## Polarization Optics

Nicolas Fressengeas

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# UE SPM-PHY-S07-101 Polarization Optics 

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Document à télécharger sur http://moodle.univ-metz.fr/

## Further reading

S. Huard.

Polarisation de la lumière.
Masson, 1994.
围 G. P. Können.
Polarized light in Nature.
Cambridge University Press, 1985.

## Course Outline

(1) The physics of polarization optics

- Polarization states
- Jones Calculus
- Stokes parameters and the Poincare Sphere
(2) Polarized light propagation
- Jones Matrices
- Polarizers
- Linear and Circular Anisotropy
- Jones Matrices Composition
(3) Partially polarized light
- Formalisms used
- Propagation through optical devices

The physics of polarization optics Polarized light propagation Partially polarized light
(1) The physics of polarization optics

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## The vector nature of light

Optical wave can be polarized, sound waves cannot

## The scalar monochromatic plane wave <br> The electric field reads:

- Electric field is orthogonal to wave and Poynting vectors


## The vector nature of light

Optical wave can be polarized, sound waves cannot

## The scalar monochromatic plane wave

The electric field reads:

$$
A \cos (\omega t-k z-\varphi)
$$

A vector monochromatic plane wave

- Electric field is orthogonal to wave and Poynting vectors
- Lies in the wave vector normal plane
- Needs 2 components


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A vector monochromatic plane wave

- Electric field is orthogonal to wave and Poynting vectors
- Lies in the wave vector normal plane
- Needs 2 components
- $E_{x}=A_{x} \cos \left(\omega t-k z-\varphi_{x}\right)$
- $E_{y}=A_{y} \cos \left(\omega t-k z-\varphi_{y}\right)$

The physics of polarization optics
Polarized light propagation Partially polarized light

Polarization states
Jones Calculus
Stokes parameters and the Poincare Sphere

## Linear and circular polarization states

## In phase components $\quad \varphi_{y}=\varphi_{x}$


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The physics of polarization optics

Polarization states
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## Linear and circular polarization states


$\pi$ shift $\quad \varphi_{y}=\varphi_{x}+\pi$


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Polarization states
Jones Calculus
Stokes parameters and the Poincare Sphere

## Linear and circular polarization states




$\pi / 2$ shift

$$
\varphi_{y}=\varphi_{x} \pm \pi / 2
$$



Left or Right
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The physics of polarization optics
Polarized light propagation Partially polarized light

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Left or Right

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The physics of polarization optics
Polarized light propagation Partially polarized light

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## Linear and circular polarization states

In phase components $\quad \varphi_{y}=\varphi_{x}$

$\pi$ shift $\quad \varphi_{y}=\varphi_{x}+\pi$

$\pi / 2$ shift

$\varphi_{y}=\varphi_{x} \pm \pi / 2$

Left or Right
UE SPM-PHY-S07-109, version 1.1, frame 5

## The elliptic polarization state

The polarization state of ANY monochromatic wave



## Electric field

- $E_{x}=A_{x} \cos \left(\omega t-k z-\varphi_{x}\right)$
- $E_{y}=A_{y} \cos \left(\omega t-k z-\varphi_{y}\right)$


## The elliptic polarization state

The polarization state of ANY monochromatic wave

$$
\varphi_{y}-\varphi_{x}= \pm \pi / 8
$$



## Electric field

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The polarization state of ANY monochromatic wave

$$
\varphi_{y}-\varphi_{x}= \pm \pi / 4
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## The elliptic polarization state

The polarization state of ANY monochromatic wave

$$
\varphi_{y}-\varphi_{x}= \pm 3 \pi / 4
$$



## Electric field

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- $E_{y}=A_{y} \cos \left(\omega t-k z-\varphi_{y}\right)$


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- $E_{x}=A_{x} \cos \left(\omega t-k z-\varphi_{x}\right)$
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## 4 real numbers

- $A_{x}, \varphi_{x}$
- $A_{y}, \varphi_{y}$


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Electric field

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4 real numbers

- $A_{x}, \varphi_{x}$
- $A_{y}, \varphi_{y}$

2 complex numbers

- $A_{x} \exp \left(i \varphi_{x}\right)$
- $A_{y} \exp \left(i \varphi_{y}\right)$

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## Polarization states are vectors

Monochromatic polarizations belong to a 2D vector space based on the Complex Ring
ANY elliptic polarization state $\Longleftrightarrow$ Two complex numbers
A set of two ordered complex numbers is one 2D complex vector


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## Canonical Basis

$$
\left(\left[\begin{array}{l}
1 \\
0
\end{array}\right],\left[\begin{array}{l}
0 \\
1
\end{array}\right]\right)
$$


$\square$ These two vectors represent time nolarization statac

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## Link with optics ?

- These two vectors represent two polarization states
- We must decide which ones !


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## Link with optics ?

- These two vectors represent two polarization states
- We must decide which ones !


## Polarization Basis

Two independent polarizations :

- Crossed Linear
- Reversed circular
- ...
- YOUR choice



## Examples: Linear Polarizations

## Canonical Basis Choice

- $\left[\begin{array}{l}1 \\ 0\end{array}\right]$ : horizontal linear polarization
- $\left[\begin{array}{l}0 \\ 1\end{array}\right]$ : vertical linear polarization


## Examples: Linear Polarizations

## Canonical Basis Choice

- $\left[\begin{array}{l}1 \\ 0\end{array}\right]$ : horizontal linear polarization
- $\left[\begin{array}{l}0 \\ 1\end{array}\right]$ : vertical linear polarization

$$
\begin{array}{ll}
\text { Tilt } & \theta=\pi / 4 \\
& \frac{1}{\sqrt{2}}\left[\begin{array}{l}
1 \\
1
\end{array}\right]
\end{array}
$$

## Examples: Linear Polarizations

## Canonical Basis Choice

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## Tilt <br> $$
\theta=3 \pi / 4
$$

$$
\frac{1}{\sqrt{2}}\left[\begin{array}{c}
-1 \\
1
\end{array}\right]
$$

## Examples: Linear Polarizations

## Canonical Basis Choice

- $\left[\begin{array}{l}1 \\ 0\end{array}\right]$ : horizontal linear polarization
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## Tilt

$$
\left[\begin{array}{c}
\cos (\theta) \\
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## Examples: Linear Polarizations

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## Tilt

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Linear polarization Jones vector

- Linear Polarization : two in phase components
- Two real numbers

In a linear polarization basis

## Examples: Circular Polarizations

In the same canonical basis choice : linear polarizations

$$
\varphi_{y}-\varphi_{x}= \pm \pi / 2
$$



## Electric field

- $E_{x}=A_{x} \cos \left(\omega t-k z-\varphi_{x}\right)$
- $E_{y}=A_{y} \cos \left(\omega t-k z-\varphi_{y}\right)$
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Jones vector

$$
\frac{1}{\sqrt{2}}\left[\begin{array}{c}
1 \\
\pm i
\end{array}\right]
$$

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## About changing basis

A polarization state Jones vector is basis dependent

## Some elementary algebra

- The polarization vector space dimension is 2
- Therefore : two non colinear vectors form a basis
- Any polarization state can be expressed as the sum of two non colinear other states


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## Find the transformation matrix between between the two following

 bases- Horizontal and Vertical Linear Polarizations
- Right and Left Circular Polarizations


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## Homework

Find the transformation matrix between between the two following bases :

- Horizontal and Vertical Linear Polarizations
- Right and Left Circular Polarizations


## Relationship between Jones and Poynting vectors

 Jones vectors also provide information about intensity
## Choose an orthonormal basis

- Hermitian product is null : $\overline{J_{1}} \cdot J_{2}=0$
- Each vector norm is unity : $\overline{J_{1}} \cdot J_{1}=\overline{J_{2}} \cdot J_{2}=1$

Simple calculations show that

- If each Jones component is one complex electric field
component
- The Hermitian norm is proportional to beam intensity


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Hermitian Norm is Intensity
Simple calculations show that:

- If each Jones component is one complex electric field component
- The Hermitian norm is proportional to beam intensity

The physics of polarization optics

## Polarization as a unique complex number

If the intensity information disappears, polarization is summed up in one complex number

Rule out the intensity
Norm the Jones vector to unity

- Multiplying Jones vector by a complex number does not
change the polarization state
- Norm the first component to 1 : $\left[\begin{array}{l}1 \\ \xi\end{array}\right]$
- The sole $\xi$ describes the polarization state


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Either you norm the vector, or its first component.

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The physics of polarization optics
(1) The physics of polarization optics

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The physics of polarization optics
Polarized light propagation Partially polarized light

## The Stokes parameters

A set of 4 dependent real parameters that can be measured

## Sample Jones Vector

$$
\left[\begin{array}{l}
A_{x} \exp (+i \varphi / 2) \\
A_{y} \exp (-i \varphi / 2)
\end{array}\right]
$$

$P_{0}$

## Overall Intensity

$$
P_{0}=A_{x}^{2}+A_{y}^{2}=I
$$

The physics of polarization optics
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$P_{0} \quad$ Overall Intensity

$$
P_{0}=A_{x}^{2}+A_{y}^{2}=I
$$

## $P_{1}$

 Intensity Différence$$
P_{1}=A_{x}^{2}-A_{y}^{2}=I_{x}-I_{y}
$$

## The Stokes parameters

A set of 4 dependent real parameters that can be measured

## Sample Jones Vector

$$
\left[\begin{array}{l}
A_{x} \exp (+i \varphi / 2) \\
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$P_{0} \quad$ Overall Intensity
$P_{0}=A_{x}^{2}+A_{y}^{2}=I$
$\pi / 4$ Tilted Basis
$P_{2} \quad \frac{1}{\sqrt{2}}\left[\begin{array}{l}A_{x} e^{-i \varphi / 2}+A_{y} e^{+i \varphi / 2} \\ A_{x} e^{-i \varphi / 2}-A_{y} e^{+i \varphi / 2}\end{array}\right]$
$P_{2}=I_{\pi / 4}-I_{-\pi / 4}=2 A_{x} A_{y} \cos (\varphi)$

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$\pi / 4$ Tilted Basis
$J_{\pi / 4}=\frac{1}{\sqrt{2}}\left[\begin{array}{l}A_{x} e^{-i \varphi / 2}+A_{y} e^{+i \varphi / 2} \\ A_{x} e^{-i \varphi / 2}-A_{y} e^{+i \varphi / 2}\end{array}\right]$
$P_{2}=I_{\pi / 4}-I_{-\pi / 4}=2 A_{x} A_{y} \cos (\varphi)$

## Intensity Différence

$$
P_{1}=A_{x}^{2}-A_{y}^{2}=I_{x}-I_{y}
$$

$P_{3}$
Circular Basis

$$
\begin{gathered}
J_{\mathrm{cir}}=\frac{1}{\sqrt{2}}\left[\begin{array}{c}
A_{x} e^{-i \varphi / 2}-i A_{y} e^{+i \varphi / 2} \\
A_{x} e^{-i \varphi / 2}+i A_{y} e^{+i \varphi / 2}
\end{array}\right] \\
P_{3}=I_{\mathrm{L}}-I_{\mathrm{R}}=2 A_{x} A_{y} \sin (\varphi)
\end{gathered}
$$

## The Stokes parameters

A set of 4 dependent real parameters that can be measured

## Sample Jones Vector

$$
\left[\begin{array}{l}
A_{x} \exp (+i \varphi / 2) \\
A_{y} \exp (-i \varphi / 2)
\end{array}\right]
$$

Overall Intensity

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P_{0}=A_{x}^{2}+A_{y}^{2}=I
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$\pi / 4$ Tilted Basis
$J_{\pi / 4}=\frac{1}{\sqrt{2}}\left[\begin{array}{l}A_{x} e^{-i \varphi / 2}+A_{y} e^{+i \varphi / 2} \\ A_{x} e^{-i \varphi / 2}-A_{y} e^{+i \varphi / 2}\end{array}\right]$
$P_{2}=I_{\pi / 4}-I_{-\pi / 4}=2 A_{x} A_{y} \cos (\varphi)$

## $P_{1} \quad$ Intensity Différence

$$
P_{1}=A_{x}^{2}-A_{y}^{2}=I_{x}-I_{y}
$$

$$
\begin{gathered}
J_{\mathrm{cir}}=\frac{1}{\sqrt{2}}\left[\begin{array}{c}
A_{x} e^{-i \varphi / 2}-i A_{y} e^{+i \varphi / 2} \\
A_{x} e^{-i \varphi / 2}+i A_{y} e^{+i \varphi / 2}
\end{array}\right] \\
P_{3}=I_{\mathrm{L}}-I_{\mathrm{R}}=2 A_{x} A_{y} \sin (\varphi)
\end{gathered}
$$

## 4 dependent parameters

$$
P_{0}^{2}=P_{1}^{2}+P_{2}^{2}+P_{3}^{2}
$$

## The Stokes parameters

A set of 4 dependent real parameters that can be measured

## Sample Jones Vector

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Overall Intensity

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$$
P_{1}=A_{x}^{2}-A_{y}^{2}=I_{x}-I_{y}
$$

$$
\begin{array}{l|c}
P_{2} & \pi / 4 \text { Tilted Basis } \\
J_{\pi / 4}=\frac{1}{\sqrt{2}}\left[\begin{array}{cc}
A_{x} e^{-i \varphi / 2}+A_{y} e^{+i \varphi / 2} \\
A_{x} e^{-i \varphi / 2}-A_{y} e^{+i \varphi / 2}
\end{array}\right] & \text { Circular Basis } \\
P_{2}=I_{\pi / 4}-I_{-\pi / 4}=2 A_{x} A_{y} \cos (\varphi) & I_{3}=\left[\begin{array}{c}
A_{x} e^{-i \varphi / 2}-i A_{y} e^{+i \varphi / 2} \\
A_{x} e^{-i \varphi / 2}+i A_{y} e^{+i \varphi / 2}
\end{array}\right] \\
P_{3}=I_{\mathrm{L}}-I_{\mathrm{R}}=2 A_{x} A_{y} \sin (\varphi)
\end{array}
$$

## Homework

Find the reverse relationship : $\varphi, A_{x}, A_{y}$ from the Stokes parameters

## The Poincare Sphere

Polarization states can be described geometrically on a sphere

## Recall the Stokes parameters

$$
P_{0}^{2}=P_{1}^{2}+P_{2}^{2}+P_{3}^{2}
$$

## Normalized Stokes parameters

## The Poincare Sphere

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## Recall the Stokes parameters

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## Normalized Stokes parameters

$$
S_{i}=P_{i} / P_{0}
$$



## The Poincare Sphere

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## Normalized Stokes parameters

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## Unit Radius Sphere

$$
\sum_{i=1}^{3} S_{i}=1
$$

The physics of polarization optics

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## Normalized Stokes parameters

$$
S_{i}=P_{i} / P_{0}
$$

## Unit Radius Sphere

$$
\sum_{i=1}^{3} S_{i}=1
$$

## ( $S_{1}, S_{2}, S_{3}$ ) on a unit radius sphere



Figures from [Hua94]

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The physics of polarization optics

## Polarization states

Jones Calculus
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## The Poincare Sphere

Polarization states can be described geometrically on a sphere

## Normalized Stokes parameters

$$
S_{i}=P_{i} / P_{0}
$$

## Unit Radius Sphere

$$
\sum_{i=1}^{3} S_{i}=1
$$

## General Polarisation



## $\left(S_{1}, S_{2}, S_{3}\right)$ on a unit radius sphere

[^0]Figures from [Hua94]


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## Eigen Polarization states

## Polarization states that do not change after propagation in an anisotropic medium

## Eigen Polarization states <br> - Do not change <br> - Except for Intensity

## Eigen Polarization states

Polarization states that do not change after propagation in an anisotropic medium

## Eigen Polarization states

- Do not change
- Except for Intensity


## Linear Anisotropy Eigen Polarizations

- Quarter and half wave plates and Birefringent materials
- Eigen Polarizations are linear along the eigen axes


## Eigen Polarization states

Polarization states that do not change after propagation in an anisotropic medium

## Eigen Polarization states

- Do not change
- Except for Intensity


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Hermitian operator
2 eigen polarization states are orthonormal

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## Jones Matrices

2D Linear Algebra to compute polarization propagation through devices

## Jones matrices in the eigen basis

- Let $\lambda_{1}$ and $\lambda_{2}$ be the two eigenvalues of a given device
- e.g. for linear anisotropy : $\lambda_{i}=e^{n_{i} k_{0} \Delta z}$


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## Homework

Find half and quarter wave plates Jones Matrices

## Jones Matrices

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## In another basis

- Let $\overrightarrow{J_{1}}=\left[\begin{array}{l}u \\ v\end{array}\right]$ and $\overrightarrow{J_{2}}=\left[\begin{array}{c}-\bar{v} \\ \bar{u}\end{array}\right]$ be the orthonormal eigen vectors
- $\left\{\begin{array}{l}\mathbf{M} \overrightarrow{\jmath_{1}}=\lambda_{1} \overrightarrow{\jmath_{1}} \\ \mathbf{M} \overrightarrow{\jmath_{2}}=\lambda_{2} \overrightarrow{J_{2}}\end{array} \Rightarrow \mathbf{M}=\left[\begin{array}{ll}\lambda_{1} u \bar{u}+\lambda_{2} v \bar{v} & \left(\lambda_{1}-\lambda_{2}\right) u \bar{v} \\ \left(\lambda_{1}-\lambda_{2}\right) v \bar{u} & \lambda_{2} v \bar{u}+\lambda_{1} v \bar{v}\end{array}\right]\right.$


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## The particular case of non absorbing devices

 Jones matrix is a unitary operator when $\lambda_{1}=\lambda_{2}=1$
## Nor absorbing neither amplifying devices

- $\left|\lambda_{1}\right|=\left|\lambda_{2}\right|=1$
- $\mathbf{M} \cdot \overline{\mathbf{M}^{\boldsymbol{t}}}=\overline{\mathbf{M}^{\mathrm{t}}} \cdot \mathbf{M}=\mathbf{I}$
- $\mathbf{M}$ is a unitary operator
- Norm is conserved: Intensity is unchanged after propagation
- Orthogonality is conserved : two initially orthogonal states mill remain so after propagation


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$\square$


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## Unitary operator properties

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## Jones Matrix of a polarizer

## In its eigen basis

- A polarized is designed for :
- Full transmission of one linear polarization
- Zero transmission of its orthogonal counterpart



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When transmitted polarization is $\theta$ tilted
Change base through $\theta$ rotation Transformation Matrix

$$
\mathbf{R}(\theta)=\left[\begin{array}{cc}
\cos (\theta) & -\sin (\theta) \\
\sin (\theta) & \cos (\theta)
\end{array}\right]
$$

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& \mathbf{P}(\theta)=\mathbf{R}(\theta)\left[\begin{array}{ll}
1 & 0 \\
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$$

## Homework

Find again the $\theta$ tilted polarizer Jones Matrix by using physics arguments only

## Intensity transmitted through a polarizer

## From natural or non polarized light

Half the intensity is transmitted


## Intensity transmitted through a polarizer

## From natural or non polarized light

## Half the intensity is transmitted

## From linearly polarized light

- Transmitted Jones vector in polarizer eigen basis:

$$
\left[\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right]\left[\begin{array}{c}
\cos (\theta) \\
\sin (\theta)
\end{array}\right]=\left[\begin{array}{c}
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\end{array}\right]
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- Transmitted Intensity : $\cos ^{2}(\theta)$

Show that whatever the polarizer orientation, the transmitted intensity is half the incident intensity.

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- Transmitted Intensity : $\cos ^{2}(\theta)$

From circularly polarized light
Show that whatever the polarizer orientation, the transmitted intensity is half the incident intensity.
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## Linear anisotropy eigen polarization vectors

- Two orthogonal polarization directions
- Two different refraction indexes $n_{1}$ and $n_{2}$
- Two linear eigen modes along the eigen directions

- Find the Jones Matrices of Quarter and Half wave plates




## Linear anisotropy eigen polarization vectors

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Jones Matrix in the eigen basis Express phase delay only

$$
\left[\begin{array}{cc}
e^{i n_{1} k \Delta z} & 0 \\
0 & e^{i n_{2} k \Delta z}
\end{array}\right]=e^{i \psi}\left[\begin{array}{cc}
e^{i \phi / 2} & 0 \\
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$$

## - Find the Jones Matrices of Quarter and Half wave plates

- Find their action on tilted linear polarization (special case fo



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Quarter and Half wave plates
Homework

- Find the Jones Matrices of Quarter and Half wave plates
- Find their action on tilted linear polarization (special case for $\pi / 4$ tilt)
- Find their action on circular polarization


## What is circular anisotropy ?

- Two orthogonal circular eigen polarization states
- Two different refraction indexes $n_{L}$ and $n_{R}$

transformation matrix


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Jones Matrix in the circular eigen basis Express phase delay only

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0 & e^{i n_{R} k \Delta z}
\end{array}\right]=e^{i \psi}\left[\begin{array}{cc}
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Jones Matrix in a linear polarization basis Transformation matrix

- use $\mathbf{P}_{\text {Cir } \rightarrow \text { Lin }}=\frac{1}{\sqrt{2}}\left[\begin{array}{cc}1 & 1 \\ i & -i\end{array}\right]$ transformation matrix
- $\mathbf{P}_{\mathrm{Cir} \rightarrow \operatorname{Lin}} M\left(\mathbf{P}_{\mathrm{Cir} \rightarrow \operatorname{Lin}}\right)^{-1}=e^{i \boldsymbol{\psi}}\left[\begin{array}{cc}\cos (\phi / 2) & \sin (\phi / 2) \\ -\sin (\phi / 2) & \cos (\phi / 2)\end{array}\right]$

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## Homework

Show that an incoming linear polarisation is simply rotated by an angle proportional to the propagation distance
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## Jones Matrices Composition

The Jones matrices of cascaded optical elements can be composed through Matrix multiplication

## Matrix composition

- If a $\overrightarrow{J_{0}}$ incident light passes through $\mathbf{M}_{\mathbf{1}}$ and $\mathbf{M}_{\mathbf{2}}$ in that order
- First transmission: $\mathbf{M}_{\mathbf{1}} \overrightarrow{\mathrm{J}_{0}}$
- Second transmission: $\mathbf{M}_{\mathbf{2}} \mathbf{M}_{\mathbf{1}} \overrightarrow{J_{0}}$
- Composed Jones Matrix : $\mathbf{M}_{\mathbf{2}} \mathbf{M}_{\mathbf{1}}$ Reversed order


## - Matrix product does not commute in general <br> - Think of the case of a linear anisotropy followed by optical

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Beware of non commutativity

- Matrix product does not commute in general
- Think of the case of a linear anisotropy followed by optical activity
- in that order
- in the reverse order
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## Stokes parameters for partially polarized light

Generalize the coherent definition using the statistical average intensity

## Stokes Vector

$$
\vec{S}=\left[\begin{array}{c}
P_{0} \\
P_{1} \\
P_{2} \\
P_{3}
\end{array}\right]=\left[\begin{array}{c}
\left\langle I_{x}+I_{y}\right\rangle \\
\left\langle I_{x}-I_{y}\right\rangle \\
\left\langle I_{\pi / 4}-I_{-\pi / 4}\right\rangle \\
\left\langle I_{L}-I_{R}\right\rangle
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$$
p=\frac{\sqrt{P_{1}^{2}+P_{2}^{2}+P_{3}^{2}}}{P_{0}}
$$



## Stokes parameters for partially polarized light

Generalize the coherent definition using the statistical average intensity

## Stokes Vector

## Polarization degree $0 \leq p \leq 1$

$$
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Stokes decomposition
Polarized and depolarized sum

$$
\vec{S}=\left[\begin{array}{l}
P_{0} \\
P_{1} \\
P_{2} \\
P_{3}
\end{array}\right]=\left[\begin{array}{c}
p P_{0} \\
P_{1} \\
P_{2} \\
P_{3}
\end{array}\right]+\left[\begin{array}{c}
(1-p) P_{0} \\
0 \\
0 \\
0
\end{array}\right]=\overrightarrow{S_{P}}+\overrightarrow{S_{N P}}
$$

## The Jones Coherence Matrix

## Jones Coherence Matrix

$$
\text { - If } \vec{J}=\left[\begin{array}{l}
A_{x}(t) e^{i \varphi_{x}(t)} \\
A_{y}(t) e^{i \varphi_{y}(t)}
\end{array}\right]
$$

- Meaningless when not monochromatic


## The Jones Coherence Matrix

## Jones Coherence Matrix

- If $\vec{J}=\left[\begin{array}{l}A_{x}(t) e^{i \varphi_{x}(t)} \\ A_{y}(t) e^{i \varphi_{y}(t)}\end{array}\right]$
- $\Gamma_{i j}=\left\langle\vec{J}_{i}(t) \vec{J}_{j}(t)\right\rangle$


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- $\Gamma_{i j}=\left\langle\vec{J}_{i}(t) \overline{\vec{J}}_{j}(t)\right\rangle$
- $\Gamma=\left\langle\overrightarrow{J(t)} \overrightarrow{J(t)}^{t}\right\rangle$


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- $\Gamma=\left\langle\overrightarrow{J(t)} \overrightarrow{J(t)} \overrightarrow{\mathrm{J}}^{\mathrm{J}}\right\rangle$


## Coherence Matrix: explicit formulation

$$
\Gamma=\left[\begin{array}{ll}
\Gamma_{x x} & \Gamma_{x y} \\
\Gamma_{y x} & \Gamma_{y y}
\end{array}\right]
$$

## The Jones Coherence Matrix

## Jones Coherence Matrix

## Jones Vectors are out

- They describe phase differences
- Meaningless when not monochromatic

$$
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A_{x}(t) e^{i \varphi_{x}(t)} \\
A_{y}(t) e^{i \varphi_{y}(t)}
\end{array}\right]
$$

$$
-\Gamma_{i j}=\left\langle\vec{J}_{i}(t) \vec{J}_{j}(t)\right\rangle
$$

$$
\cdot \Gamma=\left\langle\overrightarrow{J(t)} \overrightarrow{\overrightarrow{J(t)}^{t}}\right\rangle
$$

Coherence Matrix: explicit formulation

$$
\Gamma=\left[\begin{array}{cc}
\left.\left.\langle | A_{x}(t)\right|^{2}\right\rangle & \left\langle A_{x}(t) \overline{A_{y}(t)} e^{i\left(\varphi_{x}-\varphi_{y}\right)}\right\rangle \\
\left\langle\overline{A_{x}(t)} A_{y}(t) e^{-i\left(\varphi_{x}-\varphi_{y}\right)}\right\rangle & \left.\left.\langle | A_{y}(t)\right|^{2}\right\rangle
\end{array}\right]
$$

## Jones Coherence Matrix: properties

## The Coherence Matrix

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\Gamma=\left[\begin{array}{cc}
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Trace is Intensity

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\operatorname{Tr}(\Gamma)=1
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Base change Transformation $\mathbf{P}$
$\mathbf{P}^{-1}$ Г $\mathbf{P}$


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\end{array}\right]
$$

## Trace is Intensity

Base change Transformation $\mathbf{P}$

$$
\operatorname{Tr}(\Gamma)=1 \quad \mathbf{P}^{-1} \Gamma \mathbf{P}
$$

Relationship with Stokes parameters
from definition

$$
\left[\begin{array}{l}
P_{0} \\
P_{1} \\
P_{2} \\
P_{3}
\end{array}\right]=\left[\begin{array}{cccc}
1 & 1 & 0 & 0 \\
1 & -1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & i & -i
\end{array}\right]\left[\begin{array}{l}
\Gamma_{x x} \\
\Gamma_{y y} \\
\Gamma_{x y} \\
\Gamma_{y x}
\end{array}\right]
$$

## Jones Coherence Matrix: properties

Trace is Intensity

$$
\operatorname{Tr}(\Gamma)=1
$$

Transformation P

## $\mathbf{P}^{-1}$ Г $\mathbf{P}$

Relationship with Stokes parameters
from definition

$$
\left[\begin{array}{l}
P_{0} \\
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1 & 1 & 0 & 0 \\
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\end{array}\right]\left[\begin{array}{c}
\Gamma_{x x} \\
\Gamma_{y y} \\
\Gamma_{x y} \\
\Gamma_{y x}
\end{array}\right]
$$

Inverse relationship

$$
\left[\begin{array}{c}
\Gamma_{x x} \\
\Gamma_{y y} \\
\Gamma_{x y} \\
\Gamma_{y x}
\end{array}\right]=\frac{1}{2}\left[\begin{array}{cccc}
1 & 1 & 0 & 0 \\
1 & -1 & 0 & 0 \\
0 & 0 & 1 & -i \\
0 & 0 & -1 & i
\end{array}\right]\left[\begin{array}{c}
P_{0} \\
P_{1} \\
P_{2} \\
P_{3}
\end{array}\right]
$$

## Coherence Matrix: further properties

## Polarization degree

$$
p=\sqrt{\frac{P_{1}^{2}+P_{2}^{2}+P_{3}^{2}}{P_{0}^{2}}}=\sqrt{1-\frac{4\left(\Gamma_{x x} \Gamma_{y y}-\Gamma_{x y} \Gamma_{y x}\right)}{\left(\Gamma_{x x}+\Gamma_{y y}\right)^{2}}}=\sqrt{1-\frac{4 \operatorname{Det}(\Gamma)}{\operatorname{Tr}(\Gamma)^{2}}}
$$



[^1]
## Coherence Matrix: further properties

## Polarization degree

$$
p=\sqrt{\frac{P_{1}^{2}+P_{2}^{2}+P_{3}^{2}}{P_{0}^{2}}}=\sqrt{1-\frac{4\left(\Gamma_{x x} \Gamma_{y y}-\Gamma_{x y} \Gamma_{y x}\right)}{\left(\Gamma_{x x}+\Gamma_{y y}\right)^{2}}}=\sqrt{1-\frac{4 \operatorname{Det}(\Gamma)}{\operatorname{Tr}(\Gamma)^{2}}}
$$

$\Gamma$ Decomposition in polarized and depolarized components

- $\Gamma=\Gamma_{P}+\Gamma_{N P}$
- Find $\Gamma_{P}$ and $\Gamma_{N P}$ using the relationship with the Stokes parameters
(1) The physics of polarization optics
- Polarization states
- Jones Calculus
- Stokes parameters and the Poincare Sphere
(2) Polarized light propagation
- Jones Matrices
- Polarizers
- Linear and Circular Anisotropy
- Jones Matrices Composition
(3) Partially polarized light
- Formalisms used
- Propagation through optical devices


## Propagation of the Coherence Matrix

## Jones Calculus

- If incoming polarization is $\overrightarrow{J(t)}$
- Output one is $\overrightarrow{J^{\prime}(t)}=\mathbf{M} \overrightarrow{J(t)}$


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Coherence Matrix if $\mathbf{M}$ is unitary

- M unitary means : linear and/or circular anisotropy only


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- $\Gamma^{\prime}=\left\langle\overrightarrow{J^{\prime}(t)} \overrightarrow{\bar{J}^{\prime}(t)}{ }^{\mathrm{t}}\right\rangle$


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- $\Gamma^{\prime}=\left\langle\overrightarrow{J^{\prime}(t)} \overline{\bar{J}^{\prime}(t)}{ }^{\mathrm{t}}\right\rangle$
- $\Gamma^{\prime}=\mathbf{M}\left\langle\overrightarrow{J(t)} \overline{\overline{J(t)}}{ }^{\mathrm{J}}\right\rangle \mathbf{M}^{-1}$

Basis change

## Propagation of the Coherence Matrix

## Jones Calculus

- If incoming polarization is $\overrightarrow{J(t)}$
- Output one is $\overrightarrow{J^{\prime}(t)}=\mathbf{M} \overrightarrow{J(t)}$

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Basis change
Polarization degree

- Unaltered for unitary operators $\quad \mathrm{Tr}$ and Det are unaltered


## Propagation of the Coherence Matrix

## Jones Calculus

- If incoming polarization is $\overrightarrow{J(t)}$
- Output one is $\overrightarrow{J^{\prime}(t)}=\mathbf{M} \overrightarrow{J(t)}$

Coherence Matrix if $\mathbf{M}$ is unitary

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- $\Gamma^{\prime}=\left\langle\overrightarrow{J^{\prime}(t)} \overline{\bar{J}^{\prime}(t)}{ }^{\mathrm{t}}\right\rangle$
- $\Gamma^{\prime}=\mathbf{M}\langle\overrightarrow{J(t)} \overline{\overline{J(t)}}\rangle \mathbf{M}^{-1}$

Basis change
Polarization degree

- Unaltered for unitary operators $\quad$ Tr and Det are unaltered
- Not the case if a polarizer is present: $p$ becomes 1


## Mueller Calculus

Propagating the Jones coherence matrix is difficult if the operator is not unitary

Jones Calculus raises some difficulties

- Coherence matrix OK for partially polarized light
- Propagation through unitary optical devices
(linear or circular anisotropy only)
$\square$
- Hard Times if Polarizers are present
- Describing intensity, they can be readily measurered


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The Stokes parameters may be an alternative

- Describing intensity, they can be readily measurered


## Mueller Calculus <br> Propagating the Jones coherence matrix is difficult if the operator is not unitary

## Jones Calculus raises some difficulties

- Coherence matrix OK for partially polarized light
- Propagation through unitary optical devices
(linear or circular anisotropy only)
- Hard Times if Polarizers are present


## The Stokes parameters may be an alternative

- Describing intensity, they can be readily measurered
- We will show they can be propagated using $4 \times 4$ real matrices
- They are the Mueller matrices


## The projection on a polarization state

Matrix of the polariazer with axis parallel to $\vec{V}$

## Projection on $\vec{V}$ in Jones Basis

- Orthogonal Linear Polarizations Basis: $\vec{X}$ and $\vec{Y}$
- Normed Projection Base Vector :
- $\vec{V}=A_{x} e^{-i \frac{\varphi}{2}} \vec{X}+A_{y} e^{i \frac{\varphi}{2}} \vec{Y}$
- $\overline{\vec{V}}^{\mathrm{t}} \vec{V}=1$
${ }^{3}$ Easy to check in the projection eigen basis



## The projection on a polarization state

Matrix of the polariazer with axis parallel to $\vec{V}$

Projection on $\vec{V}$ in Jones Basis

- Orthogonal Linear Polarizations Basis: $\vec{X}$ and $\vec{Y}$
- Normed Projection Base Vector :
- $\vec{V}=A_{x} e^{-i \frac{\varphi}{2}} \vec{X}+A_{y} e^{i \frac{\varphi}{2}} \vec{Y}$
- $\overline{\vec{V}}^{\mathrm{t}} \vec{V}=1$
- $\mathrm{P}_{\mathrm{V}}=\vec{V} \overline{\vec{V}}^{\mathrm{t}}{ }^{a}$
${ }^{\text {a }}$ Easy to check in the projection eigen basis : $\left(\vec{V}, \overline{\vec{V}}^{\mathrm{t}}\right)$


## The Pauli Matrices

## A base for the 4D $2 \times 2$ matrix vector space

$$
\sigma_{1}=\left[\begin{array}{ll}
1 & 0 \\
0 & 1
\end{array}\right], \sigma_{2}=\left[\begin{array}{cc}
1 & 0 \\
0 & -1
\end{array}\right], \sigma_{3}=\left[\begin{array}{ll}
0 & 1 \\
1 & 0
\end{array}\right], \sigma_{1}=\left[\begin{array}{cc}
0 & -i \\
i & 0
\end{array}\right]
$$

Pv decomposition

$$
\mathbf{P}_{\mathbf{v}}=\frac{1}{2}\left(p_{0} \sigma_{0}+p_{1} \sigma_{1}+p_{2} \sigma_{2}+p_{3} \sigma_{3}\right)
$$

## Pv composition and Trace property

Trace is the eigen values sum

## Projection property

## Pv composition and Trace property

Trace is the eigen values sum

## Projection property

$$
\overline{\vec{V}^{\mathrm{t}}} \cdot \sigma_{\mathbf{j}} \vec{V}=\left(\overline{\vec{V}^{\mathrm{t}}} \vec{V}\right) \overline{\vec{V}}^{\mathrm{t}} \cdot \sigma_{\mathbf{j}} \vec{V}=\overline{\vec{V}}^{\mathrm{t}}\left(\vec{V} \overline{\vec{V}}^{\mathrm{t}}\right) \sigma_{j} \vec{V}
$$

Projection Trace in its eigen

## Pv composition and Trace property

Trace is the eigen values sum

## Projection property

$$
\overline{\vec{V}^{\mathrm{t}}} \cdot \sigma_{\mathbf{j}} \vec{V}=\left(\overline{\vec{V}^{\mathrm{t}}} \vec{V}\right) \overline{\vec{V}^{\mathrm{t}}} \cdot \sigma_{\mathbf{j}} \vec{V}=\overline{\vec{V}}^{\mathrm{t}} \cdot \mathrm{P}_{\mathbf{v}} \sigma_{\mathrm{j}} \vec{V}
$$

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$$
\overline{\vec{V}}^{\mathrm{t}} \cdot \sigma_{\mathbf{j}} \vec{V}=\left(\overline{\vec{V}^{\mathrm{t}}} \vec{V}\right) \overline{\vec{V}}^{\mathrm{t}} \cdot \sigma_{\mathbf{j}} \vec{V}=\overline{\vec{V}}^{\mathrm{t}} \cdot \mathbf{P}_{\mathbf{v}} \sigma_{\mathbf{j}} \vec{V}
$$

Projection Trace in its eigen basis

- Pv eigenvalues : 0 \& 1

$$
\operatorname{Tr}\left(P_{V}\right)=1
$$

## Pv composition and Trace property

Trace is the eigen values sum

Projection property

$$
\overline{\vec{V}}^{\mathrm{t}} \cdot \sigma_{\mathbf{j}} \vec{V}=\left(\overline{\vec{V}^{\mathrm{t}}} \vec{V}\right) \overline{\vec{V}}^{\mathrm{t}} \cdot \sigma_{\mathbf{j}} \vec{V}=\overline{\vec{V}}^{\mathrm{t}} \cdot \mathbf{P}_{\mathbf{v}} \sigma_{\mathbf{j}} \vec{V}
$$

Projection Trace in its eigen basis

- $\mathbf{P V}_{\mathbf{v}}$ eigenvalues : 0 \& 1
- $\mathbf{P}_{\mathbf{V}} \sigma_{\mathbf{j}}$ eigenvalues : 0 \& $\alpha$
$\alpha \leq 1$
$\operatorname{Tr}\left(P_{V}\right)=1$
$\operatorname{Tr}\left(P_{V} \sigma_{j}\right)=\alpha$


## Pv composition and Trace property

Trace is the eigen values sum

Projection property

$$
\overline{\vec{V}}^{\mathrm{t}} \cdot \sigma_{\mathbf{j}} \vec{V}=\left(\overline{\vec{V}^{\mathrm{t}}} \vec{V}\right) \overline{\vec{V}}^{\mathrm{t}} \cdot \sigma_{\mathbf{j}} \vec{V}=\overline{\vec{V}}^{\mathrm{t}} \cdot \mathbf{P}_{\mathbf{v}} \sigma_{\mathbf{j}} \vec{V}
$$

Projection Trace in its eigen basis

- $\mathbf{P v}_{\mathbf{v}}$ eigenvalues: 0 \& 1
- $\mathbf{P}_{\mathbf{V}} \sigma_{\mathbf{j}}$ eigenvalues : 0 \& $\alpha \quad \alpha \leq 1$
- $\mathbf{P}_{\mathbf{V}} \sigma_{\mathbf{j}}$ eigenvectors are the same as $\mathbf{P}_{\mathbf{V}}$ :
- $\vec{V}$ associated to eigenvalue $\alpha$
$\operatorname{Tr}\left(P_{V}\right)=1$
$\operatorname{Tr}\left(P_{\vee} \sigma_{j}\right)=\alpha$

Project the projection


## Pv composition and Trace property

Trace is the eigen values sum

Projection property

$$
\overline{\vec{V}}^{\mathrm{t}} \cdot \sigma_{\mathbf{j}} \vec{V}=\left(\overline{\vec{V}^{\mathrm{t}}} \vec{V}\right) \overline{\vec{V}}^{\mathrm{t}} \cdot \sigma_{\mathbf{j}} \vec{V}=\overline{\vec{V}}^{\mathrm{t}} \cdot \mathbf{P}_{\mathbf{v}} \sigma_{\mathbf{j}} \vec{V}
$$

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\overline{\vec{V}}^{\mathrm{t}} \cdot \sigma_{\mathbf{j}} \vec{V}=\left(\overline{\vec{V}^{\mathrm{t}}} \vec{V}\right) \overline{\vec{V}}^{\mathrm{t}} \cdot \sigma_{\mathbf{j}} \vec{V}=\overline{\vec{V}}^{\mathrm{t}} \cdot \mathbf{P}_{\mathbf{v}} \sigma_{\mathbf{j}} \vec{V}
$$

Projection Trace in its eigen basis

- $\mathbf{P v}_{\mathbf{v}}$ eigenvalues: 0 \& 1
- $\mathbf{P}_{\mathbf{v}} \sigma_{\mathbf{j}}$ eigenvalues: 0 \& $\alpha$
- $\mathbf{P}_{\mathbf{v}} \sigma_{\mathbf{j}}$ eigenvectors are the same as $\mathbf{P}_{\mathbf{v}}$ :
- $\vec{V}$ associated to eigenvalue $\alpha$
- Project the projection

$$
\overline{\vec{V}^{\mathrm{t}}} \cdot \mathbf{P}_{\mathbf{v}} \sigma_{\mathbf{j}} \vec{V}=\alpha=\operatorname{Tr}\left(\mathbf{P}_{\mathbf{v}} \sigma_{\mathbf{j}}\right)
$$

$$
\alpha \leq 1
$$

$$
\operatorname{Tr}\left(P_{V} \sigma_{j}\right)=\alpha
$$

## Pv composition and Trace property

Trace is the eigen values sum

## Projection property

$$
\overline{\vec{V}}^{\mathrm{t}} \cdot \sigma_{\mathbf{j}} \vec{V}=\left(\overline{\vec{V}^{\mathrm{t}}} \vec{V}\right) \overline{\vec{V}}^{\mathrm{t}} \cdot \sigma_{\mathbf{j}} \vec{V}=\overline{\vec{V}}^{\mathrm{t}} \cdot \mathbf{P}_{\mathbf{v}} \sigma_{\mathbf{j}} \vec{V}
$$

Projection Trace in its eigen basis

- $\mathbf{P}_{\mathbf{V}}$ eigenvalues : 0 \& 1
- $\mathbf{P}_{\mathbf{V}} \sigma_{\mathbf{j}}$ eigenvalues: 0 \& $\alpha$ $\alpha \leq 1$
$\operatorname{Tr}\left(P_{V}\right)=1$
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- $\mathbf{P}_{\mathbf{V}} \sigma_{\mathbf{j}}$ eigenvectors are the same as $\mathbf{P}_{\mathbf{v}}$ :
- $\vec{V}$ associated to eigenvalue $\alpha$
- Project the projection


## $\mathbf{P}_{\mathbf{V}}$ Pauli components and physical meaning

Express $p_{i}$ as a function of $\vec{V}$ and the Pauli matrices, then find their signification

$$
\overline{\vec{V}}^{t} \cdot \sigma_{\mathrm{j}} \vec{V}=\operatorname{Tr}\left(P_{V} \sigma_{j}\right) \quad \operatorname{Tr}\left(\sigma_{i} \sigma_{j}\right)=2 \delta_{i j}
$$

$$
\operatorname{Tr}\left(P_{V} \sigma_{j}\right)=\frac{1}{2} \sum_{i} \operatorname{Tr}\left(\sigma_{i} \sigma_{j}\right) p_{i}
$$

## $\mathbf{P}_{\mathbf{V}}$ Pauli components and physical meaning

Express $p_{i}$ as a function of $\vec{V}$ and the Pauli matrices, then find their signification

$$
\begin{aligned}
& \overline{\vec{V}^{t}} \cdot \sigma_{\mathrm{j}} \overrightarrow{\mathrm{~V}}=\operatorname{Tr}\left(P_{V} \sigma_{j}\right) \\
& \operatorname{Tr}\left(\sigma_{i} \sigma_{j}\right)=2 \delta_{i j} \\
& \operatorname{Tr}\left(P_{V} \sigma_{j}\right)=\frac{1}{2} \sum_{i} \operatorname{Tr}\left(\sigma_{i} \sigma_{j}\right) p_{i}=\frac{1}{2} \sum_{i} 2 \delta_{i j} p_{i}
\end{aligned}
$$

## $\mathbf{P}_{\mathbf{V}}$ Pauli components and physical meaning

Express $p_{i}$ as a function of $\vec{V}$ and the Pauli matrices, then find their signification

$$
\begin{aligned}
& {\overline{\vec{V}^{\mathrm{t}}} \cdot \sigma_{\mathrm{j}} \vec{V}=\operatorname{Tr}\left(P_{V} \sigma_{j}\right) \quad \operatorname{Tr}\left(\sigma_{i} \sigma_{j}\right)=2 \delta_{i j}}_{\overline{\vec{V}}^{\mathrm{t}} \cdot \sigma_{\mathrm{j}} \vec{V}=\operatorname{Tr}\left(P_{V} \sigma_{j}\right)=\frac{1}{2} \sum_{i} \operatorname{Tr}\left(\sigma_{i} \sigma_{j}\right) p_{i}=\frac{1}{2} \sum_{i} 2 \delta_{i j} p_{i}=p_{j}}
\end{aligned}
$$

## $\mathbf{P}_{\mathbf{V}}$ Pauli components and physical meaning

Express $p_{i}$ as a function of $\vec{V}$ and the Pauli matrices, then find their signification

$$
\begin{aligned}
& \overline{\vec{V}^{\mathrm{t}}} \cdot \sigma_{\mathrm{j}} \vec{V}=\operatorname{Tr}\left(P_{V} \sigma_{j}\right) \quad \operatorname{Tr}\left(\sigma_{i} \sigma_{j}\right)=2 \delta_{i j} \\
& \overline{\vec{V}^{\mathrm{t}}} \cdot \sigma_{\mathrm{j}} \vec{V}=\operatorname{Tr}\left(P_{V} \sigma_{j}\right)=\frac{1}{2} \sum_{i} \operatorname{Tr}\left(\sigma_{i} \sigma_{j}\right) p_{i}=\frac{1}{2} \sum_{i} 2 \delta_{i j} p_{i}=p_{j}
\end{aligned}
$$

Project the base vectors on $\vec{V}$

- Using $\vec{V}=A_{x} e^{-i \frac{\varphi}{2}} \vec{X}+A_{y} e^{i \frac{\varphi}{2}} \vec{Y}$
- $\mathrm{P}_{\mathbf{v}} \vec{X}=A_{x}^{2} \vec{X}+A_{x} A_{y} e^{i \varphi} \vec{Y}$
- $\mathbf{P}_{\mathbf{v}} \vec{Y}=A_{y}^{2} \vec{Y}+A_{x} A_{y} e^{-i \varphi} \vec{X}$


## - Using the $\mathrm{P}_{\mathrm{V}}$ decomposition on the Pauli Basis

## $\mathbf{P}_{\mathbf{V}}$ Pauli components and physical meaning

Express $p_{i}$ as a function of $\vec{V}$ and the Pauli matrices, then find their signification

$$
\begin{aligned}
& \overline{\vec{V}}^{\mathrm{t}} \cdot \sigma_{\mathrm{j}} \vec{V}=\operatorname{Tr}\left(P_{V} \sigma_{j}\right) \quad \operatorname{Tr}\left(\sigma_{i} \sigma_{j}\right)=2 \delta_{i j} \\
& \vec{V}^{\mathrm{t}} \cdot \sigma_{\mathrm{j}} \vec{V}=\operatorname{Tr}\left(P_{V} \sigma_{j}\right)=\frac{1}{2} \sum_{i} \operatorname{Tr}\left(\sigma_{i} \sigma_{j}\right) p_{i}=\frac{1}{2} \sum_{i} 2 \delta_{i j} p_{i}=p_{j}
\end{aligned}
$$

Project the base vectors on $\vec{V}$

- Using $\vec{V}=A_{x} e^{-i \frac{\varphi}{2}} \vec{X}+A_{y} e^{i \frac{\varphi}{2}} \vec{Y}$
- $\mathrm{P}_{\mathbf{v}} \vec{X}=A_{x}^{2} \vec{X}+A_{x} A_{y} e^{i \varphi} \vec{Y}$
- $\mathbf{P}_{v} \vec{Y}=A_{y}^{2} \vec{Y}+A_{x} A_{y} e^{-i \varphi} \vec{X}$
- Using the $\mathbf{P}_{\mathbf{V}}$ decomposition on the Pauli Basis
- $\mathbf{P}_{\mathbf{v}} \vec{X}=\frac{1}{2}\left(p_{0}+p_{1}\right) \vec{X}+\frac{1}{2}\left(p_{2}+i p_{3}\right) \vec{Y}$
- $\mathbf{P}_{\mathbf{v}} \vec{Y}=\frac{1}{2}\left(p_{0}-p_{1}\right) \vec{Y}+\frac{1}{2}\left(p_{2}-i p_{3}\right) \vec{X}$


## $\mathbf{P}_{\mathbf{V}}$ Pauli components and physical meaning

Express $p_{i}$ as a function of $\vec{V}$ and the Pauli matrices, then find their signification

$$
\begin{aligned}
& \overline{\vec{V}^{\mathrm{t}}} \cdot \sigma_{\mathrm{j}} \vec{V}=\operatorname{Tr}\left(P_{V} \sigma_{j}\right) \quad \operatorname{Tr}\left(\sigma_{i} \sigma_{j}\right)=2 \delta_{i j} \\
& \overline{\vec{V}^{\mathrm{t}}} \cdot \sigma_{\mathrm{j}} \vec{V}=\operatorname{Tr}\left(P_{V} \sigma_{j}\right)=\frac{1}{2} \sum_{i} \operatorname{Tr}\left(\sigma_{i} \sigma_{j}\right) p_{i}=\frac{1}{2} \sum_{i} 2 \delta_{i j} p_{i}=p_{j}
\end{aligned}
$$

Project the base vectors on $\vec{V}$

- Using $\vec{V}=A_{x} e^{-i \frac{\varphi}{2}} \vec{X}+A_{y} e^{i \frac{\varphi}{2}} \vec{Y}$
- $\mathrm{P}_{\mathbf{v}} \vec{X}=A_{x}^{2} \vec{X}+A_{x} A_{y} e^{i \varphi} \vec{Y}$
- $\mathbf{P}_{v} \vec{Y}=A_{y}^{2} \vec{Y}+A_{x} A_{y} e^{-i \varphi} \vec{X}$
- Using the $\mathbf{P}_{\mathbf{V}}$ decomposition on the Pauli Basis
- $\mathbf{P}_{\mathbf{V}} \vec{X}=\frac{1}{2}\left(p_{0}+p_{1}\right) \vec{X}+\frac{1}{2}\left(p_{2}+i p_{3}\right) \vec{Y}$
- $\mathbf{P}_{\mathbf{V}} \vec{Y}=\frac{1}{2}\left(p_{0}-p_{1}\right) \vec{Y}+\frac{1}{2}\left(p_{2}-i p_{3}\right) \vec{X}$
- Identify


## Pv Pauli composition and Stokes parameters

## Stokes parameters as $\mathrm{P}_{\mathbf{V}}$ decomposition on the Pauli base

- $p_{0}=P_{0}=A_{x}^{2}-A_{y}^{2}=I_{x}-I_{y}$
- $p_{1}=P_{1}=A_{x}^{2}-A_{y}^{2}=I_{x}-I_{y}$
- $p_{2}=P_{2}=2 A_{x} A_{y} \cos (\varphi)=I_{\pi / 4}-I_{-\pi / 4}$
- $p_{3}=P_{3}=2 A_{x} A_{y} \sin (\varphi)=I_{\mathrm{L}}-I_{\mathrm{R}}$


## Propagating through devices: Mueller matrices $\overrightarrow{V^{\prime}}=M_{\jmath} \vec{V}$

Projection on $\overrightarrow{V^{\prime}}$

$$
\mathbf{P}_{\mathbf{V}^{\prime}}=\overrightarrow{V^{\prime}} \overrightarrow{{\overline{V^{\prime}}}^{\mathrm{t}}}
$$

## Propagating through devices: Mueller matrices $\overrightarrow{V^{\prime}}=M_{\jmath} \vec{V}$

Projection on $\overrightarrow{V^{\prime}}$

$$
\mathbf{P}_{\mathbf{v}^{\prime}}=\overrightarrow{V^{\prime}} \overline{{\overrightarrow{V^{\prime}}}^{\mathrm{t}}}=\mathbf{M}_{\mathbf{J}} \vec{V} \overline{\vec{V}^{t}} \overline{\mathbf{M}_{\mathbf{J}}^{\mathrm{t}}}
$$

## Propagating through devices: Mueller matrices $\overrightarrow{V^{\prime}}=M_{\mathrm{J}} \vec{V}$

Projection on $\overrightarrow{V^{\prime}}$

$$
\mathbf{P}_{\mathbf{V}^{\prime}}=\overrightarrow{V^{\prime}} \overline{{\overline{V^{\prime}}}^{t}}=\mathbf{M}_{\mathbf{J}} \overrightarrow{V^{\prime}} \overline{\vec{V}^{t}} \overline{\mathbf{M}_{\mathbf{J}}^{\mathrm{t}}}=\mathbf{M}_{\mathbf{J}} \mathbf{P}_{\mathbf{V}} \overline{\mathbf{M}_{\mathbf{J}}^{\mathrm{t}}}
$$

## Propagating through devices: Mueller matrices $\overrightarrow{V^{\prime}}=M_{\jmath} \vec{V}$

Projection on $\overrightarrow{V^{\prime}}$

$$
\mathbf{P}_{\mathbf{V}^{\prime}}=\overrightarrow{\vec{V}^{\prime}} \overline{\overline{V^{\prime}}}=\mathbf{M}_{\mathbf{J}} \vec{V} \overline{\vec{V}^{\mathrm{t}}} \overline{\mathbf{M}_{\mathbf{J}}^{\mathrm{t}}}=\mathbf{M}_{\mathbf{J}} \mathbf{P}_{\mathbf{v}} \overline{\mathbf{M}_{\mathbf{J}}^{\mathrm{t}}}
$$

Trace relationship

$$
P_{i}^{\prime}=\operatorname{Tr}\left(\mathbf{P}_{\mathbf{v}^{\prime}} \sigma_{i}\right)
$$

## Propagating through devices: Mueller matrices $\overrightarrow{V^{\prime}}=M_{\jmath} \vec{V}$

Projection on $\overrightarrow{V^{\prime}}$

$$
\mathbf{P}_{\mathbf{V}^{\prime}}=\overrightarrow{V^{\prime}} \overline{{\overline{V^{\prime}}}^{\mathrm{t}}}=\mathbf{M}_{\mathbf{J}} \overrightarrow{\vec{V}} \overline{\vec{V}^{t}} \overline{\mathbf{M}_{\mathbf{J}}^{\mathrm{t}}}=\mathbf{M}_{\mathbf{J}} \mathbf{P}_{\mathbf{V}} \overline{\mathbf{M}_{\mathbf{J}}^{\mathrm{t}}}
$$

## Trace relationship

$$
P_{i}^{\prime}=\operatorname{Tr}\left(\mathbf{P}_{\mathbf{v}^{\prime}} \sigma_{i}\right)=\operatorname{Tr}\left(\mathbf{M}_{\mathbf{J}} \mathbf{P}_{\mathbf{v}} \overline{\left.\left.\mathbf{M}_{\mathbf{J}}{ }^{\mathrm{t}} \sigma_{\mathbf{i}}\right), ~\right)}\right.
$$



## Propagating through devices: Mueller matrices $\overrightarrow{V^{\prime}}=M_{\jmath} \vec{V}$

Projection on $\overrightarrow{V^{\prime}}$

$$
\mathbf{P}_{\mathbf{V}^{\prime}}=\overrightarrow{\vec{V}^{\prime}} \overrightarrow{{\overrightarrow{V^{\prime}}}^{t}}=\mathbf{M}_{\mathbf{J}} \vec{V} \overline{\bar{V}^{t}} \overline{\mathbf{M}_{\mathbf{J}}{ }^{t}}=\mathbf{M}_{\mathbf{J}} \mathbf{P}_{\mathbf{V}} \overline{\mathbf{M}_{\mathbf{J}}^{\mathrm{t}}}
$$

Trace relationship

$$
\begin{gathered}
P_{i}^{\prime}=\operatorname{Tr}\left(\mathbf{P}_{\mathbf{v}} \sigma_{i}\right)=\operatorname{Tr}\left(\mathbf{M}_{\mathbf{\jmath}} \mathbf{P}_{\mathbf{v}} \overline{\mathbf{M}_{\mathbf{J}}{ }^{\mathbf{t}}} \sigma_{\mathbf{i}}\right)= \\
\frac{1}{2} \sum_{j=0}^{3} \operatorname{Tr}\left(\mathbf{M}_{\mathbf{J}} \sigma_{\mathbf{j}} \overline{\left.\mathbf{M}_{\mathbf{\prime}}{ }^{\mathbf{t}} \sigma_{\mathbf{i}}\right) P_{j}}\right.
\end{gathered}
$$



Propagating through devices: Mueller matrices $\overrightarrow{V^{\prime}}=M_{\mathrm{J}} \vec{V}$

Projection on $\overrightarrow{V^{\prime}}$

$$
\mathbf{P}_{\mathbf{V}^{\prime}}=\overrightarrow{V^{\prime}} \overrightarrow{{\overrightarrow{V^{\prime}}}^{\mathrm{t}}}=\mathbf{M}_{\mathbf{J}} \vec{V} \overline{\vec{V}^{\mathrm{t}}} \overline{\mathbf{M}_{\mathbf{J}}^{\mathrm{t}}}=\mathbf{M}_{\mathbf{J}} \mathbf{P}_{\mathbf{V}} \overline{\mathbf{M}_{\mathbf{J}}^{\mathrm{t}}}
$$

Trace relationship

$$
\begin{gathered}
P_{i}^{\prime}=\operatorname{Tr}\left(\mathbf{P}_{\mathbf{V}^{\prime}} \sigma_{i}\right)=\operatorname{Tr}\left(\mathbf{M}_{\mathbf{J}} \mathbf{P}_{\mathbf{v}} \overline{\left.\mathbf{M}_{\mathbf{J}}{ }^{\mathrm{t}} \sigma_{\mathbf{i}}\right)=}\right. \\
\frac{1}{2} \sum_{j=0}^{3} \operatorname{Tr}\left(\mathbf{M}_{\mathbf{J}} \sigma_{\mathbf{j}} \overline{\left.\mathbf{M}_{\mathbf{J}}{ }^{\mathrm{t}} \sigma_{\mathbf{i}}\right) P_{j}}\right.
\end{gathered}
$$

Mueller matrix

$$
\overrightarrow{S^{\prime}}=\mathbf{M}_{\mathbf{M}} \vec{S}
$$

$$
\left(M_{M}\right)_{i j}=\frac{1}{2} \operatorname{Tr}\left(\mathbf{M}_{\mathbf{J}} \sigma_{\mathbf{j}} \overline{\left.\mathbf{M}_{\mathbf{J}}{ }^{\mathrm{t}} \sigma_{\mathbf{i}}\right), ~}\right.
$$

## Mueller matrices and partially polarized light

Time average of the previous study
Mueller matrices are time independent

$$
\left\langle\overrightarrow{S^{\prime}}\right\rangle=\mathbf{M}_{\mathbf{M}}\langle\vec{S}\rangle
$$



- Partially coherent light
$\square$
 - half and quarter wave plates


# - linearly and circularly hirefringent crystal 

## Mueller matrices and partially polarized light <br> Time average of the previous study

Mueller matrices are time independent

$$
\left\langle\overrightarrow{S^{\prime}}\right\rangle=\mathbf{M}_{\mathbf{M}}\langle\vec{S}\rangle
$$

## Mueller calculus can be extended to...

- Partially coherent light
- Cascaded optical devices



## Mueller matrices and partially polarized light

Time average of the previous study

## Mueller matrices are time independent

$$
\left\langle\overrightarrow{S^{\prime}}\right\rangle=\mathbf{M}_{\mathbf{M}}\langle\vec{S}\rangle
$$

## Mueller calculus can be extended to...

- Partially coherent light
- Cascaded optical devices


## Final homework

Find the Mueller matrix of each :

- Polarizers along eigen axis or $\theta$ tilted
- half and quarter wave plates
- linearly and circularly birefringent crystal


[^0]:    N. Fressengeas

[^1]:    parameters

