E'_G)/G \to E_G/G \simeq B_G$ and $q_2\colon (E_G\times E'_G)/G \to E'_G/G \simeq B'_G$ are the obvious maps, cf. 1.4. Now 1.7(2) shows that

$$q_1^*\mathcal{L}(\lambda; E_G) \simeq \mathcal{L}(\lambda; E_G\times E'_G) \simeq q_2^*\mathcal{L}(\lambda; E'_G)$$

which yields the claimed independence. (Note that $E_G\times E'_G \to (E_G\times E'_G)/G$ is a universal principal $G$–bundle.)

Return now to our torus $T = \mathbf{C}^\times \times \mathbf{C}^\times \times\cdots\times \mathbf{C}^\times$ ($d$ factors). Denote by $\varepsilon_i$ the projection onto the $i$–th factor of the product. So $\varepsilon_i$ is a continuous group homomorphism $T\to\mathbf{C}^\times$. We claim that we can choose the isomorphism in 1.6(3) such that $x_i$ is mapped to $c_1(-\varepsilon_i)$ for all $i$, $1\le i\le d$.

Let $\iota$ denote the inclusion of $B_T^1 = \mathbf{P}^1(\mathbf{C})^d$ into $B_T = \mathbf{P}^\infty(\mathbf{C})^d$ from the inductive limit construction of $B_T$. Then $\iota^*$ maps $c_1(-\varepsilon_i)$ taken in $H^2(B_T;\mathbf{C})$ to $c_1(-\varepsilon_i)$ taken in $H^2(B_T^1;\mathbf{C})$. (Use 1.7(2).) Therefore it suffices to show that we can choose the isomorphism in 1.6(2) for $n=1$ such that the coset of $x_i$ is mapped to $c_1(-\varepsilon_i)$ for all $i$, $1\le i\le d$.

Let $\pi_i$ denote the projection from $B_T^1 = \mathbf{P}^1(\mathbf{C})^d$ to the $i$–th factor. The isomorphism in 1.6(2) arises from the Künneth theorem and maps the coset of $x_i$ to the image under $\iota^*$ of a standard generator of $H^2(\mathbf{P}^1(\mathbf{C});\mathbf{C})$. Such a generator is the Chern class $c_1(\mathcal{L})$ where $\mathcal{L}$ is the tautological bundle on $\mathbf{P}^1(\mathbf{C})$. Now 1.6(4) combined with 1.6(2) shows that

$$\iota^* c_1(\mathcal{L}) = c_1(\iota^*\mathcal{L}) = c_1(\mathcal{L}(-\varepsilon_i)) = c_1(-\varepsilon_i)$$

as claimed.

Let $X(T)$ denote the subgroup generated by $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_d$ in the (additive) group of all continuous group homomorphisms from $T$ to $\mathbf{C}^\times$. This is a free abelian group of rank $d$ with the $\varepsilon_i$ as a basis. Our result above shows that the $c_1(\varepsilon_i)$ are a basis for $H^2(B_T;\mathbf{C})$. So the group homomorphism $c_1\colon X(T)\to H^2(B_T;\mathbf{C})$ induces an isomorphism of vector spaces $X(T)\otimes_{\mathbf{Z}}\mathbf{C} \xrightarrow{\sim} H^2(B_T;\mathbf{C})$. Now we can restate 1.6(3): We have an algebra isomorphism

$$S(X(T)\otimes_{\mathbf{Z}}\mathbf{C}) \xrightarrow{\sim} H^\bullet(B_T;\mathbf{C}) \tag{2}$$

where we use the notation $S(V)$ for the symmetric algebra of a vector space $V$. This is an isomorphism of graded algebras if we double the usual grading on the symmetric algebra putting $X(T)\otimes_{\mathbf{Z}}\mathbf{C}$ into degree 2.

**1.9.** (**Homogeneous spaces for tori**) Let $G$ be a Lie group and $H$ a closed Lie subgroup of $G$. Fix a universal principal $G$–bundle $(E_G, p, B_G)$. As observed in 1.3 we get a universal principal $H$–bundle $(E_H, p', B_H)$ by setting $E_H = E_G$ and $B_H = E_G/H$ with $p'(x) = Hx$ for all $x\in E_G$. Denote by $q\colon B_H\to B_G$ the map given by $q(Hx) = p(x)$ such that $q\circ p' = p$. Then $q^*$ yields the $H^\bullet_G(\mathrm{pt};\mathbf{C})$–algebra structure on $H^\bullet_G(G/H;\mathbf{C})$, see 1.5(3).

If $\lambda\colon G \to \mathbf{C}^\times$ is a continuous group homomorphism, then we get from 1.7 (2)

$$q^*(c_1(\lambda)) = c_1(\lambda_{|H}) \tag{1}$$

taking $\psi = \mathrm{id}_{E_G}$ and $\varphi = q$ in 1.7 together with $\alpha$ equal to the inclusion of $H$ into $G$.

Denote by $H^0$ the connected component of the identity in $H$. This is a normal closed Lie subgroup of $H$. Note that we can factor $p$ as follows:

$$E_G \xrightarrow{p_1} E_G/H^0 = B_{H_0} \xrightarrow{p_2} E_G/H = B_H \xrightarrow{q} B_G$$

such that $p_2 \circ p_1 = p$. The group $H/H^0$ acts freely on $E_G/H^0$ via $(hH^0)\cdot H^0x = H^0hx$ for all $h \in H$ and $x \in E_G$. The orbits of $H/H^0$ are precisely the fibres of $p_2$. The local triviality of $p$ implies that of $p_2$.

Assume for the moment that $H/H^0$ is *finite*. Then $p_2$ is a covering with group $H/H^0$. Since we are working with coefficients in a field of characteristic 0, we get now that $p_2^*$ induces an isomorphism

$$p_2^*\colon H^\bullet(B_H;\mathbf{C}) \xrightarrow{\sim} H^\bullet(B_{H^0};\mathbf{C})^{H/H^0} \subset H^\bullet(B_{H^0};\mathbf{C}) \tag{2}$$

where the exponent $H/H^0$ means that we take the fixed points under this group.

Let $h \in H$ and denote by $\varphi_h$ the action of $hH^0$ on $E_G/H^0 = B_{H^0}$, i.e., $\varphi_h(H^0x) = H^0hx$. Then the action of $hH^0$ is given by $(\varphi_h^{-1})^*$. If $\mu\colon H^0 \to \mathbf{C}^\times$ is a continuous group homomorphism, then this action satisfies

$$(\varphi_h^{-1})^*(c_1(\mu)) = c_1(\mu \circ \mathrm{Int}(h^{-1})_{|H^0}), \tag{3}$$

cf. 1.7(2).

Suppose now that $G = T$ is an (algebraic) torus as in 1.6 and that $H$ is a Zariski closed subgroup. Then the theory of algebraic groups tells us that $H/H^0$ is finite and that $H^0$ is a torus. It follows that the Chern class $c_1$ induces an isomorphism between $S(X(H^0)\otimes_{\mathbf{Z}}\mathbf{C})$ and $H^\bullet(B_{H^0};\mathbf{C})$. Since $H$ is commutative, (3) implies that $H/H^0$ acts trivially on $H^\bullet(B_{H^0};\mathbf{C})$. So (2) and 1.5(3) imply that we have an isomorphism

$$S(X(H^0)\otimes_{\mathbf{Z}}\mathbf{C}) \xrightarrow{\sim} H_T^\bullet(T/H;\mathbf{C}). \tag{4}$$

The theory of algebraic groups says also that $\lambda \mapsto \lambda_{|H^0}$ is a surjective group homomorphism $X(T) \to X(H^0)$. The map $H_T^\bullet(\mathrm{pt};\mathbf{C}) \to H_T^\bullet(T/H;\mathbf{C})$ defining the algebra structure identifies with $q^*$, hence by (1) with the map

$$S(X(T)\otimes_{\mathbf{Z}}\mathbf{C}) \longrightarrow S(X(H^0)\otimes_{\mathbf{Z}}\mathbf{C}) \tag{5}$$

coming from the restriction map $X(T) \to X(H^0)$.

Any $\lambda \in X(T)$ is a homomorphism of Lie groups; its differential is then a linear form on the Lie algebra $\mathrm{Lie}\,T$ of $T$. Mapping $\lambda$ to its differential induces an isomorphism $X(T)\otimes_{\mathbf{Z}}\mathbf{C} \xrightarrow{\sim} (\mathrm{Lie}\,T)^*$. The same applies to $H^0$ and we can identify (5) with the natural map $S((\mathrm{Lie}\,T)^*) \to S((\mathrm{Lie}\,H^0)^*)$.

**1.10.** (**Simple examples**) Let again $T$ be a torus. We set $S = S(X(T) \otimes_{\mathbf{Z}} \mathbf{C})$ and identify this graded algebra with $H_T^\bullet(\mathrm{pt}; \mathbf{C})$. Here the grading on $S$ is chosen such that $X(T) \otimes_{\mathbf{Z}} \mathbf{C}$ is the homogeneous component of degree 2.

Consider the $T$–space $\mathbf{C}_\lambda^\times = \mathbf{C}_\lambda \setminus \{0\}$ for some $\lambda \in X(T)$, $\lambda \neq 0$. Then $t \mapsto t\,1$ induces an isomorphism of $G$–spaces $T/H \xrightarrow{\sim} \mathbf{C}_\lambda^\times$ where $H = \ker\lambda$ is Zariski closed. The restriction map $X(T) \to X(H^0)$ is surjective with kernel equal to $X(T) \cap \mathbf{Q}\lambda$ where the intersection is taken inside $X(T) \otimes_{\mathbf{Z}} \mathbf{Q}$ identifying any $\mu \in X(T)$ with $\mu \otimes 1$. Therefore the induced map $X(T) \otimes_{\mathbf{Z}} \mathbf{C} \to X(H^0) \otimes_{\mathbf{Z}} \mathbf{C}$ is surjective with kernel equal to $\mathbf{C}\lambda$. Now 1.9(4),(5) say that we have an isomorphism

$$S/S\lambda \xrightarrow{\sim} H_T^\bullet(\mathbf{C}_\lambda^\times; \mathbf{C}) \tag{1}$$

of $S$–algebras.

Consider next the $T$–space $\mathbf{C}_\lambda$ with $\lambda$ as above. Then $(\{0\} \times E_T)/T$ is a deformation retract of $(\mathbf{C}_\lambda \times E_T)/T$. (There is a map $\Phi\colon (\mathbf{C}_\lambda \times E_T)/T \times [0,1] \to (\mathbf{C}_\lambda \times E_T)/T$ such that $\Phi(T(x,y),a) = T(ax,y)$ for all $a \in [0,1]$, $x \in \mathbf{C}$, and $y \in E_T$.) Therefore the inclusion $\iota$ of $(\{0\} \times E_T)/T \simeq B_T$ into $(\mathbf{C}_\lambda \times E_T)/T$ induces an isomorphism in cohomology. The inverse of $\iota^*$ is equal to $\pi^*$ with $\pi(T(x,y)) = T(0,y)$ since $\pi \circ \iota$ is the identity on $(\{0\} \times E_T)/T$. Since $\pi^*$ is also the map defining the $S$–algebra structure on $H_T^\bullet(\mathbf{C}_\lambda; \mathbf{C})$, we see that we this map is an isomorphism

$$S \xrightarrow{\sim} H_T^\bullet(\mathbf{C}_\lambda; \mathbf{C}). \tag{2}$$

The inclusion $j$ of $\mathbf{C}_\lambda^\times$ into $\mathbf{C}_\lambda$ induces a homomorphism of $S$–algebras $H_T^\bullet(\mathbf{C}_\lambda; \mathbf{C}) \to H_T^\bullet(\mathbf{C}_\lambda^\times; \mathbf{C})$. Under the identifications (1) and (2) this is just the natural map $S \to S/S\lambda$. (The identifications are induced by $q_1\colon \mathbf{C}_\lambda \to \{\mathrm{pt}\}$ and $q_2\colon \mathbf{C}_\lambda^\times \to \{\mathrm{pt}\}$; we have $q_2 = q_1 \circ j$.)

As a third example consider (for $\lambda$ as above) the $T$–space $X = \mathbf{P}^1(\mathbf{C})$ with the action given by $t\,[x:y] = [\lambda(t)\,x : y]$ in homogeneous coordinates. Then $U_1 := \mathbf{P}^1(\mathbf{C}) \setminus \{[1:0]\}$ and $U_2 := \mathbf{P}^1(\mathbf{C}) \setminus \{[0:1]\}$ and $V := U_1 \cap U_2$ are $T$–stable open subsets of $X$. It follows that $(U_1 \times E_T)/T$ and $(U_2 \times E_T)/T$ form an open covering of $(X \times E_T)/T$ with intersection equal to $(V \times E_T)/T$. Therefore we have a long exact Mayer-Vietoris sequence

$$\cdots \to H_T^{i-1}(V) \to H_T^i(X) \to H_T^i(U_1) \oplus H_T^i(U_2) \to H_T^i(V) \to H_T^{i+1}(X) \to \cdots$$

where we have dropped the coefficients equal to $\mathbf{C}$.

We have obvious identifications $U_1 \simeq \mathbf{C}_\lambda$ and $U_2 \simeq \mathbf{C}_{-\lambda}$ and $V \simeq \mathbf{C}_\lambda^\times$. So we get the equivariant cohomology of these spaces from (1) or (2). Since the odd degree parts of $S$ vanish, it follows that the long exact sequence above breaks up into finite exact sequences

$$0 \to H_T^{2i}(X) \longrightarrow S^{2i} \oplus S^{2i} \xrightarrow{\varphi} (S/S\lambda)^{2i} \to H_T^{2i+1}(X) \to 0$$

where $\varphi$ maps any pair $(a,b)$ to the residue class $a - b + S\lambda$. This map is clearly surjective. It follows that the odd equivariant cohomology of $X$ vanishes. So we get an isomorphism of graded algebras

$$H_T^\bullet(\mathbf{P}^1(\mathbf{C}); \mathbf{C}) \xrightarrow{\sim} \{\, (a,b) \in S^2 \mid a \equiv b \pmod{S\lambda} \,\}. \tag{3}$$

Arguing as in 1.1 one checks that

$$H_T^\bullet(\mathbf{P}^1(\mathbf{C}); \mathbf{C}) \xrightarrow{\sim} S\,(1,1) \oplus S\,(\lambda,0) \simeq S \oplus S\langle 2\rangle$$

is free of rank 2 as an $S$–module.

**1.11.** (**Some relative equivariant cohomology**) Keep the assumptions on $T$ and $S$. If $Z$ is a $T$–stable subset of a $T$–space $X$, then we can define the relative equivariant cohomology $H_T^\bullet(X,Z;\mathbf{C})$ as $H^\bullet((X\times E_T)/T,(Z\times E_T/T);\mathbf{C})$. One has then the usual long exact sequence linking $H_T^\bullet(X,Z;\mathbf{C})$ to $H_T^\bullet(X;\mathbf{C})$ and $H_T^\bullet(Z;\mathbf{C})$. (This remark generalises of course from $T$ to all topological groups.)

Take for example $X=\mathbf{P}^1(\mathbf{C})$ with the $T$–action given by some $\lambda\in X(T)$, $\lambda\neq 0$ as in 1.10. Set $Z=\{[1:0],[0:1]\}$. Both $H_T^\bullet(X;\mathbf{C})$ and $H_T^\bullet(Z;\mathbf{C})\simeq S\oplus S$ vanish in odd degrees. The natural map $H_T^\bullet(X;\mathbf{C})\to H_T^\bullet(Z;\mathbf{C})$ identifies with the embedding of $H_T^\bullet(X;\mathbf{C})$ into $S^2$ as in 1.10(3). Therefore all $H_T^{2i}(X,Z;\mathbf{C})$ vanish whereas there are short exact sequences

$$0\to H_T^{2i}(X;\mathbf{C})\xrightarrow{\psi} S^{2i}\oplus S^{2i}\xrightarrow{\varphi} H_T^{2i+1}(X,Z;\mathbf{C})\to 0.$$

Here $\psi$ is the component of degree $2i$ of the map from 1.10(3). Therefore $H_T^{2i+1}(X,Z;\mathbf{C})$ is isomorphic to $(S/S_\lambda)^{2i}$ and $\varphi$ identifies with the map $(a,b)\mapsto a-b+S\lambda$. We get thus

$$H_T^\bullet(X,Z;\mathbf{C})\simeq (S/S\lambda)\langle 1\rangle. \tag{1}$$

Let us generalise this example. Consider a $T$–space $X$ such that $X^T$ is finite, say $X^T=\{x_1,x_2,\dots,x_r\}$, and such that $X$ is a union $X^T\cup P_1\cup P_2\cup\cdots\cup P_s$ where each $P_i$ is a closed $T$–stable subspace isomorphic as a $T$–space to $\mathbf{P}^1(\mathbf{C})$ with $T$ acting on $\mathbf{P}^1(\mathbf{C})$ via $t\,[x:y]=[\lambda_i(t)\,x:y]$ for some $\lambda_i\in X(T)$, $\lambda_i\neq 0$. Each $P_i$ contains exactly two fixed points, say $x_{a(i)}$ and $x_{z(i)}$; we assume that $x_{a(i)}$ goes to $[0:1]$ and $x_{z(i)}$ to $[1:0]$ under the isomorphism between $P_i$ and $\mathbf{P}^1(\mathbf{C})$.

We claim that in this case the inclusions $(P_i,P_i^T)\subset(X,X^T)$ induce an isomorphism of $S$–modules

$$H_T^\bullet(X,X^T;\mathbf{C})\xrightarrow{\sim}\bigoplus_{i=1}^{s}H^\bullet(P_i,P_i^T;\mathbf{C})\xrightarrow{\sim}\bigoplus_{i=1}^{s}(S/S\lambda_i)\langle 1\rangle \tag{2}$$

where the second isomorphism follows from (1). We prove this by induction on $r=|X^T|$. If $r=1$, then $s=0$ and $X=X^T$. So all terms in (2) are equal to 0.

Suppose now that $r>1$. Pick an arbitrary fixed point $x\in X^T$. Set $X'$ equal to the union of $X^T\setminus\{x\}$ and of all $P_i$ with $x\notin P_i$. This is a closed $T$–stable subspace of $X$ satisfying the same assumptions as $X$, but with $|(X')^T|=r-1$. So we may assume that we have an isomorphism for $H_T^\bullet(X',(X')^T;\mathbf{C})$ as in (2). Set $U_1=X\setminus X'$ and $U_2=X\setminus\{x\}$ and $V=U_1\cap U_2$. These are open $T$–stable subsets of $X$ with $X=U_1\cup U_2$. We get now a long exact Mayer-Vietoris sequence of the form

$$\cdots\to H_T^{j-1}(V)\to H_T^j(X,X^T)\to H_T^j(U_1,U_1^T)\oplus H_T^j(U_2,U_2^T)\to H_T^j(V)\to\cdots$$

where we have dropped the coefficients equal to $\mathbf{C}$. Here we have used that $V^T=\emptyset$.

As a $T$-space $V$ is isomorphic the disjoint union of all $\mathbf{C}_{\lambda_i}^\times$ over all $i$ with $x\in P_i$. Therefore we get $H_T^\bullet(V;\mathbf{C})\simeq\bigoplus_{x\in P_i}S/S\lambda_i$. In particular $H_T^\bullet(V;\mathbf{C})$ vanishes in odd degrees.

On the other hand, we have $U_1^T = \{x\}$ and $U_2^T = (X')^T$. Furthermore, we can construct retracting homotopies $(U_1 \times E_T)/T \to (\{x\} \times E_T)/T$ and $(U_2 \times E_T)/T \to (X' \times E_T)/T$. We get thus isomorphisms $H_T^\bullet(U_1, U_1^T; \mathbf{C}) \simeq H_T^\bullet(\{x\}, \{x\}; \mathbf{C}) = 0$ and $H_T^\bullet(U_2, U_2^T; \mathbf{C}) \simeq H_T^\bullet(X', (X')^T; \mathbf{C})$. By our induction assumption we know that $H_T^\bullet(X', (X')^T; \mathbf{C})$ vanishes in even degrees.

Now our long exact sequence shows that also $H_T^\bullet(X, X^T; \mathbf{C})$ vanishes in even degrees. We get now short exact sequences and a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & H_T^{2j}(V) & \to & H_T^{2j+1}(X, X^T) & \to & H_T^{2j+1}(X', (X')^T) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & \bigoplus_{i=1}^{s} H_T^{2j}(V \cap P_i) & \to & \bigoplus_{i=1}^{s} H_T^{2j+1}(P_i, P_i^T) & \to & \bigoplus_{x \notin P_i} H_T^{2j+1}(P_i, P_i^T) & \to & 0 \end{array}$$

Here the lower row arises from the Mayer-Vietoris sequences corresponding to the covering of each $P_i$ by $U_1 \cap P_i$ and $U_2 \cap P_i$.

Now the right vertical map is an isomorphism by induction. Since $V$ is the disjoint union of all $V \cap P_i \simeq \mathbf{C}_{\lambda_i}^\times$ with $x \in P_i$ whereas $V \cap P_i = \emptyset$ for $x \notin P_i$, also the first vertical map is an isomorphism. Now the five lemma yields the desired isomorphism in (2).

Let us look at the long exact (equivariant) cohomology sequence for the pair $(X, X^T)$. Since $H_T^\bullet(X^T; \mathbf{C}) \simeq S^r$ vanishes in odd degrees and since $H_T^\bullet(X, X^T; \mathbf{C}) \simeq S^r$ vanishes by (2) in even degrees, the long exact sequence breaks up into exact sequences of the form

$$0 \to H_T^{2j}(X; \mathbf{C}) \longrightarrow H_T^{2j}(X^T; \mathbf{C}) \longrightarrow H_T^{2j+1}(X, X^T; \mathbf{C}) \longrightarrow H_T^{2j+1}(X; \mathbf{C}) \to 0. \qquad (3)$$

The maps in the middle add up to a map

$$S^r \simeq H_T^\bullet(X^T; \mathbf{C}) \longrightarrow H_T^\bullet(X, X^T; \mathbf{C})\langle -1 \rangle \simeq \bigoplus_{i=1}^{s} (S/S\lambda_i). \qquad (4)$$

It maps any $r$–tuple $(a_1, a_2, \ldots, a_r) \in S^r$ to the family of all $a_{a(i)} - a_{z(i)} + S\lambda_i$ with $1 \leq i \leq s$. This follows from the corresponding result for $\mathbf{P}^1(\mathbf{C})$ above and the commutativity of the diagram

$$\begin{array}{ccc} H_T^\bullet(X^T; \mathbf{C}) & \longrightarrow & H_T^\bullet(X, X^T; \mathbf{C})\langle -1 \rangle \\ \downarrow & & \downarrow \\ \bigoplus_{i=1}^{s} H_T^\bullet(P_i^T; \mathbf{C}) & \longrightarrow & \bigoplus_{i=1}^{s} H_T^\bullet(P_i, P_i^T; \mathbf{C}) \end{array}$$

It follows that the even equivariant cohomology $H_T^{\mathrm{ev}}(X; \mathbf{C})$ (the direct sum of all $H^{2j}(X; \mathbf{C})$) is given by

$$H_T^{\mathrm{ev}}(X; \mathbf{C}) \simeq \{\, (a_1, a_2, \ldots, a_r) \in S^r \mid a_{a(i)} \equiv a_{z(i)} \ (\mathrm{mod}\, S\lambda_i) \text{ for all } i,\ 1 \leq i \leq s \,\}. \qquad (5)$$

And the odd equivariant cohomology is a torsion module for $S$ being a homomorphic image of $\bigoplus_{i=1}^{s} (S/S\lambda_i)\langle 1 \rangle$.

**1.12.** Suppose now that $T$ acts algebraically on a projective variety $X$ over $\mathbf{C}$ such that both the set $X^T$ of fixed points and the set of (complex) one dimensional orbits of $T$ on $X$ are finite. Write $X^T = \{ x_1, x_2, \dots, x_r \}$ and denote by $P_1, P_2, \dots, P_s$ the closures of the one dimensional $T$–orbits on $X$. Assume* also that each $P_i$ is isomorphic as a $T$–space to $\mathbf{P}^1(\mathbf{C})$ with $T$ acting as above via some $\lambda_i \in X(T)$, $\lambda_i \neq 0$. (This is an isomorphism in the category of topological spaces, not of algebraic varieties.) We use again the notation $x_{a(i)}$ and $x_{z(i)}$ for the points in $P_i^T$. Now Theorem 7.2 in [GKM] states:

**Theorem:** *If $H_T^\bullet(X;\mathbf{C})$ is a free $S$–module, then the map $H_T^\bullet(X;\mathbf{C}) \to H_T^\bullet(X^T;\mathbf{C})$ induced by the inclusion of $X^T$ into $X$ is injective and induces an isomorphism*

$$H_T^\bullet(X;\mathbf{C}) \xrightarrow{\sim} \{ (a_1, a_2, \dots, a_r) \in S^r \mid a_{a(i)} \equiv a_{z(i)} \pmod{S\lambda_i} \text{ for all } i,\ 1 \le i \le s \} \tag{1}$$

*of $S$–algebras.*

This result follows in [GKM] from the calculations described in 1.11 and from the existence of an exact sequence

$$0 \longrightarrow H_T^\bullet(X;\mathbf{C}) \longrightarrow H_T^\bullet(X^T;\mathbf{C}) \longrightarrow H_T^\bullet(X_1, X^T;\mathbf{C}) \tag{2}$$

where $X_1$ is the union of $X^T$ and all $P_i$. The exactness of (2) is a special case of the more general Theorem 6.3 in [GKM].

If we want to apply this theorem, we have to know in advance that $H_T^\bullet(X;\mathbf{C})$ is a free $S$–module. This condition is (e.g.) satisfied if the ordinary cohomology $H^\bullet(X;\mathbf{C})$ vanishes in odd degrees: The Serre spectral sequence associated to the fibration $(X \times E_T)/T \to B_T$ with fibre $X$ has the form

$$E_2^{p,q} = H^p(B_T;\mathbf{C}) \otimes H^q(X;\mathbf{C}) \Rightarrow H_T^{p+q}(X;\mathbf{C}) \tag{3}$$

because $B_T$ is simply connected and because we are working with coefficients in a field of characteristic 0. Since also $H^\bullet(B_T;\mathbf{C})$ vanishes in odd degrees and since the $r$–th differential in the spectral sequence has bidegree $(r, 1-r)$, we see that all differentials are 0 so that $E_\infty^{p,q} = E_2^{p,q}$. Now the abutment $H_T^\bullet(X;\mathbf{C})$ has a filtration with factors $E_\infty^{\bullet,q} = \bigoplus_p E_\infty^{p,q} \simeq H_T^\bullet(B_T;\mathbf{C}) \otimes H^q(X;\mathbf{C})$. Each factor is free over $S \simeq H_T^\bullet(B_T;\mathbf{C})$, hence so is the total module.

Take for example $T = (\mathbf{C}^\times)^3$ acting on $X = \mathbf{P}^2(\mathbf{C})$ as via $(t_1, t_2, t_3) \cdot [x:y:z] = [t_1 x : t_2 y : t_3 z]$. The fixed points for this action are $[1:0:0]$, $[0:1:0]$, and $[0:0:1]$. There are three one dimensional orbits. For example all $[x:y:0]$ with $x \neq 0 \neq y$ form one such orbit; the fixed points in its closure are $[1:0:0]$ and $[0:1:0]$. The action of $T$ on this orbit closure is given by $(t_1, t_2, t_3) \cdot [x:y:0] = [t_1 x : t_2 y : 0] = [t_1 t_2^{-1} x : y : 0]$. So this closure is isomorphic to $\mathbf{P}^1(\mathbf{C})$ with $T$ acting via $\varepsilon_1 - \varepsilon_2$ where $\varepsilon_1, \varepsilon_2, \varepsilon_3$ are the three coordinate functions. The other two orbits arise by permuting the indices.

Since $H^\bullet(\mathbf{P}^2(\mathbf{C});\mathbf{C})$ vanishes in odd degrees, we can now apply the theorem. A comparison of (1) in this case with 1.1(1) shows that the algebra $Z$ defined there in case $k = \mathbf{C}$ is isomorphic to $H_T^\bullet(\mathbf{P}^2(\mathbf{C});\mathbf{C})$. (Identify the algebra $S$ in 1.1 with our present $S$ by mapping each $x_i$ to $\varepsilon_i$.)

This example can of course be generalised to any $\mathbf{P}^n(\mathbf{C})$.

* This condition holds automatically when there exists a $T$–equivariant embedding of $X$ into some $\mathbf{P}^n(\mathbf{C})$ such that the $T$–action on $\mathbf{P}^n(\mathbf{C})$ arises from a linear action of $T$ on $\mathbf{C}^{n+1}$.

**1.13.** (**Flag varieties**) Let $G$ be a connected reductive algebraic group over $\mathbf{C}$. Choose a Borel subgroup $B$ in $G$ and a maximal torus $T \subset B$. Denote by $\Phi \subset X(T)$ the set of roots of $T$ in $G$ and by $\Phi^+$ the set of roots of $T$ in $B$. Then one has $0 \notin \Phi$ and $\Phi$ is the disjoint union of $\Phi^+$ and $-\Phi^+$.

For each $\alpha \in \Phi$ choose a corresponding root homomorphism $x_\alpha$ from the additive group of $\mathbf{C}$ into $G$. This homomorphism is injective and satisfies $t\, x_\alpha(a)\, t^{-1} = x_\alpha(\alpha(t)a)$ for all $t \in T$ and $a \in \mathbf{C}$. Set $U_\alpha = x_\alpha(\mathbf{C})$. One has $U_\alpha \subset B$ if and only if $\alpha \in \Phi^+$.

Set $W = N_G(T)/T$; this is a finite group called the Weyl group of $G$. It acts on $T$ by conjugation and hence on $X(T)$. This action permutes $\Phi$. For each $w \in W$ let $\dot{w} \in N_G(T)$ denote a representative for $w$. We have then $\dot{w} U_\alpha \dot{w}^{-1} = U_{w\,\alpha}$ for all $\alpha \in \Phi$.

If one chooses the root homomorphisms suitably, then one gets for each $\alpha \in \Phi$ a homomorphism $\varphi_\alpha \colon \mathrm{SL}_2(\mathbf{C}) \to G$ such that for all $a \in \mathbf{C}$ and $b \in \mathbf{C}^\times$

$$\varphi_\alpha(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}) = x_\alpha(a), \qquad \varphi_\alpha(\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}) = x_{-\alpha}(a), \qquad \varphi_\alpha(\begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}) \in T \tag{1}$$

and such that

$$\varphi_\alpha(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}) \in N_G(T). \tag{2}$$

Denote by $s_\alpha$ the class in $W$ of the element in (2). It is an element of order 2 and satisfies $s_\alpha(\alpha) = -\alpha$.

One calls $X := G/B$ the *flag variety* of $G$. This is a projective algebraic variety. For each $w \in W$ set $C_w = B\dot{w}B/B \subset X$; these subsets are called the *Bruhat cells* in $X$. Now $X$ is the disjoint union of all $C_w$ with $w \in W$. So mapping $w$ to $C_w$ is a bijection from $W$ onto the set of all $B$–orbits in $X$. The closure of a $B$–orbit is the union of that orbit and of some $B$–orbits of strictly smaller dimension. Therefore one can define a partial ordering on $W$ such that $w' \le w$ if and only if $C_{w'} \subset \overline{C}_w$. This ordering was determined by Chevalley and is usually called the Bruhat order.

For each $w \in W$ set $\Phi(w) = \{\, \alpha \in \Phi^+ \mid w^{-1}\alpha \in -\Phi^+ \,\}$ and $n(w) = |\Phi(w)|$. For any ordering $\alpha_1, \alpha_2, \dots, \alpha_{n(w)}$ of the roots in $\Phi(w)$, the map

$$(a_1, a_2, \dots a_{n(w)}) \mapsto x_{\alpha_1}(a_1)\, x_{\alpha_2}(a_2)\, \dots\, x_{\alpha_{n(w)}}(a_{n(w)})\, \dot{w}B$$

is an isomorphism $\mathbf{C}^{n(w)} \to C_w$ of varieties. Therefore the action of $T$ on $C_w$ is given by

$$t\, x_{\alpha_1}(a_1)\, \dots\, x_{\alpha_{n(w)}}(a_{n(w)})\, \dot{w}B = x_{\alpha_1}(\alpha_1(t)\, a_1)\, \dots\, x_{\alpha_{n(w)}}(\alpha_{n(w)}(t)\, a_{n(w)})\, \dot{w}B. \tag{3}$$

Since $x_\alpha$ is injective and since $\alpha \neq 0$ for all $\alpha \in \Phi$, we see: *The fixed points of $T$ on $X$ are exactly all $\dot{w}B$ with $w \in W$.*

The equation (3) implies for each $w \in W$ and each $\alpha \in \Phi(w)$ that all $x_\alpha(a)\dot{w}B$ with $a \in \mathbf{C}^\times$ form a one dimensional $T$–orbit. These are in fact *all* one dimensional $T$–orbits on $X$: Any $T$–orbit is contained in a $B$–orbit, hence in some $C_w$. We have to show: Take an element as in (3); if we have $a_i \neq 0 \neq a_j$ for some $i \neq j$, then the dimension of its $T$–orbit is at least 2. This follows from the fact that $\alpha_i \neq \alpha_j$ implies that $\mathbf{C}\alpha_i \neq \mathbf{C}\alpha_j$ by a property of the root system and that therefore $t \mapsto (\alpha_i(t), \alpha_j(t))$ maps $T$ onto $\mathbf{C}^\times \times \mathbf{C}^\times$.

So $X = G/B$ satisfies the assumptions at the beginning of 1.12. Furthermore, the cohomology $H^\bullet(G/B;\mathbf{C})$ vanishes in odd degrees; this follows from the paving of $X$ by the affine spaces $C_w$, $w \in W$. Therefore we can apply Theorem 1.12 once we know the closures of the one dimensional $T$–orbits.

Let $w \in W$ and $\alpha \in \Phi(w)$. The closure of the orbit consisting of all $x_\alpha(a)\,\dot{w}B$ with $a \in \mathbf{C}^\times$ contains certainly $x_\alpha(0)\,\dot{w}B = \dot{w}B$. In order to find the other fixed point in the closure look at the equation in $\mathrm{SL}_2(\mathbf{C})$

$$\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ a^{-1} & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} 1 & a^{-1} \\ 0 & 1 \end{pmatrix}$$

for all $a \in \mathbf{C}^\times$. Applying $\varphi_\alpha$ we get $x_\alpha(a^{-1}) \in x_{-\alpha}(a)\,\dot{s}_\alpha\, x_{-\alpha}(a^{-1})\,T$, hence

$$x_\alpha(a^{-1})\,\dot{w}B = x_{-\alpha}(a)\,\dot{s}_\alpha\, x_{-\alpha}(a^{-1})\,\dot{w}B.$$

Now $\alpha \in \Phi(w)$ implies that $\dot{w}^{-1}\,x_{-\alpha}(a^{-1})\,\dot{w} \in U_{-w^{-1}\alpha} \subset B$ and thus

$$x_\alpha(a^{-1})\,\dot{w}B = x_{-\alpha}(a)\,\dot{s}_\alpha \dot{w}B.$$

Therefore $\dot{s}_\alpha \dot{w}B$ is the other fixed point in the closure of our orbit.

Note that this result means that each fixed point $\dot{w}B$ of $T$ on $X$ belongs to the closure of exactly $|\Phi^+|$ one dimensional $T$–orbits: For each $\alpha \in \Phi^+$ there is one such $T$–orbit containing $\dot{w}B$ and $\dot{s}_\alpha\dot{w}B$ as the fixed points in its closure. This is clear by the result above in case $\alpha \in \Phi(w)$, i.e., when $w^{-1}\alpha \in -\Phi^+$. In case $w^{-1}\alpha \in \Phi^+$ one observes that $\alpha \in \Phi(s_\alpha w)$ since $(s_\alpha w)^{-1}\alpha = w^{-1}s_\alpha\alpha = -w^{-1}\alpha \in -\Phi^+$; so there exists a one dimensional $T$–orbit containing $\dot{s}_\alpha\dot{w}B$ and $\dot{s}_\alpha\dot{s}_\alpha\dot{w}B = \dot{w}B$ in its closure.

Theorem 1.12 therefore implies that

$$H^\bullet_T(G/B;\mathbf{C}) \simeq \{\, (a_w)_{w\in W} \in S^{|W|} \mid a_w \equiv a_{s_\alpha w} \bmod S\alpha \text{ for all } w \in W \text{ and } \alpha \in \Phi^+ \,\}. \quad (4)$$

Here we use the notation $S^{|W|}$ to denote the direct product of $|W|$ copies of $S$ while we reserve the notation $S^W$ for the algebra of $W$–invariants in $S$. A more classical approach to the equivariant cohomology of $G/B$ (cf. [Br], Prop. 1) yields an isomorphism

$$H^\bullet_T(G/B;\mathbf{C}) \simeq S \otimes_{S^W} S. \quad (5)$$

These two results are of course compatible: There exists an isomorphism from $S \otimes_{S^W} S$ onto the right hand side of (4) mapping any $a \otimes b$ with $a, b \in S$ to the family of all $w(a)b$, $w \in W$.

**1.14.** (**Equivariant Intersection Cohomology**) In a 1980 paper Goresky and MacPherson constructed intersection (co)homology groups for pseudomanifolds. This class of topological spaces includes algebraic varieties over $\mathbf{C}$. In the following we consider only intersection cohomology groups with respect to middle perversity and we work with coefficients in $\mathbf{C}$.

Let $X$ be an algebraic variety over $\mathbf{C}$. The intersection cohomology groups $IH^d(X)$ can be described as the hypercohomology groups $IH^d(X) = \mathbf{H}^d(\mathbf{IC}^\bullet(X))$ of a complex $\mathbf{IC}^\bullet(X)$ in the bounded derived category of sheaves on $X$. The direct sum $IH^\bullet(X)$ of these groups has a natural structure as a graded module over the usual cohomology $H^\bullet(X;\mathbf{C})$.

If $i\colon Y \to X$ is the inclusion of a subvariety of $X$, then we set

$$\mathbf{IC}^\bullet(X)_Y = i^*\mathbf{IC}^\bullet(X) \qquad \text{and} \qquad IH^d(X)_Y = \mathbf{H}^d(\mathbf{IC}^\bullet(X)_Y) \tag{1}$$

where $i^*$ is the induced map on the derived categories. We use $i_*$ in a similar sense. Since $i_*$ is right adjoint to $i^*$, we have a natural adjunction morphism $\mathbf{IC}^\bullet(X) \to i_*i^*\mathbf{IC}^\bullet(X)$; it induces natural maps

$$IH^d(X) = \mathbf{H}^d(\mathbf{IC}^\bullet(X)) \longrightarrow \mathbf{H}^d(i_*i^*\mathbf{IC}^\bullet(X)) = IH^d(X)_Y \tag{2}$$

where the second equality follows from the exactness of $i_*$.

If $j\colon Z \to Y$ is another inclusion of varieties, then $(i \circ j)^* = j^* \circ i^*$ implies that

$$\mathbf{IC}^\bullet(X)_Z = j^*\mathbf{IC}^\bullet(X)_Y.$$

Therefore the adjunction $\mathbf{IC}^\bullet(X)_Y \to j_*j^*\mathbf{IC}^\bullet(X)_Y$ induces a homomorphism of intersection cohomology groups

$$IH^d(X)_Y = \mathbf{H}^d(\mathbf{IC}^\bullet(X)_Y) \longrightarrow \mathbf{H}^d(\mathbf{IC}^\bullet(X)_Z) = IH^d(X)_Z. \tag{3}$$

The inclusion $i\colon Y \to X$ is called *normally nonsingular* if there exists an open neighbourhood $W$ of $Y$ in $X$ (in the complex topology) that admits a projection $\pi\colon W \to Y$ that is a locally trivial fibration with fibres isomorphic to some $\mathbf{C}^s$.

If $i$ is normally nonsingular, then there exists a canonical isomorphism $i^*\mathbf{IC}(X) \simeq \mathbf{IC}(Y)$. So in this case (2) yields canonical maps $IH^\bullet(X) \to IH^\bullet(Y)$.

Note: If $X$ and $Y$ are non-singular projective varieties, then the inclusion $i$ is normally nonsingular; in this case one has always what is usually called a *tubular neighbourhood*.

Now suppose that our algebraic torus $T$ acts on $X$. Recall the principal bundles $p_T^n\colon E_T^n \to B_T^n$ from 1.6 with $E_T^n = (\mathbf{C}^{n+1} \setminus \{0\})^d$ and $B_T^n = \mathbf{P}^n(\mathbf{C})^d$ where $d = \dim T$. Recall also the embeddings $E_T^n \hookrightarrow E_T^{n+1}$ and $B_T^n \hookrightarrow B_T^{n+1}$ induced by the embedding of $\mathbf{C}^{n+1}$ into $\mathbf{C}^{n+2}$ as the subspace where the last coordinate is equal to 0. It is easy to see that all embeddings $B_T^n \hookrightarrow B_T^{n+1}$ are normally nonsingular. Set $X_T^n = (X \times E_T^n)/T$ for all $n \in \mathbf{N}$.

Using the local triviality of the maps $X_T^n \to B_T^n$ one can show that also the embeddings $X_T^n \hookrightarrow X_T^{n+1}$ are normally nonsingular. So we get canonical maps

$$IH^d(X_T^{n+1}) \longrightarrow IH^d(X_T^n) \tag{4}$$

that we use to define the *equivariant intersection cohomology*:

$$IH_T^d(X) = \varprojlim IH^d(X_T^n). \tag{5}$$

Each $IH^\bullet(X_T^n)$ is a graded module over $H^\bullet(X_T^n; \mathbf{C})$, hence via the map $X_T^n \to B_T^n$ also a graded module over $H^\bullet(B_T^n; \mathbf{C})$. The maps in (4) are compatible with the natural maps $H^\bullet(B_T^{n+1}; \mathbf{C}) \to H^\bullet(B_T^n; \mathbf{C})$. In this way the (graded) inverse limit $IH_T^\bullet(X)$ becomes a module over $S \simeq H^\bullet(B_T; \mathbf{C})$, the inverse limit of all $H^\bullet(B_T^n; \mathbf{C})$. Note that $E_T^0$ identifies with $T$, hence $X_T^0$ with $X$. So our equivariant intersection cohomology comes with a natural map

$$IH_T^\bullet(X) \longrightarrow IH^\bullet(X_T^0) \simeq IH^\bullet(X)$$

which yields a homomorphism

$$IH_T^\bullet(X) \otimes_S \mathbf{C} \longrightarrow IH^\bullet(X) \tag{6}$$

where we identify $\mathbf{C} \simeq S/\mathfrak{m}$.

Let $Y$ be a $T$–stable subvariety of $X$. Consider the inclusions $i_n\colon X_T^n \hookrightarrow X_T^{n+1}$ and $j_n\colon Y_T^n \hookrightarrow Y_T^{n+1}$ and $\alpha_n\colon Y_T^n \hookrightarrow X_T^n$; we have $\alpha_{n+1} \circ j_n = i_n \circ \alpha_n$. Using adjunction maps we get now morphisms

$$\begin{aligned}\mathbf{IC}^\bullet(X_T^{n+1})_{Y_T^{n+1}} = \alpha_{n+1}^*\mathbf{IC}^\bullet(X_T^{n+1}) &\longrightarrow (j_n)_* j_n^* \alpha_{n+1}^* \mathbf{IC}^\bullet(X_T^{n+1}) = (j_n)_* \alpha_n^* i_n^* \mathbf{IC}^\bullet(X_T^{n+1}) \\ &\xrightarrow{\sim} (j_n)_* \alpha_n^* \mathbf{IC}^\bullet(X_T^n) = (j_n)_* \mathbf{IC}^\bullet(X_T^n)_{Y_T^n}\end{aligned}$$

hence a homomorphism

$$IH^\bullet(X_T^{n+1})_{Y_T^{n+1}} = \mathbf{H}^\bullet(\mathbf{IC}^\bullet(X_T^{n+1})_{Y_T^{n+1}}) \longrightarrow \mathbf{H}^\bullet(\mathbf{IC}^\bullet(X_T^n)_{Y_T^n}) = IH^\bullet(X_T^n)_{Y_T^n}.$$

We can now take the limit and set

$$IH_T^\bullet(X)_Y = \varprojlim IH^\bullet(X_T^n)_{Y_T^n}. \tag{7}$$

We have by (2) natural maps $IH^\bullet(X_T^n) \to IH^\bullet(X_T^n)_{Y_T^n}$. If $Z \subset Y$ is another $T$–stable subvariety, then we have by (3) natural maps $IH^\bullet(X_T^n)_{Y_T^n} \to IH^\bullet(X_T^n)_{Z_T^n}$. We can then take limits and get natural maps

$$IH_T^\bullet(X) \to IH_T^\bullet(X)_Y \qquad \text{and} \qquad IH_T^\bullet(X)_Y \to IH_T^\bullet(X)_Z. \tag{8}$$

Also each $IH_T^\bullet(X)_Y$ has a natural structure as a graded module over $S \simeq H^\bullet(B_T; \mathbf{C})$; the maps in (8) are homomorphisms of graded $S$–modules.

**1.15.** Consider as in 1.12 a projective variety $X$ over $\mathbf{C}$ with an algebraic action of $T$ such that both the set $X^T$ of fixed points and the set of (complex) one dimensional orbits of $T$ on $X$ are finite. Let us make two additional assumptions:

(A) Each fixed point $x \in X^T$ is *contracting.* This means that there exists a homomorphism $\nu_x\colon \mathbf{C}^\times \to T$ of algebraic groups and a (Zariski) open neighbourhood $U$ of $x$ in $X$ such that $\lim_{a\to 0} \nu_x(a)\, u = x$ for all $u \in U$.

(B) There exists a Whitney stratification $X = \bigcup_{x\in X^T} C_x$ where the $C_x$ are $T$–stable subvarieties isomorphic to $\mathbf{C}^{n(x)}$ for some $n(x) \in \mathbf{N}$ such that $x \in C_x$ for each $x \in X^T$.

A theorem of Bialynicki-Birula implies then that one can choose the isomorphism $C_x \xrightarrow{\sim} \mathbf{C}^{n(x)}$ such that $x$ is mapped to 0 and such that the action of $T$ on $C_x$ corresponds to a linear action on $\mathbf{C}^{n(x)}$. Then one checks that the homomorphism $\nu$ as in (A) satisfies $\lim_{a\to 0} \nu(a)\, u = x$ for all $u \in C_x$.

Any one dimensional orbit $P$ of $T$ on $X$ is contained in one of the strata $C_x$. The preceding remarks show that $x \in \overline{P}$. On the other hand, the closure of $P$ is a projective variety and cannot be contained in the affine space $C_x$. So there exists a second fixed point $y$ in the closure of $P$. It satisfies $y \in \overline{C}_x \setminus C_x$. So $\overline{P}$ is homeomorphic to $\mathbf{P}^1(\mathbf{C})$.

Under these assumptions Braden and MacPherson give in [BM] a combinatorial description not only of the equivariant cohomology, but also of the equivariant intersection cohomology. Their procedure involves besides $IH^\bullet_T(X)$ also all $IH^\bullet_T(X)_{\{x\}}$ with $x \in X^T$ and all $IH^\bullet_T(X)_P$ with $P$ a one dimensional $T$–orbit. Furthermore one needs maps

$$IH^\bullet_T(X)_{\{x\}} \longrightarrow IH^\bullet_T(X)_P \tag{1}$$

whenever a fixed point $x$ belongs to the closure of a one dimensional orbit $P$. In this situation we have by 1.14(8) natural maps

$$IH^\bullet_T(X)_{P\cup\{x\}} \longrightarrow IH^\bullet_T(X)_{\{x\}} \qquad \text{and} \qquad IH^\bullet_T(X)_{P\cup\{x\}} \longrightarrow IH^\bullet_T(X)_P.$$

Here the first map turns out to be an isomorphism; so we can compose its inverse with the second map to get (1).

This set-up leads to the notion of a sheaf on a moment graph to be discussed in the next section. Afterwards we return to the theorem proved by Braden and MacPherson.

**1.16.** Return to the flag variety $X = G/B$ as in 1.13. We want to show that the conditions (A) and (B) from 1.15 are satisfied in this case. We know that $X^T = \{\dot{w}B \mid w \in W\}$.

Denote by $Y(T)$ the group of all homomorphisms $\nu\colon \mathbf{C}^\times \to T$ of algebraic groups. For each $\lambda \in X(T)$ and $\nu \in Y(T)$ there exists an integer $\langle \lambda, \nu\rangle \in \mathbf{Z}$ such that

$$\lambda(\nu(a)) = a^{\langle\lambda,\nu\rangle} \qquad \text{for all } a \in \mathbf{C}^\times.$$

The Weyl group $W$ acts on $Y(T)$ such that $(w\nu)\,(a) = \dot{w}\,\nu(a)\,\dot{w}^{-1}$ for all $a \in \mathbf{C}^\times$. One gets then $\langle\lambda,\nu\rangle = \langle w\lambda, w\nu\rangle$ for all $\lambda \in X(T)$, $\nu \in Y(T)$, and $w \in W$. The theory of root systems shows that there exists $\delta \in Y(T)$ with $\langle \alpha, \delta\rangle > 0$ for all $\alpha \in \Phi^+$.

Choose some ordering $\alpha_1, \alpha_2, \dots, \alpha_N$ of $\Phi^+$. Then the map $\mathbf{C}^N \to X$ with

$$(a_1, a_2, \dots, a_N) \mapsto \dot{w} x_{-\alpha_1}(a_1)\, x_{-\alpha_2}(a_2)\, \dots\, x_{-\alpha_N}(a_N) B \tag{1}$$

is an isomorphism onto an open neighbourhood of $\dot{w}B$ in $X$. If we apply any $\nu(a)$ with $\nu \in Y(T)$ and $a \in \mathbf{C}^\times$ to the right hand side in (1), then we get

$$\dot{w} x_{-\alpha_1}(a^{-\langle w\alpha_1, \nu\rangle} a_1)\, x_{-\alpha_2}(a^{-\langle w\alpha_2, \nu\rangle} a_2)\, \dots\, x_{-\alpha_N}(a^{-\langle w\alpha_N, \nu\rangle} a_N) B. \tag{2}$$

If we choose $\nu = -w\delta$, then all exponents of $a$ in (2) are positive and the limit for $a$ going to 0 is equal to $\dot{w}B$. Therefore (A) is satisfied.

In order to check (B) we want to take $C_{\dot{w}B} = C_w = B\dot{w}B/B$ for each $w \in W$. We observed in 1.13 that $X$ is the disjoint union of all $C_w$ and that the closure of any $C_w$ is the union of certain $C_v$ with $v \in W$. Since clearly $\dot{w}B \in C_w$, it suffices to check the Whitney property.

Suppose that $C_w \subset \overline{C}_{w'}$ for some $w, w' \in W$. The set $Y$ of all points in $C_w$ not satisfying the Whitney condition with respect to $C_{w'}$ has codimension at least 1, see Thm. 2 in [Ka]. On the other hand, both $C_w$ and $C_{w'}$ are $B$–orbits. Therefore $Y$ is $B$–stable, hence either equal to $C_w$ or empty. The first possibility is excluded by the result on codimension. So the Whitney condition holds everywhere.

## 2 Sheaves on moment graphs

**2.1.** (**Moment graphs**) Let $k$ be a field and $V$ a finite dimensional vector space over $k$. Denote by $S = S(V)$ the symmetric algebra of the vector space $V$. We consider $S$ as a graded algebra with the usual grading doubled such that $V$ is the homogeneous component of degree 2 and such that the components of odd degree are 0.

An *unordered moment graph* over $V$ is a triple $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \alpha)$ where $(\mathcal{V}, \mathcal{E})$ is a graph with set of vertices $\mathcal{V}$ and set of edges $\mathcal{E}$ and where $\alpha$ is a map that associates to each edge a one dimensional subspace in $V$, i.e., an element in the projective space $\mathbf{P}(V)$. We usually denote the line $\alpha(E)$ associated to some $E \in \mathcal{E}$ by $k\,\alpha_E$ with a suitable $\alpha_E \in V$, $\alpha_E \neq 0$.

Each $E \in \mathcal{E}$ joins two vertices in $\mathcal{V}$; we usually denote these vertices by $a_E$ and $z_E$. We say that an edge $E$ is adjacent to a vertex $x$ if $x \in \{a_E, z_E\}$.

For example, in the set-up at the beginning of 1.12 we get an unordered moment graph over $X(T) \otimes_{\mathbf{Z}} \mathbf{C}$ such that $\mathcal{V} = X^T$ and such that $\mathcal{E}$ is in bijection with the one dimensional orbit closures $P_i$ as in 1.12. If $E \in \mathcal{E}$ corresponds to $P_i$, then $E$ joins $x_{a(i)}$ and $x_{z(i)}$. And we set then $\mathbf{C}\alpha_E = \mathbf{C}\lambda_i$.

**2.2.** (**Sheaves on moment graphs**) Keep $k$, $V$ and $S$ as in 2.1 and let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \alpha)$ be an unordered moment graph over $V$. Let $A$ be an $S$–algebra. An *$A$–sheaf* $\mathcal{M}$ on $\mathcal{G}$ is a collection of the following type of data:

(A) For each $x \in \mathcal{V}$ an $A$–module $\mathcal{M}_x$.
(B) For each $E \in \mathcal{E}$ an $A$–module $\mathcal{M}_E$ such that $\alpha_E\,\mathcal{M}_E = 0$.
(C) For each $x \in \mathcal{V}$ and for each $E \in \mathcal{E}$ adjacent to $x$ a homomorphism $\rho^{\mathcal{M}}_{x,E}\colon \mathcal{M}_x \to \mathcal{M}_E$ of $A$–modules. (We usually write just $\rho_{x,E}$ instead of $\rho^{\mathcal{M}}_{x,E}$.)

In (B) we have to interpret $\alpha_E$ as its image in $A$ under the homomorphism $S \to A$ that makes $A$ into an $S$–algebra.

For example, we get an $A$–sheaf $\mathcal{A} = \mathcal{A}_{\mathcal{G}}$, called the *structure sheaf* of $\mathcal{G}$, setting $\mathcal{A}_x = A$ for all $x$, setting $\mathcal{A}_E = A/A\,\alpha_E$ for all $E$, and setting any $\rho^{\mathcal{A}}_{x,E}\colon A \to A/A\alpha_E$ equal to the canonical map.

In the following we keep $A$ fixed and say just *sheaf* instead of *$A$–sheaf* when it is clear which $A$ we consider.

We say that a sheaf $\mathcal{M}$ on $\mathcal{G}$ has *finite type* if all $\mathcal{M}_x$ and all $\mathcal{M}_E$ are finitely generated graded $A$–modules.

The sheaves on $\mathcal{G}$ form a category. A morphism $f\colon \mathcal{M} \to \mathcal{N}$ between two sheaves is by definition a pair of families $((f_x)_{x\in\mathcal{V}}, (f_E)_{E\in\mathcal{E}})$ where $f_x\colon \mathcal{M}_x \to \mathcal{N}_x$ and $f_E\colon \mathcal{M}_E \to \mathcal{N}_E$ are homomorphisms of $A$–modules such that

$$f_E \circ \rho^{\mathcal{M}}_{x,E} = \rho^{\mathcal{N}}_{x,E} \circ f_x \tag{1}$$

for any $x \in \mathcal{V}$ and any edge $E$ adjacent to $x$.

We denote by $\mathrm{Hom}(\mathcal{M}, \mathcal{N})$ the set of all morphisms $\mathcal{M} \to \mathcal{N}$ of sheaves on $\mathcal{G}$; this set has a natural structure as an $A$–module.

**2.3.** (**Sections**) Let $\mathcal{M}$ be a sheaf on $\mathcal{G}$. For any subset $Z$ of $\mathcal{V}\cup\mathcal{E}$ set $\mathcal{M}(Z)$ equal to the set of all pairs of families $((u_x)_{x\in Z\cap\mathcal{V}}, (v_E)_{E\in Z\cap\mathcal{E}})$ with $u_x \in \mathcal{M}_x$ for all $x$ and $v_E \in \mathcal{M}_E$ for all $E$ such that $\rho_{x,E}(u_x) = v_E$ for all $x$ and $E$ with $E$ adjacent to $x$. Elements of $\mathcal{M}(Z)$ are called *sections* of $\mathcal{M}$ over $Z$.

Any $\mathcal{M}(Z)$ is an $A$–submodule of the direct product $\prod_{x\in Z\cap\mathcal{V}} \mathcal{M}_x \times \prod_{E\in Z\cap\mathcal{E}} \mathcal{M}_E$. If $\mathcal{M}$ is of finite type, if $A$ is noetherian, and if $Z$ is finite, then $\mathcal{M}(Z)$ is a finitely generated $A$–module.

In the case of the structural sheaf $\mathcal{A}$ each $\mathcal{A}(Z)$ is an $A$–algebra. For arbitrary $\mathcal{M}$ then any $\mathcal{M}(Z)$ is an $\mathcal{A}(Z)$–module under componentwise action, i.e., each $u \in \mathcal{A}_x = A$ acts with the given $A$–module structure on $\mathcal{M}_x$ and each $v \in \mathcal{M}_E = A/A\alpha_E$ acts as given on $\mathcal{M}_E$.

If $f\colon \mathcal{M} \to \mathcal{N}$ is a morphism of sheaves on $\mathcal{G}$, then $f$ induces for each subset $Z$ of $\mathcal{V}\cup\mathcal{E}$ a homomorphism $f(Z)\colon \mathcal{M}(Z) \to \mathcal{N}(Z)$ of $A$–modules, in fact: of $\mathcal{A}(Z)$–modules. It is the restriction of the product of all $f_z$ with $z \in Z\cap\mathcal{V}$ and of all $f_E$ with $E \in Z\cap\mathcal{E}$.

We call

$$\mathcal{Z}(\mathcal{G}) = \{\, (u_x)_{x\in\mathcal{V}} \in A^{\mathcal{V}} \mid u_{a_E} \equiv u_{z_E} \ (\operatorname{mod} A\alpha_E) \text{ for all } E \in \mathcal{E} \,\} \tag{1}$$

the *structure algebra* of $\mathcal{G}$ over $A$.

If $\mathcal{G}$ is constructed as in 2.1 from a $T$–space $X$ as in 1.12 and if $A = S$, then we get by Thm. 1.12 $\mathcal{Z}(\mathcal{G}) \simeq H_T^\bullet(X;\mathbf{C})$ if the right hand side is a free module over $H^\bullet(B_T;\mathbf{C})$.

Note that we have for general $\mathcal{G}$ a natural isomorphism of $A$–algebras

$$\mathcal{Z}(\mathcal{G}) \xrightarrow{\sim} \mathcal{A}(\mathcal{V}\cup\mathcal{E}) \tag{2}$$

mapping any family $(u_x)_{x\in\mathcal{V}}$ to the pair of families $((u_x)_{x\in\mathcal{V}}, (\rho_{a_E,E}(u_{a_E}))_{E\in\mathcal{E}})$.

**2.4.** (**Sheaves and sheaves**) Call a subset $Y$ of $\mathcal{V}\cup\mathcal{E}$ *open* if $Y$ contains with any $x \in Y\cap\mathcal{V}$ also all edges adjacent to $x$. One checks easily that $\mathcal{V}\cup\mathcal{E}$ becomes a topological space with this definition. A subset $Z$ of $\mathcal{V}\cup\mathcal{E}$ is then *closed* if $Z$ contains with any $E \in Z\cap\mathcal{E}$ also the two vertices joined by $E$. In this case we can regard $Z$ again as an unordered moment graph equal to $(Z\cap\mathcal{V}, Z\cap\mathcal{E}, \alpha_{|Z\cap\mathcal{E}})$.

We can now regard any sheaf $\mathcal{M}$ on $\mathcal{G}$ as a sheaf of $A$–modules on the topological space $\mathcal{V}\cup\mathcal{E}$ mapping any open subset $Y$ to $\mathcal{M}(Y)$. For $Y' \subset Y$ we get a natural restriction map induced by the obvious projection

$$\prod_{x\in Y\cap\mathcal{V}} \mathcal{M}_x \times \prod_{E\in Y\cap\mathcal{E}} \mathcal{M}_E \longrightarrow \prod_{x\in Y'\cap\mathcal{V}} \mathcal{M}_x \times \prod_{E\in Y'\cap\mathcal{E}} \mathcal{M}_E.$$

For any open covering $Y = \bigcup_{i\in I} Y_i$ one checks easily that the sequence

$$\mathcal{M}(Y) \longrightarrow \prod_{i\in I} \mathcal{M}(Y_i) \rightrightarrows \prod_{i,j\in I} \mathcal{M}(Y_i\cap Y_j)$$

is exact.

Note that we can recover $\mathcal{M}$ from this sheaf on $\mathcal{V}\cup\mathcal{E}$: For each $E\in\mathcal{E}$ the subset $\{E\}$ of $\mathcal{V}\cup\mathcal{E}$ is open and we have $\mathcal{M}_E=\mathcal{M}(\{E\})$. For any $x\in\mathcal{V}$ let $x^o\subset\mathcal{V}\cup\mathcal{E}$ denote the set consisting of $x$ and all $E\in\mathcal{E}$ adjacent to $x$. Then $x^o$ is open in $\mathcal{V}\cup\mathcal{E}$ and we have an isomorphism

$$\mathcal{M}_x \xrightarrow{\sim} \mathcal{M}(x^o), \qquad u\mapsto (u,(\rho_{x,E}(u))_E)$$

where $E$ runs over the edges adjacent to $x$. Finally we recover $\rho_{x,E}$ under this identification as the restriction map $\mathcal{M}(x^o)\to\mathcal{M}(\{E\})$.

Applying this construction to the structural sheaf $\mathcal{A}$ we get a sheaf of $A$–algebras on $\mathcal{V}\cup\mathcal{E}$. The action of any $\mathcal{A}(Y)$ on $\mathcal{M}(Y)$ turns the sheaf on $\mathcal{V}\cup\mathcal{E}$ corresponding to a sheaf $\mathcal{M}$ on $\mathcal{G}$ into an $\mathcal{A}$–module. It is now easy to see that this construction induces an equivalence of categories between sheaves on $\mathcal{G}$ and $\mathcal{A}$–modules.

**2.5.** (**Subsheaves & Co**) A *subsheaf* of a sheaf $\mathcal{M}$ on $\mathcal{G}$ is a sheaf $\mathcal{N}$ such that each $\mathcal{N}_x$ with $x\in\mathcal{V}$ is a submodule of $\mathcal{M}_x$, each $\mathcal{N}_E$ with $E\in\mathcal{E}$ a submodule of $\mathcal{M}_E$, and any $\rho^{\mathcal{N}}_{x,E}$ the restriction of $\rho^{\mathcal{M}}_{x,E}$.

For example, if $f\colon\mathcal{M}\to\mathcal{M}'$ is a homomorphism of sheaves, then we get subsheaves $\ker f\subset\mathcal{M}$ and $f(\mathcal{M})\subset\mathcal{M}'$ defined by

$$(\ker f)_x=\ker(f_x) \qquad\text{and}\qquad f(\mathcal{M})_x=f_x(\mathcal{M}_x) \qquad\text{for all } x\in\mathcal{V}$$

and

$$(\ker f)_E=\ker(f_E) \qquad\text{and}\qquad f(\mathcal{M})_E=f_E(\mathcal{M}_E) \qquad\text{for all } E\in\mathcal{E}.$$

Using 2.2(1) one checks that $\rho^{\mathcal{M}}_{x,E}(\ker f_x)\subset\ker f_E$ and $\rho^{\mathcal{M}'}_{x,E}(f_x(\mathcal{M}_x))\subset f_E(\mathcal{M}_E)$ for any vertex $x$ and any edge $E$ adjacent to $x$.

If $\mathcal{N}$ is a subsheaf of $\mathcal{M}$, then we can define a *factor sheaf* $\mathcal{M}/\mathcal{N}$ setting

$$(\mathcal{M}/\mathcal{N})_x=\mathcal{M}_x/\mathcal{N}_x \qquad\text{and}\qquad (\mathcal{M}/\mathcal{N})_E=\mathcal{M}_E/\mathcal{N}_E$$

for all vertices $x$ and all edges $E$. Any $\rho^{\mathcal{M}/\mathcal{N}}_{x,E}$ is defined by

$$\rho^{\mathcal{M}/\mathcal{N}}_{x,E}(u+\mathcal{N}_x)=\rho^{\mathcal{M}}_{x,E}(u)+\mathcal{N}_E \qquad\text{for all } u\in\mathcal{M}_x.$$

This makes sense since $\rho^{\mathcal{M}}_{x,E}(\mathcal{N}_x)\subset\mathcal{N}_E$.

We get then a canonical homomorphism $\pi\colon\mathcal{M}\to\mathcal{M}/\mathcal{N}$ with $\pi_x(u)=u+\mathcal{N}_x$ for all $u\in\mathcal{M}_x$ and $\pi_E(v)=v+\mathcal{N}_E$ for all $v\in\mathcal{M}_E$. It is clear that then $\ker\pi=\mathcal{N}$. And we have the usual universal property: If $f\colon\mathcal{M}\to\mathcal{M}'$ is a homomorphism of sheaves with $\mathcal{N}\subset\ker f$, then there exists a unique homomorphism $\overline{f}\colon\mathcal{M}/\mathcal{N}\to\mathcal{M}'$ with $f=\overline{f}\circ\pi$.

**2.6.** (**Truncation**) Let $\mathcal{M}$ be a sheaf on $\mathcal{G}$. For any subset $Z \subset \mathcal{V} \cup \mathcal{E}$ we define a sheaf $\mathcal{M}[Z]$ on $\mathcal{G}$ setting for all $x \in \mathcal{V}$ and $E \in \mathcal{E}$

$$\mathcal{M}[Z]_x = \begin{cases} \mathcal{M}_x, & \text{if } x \in Z \cap \mathcal{V}, \\ 0, & \text{otherwise,} \end{cases} \qquad \text{and} \qquad \mathcal{M}[Z]_E = \begin{cases} \mathcal{M}_E, & \text{if } E \in Z \cap \mathcal{E}, \\ 0, & \text{otherwise;} \end{cases}$$

we set $\rho_{x,E}^{\mathcal{M}[Z]} = \rho_{x,E}^{\mathcal{M}}$ in case $x \in Z \cap \mathcal{V}$ and $E \in Z \cap \mathcal{E}$; otherwise $\rho_{x,E}^{\mathcal{M}[Z]} = 0$. We call $\mathcal{M}[Z]$ the *truncation* of $\mathcal{M}$ to $Z$.

Note that $\mathcal{M} \mapsto \mathcal{M}[Z]$ is a functor: Any morphism $f\colon \mathcal{M} \to \mathcal{N}$ of sheaves on $\mathcal{G}$ induces a morphism $f[Z]\colon \mathcal{M}[Z] \to \mathcal{N}[Z]$ setting $f[Z]_x = f_x$ (resp. $f[Z]_E = f_E$) for all $x \in Z \cap \mathcal{V}$ (resp. $E \in Z \cap \mathcal{E}$) whereas all other components of $f[Z]$ are equal to 0. (We have to check equations of the form $\rho_{x,E}^{\mathcal{N}[Z]} \circ f[Z]_x = f[Z]_E \circ \rho_{x,E}^{\mathcal{M}[Z]}$. Well, if $x \notin Z$ or $E \notin Z$, then both $\rho$–terms are equal to 0; if $x \in Z$ and $E \in Z$, then the equation reduces to 2.2(1).)

If $Z$ is open in $\mathcal{V} \cup \mathcal{E}$, then $\mathcal{M}[Z]$ is a subsheaf of $\mathcal{M}$. If $Z$ is closed in $\mathcal{V} \cup \mathcal{E}$, then we have a natural homomorphism

$$\pi^{\mathcal{M}}[Z]\colon \mathcal{M} \longrightarrow \mathcal{M}[Z] \tag{1}$$

such that any $\pi^{\mathcal{M}}[Z]_x$ with $x \in Z \cap \mathcal{V}$ and any $\pi^{\mathcal{M}}[Z]_E$ with $E \in Z \cap \mathcal{E}$ is the identity while all remaining $\pi^{\mathcal{M}}[Z]_x$ and $\pi^{\mathcal{M}}[Z]_E$ are equal to 0. It induces an isomorphism

$$\mathcal{M}/\mathcal{M}[(\mathcal{V} \cup \mathcal{E}) \setminus Z] \xrightarrow{\sim} \mathcal{M}[Z] \qquad (Z \text{ closed}). \tag{2}$$

One has in this case for any $Y \subset \mathcal{V} \cup \mathcal{E}$ an obvious isomorphism

$$\mathcal{M}[Z](Y) \xrightarrow{\sim} \mathcal{M}(Y \cap Z) \qquad (Z \text{ closed}). \tag{3}$$

**2.7.** (**Base change**) Let $A'$ be a commutative $A$–algebra. It is clear that we have a forgetful functor from the category of $A'$–sheaves on $\mathcal{G}$ to the category of $A$–sheaves on $\mathcal{G}$ by regarding all $A'$–modules as $A$–modules.

On the other hand we can associate to any $A$–sheaf $\mathcal{M}$ on $\mathcal{G}$ an $A'$–sheaf $\mathcal{M} \otimes A'$ on $\mathcal{G}$ setting $(\mathcal{M} \otimes A')_x = \mathcal{M}_x \otimes_A A'$ for all $x$ and $(\mathcal{M} \otimes A')_E = \mathcal{M}_E \otimes_A A'$ for all $E$ and $\rho_{x,E}^{\mathcal{M}\otimes A'} = \rho_{x,E}^{\mathcal{M}} \otimes \mathrm{id}_{A'}$ for all $x$ and all $E$ adjacent to $x$.

**Lemma:** *Suppose that $A'$ is flat as an $A$–module. Let $M$ be an $A$–sheaf on $\mathcal{G}$.*

(a) *We have a natural isomorphism $\mathcal{M}(Y) \otimes_A A' \xrightarrow{\sim} (\mathcal{M} \otimes A')(Y)$ of $A'$–modules for any finite subset $Y \subset \mathcal{V} \cup \mathcal{E}$.*

(b) *Suppose that $\mathcal{V} \cup \mathcal{E}$ is finite, that $A$ is noetherian, and that $\mathcal{M}$ has finite type. Then we have for any $A$–sheaf $\mathcal{N}$ on $\mathcal{G}$ a natural isomorphism*

$$\mathrm{Hom}(\mathcal{M}, \mathcal{N}) \otimes_A A' \xrightarrow{\sim} \mathrm{Hom}(\mathcal{M} \otimes_A A', \mathcal{N} \otimes_A A')$$

*of $A'$–modules.*

*Proof*: (a) We have by definition of $\mathcal{M}(Y)$ an exact sequence of $A$–modules

$$0 \longrightarrow \mathcal{M}(Y) \longrightarrow \prod_{x \in Y \cap \mathcal{V}} \mathcal{M}_x \times \prod_{E \in Y \cap \mathcal{E}} \mathcal{M}_E \xrightarrow{\delta} \prod_{x,E} \mathcal{M}_E$$

where the last product is over all pairs $(x, E)$ with $x \in Y \cap \mathcal{V}$ and $E$ an edge in $Y$ adjacent to $x$; the map $\delta$ takes a pair of families $((u_x)_x, (v_E)_E)$ to the family of all $\rho_{x,E}(u_x) - v_E$.

This sequence remains exact when tensoring with $A'$ over $A$ because $A'$ is flat over $A$. Since $Y$ is finite, tensoring with $A'$ commutes with the direct product. We get thus an exact sequence

$$0 \longrightarrow \mathcal{M}(Y) \otimes_A A' \longrightarrow \prod_{x \in Y \cap \mathcal{V}} (\mathcal{M} \otimes A')_x \times \prod_{E \in Y \cap \mathcal{E}} (\mathcal{M} \otimes A')_E \xrightarrow{\delta'} \prod_{x,E} (\mathcal{M} \otimes A')_E.$$

Now $\delta'$ can be described like $\delta$; in particular the kernel of $\delta'$ identifies with $(\mathcal{M} \otimes A')(Y)$. The claim follows.

(b) By definition $\operatorname{Hom}(\mathcal{M}, \mathcal{N})$ is the kernel of the homomorphism of $A$–modules

$$\prod_{x \in \mathcal{V}} \operatorname{Hom}_A(\mathcal{M}_x, \mathcal{N}_x) \times \prod_{E \in \mathcal{E}} \operatorname{Hom}_A(\mathcal{M}_E, \mathcal{N}_E) \to \prod_{E \in \mathcal{E}} \Big( \operatorname{Hom}_A(\mathcal{M}_{a_E}, \mathcal{N}_E) \times \operatorname{Hom}_A(\mathcal{M}_{z_E}, \mathcal{N}_E) \Big)$$

mapping a pair of families $((f_x)_{x \in \mathcal{V}}, (f_E)_{E \in \mathcal{E}})$ to the family of all

$$(f_E \circ \rho^{\mathcal{M}}_{a_E,E} - \rho^{\mathcal{N}}_{a_E,E} \circ f_{a_E}, f_E \circ \rho^{\mathcal{M}}_{z_E,E} - \rho^{\mathcal{N}}_{z_E,E} \circ f_{z_E}).$$

There is a similar description for $\operatorname{Hom}(\mathcal{M} \otimes_A A', \mathcal{N} \otimes_A A')$.

Now the claim follows from the flatness of $A'$ and the fact that

$$\operatorname{Hom}_A(M, N) \otimes_A A' \simeq \operatorname{Hom}_{A'}(M \otimes_A A', N \otimes_A A')$$

for all $A$–modules $M$ and $N$ with $M$ finitely generated.

*Remark*: Consider $Y$ and $A'$ as in part (a) of the lemma. Assume in addition that the image in $A'$ of each $\alpha_E$ with $E \in Y \cap \mathcal{E}$ is a unit in $A'$. (For example, this holds when $A'$ is the field of fractions of $S$.) Then $\alpha_E \mathcal{M}_E = 0$ implies $(\mathcal{M} \otimes A')_E = \mathcal{M}_E \otimes_A A' = 0$ for all $E \in Y \cap \mathcal{E}$. It follows that $(\mathcal{M} \otimes A')\,(Y) \xrightarrow{\sim} \prod_{x \in Y \cap \mathcal{V}} (\mathcal{M} \otimes A')_x$. So the lemma implies that the natural map $\mathcal{M}(Y) \to \prod_{x \in Y \cap \mathcal{V}} \mathcal{M}_x$ induces an isomorphism

$$\mathcal{M}(Y) \otimes_A A' \to \prod_{x \in Y \cap \mathcal{V}} (\mathcal{M}_x \otimes_A A'). \tag{1}$$

**2.8.** (**Global sections**) Assume from now on that the unordered moment graph $\mathcal{G}$ is *finite*. The general case requires extra care and extra assumptions that I do not want to discuss here. We also assume from now on that $A$ is the localisation of $S$ with respect to some multiplicative subset. So $A$ is an integral domain contained in the field $Q$ of fractions of $S$; it is integrally closed and noetherian.

Let $\mathcal{M}$ be a sheaf on $\mathcal{G}$. For any closed subset $Z \subset \mathcal{V} \cup \mathcal{E}$ set $\Gamma(Z, \mathcal{M})$ equal to the set of all families $(u_x)_{x \in Z \cap \mathcal{V}}$ with each $u_x \in \mathcal{M}_x$ such that $\rho_{a_E}(u_{a_E}) = \rho_{z_E}(u_{z_E})$ for all $E \in Z \cap \mathcal{E}$. We have obviously an isomorphism $\mathcal{M}(Z) \xrightarrow{\sim} \Gamma(Z, \mathcal{M})$ forgetting the components in all $\mathcal{M}_E$ with $E \in Z \cap \mathcal{E}$.

We simply write $\Gamma(\mathcal{M}) = \Gamma(\mathcal{V} \cup \mathcal{E}, \mathcal{M})$. We call elements in $\Gamma(\mathcal{M})$ the *global sections* of $\mathcal{M}$. In case of the structure sheaf we get thus $\Gamma(\mathcal{A}) = \mathcal{Z}(\mathcal{G})$, cf. 2.1(1). For general $\mathcal{M}$ each $\Gamma(Z, \mathcal{M})$ is a module over $\Gamma(Z, \mathcal{A})$. In particular, $\Gamma(\mathcal{M})$ is a module over $\mathcal{Z}(\mathcal{G})$. It is clear that $\Gamma$ defines a functor from sheaves on $\mathcal{G}$ to $\mathcal{Z}(\mathcal{G})$–modules.

Since $\Gamma(\mathcal{M})$ is an $A$–submodule of $\prod_{x \in \mathcal{V}} \mathcal{M}_x$, the following properties of $\Gamma$ are clear: If each $\mathcal{M}_x$ is torsion free as an $A$–module, then so is $\Gamma(\mathcal{M})$. If each $\mathcal{M}_x$ is finitely generated as an $A$–module, then so is $\Gamma(\mathcal{M})$. (Here we use that $\mathcal{G}$ is finite and that $A$ is Noetherian.)

In general, a sheaf on $\mathcal{G}$ is not determined by its global sections. For example, we can add to any $\mathcal{M}_E$ a direct summand without changing $\Gamma(\mathcal{M})$. On the other hand, if $E$ joins the vertices $x$ and $y$, then we can replace $\mathcal{M}_x$ by $\rho_{x,E}^{-1}(\rho_{y,E}(\mathcal{M}_y))$ without changing $\Gamma(\mathcal{M})$.

We want to associate to any $\mathcal{Z}(\mathcal{G})$–module $M$ (torsion free over $A$) a sheaf $\mathcal{L}(M)$ on $\mathcal{G}$. By the observations above, this cannot lead to an equivalence of categories. But it will turn out that we get such an equivalence once we restrict to suitable subcategories.

First an observation that will motivate part of the construction later on. Consider an edge $E$ of $\mathcal{G}$. Denote by $x$ and $y$ the vertices joined by $E$. Then $\overline{E} = \{x, y, E\}$ is the closure of $\{E\}$ in $\mathcal{V} \cup \mathcal{E}$. We have

$$\Gamma(\overline{E}, \mathcal{M}) = \{ (u_x, u_y) \in \mathcal{M}_x \times \mathcal{M}_y \mid \rho_{x,E}(u_x) = \rho_{y,E}(u_y) \}. \tag{1}$$

Denote by $\pi_x: \Gamma(\overline{E}, \mathcal{M}) \to \mathcal{M}_x$ and $\pi_y: \Gamma(\overline{E}, \mathcal{M}) \to \mathcal{M}_y$ the two projections.

**Claim:** *We have an isomorphism of $A$–modules*

$$(\mathcal{M}_x \oplus \mathcal{M}_y)/\{ (\pi_x(v), -\pi_y(v)) \mid v \in \Gamma(\overline{E}, \mathcal{M}) \} \xrightarrow{\sim} \rho_{x,E}(\mathcal{M}_x) + \rho_{y,E}(\mathcal{M}_y) \subset \mathcal{M}_E \tag{2}$$

*mapping the class of any $(u_x, u_y) \in \mathcal{M}_x \oplus \mathcal{M}_y$ to $\rho_{x,E}(u_x) + \rho_{y,E}(u_y)$.*

*Proof*: Consider the homomorphism

$$\varphi: \mathcal{M}_x \oplus \mathcal{M}_y \to \mathcal{M}_E, \qquad (u_x, u_y) \mapsto \rho_{x,E}(u_x) + \rho_{y,E}(u_y).$$

The image of $\varphi$ is equal to $\rho_{x,E}(\mathcal{M}_x) + \rho_{y,E}(\mathcal{M}_y)$. An element $(u_x, u_y) \in \mathcal{M}_x \oplus \mathcal{M}_y$ belongs to the kernel of $\varphi$ if and only if $\rho_{x,E}(u_x) = \rho_{y,E}(-u_y)$ if and only if $(u_x, -u_y) \in \Gamma(\overline{E}, \mathcal{M})$ if and only if there exists $v \in \Gamma(\overline{E}, \mathcal{M})$ with $(u_x, u_y) = (\pi_x(v), -\pi_y(v))$.

**2.9.** (**Pushouts**) In the situation of Claim 2.8 we can replace $\mathcal{M}_E$ by $\rho_{x,E}(\mathcal{M}_x) + \rho_{y,E}(\mathcal{M}_y)$ without changing $\Gamma(\mathcal{M})$. If we do so, then this claim says that $\mathcal{M}_E$ is the pushout module determined by $\pi_x$ and $\pi_y$. Let me recall some properties of pushout modules.

Consider two homomorphisms $f: L \to M$ and $g: L \to N$ of modules over some ring. The pushout module of these data is the module

$$P = (M \oplus N)/\{ (f(x), -g(x)) \mid x \in L \}.$$

It comes with two homomorphisms $\overline{f}: N \to P$ and $\overline{g}: M \to P$ given by $\overline{f}(z) = [0, z]$ and $\overline{g}(y) = [y, 0]$ where we denote by $[y, z]$ the class in $P$ of $(y, z) \in M \oplus N$. Then the diagram

$$\begin{array}{ccc} L & \xrightarrow{f} & M \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \overline{g}} \\ N & \xrightarrow{\overline{f}} & P \end{array} \tag{1}$$

is commutative and has the following universal property: If $\varphi\colon M \to Q$ and $\psi\colon N \to Q$ are homomorphisms with $\varphi \circ f = \psi \circ g$, then there exists a unique homomorphism $\xi\colon P \to Q$ such that $\varphi = \xi \circ \overline{g}$ and $\psi = \xi \circ \overline{f}$. In fact, the homomorphism $M \oplus N \to Q$ with $(y, z) \mapsto \varphi(y) + \psi(z)$ annihilates all $(f(x), -g(x))$ with $x \in L$ and induces $\xi$.

Note that

$$\ker \overline{f} = g\,(\ker f) \qquad \text{and} \qquad \ker \overline{g} = f\,(\ker g). \tag{2}$$

For example, consider $y \in M$. We have $y \in \ker \overline{g}$ if and only if there exists $x \in L$ with $(y, 0) = (f(x), -g(x))$, i.e., with $y = f(x)$ and $g(x) = 0$. This is equivalent to $y \in f(\ker g)$.

Note next that

$$f \text{ surjective } \Rightarrow \overline{f} \text{ surjective} \qquad \text{and} \qquad g \text{ surjective } \Rightarrow \overline{g} \text{ surjective}. \tag{3}$$

Indeed, consider $[y, z] \in P$ and assume, for example, that $f$ is surjective. Then there exists $x \in L$ with $y = f(x)$. We get then $[y, z] = [y - f(x), z + g(x)] = [0, z + g(x)] = \overline{f}(z + g(x))$.

Consider finally the homomorphism $\delta\colon L \to M \oplus N$ with $\delta(x) = (f(x), g(x))$. We claim that

$$\delta(L) \;=\; \{\, (y, z) \in M \oplus N \mid \overline{g}(y) = \overline{f}(z) \,\}. \tag{4}$$

Well a pair $(y, z) \in M \oplus N$ belongs to the right hand side in (4) if and only if $[y, 0] = [0, z]$ if and only if $[y, -z] = 0$, hence if and only if there exists $x \in L$ with $(y, -z) = (f(x), -g(x))$. The last identity is equivalent to $(y, z) = (f(x), g(x))$. The claim follows.

**2.10.** (**Localisation**) In the next subsections we write $\otimes$ short for $\otimes_A$. Let $Q$ denote the field of fractions of $A$. Set $\mathcal{Z} = \mathcal{Z}(\mathcal{G})$. Recall that we assume $\mathcal{G}$ to be finite. So we get from 2.7(1) an isomorphism

$$\mathcal{Z} \otimes Q \xrightarrow{\sim} \prod_{x \in \mathcal{V}} (\mathcal{A}_x \otimes Q) = \prod_{x \in \mathcal{V}} Q. \tag{1}$$

Decompose the one element $1 = 1 \otimes 1$ in $\mathcal{Z} \otimes Q$ as $1 = \sum_{x \in \mathcal{V}} e_x$ with each $e_x \in \mathcal{A}_x \otimes Q$. This is a decomposition into orthogonal idempotents. We have $e_x(\mathcal{Z} \otimes Q) = \mathcal{A}_x \otimes Q$ for all $x \in \mathcal{V}$. If $N$ is a $(\mathcal{Z} \otimes Q)$–module, then $N = \bigoplus_{x \in \mathcal{V}} e_x N$.

Let $M$ be a $\mathcal{Z}$–module that is torsion free as an $A$–module. We want to associate to $M$ a sheaf $\mathcal{L}(M)$ on $\mathcal{G}$ that we call the *localisation* of $M$. Since $M$ is torsion free, we can identify $M$ with $M \otimes 1 \subset M \otimes Q$. Now $M \otimes Q$ is a $(\mathcal{Z} \otimes Q)$–module, so we have a direct sum decomposition $M \otimes Q = \bigoplus_{x \in \mathcal{V}} e_x\,(M \otimes Q)$. Set for each $x \in \mathcal{V}$

$$\mathcal{L}(M)_x = e_x M = e_x(M \otimes 1) \subset e_x(M \otimes Q). \tag{2}$$

Since $e_x$ commutes with the action of $A$ (after all $\mathcal{Z} \otimes Q$ is commutative), we see that $e_x$ is an $A$–submodule of $M \otimes Q$, hence torsion free. It is also a homomorphic image of $M$. So, if $M$ is finitely generated over $A$, then so is each $\mathcal{L}(M)_x$.

Consider now $E \in \mathcal{E}$; write $\alpha = \alpha_E$. Denote by $x$ and $y$ the vertices joined by $E$. Then $\overline{E} = \{x, y, E\}$ is the closure of $\{E\}$ in $\mathcal{V} \cup \mathcal{E}$.

Set

$$M(E) = (e_x + e_y)\,M + \alpha e_x M = (e_x + e_y)\,M + \alpha e_y M \subset e_x(M \otimes Q) \oplus e_y(M \otimes Q). \quad (3)$$

Denote by $\pi_x$ and $\pi_y$ the projections from $e_x(M \otimes Q) \oplus e_y(M \otimes Q)$ onto the two summands; they are given by $\pi_x(z) = e_x\, z$ and $\pi_y(z) = e_y\, z$. We have $e_x(e_x + e_y) = e_x$ and $e_y(e_x + e_y) = e_y$, hence

$$\pi_x(M(E)) = \mathcal{L}(M)_x \qquad \text{and} \qquad \pi_y(M(E)) = \mathcal{L}(M)_y. \quad (4)$$

We now define $\mathcal{L}(M)_E$ and $\rho_{x,E} = \rho_{x,E}^{\mathcal{L}(M)}$ and $\rho_{y,E} = \rho_{y,E}^{\mathcal{L}(M)}$ by the pushout diagram

$$\begin{array}{ccc} M(E) & \xrightarrow{\pi_x} & \mathcal{L}(M)_x \\ {\scriptstyle \pi_y}\Big\downarrow & & \Big\downarrow{\scriptstyle \rho_{x,E}} \\ \mathcal{L}(M)_y & \xrightarrow{\rho_{y,E}} & \mathcal{L}(M)_E \end{array} \quad (5)$$

It remains to show that our construction satisfies the condition (B) in 2.2, i.e., that $\alpha\, \mathcal{L}(M)_E = 0$. Since $\pi_x$ and $\pi_y$ are surjective by (4), we get from 2.9(3) that

$$\rho_{x,E} \text{ and } \rho_{y,E} \text{ are surjective.} \quad (6)$$

So it suffices to show (e.g.) that $\alpha\, \mathcal{L}(M)_y = \alpha e_y M \subset \ker \rho_{y,E} = \pi_y(\ker \pi_x)$, cf. 2.9(2). But this is clear since $\alpha e_y M \subset M(E)$ and $\pi_x(\alpha e_y M) = 0$ while $\pi_y(\alpha e_y M) = \alpha e_y M$.

Note that (6) implies: If all $\mathcal{L}(M)_x$ are finitely generated over $A$, then so are all $\mathcal{L}(M)_E$. It follows that $\mathcal{L}(M)$ has finite type if $M$ is finitely generated over $A$.

*Remarks*: 1) We get from 2.9(4) that $u \mapsto (\pi_x(u), \pi_y(u))$ is an isomorphism

$$M(E) \xrightarrow{\sim} \Gamma(\overline{E}, \mathcal{L}(M)), \quad (7)$$

cf. 2.8(1). Since $M(E)$ contains $(e_x + e_y)\,M$, this implies that $(e_x v, e_y v) \in \Gamma(\overline{E}, \mathcal{L}(M))$ for all $v \in M$. Applying this to all $E \in \mathcal{E}$ we get that we have a homomorphism of $\mathcal{Z}$–modules

$$g^M \colon M \longrightarrow \Gamma(\mathcal{L}(M)), \qquad v \mapsto (e_x v)_{x \in \mathcal{V}}. \quad (8)$$

This map is injective since $v = \sum_{x \in \mathcal{V}} e_x v$.

2) Consider a homomorphism $\varphi \colon M \to N$ of $\mathcal{Z}$–modules, both torsion free over $A$. Then $\varphi$ induces a homomorphism $\varphi_Q = \varphi \otimes \mathrm{id}_Q \colon M \otimes Q \to N \otimes Q$ of $(\mathcal{Z} \otimes Q)$–modules. It maps each $e_x(u \otimes 1)$ with $u \in M$ and $x \in \mathcal{V}$ to $e_x(\varphi(u) \otimes 1)$ and induces thus a homomorphism

$$\mathcal{L}(\varphi)_x \colon \mathcal{L}(M)_x = e_x\,(M \otimes 1) \longrightarrow e_x\,(N \otimes 1) = \mathcal{L}(N)_x$$

of $A$–modules.

One gets similarly that $\varphi_Q$ maps any $M(E)$ with $E \in \mathcal{E}$ to $N(E)$. Using the universal property of the pushout one checks now that $\varphi_Q$ induces a homomorphism of $A$–modules $\mathcal{L}(\varphi)_E \colon \mathcal{L}(M)_E \to \mathcal{L}(N)_E$ that is compatible with the $\rho$–maps.

This shows that $M \mapsto \mathcal{L}(M)$ is a functor. It is now elementary to check that the map $M \mapsto g^M$ as in (8) is a natural transformation from the identity functor to the composition $\Gamma \circ \mathcal{L}$, i.e., that

$$g^N \circ \varphi = \Gamma(\mathcal{L}(\varphi)) \circ g^M \quad (9)$$

for any $\varphi$ as above. But this is just the fact that $\Gamma(\mathcal{L}(\varphi)) = \prod_{x \in \mathcal{V}} \mathcal{L}(\varphi)_x$ maps any family $(e_x v)_{x \in \mathcal{V}}$ with $v \in M$ to the family $(e_x \varphi(v))_{x \in \mathcal{V}}$.

**2.11.** (**Localisation and truncation**) Let $Z \subset \mathcal{V} \cup \mathcal{E}$ be closed. Set $e_Z = \sum_{c \in Z \cap \mathcal{V}} e_x$. Let $M$ be a $\mathcal{Z}$–module that is torsion free as an $A$–module. Then $e_Z M$ is a $\mathcal{Z}$–submodule of $M \otimes Q$; it is of course again torsion free as an $A$–module.

**Lemma:** *Suppose that $Z$ contains with two vertices $x$ and $y$ also all edges joining $x$ and $y$. Then we have $\mathcal{L}(e_Z M) = \mathcal{L}(M)[Z]$.*

*Proof*: Given $x \in \mathcal{V}$ we have $e_x e_Z = e_x$ if $x \in Z$ and $e_x e_Z = 0$ otherwise. This shows that

$$\mathcal{L}(e_Z M)_x = e_x e_Z M = \begin{cases} e_x M = \mathcal{L}(M)[Z]_x & \text{if } x \in Z, \\ 0 = \mathcal{L}(M)[Z]_x & \text{if } x \notin Z. \end{cases}$$

Consider now $E \in \mathcal{E}$; write $x = a_E$ and $y = z_E$. In case $E \in Z$ both $x$ and $y$ belong to $Z$ because $Z$ is closed. It follows that $(e_x + e_y) e_Z M = (e_x + e_y) M$ and $e_x e_Z M = e_x M$, hence $(e_Z M)(E) = M(E)$ and finally $\mathcal{L}(e_Z M)_E = \mathcal{L}(M)_E = \mathcal{L}(M)[Z]_E$.

Suppose on the other hand that $E \notin Z$. Then our assumption on $Z$ implies that $x \notin Z$ or $y \notin Z$, hence $\mathcal{L}(e_Z M)_x = 0$ or $\mathcal{L}(e_Z M)_y = 0$. Now 2.10(6) applied to $e_Z M$ yields that $\mathcal{L}(e_Z M)_E = 0 = \mathcal{L}(M)[Z]_E$.

*Remark*: The map $M \to e_Z M$, $v \mapsto e_Z v$, is a homomorphism of $\mathcal{Z}$–modules. It is mapped under the functor $\mathcal{L}$ to the morphism $\pi^{\mathcal{L}(M)}[Z] \colon \mathcal{L}(M) \to \mathcal{L}(M)[Z]$. (Note for each $x \in Z \cap \mathcal{V}$ that the induced map $e_x M \to e_x e_Z M$ is the identity.)

**2.12.** (**The image of $\mathcal{L}$**) We say that a sheaf $\mathcal{M}$ on $\mathcal{G}$ is *generated by global sections** if the projection $\Gamma(\mathcal{M}) \to \mathcal{M}_x$ is surjective for each $x \in \mathcal{V}$.

Let $\mathcal{M}$ be a sheaf on $\mathcal{G}$ such that each $\mathcal{M}_x$ with $x \in \mathcal{V}$ is torsion free over $A$. If $\mathcal{M}$ is isomorphic to $\mathcal{L}(M)$ for some $\mathcal{Z}$–module $M$ as in 2.10, then $\mathcal{M}$ has the following properties:

(A) $\mathcal{M}$ is generated by global sections.

(B) Any map $\rho^{\mathcal{M}}_{x,E} \colon \mathcal{M}_x \to \mathcal{M}_E$ is surjective.

(C) We have
$$\Gamma(\overline{E}, \mathcal{M}) = (\pi_x + \pi_y)(\Gamma(\mathcal{M})) + \alpha_E \pi_x(\Gamma(\mathcal{M}))$$
for each $E \in \mathcal{E}$. Here $x$ and $y$ are the vertices joined by $E$ while $\pi_x \colon \Gamma(\mathcal{M}) \to \mathcal{M}_x$ and $\pi_y \colon \Gamma(\mathcal{M}) \to \mathcal{M}_y$ denote the natural projections.

Indeed, 2.10(6) yields (B), we get (A) using 2.10(2) and 2.10(8), and (C) follows from 2.10(7) and 2.10(3).

**Proposition:** *Let $\mathcal{M}$ be a sheaf on $\mathcal{G}$ such that each $\mathcal{M}_x$ with $x \in \mathcal{V}$ is torsion free over $A$. There exists a natural homomorphism $f^{\mathcal{M}} \colon \mathcal{L}(\Gamma(\mathcal{M})) \to \mathcal{M}$ of sheaves on $\mathcal{G}$. If $\mathcal{M}$ satisfies* (A)–(C) *above, then $f^{\mathcal{M}}$ is an isomorphism.*

*Proof*: Recall the decomposition $1 = \sum_{x \in \mathcal{V}} e_x$ in $\mathcal{Z} \otimes Q$. A family $(u_x)_{x \in \mathcal{V}}$ in $\mathcal{Z}$ acting on a family $(v_x)_{x \in \mathcal{V}}$ in $\Gamma(\mathcal{M})$ yields the family of all $u_x v_x$. So a family $(u_x \otimes r_x)_{x \in \mathcal{V}}$ in $\mathcal{Z} \otimes Q$ (with each $u_x \in \mathcal{A}_x$ and $r_x \in Q$) acting on a family $(b_x \otimes s_x)_{x \in \mathcal{V}}$ in $\Gamma(\mathcal{M}) \otimes Q$ (with each

* Here my terminology differs from Fiebig's.

$v_x \in \mathcal{M}_x$ and $s_x \in Q$) yields the family of all $u_x v_x \otimes r_x s_x$. Therefore any $e_y$ with $y \in \mathcal{V}$ acts on $\Gamma(\mathcal{M}) \otimes Q \subset \prod_{x\in\mathcal{V}}(\mathcal{M}_x \otimes Q)$ as the projection to the factor $\mathcal{M}_y \otimes Q$.

This shows that

$$\mathcal{L}(\Gamma(\mathcal{M}))_x = e_x\,\Gamma(\mathcal{M}) \subset \mathcal{M}_x \qquad \text{for each } x \in \mathcal{V}. \tag{1}$$

We set $f_x^{\mathcal{M}} = (f^{\mathcal{M}})_x$ equal to the inclusion of $\mathcal{L}(\Gamma(\mathcal{M}))_x$ into $\mathcal{M}_x$. If $\mathcal{M}$ satisfies (A), then each $f_x^{\mathcal{M}}$ is an isomorphism.

Consider an edge $E \in \mathcal{E}$; denote by $x$ and $y$ the vertices joined by $E$. We have

$$\Gamma(\mathcal{M})\,(E) = (\pi_x + \pi_y)(\Gamma(\mathcal{M})) + \alpha_E \pi_x(\Gamma(\mathcal{M})) \subset \Gamma(\overline{E}, \mathcal{M}). \tag{2}$$

If $\mathcal{M}$ satisfies (C), then we get equality in (2).

It follows that the inclusion $f_x^{\mathcal{M}} \oplus f_y^{\mathcal{M}}\colon \mathcal{L}(\Gamma(\mathcal{M}))_x \oplus \mathcal{L}(\Gamma(\mathcal{M}))_y \to \mathcal{M}_x \oplus \mathcal{M}_y$ maps

$$I := \{\, (\pi_x(u), -\pi_y(u)) \mid u \in \Gamma(\mathcal{M})\,(E) \,\} \quad \text{into} \quad J := \{\, (\pi_x(u), -\pi_y(u)) \mid u \in \Gamma(\overline{E}, \mathcal{M}) \,\}$$

and thus induces a homomorphism $f_E^{\mathcal{M}}$ from $\mathcal{L}(\Gamma(\mathcal{M}))_E = (\mathcal{L}(\Gamma(\mathcal{M}))_x \oplus \mathcal{L}(\Gamma(\mathcal{M}))_y)/I$ to

$$(\mathcal{M}_x \oplus \mathcal{M}_y)/J \xrightarrow{\sim} \rho_{x,E}^{\mathcal{M}}(\mathcal{M}_x) + \rho_{y,E}^{\mathcal{M}}(\mathcal{M}_y) \subset \mathcal{M}_E,$$

cf. 2.8(2). If $\mathcal{M}$ satisfies (A)–(C), then $f_E^{\mathcal{M}}$ is an isomorphism.

It remains to check that $f_E^{\mathcal{M}} \circ \rho_{x,E}^{\mathcal{L}(\Gamma(\mathcal{M}))} = \rho_{x,E}^{\mathcal{M}} \circ f_x^{\mathcal{M}}$. Consider $u \in \mathcal{L}(\Gamma(\mathcal{M}))_x$. Then $\rho_{x,E}^{\mathcal{L}(\Gamma(\mathcal{M}))}(u)$ is the coset $(u,0)+I$ in $\mathcal{L}(\Gamma(\mathcal{M}))_E$. It is mapped under $f_E^{\mathcal{M}}$ to the image if the coset $(u,0)+J = (f_x^{\mathcal{M}}(u),0)+J$ in $\mathcal{M}_E$, hence to $\rho_{x,E}^{\mathcal{M}} \circ f_x^{\mathcal{M}}(u)$. The claim follows.

*Remarks*: 1) Take for example the structure sheaf $\mathcal{A}$. It satisfies (A): For any $u \in A$ the family $(u_x)_{x\in\mathcal{V}}$ with $u_x = u$ for all $x \in \mathcal{V}$ belongs to $\Gamma(\mathcal{A}) = \mathcal{Z}$. The condition (B) holds obviously for $\mathcal{A}$. Finally (C) is satisfied since $(1,1) \in (\pi_x + \pi_y)(\Gamma(\mathcal{A}))$ yields

$$\Gamma(\overline{E}, \mathcal{A}) = \{\, (u,u) \mid u \in A \,\} + \{\, (\alpha u, 0) \mid u \in A \,\} = (\pi_x + \pi_y)(\Gamma(\mathcal{A})) + \alpha\pi_x(\Gamma(\mathcal{A})).$$

So we have a natural isomorphism $f^{\mathcal{A}}\colon \mathcal{L}(\mathcal{Z}) \xrightarrow{\sim} \mathcal{A}$.

2) Let $\psi\colon \mathcal{M} \to \mathcal{N}$ be a homomorphism of sheaves on $\mathcal{G}$ such that all $\mathcal{M}_x$ and $\mathcal{N}_x$ are torsion free over $S$. One checks now that

$$\psi \circ f^{\mathcal{M}} = f^{\mathcal{N}} \circ \mathcal{L}(\Gamma(\psi)). \tag{3}$$

This means that $\mathcal{M} \mapsto f^{\mathcal{M}}$ is a natural transformation from the composition $\mathcal{L} \circ \Gamma$ to the identity functor.

**2.13.** (**Adjointness**) Let $\mathcal{C}$ denote the category of all $\mathcal{Z}$–modules that are torsion free over $A$. Let $\mathcal{S}$ denote the category of all sheaves $\mathcal{M}$ on $\mathcal{G}$ such that each $\mathcal{M}_x$ with $x \in \mathcal{V}$ is torsion free over $A$.

We have functors $\Gamma: \mathcal{S} \to \mathcal{C}$ and $\mathcal{L}: \mathcal{C} \to \mathcal{S}$ together with natural transformations $f: \mathcal{L} \circ \Gamma \to \mathrm{id}_{\mathcal{S}}$ and $g: \mathrm{id}_{\mathcal{C}} \to \Gamma \circ \mathcal{L}$.

Proposition 2.12 shows that $f^{\mathcal{L}(M)}: \mathcal{L}(\Gamma(\mathcal{L}(M))) \to \mathcal{L}(M)$ is an isomorphism for any $M$ in $\mathcal{C}$. Note that its inverse is given by

$$(f^{\mathcal{L}(M)})^{-1} = \mathcal{L}(g^M). \tag{1}$$

For example, any $\mathcal{L}(g^M)_x$ with $x \in \mathcal{V}$ maps $\mathcal{L}(M)_x = e_x M$ to $\mathcal{L}(\Gamma(\mathcal{L}(M)))_x = e_x \Gamma(\mathcal{L}(M))$ sending any $e_x v$ with $v \in M$ to $e_x g^M(v)$. Now $g^M(v)$ is the family of all $e_y v$ with $y \in \mathcal{V}$, and $e_x$ acts on this family as the projection to the $x$–component. We get thus $e_x g^M(v) = e_x v$. So $\mathcal{L}(g^M)_x$ is just the identity map, hence equal to the inverse of the inclusion $f_x^{\mathcal{L}(M)}$. One argues similarly for any $\mathcal{L}(g^M)_E$.

**Lemma:** *For any $\mathcal{M}$ in $\mathcal{S}$ the morphism $g^{\Gamma(\mathcal{M})}: \Gamma(\mathcal{M}) \to \Gamma(\mathcal{L}(\Gamma(\mathcal{M})))$ is an isomorphism with inverse equal to $\Gamma(f^{\mathcal{M}})$.*

*Proof*: Consider the commutative diagram

$$\begin{array}{ccccc}
\Gamma(\mathcal{M}) & \hookrightarrow & \prod_{x\in\mathcal{V}} \mathcal{M}_x & \hookrightarrow & \prod_{x\in\mathcal{V}} (\mathcal{M}_x \otimes Q) \\
{\scriptstyle g^{\Gamma(\mathcal{M})}}\big\downarrow & & & & \big\downarrow{\scriptstyle \mathrm{id}} \\
\Gamma(\mathcal{L}(\Gamma(\mathcal{M}))) & \hookrightarrow & \prod_{x\in\mathcal{V}} \mathcal{L}(\Gamma(\mathcal{M}))_x & \hookrightarrow & \prod_{x\in\mathcal{V}} (\mathcal{M}_x \otimes Q) \\
{\scriptstyle \Gamma(f^{\mathcal{M}})}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}} \\
\Gamma(\mathcal{M}) & \hookrightarrow & \prod_{x\in\mathcal{V}} \mathcal{M}_x & \hookrightarrow & \prod_{x\in\mathcal{V}} (\mathcal{M}_x \otimes Q)
\end{array}$$

The middle vertical map is equal to the product $\prod_{x\in\mathcal{V}} f_x^{\mathcal{M}}$. The lower square on the right hand side commutes by the definition of the $f_x^{\mathcal{M}}$, the lower square on the left hand side commutes by the definition of $\Gamma(f^{\mathcal{M}})$. An element in $\Gamma(\mathcal{M})$ is a family $u = (u_x)_{x\in\mathcal{V}}$ with each $u_x \in \mathcal{M}_x$. One has $e_x\, u = u_x$ for all $x \in \mathcal{V}$. On the other hand, we have $g^{\Gamma(\mathcal{M})}(u) = (e_x\, u)_{x\in\mathcal{V}}$ by definition of $g$. This shows that the upper half of our diagram is commutative.

Looking at the total diagram we get now that $\Gamma(f^{\mathcal{M}}) \circ g^{\Gamma(\mathcal{M})} = \mathrm{id}_{\Gamma(\mathcal{M})}$. Since each $f_x^{\mathcal{M}}$ is injective, so is $\Gamma(f^{\mathcal{M}})$. The claim follows.

**2.14. Proposition:** *The functor $\mathcal{L}: \mathcal{C} \to \mathcal{S}$ is left adjoint to the functor $\Gamma: \mathcal{S} \to \mathcal{C}$.*

*Proof*: We have for any $M$ in $\mathcal{C}$ and any $\mathcal{N}$ in $\mathcal{S}$ natural maps

$$\mathrm{Hom}_{\mathcal{S}}(\mathcal{L}(M), \mathcal{N}) \longrightarrow \mathrm{Hom}_{\mathcal{Z}}(M, \Gamma(\mathcal{N})), \qquad \varphi \mapsto \Gamma(\varphi) \circ g^M \tag{2}$$

and

$$\mathrm{Hom}_{\mathcal{Z}}(M, \Gamma(\mathcal{N})) \longrightarrow \mathrm{Hom}_{\mathcal{S}}(\mathcal{L}(M), \mathcal{N}), \qquad \psi \mapsto f^{\mathcal{N}} \circ \mathcal{L}(\psi). \tag{3}$$

The compositions of these maps are given by

$$\varphi \mapsto f^{\mathcal{N}} \circ \mathcal{L}(\Gamma(\varphi)) \circ \mathcal{L}(g^M) = \varphi \circ f^{\mathcal{L}(M)} \circ \mathcal{L}(g^M) = \varphi$$

and

$$\psi \mapsto \Gamma(f^{\mathcal{N}}) \circ \Gamma(\mathcal{L}(\psi)) \circ g^M = \Gamma(f^{\mathcal{N}}) \circ g^{\Gamma(\mathcal{N})} \circ \psi = \psi.$$

Here we use the natural transformation property of $f$ or $g$ (see 2.12(3) and 2.10(9)) together with (1) or the lemma.

It follows that the maps in (2) and (3) are inverse isomorphisms. Since they induce natural transformations between $\mathrm{Hom}_{\mathcal{S}}(\mathcal{L}(\cdot), \cdot\cdot)$ and $\mathrm{Hom}_{\mathcal{Z}}(\cdot, \Gamma(\cdot\cdot))$, we get the claim.

*Remark*: Let $\mathcal{S}'$ denote the subcategory of all $\mathcal{M}$ in $\mathcal{S}$ isomorphic to some $\mathcal{L}(M)$ with $M$ in $\mathcal{C}$. Let $\mathcal{C}'$ denote the subcategory of all $M$ in $\mathcal{C}$ isomorphic to some $\Gamma(\mathcal{M})$ with $\mathcal{M}$ in $\mathcal{S}$. Then the results in this subsection show that $\Gamma$ and $\mathcal{L}$ induce equivalences of categories between $\mathcal{S}'$ and $\mathcal{C}'$.

**2.15.** (**GKM-graphs**) Let $\mathfrak{p}$ be a prime ideal in $A$ of height 1, i.e., an ideal that is minimal among the non-zero prime ideals. Recall that we assume that $A$ is the localisation of $S$ with respect to some multiplicative subset. This implies that $\mathfrak{p} \cap S$ is a prime ideal in $S$ of height 1 and that $\mathfrak{p} = A(\mathfrak{p} \cap S)$. Because $S$ is a unique factorisation domain, there exists an irreducible element $\gamma \in S$ such that $\mathfrak{p} \cap S = S\gamma$, hence with $\mathfrak{p} = A\gamma$.

Denote by $A_{\mathfrak{p}}$ the local ring of $A$ at $\mathfrak{p}$. So this is the localisation of $A$ where we invert all elements in $A \setminus \mathfrak{p}$.

Consider a sheaf $\mathcal{M}$ on $\mathcal{G}$. If $E$ is an edge of $\mathcal{G}$ with $\alpha_E \notin k\,\gamma$ then we get (by the irreducibility of $\alpha_E$) that $\alpha_E \notin S\,\gamma = \mathfrak{p} \cap S$, hence $\alpha_E \notin \mathfrak{p}$. So $\alpha_E$ is a unit in $A_{\mathfrak{p}}$. Now $\alpha_E \mathcal{M}_E = 0$ implies $\mathcal{M}_E \otimes A_{\mathfrak{p}} = 0$. This shows: When we calculate $\Gamma(\mathcal{M} \otimes A_\gamma)$, then the condition $\rho_{a_E,E}(u_{a_E}) = \rho_{z_E,E}(u_{z_E})$ is automatically satisfied for all $E$ with $\alpha_E \notin k\,\gamma$.

*Definition*: We call a moment graph a *GKM-graph* if $k\alpha_E \neq k\alpha_{E'}$ for all pairs $(E, E')$ of edges, $E \neq E'$, such that there exists a vertex $x$ adjacent to both $E$ and $E'$.

For example, the moment graph associated as in 2.1 to a flag variety as in 1.13 is a GKM-graph: Recall that the vertices $\dot{w}B$ correspond to the elements $w$ of the Weyl group. There is then a bijection between $\Phi^+$ and the set of edges adjacent to $\dot{w}B$. The edge $E$ corresponding to $\alpha \in \Phi^+$ satisfies $\alpha(E) = \mathbf{C}\alpha$. So the GKM-property follows from the fact that $\mathbf{C}\alpha \neq \mathbf{C}\beta$ for any $\alpha, \beta \in \Phi^+$ with $\alpha \neq \beta$.

**Lemma:** *Suppose that $\mathcal{G}$ is a GKM-graph. Let $M$ be a $\mathcal{Z}$–module that is torsion free over $A$. Then $g^M$ induces an isomorphism $g^M \otimes \mathrm{id}\colon M \otimes A_{\mathfrak{p}} \xrightarrow{\sim} \Gamma(\mathcal{L}(M)) \otimes A_{\mathfrak{p}}$.*

*Proof*: Denote by $E_1, E_2, \ldots, E_r$ the edges of $\mathcal{G}$ with $\alpha_{E_i} \in k\gamma$. Denote by $x_i$ and $y_i$ the vertices joined by $E_i$. We have $\{x_i, y_i\} \cap \{x_j, y_j\} = \emptyset$ whenever $i \neq j$; this follows from the assumption that $\mathcal{G}$ is a GKM-graph.

This implies: If $\mathcal{M}$ is an $A_{\mathfrak{p}}$–sheaf on $\mathcal{G}$, then $\Gamma(\mathcal{M})$ consists of all $(u_x)_{x \in \mathcal{V}} \in \prod_{x \in \mathcal{V}} \mathcal{M}_x$ such that $\rho_{x_i,E_i}(u_{x_i}) = \rho_{y_i,E_i}(u_{y_i})$ for all $i$. Setting $\mathcal{V}' = \mathcal{V} \setminus \{x_i, y_i \mid 1 \leq i \leq r\}$ we get thus

$$\Gamma(\mathcal{M}) = \prod_{i=1}^{r} \Gamma(\overline{E}_i, \mathcal{M}) \times \prod_{x \in \mathcal{V}'} \mathcal{M}_x. \tag{1}$$

Let us apply (1) to $\mathcal{M} = \mathcal{L}(M) \otimes A_{\mathfrak{p}}$. We have by 2.7 isomorphisms

$$\Gamma(\mathcal{L}(M)) \otimes A_{\mathfrak{p}} \xrightarrow{\sim} \Gamma(\mathcal{L}(M) \otimes A_{\mathfrak{p}}) \qquad \text{and} \qquad \Gamma(\overline{E}_i, \mathcal{L}(M)) \otimes A_{\mathfrak{p}} \xrightarrow{\sim} \Gamma(\overline{E}_i, \mathcal{L}(M) \otimes A_{\mathfrak{p}}).$$

So we get by (1) an isomorphism

$$\Gamma(\mathcal{L}(M)) \otimes A_{\mathfrak{p}} \xrightarrow{\sim} \prod_{i=1}^{r} \Gamma(\overline{E}_i, \mathcal{L}(M)) \otimes A_{\mathfrak{p}} \times \prod_{x \in \mathcal{V}'} \mathcal{L}(M)_x \otimes A_{\mathfrak{p}} \tag{2}$$

mapping any $u \otimes 1$ with $u \in \Gamma(\mathcal{L}(M))$ to the family of all $(e_{x_i} + e_{y_i})\,(u \otimes 1)$, $1 \le i \le r$, and of all $e_x(u \otimes 1)$, $x \in \mathcal{V}'$.

Set $\mathcal{Z}_i = \Gamma(\overline{E}_i, \mathcal{A}) = (e_{x_i} + e_{y_i})\,\mathcal{Z} + \alpha e_{x_i} \mathcal{Z}$. We apply (2) to $M = \mathcal{Z}$ where $\mathcal{L}(\mathcal{Z}) = \mathcal{A}$ and $\Gamma(\overline{E}_i, \mathcal{L}(\mathcal{Z})) = \mathcal{Z}_i$; we get an isomorphism

$$\mathcal{Z} \otimes A_{\mathfrak{p}} \xrightarrow{\sim} \prod_{i=1}^{r} \mathcal{Z}_i \otimes A_{\mathfrak{p}} \times \prod_{x \in \mathcal{V}'} \mathcal{A}_x \otimes A_{\mathfrak{p}}. \tag{3}$$

It follows that $\mathcal{Z}_i \otimes A_{\mathfrak{p}} = (e_{x_i} + e_{y_i})\,(\mathcal{Z} \otimes A_{\mathfrak{p}})$.

We know by 2.10(7) that $M(E_i) \xrightarrow{\sim} \Gamma(\overline{E}_i, \mathcal{L}(M))$. On the other hand 2.10(3) yields $M(E_i) = \mathcal{Z}_i (e_{x_i} + e_{y_i})\, M$. This implies that

$$\begin{aligned}
\Gamma(\overline{E}_i, \mathcal{L}(M)) \otimes A_{\mathfrak{p}} &\simeq M(E_i) \otimes A_{\mathfrak{p}} = \mathcal{Z}_i\,(e_{x_i} + e_{y_i})\, M \otimes A_{\mathfrak{p}} \\
&\simeq (\mathcal{Z}_i \otimes A_{\mathfrak{p}})\,(e_{x_i} + e_{y_i})\,(M \otimes A_{\mathfrak{p}}) \\
&\simeq (e_{x_i} + e_{y_i})\,(\mathcal{Z} \otimes A_{\mathfrak{p}})\,(e_{x_i} + e_{y_i})\,(M \otimes A_{\mathfrak{p}}) \\
&\simeq (e_{x_i} + e_{y_i})\,(\mathcal{Z} M \otimes A_{\mathfrak{p}}) = (e_{x_i} + e_{y_i})\,(M \otimes A_{\mathfrak{p}}).
\end{aligned}$$

We therefore can rewrite (2) as

$$\Gamma(\mathcal{L}(M)) \otimes A_{\mathfrak{p}} \xrightarrow{\sim} \prod_{i=1}^{r} (e_{x_i}\,(M \otimes A_{\mathfrak{p}}) \times e_{y_i}\,(M \otimes A_{\mathfrak{p}})) \times \prod_{x \in \mathcal{V}'} e_x\,(M \otimes A_{\mathfrak{p}}). \tag{4}$$

Recall that $g^M$ maps any $u \in M$ to the family of all $e_x u$ with $x \in \mathcal{V}$. So $g^M \otimes \mathrm{id}$ maps any $v \in M \otimes A_{\mathfrak{p}}$ to the family of all $e_x v$ with $x \in \mathcal{V}$. Therefore $M \otimes A_{\mathfrak{p}}$ is mapped onto the right hand side of (4). The claim follows, because with $g^M$ also $g^M \otimes \mathrm{id}$ is injective.

**2.16.** (**Reflexive modules**) An $A$–module $M$ is called *reflexive* if the natural map $M \to (M^*)^* = \mathrm{Hom}_A(\mathrm{Hom}_A(M, A), A)$ is an isomorphism. Any free $A$–module of finite rank is reflexive. Any reflexive $A$–module is torsion free. Let $\mathfrak{P}(A)$ denote the set of all prime ideals of height 1 in $A$. A finitely generated torsion free $A$–module $M$ is reflexive if and only if

$$M = \bigcap_{\mathfrak{p} \in \mathfrak{P}(A)} (M \otimes A_{\mathfrak{p}}) \tag{1}$$

where the intersection is taken inside $M \otimes Q$, cf. [Bourbaki, Alg. comm., ch. VII], § 4, n° 2.

**Proposition:** *Let $M$ be a $\mathcal{Z}$–module that is finitely generated and reflexive over $A$. If $\mathcal{G}$ is a GKM-graph, then $g^M\colon M \to \Gamma(\mathcal{L}(M))$ is an isomorphism.*

*Proof*: We have a commutative diagram:

$$\begin{array}{ccc} M & \hookrightarrow & \bigcap_{\mathfrak{p}\in\mathfrak{P}(A)} (M \otimes A_{\mathfrak{p}}) \\ {\scriptstyle g^M}\big\downarrow & & \big\downarrow \\ \Gamma(\mathcal{L}(M)) & \hookrightarrow & \bigcap_{\mathfrak{p}\in\mathfrak{P}(A)} (\Gamma(\mathcal{L}(M)) \otimes A_{\mathfrak{p}}) \end{array}$$

where the vertical map on the right hand side is the restriction of $g^M \otimes \mathrm{id}_Q$. By Lemma 2.15 this map is an isomorphism. Since we assume $M$ to be reflexive over $A$, the upper inclusion is an identity. This shows that $g^M$ has to be surjective, hence bijective.

**2.17. Lemma:** *Let $\mathcal{M}$ be a sheaf on $\mathcal{G}$ such that all $\mathcal{M}_x$ with $x \in \mathcal{V}$ and all $\Gamma(\overline{E}, \mathcal{M})$ with $E \in \mathcal{E}$ are finitely generated reflexive $A$–modules. Then $\Gamma(\mathcal{M})$ is a finitely generated reflexive $A$–module.*

*Proof*: Since $\Gamma(\mathcal{M})$ is an $A$–submodule of $\prod_{x\in\mathcal{V}} \mathcal{M}_x$, it is finitely generated and torsion free over $A$. So it suffices to show that $\Gamma(\mathcal{M}) = \bigcap_{\mathfrak{p}\in\mathfrak{P}(A)}(\Gamma(\mathcal{M}) \otimes A_{\mathfrak{p}})$. This intersection is taken inside $\Gamma(\mathcal{M}) \otimes Q = \prod_{x\in\mathcal{V}}(\mathcal{M}_x \otimes Q)$.

Consider an element $u$ in the intersection of all $\Gamma(\mathcal{M})\otimes A_{\mathfrak{p}}$. So this is a family $(u_x)_{x\in\mathcal{V}}$ with each $u_x \in \mathcal{M}_x \otimes Q$ and there exists for each $\mathfrak{p} \in \mathfrak{P}(A)$ an element $a_{\mathfrak{p}} \in A \setminus \mathfrak{p}$ such that the family $a_{\mathfrak{p}} u$ of all $a_{\mathfrak{p}} u_x$, $x \in \mathcal{V}$, belongs to $\Gamma(\mathcal{M})$.

This implies for each $x \in \mathcal{V}$ that $a_{\mathfrak{p}} u_x \in \mathcal{M}_x$, hence that $u_x \in \mathcal{M}_x \otimes A_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathfrak{P}(A)$. Since $\mathcal{M}_x$ is reflexive, this implies that $u_x \in \mathcal{M}_x$.

Consider now an edge $E \in \mathcal{E}$. Denote by $x$ and $y$ the vertices joined by $E$. Since $a_{\mathfrak{p}} u$ belongs to $\Gamma(\mathcal{M})$, we get $(a_{\mathfrak{p}} u_x, a_{\mathfrak{p}} u_y) \in \Gamma(\overline{E}, \mathcal{M})$. It follows that $(u_x, u_y) \in \Gamma(\overline{E}, \mathcal{M}) \otimes A_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathfrak{P}(A)$. Since $\Gamma(\overline{E}, \mathcal{M})$ is reflexive, this implies $(u_x, u_y) \in \Gamma(\overline{E}, \mathcal{M})$. Because this holds for all $E \in \mathcal{E}$, we get $u \in \Gamma(\mathcal{M})$.

*Remarks*: 1) The same proof shows for any closed subset $Z$ of $\mathcal{V} \cup \mathcal{E}$ that $\Gamma(Z, \mathcal{M})$ is a finitely generated reflexive $A$–module.

2) Consider an edge $E \in \mathcal{E}$ and denote by $x$ and $y$ the vertices joined by $E$. We claim: If $\mathcal{M}_x$ and $\mathcal{M}_y$ are free of finite rank over $A$ and if $\rho_{x,E}$ induces an isomorphism $\mathcal{M}_x/\alpha_E\mathcal{M}_x \xrightarrow{\sim} \mathcal{M}_E$, then also $\Gamma(\overline{E}, \mathcal{M})$ is free of finite rank over $A$. (We shall use this claim later on to check the assumptions in the lemma in a special case.)

Because $\rho_{x,E}$ is surjective and because $\mathcal{M}_y$ is a projective $A$–module, there exists an $A$–linear map $f\colon \mathcal{M}_y \to \mathcal{M}_x$ with $\rho_{x,E} \circ f = \rho_{y,E}$. It follows that

$$\Gamma(\overline{E}, \mathcal{M}) = \{\, (\alpha_E u + f(v), v) \mid v \in \mathcal{M}_y,\, u \in \mathcal{M}_x \,\}. \tag{1}$$

The map $(u, v) \mapsto (\alpha_E u + f(v), v)$ is an isomorphism $\mathcal{M}_x \oplus \mathcal{M}_y \xrightarrow{\sim} \Gamma(\overline{E}, \mathcal{M})$ of $A$–modules. The claim follows.

**2.18.** (**Localisation and base change**) Let $A' \subset Q$ be the localisation of $A$ with respect to some multiplicative subset. It follows that $A'$ is flat when considered as an $A$–module. If $X$ is an $A$–submodule of a $Q$–module $Y$, then the map $X \otimes A' \to Y$, $u \otimes t \mapsto t\,u$ is injective with image equal to the $A'$–submodule of $Y$ generated by $X$. (Both this image and $X \otimes A'$ are the localisation of the $A$–module $X$ with respect to the multiplicative set used to define $A'$.)

Let $M$ be a $\mathcal{Z}$–module that is torsion free as an $A$–module. Then $M \otimes A'$ is a $(\mathcal{Z} \otimes A')$–module that is torsion free over $A'$.

**Lemma:** *We have a natural isomorphism*

$$\mathcal{L}(M) \otimes A' \xrightarrow{\sim} \mathcal{L}(M \otimes A')$$

*of $A'$–sheaves on $\mathcal{G}$.*

*Proof*: We have for any $x \in \mathcal{V}$

$$\mathcal{L}(M \otimes A')_x = e_x(M \otimes A') = A' e_x(M \otimes 1) = A' \mathcal{L}(M)_x \subset M \otimes Q.$$

Therefore the natural map $\mathcal{L}(M)_x \otimes A' \to M \otimes Q$ induces an isomorphism $\mathcal{L}(M)_x \otimes A' \xrightarrow{\sim} \mathcal{L}(M \otimes A')_x$.

For any edge $E$ (joining vertices $x$ and $y$) we see that

$$(M \otimes A')\,(E) = (e_x + e_y)(M \otimes A') + \alpha_E e_x(M \otimes A')$$

is the $A'$–submodule generated by $M(E) = (e_x + e_y) + \alpha_E e_x M$ in $M \otimes Q$. We get thus an isomorphism $M(E) \otimes A' \xrightarrow{\sim} M(E \otimes A')$ induced by the natural map $M(E) \otimes A' \xrightarrow{\sim} M \otimes Q$. We get next an isomorphism

$$\{(\pi_x(u), -\pi_y(u)) \mid u \in M(E)\} \otimes A' \xrightarrow{\sim} \{(\pi_x(v), -\pi_y(v)) \mid v \in (M \otimes A')(E)\}$$

and finally an isomorphism

$$\mathcal{L}(M)_E \otimes A' \xrightarrow{\sim} \mathcal{L}(M \otimes A')_E$$

compatible with the $\rho$–maps.

*Remark*: Under the identifications $\Gamma(\mathcal{L}(M \otimes A')) \xrightarrow{\sim} \Gamma(\mathcal{L}(M) \otimes A') \xrightarrow{\sim} \Gamma(\mathcal{L}(M)) \otimes A'$ arising from the lemma and from Lemma 2.7.a the homomorphism $g^{M \otimes A'} \colon M \otimes A' \to \Gamma(\mathcal{L}(M \otimes A'))$ is identified with $g^M \otimes \mathrm{id}_{A'}$. Similarly $f^{\mathcal{M} \otimes A'}$ identifies with $f^{\mathcal{M}} \otimes \mathrm{id}_{A'}$ for any sheaf $\mathcal{M}$ on $\mathcal{G}$ such that all $\mathcal{M}_x$ are torsion free over $A$.

**2.19.** (**Graded objects**) In this and the next two sections we work with $A = S$. A *graded sheaf* on $\mathcal{G}$ is a sheaf $\mathcal{M}$ on $\mathcal{G}$ such that all $\mathcal{M}_x$ and all $\mathcal{M}_E$ are graded $S$–modules and such that each $\rho_{x,E}$ is a homomorphism of graded $S$–modules. For example, we can regard the structure sheaf $\mathcal{A}$ as a graded sheaf giving each $\mathcal{A}_x = S$ the same grading as $S$ and giving each $\mathcal{A}_E = S/S\alpha_E$ the induced grading. (Recall that we consider $S$ as a graded algebra with the usual grading doubled.)

If $\mathcal{M}$ is a graded sheaf on $\mathcal{G}$, then each $\mathcal{M}(Y)$ with $Y \subset \mathcal{V} \cup \mathcal{E}$ has an induced grading as an $S$–submodule of $\prod_{x \in Y \cap \mathcal{V}} \mathcal{M}_x \times \prod_{E \in Y \cap \mathcal{E}} \mathcal{M}_E$. And for each closed subset $Z \subset \mathcal{V} \cup \mathcal{E}$ the $S$–module $\Gamma(E, \mathcal{M})$ is a graded submodule of $\prod_{x \in Z \cap \mathcal{V}} \mathcal{M}_x$.

In the case of the structure algebra $\mathcal{A}$ any $\mathcal{A}(Y)$ and any $\Gamma(Z, \mathcal{A})$ is a graded $S$–algebra. For general graded $\mathcal{M}$ any $\mathcal{M}(Y)$ is a graded module over $\mathcal{A}(Y)$, any $\Gamma(Z, \mathcal{M})$ a graded module over $\Gamma(Z, \mathcal{M})$. In particular, $\Gamma(\mathcal{M})$ is a graded $\mathcal{Z}(\mathcal{G})$–module.

A homomorphism of graded sheaves is a homomorphism $f : \mathcal{M} \to \mathcal{N}$ of sheaves such that all $f_x$ and all $f_E$ are homomorphisms of graded $S$–modules. If so, then $\Gamma(f) : \Gamma(\mathcal{M}) \to \Gamma(\mathcal{N})$ is a homomorphisms of graded $\mathcal{Z}$–modules. We can regard $\Gamma$ as a functor from graded sheaves on $\mathcal{G}$ to graded $\mathcal{Z}$–modules.

**Lemma:** *Let $M$ be a graded $\mathcal{Z}$–module that is torsion free as an $S$–module. Then $\mathcal{L}(M)$ has a unique structure as a graded sheaf on $\mathcal{G}$ such that the map $M \to \mathcal{L}(M)_x$, $u \mapsto e_x u$, is a homomorphism of graded $S$–modules for each $x \in \mathcal{V}$.*

*Proof*: The uniqueness is obvious: Since $M \to \mathcal{L}(M)_x$ is surjective for all $x$, our condition determines the grading on each $\mathcal{L}(M)_x$. The same follows for all $\mathcal{L}(M)_E$ because all $\rho_{x,E}$ are surjective.

In order to prove the existence of the grading, we look first at each $\mathcal{L}(M)_x$. We can give $\mathcal{L}(M)_x$ a grading such that $u \mapsto e_x u$ is a homomorphism of graded $S$–modules if and only if the kernel of this map is a graded submodule of $M$.

Set $\beta = \prod_{E \in \mathcal{E}} \alpha_E \in S$. This is a non-zero and homogeneous element in $S$ because each $\alpha_E$ is non-zero and homogeneous. For each $x \in \mathcal{V}$ the element $\widetilde{e}_x := \beta e_x$ is the family in $\mathcal{Z} \otimes Q = \prod_{y \in \mathcal{V}} \mathcal{A}_y \otimes Q$ with $x$–component equal to $\beta$ and all other components equal to 0. The definition of $\mathcal{Z}$ implies that $\widetilde{e}_x \in \mathcal{Z}$. Since $M$ is torsion free over $S$, we get

$$\{ u \in M \mid e_x u = 0 \} = \{ u \in M \mid \widetilde{e}_x u = 0 \}.$$

Here the right side is a graded submodule of $M$ because $\widetilde{e}_x$ is a homogeneous element in $\mathcal{Z}$. So we get our graded structure on each $\mathcal{L}(M)_x$.

Consider now an edge $E$ joining two vertices $x$ and $y$. The same argument as above shows that $(e_x + e_y)M$ has a grading such that $u \mapsto (e_x + e_y)u$ is a graded homomorphism.Then $(e_x + e_y)M$ and $\alpha_E e_x M$ are graded submodules of $\mathcal{L}(M)_x \oplus \mathcal{L}(M)_y$, hence so is $M(E)$. The projection maps $\pi_x$ and $\pi_y$ are compatible with the gradings. So also $\{(\pi_x(v), -\pi_y(v)) \mid v \in M(E)\}$ is a graded submodule. This yields now the grading on $\mathcal{L}(M)_E$ compatible with the $\rho$–maps.

*Remarks*: It now follows that any isomorphism $M(E) \xrightarrow{\sim} \Gamma(\overline{E}, \mathcal{L}(M))$ as in 2.10(7) is compatible with the grading. So is the homomorphism $g^M$ from 2.10(8).

If $\varphi: M \to N$ is a graded homomorphism of graded $\mathcal{Z}$–modules, both torsion free over $S$, then $\mathcal{L}(\varphi)$ is easily checked to be a homomorphism of graded sheaves. So we can regard $\mathcal{L}$ as functor between the graded categories.

Let $\mathcal{M}$ be a graded sheaf on $\mathcal{G}$. Running through the construction one sees that $f^{\mathcal{M}}$ as in Proposition 2.12 is a homomorphism of graded sheaves.

It follows that the isomorphisms in 2.13(2) and 2.13(3) map graded homomorphisms to graded homomorphisms. So $\mathcal{L}$ is also left adjoint to $\Gamma$ when we regard both as functors between the graded versions of the categories $\mathcal{C}$ and $\mathcal{S}$ from 2.13.

**2.20.** (**Krull-Schmidt**) If $M$ and $N$ are graded $S$–modules, then we write $\mathrm{Hom}^0_S(M,N)$ for the space of all homomorphisms $M \to N$ as graded $S$–modules. If $\mathcal{M}$ and $\mathcal{N}$ are graded sheaves on $\mathcal{G}$, then we write $\mathrm{Hom}^0(\mathcal{M},\mathcal{N})$ for the space of all homomorphisms $\mathcal{M} \to \mathcal{N}$ as graded sheaves. We write $\mathrm{End}^0_S(M) = \mathrm{Hom}^0_S(M,M)$ and $\mathrm{End}^0(\mathcal{M}) = \mathrm{Hom}^0(\mathcal{M},\mathcal{M})$.

If $M$ is a finitely generated graded $S$–module, then all homogeneous components $M_i$, $i \in \mathbf{Z}$, are finite dimensional. This implies: If $M$ and $N$ are finitely generated graded $S$–modules, then $\mathrm{Hom}^0_S(M,N)$ is finite dimensional. Indeed, choose generators $v_1, v_2, \dots, v_r$ for $M$. We may assume that each $v_i$ is homogeneous, i.e., that $v_i \in M_{s(i)}$ for suitable $s(i) \in \mathbf{Z}$. We get now an injective $k$–linear map from $\mathrm{Hom}^0_S(M,N)$ into the direct sum of all $N_{s(i)}$, $1 \le i \le r$, mapping any homomorphism $\varphi$ to the $r$–tuple of all $\varphi(v_i)$.

Recall that a sheaf $\mathcal{M}$ on $\mathcal{G}$ has finite type if all $\mathcal{M}_x$ and all $\mathcal{M}_E$ are finitely generated graded $S$–modules. Using the fact that $\mathcal{V} \cup \mathcal{E}$ is finite, we get now: If $\mathcal{M}$ and $\mathcal{N}$ are graded sheaves of finite type on $\mathcal{G}$, then $\mathrm{Hom}^0(\mathcal{M},\mathcal{N})$ is finite dimensional.

Let $M$ be a finitely generated graded $S$–module. Consider $\varphi \in \mathrm{End}^0_S(M)$. We claim that there exists an integer $m > 0$ such that $\varphi^m(M) = \varphi^n(M)$ for all $n \ge m$ and that $M = \ker \varphi^m \oplus \varphi^m(M)$ for any such $m$. Indeed: For any homogeneous component $M_i$ of $M$ the chain of subspaces $\varphi(M_i) \supset \varphi^2(M_i) \supset \varphi^3(M_i) \supset \cdots$ stabilises since $\dim M_i < \infty$. So there exists an integer $m(i) > 0$ such that $\varphi^{m(i)}(M_i) = \varphi^n(M_i)$ for all $n \ge m(i)$. There exist integers $r \le s$ such that $M = \sum_{i=r}^s S\, M_i$. Set $m$ equal to the maximum of all $m(i)$ with $r \le i \le s$. We get then $\varphi^m(M) = \sum_{i=r}^s S\, \varphi^m(M_i) = \sum_{i=r}^s S\, \varphi^n(M_i) = \varphi^n(M)$ for all $n \ge m$. On the other hand suppose that we have $\varphi^m(M) = \varphi^n(M)$ for all $n \ge m$. We have then $\varphi^m(M_i) = \varphi^n(M_i)$ for all $i$ and all $n \ge m$. Therefore the restriction of $\varphi^m$ is a surjective linear map $\varphi^m(M_i) \to \varphi^{2m}(M_i) = \varphi^m(M_i)$. So the kernel of $\varphi^m$ intersects any $\varphi^m(M_i)$ in 0. This implies by dimension reasons that $M_i = (\ker \varphi^m \cap M_i) \oplus \varphi^m(M_i)$. Taking the sum over all $i$ we get $M = \ker \varphi^m \oplus \varphi^m(M)$.

If $M$ is indecomposable as a graded $S$–module, then we get that any graded endomorphism of $M$ is either nilpotent or bijective. So $\mathrm{End}^0_S(M)$ is a local ring. It follows that the Krull-Schmidt theorem holds for the decomposition of finitely generated graded $S$–modules into indecomposables.

Let $\mathcal{M}$ be a graded sheaf of finite type on $\mathcal{G}$. Given $\varphi \in \mathrm{End}^0(\mathcal{M})$ we can find an integer $m > 0$ such that $\mathcal{M}_x = \ker(\varphi_x)^m \oplus \varphi_x^m(\mathcal{M}_x)$ for all $x \in \mathcal{V}$ and $\mathcal{M}_E = \ker(\varphi_E)^m \oplus \varphi_E^m(\mathcal{M}_E)$ for all $E \in \mathcal{E}$. (Here we use that $\mathcal{V} \cup \mathcal{E}$ is finite.) This implies that $\mathcal{M} = \ker \varphi^m \oplus \varphi^m(\mathcal{M})$.

This implies: If $\mathcal{M}$ is indecomposable as a graded sheaf on $\mathcal{G}$, then any graded endomorphism of $\mathcal{M}$ is either nilpotent or bijective. So $\mathrm{End}^0(\mathcal{M})$ is a local ring. It follows that the Krull-Schmidt theorem holds for the decomposition of graded sheaves of finite type on $\mathcal{G}$ into indecomposables.

**2.21.** (**Projective covers**) In this subsection let $\mathcal{C}$ denote the category of all finitely generated graded $S$–modules. It is clear that $S$ and any $S\langle r\rangle$ with $r \in \mathbf{Z}$ is a projective object in $\mathcal{C}$. Any $M$ in $\mathcal{C}$ is a homomorphic image of a finite direct sum of modules of the form $S\langle r_i\rangle$, hence of a projective object in $\mathcal{C}$.

A *projective cover* of an object $M$ in $\mathcal{C}$ is a pair $(P, \pi)$ where $P$ is a projective object in $\mathcal{C}$ and $\pi\colon P \to M$ a surjective homomorphism in $\mathcal{C}$ such that $\pi(N) \neq M$ for any *graded* submodule $N$ of $P$ with $N \neq P$.

For example, the natural map $S \to k$ with kernel $\mathfrak{m} = \bigoplus_{i>0} S_i$ is a projective cover in $\mathcal{C}$ because any graded submodule $N$ of $S$, $N \neq S$ is contained in $\mathfrak{m}$.

If $(P, \pi)$ is a projective cover of some $M$ in $\mathcal{C}$, then any $(P\langle r\rangle, \pi)$ with $r \in \mathbf{Z}$ is a projective cover of $M\langle r\rangle$.

For any $M$ in $\mathcal{C}$ denote by $\operatorname{rad}_{\mathcal{C}} M$ the intersection of all maximal *graded* submodules of $M$. For example, we have $\operatorname{rad}_{\mathcal{C}} S = \mathfrak{m}$. Any proper graded submodule of $M$ is contained in a maximal graded submodule of $M$. This implies: If $N$ is a graded submodule of $M$ with $M = N + \operatorname{rad}_{\mathcal{C}} M$, then $M = N$.

**Lemma:** (a) *We have* $\operatorname{rad}_{\mathcal{C}} M = \mathfrak{m}M$ *for any* $M$ *in* $\mathcal{C}$.

(b) *Let* $\pi\colon P \to M$ *be a homomorphism in* $\mathcal{C}$ *such that* $P$ *is projective. Then* $(P, \pi)$ *is a projective cover of* $M$ *in* $\mathcal{C}$ *if and only if the induced map*

$$\overline{\pi}\colon P/\mathfrak{m}P \longrightarrow M/\mathfrak{m}M$$

*is an isomorphism.*

*Proof*: (a) We have $\mathfrak{m}M \neq M$ for any non-zero $M$ in $\mathcal{C}$. (If $r$ is minimal for $M_r \neq 0$, then $M_r \cap \mathfrak{m}M = 0$.) This implies: If $M$ in $\mathcal{C}$ is simple, then $\mathfrak{m}M = 0$. So $M$ is a graded module over $S/\mathfrak{m}S$, i.e., a graded vector space over $k$. Now the simplicity means that $\dim_k M = 1$. So the simple objects in $\mathcal{C}$ are all $k\langle r\rangle$ with $r \in \mathbf{Z}$.

If $N$ is a maximal graded submodule of some $M$ in $\mathcal{C}$, then $M/N$ is simple in $\mathcal{C}$, hence $\mathfrak{m}(M/N) = 0$ and $N \supset \mathfrak{m}M$. This yields the inclusion "$\supset$" in (a). On the other hand, $M/\mathfrak{m}M$ is an $S/\mathfrak{m}S$–module, hence a semisimple module. This yields the other inclusion in (a).

(b) If $(P, \pi)$ is a projective cover, then $\pi$ is surjective, hence so is $\overline{\pi}$. On the other hand, if $\overline{\pi}$ is surjective, then $M = \pi(P) + \mathfrak{m}M = \pi(P) + \operatorname{rad}_{\mathcal{C}} M$, hence $\pi(P) = M$ and $\pi$ is surjective.

Suppose now that $\pi$ is surjective. The usual arguments show now that $(P, \pi)$ is a projective cover if and only if $\ker \pi \subset \operatorname{rad}_{\mathcal{C}} P$. (Assume that $\pi$ is surjective. Any proper graded submodule of $P$ is contained in a maximal graded submodule of $P$. This shows that $(P, \pi)$ is a projective cover if and only if $\pi(N) \neq M$ for any maximal graded submodule $N$ of $P$. This is then equivalent to $N + \ker \pi \neq P$, hence by the maximality of $N$ to $\ker \pi \subset N$.)

So we have to show for surjective $\pi$ that $\overline{\pi}$ is injective if and only if $\ker \pi \subset \operatorname{rad}_{\mathcal{C}} P$. Well, assume first that $\overline{\pi}$ is injective. Then any $x \in \ker \pi$ satisfies $x + \mathfrak{m}P \in \ker \overline{\pi}$, hence $x \in \mathfrak{m}P$; this yields $\ker \pi \subset \operatorname{rad}_{\mathcal{C}} P$ by (a).

Suppose on the other hand that $\ker \pi \subset \operatorname{rad}_{\mathcal{C}} P$. Consider $x \in P$ with $\overline{\pi}(x + \mathfrak{m}P) = 0$. We get then $\pi(x) \in \mathfrak{m}M$. Since $\pi$ is surjective, we have $\mathfrak{m}M = \mathfrak{m}\pi(P) = \pi(\mathfrak{m}P)$, and get thus $y \in \mathfrak{m}P$ with $\pi(x) = \pi(y)$. Now $x - y \in \ker \pi$ implies $x - y \in \operatorname{rad}_{\mathcal{C}} P = \mathfrak{m}P$. As already $y \in \mathfrak{m}P$, we get $x + \mathfrak{m}P = 0$.

*Remarks*: 1) It is now clear that projective covers exist and how to construct them: Given $M$ in $\mathcal{C}$ we can find integers $n_1, n_2, \dots, n_r$ for some $r \geq 0$ such that $M/\mathfrak{m}M$ is isomorphic to the direct sum of all $k\langle n_i \rangle$. Fix a surjective homomorphism $f\colon M \to \bigoplus_{i=1}^r k\langle n_i \rangle$ such that $\ker f = \mathfrak{m}M$. We have a natural surjective homomorphism $g$ from the direct sum $P = \bigoplus_{i=1}^r S\langle n_i \rangle$ onto $\bigoplus_{i=1}^r k\langle n_i \rangle$ with $\ker g = \mathfrak{m}P = \operatorname{rad}_{\mathcal{C}} P$. Since $P$ is projective, there exists a homomorphism $\pi\colon P \to M$ such that $g = f \circ \pi$.Then $\overline{\pi}$ is an isomorphism by construction.

2) If $(P, \pi)$ and $(P', \pi')$ are projective covers of some $M$ in $\mathcal{C}$, then there exists an isomorphism $\gamma\colon P \xrightarrow{\sim} P'$ with $\pi' \circ \gamma = \pi$.

## 3 More sheaves on moment graphs

**3.1.** (**Moment graphs**) An (ordered) moment graph is a quadruple $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \alpha, \leq)$ where $(\mathcal{V}, \mathcal{E}, \alpha)$ is an unordered moment graph as in 2.1 and where $\leq$ is a partial ordering on $\mathcal{V}$ such that for any edge $E \in \mathcal{E}$ the two vertices joined by $E$ are comparable with respect to $\leq$. From now on we denote these two vertices by $a_E$ and $z_E$ such that $a_E < z_E$.

*Example*: In the set-up of 1.15 (which is a special case of the set-up of 1.12) we get such an (ordered) moment graph. As in 2.1 the vertices are the fixed points of $T$ on $X$ and the edges are in bijection with the one dimensional $T$–orbits on $X$. We now define a partial ordering $\leq$ on the set of fixed points setting $x \leq y$ if and only if the corresponding strata satisfy $C_x \subset \overline{C_y}$. If an edge $E$ corresponds to a one dimensional $T$–orbit $P$, then there exists a stratum $C_x$ with $P \subset C_x$. Then $x$ is one of the fixed points in the closure of $P$. Since $\overline{P} \subset \overline{C_x}$ the other fixed point $y$ in the closure of $P$ satisfies $y \in \overline{C_x}$, hence $C_y \subset \overline{C_x}$ and $y \leq x$. Because $x$ and $y$ are the two vertices joined by $E$, we see that the condition on $\leq$ is satisfied. (We could alternatively have defined $x \leq y$ if and only if $C_y \subset \overline{C_x}$.)

A *full subgraph* of a moment graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \alpha, \leq)$ is a moment graph $\mathcal{H} = (\mathcal{V}', \mathcal{E}', \alpha', \leq')$ such that $\mathcal{V}' \subset \mathcal{V}$ and $\mathcal{E}' \subset \mathcal{E}$, such that $\alpha'$ (resp. $\leq'$) is the restriction of $\alpha$ (resp. of $\leq$) to $\mathcal{E}'$ (resp. to $\mathcal{V}'$) and such that for all $x, y \in \mathcal{V}'$ any edge $E \in \mathcal{E}$ joining $x$ and $y$ actually belongs to $\mathcal{E}'$. (The inclusion $\mathcal{E}' \subset \mathcal{E}$ should be interpreted as follows: If $E \in \mathcal{E}'$ joins $x$ and $y$ in $\mathcal{H}$, then it joins the same vertices in $\mathcal{G}$.)

It is clear that a full subgraph $\mathcal{H} = (\mathcal{V}', \mathcal{E}', \alpha', \leq')$ of $\mathcal{G}$ is completely determined by the subset $\mathcal{V}'$ of $\mathcal{V}$. For any $x \in \mathcal{V}$ denote by $\mathcal{G}_{<x}$ the full subgraph with $\mathcal{V}' = \{y \in \mathcal{V} \mid y < x\}$ and by $\mathcal{G}_{\leq x}$ the full subgraph with $\mathcal{V}' = \{y \in \mathcal{V} \mid y \leq x\}$.

If $\mathcal{H} = (\mathcal{V}', \mathcal{E}', \alpha', \leq')$ is a full subgraph of $\mathcal{G}$, then $\mathcal{V}' \cup \mathcal{E}'$ is a closed subset of $\mathcal{V} \cup \mathcal{E}$ in the sense of 2.4. We call now $\mathcal{H}$ an *F-open subgraph* if it has the following property: If $x \in \mathcal{V}'$ and $y \in \mathcal{V}$ with $y \leq x$, then also $y \in \mathcal{V}'$. In other words: We require that $\mathcal{H}$ is the union of all $\mathcal{G}_{\leq x}$ with $x \in \mathcal{V}'$. Clearly all $\mathcal{G}_{<x}$ and all $\mathcal{G}_{\leq x}$ are F-open. (This is of course the usual topology on the partially ordered set $\mathcal{V}$. I want to use a slightly different terminology in order to avoid confusion with the topology from 2.4.)

**3.2.** (**Flabby Sheaves**) Fix a moment graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \alpha, \leq)$ and an $S$–algebra $A$ that is the localisation of $S$ with respect to a multiplicative subset; we say sheaf instead of $A$–sheaf.

If $\mathcal{M}$ is a sheaf on $\mathcal{G}$, then set

$$\Gamma(\mathcal{H}, \mathcal{M}) = \Gamma(\mathcal{V}', \mathcal{M}) \tag{1}$$

for any full subgraph $\mathcal{H} = (\mathcal{V}', \mathcal{E}', \alpha', \leq')$ of $\mathcal{G}$.

A sheaf $\mathcal{M}$ on $\mathcal{G}$ is called *flabby* if for each F-open subgraph $\mathcal{H}$ of $\mathcal{G}$ the restriction map

$$\Gamma(\mathcal{M}) \longrightarrow \Gamma(\mathcal{H}, \mathcal{M}) \tag{2}$$

is surjective. If so, then the restriction map

$$\Gamma(\mathcal{G}_1, \mathcal{M}) \longrightarrow \Gamma(\mathcal{G}_2, \mathcal{M}) \tag{3}$$

is surjective for any two F-open subgraphs $\mathcal{G}_1$ and $\mathcal{G}_2$ with $\mathcal{G}_2 \subset \mathcal{G}_1$.

**Lemma:** *A sheaf $\mathcal{M}$ on $\mathcal{G}$ is flabby if and only if for each $x \in \mathcal{V}$ the restriction map*

$$\Gamma(\mathcal{G}_{\leq x}, \mathcal{M}) \longrightarrow \Gamma(\mathcal{G}_{<x}, \mathcal{M}) \tag{4}$$

*is surjective.*

*Proof*: If $\mathcal{M}$ is flabby, then the surjectivity in (4) is a special case of the surjectivity in (3). One direction in the lemma is therefore obvious. So let us assume that the map in (4) is surjective for all $x$.

We want to show that $\mathcal{M}$ is flabby. Working inductively, we see that is enough to check that the map in (3) is surjective whenever $\mathcal{G}_1$ contains one vertex more than $\mathcal{G}_2$. In this case denote by $\mathcal{V}_i$ the set of vertices of $\mathcal{G}_i$ for $i = 1, 2$ and let $x \in \mathcal{V}_1$ denote the vertex with $\mathcal{V}_2 = \mathcal{V}_1 \setminus \{x\}$.

Consider a family $u = (u_y)_{y \in \mathcal{V}_2}$ in $\Gamma(\mathcal{G}_2, \mathcal{M}) = \Gamma(\mathcal{V}_2, \mathcal{M})$. Then the family $(u_y)_{y<x}$ belongs to $\Gamma(\mathcal{G}_{<x}, \mathcal{M})$. By assumption there exists $u_x \in \mathcal{M}_x$ such that the family $(u_y)_{y \leq x}$ belongs to $\Gamma(\mathcal{G}_{\leq x}, \mathcal{M})$. We claim that the family $u' = (u_y)_{y \in \mathcal{V}_1}$ belongs to $\Gamma(\mathcal{G}_1, \mathcal{M})$. This will yield the desired surjectivity since $u'$ restricts to $u$.

We have to check that

$$\rho_{a_E, E}(u_{a_E}) = \rho_{z_E, E}(u_{z_E}) \tag{5}$$

for any edge $E$ of $\mathcal{G}_1$. In case $E$ belongs to $\mathcal{G}_2$ this equality follows from $u \in \Gamma(\mathcal{G}_2, \mathcal{M})$. So suppose that $E$ does not belong to $\mathcal{G}_2$. Then $x$ has to be one of the vertices joined by $E$. We cannot have $x = a_E$ because otherwise $x < z_E$ contradicts the assumption that $\mathcal{G}_2$ is F-open since $z_E \in \mathcal{V}_2$ and $x \notin \mathcal{V}_2$. So we have $x = z_E$; now $a_E < x$ implies that $E$ belongs to $\mathcal{G}_{\leq x}$. Therefore in this case (5) follows from $(u_y)_{y \leq x} \in \Gamma(\mathcal{G}_{\leq x}, \mathcal{M})$.

*Remark*: A direct sum of two sheaves is flabby if and only if both summands are flabby.

**3.3.** For any $x \in \mathcal{V}$ set $D_x$ equal to the set of all edges $E$ with $x = z_E$ and $U_x$ equal to the set of all edges $E$ with $x = a_E$. (The letters $D$ and $U$ stand for *down* and *up*.)

Let $\mathcal{M}$ be a sheaf on $\mathcal{G}$. Consider for each $x \in \mathcal{V}$ the homomorphism of $A$–modules

$$\rho_x^D \colon \Gamma(\mathcal{G}_{<x}, \mathcal{M}) \longrightarrow \bigoplus_{E \in D_x} \mathcal{M}_E \tag{1}$$

that maps any family $(u_y)_{y<x}$ in $\Gamma(\mathcal{G}_{<x}, \mathcal{M})$ to the family of all $\rho_{a_E, E}(u_{a_E})$ with $E \in D_x$. Denote the image of this map by

$$\mathcal{M}_{\partial x} \subset \bigoplus_{E \in D_x} \mathcal{M}_E. \tag{2}$$

Consider also the homomorphism of $A$–modules

$$\rho_{x,D} = \rho_{x,D}^{\mathcal{M}} \colon \mathcal{M}_x \longrightarrow \bigoplus_{E \in D_x} \mathcal{M}_E, \qquad u \mapsto (\rho_{x,E}(u))_{E \in D_x}. \tag{3}$$

**Lemma:** (a) *The restriction map* $\Gamma(\mathcal{G}_{\le x}, \mathcal{M}) \longrightarrow \Gamma(\mathcal{G}_{<x}, \mathcal{M})$ *is surjective if and only if* $\mathcal{M}_{\partial x} \subset \rho_{x,D}(\mathcal{M}_x)$.

(b) *The restriction map* $\Gamma(\mathcal{G}_{\le x}, \mathcal{M}) \longrightarrow \mathcal{M}_x$ *is surjective if and only* $\rho_{x,D}(\mathcal{M}_x) \subset \mathcal{M}_{\partial x}$.

(c) *We have* $\rho_{x,D}(\mathcal{M}_x) = \mathcal{M}_{\partial x}$ *for all* $x \in \mathcal{V}$ *if and only if* $\mathcal{M}$ *is flabby and generated by global sections.*

*Proof*: Note that $\Gamma(\mathcal{G}_{\le x}, \mathcal{M})$ consists of all families $(u_y)_{y\le x}$ in $\prod_{y\le x} \mathcal{M}_y$ such that $(u_y)_{y<x}$ belongs to $\Gamma(\mathcal{G}_{<x}, \mathcal{M})$ and such that

$$\rho_{x,D}(u_x) = \rho_x^D((u_y)_{y<x}).$$

This yields easily the claims in (a) and (b).

If $\mathcal{M}$ is flabby, then $\mathcal{M}$ is generated by global sections if and only if the restriction map $\Gamma(\mathcal{G}_{\le x}, \mathcal{M}) \longrightarrow \mathcal{M}_x$ is surjective for all $x \in \mathcal{V}$. Therefore (c) follows from (a), (b), and Lemma 3.2.

**3.4. Lemma:** *Let* $\mathcal{M}$ *be a sheaf on* $\mathcal{G}$ *such that each* $\mathcal{M}_x$ *with* $x \in \mathcal{V}$ *is torsion free over* $A$. *Suppose that* $\mathcal{M}$ *is generated by global sections and that* $\rho_{x,E}$ *induces an isomorphism*

$$\mathcal{M}_x/\alpha_E\,\mathcal{M}_x \xrightarrow{\sim} \mathcal{M}_E \tag{1}$$

*for each* $x \in \mathcal{V}$ *and each* $E \in U_x$. *Then* $f^{\mathcal{M}}\colon \mathcal{L}(\Gamma(\mathcal{M})) \to \mathcal{M}$ *is an isomorphism.*

*Proof*: We see as in the proof of Proposition 2.12 that $f_x^{\mathcal{M}}$ is an isomorphism $\mathcal{L}(\Gamma(\mathcal{M}))_x \to \mathcal{M}_x$ for each $x \in \mathcal{V}$. (In fact, it is the identity.)

Consider now an edge $E$ and set $x = a_E$. We have a commutative diagram

$$\begin{tikzcd}
\mathcal{L}(\Gamma(\mathcal{M}))_x \arrow[r, "f_x^{\mathcal{M}}"] \arrow[d, "\rho_{x,E}^{\mathcal{L}(\Gamma(\mathcal{M}))}"'] & \mathcal{M}_x \arrow[d, "\rho_{x,E}^{\mathcal{M}}"] \\
\mathcal{L}(\Gamma(\mathcal{M}))_E \arrow[r, "f_E^{\mathcal{M}}"] & \mathcal{M}_E
\end{tikzcd}$$

The assumption (1) says that $\rho_{x,E}^{\mathcal{M}}$ is surjective and has kernel $\alpha_E\,\mathcal{M}_E$. Since $f_x^{\mathcal{M}}$ is bijective, also $f_E^{\mathcal{M}} \circ \rho_{x,E}^{\mathcal{L}(\Gamma(\mathcal{M}))}$ is surjective and it has kernel $\alpha_E\,\mathcal{L}(\Gamma(\mathcal{M}))_x$. It follows that also $f_E^{\mathcal{M}}$ is surjective. Furthermore, since $\rho_{x,E}^{\mathcal{L}(\Gamma(\mathcal{M}))}$ is surjective (cf. 2.12), we get that the kernel of $f_E^{\mathcal{M}}$ is the image under $\rho_{x,E}^{\mathcal{L}(\Gamma(\mathcal{M}))}$ of the kernel of $f_E^{\mathcal{M}} \circ \rho_{x,E}^{\mathcal{L}(\Gamma(\mathcal{M}))}$, hence equal to

$$\rho_{x,E}^{\mathcal{L}(\Gamma(\mathcal{M}))}(\alpha_E\,\mathcal{L}(\Gamma(\mathcal{M}))_x) = \alpha_E\,\rho_{x,E}^{\mathcal{L}(\Gamma(\mathcal{M}))}(\mathcal{L}(\Gamma(\mathcal{M}))_x) = \alpha_E\,\mathcal{L}(\Gamma(\mathcal{M}))_E = 0.$$

So $f_E^{\mathcal{M}}$ is also injective, hence an isomorphism.

**3.5.** (**Braden-MacPherson sheaves**) In the next two subsections we are going to work with $A = S$. We are now going to construct for each vertex a graded sheaf $\mathcal{B}(z)$ of finite type on $\mathcal{G}$, the *Braden-MacPherson sheaf* associated to $z$. This sheaf will have the following properties:

$$\text{We have } \mathcal{B}(z)_z = S \quad \text{(with the usual grading).} \tag{1}$$

$$\text{If } x \in \mathcal{V} \text{ with } x \not\geq z, \text{ then } \mathcal{B}(z)_x = 0. \tag{2}$$

$$\text{Each } \mathcal{B}(z)_x \text{ is a free } S\text{–module of finite rank.} \tag{3}$$

$$\rho_{x,E} \text{ induces an isomorphism } \mathcal{B}(z)_x/\alpha_E \mathcal{B}(z)_x \xrightarrow{\sim} \mathcal{B}(z)_E \text{ for any } x \in \mathcal{V} \text{ and } E \in U_x. \tag{4}$$

We construct inductively the restriction of $\mathcal{B}(z)$ to each $\mathcal{G}_{\leq x}$ assuming that we know already the restriction of $\mathcal{B}(z)$ to all $\mathcal{G}_{\leq y}$ with $y < x$. This assumption implies of course that we already know the restriction of $\mathcal{B}(z)$ to $\mathcal{G}_{<x}$.

The edges in $\mathcal{G}_{\leq x}$ that do not belong to $\mathcal{G}_{<x}$ are the edges $E$ in $D_x$. For these we have $a_E < x = z_E$, so $\mathcal{B}(z)_{a_E}$ is already known. We can thus define

$$\mathcal{B}(z)_E = \mathcal{B}(z)_{a_E}/\alpha_E \mathcal{B}(z)_{a_E}$$

with the obvious grading and set $\rho_{a_E,E}$ equal to the canonical map to the factor module. This part of the construction ensures that (4) is satisfied for all $E \in D_x$ (with $a_E$ instead of $x$).

If $x = z$, then we use (1) to define $\mathcal{B}(z)_x$ and we set $\rho_{z,E} = 0$ for all $E \in D_z$. Suppose next that $x \neq z$. Since we know the restriction of $\mathcal{B}(z)$ to $\mathcal{G}_{<x}$, we know also $\Gamma(\mathcal{G}_{<x}, \mathcal{B}(z))$. Using the first step of the inductive construction we also know the map $\rho_x^D$ as in 3.3(1), hence its image $\mathcal{B}(z)_{\partial x}$. Using (3) for the vertices $y < x$, we see that $\mathcal{B}(z)_{\partial x}$ is a finitely generated graded $S$–module. We now choose $\mathcal{B}(z)_x$ as a projective cover of $\mathcal{B}(z)_{\partial x}$ in the category of graded $S$–modules. Then $\mathcal{B}(z)_x$ is a free $S$–module of finite rank and it comes with a surjective homomorphism $\pi_x : \mathcal{B}(z)_x \to \mathcal{B}(z)_{\partial x}$. Recall that $\mathcal{B}(z)_{\partial x}$ is a submodule of $\bigoplus_{E \in D_x} \mathcal{B}(z)_E$. We finally define any $\rho_{x,E}$ with $E \in D_x$ as the composition of $\pi_x$ with the projection $\mathcal{B}(z)_{\partial x} \to \mathcal{B}(z)_E$.

Note: If $x \not\geq z$, then $y \not\geq z$ for all $y < x$. Then we have by induction $\mathcal{B}(z)_y = 0$ for all these $y$, hence $\Gamma(\mathcal{G}_{<x}, \mathcal{B}(z)) = 0$ and then $\mathcal{B}(z)_{\partial x} = 0$ and finally $\mathcal{B}(z)_x = 0$. So (2) holds also for $x$.

We have by construction

$$\rho_{x,D}(\mathcal{B}(z)_x) = \mathcal{B}(z)_{\partial x} \tag{5}$$

for all $x \neq z$. But this equation holds also for $x = z$, where $\mathcal{B}(z)_{\partial z} = 0$ by (2) and where $\rho_{x,D} = 0$ by construction.

**Proposition:** (a) *The sheaf $\mathcal{B}(z)$ is flabby and generated by global sections.*

(b) *The map $f^{\mathcal{B}(z)} : \mathcal{L}(\Gamma(\mathcal{B}(z))) \longrightarrow \mathcal{B}(z)$ is an isomorphism.*

(c) *For each closed subset $Z \subset \mathcal{V} \cup \mathcal{E}$ the $S$–module $\Gamma(Z, \mathcal{B}(z))$ is finitely generated and reflexive.*

(d) *The sheaf $\mathcal{B}(z)$ is indecomposable. So is the $\mathcal{Z}$–module $\Gamma(\mathcal{B}(z))$.*

*Proof*: Here (a) follows from (5) and Lemma 3.3.c whereas (b) follows from (4) and Lemma 3.4. We get (c) from Lemma 2.17 and its remarks. In (d) the second claim follows from the first one using (b).

It remains to prove that $\mathcal{B}(z)$ is indecomposable. Suppose that $\mathcal{B}(z) = \mathcal{M} \oplus \mathcal{N}$ for some sheaves $\mathcal{M}$ and $\mathcal{N}$ on $\mathcal{G}$. We have $S = \mathcal{B}(z)_z = \mathcal{M}_z \oplus \mathcal{N}_z$. Because $S$ is an integral domain, it is indecomposable as an $S$–module. So we may assume that $\mathcal{M}_z = S$ and $\mathcal{N}_z = 0$. We want to show that $\mathcal{N} = 0$ and $\mathcal{M} = \mathcal{B}(z)$.

It suffices to show inductively for all $x \in \mathcal{V}$ that $\mathcal{N}_x = 0$ and $\mathcal{N}_E = 0$ for all $E \in D_x$. This is obvious for $x = z$ as $\mathcal{B}(z)_E = 0$ for all $E \in D_z$. Let now $x \neq z$ and suppose that we know our claim already for all $y < x$.

If $E \in D_x$, then $y = a_E$ satisfies $y < x$, hence

$$\mathcal{B}(z)_E = \rho_{y,E}(\mathcal{B}(z)_y) = \rho_{y,E}(\mathcal{M}_y) \subset \mathcal{M}_E.$$

It follows that $\mathcal{M}_E = \mathcal{B}(z)_E$ and $\mathcal{N}_E = 0$.

Furthermore, we have $\Gamma(\mathcal{G}_{<x}, \mathcal{M}) = \Gamma(\mathcal{G}_{<x}, \mathcal{B}(z))$ and conclude that $\mathcal{B}(z)_{\partial x} = \mathcal{M}_{\partial x}$, hence $\mathcal{N}_{\partial x} = 0$. This implies that $\rho_{x,D}(\mathcal{N}_x) = 0$. Since $\rho_{x,D}$ induces an isomorphism $\mathcal{B}(z)_x/\mathfrak{m}\mathcal{B}(z)_x \xrightarrow{\sim} \mathcal{B}(z)_{\partial x}/\mathfrak{m}\mathcal{B}(z)_{\partial x}$ we get now $\mathcal{N}_x/\mathfrak{m}\mathcal{N}_x = 0$. Because $\mathcal{N}_x$ is a finitely generated projective $S$–module, this implies $\mathcal{N}_x = 0$.

**3.6.** (**Sheaves and intersection cohomology**) Let us return for the moment to the set-up from 1.15. So we consider a complex projective variety $X$ with an algebraic action of a complex torus $T$ such that the set $X^T$ of fixed points and the set of one dimensional orbits are finite. The fixed points are supposed to be contracting and there is a Whitney stratification $X = \bigcup_{x \in X^T} C_x$ with certain properties.

As described in 2.1 and 3.1 we get in this situation a moment graph $\mathcal{G}$ with $\mathcal{V} = X^T$ and with $\mathcal{E}$ equal to the set of one dimensional $T$–orbits on $X$. The order relation we consider now is the one where $x \leq y$ if and only if $C_y \subset \overline{C_x}$. Note that we have

$$H_T^\bullet(X) \simeq \mathcal{Z}(\mathcal{G}) \tag{1}$$

by Theorem 1.12. The main result in [BM] says now:

**Theorem:** *We have for each $x \in X^T$ isomorphisms*

$$IH_T^\bullet(\overline{C}_x) \simeq \Gamma(\mathcal{B}(x)) \tag{2}$$

*and*

$$IH_T^\bullet(\overline{C}_x)_{\{y\}} \simeq \mathcal{B}(x)_y \qquad \textit{and} \qquad IH_T^\bullet(\overline{C}_x)_E \simeq \mathcal{B}(x)_E \tag{3}$$

*for all $y \in X^T$ and $E \in \mathcal{E}$. Regarding $\mathbf{C}$ as an $S$–algebra isomorphic to $S/\mathfrak{m}$ we have further isomorphisms*

$$IH_T^\bullet(\overline{C}_x) \otimes_S \mathbf{C} \simeq IH^\bullet(\overline{C}_x) \tag{4}$$

*and*

$$IH_T^\bullet(\overline{C}_x)_{\{y\}} \otimes_S \mathbf{C} \simeq IH^\bullet(\overline{C}_x)_{\{y\}} \qquad \textit{and} \qquad IH_T^\bullet(\overline{C}_x)_E \otimes_S \mathbf{C} \simeq IH^\bullet(\overline{C}_x)_E \tag{5}$$

*for all $y \in X^T$ and $E \in \mathcal{E}$.*

In fact, the maps in (4) and (5) come from maps as in 1.14(6).

**3.7.** (**Truncation**) Return to more general $A$ as in 3.2. Recall from 2.6 the definition of the truncated sheaf $\mathcal{M}[Z]$ for any subset $Z \subset \mathcal{V} \cup \mathcal{E}$ and any sheaf $\mathcal{M}$ on $\mathcal{G}$. If $\mathcal{H} = (\mathcal{V}', \mathcal{E}', \alpha', \leq')$ is a full subgraph of $\mathcal{G}$, then we now write $\mathcal{M}[\mathcal{H}] = \mathcal{M}[\mathcal{V}' \cup \mathcal{E}']$. Since $\mathcal{V}' \cup \mathcal{E}'$ is closed in $\mathcal{V} \cup \mathcal{E}$, we have by 2.6(3) a canonical isomorphism

$$\Gamma(\mathcal{M}[\mathcal{H}]) \xrightarrow{\sim} \Gamma(\mathcal{H}, \mathcal{M}). \tag{1}$$

If $\mathcal{M}$ is generated by global sections, then so is $\mathcal{M}[\mathcal{H}]$. If $\mathcal{M}$ is flabby and if $\mathcal{H}$ is F-open, then also $\mathcal{M}[\mathcal{H}]$ is flabby.

We have by 2.6(1) a natural morphism

$$\pi^{\mathcal{M}}[\mathcal{H}] : \mathcal{M} \longrightarrow \mathcal{M}[\mathcal{H}]. \tag{2}$$

By 2.6(4) it has the following universal property: For any sheaf $\mathcal{N}$ on $\mathcal{G}$ the map

$$\operatorname{Hom}(\mathcal{M}[\mathcal{H}], \mathcal{N}) \longrightarrow \operatorname{Hom}(\mathcal{M}, \mathcal{N}), \qquad \psi \mapsto \psi \circ \pi^{\mathcal{M}}[\mathcal{H}] \tag{3}$$

is injective; its image consists of all morphisms $\varphi : \mathcal{M} \to \mathcal{N}$ with $\varphi_x = 0$ for all vertices $x \notin \mathcal{V}'$ and with $\varphi_E = 0$ for all edges $E \notin \mathcal{E}'$. We get in particular that $\psi \mapsto \psi \circ \pi^{\mathcal{M}}[\mathcal{H}]$ is an isomorphism

$$\operatorname{Hom}(\mathcal{M}[\mathcal{H}], \mathcal{N}[\mathcal{H}]) \xrightarrow{\sim} \operatorname{Hom}(\mathcal{M}, \mathcal{N}[\mathcal{H}]). \tag{4}$$

More generally, if $\mathcal{G}_1$ and $\mathcal{G}_2$ are full subgraphs of $\mathcal{G}$ with $\mathcal{G}_2$ contained in $\mathcal{G}_1$, then we have a natural restriction map $\mathcal{M}[\mathcal{G}_1] \to \mathcal{M}[\mathcal{G}_2]$. Since $\mathcal{M}[\mathcal{G}_2] = (\mathcal{M}[\mathcal{G}_1])[\mathcal{G}_2]$, this reduces to the case already considered.

If $\mathcal{M}$ is a graded sheaf, then $\mathcal{M}[\mathcal{H}]$ has a natural grading such that $\pi^{\mathcal{M}}[\mathcal{H}]$ is a morphism of graded sheaves. Then $(\mathcal{M}[\mathcal{H}], \pi^{\mathcal{M}}[\mathcal{H}])$ has a corresponding universal property for morphisms of graded sheaves. We can replace Hom in (3) and (4) by $\operatorname{Hom}^0$.

Recall that $\mathcal{M} \mapsto \mathcal{M}[\mathcal{H}]$ is a functor. If $f : \mathcal{M} \to \mathcal{N}$ is a morphism of sheaves on $\mathcal{G}$, then $f[\mathcal{H}] : \mathcal{M}[\mathcal{H}] \longrightarrow \mathcal{N}[\mathcal{H}]$ is by(4) uniquely determined by the condition $f[\mathcal{H}] \circ \pi^{\mathcal{M}}[\mathcal{H}] = \pi^{\mathcal{N}}[\mathcal{H}] \circ f$. In case $f$ is a graded morphism of graded sheaves, so is $f[\mathcal{H}]$.

**3.8.** We want to show that the sheaves $\mathcal{B}(z) \otimes_S A$ are projective in a suitable category of sheaves. For the time being, we say that a sheaf $\mathcal{P}$ on $\mathcal{G}$ is *F-projective* if

(A) $\mathcal{P}$ is flabby and generated by global sections.
(B) Each $\mathcal{P}_x$ with $x \in \mathcal{V}$ is a projective $A$–module.
(C) Any $\rho_{x,E}$ with $x \in \mathcal{V}$ and $E \in U_x$ induces an isomorphism $\mathcal{P}_x / \alpha_E \mathcal{P}_x \xrightarrow{\sim} \mathcal{P}_E$.

(F stands for Fiebig.) It follows from 3.5(3),(4),(a) that any $\mathcal{B}(z) \otimes_S A$ is F-projective. A direct sum of two sheaves is F-projective if and only if both summands are F-projective. If $\mathcal{P}$ is an F-projective sheaf, then $f^{\mathcal{P}} : \mathcal{L}(\Gamma(\mathcal{P})) \to \mathcal{P}$ is by Lemma 3.4 an isomorphism.

**Lemma:** *Let $\mathcal{P}$ be an F-projective sheaf on $\mathcal{G}$. Let $\mathcal{M}$ be a sheaf on $\mathcal{G}$ that is flabby and generated by global sections. Fix a vertex $x \in \mathcal{V}$ and denote by $\pi$ the natural morphism $\mathcal{M}[\mathcal{G}_{\leq x}] \to \mathcal{M}[\mathcal{G}_{<x}]$. Then the map*

$$\operatorname{Hom}(\mathcal{P}, \mathcal{M}[\mathcal{G}_{\leq x}]) \longrightarrow \operatorname{Hom}(\mathcal{P}, \mathcal{M}[\mathcal{G}_{<x}]), \qquad \varphi \mapsto \pi \circ \varphi$$

*is surjective.*

*Proof*: Let $\psi \in \mathrm{Hom}(\mathcal{P}, \mathcal{M}[\mathcal{G}_{<x}])$. We have to find $\varphi \in \mathrm{Hom}(\mathcal{P}, \mathcal{M}[\mathcal{G}_{\le x}])$ with $\psi = \pi \circ \varphi$. It is clear that we have to take $\varphi_y = \psi_y$ for all $y < x$ (since $\pi_y = \mathrm{id}$) and we have to take $\varphi_E = \psi_E$ for all edges $E$ belonging to $\mathcal{G}_{<x}$ (since $\pi_E = \mathrm{id}$). It is also clear that $\varphi_y = 0$ for all $y \not\le x$ (since $\mathcal{M}[\mathcal{G}_{\le x}]_y = 0$) and that $\varphi_E = 0$ for all edges $E$ not belonging to $\mathcal{G}_{\le x}$ (since $\mathcal{M}[\mathcal{G}_{\le x}]_E = 0$). So we only have to define $\varphi_x$ and all $\varphi_E$ with $E \in D_x$.

Consider first $E \in D_x$. Set $y = a_E$. The composition $\rho^{\mathcal{M}}_{y,E} \circ \psi_y \colon \mathcal{P}_y \to \mathcal{M}_E$ maps $\alpha_E\, \mathcal{P}_y$ into $\alpha_E\, \mathcal{M}_E = 0$. Since $\mathcal{P}$ satisfies (C) we get now a unique morphism $\varphi_E \colon \mathcal{P}_E \to \mathcal{M}_E$ making the following diagram commutative:

$$\begin{array}{ccc} \mathcal{P}_y & \xrightarrow{\psi_y = \varphi_y} & \mathcal{M}_y \\ {\scriptstyle \rho^{\mathcal{P}}_{y,E}} \big\downarrow & & \big\downarrow {\scriptstyle \rho^{\mathcal{M}}_{y,E}} \\ \mathcal{P}_E & \xrightarrow{\varphi_E} & \mathcal{M}_E \end{array} \tag{1}$$

Combining these diagrams for all $E \in D_x$ we get a commutative diagram

$$\begin{array}{ccc} \Gamma(\mathcal{G}_{<x}, \mathcal{P}) & \xrightarrow{\varphi_{<x}} & \Gamma(\mathcal{G}_{<x}, \mathcal{M}) \\ \big\downarrow & & \big\downarrow \\ \bigoplus_{E \in D_x} \mathcal{P}_E & \xrightarrow{\bigoplus \varphi_E} & \bigoplus_{E \in D_x} \mathcal{M}_E \end{array}$$

where $\varphi_{<x}$ is the restriction of $\bigoplus_{z<x} \varphi_z$ to $\Gamma(\mathcal{G}_{<x}, \mathcal{P})$ and where the vertical maps are the $\rho^D_x$ for the two sheaves involved. It follows that $\bigoplus \varphi_E$ maps $\mathcal{P}_{\partial x}$ (the image of the left vertical map) into $\mathcal{M}_{\partial x}$ (the image of the right vertical map). The assumption that $\mathcal{M}$ is flabby and generated by global sections implies that $\rho^{\mathcal{M}}_{x,D}(\mathcal{M}_x) = \mathcal{M}_{\partial x}$. As $\mathcal{P}_x$ is projective, there exists a homomorphism $\varphi_x \colon \mathcal{P}_x \to \mathcal{M}_x$ such that the diagram

$$\begin{array}{ccc} \mathcal{P}_x & \xrightarrow{\varphi_x} & \mathcal{M}_x \\ {\scriptstyle \rho^{\mathcal{P}}_{x,D}} \big\downarrow & & \big\downarrow {\scriptstyle \rho^{\mathcal{M}}_{x,D}} \\ \mathcal{P}_{\partial x} & \xrightarrow{\bigoplus \varphi_E} & \mathcal{M}_{\partial x} \end{array}$$

is commutative. This implies in particular that $\rho^{\mathcal{M}}_{x,E} \circ \varphi_x = \varphi_E \circ \rho^{\mathcal{P}}_{x,E}$ for all $E \in D_x$. Together with the commutativity of (1) this shows that $\varphi$ is a morphism of sheaves.

**3.9.** If $\mathcal{M}$ is a sheaf on $\mathcal{G}$, then we set for all $x \in \mathcal{V}$

$$\mathcal{M}_{[x]} = \ker \rho_{x,D} = \bigcap_{E \in D_x} \ker \rho_{x,E} \tag{1}$$

cf. 3.3(3). This is an $A$–submodule of $\mathcal{M}_x$; if $\mathcal{M}$ is a graded sheaf (in case $A = S$), then $\mathcal{M}_{[x]}$ is a graded submodule.

If $f\colon \mathcal{M} \to \mathcal{N}$ is a morphism of sheaves on $\mathcal{G}$, then $f$ induces for each $x \in \mathcal{V}$ a homomorphism

$$f_{[x]}\colon \mathcal{M}_{[x]} \longrightarrow \mathcal{N}_{[x]} \tag{2}$$

of $A$–modules. Indeed, $\rho^{\mathcal{N}}_{x,E} \circ f_x = f_E \circ \rho^{\mathcal{M}}_{x,E}$ implies $f_x(\ker \rho^{\mathcal{M}}_{x,E}) \subset \ker \rho^{\mathcal{N}}_{x,E}$ for each $E \in D_x$, hence $f_x(\mathcal{M}_{[x]}) \subset \mathcal{N}_{[x]}$. So we define $f_{[x]}$ as the restriction of $f_x$. In case $f$ is a graded morphism of graded sheaves on $\mathcal{G}$ (for $A = S$), then $f_{[x]}$ is a graded homomorphism of $S$–modules.

**Proposition:** *Let $\mathcal{P}$ be an F-projective sheaf on $\mathcal{G}$. Let $f\colon \mathcal{M} \to \mathcal{N}$ be a morphism of sheaves on $\mathcal{G}$ that are flabby and generated by global sections. If $f_{[x]}\colon \mathcal{M}_{[x]} \to \mathcal{N}_{[x]}$ is surjective for all $x \in \mathcal{V}$, then*

$$\operatorname{Hom}(\mathcal{P}, \mathcal{M}) \to \operatorname{Hom}(\mathcal{P}, \mathcal{N}), \qquad \varphi \mapsto f \circ \varphi \tag{3}$$

*is surjective.*

*Proof*: Let $\psi \in \operatorname{Hom}(\mathcal{P}, \mathcal{N})$. We have to find $\varphi \in \operatorname{Hom}(\mathcal{P}, \mathcal{M})$ with $\psi = f \circ \varphi$. For any $x \in \mathcal{V}$ let $f_{\le x} = f[\mathcal{G}_{\le x}]$ denote the morphism $\mathcal{M}[\mathcal{G}_{\le x}] \to \mathcal{N}[\mathcal{G}_{\le x}]$ induced by $f$. Similarly, $\psi_{\le x} = \psi[\mathcal{G}_{\le x}]$ denotes the morphism $\mathcal{P}[\mathcal{G}_{\le x}] \to \mathcal{N}[\mathcal{G}_{\le x}]$ induced by $\psi$. We want to construct inductively a morphism $\varphi_{\le x}\colon \mathcal{P}[\mathcal{G}_{\le x}] \to \mathcal{M}[\mathcal{G}_{\le x}]$ such that $\psi_{\le x} = f_{\le x} \circ \varphi_{\le x}$. In order to glue these maps together to a morphism $\varphi\colon \mathcal{P} \to \mathcal{M}$, we require for all $y < x$ that the diagram

$$\begin{array}{ccc} \mathcal{P}[\mathcal{G}_{\le x}] & \xrightarrow{\varphi_{\le x}} & \mathcal{M}[\mathcal{G}_{\le x}] \\ \downarrow & & \downarrow \\ \mathcal{P}[\mathcal{G}_{\le y}] & \xrightarrow{\varphi_{\le y}} & \mathcal{M}[\mathcal{G}_{\le y}] \end{array} \tag{4}$$

is commutative. Here the vertical maps are the natural restrictions.

Fix $x \in \mathcal{V}$ and assume by induction that we already have $\varphi_{\le y}$ for all vertices $y < x$. We glue these maps together to get a morphism $\varphi_{<x}\colon \mathcal{P}[\mathcal{G}_{<x}] \to \mathcal{M}[\mathcal{G}_{<x}]$. By Lemma 3.8 we can extend $\varphi_{<x}$ to a morphism $\widehat{\varphi}\colon \mathcal{P}[\mathcal{G}_{\le x}] \to \mathcal{M}[\mathcal{G}_{\le x}]$. Set

$$\psi' = f_{\le x} \circ \widehat{\varphi} - \psi_{\le x}\colon\ \mathcal{P}[\mathcal{G}_{\le x}] \longrightarrow \mathcal{N}[\mathcal{G}_{\le x}].$$

We want to find a morphism $\varphi'\colon \mathcal{P}[\mathcal{G}_{\le x}] \to \mathcal{M}[\mathcal{G}_{\le x}]$ with $\psi' = f_{\le x} \circ \varphi'$; then $\varphi_{\le x} := \widehat{\varphi} - \varphi'$ satisfies $\psi_{\le x} = f_{\le x} \circ \varphi_{\le x}$.

We have $\psi'_y = 0$ for all $y < x$ since $(f_{\le x})_y \circ \widehat{\varphi}_y - (\psi_{\le x})_y = f_y \circ (\varphi_{\le y})_y - \psi_y = 0$ since $\psi_{\le y} = f_{\le y} \circ \varphi_{\le y}$ by induction. Using 3.8(C) we get that also $\psi'_E = 0$ for all edges $E$ belonging to $\mathcal{G}_{<x}$ or to $D_x$. So we set $\varphi'_y = 0$ for all $y < x$ and $\varphi'_E = 0$ for all $E \in D_x$ and for all $E$ belonging to $\mathcal{G}_{<x}$. These definitions ensure also later on that the restriction of $\varphi_{\le x}$ to any $\mathcal{G}_{\le y}$ with $y < x$ is equal to the restriction of $\widehat{\varphi}$, hence equal to $\varphi_{\le y}$. Therefore (4) will commute.

It remains to define $\varphi'_x\colon \mathcal{P}_x \to \mathcal{M}_x$. It has to satisfy $f_x \circ \varphi'_x = \psi'_x$ and $\rho^{\mathcal{M}}_{x,E} \circ \varphi'_x = \varphi'_E \circ \rho^{\mathcal{P}}_{x,E} = 0$ for each $E \in D_x$. Since $\rho^{\mathcal{N}}_{x,E} \circ \psi'_x = \psi'_E \circ \rho^{\mathcal{P}}_{x,E} = 0$ for all $E \in D_x$, we have $\psi'_x(\mathcal{P}_x) \subset \mathcal{N}_{[x]}$. As we assume that $f_{[x]}(\mathcal{M}_{[x]}) = \mathcal{N}_{[x]}$, we now use the projectivity of $\mathcal{P}_x$ to find a morphism $\varphi'_x\colon \mathcal{P}_x \to \mathcal{M}_{[x]}$ with $f_{[x]} \circ \varphi'_x = \psi'_x$. Since $f_{[x]}$ is the restriction of $f_x$, we can rewrite this equation as $f_x \circ \varphi'_x = \psi'_x$. And we get $\rho^{\mathcal{M}}_{x,E} \circ \varphi'_x = 0$ for all $E \in D_x$ from $\varphi'_x(\mathcal{P}_x) \subset \mathcal{M}_{[x]}$.

*Remark*: If $\mathcal{P}$, $\mathcal{M}$, and $\mathcal{N}$ are graded sheaves, then we can replace Hom by $\mathrm{Hom}^0$ in the proposition; a similar statement holds for Lemma 3.8.

**3.10. Lemma:** *Let $\mathcal{H}$ be an F-open full subgraph of $\mathcal{G}$. Let $\mathcal{M}$ be a sheaf on $\mathcal{G}$. Then $\pi^{\mathcal{M}}[\mathcal{H}]_{[x]}: \mathcal{M}_{[x]} \to \mathcal{M}[\mathcal{H}]_{[x]}$ is surjective for each $x \in \mathcal{V}$.*

*Proof*: If $x$ does not belong to $\mathcal{H}$, then $\mathcal{M}[\mathcal{H}]_{[x]} = 0$ and the claim is obvious. If $x$ belongs to $\mathcal{H}$, then so do all $E \in D_x$ since $\mathcal{H}$ is F-open. It follows that $\mathcal{M}_x = \mathcal{M}[\mathcal{H}]_x$ and $\mathcal{M}_E = \mathcal{M}[\mathcal{H}]_E$ for all $E \in D_x$, hence $\mathcal{M}_{[x]} = \mathcal{M}[\mathcal{H}]_{[x]}$. And $\pi^{\mathcal{M}}[\mathcal{H}]_{[x]}$ is the identity.

*Remark*: More generally: Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be F-open full subgraphs of $\mathcal{G}$ with $\mathcal{G}_2$ contained in $\mathcal{G}_1$. Denote by $\pi: \mathcal{M}[\mathcal{G}_1] \to \mathcal{M}[\mathcal{G}_2]$ the natural restriction. Then $\pi_{[x]}$ is surjective for all $x \in \mathcal{V}$. This observation shows that Lemma 3.8 is a special case of Proposition 3.9.

**3.11.** (**Verma sheaves**) For each $x \in \mathcal{V}$ let $\mathcal{V}_A(x)$ denote the (skyscraper) sheaf on $\mathcal{G}$ with $\mathcal{V}_A(x)_x = A$ and $\mathcal{V}_A(x)_y = 0$ for all vertices $y \neq x$ and with $\mathcal{V}_A(x)_E = 0$ for all edges $E \in \mathcal{E}$. All $\rho_{y,E}^{\mathcal{V}_A(x)}$ are of course equal to 0. We call $\mathcal{V}_A(x)$ the *Verma sheaf* at $x$.

It is clear that $\Gamma(\mathcal{V}_A(x)) = A$. Any family $(u_y)_{y \in \mathcal{V}_A} \in \mathcal{Z}$ acts on $\Gamma(\mathcal{V}_A(x))$ as multiplication by $u_x$. One checks easily that $\mathcal{V}_A(x)$ satisfies the conditions (A)–(C) in 2.12. So $f^{\mathcal{V}_A(x)}$ is an isomorphism $\mathcal{L}(\Gamma(\mathcal{V}_A(x))) \xrightarrow{\sim} \mathcal{V}_A(x)$.

Note that $\rho_{x,E}^{\mathcal{V}_A(x)} = 0$ for all $E \in D_x$ implies that $\mathcal{V}_A(x)_{[x]} = \mathcal{V}_A(x)_x = A$; we have of course $\mathcal{V}_A(x)_{[y]} = 0$ for all $y \neq x$.

If $\mathcal{M}$ is any sheaf on $\mathcal{G}$, then we have an isomorphism

$$\mathrm{Hom}\,(\mathcal{M}, \mathcal{V}_A(x)) \xrightarrow{\sim} \mathrm{Hom}_A(\mathcal{M}_x, A), \qquad f \mapsto f_x. \tag{1}$$

Indeed, the map is injective since any $f \in \mathrm{Hom}\,(\mathcal{M}, \mathcal{V}_A(x))$ satisfies $f_y = 0$ for all vertices $y \neq x$ and $f_E = 0$ for all edges $E \in \mathcal{E}$. On the other hand, any $g \in \mathrm{Hom}_A(\mathcal{M}_x, A)$ can be extended to a morphism $f: \mathcal{M} \to \mathcal{V}_A(x)$ with $f_x = g$ setting all other components of $f$ equal to 0. We do not have to worry about conditions of the form $f_E \circ \rho_{y,E}^{\mathcal{M}} = \rho_{y,E}^{\mathcal{V}_A(x)} \circ f_y$ since both sides always are 0.

In case $A = S$ we regard $\mathcal{V}(x) = \mathcal{V}_S(x)$ as a graded sheaf on $\mathcal{G}$ giving $\mathcal{V}(x)_x = S$ the here usual grading. Then (1) restricts for graded $\mathcal{M}$ to an isomorphism

$$\mathrm{Hom}^0\,(\mathcal{M}, \mathcal{V}(x)) \xrightarrow{\sim} \mathrm{Hom}^0_S(\mathcal{M}_x, S), \qquad f \mapsto f_x. \tag{2}$$

**3.12. Proposition:** *Suppose that $A = S$. Let $\mathcal{P}$ be a graded F-projective sheaf of finite type on $\mathcal{G}$. Then there exists an isomorphism of graded sheaves*

$$\mathcal{P} \simeq \mathcal{B}(z_1)\langle r_1\rangle \oplus \mathcal{B}(z_2)\langle r_2\rangle \oplus \cdots \oplus \mathcal{B}(z_s)\langle r_s\rangle \tag{1}$$

*with suitable vertices $z_1, z_2, \ldots, z_s$ and integers $r_1, r_2, \ldots, r_s$. The pairs $(z_i, r_i)$ are determined uniquely up to order by $\mathcal{P}$.*

*Proof*: The uniqueness part follows from 2.20. For the existence we use induction over the sum over $x \in \mathcal{V}$ of the rank of $\mathcal{P}_x$ as a free $S$–module. If the sum is 0, then $\mathcal{P} = 0$ and the claim holds with $s = 0$.

If $\mathcal{P} \neq 0$, then we choose $z \in \mathcal{V}$ minimal for $\mathcal{P}_z \neq 0$. Choose a direct sum decomposition $\mathcal{P}_z = A \oplus B$ as graded $S$–module such that $A \simeq S\langle r\rangle$ for some $r \in \mathbf{Z}$. In order to simplify notation, let us assume that $r = 0$. One gets the general case by replacing $\mathcal{P}$ by $\mathcal{P}\langle -r\rangle$.

We get now from 3.11(2) a morphism $f \in \mathrm{Hom}^0(\mathcal{P}, \mathcal{V}(z))$ such that

$$f_z\colon \mathcal{P}_z = A \oplus B \longrightarrow \mathcal{V}(z)_z = S$$

is the projection to the summand $A$ followed by an isomorphism $A \xrightarrow{\sim} S$. Note that all $f_{[x]}\colon \mathcal{P}_{[x]} \to \mathcal{V}(z)_{[x]}$ with $x \in \mathcal{V}$ are surjective: This is clear for $x \neq z$ where $\mathcal{V}(z)_{[x]} = 0$. For $x = z$ the minimality of $z$ with $\mathcal{P}_z \neq 0$ implies $\mathcal{P}_E = 0$ for all $E \in D_z$. (Recall that $\mathcal{P}$ satisfies (C) in 3.8.) It follows that $\mathcal{P}_{[z]} = \mathcal{P}_z$, hence that $\mathcal{V}(z)_{[z]} = \mathcal{V}(z)_z = f_z(\mathcal{P}_z) = f_z(\mathcal{P}_{[z]})$.

We have similarly a morphism $g\colon \mathcal{B}(z) \to \mathcal{V}(z)$ such that $g_z\colon \mathcal{B}(z)_z \to \mathcal{V}(z)_z$ is an isomorphism. Furthermore, each $g_{[x]}$ is surjective, by the same argument as for $f$. Now Proposition 3.8 and its remark yield morphisms $\varphi \in \mathrm{Hom}^0(\mathcal{B}(z), \mathcal{P})$ and $\psi \in \mathrm{Hom}^0(\mathcal{P}, \mathcal{B}(z))$ with $g \circ \psi = f$ and $f \circ \varphi = g$. It follows that $g \circ \psi \circ \varphi = g$, hence $g \circ (\psi \circ \varphi)^n = g$ for all $n \in \mathbf{N}$. Since $g \neq 0$ this implies that $\psi \circ \varphi$ is not nilpotent. It follows therefore by Proposition 3.5.d and Section 2.20 that $\psi \circ \varphi$ is bijective. This yields that $\mathcal{P} = \ker \varphi \oplus \psi(\mathcal{B}(z))$ and that $\psi(\mathcal{B}(z))$ is isomorphic to $\mathcal{B}(z)$. Now apply the induction hypothesis to $\ker \varphi$.

**3.13.** Let $\mathcal{H} = (\mathcal{V}', \mathcal{E}', \alpha', \leq')$ be an F-open full subgraph of $\mathcal{G}$. Let $x \in \mathcal{V}'$ be a maximal element in $\mathcal{V}'$. Set $\mathcal{H}_{\neq x}$ denote the full subgraph of $\mathcal{G}$ with set of vertices equal to $\mathcal{V}' \setminus \{x\}$. The maximality of $x$ in $\mathcal{V}'$ implies that $\mathcal{H}_{\neq x}$ again is F-open. If $\mathcal{M}$ is a flabby sheaf on $\mathcal{G}$, then we get now a short exact sequence

$$0 \to \mathcal{M}_{[x]} \longrightarrow \Gamma(\mathcal{H}, \mathcal{M}) \longrightarrow \Gamma(\mathcal{H}_{\neq x}, \mathcal{M}) \to 0 \tag{1}$$

of $S$–modules.

**Lemma:** (a) *Let $\mathcal{M}$ be a flabby sheaf on $\mathcal{G}$. Then all $\mathcal{M}_{[x]}$ with $x \in \mathcal{V}$ are projective as modules over $A$ if and only if $\Gamma(\mathcal{H}, \mathcal{M})$ is a projective $A$–module for each F-open full subgraph $\mathcal{H}$ of $\mathcal{G}$.*

(b) *Let $f\colon \mathcal{L} \to \mathcal{M}$ and $g\colon \mathcal{M} \to \mathcal{N}$ be morphisms of flabby sheaves on $\mathcal{G}$ with $g \circ f = 0$. Then*

$$0 \to \mathcal{L}_{[x]} \longrightarrow \mathcal{M}_{[x]} \longrightarrow \mathcal{N}_{[x]} \to 0 \tag{2}$$

*is exact for all $x \in \mathcal{V}$ if and only if*

$$0 \to \Gamma(\mathcal{H}, \mathcal{L}) \longrightarrow \Gamma(\mathcal{H}, \mathcal{M}) \longrightarrow \Gamma(\mathcal{H}, \mathcal{N}) \to 0 \tag{3}$$

*is exact for all F-open full subgraphs $\mathcal{H}$ of $\mathcal{G}$.*

*Proof*: (a) If all $\mathcal{M}_{[x]}$ are projective, then we prove the projectivity of $\Gamma(\mathcal{H}, \mathcal{M})$ using induction on the number of vertices in $\mathcal{H}$. Given a non-empty $\mathcal{H}$ we choose a maximal vertex $x$ and get a short exact sequence as in (1). By induction $\Gamma(\mathcal{H}_{\neq x}, \mathcal{M})$ is projective; so is $\mathcal{M}_{[x]}$ by assumption. It follows that also $\Gamma(\mathcal{H}, \mathcal{M})$ is projective.

On the other hand, suppose that all $\Gamma(\mathcal{H}, \mathcal{M})$ are projective. Then we apply (1) with $\mathcal{H} = \mathcal{G}_{\leq x}$ and get the projectivity of $\mathcal{M}_{[x]}$.

(b) If $\mathcal{H}$ is an F-open full subgraph of $\mathcal{G}$ and if $x$ is a maximal vertex of $\mathcal{H}$, then we have a commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{L}_{[x]} & \longrightarrow & \Gamma(\mathcal{H}, \mathcal{L}) & \longrightarrow & \Gamma(\mathcal{H}_{\neq x}, \mathcal{L}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{M}_{[x]} & \longrightarrow & \Gamma(\mathcal{H}, \mathcal{M}) & \longrightarrow & \Gamma(\mathcal{H}_{\neq x}, \mathcal{M}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{N}_{[x]} & \longrightarrow & \Gamma(\mathcal{H}, \mathcal{N}) & \longrightarrow & \Gamma(\mathcal{H}_{\neq x}, \mathcal{N}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where the rows are exact by (1).

If we assume that all sequences as in (3) are exact, then the second and third columns in our diagram are exact. Then the 9-lemma yields the exactness of the first column. We get thus (2) working with $\mathcal{H} = \mathcal{G}_{\leq x}$.

If we assume that all sequences as in (2) are exact, then we want to prove the exactness of (3) using induction on the number of vertices in $\mathcal{H}$. Given a non-empty $\mathcal{H}$ we choose a maximal vertex $x$ and apply induction to $\mathcal{H}_{\neq x}$. Now the first and third columns in our diagram are exact. Then the 9-lemma yields the exactness of the second column because we assume that $g \circ f = 0$.

**3.14.** If $\mathcal{H} = (\mathcal{V}', \mathcal{E}', \alpha', \leq')$ is a full subgraph of $\mathcal{G}$, then we set $e_{\mathcal{H}} = \sum_{x \in \mathcal{V}'} e_x$. So we have by Lemma 2.11

$$\mathcal{L}(e_{\mathcal{H}} M) = \mathcal{L}(M)[\mathcal{H}] \tag{1}$$

for any $\mathcal{Z}$–module $M$ that is torsion free over $A$.

**Proposition:** *Suppose that $\mathcal{G}$ is a GKM-graph. Let $M$ be a finitely generated $\mathcal{Z}$–module that is torsion free over A. Suppose that $e_{\mathcal{H}} M$ is a reflexive $A$–module for each F-open full subgraph $\mathcal{H}$ of $\mathcal{G}$. Then $\mathcal{L}(M)$ is a flabby sheaf on $\mathcal{G}$; we have a natural isomorphism*

$$e_{\mathcal{H}} M \xrightarrow{\sim} \Gamma(\mathcal{H}, \mathcal{L}(M)) \tag{2}$$

*for each F-open full subgraph $\mathcal{H}$ of $\mathcal{G}$.*

*Proof*: Proposition 2.16 says that we have for each F-open full subgraph $\mathcal{H}$ of $\mathcal{G}$ an isomorphism

$$g^{e_{\mathcal{H}}} \colon e_{\mathcal{H}} M \xrightarrow{\sim} \Gamma(\mathcal{L}(e_{\mathcal{H}} M)).$$

By (1) and 3.7(1) we have a natural isomorphism

$$\Gamma(\mathcal{G}, \mathcal{L}(e_{\mathcal{H}}M)) = \Gamma(\mathcal{L}(M)[\mathcal{H}]) \xrightarrow{\sim} \Gamma(\mathcal{H}, \mathcal{L}(M)).$$

Composing these maps we get (2).

Going through the construction one checks that the map in (2) sends any $e_{\mathcal{H}}v$ with $v \in M$ to the family of all $e_x e_{\mathcal{H}} v = e_x v$ with $x$ running over all vertices belonging to $\mathcal{H}$. So we have a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{g^M} & \Gamma(\mathcal{L}(M)) \\ \downarrow & & \downarrow \\ e_{\mathcal{H}}M & \longrightarrow & \Gamma(\mathcal{H}, \mathcal{L}(M)) \end{array}$$

where the lower horizontal map is the one from (2), the left vertical map sends any $v \in M$ to $e_{\mathcal{H}}v$, and the right vertical map is the restriction of sections for the sheaf $\mathcal{L}(M)$. The two horizontal maps are isomorphisms and the left vertical map is clearly surjective. It follows that also the right vertical map is onto for all $\mathcal{H}$. So $\mathcal{L}(M)$ is flabby.

**3.15.** (**Base change**) Let $A' \subset Q$ be the localisation of $A$ with respect to some multiplicative subset of $A$. Let $\mathcal{M}$ be an $A$–sheaf on $\mathcal{G}$. If $\mathcal{M}$ is flabby, then Lemma 2.7.a shows that $\mathcal{M} \otimes A'$ is flabby. Furthermore $\rho_x^D$ and $\rho_{x,D}$ for $\mathcal{M} \otimes A'$ identify with $\rho_x^D \otimes \mathrm{id}_{A'}$ and $\rho_{x,D} \otimes \mathrm{id}_{A'}$ where $\rho_x^D$ and $\rho_{x,D}$ denote the maps for $\mathcal{M}$. We get for all vertices $x$ that $(\mathcal{M} \otimes A')_{\partial x} = \mathcal{M}_{\partial x} \otimes A'$ and $(\mathcal{M} \otimes A')_{[x]} = \mathcal{M}_{[x]} \otimes A'$

If $\mathcal{M}$ is an $A$–sheaf, then $(\mathcal{M} \otimes A')[\mathcal{H}] = \mathcal{M}[\mathcal{H}] \otimes A'$. If an $A$–sheaf $\mathcal{P}$ is F-projective, then $\mathcal{P} \otimes A'$ is an F-projective $A'$–sheaf. We get $\mathcal{V}_{A'}(x) \simeq \mathcal{V}(x) \otimes A'$.

**3.16.** Set $S_{\mathfrak{m}}$ equal to the localisation of $S$ at the maximal ideal $\mathfrak{m} = S(V)V$.

**Lemma:** *Let $z \in \mathcal{V}$ and let $f \in \mathrm{Hom}(\mathcal{B}(z) \otimes S_{\mathfrak{m}}, \mathcal{B}(z) \otimes S_{\mathfrak{m}})$. If $f_z$ is bijective, then $f$ is an isomorphism.*

*Proof*: We want to show inductively for all $x \in \mathcal{V}$ that $f_x$ and all $f_E$ with $E \in D_x$ are bijective. By assumption this holds for $x = z$ since $(\mathcal{B}(z) \otimes S_{\mathfrak{m}})_E = 0$ for all $E \in D_z$.

Take now $x \neq z$ and suppose that the claim holds for all $y < x$. Consider some $E \in D_x$ and set $y = a_E$; we have $y < x$. The construction of $\mathcal{B}(z)$ shows that $\rho_{y,E}$ induces an isomorphism

$$(\mathcal{B}(z) \otimes S_{\mathfrak{m}})_y / \alpha_E (\mathcal{B}(z) \otimes S_{\mathfrak{m}})_y \xrightarrow{\sim} (\mathcal{B}(z) \otimes S_{\mathfrak{m}})_E, \tag{1}$$

cf. 3.5(4). By induction $f_y$ is bijective. Therefore $f_y$ induces a bijection on the left hand side of (1). This bijection corresponds to $f_E$ under the isomorphism in (1). Therefore also $f_E$ is bijective.

We get also that $f$ induces bijections on $\Gamma(\mathcal{G}_{<x}, \mathcal{B}(z) \otimes S_{\mathfrak{m}})$ and on $(\mathcal{B}(z) \otimes S_{\mathfrak{m}})_{\partial x}$. Now $\rho_{x,D}$ induces an isomorphism

$$(\mathcal{B}(z) \otimes S_{\mathfrak{m}})_x / \mathfrak{m} (\mathcal{B}(z) \otimes S_{\mathfrak{m}})_x \xrightarrow{\sim} (\mathcal{B}(z) \otimes S_{\mathfrak{m}})_{\partial x} / \mathfrak{m} (\mathcal{B}(z) \otimes S_{\mathfrak{m}})_{\partial x}. \tag{2}$$

This isomorphism is compatible with the maps induced by $f$ on both sides. We know already that $f$ induces a bijection on the right hand side. Therefore $f_x$ induces a bijection on the left hand side. It follows that $f_x$ is bijective because $S_{\mathfrak{m}}$ is a local ring with maximal ideal $\mathfrak{m}S_{\mathfrak{m}}$ and because $(\mathcal{B}(z) \otimes S_{\mathfrak{m}})_x$ is a free $S_{\mathfrak{m}}$–module of finite rank.

*Remark*: The lemma implies that $\mathcal{B}(z) \otimes S_{\mathfrak{m}}$ is an indecomposable $S_{\mathfrak{m}}$–sheaf. Indeed, if $\mathcal{B}(z) \otimes S_{\mathfrak{m}} = \mathcal{M} \oplus \mathcal{N}$, then we may assume that $\mathcal{M}_z = (\mathcal{B}(z) \otimes S_{\mathfrak{m}})_z$ and $\mathcal{N}_z = 0$ because $S_{\mathfrak{m}} \simeq (\mathcal{B}(z) \otimes S_{\mathfrak{m}})_z$ is indecomposable. Define now $f \in \mathrm{Hom}(\mathcal{B}(z) \otimes S_{\mathfrak{m}}, \mathcal{B}(z) \otimes S_{\mathfrak{m}})$ as the projection $\mathcal{B}(z) \otimes S_{\mathfrak{m}} \to \mathcal{M}$ with kernel $\mathcal{N}$ followed by the inclusion of $\mathcal{M}$ into $\mathcal{B}(z) \otimes S_{\mathfrak{m}}$. The lemma implies that $f$ is an isomorphism. It follows that $\mathcal{N} = 0$.

**3.17. Proposition:** *Let $\mathcal{P}$ be an F-projective $S_{\mathfrak{m}}$–sheaf of finite type on $\mathcal{G}$. Then there exists an isomorphism of sheaves*

$$\mathcal{P} \simeq \mathcal{B}(z_1) \otimes S_{\mathfrak{m}} \ \oplus\ \mathcal{B}(z_2) \otimes S_{\mathfrak{m}} \ \oplus \cdots \oplus\ \mathcal{B}(z_s) \otimes S_{\mathfrak{m}} \tag{1}$$

*with suitable vertices $z_1, z_2, \ldots, z_s$.*

*Proof*: One proceeds as in the proof of Proposition 3.12 ignoring all statements involving the grading. In the last paragraph a modification is needed. Since $g_z$ is an isomorphism, the equation $g = g \circ \psi \circ \varphi$ implies that $(\psi \circ \varphi)_x$ is an isomorphism. Then Lemma 3.16 yields that $\psi \circ \varphi$ is an isomorphism and we can conclude as in 3.12.

*Remark*: One can show that the $z_i$ in the proposition are uniquely determined by $\mathcal{P}$ up to order: Look at the ranks of all $\mathcal{P}_z$.

## 4 Representations

For the background on semi-simple Lie algebras assumed in 4.1 you can consult [Hu1] or ch. 1 of [Di]. For Verma modules, see ch. 7 in [Di] or ch. 1 in [Ja] or the forthcoming book [Hu2] by Humphreys.

**4.1.** (**Semi-simple Lie algebras**) Let $\mathfrak{g}$ be a finite dimensional semi-simple Lie algebra over $\mathbf{C}$ and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. If $\mathfrak{a}$ is a Lie subalgebra of $\mathfrak{g}$, then we denote by $U(\mathfrak{a})$ its universal enveloping algebra.

If $M$ is an $\mathfrak{h}$–module and if $\lambda \in \mathfrak{h}^*$, then we call

$$M_\lambda := \{\, v \in M \mid h\, v = \lambda(h)\, v \text{ for all } h \in \mathfrak{h} \,\}$$

the *weight space* of $M$ for the weight $\lambda$. We say that $\lambda$ is a *weight of* $M$ if $M_\lambda \neq 0$.

Let $\Phi \subset \mathfrak{h}^*$ denote the *root system* of $\mathfrak{g}$ with respect to $\mathfrak{h}$. So these are the non-zero weights of $\mathfrak{g}$ considered as an $\mathfrak{h}$–module under the adjoint action. We fix a positive system $\Phi^+$ in $\Phi$ and set $\mathfrak{n}^+ = \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_\alpha$ and $\mathfrak{n}^- = \bigoplus_{\alpha\in\Phi^+} \mathfrak{g}_{-\alpha}$ and $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$. These subspaces are Lie subalgebras of $\mathfrak{g}$; we have $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ (as a vector space) and $[\mathfrak{b}, \mathfrak{b}] = \mathfrak{n}^+$. We denote by $\leq$ the partial ordering on $\mathfrak{h}^*$ such that $\lambda \leq \mu$ if and only if there exist non-negative integers $n_\alpha$, $\alpha \in \Phi^+$, such that $\mu - \lambda = \sum_{\alpha\in\Phi^+} n_\alpha \alpha$.

Let $W$ denote the *Weyl group* of $\mathfrak{g}$ with respect to $\mathfrak{h}$. This is a group acting on $\mathfrak{h}$ and $\mathfrak{h}^*$. It is generated by reflections $s_\alpha$, $\alpha \in \Phi$. The action of $W$ on $\mathfrak{h}^*$ is determined by $s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$ for all $\alpha \in \Phi$ and $\lambda \in \mathfrak{h}^*$ where $\alpha^\vee \in \mathfrak{h}$ is the coroot corresponding to $\alpha$. We often consider the *dot action* given by

$$w\bullet\lambda = w(\lambda + \rho) - \rho \qquad \text{for all } w \in W \text{ and } \lambda \in \mathfrak{h}^*$$

where $\rho = (1/2) \sum_{\alpha\in\Phi^+} \alpha$.

**4.2.** (**Verma modules**) For any $\lambda \in \mathfrak{h}^*$ let $\mathbf{C}_\lambda$ denote the $\mathfrak{b}$–module that is equal to $\mathbf{C}$ as a vector space, where each $h \in \mathfrak{h}$ acts as multiplication by $\lambda(h)$ and any $x \in \mathfrak{n}^+$ as 0. The induced $\mathfrak{g}$–module

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbf{C}_\lambda$$

is called the *Verma module with highest weight* $\lambda$. It is the direct sum of its weight spaces $M(\lambda)_\mu$, $\mu \in \mathfrak{h}^*$, and all weights $\mu$ of $M(\lambda)$ satisfy $\mu \leq \lambda$. All weight spaces are finite dimensional and $M(\lambda)_\lambda = \mathbf{C}(1 \otimes 1)$ has dimension 1. The map $u \mapsto u \otimes 1$ is an isomorphism of $U(\mathfrak{n}^-)$–modules $U(\mathfrak{n}^-) \xrightarrow{\sim} M(\lambda)$.

It is easy to show that $M(\lambda)$ has a unique simple factor module; it will be denoted by $L(\lambda)$. This simple $\mathfrak{g}$–module is characterised by the fact that it is the direct sum of its weight spaces, that all weights $\mu$ of $L(\lambda)$ satisfy $\mu \leq \lambda$ and that $L(\lambda)_\lambda$ has dimension 1.

Using the structure of the centre of $U(\mathfrak{g})$ it is not difficult to show that each Verma module has finite length. All composition factors have the form $L(\mu)$ with $\mu \leq \lambda$ and $\mu \in W\bullet\lambda$. We denote by $[M(\lambda) : L(\mu)]$ the multiplicity of $L(\mu)$ as a composition factor of $M(\lambda)$. We have $[M(\lambda) : L(\lambda)] = 1$.

A theorem, conjectured by Verma and proved by Bernstein, Gel′fand, and Gel′fand, gives precise information as to when $[M(\lambda) : L(\mu)] \neq 0$. For the sake of simplicity I shall formulate the result only in a special case.

One calls $\lambda \in \mathfrak{h}^*$ *integral* if $\langle \lambda, \alpha^\vee \rangle \in \mathbf{Z}$ for all $\alpha \in \Phi$. One calls $\lambda \in \mathfrak{h}^*$ *regular* if $\langle \lambda + \rho, \alpha^\vee \rangle \neq 0$ for all $\alpha \in \Phi$; this is equivalent to the condition that the map $w \mapsto w{\bullet}\lambda$ is a bijection $W \to W{\bullet}\lambda$. If $\lambda$ is regular (resp. integral), then so are all elements in $W{\bullet}\lambda$. An integral and regular element $\lambda \in \mathfrak{h}^*$ is called *antidominant* if $\langle \lambda + \rho, \alpha^\vee \rangle < 0$ for all $\alpha \in \Phi^+$. If $\mu \in \mathfrak{h}^*$ is integral and regular, then $W{\bullet}\mu$ contains exactly one antidominant element.

Fix $\lambda \in \mathfrak{h}^*$ that is integral, regular, and antidominant. The result proved by Bernstein, Gel′fand, and Gel′fand says for any $w, w' \in W$ that $[M(w{\bullet}\lambda) : L(w'{\bullet}\lambda)]$ is non-zero if and only if $w' \leq w$ in the Bruhat ordering $\leq$ on $W$, cf. 1.13.

While this result was proved using only methods from representation theory, the determination of the exact values of the multiplicities required quite different techniques. Kazhdan and Lusztig conjectured that $[M(w{\bullet}\lambda) : L(w'{\bullet}\lambda)]$ should be the value at 1 of a certain *Kazhdan-Lusztig polynomial.* This conjecture was proved by Brylinski & Kashiwara and independently by Beilinson & Bernstein. We shall return to this point in 4.23 and shall not use the result until then.

One special case of the conjecture was easy: If $\lambda \in \mathfrak{h}^*$ is integral, regular, and antidominant, then

$$[M(w{\bullet}\lambda) : L(\lambda)] = 1 \tag{1}$$

for all $w \in W$. For example, this follows from Satz 2.23.b in [Ja].

**4.3.** (**Category $\mathcal{O}$**) Bernstein, Gel′fand, and Gel′fand introduced a nice category, the category $\mathcal{O}$, containing all $M(\mu)$ and $L(\mu)$. It consists of all $\mathfrak{g}$–modules $M$ such that

(A) $M$ is finitely generated over $U(\mathfrak{g})$.

(B) $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$.

(C) We have $\dim U(\mathfrak{n}^+)\, v < \infty$ for all $v \in M$.

These conditions imply that $M$ has a finite chain of submodules $M = M_1 \supset M_2 \supset \cdots \supset M_r \supset M_{r+1} = 0$ such that each factor $M_i/M_{i+1}$, $1 \leq i \leq r$, is a homomorphic image of a Verma module. It follows that $M$ has finite length and that the simple modules in $\mathcal{O}$ are exactly all $L(\mu)$ with $\mu \in \mathfrak{h}^*$.

One says that a module $M$ in $\mathcal{O}$ has a *Verma flag* if there exists a finite chain of submodules

$$M = M_1 \supset M_2 \supset \cdots \supset M_r \supset M_{r+1} = 0$$

such that there exists for each $i$ a weight $\mu_i \in \mathfrak{h}^*$ with $M_i/M_{i+1} \simeq M(\mu_i)$. Looking at dimensions of weight spaces one checks for each $\mu \in \mathfrak{h}^*$ that the number of $i$ with $\mu_i = \mu$ is independent of the choice of the chain. We denote this number by $(M : M(\mu))$.

The main result in [BGG] says that the category $\mathcal{O}$ contains enough projectives. For each $\lambda \in \mathfrak{h}^*$ there exists a projective cover $P(\lambda)$ of $L(\lambda)$ in $\mathcal{O}$. Furthermore, each $P(\lambda)$ admits a Verma flag and we have the reciprocity law

$$(P(\lambda) : M(\mu)) = [M(\mu) : L(\lambda)] \tag{1}$$

for all $\lambda, \mu \in \mathfrak{h}^*$.

**4.4.** (**Deformed Verma modules**) If $A$ is a (commutative and associative) $\mathbf{C}$–algebra, then we set $\mathfrak{g}_A = \mathfrak{g} \otimes_{\mathbf{C}} A$ and, more generally, $\mathfrak{a}_A = \mathfrak{a} \otimes_{\mathbf{C}} A$ for any Lie subalgebra $\mathfrak{a}$ of $\mathfrak{g}$. The enveloping algebra $U(\mathfrak{g}_A)$ of the Lie algebra $\mathfrak{a}_A$ over $A$ can then be identified with $U(\mathfrak{a}) \otimes_{\mathbf{C}} A$. We identify $\mathfrak{h}_A^* = \mathrm{Hom}_A(\mathfrak{h}_A, A)$ with $\mathfrak{h}^* \otimes_{\mathbf{C}} A$.

Set $S = U(\mathfrak{h})$; since $\mathfrak{h}$ is commutative, $S$ coincides with the symmetric algebra of $\mathfrak{h}$. Let $A$ be a (commutative) $S$–algebra (and hence by transitivity a $\mathbf{C}$–algebra). Let $\tau\colon \mathfrak{h} \to A$ denote the composition of the inclusion $\mathfrak{h} \hookrightarrow S(\mathfrak{h}) = S$ with the homomorphism $S \to A$ that makes $A$ into an $S$–algebra. Then $\tau$ is $\mathbf{C}$–linear and we extend $\tau$ to an $A$–linear map $\mathfrak{h}_A = \mathfrak{h} \otimes_{\mathbf{C}} A \to A$; we denote this extension again by $\tau$.

Any $\lambda \in \mathfrak{h}^*$ defines an $A$–linear map $\lambda \otimes \mathrm{id}_A\colon \mathfrak{h}_A = \mathfrak{h} \otimes_{\mathbf{C}} A \to \mathbf{C} \otimes_{\mathbf{C}} A = A$. By abuse of notation we write again $\lambda$ for this element of $\mathfrak{h}_A^*$. We then denote by $A_\lambda$ the $\mathfrak{b}_A$–module that is equal to $A$ as an $A$–module, where each $h \in \mathfrak{h}_A$ acts as multiplication by $(\lambda+\tau)(h)$ and where each $x \in \mathfrak{n}_A^+$ acts as 0. Note the occurrence of $\tau$; so we do *not* get $A_\lambda$ from $\mathbf{C}_\lambda$ by extension of scalars. We then call

$$M_A(\lambda) = U(\mathfrak{g}_A) \otimes_{U(\mathfrak{b}_A)} A_\lambda$$

a *deformed* Verma module. The map $u \mapsto u \otimes 1$ is an isomorphism of $U(\mathfrak{n}_A^-)$–modules $U(\mathfrak{n}_A^-) \xrightarrow{\sim} M_A(\lambda)$.

In this set-up it is appropriate to define weight spaces in an $\mathfrak{h}_A$–module $M$ by setting

$$M_\lambda = \{\, v \in M \mid h\, v = (\lambda + \tau)(h)\, v \text{ for all } h \in \mathfrak{h}_A \,\}.$$

Then $M_A(\lambda)$ is the direct sum of all $M_A(\lambda)_\mu$ with $\mu \in \mathfrak{h}^*$, and $M_A(\lambda)_\mu \neq 0$ implies $\mu \leq \lambda$. All $M_A(\lambda)_\mu$ are free $A$–modules of finite rank; in particular $M_A(\lambda)_\lambda = A(1 \otimes 1)$ is free of rank 1. We have $M_A(\lambda) = U(\mathfrak{g}_A) M_A(\lambda)_\lambda$; therefore any endomorphism of $M_A(\lambda)$ is determined by its restriction to $M_A(\lambda)_\lambda$. It follows that

$$\mathrm{End}_{\mathfrak{g}_A} M_A(\lambda) = A\, \mathrm{id}_{M_A(\lambda)}$$

as $M_A(\lambda)_\lambda \simeq A$.

Note: If we take $A = S/S\mathfrak{h} \simeq \mathbf{C}$, then $\tau = 0$. In this case the present definition of $M_\lambda$ coincides with that from 4.1 and $M_A(\lambda)$ is just the old Verma module from 4.2.

**4.5.** (**Deforming $\mathcal{O}$**) Let again $A$ be an $S$–algebra. We generalise the definition of the category $\mathcal{O}$ and define now a category $\mathcal{O}_A$ that contains all $M_A(\lambda)$. It consists of all $\mathfrak{g}_A$–modules $M$ such that

(A) $M$ is finitely generated over $U(\mathfrak{g}_A)$.
(B) $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$.
(C) Each $\dim U(\mathfrak{n}_A^+)\, v < \infty$ with $v \in M$ is finitely generated over $A$.

These conditions imply that $M$ has a finite chain of submodules $M = M_1 \supset M_2 \supset \cdots \supset M_r \supset M_{r+1} = 0$ such that each factor $M_i/M_{i+1}$, $1 \leq i \leq r$, is a homomorphic image of a deformed Verma module $M_A(\lambda)$. It follows that all $M_\lambda$ are finitely generated over $A$. If $M$ and $N$ are two modules in $\mathcal{O}_A$, then $\mathrm{Hom}_{\mathfrak{g}_A}(M, N)$ is a finitely generated $A$–module.

The category $\mathcal{O}_A$ is closed under submodules, factor modules, and finite direct sums. An extension of two modules in $\mathcal{O}_A$ belongs again to $\mathcal{O}_A$ if the extension satisfies (B).

We denote by $\mathcal{O}_A^{\mathrm{VF}}$ the full subcategory of all $M$ in $\mathcal{O}_A$ admitting a Verma flag, i.e., a finite chain of submodules

$$M = M_1 \supset M_2 \supset \cdots \supset M_r \supset M_{r+1} = 0$$

such that there exists for each $i$ a weight $\mu_i \in \mathfrak{h}^*$ with $M_i/M_{i+1} \simeq M_A(\mu_i)$. If so, then we have $(M_{i+1})_\lambda \subset (M_i)_\lambda$ for all $i$ and all $\lambda \in \mathfrak{h}^*$, hence $M_i/M_{i+1} = \bigoplus_{\lambda\in\mathfrak{h}^*}(M_i)_\lambda/(M_{i+1})_\lambda$ and thus $(M_i/M_{i+1})_\lambda = (M_i)_\lambda/(M_{i+1})_\lambda$ for all $\lambda$. It follows that each $M_\lambda$ is a free $A$–module of finite rank and that each short exact sequence

$$0 \to M_{i+1} \longrightarrow M_i \longrightarrow M_A(\mu_i) \to 0$$

splits as a sequence of $A$–modules. Furthermore, one gets as in 4.3 for each $\lambda \in \mathfrak{h}^*$ that the number of $i$ with $\lambda_i = \lambda$ is independent of the choice of the chain. We denote this number by $(M : M_A(\lambda))$.

If $A$ is an integral domain, then $\mathrm{Hom}_{\mathfrak{g}_A}(M, N)$ is a torsion free $A$–module for any $M$ and $N$ in $\mathcal{O}_A^{\mathrm{VF}}$ because $N$ is torsion free.

If $A'$ is a (commutative) $A$–algebra, then $A'$ is naturally an $S$–algebra. We can then identify $\mathfrak{a}_A \otimes_A A'$ with $\mathfrak{a}_{A'}$ for any Lie subalgebra $\mathfrak{a}$ of $\mathfrak{g}$. We get then for each $\lambda \in \mathfrak{h}^*$ a natural isomorphism $M_A(\lambda)\otimes_A A' \simeq M_{A'}(\lambda)$ of $\mathfrak{g}_{A'}$–modules. If $M$ is a $\mathfrak{g}_A$–module in $\mathcal{O}_A$, then $M\otimes_A A'$ is a $\mathfrak{g}_{A'}$–module in $\mathcal{O}_{A'}$ and we have $(M\otimes_A A')_\mu = M_\mu \otimes_A A'$ for all $\mu \in \mathfrak{h}^*$. If $M$ belongs to $\mathcal{O}_A^{\mathrm{VF}}$, then $M \otimes_A A'$ belongs to $\mathcal{O}_{A'}^{\mathrm{VF}}$ and we have $(M \otimes_A A' : M_{A'}(\lambda)) = (M : M_A(\lambda))$ for all $\lambda \in \mathfrak{h}^*$. (Here we use that any Verma flag splits over $A$.)

**4.6.** (**The field case**) Suppose that $K$ is a field that is an $A$–algebra. In this case one can proceed as over $\mathbf{C}$. The point is that everything in 4.1–4.3 works equally well over any field of characteristic 0 if one replaces $\mathfrak{g}$ with a split semi-simple Lie algebra over that field, cf. [Ja].

So one gets that each $M_K(\lambda)$ has a unique simple factor module; it will be denoted by $L_K(\lambda)$. Each $M_K(\mu)$ and each $M$ in $\mathcal{O}_K$ has finite length and all its composition factors have the form $L_K(\lambda)$ with $\lambda \in \mathfrak{h}^*$. The category $\mathcal{O}_K$ contains enough projectives. The projective cover $P_K(\lambda)$ of $L_K(\lambda)$ belongs to $\mathcal{O}_K^{\mathrm{VF}}$ and satisfies

$$(P_K(\lambda) : M_K(\mu)) = [M_K(\mu) : L_K(\lambda)] \tag{1}$$

for all $\mu \in \mathfrak{h}^*$.

We can in particular take $K = S/S\mathfrak{h} \simeq \mathbf{C}$. In this case $\mathcal{O}_K$ is just the category $\mathcal{O}$ from 4.3; we have $L_K(\mu) = L(\mu)$ and $P_K(\mu) = P(\mu)$ for all $\mu \in \mathfrak{h}^*$.

As another extreme, consider a field $K$ such that for each $\alpha \in \Phi$ the image of $\alpha^\vee$ in $K$ does not belong to $\mathbf{C}$. Then one gets $M_K(\mu) = L_K(\mu) = P_K(\mu)$ for all $\mu \in \mathfrak{h}^*$. Any module in $\mathcal{O}_K$ is semi-simple, and we have $\mathcal{O}_K^{\mathrm{VF}} = \mathcal{O}_K$. This applies in particular to $K = Q$, the fraction field of $S$.

Consider finally the following case: Suppose that there exists a positive root $\alpha$ such that the image of $\alpha^\vee$ in $K$ is 0 whereas for any other positive root $\beta \neq \alpha$ the image of $\beta^\vee$ in $K$ does not belong to $\mathbf{C}$. In this case one gets: If $\lambda \in \mathfrak{h}^*$ satisfies $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbf{Z}$ or $\langle \lambda + \rho, \alpha^\vee \rangle = 0$, then $M_K(\lambda) = L_K(\lambda) = P_K(\lambda)$. If $\langle \lambda + \rho, \alpha^\vee \rangle$ is a negative integer, then $M_K(\lambda) = L_K(\lambda)$ and we have a short exact sequence of $\mathfrak{g}_K$–modules

$$0 \to M_K(s_\alpha \bullet \lambda) \longrightarrow P_K(\lambda) \longrightarrow M_K(\lambda) \to 0. \tag{2}$$

If $\langle \lambda + \rho, \alpha^\vee \rangle$ is a positive integer, then $M_K(\lambda) = P_K(\lambda)$ and we have a short exact sequence of $\mathfrak{g}_K$–modules

$$0 \to L_K(s_\alpha \bullet \lambda) \longrightarrow M_K(\lambda) \longrightarrow L_K(\lambda) \to 0. \tag{3}$$

Here the description of the composition factors of $M_K(\lambda)$ follows from [Ja], Satz 1.8 and Satz 2.16.b. Then one uses (1) to get the $P_K(\lambda)$.

**4.7.** (**The local case**) We now consider the case where our $S$–algebra $A$ is a local ring. We also assume that $A$ is noetherian and an integral domain. Let $K$ denote the residue field of $A$.

**Proposition:** (a) *There exists for each $\lambda \in \mathfrak{h}^*$ a projective object $P_A(\lambda)$ in $\mathcal{O}_A$ such that $P_A(\lambda) \otimes_A K \simeq P_K(\lambda)$. This module is indecomposable; it has a Verma flag and satisfies*

$$(P_A(\lambda) : M_A(\mu)) = [M_K(\mu) : L_K(\lambda)] \tag{1}$$

*for all $\mu \in \mathfrak{h}^*$.*

(b) *Each projective object in $\mathcal{O}_A$ is isomorphic to a finite direct sum of modules of the form $P_A(\lambda)$ with $\lambda \in \mathfrak{h}^*$.*

(c) *Let $A'$ be an $A$–algebra. If $P$ is a projective object in $\mathcal{O}_A$, then $P \otimes_A A'$ is a projective object in $\mathcal{O}_{A'}$ and the natural map*

$$\mathrm{Hom}_{\mathfrak{g}_A}(P, M) \otimes_A A' \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{g}_{A'}}(P \otimes_A A', M \otimes_A A') \tag{2}$$

*is an isomorphism for any $M$ in $\mathcal{O}_A$.*

For a proof let me refer you to [F1], Thm. 2.7 and Prop. 2.4. Actually there one deals with more general Kac-Moody algebras instead of just with our finite dimensional $\mathfrak{g}$. This makes it necessary to work in [F1] with certain truncated categories $\mathcal{O}_A^{\leq \nu}$. We can get rid of them here because the category $\mathcal{O}_A$ splits into blocks and because in our finite dimensional case each block is contained in a suitable $\mathcal{O}_A^{\leq \nu}$.

**Corollary:** *Each object in $\mathcal{O}_A$ is a homomorphic image of a projective object.*

*Proof*: Let $\mathfrak{m}$ denote the maximal ideal in $A$. We have for each $\mu \in \mathfrak{h}^*$ surjective homomorphisms

$$f \colon M_A(\mu) \longrightarrow M_A(\mu)/\mathfrak{m} M_A(\mu) \xrightarrow{\sim} M_K(\mu) \longrightarrow L_K(\mu)$$

and

$$g \colon P_A(\mu) \longrightarrow P_A(\mu)/\mathfrak{m} P_A(\mu) \xrightarrow{\sim} P_K(\mu) \longrightarrow L_K(\mu).$$

So there exists a homomorphism $h\colon P_A(\mu) \to M_A(\mu)$ with $f \circ h = g$. As $f$ induces an isomorphism $M_A(\mu)_\mu/\mathfrak{m}M_A(\mu)_\mu \xrightarrow{\sim} L_K(\mu)_\mu$ and as $L_K(\mu)_\mu = g(P_A(\mu)_\mu) = f(h(P_A(\mu)_\mu))$ we get $M_A(\mu)_\mu = h(P_A(\mu)_\mu) + \mathfrak{m}M_A(\mu)_\mu$, hence $M_A(\mu)_\mu = h(P_A(\mu)_\mu)$ by the Nakayama lemma. It follows that $h$ is surjective because of $M_A(\mu) = U(\mathfrak{g}_A)\, M_A(\mu)_\mu$.

Let now $M$ be an arbitrary module in $\mathcal{O}_A$. Recall that $M$ has a finite chain of submodules $M = M_1 \supset M_2 \supset \cdots \supset M_r \supset M_{r+1} = 0$ such that each factor $M_i/M_{i+1}$, $1 \le i \le r$, is a homomorphic image of a deformed Verma module $M_A(\mu_i)$, cf. 4.5. We get now a surjective homomorphism $f_1\colon P_A(\mu_1) \to M_A(\mu_1) \to M_1/M_2$, hence a homomorphism $g_1\colon P_A(\mu_1) \to M_1$ with $g_1(v) + M_2 = f_1(v)$ for all $v \in P_A(\mu_1)$. It follows that $M = g_1(P_A(\mu_1)) + M_2$. Now we use induction on $r$ to get homomorphisms $g_i\colon P_A(\mu_i) \to M_i$ with $M = \sum_{i=1}^r g_i(P_A(\mu_i))$.

*Examples*: Suppose that $A = S_\mathfrak{p}$ is the local ring of $S$ at a prime ideal $\mathfrak{p}$ of height 1. So we have $\mathfrak{p} = S\gamma$ for some irreducible polynomial $\gamma \in S$. Assume in addition that *the constant term of* $\gamma$ *is* 0. Recall that $A\mathfrak{p}$ is the maximal ideal in $A$.

We look first at the case where $\gamma \notin \mathbf{C}\alpha^\vee$ for all $\alpha \in \Phi$. Then each $\alpha^\vee$ is a unit in $A$. Let us show that the residue class of $\alpha^\vee$ in $K$ does not belong to $\mathbf{C}$. Well, assume that $z \in \mathbf{C}$ with $\alpha^\vee - z \in A\mathfrak{p}$. We get then $\alpha^\vee - z \in S \cap A\mathfrak{p} = \mathfrak{p} = S\gamma$. Both $\alpha^\vee$ and $\gamma$ have constant term 0. This implies that $z = 0$ and hence $\alpha^\vee \in S\gamma$ contradicting our assumption. Now we get from 4.6 that $P_K(\mu) = M_K(\mu)$, hence that $P_A(\mu) = M_A(\mu)$ for all $\mu \in \mathfrak{h}^*$.

Consider now the case where $\mathfrak{p} = S\alpha^\vee$ for some $\alpha \in \Phi^+$. Now the image of $\alpha^\vee$ in the residue field $K$ of $A$ is 0. On the other hand the argument above shows for all $\beta \in \Phi^+$ with $\beta \neq \alpha$ that the image of $\beta^\vee$ in $K$ does not belong to $\mathbf{C}$.

Now the discussion at the end of 4.6 yields: Let $\mu \in \mathfrak{h}^*$. If $\langle \mu + \rho, \alpha^\vee \rangle$ is a negative integer, then there is an exact sequence

$$0 \to M_A(s_\alpha \bullet \mu) \longrightarrow P_A(\mu) \longrightarrow M_A(\mu) \to 0. \tag{3}$$

In all other cases one has $P_A(\mu) = M_A(\mu)$.

**4.8.** (**Hom spaces**) Suppose that $A \subset Q$ is the localisation of $S$ with respect to some multiplicative subset. So $A$ is noetherian and integrally closed. If $M$ is an object in $\mathcal{O}_A^{\mathrm{VF}}$, then $M_\lambda$ is a free $A$–module of finite rank for each $\lambda \in \mathfrak{h}^*$. So we can identify $M_\lambda$ with the $A$–submodule $M_\lambda \otimes 1$ of $M_\lambda \otimes_A Q$ and we have, cf. 2.16(1)

$$M_\lambda = \bigcap_{\mathfrak{p} \in \mathfrak{P}(A)} M_\lambda \otimes_A A_\mathfrak{p}$$

where $\mathfrak{P}(A)$ is the set of all prime ideals in $A$ of height 1 and where $A_\mathfrak{p}$ denotes the local ring at $\mathfrak{p}$. Here also each $M_\lambda \otimes_A A_\mathfrak{p}$ is identified with a submodule of $M_\lambda \otimes_A Q$.

It follows that we can identify $M$ and each $M \otimes_A A_\mathfrak{p}$ with a submodule of $M \otimes_A Q$ and that

$$M = \bigcap_{\mathfrak{p} \in \mathfrak{P}(A)} M \otimes_A A_\mathfrak{p}. \tag{1}$$

Let also $N$ be an object in $\mathcal{O}_A^{\mathrm{VF}}$. We get then

$$\mathrm{Hom}_{\mathfrak{g}_A}(M,N) = \{\, \varphi \in \mathrm{Hom}_{\mathfrak{g}_Q}(M \otimes_A Q, N \otimes_A Q) \mid \varphi(M) \subset N \,\}$$

and for all $\mathfrak{p}$

$$\mathrm{Hom}_{\mathfrak{g}_{A_\mathfrak{p}}}(M\otimes_A A_\mathfrak{p}, N\otimes_A A_\mathfrak{p}) = \{\, \varphi \in \mathrm{Hom}_{\mathfrak{g}_Q}(M\otimes_A Q, N\otimes_A Q) \mid \varphi(M\otimes_A A_\mathfrak{p}) \subset N\otimes_A A_\mathfrak{p} \,\}.$$

Now (1) shows that

$$\mathrm{Hom}_{\mathfrak{g}_A}(M,N) = \bigcap_{\mathfrak{p}\in\mathfrak{P}(A)} \mathrm{Hom}_{\mathfrak{g}_{A_\mathfrak{p}}}(M \otimes_A A_\mathfrak{p}, N \otimes_A A_\mathfrak{p}) \tag{2}$$

where the intersection is taken inside $\mathrm{Hom}_{\mathfrak{g}_Q}(M \otimes_A Q, N \otimes_A Q)$.

Note also that we have natural isomorphisms

$$\mathrm{Hom}_{\mathfrak{g}_A}(M,N) \otimes_A A_\mathfrak{p} \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{g}_{A_\mathfrak{p}}}(M \otimes_A A_\mathfrak{p}, N \otimes_A A_\mathfrak{p}) \tag{3}$$

and

$$\mathrm{Hom}_{\mathfrak{g}_A}(M,N) \otimes_A Q \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{g}_Q}(M \otimes_A Q, N \otimes_A Q). \tag{4}$$

Here one has to be a bit careful since $M$ is not finitely generated over $A$. However, we can find finitely many weights $\lambda_1, \lambda_2, \dots, \lambda_s \in \mathfrak{h}^*$ such that $M = \sum_{i=1}^s U(\mathfrak{g}_A) M_{\lambda_i}$. If we have now $\varphi \in \mathrm{Hom}_{\mathfrak{g}_Q}(M\otimes_A Q, N\otimes_A Q)$, then we can find $t \in A$, $t \neq 0$ such that $t\,\varphi(M_{\lambda_i}) \subset N_{\lambda_i}$ for all $i$. It follows that $t\varphi(M) \subset N$ and $\varphi = (t\varphi) \otimes t^{-1} \in \mathrm{Hom}_{\mathfrak{g}_A}(M,N) \otimes_A Q$. One argues similarly with the $A_\mathfrak{p}$.

*Remark*: Consider for example $A = S_\mathfrak{p}$ where $\mathfrak{p} = S\alpha^\vee$ for some $\alpha \in \Phi^+$. Choose $\mu \in \mathfrak{h}^*$ such that $\langle \mu + \rho, \alpha^\vee \rangle$ is a negative integer. Then 4.7(3) implies that $P_A(\mu) \otimes_A Q$ is an extension of $M_Q(\mu) = P_Q(\mu)$ by $M_Q(s_\alpha{\scriptstyle\bullet}\mu) = P_Q(s_\alpha{\scriptstyle\bullet}\mu)$. It follows that $P_A(\mu) \otimes_A Q \simeq M_Q(\mu) \oplus M_Q(s_\alpha{\scriptstyle\bullet}\mu)$, hence that

$$\Big(\mathrm{End}_{\mathfrak{g}_A} P_A(\mu)\Big) \otimes_A Q \xrightarrow{\sim} \mathrm{End}_{\mathfrak{g}_Q}\Big(M_Q(\mu) \oplus M_Q(s_\alpha{\scriptstyle\bullet}\mu)\Big) \xrightarrow{\sim} Q \times Q. \tag{5}$$

One can show that this map induces an isomorphism

$$\mathrm{End}_{\mathfrak{g}_A} P_A(\mu) \xrightarrow{\sim} \{\, (u,v) \in A \times A \mid u \equiv v \bmod A\alpha^\vee \,\} \tag{6}$$

see [F1], Cor. 3.5.

**4.9.** (**Blocks**) Suppose that $A \subset Q$ is the localisation of $S$ with respect to some multiplicative subset. Assume in addition that $A$ is a local ring (possibly the field $Q$).

The *blocks* over $A$ are the equivalence classes for the smallest equivalence relation $\sim$ on $\mathfrak{h}^*$ such that $\lambda \sim \mu$ whenever $\mathrm{Hom}_{\mathfrak{g}_A}(P_A(\lambda), P_A(\mu)) \neq 0$. The equivalence class containing a given $\mu \in \mathfrak{h}^*$ will be called the block of $\mu$.

If $\Lambda$ is a block over $A$ and if $M$ is a module in $\mathcal{O}_A$, then we set

$$M_{[\Lambda]} \;=\; \sum_{\mu\in\Lambda} \; \sum_{\varphi\in \mathrm{Hom}_{\mathfrak{g}_A}(P_A(\mu), M)} \varphi\,(P_A(\mu)). \tag{1}$$

One gets then

$$M = \bigoplus_{\Lambda} M_{[\Lambda]} \tag{2}$$

where $\Lambda$ runs through the blocks over $A$. (Of course only finitely many summands are non-zero.) If $\varphi\colon M \to N$ is a homomorphism in $\mathcal{O}_A$, then clearly $\varphi(M_{[\Lambda]}) \subset N_{[\Lambda]}$ for all blocks $\Lambda$. We get thus an isomorphism

$$\mathrm{Hom}_{\mathfrak{g}_A}(M, N) \xrightarrow{\sim} \prod_{\Lambda} \mathrm{Hom}_{\mathfrak{g}_A}(M_{[\Lambda]}, N_{[\Lambda]}). \tag{3}$$

For any block $\Lambda$ let $\mathcal{O}_{A,\Lambda}$ denote the full subcategory of all $M$ in $\mathcal{O}_A$ with $M = M_{[\Lambda]}$. Then the category $\mathcal{O}_A$ is the direct sum of all subcategories $\mathcal{O}_{A,\Lambda}$. These subcategories are usually called *the blocks of* $\mathcal{O}_A$. A module $M$ in $\mathcal{O}_A^{\mathrm{VF}}$ belongs to $\mathcal{O}_{A,\Lambda}$ if and only all factors in a Verma flag of $M$ have the form $M_A(\mu)$ with $\mu \in \Lambda$.

*Examples*: 1) In case $A = Q$ then each $P_Q(\mu) = M_Q(\mu) = L_Q(\mu)$ is a simple module. It follows that $\mathrm{Hom}_{\mathfrak{g}_Q}(P_Q(\mu), P_Q(\nu)) \neq 0$ if and only if $\mu = \nu$. Therefore the block of any $\mu \in \mathfrak{h}^*$ is equal to $\{\mu\}$. The decomposition in (2) takes the form

$$M = \bigoplus_{\mu\in\mathfrak{h}^*} M_{[\mu]} \tag{4}$$

(after a minor simplification in the notation). Each $M_{[\mu]}$ is a direct sum of copies of $M_Q(\mu)$; it is an isotypic component of the semi-simple module $M$.

2) Consider as in 4.7 the case $A = S_{\mathfrak{p}}$ where $\mathfrak{p} = S\gamma$ is the prime ideal in $S$ generated by an irreducible polynomial $\gamma$ such that the constant term of $\gamma$ is 0.

If $\gamma \notin \mathbf{C}\alpha^\vee$ for all $\alpha \in \Phi$, then we saw in 4.7 that $P_A(\mu) = M_A(\mu)$ for all $\mu \in \mathfrak{h}^*$. It follows that $P_A(\mu) \otimes_A Q \simeq M_Q(\mu)$, hence for all $\mu, \nu \in \mathfrak{h}^*$ using 4.8(4)

$$\mathrm{Hom}_{\mathfrak{g}_A}(P_A(\mu), P_A(\nu)) \hookrightarrow \mathrm{Hom}_{\mathfrak{g}_Q}(M_Q(\mu), M_Q(\nu)) = 0.$$

This shows for all $\mu \in \mathfrak{h}^*$ that the block of $\mu$ is equal to $\{\mu\}$. If $M$ is a module in $\mathcal{O}_A^{\mathrm{VF}}$, then each $M_{[\mu]}$ with $\mu \in \mathfrak{h}^*$ is a direct sum of copies of $M_A(\mu)$.

Suppose next that $\gamma = \alpha^\vee$ for some $\alpha \in \Phi^+$. Then 4.7(3) shows: If $\langle \mu + \rho, \alpha^\vee \rangle$ is a negative integer, then there is an injective homomorphism from $M_A(s_\alpha{\bullet}\mu) = P_A(s_\alpha{\bullet}\mu)$ to $P_A(\mu)$. So in this case $\mu$ and $s_\alpha{\bullet}\mu$ belong to the same block. Tensoring with $Q$ one checks more precisely: If $\langle \mu+\rho, \alpha^\vee \rangle \notin \mathbf{Z}$ or if $\langle \mu+\rho, \alpha^\vee \rangle = 0$, then the block of $\mu$ is equal to $\{\mu\}$. If $\langle \mu + \rho, \alpha^\vee \rangle$ is a non-zero integer, then the block of $\mu$ is equal to $\{\mu, s_\alpha{\bullet}\mu\}$.

*Remark*: Let $A$ again be general. Consider a block $\Lambda$ over $A$ and a module $M$ in $\mathcal{O}_A^{\mathrm{VF}}$. Then one gets

$$M_{[\Lambda]} \otimes_A Q = \bigoplus_{\mu\in\Lambda} (M \otimes_A Q)_{[\mu]} \quad \text{and} \quad M_{[\Lambda]} = M \cap \bigoplus_{\mu\in\Lambda} (M \otimes_A Q)_{[\mu]}. \tag{5}$$

**4.10.** Suppose that $A \subset Q$ is the localisation of $S$ with respect to some multiplicative subset. Assume in addition that $A$ is a local ring. Denote by $K$ the residue field of $A$.

If $M$ is a module in $\mathcal{O}_A^{\mathrm{VF}}$, then we can apply 4.9(4) to $M \otimes_A Q$ and get a decomposition

$$M \otimes_A Q = \bigoplus_{\mu\in\mathfrak{h}^*} (M \otimes_A Q)_{[\mu]} \qquad \text{with } (M \otimes_A Q)_{[\mu]} \simeq M_Q(\mu)^{(M:M_A(\mu))} \tag{1}$$

for all $\mu \in \mathfrak{h}^*$.

**Proposition:** *Let $P$ be a projective object in $\mathcal{O}_A$, let $\mu \in \mathfrak{h}^*$. Then $\mathrm{Hom}_{\mathfrak{g}_A}(P, M_A(\mu))$ is a free $A$–module of rank $(P : M_A(\mu))$.*

*Proof*: We know by Proposition 4.7 that there exists an isomorphism $P \simeq \bigoplus_{i=1}^s P_A(\lambda_i)$ with suitable $\lambda_i \in \mathfrak{h}^*$ and that $P$ has a Verma flag. Furthermore we get

$$\begin{aligned}(P : M_A(\mu)) &= \sum_{i=1}^s (P_A(\lambda_i) : M_A(\mu)) = \sum_{i=1}^s [M_K(\mu) : L_K(\lambda_i)] \\ &= \sum_{i=1}^s \dim \mathrm{Hom}_{\mathfrak{g}_K}(P_K(\lambda_i), M_K(\mu)) \\ &= \dim \mathrm{Hom}_{\mathfrak{g}_K}(P \otimes_A K, M_A(\mu) \otimes_A K).\end{aligned}$$

We get from Proposition 4.7 also that

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{g}_K}(P \otimes_A K, M_A(\mu) \otimes_A K) &\simeq \mathrm{Hom}_{\mathfrak{g}_A}(P, M_A(\mu)) \otimes_A K \\ &\simeq \mathrm{Hom}_{\mathfrak{g}_A}(P, M_A(\mu))/\mathfrak{m}\,\mathrm{Hom}_{\mathfrak{g}_A}(P, M_A(\mu))\end{aligned}$$

where $\mathfrak{m}$ is the maximal ideal of $A$. We can now choose $f_1, f_2, \dots, f_r$ in $\mathrm{Hom}_{\mathfrak{g}_A}(P, M_A(\mu))$ such that their residue classes modulo $\mathfrak{m}$ form a basis for $\mathrm{Hom}_{\mathfrak{g}_K}(P \otimes_A K, M_A(\mu) \otimes_A K)$. We have $r = (P : M_A(\mu))$; the Nakayama lemma implies that

$$\mathrm{Hom}_{\mathfrak{g}_A}(P, M_A(\mu)) = \sum_{i=1}^r A\, f_i.$$

Now 4.8(4) shows that the $f_i \otimes 1$ generate $\mathrm{Hom}_{\mathfrak{g}_Q}(P \otimes_A Q, M_A(\mu) \otimes_A Q)$. Applying (1) to $P$, we see that this Hom space has dimension $(P : M_A(\mu)) = r$ over $Q$. Therefore the $f_i \otimes 1$ are linearly independent over $Q$, hence the $f_i$ over $A$. So they form a basis for $\mathrm{Hom}_{\mathfrak{g}_A}(P, M_A(\mu))$.

**Corollary:** *Let $P$ be a projective object in $\mathcal{O}_A$ and let $M$ be a module in $\mathcal{O}_A^{\mathrm{VF}}$. Then $\mathrm{Hom}_{\mathfrak{g}_A}(P, M)$ is a free $A$–module of finite rank.*

*Proof*: This follows by induction on the length of a Verma flag. One uses that $\mathrm{Hom}_{\mathfrak{g}_A}(P, \ )$ is exact because $P$ is projective.

**4.11.** Suppose now that $A$ is the local ring of $S$ at the maximal ideal $S\,\mathfrak{h}$. So the residue field of $A$ is equal to $\mathbf{C}$ and $\mathbf{C}$ gets thus the $S$–algebra structure as $S/S\mathfrak{h}$.

Let $\lambda \in \mathfrak{h}^*$ be regular, integral, and antidominant. Then the block of $\lambda$ is equal to $W_\bullet\lambda$; this follows from the analogous result for the residue field $\mathbf{C}$. Set $\mathcal{O}_{A,\lambda}^{\mathrm{VF}}$ equal to the full subcategory of all $M$ in $\mathcal{O}_A^{\mathrm{VF}}$ that belong to the block $\mathcal{O}_{A,W_\bullet\lambda}$, i.e., such that all factors in a Verma flag of $M$ have the form $M_A(\mu)$ with $\mu \in W_\bullet\lambda$. For $M$ in $\mathcal{O}_{A,\lambda}^{\mathrm{VF}}$ we slightly change the notation from 4.10(1) and write

$$(M \otimes_A Q)_w = (M \otimes_A Q)_{[w_\bullet\lambda]} \qquad \text{for all } w \in W.$$

We have then

$$M \otimes_A Q = \bigoplus_{w\in W} (M \otimes_A Q)_w. \tag{1}$$

By Proposition 4.7.a there exists a projective indecomposable object $P_A(\lambda)$ in $\mathcal{O}_A$ with $P_A(\lambda) \otimes_A \mathbf{C} \simeq P(\lambda)$. This module has a Verma flag. Each $M_A(w_\bullet\lambda)$ with $w \in W$ occurs with multiplicity 1 as a factor in a Verma flag of $P_A(\lambda)$, cf. 4.2(1). So we get

$$(P_A(\lambda) \otimes_A Q)_w \simeq M_Q(w_\bullet\lambda) \qquad \text{for each } w \in W.$$

We get thus natural isomorphisms

$$\mathrm{End}_{\mathfrak{g}_Q}(P_A(\lambda) \otimes_A Q) \xrightarrow{\sim} \prod_{w\in W} \mathrm{End}_{\mathfrak{g}_Q}(P_A(\lambda) \otimes_A Q)_w \xrightarrow{\sim} \prod_{w\in W} Q. \tag{2}$$

Recall from 4.8(4) the isomorphism

$$(\mathrm{End}_{\mathfrak{g}_A} P_A(\lambda)) \otimes_A Q \xrightarrow{\sim} \mathrm{End}_{\mathfrak{g}_Q}(P_A(\lambda) \otimes_A Q).$$

We get now via $\varphi \mapsto \varphi \otimes \mathrm{id}_Q$ an embedding

$$\mathrm{End}_{\mathfrak{g}_A} P_A(\lambda) \hookrightarrow \prod_{w\in W} Q. \tag{3}$$

**Proposition:** *The image of* (3) *is the set of all $W$–tuples $(u_w)_{w\in W} \in \prod_{w\in W} A$ with*

$$u_w \equiv u_{s_\alpha w} \mod A\alpha^\vee \tag{4}$$

*for all $w \in W$ and $\alpha \in \Phi$.*

*Proof*: We get from 4.8(2)

$$\mathrm{End}_{\mathfrak{g}_A} P_A(\lambda) = \bigcap_{\mathfrak{p}\in\mathfrak{P}(A)} \mathrm{End}_{\mathfrak{g}_{A_\mathfrak{p}}}(P_A(\lambda) \otimes_A A_\mathfrak{p}). \tag{5}$$

A prime ideal $\mathfrak{p} \in \mathfrak{P}(A)$ has the form $\mathfrak{p} = A\gamma$ with $\gamma$ an irreducible element in $S$, cf. 2.15. The constant term of $\gamma$ is 0 since otherwise $\gamma$ is a unit in $A$. The local ring $A_\mathfrak{p}$ coincides then with $S_{\mathfrak{p}\cap S} = S_{S\gamma}$.

If $\gamma \notin \mathbf{C}\alpha^\vee$ for all $\alpha \in \Phi$, then the examples in 4.7 yield $M_{A_\mathfrak{p}}(w\bullet\lambda) = P_{A_\mathfrak{p}}(w\bullet\lambda)$ for all $w \in W$. A look at the Verma flag of $P_A(\lambda)$ shows now that the block decomposition of $P_A(\lambda) \otimes A_\mathfrak{p}$ has the form

$$P_A(\lambda) \otimes_A A_\mathfrak{p} \simeq \bigoplus_{w\in W} (P_A(\lambda) \otimes_A A_\mathfrak{p})_{[w\bullet\lambda]} \qquad \text{with } (P_A(\lambda) \otimes_A A_\mathfrak{p})_{[w\bullet\lambda]} \simeq P_{A_\mathfrak{p}}(w\bullet\lambda)$$

for all $w \in W$. Furthermore $(P_A(\lambda)\otimes_A A_\mathfrak{p})_{[w\bullet\lambda]} = P_A(\lambda)\otimes_A A_\mathfrak{p} \cap (P_A(\lambda)\otimes_A Q)_w$, cf. 4.9(5), shows that (3) induces an identification

$$\mathrm{End}_{\mathfrak{g}_{A_\mathfrak{p}}}(P_A(\lambda) \otimes_A A_\mathfrak{p}) = \prod_{w\in W} A_\mathfrak{p} \subset \prod_{w\in W} Q. \tag{6}$$

Suppose now that $\gamma \in \mathbf{C}\alpha^\vee$ for some $\alpha \in \Phi^+$. Set $W' = \{\, w \in W \mid w^{-1}\alpha \in \Phi^+ \,\}$. Then $W$ is the disjoint union of $W'$ and $s_\alpha W'$. Each $\{w\bullet\lambda, s_\alpha w\bullet\lambda\}$ is a block over $A_\mathfrak{p}$. We have $\langle w(\lambda+\rho), \alpha^\vee\rangle < 0$ and $\langle s_\alpha w(\lambda+\rho), \alpha^\vee\rangle > 0$ for all $w \in W'$. The block decomposition of $P_A(\lambda) \otimes A_\mathfrak{p}$ has the form

$$P_A(\lambda) \otimes_A A_\mathfrak{p} \simeq \bigoplus_{w\in W'} (P_A(\lambda) \otimes_A A_\mathfrak{p})_{[w\bullet\lambda, s_\alpha w\bullet\lambda]}.$$

Each $(P_A(\lambda)\otimes_A A_\mathfrak{p})_{[w\bullet\lambda, s_\alpha w\bullet\lambda]}$ has a Verma flag with factors $M_{A_\mathfrak{p}}(w\bullet\lambda)$ and $M_{A_\mathfrak{p}}(s_\alpha w\bullet\lambda)$, both occurring once. Since $P_A(\lambda) \otimes_A A_\mathfrak{p}$ is projective, the examples in 4.7 yield

$$(P_A(\lambda) \otimes_A A_\mathfrak{p})_{[w\bullet\lambda, s_\alpha w\bullet\lambda]} \simeq P_{A_\mathfrak{p}}(w\bullet\lambda) \qquad \text{for each } w \in W'.$$

We get now

$$\mathrm{End}_{\mathfrak{g}_{A_\mathfrak{p}}}(P_A(\lambda) \otimes_A A_\mathfrak{p}) \simeq \prod_{w\in W'} \mathrm{End}_{\mathfrak{g}_{A_\mathfrak{p}}} P_{A_\mathfrak{p}}(w\bullet\lambda)$$

and we know by 4.8(6) that

$$\mathrm{End}_{\mathfrak{g}_{A_\mathfrak{p}}} P_{A_\mathfrak{p}}(w\bullet\lambda) \xrightarrow{\sim} \{\, (u,v) \in A_\mathfrak{p} \times A_\mathfrak{p} \mid u \equiv v \mod A_\mathfrak{p}\alpha^\vee \,\}. \tag{7}$$

Both isomorphisms are compatible with (3).

It is now clear that any family as in the proposition belongs to the image of (3). Consider conversely a family $(u_w)_{w\in W}$ in the image of (3). Using (6) and (7) one gets $u_w \in \bigcap_{\mathfrak{p}\in\mathfrak{P}(A)} A_\mathfrak{p} = A$ for all $w \in W$. Using (7) we get for each $\alpha \in \Phi^+$ the additional condition that

$$u_w - u_{s_\alpha w} \in A_{(A\alpha^\vee)}\alpha^\vee \cap A = A\alpha^\vee$$

for all $w \in W$.

**4.12.** (**A functor to sheaves**) Keep $A$ and $\lambda$ as in 4.11. Proposition 4.11 implies that we can identify $\operatorname{End}_{\mathfrak{g}_A} P_A(\lambda)$ with the structure algebra of a moment graph $\mathcal{G}$: We set $\mathcal{V} = W$. Two vertices $w$ and $w'$ are joined by an edge if and only if there exists $\alpha \in \Phi^+$ with $w' = s_\alpha w$; if so, there is only one edge $E$ joining these vertices and we set $\alpha_E = \alpha^\vee$. We define the ordering on $\mathcal{V}$ such that $w \leq w'$ if and only if $w{\bullet}\lambda \leq w'{\bullet}\lambda$. This is a refinement of the Bruhat ordering on $W$: If $w \leq w'$ in the Bruhat ordering, then also $w \leq w'$ in our ordering that depends on $\lambda$, but the converse does not hold in general.

In the following we only consider $A$–sheaves on $\mathcal{G}$ and call them simply sheaves. As stated above we identify

$$\mathcal{Z} = \mathcal{Z}(\mathcal{G}) = \operatorname{End}_{\mathfrak{g}_A} P_A(\lambda). \tag{1}$$

For any $M$ in $\mathcal{O}^{\mathrm{VF}}_{A,\lambda}$ set

$$\mathbf{V}M = \operatorname{Hom}_{\mathfrak{g}_A}(P_A(\lambda), M). \tag{2}$$

Since $\mathcal{Z} = \operatorname{End}_{\mathfrak{g}_A} P_A(\lambda)$ is commutative, each $\mathbf{V}M$ has a natural structure as a $\mathcal{Z}$–module. It is clear that $\mathbf{V}$ is a functor from $\mathcal{O}^{\mathrm{VF}}_{A,\lambda}$ to the category of $\mathcal{Z}$–modules. This functor is exact because $P_A(\lambda)$ is projective.

By Corollary 4.10 each $\mathbf{V}M$ is free of finite rank as an $A$–module. We can therefore apply the functor $\mathcal{L}$ from 2.10 to $\mathbf{V}M$ and get a sheaf $\mathcal{L}(\mathbf{V}M)$ of finite type on $\mathcal{G}$. We get thus a functor $M \mapsto \mathcal{L}(\mathbf{V}M)$ from $\mathcal{O}^{\mathrm{VF}}_{A,\lambda}$ to the category of sheaves of finite type on $\mathcal{G}$. We are going to investigate this functor.

**Lemma:** *The moment graph $\mathcal{G}$ is a GKM-graph.*

*Proof*: As for the example following the definition in 2.15.

**4.13.** In order to calculate the functor $\mathcal{L}$ we have to know the idempotent elements $e_w$, $w \in W$. Under the isomorphism

$$\mathcal{Z} \otimes_A Q \xrightarrow{\sim} \operatorname{End}_{\mathfrak{g}_Q}(P_A(\lambda) \otimes_A Q) \xrightarrow{\sim} \prod_{x \in W} Q$$

$e_w$ corresponds to the map

$$P_A(\lambda) \otimes_A Q \longrightarrow (P_A(\lambda) \otimes_A Q)_w \hookrightarrow P_A(\lambda) \otimes_A Q$$

where the first map is the projection with kernel $\bigoplus_{x \neq w} (P_A(\lambda) \otimes_A Q)_x$. We get

$$e_w\, f = (f \otimes \mathrm{id}_Q) \circ e_w \tag{1}$$

for any $f \in \mathbf{V}M = \operatorname{Hom}_{\mathfrak{g}_A}(P_A(\lambda), M)$ and for any $M$ in $\mathcal{O}^{\mathrm{VF}}_{A,\lambda}$.

**Proposition:** (a) *Let $w \in W$. Then $\mathcal{L}(\mathbf{V}\, M_A(w{\bullet}\lambda))$ is the Verma sheaf $\mathcal{V}_A(w)$.*

(b) *Let $M$ be a module in $\mathcal{O}^{\mathrm{VF}}_{A,\lambda}$. Then the $A$–module $\mathbf{V}M$ is free of finite rank. The map $g^{\mathbf{V}M} \colon \mathbf{V}M \to \Gamma(\mathcal{L}(\mathbf{V}M))$ is an isomorphism.*

*Proof*: (a) Proposition 4.10 implies that $\mathbf{V}M_A(w{\bullet}\lambda)$ is a free $A$–module of rank 1. Choose a basis $f$. Since $\operatorname{Hom}_{\mathfrak{g}_Q}(M_Q(x{\bullet}\lambda), M_Q(w{\bullet}\lambda)) = 0$ for all $x \neq w$, we get $e_x f = (f \otimes \mathrm{id}_Q) \circ e_x = 0$ for all $x \neq w$. It follows that $e_w f = 1 f = f$. Now a comparison with 3.11 yields $\mathcal{L}(\mathbf{V}M_A(w{\bullet}\lambda)) \simeq \mathcal{V}_A(w)$.

(b) Corollary 4.10 yields the first claim. The second one follows now from Proposition 2.16 and Lemma 4.12.

**4.14.** (In this and the following subsection $A$ can be any $S$–algebra.) Let $D$ be a subset of $\mathfrak{h}^*$ with the following property: If $\lambda \in D$ and $\mu \in \mathfrak{h}^*$ with $\mu \leq \lambda$, then also $\mu \in D$. We set for any module $M$ in $\mathcal{O}_A$

$$O^D M = \sum_{\mu \notin D} U(\mathfrak{g}_A) M_\mu \qquad \text{and} \qquad M[D] = M/O^D M. \tag{1}$$

For example, we get for any $\mu \in \mathfrak{h}^*$

$$O^D M_A(\mu) = \begin{cases} M_A(\mu) & \text{if } \mu \notin D, \\ 0 & \text{if } \mu \in D, \end{cases} \quad \text{and} \quad M_A(\mu)[D] = \begin{cases} 0 & \text{if } \mu \notin D, \\ M_A(\mu) & \text{if } \mu \in D. \end{cases} \tag{2}$$

In case $\mu \notin D$ one uses that $M_A(\mu) = U(\mathfrak{g}_A) M_A(\mu)_\mu$, in case $\mu \in D$ one observes that $M_A(\mu)_\nu \neq 0$ implies $\nu \leq \mu$, hence $\nu \in D$.

Any homomorphism $f\colon M \to N$ in $\mathcal{O}$ restricts to a homomorphism $O^D M \to O^D N$ as $f(M_\mu) \subset N_\mu$ for all $\mu \in \mathfrak{h}^*$. It follows that $f$ induces a natural homomorphism $M[D] \to N[D]$. Both $M \mapsto O^D M$ and $M \mapsto M[D]$ are functors from $\mathcal{O}_A$ to itself.

**Proposition:** *Let $\mu \in \mathfrak{h}^*$, let $0 \to N \to M \to M_A(\mu) \to 0$ be a short exact sequence in $\mathcal{O}_A$. Then the sequences*

$$0 \to O^D N \longrightarrow O^D M \longrightarrow O^D M_A(\mu) \to 0 \tag{3}$$

*and*

$$0 \to N[D] \longrightarrow M[D] \longrightarrow M_A(\mu)[D] \to 0 \tag{4}$$

*are exact.*

*Proof*: We may assume that $N$ is a submodule of $M$. Then it is clear that $O^D N$ is a submodule of $O^D M$. Denote by $\varphi$ the map $M \to M_A(\mu)$ in the original short exact sequence.

Consider first the case where $\mu \in D$. Then all weights $\nu$ of $M_A(\mu)$ satisfy $\nu \in D$. Since $0 \to N_\nu \to M_\nu \to M_A(\mu)_\nu \to 0$ is exact for all $\nu \in \mathfrak{h}^*$, we get now that $M_\nu = N_\nu$ for all $\nu \notin D$, hence that $O^D N = O^D M$. Together with (2) this yields the exactness of (3) in this case. With respect to (4) observe that the map $N[D] \to M[D]$ is the inclusion of $N/O^D N$ into $M/O^D N = M/O^D M$. So its cokernel identifies with $M/N \simeq M_A(\mu) = M_A(\mu)[D]$.

We are left with the case $\mu \notin D$. Since $M_A(\mu)_\mu$ is free of rank 1 over $A$ and since $M_A(\mu)_\nu = \varphi(M_\nu)$ for all $\nu$, we can find $v \in M_\mu$ with $M_A(\mu)_\mu = A\varphi(v)$. Then the composition $u \mapsto uv \mapsto \varphi(uv) = u\varphi(v)$ is a bijection $U(\mathfrak{n}_A^-) \to M_A(\mu)$. It follows that

$$M = N \oplus U(\mathfrak{n}_A^-)\, v.$$

This is a decomposition as an $\mathfrak{h}_A$–module. We get therefore $M_\nu = N_\nu \oplus (U(\mathfrak{n}_A^-)\, v)_\nu$ for all $\nu \in \mathfrak{h}^*$. It follows that

$$O^D M = O^D N + U(\mathfrak{g}_A)\, v.$$

The triangular decomposition of $\mathfrak{g}_A$ yields $U(\mathfrak{g}_A)\, v = U(\mathfrak{n}_A^-)\, v + U(\mathfrak{g}_A)\, \mathfrak{n}^+ v$. We have $\varphi(\mathfrak{n}_A^+ v) = \mathfrak{n}_A^+ \varphi(v) = 0$, hence $\mathfrak{n}_A^+\, v \subset N$. Since $\mathfrak{n}_A^+ v \subset \bigoplus_{\alpha\in\Phi^+} M_{\mu+\alpha}$, we get even $\mathfrak{n}_A^+\, v \subset O^D N$, hence $U(\mathfrak{g}_A)\, v \subset U(\mathfrak{n}_A^-)\, v + O^D N$. It follows that

$$O^D M = O^D N \oplus U(\mathfrak{n}_A^-)\, v.$$

This shows that $N \cap O^D M = O^D N$ and that $\varphi(O^D M) = M_A(\mu)$. So we get the exactness of (3). We see also that $M = N + O^D M$. Therefore the exactness of (4) is just the fact that the map

$$N/O^D N = N/(N\cap O^D M) \longrightarrow (N + O^D M)/O^D M = M/O^D M$$

is an isomophism.

**Corollary:** *If $M$ belongs to $\mathcal{O}_A^{\mathrm{VF}}$, then both $O^D M$ and $M[D]$ belong to $\mathcal{O}_A^{\mathrm{VF}}$. We have then*

$$(O^D M : M_A(\mu)) = \begin{cases} (M : M_A(\mu)) & \text{if } \mu \notin D,\\ 0 & \text{if } \mu \in D,\end{cases}$$

*and*

$$(M[D] : M_A(\mu)) = \begin{cases} 0 & \text{if } \mu \notin D,\\ (M : M_A(\mu)) & \text{if } \mu \in D,\end{cases}$$

*for all $\mu \in \mathfrak{h}^*$.*

*Proof*: This follows immediately from (2) and the proposition using induction on the length of a Verma flag of $M$.

*Remark*: Let $D'$ be another subset of $\mathfrak{h}^*$ satisfying the same assumptions as $D$. Suppose that $D' \subset D$. We get then for each $M$ in $\mathcal{O}_A$ an inclusion $O^D M \hookrightarrow O^{D'} M$ and a surjection $\psi\colon M[D] \to M[D']$. If we divide the short exact sequence $0 \to O^{D'} M \to M \to M[D'] \to 0$ by $O^D M$, then we get a short exact sequence

$$0 \to O^{D'} M/O^D M \longrightarrow M[D] \xrightarrow{\psi} M[D'] \to 0.$$

Let $\pi\colon M \to M[D]$ denote the natural map. Since $\pi(M_\mu) = M[D]_\mu$ for all $\mu \in \mathfrak{h}^*$, we get $O^{D'}(M[D]) = \pi(O^{D'} M)$. So we can rewrite our short exact sequence as

$$0 \to O^{D'}(M[D]) \longrightarrow M[D] \xrightarrow{\psi} M[D'] \to 0. \tag{5}$$

**4.15.** Keep the assumption on $D$ from 4.14. Consider a short exact sequence

$$0 \to L \xrightarrow{\varphi} M \xrightarrow{\psi} N \to 0$$

in $\mathcal{O}_A$.

**Proposition:** *If $N$ belongs to $\mathcal{O}_A^{\mathrm{VF}}$, then the sequence*

$$0 \to L[D] \longrightarrow M[D] \longrightarrow N[D] \to 0$$

*is exact.*

*Proof*: We use induction over the length of a Verma flag of $N$. The claim is obvious for $N = 0$. If $N \neq 0$, then we choose a submodule $N'$ of $N$ such that $N'$ belongs to $\mathcal{O}_A^{\mathrm{VF}}$ and such that $N/N' \simeq M_A(\mu)$ for a suitable $\mu \in \mathfrak{h}^*$. Setting $M' = \psi^{-1}(N')$ we have a short exact sequence

$$0 \to L \longrightarrow M' \longrightarrow N' \to 0.$$

Consider now the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & L[D] \arrow[r] \arrow[d, "\mathrm{id}"'] & M'[D] \arrow[r] \arrow[d] & N'[D] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L[D] \arrow[r] \arrow[d] & M[D] \arrow[r] \arrow[d] & N[D] \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & 0 \arrow[r] \arrow[d] & M_A(\mu)[D] \arrow[r, "\mathrm{id}"] \arrow[d] & M_A(\mu)[D] \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The first column and the third row are trivially exact. The other columns are exact by Lemma 4.14, the first row is exact by induction. Now the 9-lemma yields the exactness of the middle row, since the composed map $L[D] \to N[D]$ is clearly 0.

*Remark*: In case $A$ is local and satisfies the assumptions in 4.7, one can also proceed as follows: Observe first that the functor $M \mapsto M[D]$ is right exact. Since $\mathcal{O}_A$ has enough projectives, we can use projective resolutions to compute left derived functors. Now one has to show that the higher derived functors vanish on $\mathcal{O}_A^{\mathrm{VF}}$. Here one uses that one has for each $\mu \in \mathfrak{h}^*$ a short exact sequence

$$0 \to N \longrightarrow P_A(\mu) \longrightarrow M_A(\mu) \to 0 \tag{1}$$

with $N$ in $\mathcal{O}_A^{\mathrm{VF}}$. In order to construct (1) one chooses $D = \{ \nu \in \mathfrak{h}^* \mid \nu \leq \mu \}$ and checks that $P_A(\mu)[D] \simeq M_A(\mu)$ for this choice. This follows from 4.7(1).

**4.16.** From now on $A$ is again the local ring of $S$ at $S\mathfrak{h}$. We return to the set-up from 4.12. Let $\mathcal{H} = (\mathcal{V}', \mathcal{E}', \alpha', \leq')$ be an F-open full subgraph of $\mathcal{G}$. Set $D$ equal to the set of all $\nu \in \mathfrak{h}^*$ such that there exists $x \in \mathcal{V}'$ with $\nu \leq x_\bullet\lambda$. Then $D$ satisfies the assumption in 4.14. Since $\mathcal{H}$ is F-open, we get for any $w \in W$ that $w_\bullet\lambda \in D$ if and only if $w \in \mathcal{V}'$.

For any $M$ in $\mathcal{O}_{A,\lambda}^{\mathrm{VF}}$ set

$$O^{\mathcal{H}}M = O^D M \qquad \text{and} \qquad M[\mathcal{H}] = M[D]. \tag{1}$$

Then both $O^{\mathcal{H}}M$ and $M[\mathcal{H}]$ belong to $\mathcal{O}_{A,\lambda}^{\mathrm{VF}}$. The factors in a Verma flag of $O^{\mathcal{H}}M$ have the form $M_A(w_\bullet\lambda)$ with $w \notin \mathcal{V}'$, those for $M[\mathcal{H}]$ he form $M_A(w_\bullet\lambda)$ with $w \in \mathcal{V}'$.

Set (as in 3.14) $e_{\mathcal{H}} = \sum_{x\in\mathcal{V}'} e_x$.

**Lemma:** *There is a natural isomorphism* $e_{\mathcal{H}}\mathbf{V}M \xrightarrow{\sim} \mathbf{V}(M[\mathcal{H}])$.

*Proof*: The short exact sequence

$$0 \to O^{\mathcal{H}}M \longrightarrow M \xrightarrow{\varphi} M[\mathcal{H}] \to 0 \tag{2}$$

induces a short exact sequence

$$0 \to \mathbf{V}(O^{\mathcal{H}}M) \longrightarrow \mathbf{V}M \xrightarrow{\mathbf{V}\varphi} \mathbf{V}(M[\mathcal{H}]) \to 0. \tag{3}$$

We want to show for any $f \in \mathbf{V}M$ that $\mathbf{V}\varphi(f) = 0$ if and only if $e_{\mathcal{H}}f = 0$.

The $\mathfrak{g}_Q$–module $M \otimes_A Q$ is semi-simple and $O^{\mathcal{H}}M \otimes_A Q$ is the sum of its isotypic components of type $M_Q(x_\bullet\lambda)$ with $x \notin \mathcal{V}'$. This implies $\mathrm{Hom}_{\mathfrak{g}_Q}(M_Q(w_\bullet\lambda), O^{\mathcal{H}}M \otimes_A Q) = 0$ for all $w \in \mathcal{V}'$. If $f \in \mathbf{V}O^{\mathcal{H}}M$, then we get now $(f \otimes \mathrm{id}_Q) \circ e_w = 0$, hence $e_w f = 0$ for all $w \in \mathcal{V}'$, hence $e_{\mathcal{H}}f = 0$.

Consider on the other hand $f \in \mathbf{V}M$ with $e_{\mathcal{H}}f = 0$. Then $f \otimes \mathrm{id}_Q$ annihilates all summands $(P_A(\lambda) \otimes Q)_w \simeq M_Q(w_\bullet\lambda)$ with $w \in \mathcal{V}'$. Therefore the image of $f \otimes \mathrm{id}_Q$ is contained in $O^{\mathcal{H}}M \otimes_A Q$. (Recall the description as a sum of isotypic components.) It follows that

$$f(P_A(\lambda)) \subset M \cap (O^{\mathcal{H}}M \otimes_A Q) = O^{\mathcal{H}}M$$

where the equality holds because (2) splits over $A$. (All modules are free over $A$.) We get thus $f \in \mathbf{V}O^{\mathcal{H}}M$.

Now (3) shows that we get an isomorphism $e_{\mathcal{H}}\mathbf{V}M \xrightarrow{\sim} \mathbf{V}(M[\mathcal{H}])$ mapping any $e_{\mathcal{H}}f$ with $f \in \mathbf{V}M$ to $\mathbf{V}\varphi(f)$.

**4.17. Proposition:** *Let $M$ be a module in $\mathcal{O}_A^{\mathrm{VF}}$. Then the sheaf $\mathcal{L}(\mathbf{V}M)$ is flabby. There is for each F-open full subgraph $\mathcal{H}$ of $\mathcal{G}$ an isomorphism*

$$\Gamma(\mathcal{H}, \mathcal{L}(\mathbf{V}M)) \xrightarrow{\sim} \mathbf{V}(M[\mathcal{H}]). \tag{1}$$

*For each $x \in W$ the $A$–module $\mathcal{L}(\mathbf{V}M)_{[x]}$ is free of rank $(M : M_A(x_\bullet\lambda))$*

*Proof*: We get from Proposition 4.13.b and Lemma 4.16 for any $\mathcal{H}$ as in the proposition that $e_{\mathcal{H}}\mathbf{V}M \simeq \mathbf{V}(M[\mathcal{H}])$ is a free $A$–module of finite rank. So Proposition 3.14 yields the flabbiness and the isomorphism in (1).

Fix now $x \in W$. The inclusion of the full subgraph $\mathcal{G}_{<x}$ in $\mathcal{G}_{\le x}$ yields by 4.14(5) a short exact sequence

$$0 \to O^{\mathcal{G}_{<x}}(M[\mathcal{G}_{\le x}]) \longrightarrow M[\mathcal{G}_{\le x}] \xrightarrow{\psi} M[\mathcal{G}_{<x}] \to 0.$$

Set $N = O^{\mathcal{G}_{<x}}(M[\mathcal{G}_{\le x}])$. Using Corollary 4.14 one checks that $N$ has a Verma flag of length $(M : M_A(x{\bullet}\lambda))$ with all factors isomorphic to $M_A(x{\bullet}\lambda)$.

Applying $\mathbf{V}$ we get a short exact sequence of $\mathcal{Z}$–modules

$$0 \to \mathbf{V}N \longrightarrow \mathbf{V}(M[\mathcal{G}_{\le x}]) \xrightarrow{\mathbf{V}\psi} \mathbf{V}(M[\mathcal{G}_{<x}]) \to 0.$$

We use Lemma 4.16 to identify $\mathbf{V}(M[\mathcal{G}_{\le x}])$ with $e_{\mathcal{G}_{\le x}}\mathbf{V}M$ and $\mathbf{V}(M[\mathcal{G}_{<x}])$ with $e_{\mathcal{G}_{<x}}\mathbf{V}M$. Then $\mathbf{V}\psi$ is given by $u \mapsto e_{\mathcal{G}_{<x}}u$. Therefore $\mathbf{V}\psi$ corresponds under the isomorphism (1) to the restriction map

$$\Gamma(\mathcal{G}_{\le x}, \mathcal{L}(\mathbf{V}M)) \longrightarrow \Gamma(\mathcal{G}_{<x}, \mathcal{L}(\mathbf{V}M))$$

with kernel $\mathcal{L}(\mathbf{V}M)_{[x]}$, see 3.13(1). We get thus an isomorphism $\mathcal{L}(\mathbf{V}M)_{[x]} \simeq \mathbf{V}N$. Now the last claim follows from the description of the Verma flag of $N$.

**Corollary:** *If $0 \to L \to M \to N \to 0$ is a short exact sequence in $\mathcal{O}_A^{\mathrm{VF}}$, then the sequence*

$$0 \to \Gamma(\mathcal{H}, \mathcal{L}(\mathbf{V}L)) \to \Gamma(\mathcal{H}, \mathcal{L}(\mathbf{V}M)) \to \Gamma(\mathcal{H}, \mathcal{L}(\mathbf{V}N)) \to 0$$

*is exact for each F-open full subgraph $\mathcal{H}$ of $\mathcal{G}$.*

*Proof*: This follows from (1) and Proposition 4.15

**4.18.** We want to show: If $P$ is a projective object in $\mathcal{O}_A$, then $\mathcal{L}(\mathbf{V}P)$ is an F-projective sheaf. Any image under $\mathcal{L}$ is generated by global sections, see 2.17(A). Proposition 4.17 implies that $\mathcal{L}(\mathbf{V}P)$ is flabby. So the first condition (A) in 3.8 is satisfied. We now want to prove 3.8(B).

**Lemma:** *If $P$ is a projective object in $\mathcal{O}_A$, then any $\mathcal{L}(\mathbf{V}P)_w$ with $w \in W$ is a free $A$–module of rank $(P : M_A(w{\bullet}\lambda))$.*

*Proof*: Set $\mu = w{\bullet}\lambda$ and $r = (P : M_A(w{\bullet}\lambda))$. Recall that $\mathcal{L}(\mathbf{V}P)_w = e_w\mathbf{V}P$ by definition. We want to show that there exists an isomorphism of $A$–modules

$$e_w\mathbf{V}P \xrightarrow{\sim} \mathbf{V}(M_A(\mu)^r). \tag{1}$$

Since $\mathbf{V}(M_A(\mu)^r)$ is free of rank $r$ over $A$ (cf. Proposition 4.10), this will imply the lemma.

We know by Proposition 4.10 that $\mathrm{Hom}_{\mathfrak{g}_A}(P, M_A(\mu))$ is a free $A$–module of rank $r$. Choose a basis $f_1, f_2, \dots, f_r$ for this module. Denote by $f$ the homomorphism

$$f\colon P \longrightarrow M_A(\mu)^r, \qquad v \mapsto (f_1(v), f_2(v), \dots, f_r(v)). \tag{2}$$

Then $f$ induces an $A$–linear map $\mathbf{V}f\colon \mathbf{V}P \to \mathbf{V}(M_A(\mu)^r)$; we want to show for any $g \in \mathbf{V}P$ that

$$\mathbf{V}f\,(g) = 0 \iff e_w g = 0. \tag{3}$$

It follows from 4.8(4) that the $f_i \otimes 1$ are a basis for $\mathrm{Hom}_{\mathfrak{g}_Q}(P \otimes_A Q, M_A(\mu) \otimes_A Q)$. Looking at the decomposition $P \otimes_A Q = \bigoplus_{x \in W} (P \otimes_A Q)_x$ we see that $f \otimes 1$ has kernel $\bigoplus_{x \neq w} (P \otimes_A Q)_x$ and restricts to an isomorphism $(P \otimes_A Q)_w \xrightarrow{\sim} M_Q(\mu)^r$.

Consider now $g \in \mathbf{V}P = \mathrm{Hom}_{\mathfrak{g}_A}(P_A(\lambda), P)$. We have

$$e_w g = 0 \iff (g \otimes 1)\,((P \otimes_A Q)_w) = 0.$$

Since $(g \otimes 1)\,((P \otimes_A Q)_x) \subset (P \otimes_A Q)_x$ for all $x$, this condition is equivalent to

$$(g \otimes 1)\,((P \otimes_A Q)_w) \subset \bigoplus_{x \neq w} (P \otimes_A Q)_x = \ker f \otimes 1,$$

hence to $(f \otimes 1) \circ (g \otimes 1) = 0$, hence to $0 = f \circ g = \mathbf{V}f\,(g)$. This proves (3).

We get thus a well-defined injective homomorphism of $A$–modules

$$e_w \mathbf{V}P \longrightarrow \mathbf{V}(M_A(\mu)^r) \qquad \text{with } e_w g \mapsto \mathbf{V}f\,(g) \text{ for all } g \in \mathbf{V}P. \tag{4}$$

Now (1), and hence the lemma, follow once we show that this map is surjective. But that amounts to proving that $\mathbf{V}f\colon \mathbf{V}P \to \mathbf{V}(M_A(\mu)^r)$ is surjective. By the Nakayama lemma it suffices to show that $\mathbf{V}f$ becomes surjective after reduction modulo the maximal ideal $\mathfrak{m}$ of $A$. By 4.8(3) the reduction modulo $\mathfrak{m}$ of $\mathbf{V}f$ identifies with the map

$$\mathrm{Hom}_{\mathfrak{g}}(P(\lambda), P \otimes_A \mathbf{C}) \longrightarrow \mathrm{Hom}_{\mathfrak{g}}(P(\lambda), M(\mu)^r), \qquad g \mapsto \overline{f} \circ g \tag{5}$$

where $\overline{f}\colon P \otimes_A \mathbf{C} \to M(\mu)^r$ is the reduction modulo $\mathfrak{m}$ of $f$.

By 4.8(3) the reductions modulo $\mathfrak{m}$ of the $f_i$ are a basis for $\mathrm{Hom}_{\mathfrak{g}}(P \otimes_A \mathbf{C}, M(\mu))$. Furthermore $P \otimes_A \mathbf{C}$ is a projective object in $\mathcal{O}$, see Proposition 4.7.c. Therefore our claim follows now from part (b) in the following lemma:

**4.19. Lemma:** *Let $\mu \in W_\bullet\lambda$.*

(a) *We have* $\dim \mathrm{Hom}_{\mathfrak{g}}(M, M(\mu)) \leq \dim \mathrm{Hom}_{\mathfrak{g}}(P(\lambda), M)$ *for any $M$ in $\mathcal{O}$.*

(b) *Let $P$ be a projective object in $\mathcal{O}$. Choose a basis $f_1, f_2, \ldots, f_r$ for* $\mathrm{Hom}_{\mathfrak{g}}(P, M(\mu))$. *Denote by $f$ the homomorphism*

$$f\colon P \longrightarrow M(\mu)^r, \qquad v \mapsto (f_1(v), f_2(v), \ldots, f_r(v)). \tag{1}$$

*Then the map*

$$\mathrm{Hom}_{\mathfrak{g}}(P(\lambda), P) \longrightarrow \mathrm{Hom}_{\mathfrak{g}}(P(\lambda), M(\mu)^r), \qquad g \mapsto f \circ g \tag{2}$$

*is surjective.*

*Proof*: (a) The module $M(\lambda)$ is simple. By a theorem of Verma there exists an injective homomorphism of $\mathfrak{g}$–modules $g: M(\lambda) \to M(\mu)$, cf. [Di], 7.6.23. The image of $g$ is the only simple submodule of $M(\mu)$, hence equal to its socle, cf. [Ja], Bemerkung 3 in 5.3.

Since $P(\lambda)$ is a projective cover of $L(\lambda) = M(\lambda)$, we have a surjective homomorphism $\pi: P(\lambda) \to M(\lambda)$. Then $g \circ \pi$ is a non-zero homomorphism $P(\lambda) \to M(\mu)$ with image equal to the socle of $M(\mu)$. It is in fact a basis for $\mathrm{Hom}_{\mathfrak{g}}(P(\lambda), M(\mu))$ since this Hom space has dimension $[M(\mu) : L(\lambda)] = 1$.

Consider the bilinear pairing

$$\mathrm{Hom}_{\mathfrak{g}}(P(\lambda), M) \times \mathrm{Hom}_{\mathfrak{g}}(M, M(\mu)) \longrightarrow \mathrm{Hom}_{\mathfrak{g}}(P(\lambda), M(\mu)) \simeq \mathbf{C}, \qquad (\varphi, \psi) \mapsto \psi \circ \varphi.$$

The claim follows if we can show for any $\psi \in \mathrm{Hom}_{\mathfrak{g}}(M), M(\mu))$, $\psi \neq 0$, that there exists $\varphi \in \mathrm{Hom}_{\mathfrak{g}}(P(\lambda), M)$ with $\psi \circ \varphi \neq 0$.

Well, $\psi \neq 0$ implies that $\psi(M)$ contains the socle $g(M(\lambda))$ of $M(\mu)$. Set $N = \psi^{-1}(g(M(\lambda)))$. Since $P(\lambda)$ is projective, we get a homomorphism $\varphi: P(\lambda) \to N$ with $\psi_{|N} \circ \varphi = g \circ \pi$. We have in particular that $\psi \circ \varphi \neq 0$, as desired.

(b) The map in (2) factors

$$\mathrm{Hom}_{\mathfrak{g}}(P(\lambda), P) \longrightarrow \mathrm{Hom}_{\mathfrak{g}}(P(\lambda), f(P)) \hookrightarrow \mathrm{Hom}_{\mathfrak{g}}(P(\lambda), M(\mu)^r).$$

The first map is surjective since $P(\lambda)$ is projective. So it suffices to show that

$$\mathrm{Hom}_{\mathfrak{g}}(P(\lambda), f(P)) = \mathrm{Hom}_{\mathfrak{g}}(P(\lambda), M(\mu)^r).$$

We have $\dim \mathrm{Hom}_{\mathfrak{g}}(P(\lambda), M(\mu)^r) = r$. Therefore it is enough to show that

$$\dim \mathrm{Hom}_{\mathfrak{g}}(P(\lambda), f(P)) \geq r.$$

Now (a) reduces us to showing that $\dim \mathrm{Hom}_{\mathfrak{g}}(f(P), M(\mu)) \geq r$.

Denote by $\pi_i: f(P) \to M(\mu)$, $1 \leq i \leq r$, denote the restriction to $f(P)$ of the $i$th projection $M(\mu)^r \to M(\mu)$. It suffices to prove that the $\pi_i$ are linearly independent over $\mathbf{C}$. But if $\sum_{i=1}^r a_i \pi_i = 0$ with all $a_i \in \mathbf{C}$, then $0 = \sum_{i=1}^r a_i \pi_i(f(v)) = \sum_{i=1}^r a_i f_i(v)$ for all $v \in P$, hence $\sum_{i=1}^r a_i f_i = 0$. Since the $f_i$ are a basis, this yields $a_i = 0$ for all $i$, as desired.

**4.20. Proposition:** *If $P$ is a projective object in $\mathcal{O}_A^{\mathrm{VF}}$, then $\mathcal{L}(\mathbf{V}P)$ is an F-projective sheaf.*

*Proof*: The condition 3.8(A) holds by the introductory remarks in 4.18. So does 3.8(B) by Lemma 4.18. It remains to check 3.8(C).

Consider $x \in W$ and $\alpha \in \Phi^+$ with $x{\scriptstyle\bullet}\lambda < s_\alpha x{\scriptstyle\bullet}\lambda$. Let $E$ denote the edge joining $x$ and $s_\alpha x$. We have to show that $\rho_{x,E}^{\mathcal{L}(\mathbf{V}P)}$ induces an isomorphism

$$\mathcal{L}(\mathbf{V}P)_x / \alpha^\vee \mathcal{L}(\mathbf{V}P)_x \xrightarrow{\sim} \mathcal{L}(\mathbf{V}P)_E. \tag{1}$$

Since $\rho_{x,E}^{\mathcal{L}(\mathbf{V}P)}$ is surjective by 2.10(6) and since $\alpha^\vee \mathcal{L}(\mathbf{V}P)_x$ is always contained in the kernel of $\rho_{x,E}^{\mathcal{L}(\mathbf{V}P)}$, it suffices to show that

$$\ker \rho_{x,E}^{\mathcal{L}(\mathbf{V}P)} \subset \alpha^\vee \mathcal{L}(\mathbf{V}P)_x. \tag{2}$$

Since $\mathcal{L}(\mathbf{V}P)_x$ is a free $A$–module and since $A = \bigcap_{\mathfrak{p}\in\mathfrak{P}(A)} A_\mathfrak{p}$ we have

$$\mathcal{L}(\mathbf{V}P)_x = \bigcap_{\mathfrak{p}\in\mathfrak{P}(A)} \mathcal{L}(\mathbf{V}P)_x \otimes_A A_\mathfrak{p} \subset \mathcal{L}(\mathbf{V}P)_x \otimes_A Q,$$

similarly for $\alpha^\vee \mathcal{L}(\mathbf{V}P)_x$. So it suffices to show that

$$\ker \rho_{x,E}^{\mathcal{L}(\mathbf{V}P)} \subset \alpha^\vee (\mathcal{L}(\mathbf{V}P)_x \otimes_A A_\mathfrak{p}) = \alpha^\vee e_x(\mathbf{V}P \otimes_A A_\mathfrak{p}) \tag{3}$$

for all $\mathfrak{p} \in \mathfrak{P}(A)$.

Any $\mathfrak{p} \in \mathfrak{P}(A)$ has the form $\mathfrak{p} = A\gamma$ with $\gamma \in S$ an irreducible polynomial with constant term 0. If $\mathbf{C}\gamma \neq \mathbf{C}\alpha^\vee$, then $\alpha^\vee$ is a unit in $A_\mathfrak{p}$ and (3) is trivially satisfied. So assume from now on that $\mathfrak{p} = A\alpha^\vee$.

Set $W' = \{\, w \in W \mid w^{-1}\alpha \in \Phi^+ \,\}$. For each $M$ in $\mathcal{O}_{A,\lambda}^{\mathrm{VF}}$ the block decomposition of $M \otimes_A A_\mathfrak{p}$ has the form $M \otimes_A A_\mathfrak{p} = \bigoplus_{w\in W'} (M \otimes_A A_\mathfrak{p})_w$ where each $(M \otimes_A A_\mathfrak{p})_w$ has a Verma flag with factors of the forn $M_{A_\mathfrak{p}}(w{\scriptstyle\bullet}\lambda)$ and $M_{A_\mathfrak{p}}(s_\alpha w{\scriptstyle\bullet}\lambda)$. We have

$$(M \otimes_A A_\mathfrak{p})_w = M \otimes_A A_\mathfrak{p} \cap \Big( (M \otimes_A Q)_w \oplus (M \otimes_A Q)_{s_\alpha w} \Big)$$

in the notation from 4.11(1).

Since $P \otimes_A A_\mathfrak{p}$ is projective, any indecomposable summand of $(P \otimes_A A_\mathfrak{p})_w$ is isomorphic to $P_{A_\mathfrak{p}}(w{\scriptstyle\bullet}\lambda)$ or to $P_{A_\mathfrak{p}}(s_\alpha w{\scriptstyle\bullet}\lambda)$. We have seen in the proof of 4.11 that $(P_A(\lambda) \otimes_A A_\mathfrak{p})_w \simeq P_{A_\mathfrak{p}}(w{\scriptstyle\bullet}\lambda)$ for all $w \in W$. Furthermore, we get

$$(P_A(\lambda) \otimes_A A_\mathfrak{p})_w = (e_w + e_{s_\alpha w})\,(P_A(\lambda) \otimes_A A_\mathfrak{p}).$$

Set $y = s_\alpha x$. Recall that $\mathbf{V}P(E) = (e_x + e_y)\mathbf{V}P + \alpha^\vee e_x \mathbf{V}P$ and that $\ker \rho_{x,E}^{\mathcal{L}(\mathbf{V}P)}$ is the set of all $e_x u$ with $u \in \mathbf{V}P(E)$ and $e_y u = 0$, see 2.9(2). If $u = (e_x + e_y)v_1 + \alpha^\vee e_x v_2$ with $v_1, v_2 \in \mathbf{V}P$, then $e_y u = 0$ is equivalent to $e_y(e_x + e_y)v_1 = 0$ and $e_x u \in \alpha^\vee e_x(\mathcal{L}(\mathbf{V}P) \otimes_A A_\mathfrak{p})$ is equivalent to $e_x(e_x + e_y)v_1 \in \alpha^\vee e_x(\mathcal{L}(\mathbf{V}P) \otimes_A A_\mathfrak{p})$. Therefore it suffices to show that

$$\{\, e_x u \mid u \in (e_x + e_y)\mathbf{V}P,\ e_y u = 0 \,\} \subset \alpha^\vee e_x(\mathbf{V}P \otimes_A A_\mathfrak{p}). \tag{4}$$

We can identify $\mathbf{V}P \otimes A_\mathfrak{p}$ with

$$\mathrm{Hom}_{\mathfrak{g}_{A_\mathfrak{p}}}(P_A(\lambda) \otimes_A A_\mathfrak{p}, P \otimes_A A_\mathfrak{p}) = \prod_{w\in W'} \mathrm{Hom}_{\mathfrak{g}_{A_\mathfrak{p}}}((P_A(\lambda) \otimes_A A_\mathfrak{p})_w, (P \otimes_A A_\mathfrak{p})_w),$$

hence $(e_x+e_y)\,(\mathbf{V}P\otimes A_{\mathfrak p})$ with

$$\operatorname{Hom}_{\mathfrak g_{A_{\mathfrak p}}}((P_A(\lambda)\otimes_A A_{\mathfrak p})_x,(P\otimes_A A_{\mathfrak p})_x)\simeq \operatorname{Hom}_{\mathfrak g_{A_{\mathfrak p}}}(P_{A_{\mathfrak p}}(x\bullet\lambda),(P\otimes_A A_{\mathfrak p})_x).$$

And we have

$$e_x(\mathbf{V}P\otimes_A A_{\mathfrak p})=e_x(e_x+e_y)\,(\mathbf{V}P\otimes_A A_{\mathfrak p})\simeq e_x\operatorname{Hom}_{\mathfrak g_{A_{\mathfrak p}}}(P_{A_{\mathfrak p}}(x\bullet\lambda),(P\otimes_A A_{\mathfrak p})_x).$$

So our claim will follow once we show: If $f\in\operatorname{Hom}_{\mathfrak g_{A_{\mathfrak p}}}(P_{A_{\mathfrak p}}(x\bullet\lambda),(P\otimes_A A_{\mathfrak p})_x)$ with $e_yf=0$, then $e_xf\in\alpha^\vee e_x\operatorname{Hom}_{\mathfrak g_{A_{\mathfrak p}}}(P_{A_{\mathfrak p}}(x\bullet\lambda),(P\otimes_A A_{\mathfrak p})_x)$. Here we can replace $(P\otimes_A A_{\mathfrak p})_x$ by its indecomposable summands, hence by $P_{A_{\mathfrak p}}(x\bullet\lambda)$ and $P_{A_{\mathfrak p}}(y\bullet\lambda)$. We have $P_{A_{\mathfrak p}}(y\bullet\lambda)\simeq M_{A_{\mathfrak p}}(y\bullet\lambda)$, hence $e_xf=0$ for all $f\in\operatorname{Hom}_{\mathfrak g_{A_{\mathfrak p}}}(P_{A_{\mathfrak p}}(x\bullet\lambda),P_{A_{\mathfrak p}}(y\bullet\lambda))$; so in this case the claim is obvious.

It remains to look at

$$f\in\operatorname{End}_{\mathfrak g_{A_{\mathfrak p}}}P_{A_{\mathfrak p}}(x\bullet\lambda)=A_{\mathfrak p}(e_x+e_y)+A_{\mathfrak p}\alpha^\vee e_x,$$

cf. 4.11(7). If $f=a(e_x+e_y)+b\alpha^\vee e_x$ with $a,b\in A_{\mathfrak p}$, then $e_yf=ae_y$. So $e_yf=0$ implies $a=0$, hence $e_xf=b\alpha^\vee e_x\in\alpha^\vee e_x\operatorname{End}_{\mathfrak g_{A_{\mathfrak p}}}P_{A_{\mathfrak p}}(x\bullet\lambda)$ as desired.

**4.21. Proposition:** *The functors* $\mathbf{V}$ *and* $\mathcal{L}$ *induce natural isomorphisms*

$$\operatorname{Hom}_{\mathfrak g_A}(M,N)\xrightarrow{\sim}\operatorname{Hom}_{\mathcal Z}(\mathbf{V}M,\mathbf{V}N)\xrightarrow{\sim}\operatorname{Hom}(\mathcal{L}(\mathbf{V}M),\mathcal{L}(\mathbf{V}N))$$

*for any* $M$ *and* $N$ *in* $\mathcal{O}_{A,\lambda}^{\mathrm{VF}}$.

*Proof*: Propositions 4.13.b and 2.14 yield isomorphisms

$$\operatorname{Hom}_{\mathcal Z}(\mathbf{V}M,\mathbf{V}N)\xrightarrow{\sim}\operatorname{Hom}_{\mathcal Z}(\mathbf{V}M,\Gamma(\mathcal{L}(\mathbf{V}N)))\xrightarrow{\sim}\operatorname{Hom}(\mathcal{L}(\mathbf{V}M),\mathcal{L}(\mathbf{V}N)).$$

The composition is the second isomorphism in the proposition.

Since $\mathbf{V}M$ and $\mathbf{V}N$ are free modules of finite rank over $A$, one gets

$$\operatorname{Hom}_{\mathcal Z}(\mathbf{V}M,\mathbf{V}N)=\bigcap_{\mathfrak p\in\mathfrak P(A)}\operatorname{Hom}_{\mathcal Z\otimes_A A_{\mathfrak p}}(\mathbf{V}M\otimes_A A_{\mathfrak p},\mathbf{V}N\otimes_A A_{\mathfrak p}).$$

A comparison with 4.8(2) shows that it suffices to show for all $\mathfrak p\in\mathfrak P(A)$ that $\mathbf{V}$ induces an isomorphism

$$\operatorname{Hom}_{\mathfrak g_{A_{\mathfrak p}}}(M\otimes_A A_{\mathfrak p},N\otimes_A A_{\mathfrak p})\xrightarrow{\sim}\operatorname{Hom}_{\mathcal Z\otimes_A A_{\mathfrak p}}(\mathbf{V}M\otimes_A A_{\mathfrak p},\mathbf{V}N\otimes_A A_{\mathfrak p}).$$

For this claim I have to refer to [F2], Thm 5.

*Remark*: We get in particular for any $M$ in $\mathcal{O}_{A,\lambda}^{\mathrm{VF}}$ that $\mathcal{L}\circ\mathbf{V}$ induces an algebra isomorphism $\operatorname{End}_{\mathfrak g_A}M\xrightarrow{\sim}\operatorname{End}\mathcal{L}(\mathbf{V}M)$.

**4.22. Theorem:** *We have $\mathcal{L}(\mathbf{V}P_A(w\bullet\lambda)) \simeq \mathcal{B}(w) \otimes_S A$ for all $w \in W$.*

*Proof*: We know by Proposition 4.20 that $\mathcal{L}(\mathbf{V}P_A(w\bullet\lambda))$ is F-projective, hence by Proposition 3.17 isomorphic to a direct sum of suitable $\mathcal{B}(z_i) \otimes_S A$ with $z_i \in W$.

Since $P_A(w\bullet\lambda)$ is indecomposable, the only idempotents in $\mathrm{End}_{\mathfrak{g}_A} P_A(w\bullet\lambda)$ are 0 and 1. Now the remark in 4.21 shows that 0 and 1 are the only idempotents in $\mathrm{End}\,\mathcal{L}(\mathbf{V}P_A(w\bullet\lambda))$, hence that $\mathcal{L}(\mathbf{V}P_A(w\bullet\lambda))$ is indecomposable.

So there exists $z \in W$ with $\mathcal{L}(\mathbf{V}P_A(w\bullet\lambda)) \simeq \mathcal{B}(z)\otimes_S A$. The construction of $\mathcal{B}(z)$ shows that $z$ is the smallest element $x$ in $W$ with $\mathcal{B}(z)_x \neq 0$, hence with $\mathcal{L}(\mathbf{V}P_A(w\bullet\lambda))_x \neq 0$. Now Lemma 4.18 implies that $z$ is the smallest element $x$ in $W$ with $0 \neq (P_A(w\bullet\lambda) : M_A(x\bullet\lambda)) = [M(x\bullet\lambda) : L(w\bullet\lambda)]$, hence $z = w$.

**Corollary:** *We have $[M(x\bullet\lambda) : L(w\bullet\lambda)] = \mathrm{rank}_S\,\mathcal{B}(w)_x$ for all $w, x \in W$.*

*Proof*: We have $[M(x\bullet\lambda) : L(w\bullet\lambda)] = (P_A(w\bullet\lambda) : M_A(x\bullet\lambda))$ by 4.7(1). Lemma 4.18 shows that this number is equal to the rank of the $A$–module $\mathcal{L}(\mathbf{V}P_A(w\bullet\lambda))_x$. Now apply the theorem.

**4.23.** (**The Kazhdan-Lusztig conjecture**) Let $G^\vee$ be a connected semi-simple algebraic group over $\mathbf{C}$ with a Borel subgroup $B^\vee$ and a maximal torus $T^\vee \subset B^\vee$ such that the root system of $G^\vee$ with respect to $T^\vee$ identifies with the dual of the root system of our Lie algebra $\mathfrak{g}$ with respect to $\mathfrak{h}$.

We can identify the Weyl group of $G^\vee$ with respect to $T^\vee$ with our Weyl group $W$; then $(\mathrm{Lie}T^\vee)^*$ identifies as a $W$–module with $\mathfrak{h}$, hence the symmetric algebra $S((\mathrm{Lie}T^\vee)^*)$ with $S = S(\mathfrak{h})$.

Consider the $T^\vee$–variety $G^\vee/B^\vee$ as in 1.13. We associate to this flag variety a moment graph $\mathcal{G}'$ as in 2.1 choosing its ordering $\leq'$ as in 3.6. Then $\mathcal{G}'$ identifies with $\mathcal{G}$ as an unordered moment graph, but the ordering differs. Denote by $\leq_{\mathrm{Br}}$ the Bruhat ordering on $W$. Then we have $x \leq' y$ if and only if $y \leq_{\mathrm{Br}} x$. On the other hand, the ordering introduced in 4.12 is a refinement of $\leq_{\mathrm{Br}}$.

Let $\mathcal{B}'(z)$ denote the Braden-MacPherson sheaf on $\mathcal{G}'$ associated to some $z \in W$. We claim that there is a close relationship between these $\mathcal{B}'(z)$ and our Braden-MacPherson sheaves $\mathcal{B}(w)$ on $\mathcal{G}$. More precisely, we have for all $w, x \in W$ an isomorphism of $S$–modules

$$\mathcal{B}(w)_x \simeq \mathcal{B}'(ww_0)_{xw_0} \tag{1}$$

where $w_0$ is the longest element in the Weyl group.

In order to check (1) consider an auxiliary moment graph $\mathcal{G}_{\mathrm{Br}}$ that coincides with $\mathcal{G}$ as an unordered moment graph and where we replace $\leq$ by $\leq_{\mathrm{Br}}$. The first thing to observe is that the $\mathcal{B}(w)$ are also the Braden-MacPherson sheaves for $\mathcal{G}_{\mathrm{Br}}$. This holds because we have for any two vertices $x$ and $y$ joined by an edge in $\mathcal{G}$ (or, equivalently, in $\mathcal{G}_{\mathrm{Br}}$) that $x \leq y$ is equivalent to $x \leq_{\mathrm{Br}} y$. And a look at the construction of the Braden-MacPherson sheaves shows that they only depend on the ordering of the vertices joined by edges.

Now (1) follows from the fact that we have an isomorphism $\mathcal{G}_{\mathrm{Br}} \xrightarrow{\sim} \mathcal{G}'$ of ordered moment graphs that takes any vertex $w$ to the vertex $ww_0$. In order to check that this map is compatible with the orderings one uses that $x \leq_{\mathrm{Br}} y$ if and only if $yw_0 \leq_{\mathrm{Br}} xw_0$ (for any $x, y \in W$).

Now (1) combined with Theorem 3.6 and Corollary 4.22 yields

$$[M(x\bullet\lambda) : L(w\bullet\lambda)] = \dim_{\mathbf{C}} IH^\bullet(\overline{C}_{ww_0})_{\{xw_0\}} \qquad \text{for all } x, w \in W \tag{2}$$

where $C_y = B^\vee y B^\vee / B^\vee$ for all $y \in W$. Now Kazhdan and Lusztig proved in [KL] that the right hand side in (2) is the value at 1 of a certain Kazhdan-Lusztig polynomial. If we plug this result into (2), then we get the statement of the Kazhdan-Lusztig conjecture.

**4.24.** I have followed here [F3], but made a few simplifying assumptions. For example, I have restricted myself to integral $\lambda$. One can handle the general case in the same way if one replaces $W$ by the subgroup $W_\lambda$ generated by all $s_\alpha$ with $\langle \lambda, \alpha^\vee \rangle \in \mathbf{Z}$.

Furthermore Fiebig shows that $\mathcal{L} \circ \mathbf{V}$ actually is an equivalence of categories between $\mathcal{O}_{A,\lambda}^{\mathrm{VF}}$ and a suitable category of sheaves having an analogue to a Verma flag.

Then it should be said that Fiebig works not with finite dimensional semi-simple Lie algebras, but more generally with symmetrisable Kac-Moody algebras. This leads to several complications since in general the Weyl group (and hence the moment graph) is not finite. However it turns out that one usually can restrict to finite subgraphs.

A more serious complication is the fact that in the Kac-Moody case no longer every element in $\mathfrak{h}^*$ is conjugate under the Weyl group to an antidominant one. Using a tilting functor one can also handle all weights that are conjugate to dominant weights. But weights on the so-called *critical hyperplanes* cannot be treated by this approach.

## 5 Representations in prime characteristics

**5.1.** (**Semi-simple algebraic groups**) Let $k$ be an algebraically closed field of prime characteristic $p$. Let $G$ be a connected semi-simple algebraic group over $k$ and $T$ a maximal torus in $G$. For the sake of simplicity let us assume that $G$ is almost simple and simply connected. (The results stated in 5.1–5.4 can be found in [Ja2].)

Denote by $X(T)$ the lattice of characters on $T$ and by $Y(T)$ the dual lattice of cocharacters of $T$. Then $X(T)$ contains the root system $\Phi$ of $G$ with respect to $T$. We choose a set of simple roots $\Pi$ and set $\Phi^+$ equal to the set of positive roots defined by $\Pi$. For any root $\alpha \in \Phi$ we denote by $\alpha^\vee \in Y(T)$ the corresponding dual root. Our assumption that $G$ is simply connected implies that all $\alpha^\vee$ with $\alpha \in \Pi$ form a basis for $Y(T)$ as a free module over $\mathbf{Z}$ whereas $X(T)$ has a basis consisting of the fundamental weights $\varpi_\alpha$, $\alpha \in \Pi$.

Denote by $X(T)^+ = \sum_{\alpha \in \Pi} \mathbf{N}\varpi_\alpha$ the set of *dominant* characters on $T$ (with respect to our choice of $\Pi$). We have for each $\lambda \in X(T)^+$ a simple $G$–module $L(\lambda)$ with highest weight $\lambda$. The map $\lambda \mapsto L(\lambda)$ induces a bijection between the set of dominant weights and the set of isomorphism classes of simple $G$–modules.

Set

$$X_p(T) = \{ \sum_{\alpha \in \Pi} m_\alpha \varpi_\alpha \mid 0 \le m_\alpha < p \text{ for all } \alpha \in \Pi \}.$$

Any $\lambda \in X(T)^+$ can then be written as $\lambda = \sum_{i=0}^{r} p^i \lambda_i$ with $\lambda_i \in X_p(T)$ for all $i$ (and with suitable $r$). Then Steinberg's tensor product theorem says that

$$L(\lambda) \simeq L(\lambda_0) \otimes L(\lambda_1)^{(1)} \otimes L(\lambda_2)^{(2)} \otimes \cdots \otimes L(\lambda_r)^{(r)}$$

where an exponent $(i)$ denotes a twist of the module with the $i$-th power of the Frobenius endomorphism of $G$.

**5.2.** (**The Lie algebra**) Denote by $\mathfrak{g}$ the Lie algebra of $G$ and by $\mathfrak{h} \subset \mathfrak{g}$ the Lie algebra of $T$. Then $\mathfrak{g}$ decomposes under the adjoint action of $T$ into the direct sum of $\mathfrak{h}$ and the root subspaces $\mathfrak{g}_\alpha$, $\alpha \in \Phi$. Each $\mathfrak{g}_\alpha$ has dimension 1; pick a basis element $x_\alpha$ for $\mathfrak{g}_\alpha$ over $k$.

For each $\lambda \in X(T)$ the tangent map $d\lambda\colon \mathfrak{h} \to k$ is a linear form on $\mathfrak{h}$. The map $\lambda \otimes 1 \mapsto d\lambda$ induces an isomorphism

$$X(T) \otimes_{\mathbf{Z}} k \xrightarrow{\sim} \mathfrak{h}^*.$$

We have dually an isomorphism $Y(T) \otimes_{\mathbf{Z}} k \xrightarrow{\sim} \mathfrak{h}$. Denote by $h_\alpha \in \mathfrak{h}$ the image of $\alpha^\vee \otimes 1$ under this isomorphism (for any $\alpha \in \Phi$). Then the $h_\alpha$ with $\alpha \in \Pi$ are a basis for $\mathfrak{h}$ over $k$.

The Lie algebra $\mathfrak{g}$ is restricted: It comes with a $p$–th power map $x \mapsto x^{[p]}$. For example, one has $x_\alpha^{[p]} = 0$ and $h_\alpha^{[p]} = h_\alpha$ for all $\alpha \in \Phi$.

A $\mathfrak{g}$–module $M$ is called restricted if for each $x \in \mathfrak{g}$ the action of $x^{[p]}$ on $M$ is the $p$–th power of the action of $x$ on $M$. It suffices to check this condition for all $x$ in a basis for $\mathfrak{g}$ over $k$.

Any $G$–module is a restricted $\mathfrak{g}$–module under the derived action. A theorem of Curtis says: Any $L(\lambda)$ with $\lambda \in X_p(T)$ is simple as a $\mathfrak{g}$–module; every simple restricted $\mathfrak{g}$–module is isomorphic to $L(\lambda)$ for exactly one $\lambda \in X_p(T)$.

**5.3.** (**$\mathfrak{g}$–$T$–modules**) A $\mathfrak{g}$–$T$–module is a vector space $M$ over $k$ that has both a structure as a restricted $\mathfrak{g}$–module and as a $T$–module such that the two structures are compatible in the following sense:

(A) The restriction of the $\mathfrak{g}$–action to $\mathfrak{h} = \operatorname{Lie} T$ is equal to the derived action of the $T$–action.

(B) One has $\operatorname{Ad}(t)(x)\,v = t\,x\,t^{-1}\,v$ for all $t \in T$, $x \in \mathfrak{g}$, $v \in M$. (Here $\operatorname{Ad}(t)$ denotes the adjoint action of $t \in T \subset G$ on $\mathfrak{g}$.)

Note that a $T$–module structure on a vector space $M$ over $k$ is the same as a direct sum decomposition $M = \bigoplus_{\lambda \in X(T)} M_\lambda$; then any $t \in T$ acts as multiplication by $\lambda(t)$ on each $M_\lambda$. Given this direct sum decomposition the condition (A) amounts to

$$h\,v = (d\lambda)(h)\,v \qquad \text{for all } h \in \mathfrak{h},\ v \in M_\lambda,\ \lambda \in X(T). \tag{1}$$

Furthermore (1) implies that (B) holds for all $x \in \mathfrak{h}$. Therefore (B) is equivalent to the condition that

$$x_\alpha\,M_\lambda \subset M_{\lambda+\alpha} \qquad \text{for all } \alpha \in \Phi \text{ and all } \lambda \in X(T) \tag{2}$$

provided that (1) holds. Finally, the condition that $M$ is restricted as a $\mathfrak{g}$–module means that

$$x_\alpha^p M = 0 \qquad \text{for all } \alpha \in \Phi \tag{3}$$

once (1) holds because (1) implies that $(h_\beta^p - h_\beta)\,M = 0$ for all $\beta \in \Phi$.

These considerations show: Giving a vector space $M$ over $k$ a structure as a $\mathfrak{g}$–$T$–module is the same as giving it a structure as a $\mathfrak{g}$–module with a direct sum decomposition $M = \bigoplus_{\lambda \in X(T)} M_\lambda$ as a vector space such that (1)–(3) hold.

Any $G$–module yields a $\mathfrak{g}$–$T$–module if we restrict the action of $G$ to $T$ and if we consider the derived action of $\mathfrak{g}$. A tensor product of two $\mathfrak{g}$–$T$–modules is again a $\mathfrak{g}$–$T$–module: Take the usual tensor product structures, both as a $\mathfrak{g}$–module and as a $T$–module. In [Ja2] $\mathfrak{g}$–$T$–modules appear under the name of $G_1T$–modules where $G_1$ denotes the first Frobenius kernel of $G$.

**5.4.** (**Simple and projective $\mathfrak{g}$–$T$–modules**) Any $\lambda \in X(T)$ defines a one dimensional $\mathfrak{g}$–$T$–module $k_{p\lambda}$ as follows: Any $x \in \mathfrak{g}$ acts as 0 on $k_{p\lambda}$ whereas any $t \in T$ acts as multiplication by $(p\lambda)(t) = \lambda(t)^p$. (Note that (1) is satisfied since $d(p\lambda) = 0$ in characteristic $p$.)

Any $\lambda \in X(T)$ can be uniquely decomposed $\lambda = \lambda_0 + p\mu$ with $\lambda_0 \in X_p(T)$ and $\mu \in X(T)$. We then set

$$\widehat{L}(\lambda) = L(\lambda_0) \otimes k_{p\mu}. \tag{1}$$

Then $\widehat{L}(\lambda)$ is a simple $\mathfrak{g}$–$T$–module since $L(\lambda_0)$ is a simple $\mathfrak{g}$–module. One can show that $\lambda \mapsto \widehat{L}(\lambda)$ induces a bijection from $X(T)$ to the set of isomorphism classes of simple $\mathfrak{g}$–$T$–modules.

By Steinberg's tensor product theorem we know the formal characters (i.e., the dimensions of the weight spaces for $T$) of all simple $G$–modules if we know the characters of all $L(\lambda)$ with $\lambda \in X_p(T)$. It is clearly equivalent to know the characters of all $\mathfrak{g}$–$T$–modules $\widehat{L}(\lambda)$ with $\lambda \in X(T)$. These characters are in turn determined by all compositiion multiplicities $[\widehat{Z}(\mu) : \widehat{L}(\lambda)]$ with $\lambda, \mu \in X(T)$.

Here $\widehat{Z}(\mu)$ denotes the *baby Verma module* with highest weight $\mu$. It can be described as an induced module similarly to the definition of a Verma module in 4.2. But one now has to replace enveloping algebras by restricted enveloping algebras. One gets that $\dim \widehat{Z}(\mu)_\mu = 1$; if we choose a basis vector $v$ for $\widehat{Z}(\mu)_\mu$ and pick a numbering $\alpha_1, \alpha_2, \dots, \alpha_N$ of $\Phi^+$, then all

$$x_{-\alpha_1}^{m_1} x_{-\alpha_2}^{m_2} \dots x_{-\alpha_N}^{m_N} v \qquad \text{with } 0 \le m_i < p \text{ for all } i \tag{2}$$

form a basis for $\widehat{Z}(\mu)$. It is easy to see that

$$\widehat{Z}(\mu) \otimes k_{p\nu} \simeq \widehat{Z}(\mu + p\nu) \qquad \text{for all } \mu, \nu \in X(T). \tag{3}$$

One can check that the category of all $\mathfrak{g}$–$T$–modules contains enough projective objects. Denote by $\widehat{Q}(\lambda)$ the projective cover of $\widehat{L}(\lambda)$ in this category, for any $\lambda \in X(T)$. These projective modules turn out to admit a baby Verma flag, i.e., a chain of submodules where the factor modules of subsequent terms are baby Verma modules. The number of factors isomorphic to a fixed $\widehat{Z}(\mu)$ is independent of the choice of the filtration and will be denoted by $(\widehat{Q}(\lambda) : \widehat{Z}(\mu))$. It turns out that one again has a reciprocity law:

$$(\widehat{Q}(\lambda) : \widehat{Z}(\mu)) = [\widehat{Z}(\mu) : \widehat{L}(\lambda)] \qquad \text{for all } \lambda, \mu \in X(T). \tag{4}$$

Therefore our considerations above show that the characters of all simple $G$–modules are determined if we know all $(\widehat{Q}(\lambda) : \widehat{Z}(\mu))$. It is easy to show that

$$\widehat{Q}(\lambda) \otimes k_{p\nu} \simeq \widehat{Q}(\lambda + p\nu) \qquad \text{for all } \lambda, \nu \in X(T) \tag{5}$$

and then that

$$(\widehat{Q}(\lambda) : \widehat{Z}(\mu)) = (\widehat{Q}(\lambda + p\nu) : \widehat{Z}(\mu + p\nu)) \tag{6}$$

for all $\lambda, \mu, \nu \in X(T)$. Therefore it suffices to find all $(\widehat{Q}(\lambda) : \widehat{Z}(\mu))$ with $\lambda \in X_p(T)$. Of course we could equally well take all $\lambda \in p\nu + X_p(T)$ for some fixed $\nu \in X(T)$.

**5.5.** (**Deforming $\mathfrak{g}$–$T$–modules**) Set $S = U(\mathfrak{h})$; since $\mathfrak{h}$ is commutative, $S$ coincides with the symmetric algebra of $\mathfrak{h}$. Let $A$ be a (commutative and associative) $S$–algebra (and hence by transitivity a $k$–algebra). Consider the Lie algebras $\mathfrak{g}_A := \mathfrak{g} \otimes_k A$ and $\mathfrak{h}_A = \mathfrak{h} \otimes_k A$. (The results stated in 5.5–5.14 can be found in [AJS] except for one point where we give an explicit reference.)

Let $\tau\colon \mathfrak{h} \to A$ denote the composition of the inclusion $\mathfrak{h} \hookrightarrow S(\mathfrak{h}) = S$ with the homomorphism $S \to A$ that makes $A$ into an $S$–algebra. Then $\tau$ is $k$–linear and we extend $\tau$ to an $A$–linear map $\mathfrak{h}_A = \mathfrak{h} \otimes_k A \to A$; we denote this extension again by $\tau$.

We now define a category $\mathcal{C}_A$, the "deformed" category of $\mathfrak{g}$–$T$–modules over $A$. An object in $\mathcal{C}_A$ is a $\mathfrak{g}_A$–module $M$ with a direct sum decomposition $M = \bigoplus_{\lambda \in X(T)} M_\lambda$ as an $A$–module such that:

(A) $M$ is finitely generated over $A$.
(B) We have $h\,v = (d\lambda(h) + \tau(h))\,v$ for all $h \in \mathfrak{h}$, $\lambda \in X(T)$, and $v \in M_\lambda$.
(C) We have $x_\alpha\, M_\lambda \subset M_{\lambda+\alpha}$ for all $\alpha \in \Phi$ and all $\lambda \in X(T)$.
(D) We have $x_\alpha^p M = 0$ for all $\alpha \in \Phi$.

Note: If we identify $k$ with the $S$–algebra $S/\mathfrak{h}S$, then we have $\tau = 0$ and the discussion in 5.3 shows that $\mathcal{C}_k$ is the category of all finite dimensional $\mathfrak{g}$–$T$–modules.

If $A'$ is an $A$–algebra (for an arbitrary $S$–algebra as above), then we have an obvious base change functor $\mathcal{C}_A \to \mathcal{C}_{A'}$ mapping any $M$ to $M \otimes_A A'$ with the obvious structure as a module over $\mathfrak{g}_{A'} \simeq \mathfrak{g}_A \otimes_A A'$ and the obvious grading where $(M \otimes_A A')_\mu = M_\mu \otimes_A A'$ for all $\mu$.

One can define for any $A$ and any $\mu \in X(T)$ a baby Verma module $\widehat{Z}_A(\mu)$ in $\mathcal{C}_A$. It has the property that $\widehat{Z}_A(\mu)_\mu$ is free of rank 1 over $A$; if $v$ denotes a basis for this module, then all elements as in 5.4(2) form a basis for $\widehat{Z}_A(\mu)$ over $A$. If now $A'$ is an $A$–algebra, then one has an obvious isomorphism

$$\widehat{Z}_A(\mu) \otimes_A A' \xrightarrow{\sim} \widehat{Z}_{A'}(\mu)$$

for any $\mu \in X(T)$. We denote by $\mathcal{C}_A^{\mathrm{BVF}}$ the full subcategory of $\mathcal{C}_A$ of all objects admitting a baby Verma flag (defined as in 5.4).

**5.6.** (**Lifting projectives**) We want to apply 5.5 to the case where $A$ is the completion of the localisation of $S = S(\mathfrak{h})$ at the maximal ideal generated by $\mathfrak{h}$. Then the residue field of the local ring $A$ identifies with $k = S/\mathfrak{h}S$. So the base change functor $\mathcal{C}_A \to \mathcal{C}_k$, $M \mapsto M \otimes_A k$ takes values in the undeformed category of $\mathfrak{g}$-$T$–modules from 5.3.

One shows now that the projective $\mathfrak{g}$-$T$–modules over $k$ can be lifted to $A$. For each $\lambda \in X(T)$ there exists an indecomposable projective object $\widehat{Q}_A(\lambda)$ in $\mathcal{C}_A$ with $\widehat{Q}_A(\lambda) \otimes_A k \simeq \widehat{Q}(\lambda)$. Each $\widehat{Q}_A(\lambda)$ has a baby Verma flag and we have

$$(\widehat{Q}_A(\lambda) : \widehat{Z}_A(\mu)) = [\widehat{Z}(\mu) : \widehat{L}(\lambda)] \qquad \text{for all } \lambda, \mu \in X(T). \tag{1}$$

Furthermore, any projective object in $\mathcal{C}_A$ is isomorphic to a direct sum of certain $\widehat{Q}_A(\lambda)$ with $\lambda \in X(T)$.

**5.7.** (**The affine Weyl group**) Let $W$ denote the Weyl group of the root system $\Phi$. It acts both on $X(T)$ and $Y(T)$. For each root $\alpha \in \Phi$ set $s_\alpha \in W$ equal to the corresponding reflection. Then $W$ is a Coxeter group with Coxeter generators $s_\alpha$, $\alpha \in \Pi$ (the simple roots).

Let $W_a$ denote the group of affine transformations of $X(T)$ and of the Euclidean space $X(T) \otimes_{\mathbf{Z}} \mathbf{R}$ generated by $W$ and by all translations by elements in $\Phi$, i.e., by roots. This group is usually called the *affine Weyl group* of the dual root system $\Phi^\vee$. It is isomorphic to the semi-direct product of $W$ with the normal subgroup $\mathbf{Z}\Phi$ where $W$ acts as given on $\Phi$.

The group $W_a$ is generated by the affine reflections $s_{\alpha,n}$ with $\alpha \in \Phi$ and $n \in \mathbf{Z}$ given by

$$s_{\alpha,n}\,(\lambda) = \lambda - (\langle \lambda, \alpha^\vee \rangle - n)\alpha \qquad \text{for all } \lambda \in X(T) \otimes_{\mathbf{Z}} \mathbf{R}.$$

So $s_{\alpha,n}$ is equal to $s_\alpha$ followed by the translation by $n\alpha$. Note that $s_\alpha = s_{\alpha,0}$ and $s_{-\alpha,-n} = s_{\alpha,n}$ for all $\alpha \in \Phi$ and $n \in \mathbf{Z}$. The group $W_a$ is a Coxeter group with Coxeter generators

$$\Sigma_a = \{\, s_\alpha \mid \alpha \in \Pi \,\} \cup \{s_{\alpha_0,1}\}$$

where $\alpha_0$ is the unique short root that is a dominant weight. (Here we use that $G$ is almost simple; in case all roots have the same length, all roots are short.)

We need a somewhat different action of $W_a$ on $X(T)$ that we denote by $(w, \lambda) \mapsto w\bullet_p\lambda$. If $w \in W$, then $w\bullet_p\lambda = w(\lambda+\rho)-\rho$ where $\rho$ is half the sum of the positive roots (equal to $\sum_{\alpha\in\Pi}\varpi_\alpha$). If $w$ is the translation by some $\nu \in \mathbf{Z}\Phi$, then $w\bullet_p\lambda = \lambda + p\nu$. It follows that

$$s_{\alpha,n} \bullet_p \lambda = \lambda - (\langle \lambda+\rho, \alpha^\vee\rangle - np)\alpha$$

for all $\lambda \in X(T)$, $\alpha \in \Phi$ and $n \in \mathbf{Z}$.

The point about this action is the following linkage principle:

$$[\widehat{Z}(\mu) : \widehat{L}(\lambda)] \neq 0 \implies \lambda \in W_a \bullet_p \mu. \tag{1}$$

**5.8.** ($W_a$ **as a linear group**) We get a linear action of $W_a$ on $X(T) \oplus \mathbf{Z}$ if we let any $w \in W$ act via $w\,(\lambda, a) = (w(\lambda), a)$ for all $\lambda \in X(T)$ and $a \in \mathbf{Z}$ and if we let the translation by any $\nu \in \mathbf{Z}\Phi$ act via $(\lambda, a) \mapsto (\lambda + a\nu, a)$. We get then for any $\alpha \in \Phi$ and $n \in \mathbf{Z}$

$$s_{\alpha,n}(\lambda, a) = (s_\alpha(\lambda) + an\alpha, a) \qquad \text{for all } \lambda \in X(T) \text{ and } a \in \mathbf{Z}.$$

Note that $\lambda \mapsto (\lambda, 1)$ is an $W_a$–equivariant embedding of $X(T)$ with the action of $W_a$ from 5.7 into $X(T) \times \mathbf{Z}$ with the present action.

Recall that the action of $W$ on $Y(T)$ is given by $s_\alpha(v) = v - \langle \alpha, v\rangle\alpha^\vee$ for all $v \in Y(T)$ and $\alpha \in \Phi$. We get now an action of $W_a$ on $Y(T) \oplus \mathbf{Z}$ letting any $w \in W$ act via $w\,(v, b) = (w(v), b)$ for all $v \in Y(T)$ and $b \in \mathbf{Z}$ whereas the translation by any $\mu \in \mathbf{Z}\Phi$ acts via $(v, b) \mapsto (v, \langle \mu, v\rangle + b)$. We get then for any $\alpha \in \Phi$ and $n \in \mathbf{Z}$

$$s_{\alpha,n}(v, a) = (s_\alpha(v), n\langle \alpha, v\rangle + b) \qquad \text{for all } v \in X(T) \text{ and } b \in \mathbf{Z}.$$

Write $\delta = (0, 1)$ and extend any $\mu \in \mathbf{Z}\Phi$ to a $\mathbf{Z}$–linear map $Y(T) \oplus \mathbf{Z} \to \mathbf{Z}$ by setting $\langle \mu, \delta\rangle = 0$. Then the last equation can be rewritten as

$$s_{\alpha,n}(z) = z - \langle \alpha, z\rangle\,(\alpha^\vee - n\delta) \qquad \text{for all } z \in Y(T) \oplus \mathbf{Z}.$$

So $W_a$ acts on $Y(T) \oplus \mathbf{Z}$ as the Weyl group of the extended dual root system

$$\Phi_a^\vee = \{\, \alpha^\vee + n\delta \mid \alpha \in \Phi, n \in \mathbf{Z} \,\} \subset Y(T) \oplus \mathbf{Z}.$$

It is the root system for an affine Kac-Moody algebra over $\mathbf{C}$: If $\mathfrak{g}_{\mathbf{C}}^\vee$ is the simple Lie algebra over $\mathbf{C}$ with root system $\Phi^\vee$, then $\Phi_a^\vee$ is the root system for the Kac-Moody algebra constructed as a certain central extension of $\mathfrak{g}_{\mathbf{C}}^\vee \otimes_{\mathbf{C}} \mathbf{C}[t, t^{-1}]$.

**5.9.** (**The principal block**) Let $h$ denote the Coxeter number of the root system $\Phi$; so $h-1$ is the maximum of all $\langle \rho, \alpha^\vee \rangle$ with $\alpha \in \Phi$.

Suppose from now on that $p > h$. This implies that the map

$$W_a \longrightarrow W_a \bullet_p 0, \quad w \mapsto w \bullet_p 0$$

is bijective. Furthermore, general translation principles show now that we know all multiplicities $[\widehat{Z}(\mu) : \widehat{L}(\lambda)]$ with $\lambda \in X_p(T)$ and $\mu \in X(T)$ if we know all $[\widehat{Z}(w\bullet_p 0) : \widehat{L}(x\bullet_p 0)]$ with $x, w \in W_a$ and $x\bullet_p 0 \in X_p(T)$. So, by 5.6(1) it suffices to determine all $(\widehat{Q}_A(x\bullet_p 0) : \widehat{Z}_A(w\bullet_p 0))$ for all $x$ and $w$ as before. Here $A$ is (as in 5.6) the completion of the localisation of $S$ at $S\mathfrak{h}$. Actually, it will be more convenient to consider all $x$ with $x\bullet_p 0 \in -p\rho + X_p(T)$. By 5.4(6) this does not make a difference.

Therefore we restrict ourselves from now on to modules in $\mathcal{C}_A$ having a baby Verma flag with all subsequent factors of the form $\widehat{Z}_A(w\bullet_p 0)$, $w \in W_a$. We denote the full subcategory of all these objects by $\mathcal{C}_{A,0}^{\mathrm{BVF}}$ and call it the principal block of $\mathcal{C}_A^{\mathrm{BVF}}$.

This block admits translation functors "through the walls"

$$\Theta_s : \mathcal{C}_{A,0}^{\mathrm{BVF}} \longrightarrow \mathcal{C}_{A,0}^{\mathrm{BVF}}$$

indexed by the Coxeter generators $s \in \Sigma_a$ of $W_a$. These functors are exact and take projective objects to projective objects. One has for each $w \in W_a$ a short exact sequence

$$0 \to \widehat{Z}_A(w_1\bullet_p 0) \longrightarrow \Theta_s \widehat{Z}_A(w\bullet_p 0) \longrightarrow \widehat{Z}_A(w_2\bullet_p 0) \to 0$$

where $\{w_1, w_2\} = \{w, ws\}$ and where the numbering is chosen such that $w_1\bullet_p 0 > w_2\bullet_p 0$. We get for any $M$ in $\mathcal{C}_{A,0}^{\mathrm{BVF}}$ that

$$(\Theta_s M : \widehat{Z}_A(w\bullet_p 0)) = (M : \widehat{Z}_A(w\bullet_p 0)) + (M : \widehat{Z}_A(ws\bullet_p 0)) \tag{1}$$

for all $w \in W_a$. This implies: If $x = s_r \dots s_2 s_1$ is a reduced decomposition of an element $x \in W_a$ (so all $s_i$ belong to $\Sigma_a$ and $r$ is minimal), then we get

$$(\Theta_{s_1}\Theta_{s_2}\dots\Theta_{s_r}\widehat{Z}_A(0) : \widehat{Z}_A(x\bullet_p 0)) = 1 \tag{2}$$

and for any $w \in W_a$

$$(\Theta_{s_1}\Theta_{s_2}\dots\Theta_{s_r}\widehat{Z}_A(0) : \widehat{Z}_A(w\bullet_p 0)) \neq 0 \iff w \leq x \tag{3}$$

where $\leq$ is the Bruhat ordering on $W_a$.

Consider in particular $x \in W_a$ with $x\bullet_p 0 \in -p\rho + X_p(T)$. One element with this property is $w_0$, the longest element in the finite Weyl group $W$. If our $x$ is distinct from $w_0$, then we can find $s \in \Sigma_a$ with $x\bullet_p 0 < xs\bullet_p 0$ and $xs\bullet_p 0 \in -p\rho + X_p(T)$. (The alcove containing $x\bullet_p 0$ has a wall separating it from the alcove containing $w_0\bullet_p 0$; then choose $s$ such that $ws\bullet_p 0$ is the mirror image of $x\bullet_p 0$ with respect to this wall.) This implies inductively (trivially in the case $x = w_0$) that $x$ has a reduced decomposition $x = s_r \dots s_2 s_1$ such that $w_0 = s_r \dots s_{m+1} s_m$ for a suitable $m$. In this situation Fiebig shows ([F4], Prop. 8.2):

**Lemma:** *Then $\widehat{Q}_A(x{\bullet_p}0)$ is an indecomposable direct summand of $\Theta_{s_1}\Theta_{s_2}\dots\Theta_{s_r}\widehat{Z}_A(0)$.*

More precisely, $\widehat{Q}_A(x{\bullet_p}0)$ is the unique indecomposable direct summand with $\widehat{Z}_A(x{\bullet_p}0)$ as a factor in a baby Verma flag, cf. (2). (Note that the Krull-Schmidt theorem holds in our category since $A$ is a complete local noetherian ring.)

The existence of a reduced decomposition as above shows also that the set of all $w \in W_a$ with $w \le x$ is stable under left multiplication by $W$:

$$\text{If } w \in W_a \text{ and } z \in W, \text{ then} \quad w \le x \iff zw \le x. \tag{4}$$

(It suffices to check this for $z = s_\alpha$ with $\alpha \in \Sigma$.) Furthermore, one gets by induction on the length of $w$:

$$\text{If } w \in W_a \text{ with } w \le x, \text{ then } x{\bullet_p}0 \le w{\bullet_p}0. \tag{5}$$

In fact, one has more strongly $x{\bullet_p}0 \uparrow w{\bullet_p}0$ in the notation from [Ja2], II.6.4. This is well known for $x = w_0$; for the induction step from $xs$ to $x$ as above one can use Lemma II.6.7 in [Ja2].

**5.10.** (**Further localisation**) Keep the assumptions from 5.9. Consider the subrings

$$A^\emptyset = A\,[\,h_\alpha^{-1} \mid \alpha \in \Phi^+\,]$$

and for each $\beta \in \Phi^+$

$$A^\beta = A\,[\,h_\alpha^{-1} \mid \alpha \in \Phi^+,\, \alpha \ne \beta]$$

of the fraction field of $A$. Write

$$Z^\emptyset(\mu) = \widehat{Z}_{A^\emptyset}(\mu) \qquad \text{and} \qquad Z^\beta(\mu) = \widehat{Z}_{A^\beta}(\mu)$$

for all $\mu \in X(T)$. It turns out that each $Z^\emptyset(\mu)$ is projective in $\mathcal{C}_{A^\emptyset}$; one has for all $\lambda, \mu \in X(T)$

$$\mathrm{Hom}_{\mathcal{C}_{A^\emptyset}}(Z^\emptyset(\mu), Z^\emptyset(\lambda)) \simeq \begin{cases} A^\emptyset & \text{if } \mu = \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

This implies: If $M$ is a module in $\mathcal{C}_A^{\mathrm{BVF}}$, then $\mathrm{Hom}_{\mathcal{C}_{A^\emptyset}}(Z^\emptyset(\mu), M \otimes_A A^\emptyset)$ is a free $A$–module of rank equal to $(M : \widehat{Z}_A(\mu))$. Therefore we are interested in knowing all

$$\mathrm{rk}\ \mathrm{Hom}_{\mathcal{C}_{A^\emptyset}}(Z^\emptyset(w{\bullet_p}0), \widehat{Q}_A(x{\bullet_p}0) \otimes_A A^\emptyset)$$

with $x, w \in W_a$.

Let $\beta \in \Phi^+$. For each $w \in W_a$ there are unique integers $n$ and $r$ such that

$$\langle w{\bullet_p}0 + \rho, \beta^\vee \rangle = np + r \quad \text{and } 0 \le r < p.$$

The assumption that $p > h$ actually implies that $r > 0$. Denote by $\beta\uparrow w$ the element in $W_a$ such that

$$(\beta\uparrow w){\bullet_p}0 = s_{\beta,n+1} w{\bullet_p}0 = w{\bullet_p}0 + (p-r)\beta.$$

Then there exists an extension in $\mathcal{C}_{A^\beta}$

$$0 \to Z^\beta((\beta\uparrow w){\bullet_p}0) \longrightarrow Q^\beta(w{\bullet_p}0) \longrightarrow Z^\beta(w{\bullet_p}0) \to 0 \tag{1}$$

such that $Q^\beta(w{\bullet_p}0)$ is a projective object in $\mathcal{C}_{A^\beta}$.

**5.11.** (**A functor into combinatorics**) We now define a functor $\kappa\colon \mathcal{C}_{A,0}^{\mathrm{BVF}} \to \mathcal{K}(A)$ into a "combinatorial" category $\mathcal{K}(A)$. An object in $\mathcal{K}(A)$ is a family

$$\mathcal{M} = ((\mathcal{M}_w^{\emptyset})_{w \in W_a}, (\mathcal{M}_w^{\beta})_{\beta \in \Phi^+, w \in W_a}) \tag{1}$$

where each $\mathcal{M}_w^{\emptyset}$ is an $A^{\emptyset}$–module and each $\mathcal{M}_w^{\beta}$ is an $A^{\beta}$–submodule of the $A^{\emptyset}$–module $\mathcal{M}_w^{\emptyset} \oplus \mathcal{M}_{\beta\uparrow w}^{\emptyset}$. A morphism $\mathcal{M} \to \mathcal{N}$ in $\mathcal{K}(A)$ is a family of homomorphisms $\mathcal{M}_w^{\emptyset} \to \mathcal{N}_w^{\emptyset}$ of $A^{\emptyset}$–modules that induce homomorphisms $\mathcal{M}_w^{\beta} \to \mathcal{N}_w^{\beta}$ for all $\beta$ and $w$.

Now $\kappa$ is defined by setting for all $w \in W_a$

$$\kappa(M)_w^{\emptyset} = \mathrm{Hom}_{\mathcal{C}_{A^{\emptyset}}}(Z^{\emptyset}(w\bullet_p 0), M \otimes_A A^{\emptyset})$$

and (for all $\beta \in \Phi^+$)

$$\kappa(M)_w^{\beta} = \mathrm{Hom}_{\mathcal{C}_{A^{\beta}}}(Q^{\beta}(w\bullet_p 0), M \otimes_A A^{\beta})$$

with $Q^{\beta}(w\bullet_p 0)$ as at the end of 5.10. In order to embed $\kappa(M)_w^{\beta}$ into $\kappa(M)_w^{\emptyset} \oplus \kappa(M)_{\beta\uparrow w}^{\emptyset}$ one fixes an extension as in 5.10(1). This extension splits after tensoring with $A^{\emptyset}$; this splitting leads to a well determined isomorphism

$$Z^{\emptyset}(w\bullet_p 0) \oplus Z^{\emptyset}(\beta\uparrow w\bullet_p 0) \xrightarrow{\sim} Q^{\beta}(w\bullet_p 0) \otimes_{A^{\beta}} A^{\emptyset}$$

and thus to a well determined isomorphism

$$\kappa(M)_w^{\beta} \otimes_{A^{\beta}} A^{\emptyset} \xrightarrow{\sim} \kappa(M)_w^{\emptyset} \oplus \kappa(M)_{\beta\uparrow w}^{\emptyset}.$$

We identify $\kappa(M)_w^{\beta}$ with the image of $\kappa(M)_w^{\beta} \otimes 1$ under this isomorphism.

One applies $\kappa$ to homomorphisms as one usually applies a Hom functor. Now the crucial point is:

**Theorem:** *The functor $\kappa$ is fully faithful.*

This implies in particular, that $\kappa$ takes indecomposable objects to indecomposable objects. Note also that we recover any filtration multiplicity $(M : \widehat{Z}_A(w\bullet_p 0))$ as the rank of the free $A^{\emptyset}$–module $\kappa(M)_w^{\emptyset}$. We get in particular by 5.6(1) that

$$[\widehat{Z}(w\bullet_p 0) : \widehat{L}(x\bullet_p 0)] = \mathrm{rk}_{A^{\emptyset}} \kappa(\widehat{Q}_A(x\bullet_p 0))_w^{\emptyset} \tag{2}$$

for all $x, w \in W_a$.

**5.12.** (**Examples**) It is easy to describe the image under $\kappa$ of a baby Verma module. First a notation: For any $w \in W_a$ and $\beta \in \Phi^+$ there is a unique element $\beta \downarrow w$ in $W_a$ satisfying $\beta\uparrow(\beta\downarrow w) = w$. One gets now for all $w \in W_a$

$$\kappa(\widehat{Z}_A(w\bullet_p 0))^{\emptyset}_w = A^{\emptyset} \qquad \text{and} \qquad \kappa(\widehat{Z}_A(w\bullet_p 0))^{\emptyset}_x = 0 \text{ for all } x \neq w$$

and for all $\beta \in \Phi^+$

$$\kappa(\widehat{Z}_A(w\bullet_p 0))^{\beta}_w = A^{\beta}(1,0) \qquad \text{and} \qquad \kappa(\widehat{Z}_A(w\bullet_p 0))^{\beta}_{\beta\downarrow w} = A^{\beta}(0,1)$$

and $\kappa(\widehat{Z}_A(w\bullet_p 0))^{\beta}_x = 0$ for all $x \neq w, \beta\downarrow w$.

As another example consider $Q = \widehat{Q}_A(w_0\bullet_p 0)$ where $w_0$ is the unique element in $W$ with $w_0(\Phi^+) = -\Phi^+$. One gets for all $w \in W_a$

$$\kappa(Q)^{\emptyset}_w = \begin{cases} A^{\emptyset} & \text{if } w \in W, \\ 0 & \text{if } w \notin W. \end{cases}$$

In order to describe $\kappa(Q)^{\beta}_w$ we need extra notation. Set $\Phi^+(\beta) = \{\alpha \in \Phi^+ \mid s_\beta(\alpha) \in -\Phi^+\}$ and set for each $w \in W$

$$a^{\beta}_w = \prod_{\alpha\in\Phi^+(\beta),\, w^{-1}\alpha\in\Phi^+} h_{-\alpha} \prod_{\alpha\in\Phi^+(\beta),\, w^{-1}\alpha\in-\Phi^+} h_{\alpha}^{-1}.$$

One gets now for all $\beta \in \Phi^+$ and $w \in W$

$$\kappa(Q)^{\beta}_w = \begin{cases} A^{\beta}(1,0) & \text{if } w^{-1}\beta \in \Phi^+, \\ A^{\beta}(1,0) + A^{\beta}(a^{\beta}_w,1) & \text{if } w^{-1}\beta \in -\Phi^+. \end{cases}$$

Furthermore, one gets for all $w \in W$ with $w^{-1}\beta \in -\Phi^+$ that

$$\kappa(Q)^{\beta}_{\beta\downarrow w} = A^{\beta}(0,1).$$

For all remaining $x \in W_a$ one gets $\kappa(Q)^{\beta}_x = 0$. (The proof of this result uses translation functors that are slightly more general than those from 5.9.)

**5.13.** (**Translation functors in the combinatorial category**) One can construct for each $s \in \Sigma_a$ a functor

$$\vartheta_s\colon \mathcal{K}(A) \longrightarrow \mathcal{K}(A)$$

such that there exists a natural isomorphism

$$\kappa \circ \Theta_s \xrightarrow{\sim} \vartheta_s \circ \kappa. \tag{1}$$

For example, one sets for any $\mathcal{M}$ as in 5.11(1)

$$\vartheta_s(\mathcal{M})^{\emptyset}_w = \mathcal{M}^{\emptyset}_w \oplus \mathcal{M}^{\emptyset}_{ws}. \tag{2}$$

The description of $\vartheta_s(\mathcal{M})^{\beta}_w$ is more complicated and requires some case-by-case considerations.

Let $x \in W_a$ with $x\bullet_p 0 \in -p\rho + X_p(T)$. Consider a reduced decomposition $x = s_r \dots s_2 s_1$ as in Lemma 5.9. That lemma and (1) imply that $\kappa(\widehat{Q}_A(x\bullet_p 0))$ is an indecomposable direct summand of $\vartheta_{s_1}\vartheta_{s_2}\dots\vartheta_{s_r}\kappa(\widehat{Z}_A(0))$. In fact, it is the unique indecomposable direct summand $\mathcal{M}$ of $\vartheta_{s_1}\vartheta_{s_2}\dots\vartheta_{s_r}\kappa(\widehat{Z}_A(0))$ with $\mathcal{M}^{\emptyset}_x \neq 0$. If we can find this summand, then we get all $[\widehat{Z}(w\bullet_p 0) : \widehat{L}(x\bullet_p 0)]$ with $x$ as above and with $w \in W_a$ as the rank of $\mathcal{M}^{\emptyset}_w$. And that would yield all multiplicities.

**5.14.** For any $S$–algebra $B$ that is an integral domain such that all $h_\alpha$ with $\alpha \in \Phi$ are non-zero in $B$, one sets

$$B^\emptyset = B\,[h_\alpha^{-1} \mid \alpha \in \Phi^+] \qquad \text{and} \qquad B^\beta = B\,[h_\alpha^{-1} \mid \alpha \in \Phi^+,\, \alpha \neq \beta]$$

(for all $\beta \in \Phi^+$). Then one defines a category $\mathcal{K}(B)$ analogously to $\mathcal{K}(A)$, replacing all $A$ by $B$ in the first paragraph of 5.11.

We get thus in particular the category $\mathcal{K}(S)$. We have then a base change functor $\mathcal{K}(S) \to \mathcal{K}(A)$. It takes any family $\mathcal{M} = ((\mathcal{M}_w^\emptyset)_{w\in W_a}, (\mathcal{M}_w^\beta)_{\beta\in\Phi^+, w\in W_a})$ to the family of all

$$\mathcal{M}_w^\emptyset \otimes_{S^\emptyset} A^\emptyset \qquad \text{and} \qquad \mathcal{M}_w^\beta \otimes_{S^\beta} A^\beta.$$

A look at the examples in 5.12 shows that $\kappa(\widehat{Q}_A(w_0 \bullet_p 0))$ and all $\kappa(\widehat{Z}_A(w \bullet_p 0))$ arise by base change from objects in $\mathcal{K}(S)$: Replace any $A^\emptyset$ by $S^\emptyset$ and any $A^\beta$ by $S^\beta$ in the descriptions in 5.12.

Furthermore, the explicit formulae for $\vartheta_s$ (that I did not state in 5.13) show that these functors arise by base change from corresponding functors on $\mathcal{K}(S)$. We shall denote the functors on $\mathcal{K}(S)$ again by $\vartheta_s$.

**5.15.** (**The structure algebra**) Denote by $\widetilde{S}$ the symmetric algebra of the $k$–vector space $(Y(T) \oplus \mathbf{Z}) \otimes_{\mathbf{Z}} k$ and denote by $\mathcal{Z}$ the set of all families $(u_w)_{w\in W_a}$ with each $u_w \in \widetilde{S}$ such that

$$u_{s_{\alpha,n}w} \equiv u_w \mod \widetilde{S} \cdot (\alpha^\vee - n\delta) \tag{1}$$

for all $\alpha \in \Phi^+$, $n \in \mathbf{Z}$, and $w \in W_a$. This is the structure algebra of a certain infinite moment graph; we return to this interpretation later on in 5.20.

For any $\Omega \subset W_a$ denote by $\mathcal{Z}(\Omega)$ the set of all families $(u_w)_{w\in\Omega}$ with each $u_w \in \widetilde{S}$ such that (1) holds whenever both $w$ and $s_{\alpha,n}w$ belong to $\Omega$. If $\Omega$ is finite, then $\mathcal{Z}(\Omega)$ inherits a natural grading from $\widetilde{S}$ (normalised such that $(Y(T) \oplus \mathbf{Z}) \otimes_{\mathbf{Z}} k$ sits in degree 2).

We have for any $\Omega$ a natural "forgetful" homomorphism of algebras $\mathcal{Z} \to \mathcal{Z}(\Omega)$. It need not be surjective. We say that a $\mathcal{Z}$–module $M$ has *finite support* if there exists a finite subset $\Omega \subset W_a$ such that the $\mathcal{Z}$–module structure on $M$ arises from a $\mathcal{Z}(\Omega)$–module structure via the homomorphism $\mathcal{Z} \to \mathcal{Z}(\Omega)$. If $M$ in addition has a $\mathbf{Z}$–grading that makes it into a graded $\mathcal{Z}(\Omega)$–module, then we call $M$ a graded $\mathcal{Z}$–module with finite support. In this case we denote by $M\langle n\rangle$ for any $n \in \mathbf{Z}$ the graded $\mathcal{Z}$–module with finite support that we get from $M$ by shifting the grading by $n$, cf. 1.1.

**5.16.** (**Translation functors and special modules**) Any $x \in W_a$ induces an automorphism $\sigma_x$ of the algebra $\mathcal{Z}$ mapping any family $(u_w)_{w\in W}$ to the family $(u'_w)_{w\in W}$ with $u'_w = u_{wx}$ for all $w \in W_a$. Denote by $\mathcal{Z}^x$ the subalgebra of all fixed points of $\sigma_x$. One checks for any reflection $s = s_{\alpha,n}$ with $\alpha \in \Phi$ and $n \in \mathbf{Z}$ that $\mathcal{Z}$ is free of rank 2 over $\mathcal{Z}^s$ with basis $1, c_s$ where $c_s = (c_{s,w})_{w\in W_a}$ with $c_{s,w} = w(\alpha^\vee - n\delta)$ for all $w \in W_a$, see [F4], Lemma 2.4.

For any simple reflection $s \in \Sigma_a$ we define now a functor $\theta_s$ from $\mathcal{Z}$–modules to $\mathcal{Z}$–modules by setting

$$\theta_s M = \mathcal{Z} \otimes_{\mathcal{Z}^s} M \tag{1}$$

for any $\mathcal{Z}$–module $M$. For reasons that will become clear later on, we also call these $\theta_s$ translation functors.

If $M$ is a (graded) $\mathcal{Z}$–module with finite support, then so is any $\theta_s M$, see [F4], Lemma 2.7. If $M$ is finitely generated and torsion free over $\widetilde{S}$, then so is any $\theta_s$.

Let $M_e$ denote the graded $\mathcal{Z}$–module equal to $\widetilde{S}$ as an abelian group such that any family $(u_w)_{w\in W_a}$ in $\mathcal{Z}$ acts on $\widetilde{S}$ as multiplication by $u_e$. (Here $e$ is the identity in $W_a$.) We now define a category $\mathcal{H}$ of graded $\mathcal{Z}$–modules with finite support. We take all modules of the form

$$\theta_{s_1} \circ \theta_{s_2} \circ \cdots \circ \theta_{s_r}(M_e\langle n\rangle) \tag{2}$$

with arbitrary finite sequences $s_1, s_2, \ldots, s_r$ in $\Sigma_a$ and arbitrary $n \in \mathbf{Z}$. We then add all graded direct summands of modules as in (2) as well as finite direct sums of such summands.

It follows that $\mathcal{H}$ consists of graded $\mathcal{Z}$–modules with finite support that are finitely generated and torsion free over $\widetilde{S}$. Fiebig calls objects in $\mathcal{H}$ *special* $\mathcal{Z}$–modules.

**5.17.** (**Localisation again**) In analogy to our earlier definitions we set

$$\widetilde{S}^{\emptyset} = \widetilde{S}\,[(\alpha^\vee - n\delta)^{-1} \mid \alpha \in \Phi^+,\, n \in \mathbf{Z}] \tag{1}$$

and for all $\beta \in \Phi^+$

$$\widetilde{S}^{\beta} = \widetilde{S}\,[(\alpha^\vee - n\delta)^{-1} \mid \alpha \in \Phi^+,\, \alpha \neq \beta,\, n \in \mathbf{Z}]. \tag{2}$$

These are subalgebras of the field of fractions of $\widetilde{S}$.

Let $\Omega \subset W_a$ be finite. Then $\mathcal{Z}(\Omega) \otimes_{\widetilde{S}} \widetilde{S}^{\emptyset}$ identifies naturally with $\prod_{w\in\Omega} \widetilde{S}^{\emptyset}$. For any $w \in \Omega$ denote by $\varepsilon_w$ the component of $1 \in \mathcal{Z}(\Omega) \otimes_{\widetilde{S}} \widetilde{S}^{\emptyset}$ in the factor $\widetilde{S}^{\emptyset}$ corresponding to $w$.

If $M$ is a $\mathcal{Z}(\Omega)$–module, then the $\mathcal{Z}(\Omega) \otimes_{\widetilde{S}} \widetilde{S}^{\emptyset}$–module $M \otimes_{\widetilde{S}} \widetilde{S}^{\emptyset}$ decomposes

$$M \otimes_{\widetilde{S}} \widetilde{S}^{\emptyset} = \bigoplus_{w\in\Omega} M^{\emptyset,w} \tag{3}$$

with $M^{\emptyset,w} = \varepsilon_w(M \otimes_{\widetilde{S}} \widetilde{S}^{\emptyset})$ for any $w \in \Omega$. We set $M^{\emptyset,w} = 0$ for all $w \notin \Omega$.

If $M$ is an object in $\mathcal{H}$, then one gets for all $s \in \Sigma_a$ and $w \in W_a$ that

$$(\theta_s M)^{\emptyset,w} \simeq M^{\emptyset,w} \oplus M^{\emptyset,ws} \tag{4}$$

cf. [F4], Lemma 3.6.

**5.18.** (**From $\mathcal{Z}$–modules into combinatorics**) The isomorphism $Y(T) \otimes_{\mathbf{Z}} k \xrightarrow{\sim} \mathfrak{h}$ from 5.2 (with $\alpha^\vee \otimes 1 \mapsto h_\alpha$ for all $\alpha \in \Phi$) extends to a linear map $(Y(T) \oplus \mathbf{Z}) \otimes_{\mathbf{Z}} k \xrightarrow{\sim} \mathfrak{h}$ such that $\delta \otimes 1 \mapsto 0$. This map induces a homomorphism $\widetilde{S} \to S$ of the symmetric algebras (preserving the grading) as well as of the localisations $\widetilde{S}^\emptyset \to S^\emptyset$ and $\widetilde{S}^\beta \to S^\beta$ for all $\beta \in \Phi^+$.

These homomorphisms are used by Fiebig to construct an additive functor

$$\Psi \colon \mathcal{H} \longrightarrow \mathcal{K}(A) \tag{1}$$

satisfying

$$\vartheta_s \circ \Psi = \Psi \circ \theta_s \qquad \text{for all } s \in \Sigma_a. \tag{2}$$

Here I deviate from Fiebig's notation: What I am calling $\Psi$ is the composition of Fiebig's functor $\Psi \colon \mathcal{H} \to \mathcal{K}(S)$ from [F4], 5.5 with a functor of the form $\gamma_{\mathcal{O}} \colon \mathcal{K}(S) \to \mathcal{K}(S)$ from [F4], 6.1 and finally an extension of scalars functor $\mathcal{K}(S) \to \mathcal{K}(A)$ as in 5.14.

Consider an object $M$ in $\mathcal{H}$ and choose a finite subset $\Omega$ in $W_a$ such that the $\mathcal{Z}$–module structure on $M$ comes from a $\mathcal{Z}(\Omega)$–module structure. Take the decomposition of $M \otimes_{\widetilde{S}} \widetilde{S}^\emptyset$ from 5.17(3) and set

$$(\Psi M)^\emptyset_w = (M^{\emptyset,w} \otimes_{\widetilde{S}} S) \otimes_S A \qquad \text{for all } w \in W_a. \tag{3}$$

The definition of $(\Psi M)^\beta_w$ is more complicated; here I have to refer you to [F4].

For any $M$ in $\mathcal{H}$ all $M^\emptyset_w$ are free modules over $\widetilde{S}^\emptyset$ of finite rank. It follows that each $(\Psi M)^\emptyset_w$ is a free $A^\emptyset$–module of the same rank:

$$\mathrm{rk}_{\widetilde{S}^\emptyset} M^{\emptyset,w} \;=\; \mathrm{rk}_{A^\emptyset} (\Psi M)^\emptyset_w. \tag{4}$$

**5.19.** Recall the $\mathcal{Z}$–module $M_e$ that was the starting point of the definition of the category $\mathcal{H}$ in 5.16. One gets now that

$$\Psi M_e \simeq \kappa(\widehat{Z}_A(0)), \tag{1}$$

see the proof of Thm. 5.4 in [F4].

Consider now some $x \in W_a$ with $x \bullet_p 0 \in -p\rho + X_p(T)$ and a reduced decomposition $x = s_r \dots s_2 s_1$ as in Lemma 5.9. We get from 5.13(1) and 5.18(2) that

$$\kappa(\Theta_{s_1} \Theta_{s_2} \dots \Theta_{s_r} \widehat{Z}_A(0)) \simeq \Psi(\theta_{s_1} \theta_{s_2} \dots \theta_{s_r} M_e). \tag{2}$$

Now 5.9(3) implies that $\theta_{s_1} \theta_{s_2} \dots \theta_{s_r} M_e$ has a unique indecomposable summand $M_x$ such that $M_x^{\emptyset,x} \neq 0$. Then $\Psi M_x$ is isomorphic to a direct summand of $\kappa(\Theta_{s_1} \Theta_{s_2} \dots \Theta_{s_r} \widehat{Z}_A(0))$ and satisfies $(\Psi M_x)^\emptyset_x \neq 0$. It follows that $\kappa(\widehat{Q}_A(x \bullet_p 0))$ is an indecomposable direct summand of $\Psi M_x$. This implies using 5.18(4) for all $w \in W_a$

$$\mathrm{rk}_{A^\emptyset} \kappa(\widehat{Q}_A(x \bullet_p 0))^\emptyset_w \leq \mathrm{rk}_{\widetilde{S}^\emptyset} M_x^{\emptyset,w}. \tag{3}$$

Using 5.11(2) we can restate this inequality as

$$[\widehat{Z}(w \bullet_p 0) : \widehat{L}(x \bullet_p 0)] \leq \mathrm{rk}_{\widetilde{S}^\emptyset} M_x^{\emptyset,w}. \tag{4}$$

**5.20.** (**The moment graph**) We associate to $W_a$ a moment graph $\mathcal{G}$ as follows: Its vertices are the elements of $W_a$. The ordering on the set of vertices is the reversed ordering of the Bruhat ordering on $W_a$. Two elements $w$ and $x$ in $W_a$ are joined by an edge if and only if there is a root $\alpha \in \Phi^+$ and an integer $n \in \mathbf{Z}$ such that $x = s_{\alpha,n}w$; if so, then we associate to this edge the line $k(\alpha^\vee - n\delta)$ in $(Y(T) \oplus \mathbf{Z}) \otimes_{\mathbf{Z}} k$.

This is of course an infinite moment graph whereas in the earlier chapters we have usually assumed that our moment graph is finite. Therefore we often restrict to finite subgraphs. For any subset $\Omega \subset W_a$ denote by $\mathcal{G}[\Omega]$ the full subgraph of $\mathcal{G}$ with $\Omega$ as set of vertices.

It is clear that the algebra $\mathcal{Z}$ from 5.15 is the structure algebra of $\mathcal{G}$; for any $\Omega \subset W_a$ the algebra $\mathcal{Z}(\Omega)$ is the structure algebra of $\mathcal{G}[\Omega]$.

There exists a unique element $\widehat{w}_0 \in W_a$ such that $\widehat{w}_0 \bullet_p 0$ belongs to the same alcove with respect to $W_a$ as $-p\rho$. (If $\rho$ belongs to the root lattice, then $\widehat{w}_0$ is translation by $-\rho$.) Set

$$\Omega_0 = \{\, w \in W_a \mid w \leq \widehat{w}_0 \,\}. \tag{1}$$

This is a finite subset of $W_a$. By 5.9(4) this set is stable under left multiplication by $W$. Any $w \in \Omega_0$ satisfies $\widehat{w}_0 \bullet_p 0 \leq w \bullet_p 0$, see 5.9(5). If $x \in W_a$ with $x \bullet_p 0 \in -p\rho + X_p(T)$, then $x \in \Omega_0$. (Use downward induction on the length of $x$. If $x \neq \widehat{w}_0$, then there exists a wall of the alcove containing $x \bullet_p 0$ separating this alcove from the alcove containing $\widehat{w}_0 \bullet_p 0$. Let $s \in \Sigma_a$ be the simple reflection such that $xs \bullet_p 0$ is the mirror image of $x \bullet_p 0$ with respect to this wall. Then also $xs \bullet_p 0 \in -p\rho + X_p(T)$ and $x < xs$.)

**Lemma:** *The moment graph $\mathcal{G}[\Omega_0]$ is a GKM-graph.*

This is Lemma 9.1 in [F4]. One uses that under our assumption on $p$ two distinct positive coroots remain linearly independent in $Y(T) \otimes k$. This reduces the lemma to the following claim: Given $w \in \Omega_0$, $\alpha \in \Phi^+$, and $n, m \in \mathbf{Z}$ such that $s_{\alpha,n}w, s_{\alpha,m}w \in \Omega_0$ and $n \equiv m \pmod{p}$, then $n = m$. This claim follows from the following inequality: One has

$$|\langle w \bullet_p 0 + \rho, \alpha^\vee \rangle| < p(p-1) \qquad \text{for all } w \in \Omega_0 \text{ and } \alpha \in \Phi. \tag{2}$$

In the case where $w \bullet_p 0 + \rho$ is antidominant, i.e., where $\langle w \bullet_p 0 + \rho, \alpha^\vee \rangle \leq 0$ for all $\alpha \in \Phi^+$, one has for all $\alpha \in \Phi^+$

$$\begin{aligned}\langle w \bullet_p 0 + \rho, \alpha^\vee \rangle &\geq \langle w \bullet_p 0 + \rho, \alpha_0^\vee \rangle \geq \langle \widehat{w}_0 \bullet_p 0 + \rho, \alpha_0^\vee \rangle \\ &> \langle -p\rho, \alpha_0^\vee \rangle = -p(h-1) > -p(p-1),\end{aligned}$$

and (2) follows in this case. For general $w \in \Omega_0$ we can find $z \in W$ such that $z(w \bullet_p 0 + \rho)$ is antidominant. Now observe that

$$\langle z(w \bullet_p 0 + \rho), \alpha^\vee \rangle = \langle w \bullet_p 0 + \rho, (z^{-1}\alpha)^\vee \rangle$$

for all $\alpha \in \Phi$ and that $z(w \bullet_p 0 + \rho) = (zw) \bullet_p 0 + \rho$. This shows that it suffices to prove (2) in the antidominant case.

**5.21.** Lemma 5.20 implies that we have for each $w \in \Omega_a$ a (graded) Braden-MacPherson sheaf $\mathcal{B}(w)$ on $\mathcal{G}[\Omega_0]$, see 3.5.

Consider on the other hand in $\mathcal{H}$ the subcategory $\mathcal{H}_0$ of all direct sums of graded direct summands of modules of the form $\theta_{s_1}\theta_{s_2}\dots\theta_{s_r}(M_e\langle n\rangle)$ such that $s_r \dots s_2 s_1$ is a reduced expression of an element in $\Omega_0$. We can consider $\mathcal{H}_0$ as a category of $\mathcal{Z}(\Omega_0)$–modules. Now we have according to [F4], Prop. 9.3:

**Proposition:** *The indecomposable objects in $\mathcal{H}_0$ are exactly all $\Gamma(\mathcal{B}(w)\langle n\rangle)$ with $w \in \Omega_0$ and $n \in \mathbf{Z}$.*

Note that the canonical map $M \to \Gamma(\mathcal{L}(M))$ is an isomorphism for all $M$ in $\mathcal{H}_0$, see Proposition 2.16. On the other hand, the canonical map $\mathcal{L}(\Gamma(\mathcal{P})) \to \mathcal{P}$ is an isomorphism for any F-projective graded sheaf on $\mathcal{G}[\Omega_0]$, in particular for all $\mathcal{P} = \mathcal{B}(z)$, see 3.8. So Proposition 3.12 shows that our present proposition is equivalent to the following claim: The functor $\mathcal{L}$ induces an equivalence of categories between $\mathcal{H}_0$ and the category of all F-projective graded sheaves on $\mathcal{G}[\Omega_0]$.

One starts with the observation that $\mathcal{L}(M_e) = \mathcal{B}(e)$. Consider then a module $M$ in $\mathcal{H}_0$ and $s \in \Sigma$ such that also $\theta_s M$ belongs to $\mathcal{H}_0$. Suppose we know already that $\mathcal{L}(M)$ is F-projective. Then one can use the results in Section 5 of [F5] to show that also $\mathcal{L}(\theta_s M)$ is F-projective. In this way one checks that $\mathcal{L}$ maps modules in $\mathcal{H}_0$ to F-projective sheaves. A look at supports shows then at the end that all $\mathcal{B}(w)$ with $w \in \Omega_0$ belong to the image.

Lemma 2.7.a implies for any sheaf $\mathcal{M}$ on $\mathcal{G}[\Omega_0]$ that

$$\Gamma(\mathcal{M})^{\emptyset,w} = \mathcal{M}_w \otimes_{\widetilde{S}} \widetilde{S}^{\emptyset} \qquad \text{for all } w \in \Omega_0. \tag{1}$$

From this fact and the proposition one now deduces that the direct summand $M_x$ in 5.16 is isomorphic to $\Gamma(\mathcal{B}(x)\langle n\rangle)$ for a suitable integer $n$. Combining this fact with 5.19(4) we get for all $x \in W_a$ with $x{\bullet_p}0 \in -p\rho + X_p(T)$ that

$$[\widehat{Z}(w{\bullet_p}0) : \widehat{L}(x{\bullet_p}0)] \le \mathrm{rk}_{\widetilde{S}}\mathcal{B}(x)_w \qquad \text{for all } w \in \Omega_0. \tag{2}$$

The Lusztig conjecture for the characters of the irreducible representations of $G$ is equivalent to the claim that any multiplicity $[\widehat{Z}(w{\bullet_p}0) : \widehat{L}(x{\bullet_p}0)]$ as above is equal to the value $p_{w_0 w, w_0 x}(1)$ at 1 of a suitable Kazhdan-Lusztig polynomial, cf. [Ja2], D.13(1), or [F6], Conj. 3.4 (and Thm. 3.5). A comparison with quantum groups at a $p$–th root of unity (where the analogous conjecture is known to hold) shows that

$$[\widehat{Z}(w{\bullet_p}0) : \widehat{L}(x{\bullet_p}0)] \ge p_{w_0 w, w_0 x}(1) \tag{3}$$

for all $x$ and $w$ as before. Therefore (2) and (3) imply: *The Lustig conjecture holds for $G$ if $\mathrm{rk}_{\widetilde{S}}\mathcal{B}(x)_w = p_{w_0 w, w_0 x}(1)$ for all $x$ and $w$ as above.*

**5.22.** Let $G^\vee$ be a connected semi-simple algebraic group over $\mathbf{C}$ and $T^\vee$ a maximal torus in $G^\vee$ such that the root datum of $(G^\vee, T^\vee)$ is dual to the root datum of $(G, T)$. So we can identify $Y(T)$ with $X(T^\vee)$ in a way such that $\Phi^\vee$ identifies with the root system of $G^\vee$ with respect to $T^\vee$.

Let $G' = G^\vee(\!(t)\!)$ be the corresponding loop group and $T' = T^\vee \times \mathbf{C}^\times$ the standard maximal torus of $G'$. Choose an Iwahori subgroup $B'$ of $G'$ containing $T'$ and denote by $X' = G'/B'$ the corresponding *affine flag variety.* This is an ind-variety. There is a bijection $w \mapsto \mathcal{O}_w$ between $W_a$ and the set of $B'$–orbits on $X'$. The closure $\overline{\mathcal{O}}_w$ is a projective variety, equal to the union of all $\mathcal{O}_x$ with $x \in W_a$, $x \leq w$. Each $\mathcal{O}_w$ itself is isomorphic to an affine space. The torus $T'$ has exactly one fixed point in each $\mathcal{O}_w$.

These facts show that one can apply Theorem 3.6 to compute the equivariant intersection cohomology $IH^\bullet_{T'}(\overline{\mathcal{O}}_w)$ for any $w \in W_a$. This involves a Braden-MacPherson sheaf $\mathcal{B}(w)_{\mathbf{C}}$ on a moment graph $\mathcal{G}_{\mathbf{C}}$ which is constructed in the same way as the moment graph $\mathcal{G}$ in 5.20, but with $(Y(T) \oplus \mathbf{Z}) \otimes k$ replaced by $(Y(T) \oplus \mathbf{Z}) \otimes \mathbf{C} = X(T') \otimes \mathbf{C}$. One gets then

$$\operatorname{rk} \mathcal{B}(w)_{\mathbf{C},z} = \dim IH^\bullet(\overline{\mathcal{O}}_w)_{\{z\}} \qquad \text{for all } z, w \in W_a. \tag{1}$$

A closer look at the construction in 3.5 will show for any $x \in \Omega_0$ that

$$\operatorname{rk} \mathcal{B}(x)_w = \operatorname{rk} \mathcal{B}(x)_{\mathbf{C},w} \qquad \text{for all } w \in \Omega_a \tag{2}$$

whenever $p$ is larger than a bound depending on the root system $\Phi$. Now work by Kazhdan and Lusztig in [KL] shows that

$$\dim IH^\bullet(\overline{\mathcal{O}}_w)_{\{z\}} = p_{w_0 w, w_o x}(1). \tag{3}$$

Combining (1)–(3) with the last statement in 5.21 we see: *Lusztig's conjecture holds for all $p$ larger than a bound depending on the root system* $\Phi$. So we got a new proof for the main result in [AJS].

Using deeper properties of our modules Fiebig shows for all $p > h$ (which is our standard assumption):

**Proposition:** *Let $x, w \in \Omega_0$ with $x \bullet_p 0 \in -p\rho + X_p(T)$. Then*

$$[\widehat{Z}(w \bullet_p 0) : \widehat{L}(x \bullet_p 0)] = 1 \iff p_{w_0 w, w_0 x}(1) = 1.$$

For this result one does not need the theorem in the quantum case that otherwise enters our arguments via 5.21(3).

**5.23.** Fiebig shows in Sections 7 and 8 of [F4] how one can go more directly from cohomology sheaves to $\mathcal{Z}$–modules and thus prove Lusztig's conjecture for large $p$ without mentioning moment graphs.

Using this approach Fiebig has been able to find an explicit bound on $p$ for this result, see [F7]. This bound is still extremely large: In type $A_8$ one gets a number with 40 digits whereas one expects that $p > 9$ should be good enough.

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