

UE SPM-PHY-S07-101

Polarization Optics

N. Fressengeas

Laboratoire Matériaux Optiques, Photonique et Systèmes
Unité de Recherche commune à l'Université Paul Verlaine Metz et à Supélec

Document à télécharger sur <http://moodle.univ-metz.fr/>



Further reading

[Hua94, K85]



S. Huard.

Polarisation de la lumière.

Masson, 1994.



G. P. Können.

Polarized light in Nature.

Cambridge University Press, 1985.

Course Outline

- 1 The physics of polarization optics
 - Polarization states
 - Jones Calculus
 - Stokes parameters and the Poincare Sphere
- 2 Polarized light propagation
 - Jones Matrices
 - Polarizers
 - Linear and Circular Anisotropy
 - Jones Matrices Composition
- 3 Partially polarized light
 - Formalisms used
 - Propagation through optical devices

The vector nature of light

Optical wave can be polarized, sound waves cannot

The scalar monochromatic plane wave

The electric field reads:

$$A \cos(\omega t - kz - \varphi)$$

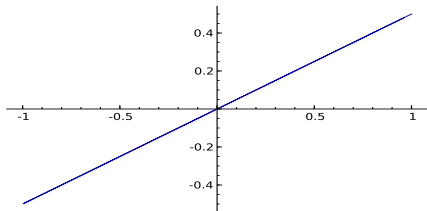
A **vector** monochromatic plane wave

- Electric field is orthogonal to wave and Poynting vectors
- Lies in the wave vector normal **plane**
- Needs **2 components**
 - $E_x = A_x \cos(\omega t - kz - \varphi_x)$
 - $E_y = A_y \cos(\omega t - kz - \varphi_y)$

Linear and circular polarization states

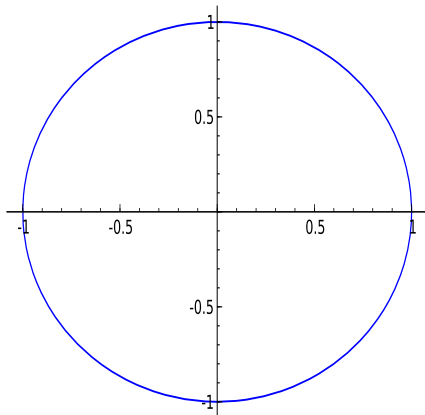
In phase components

$$\varphi_y = \varphi_x$$



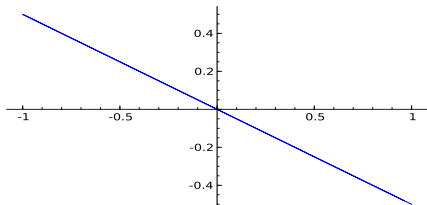
$\pi/2$ shift

$$\varphi_y = \varphi_x \pm \pi/2$$



π shift

$$\varphi_y = \varphi_x + \pi$$

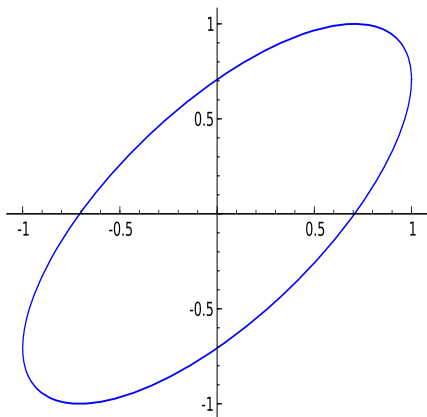


Left or Right

The elliptic polarization state

The polarization state of ANY monochromatic wave

$$\varphi_y - \varphi_x = \pm\pi/4$$



Electric field

- $E_x = A_x \cos(\omega t - kz - \varphi_x)$
- $E_y = A_y \cos(\omega t - kz - \varphi_y)$

4 real numbers

- A_x, φ_x
- A_y, φ_y

2 complex numbers

- $A_x \exp(i\varphi_x)$
- $A_y \exp(i\varphi_y)$

Polarization states are vectors

Monochromatic polarizations belong to a 2D vector space based on the Complex Ring

ANY elliptic polarization state \iff Two complex numbers

A set of **two** ordered complex numbers is **one** 2D complex vector

Canonical Basis

$$\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

Link with optics ?

- These two vectors represent two polarization states
- We must decide which ones !

Polarization Basis

Two independent polarizations :

- Crossed Linear
- Reversed circular
- ...
- **YOUR** choice

Examples : Linear Polarizations

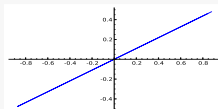
Canonical Basis Choice

- $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$: horizontal linear polarization
- $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$: vertical linear polarization

Tilt

θ

$$\begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$



Linear polarization Jones vector

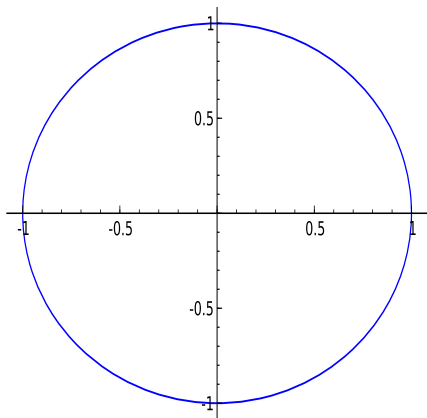
- Linear Polarization : two **in phase** components
- Two **real** numbers

In a linear polarization basis

Examples : Circular Polarizations

In the same canonical basis choice : linear polarizations

$$\varphi_y - \varphi_x = \pm\pi/2$$



Electric field

- $E_x = A_x \cos(\omega t - kz - \varphi_x)$
- $E_y = A_y \cos(\omega t - kz - \varphi_y)$

Jones vector

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ \pm i \end{bmatrix}$$

About changing basis

A polarization state Jones vector is basis dependent

Some elementary algebra

- The polarization vector space dimension is 2
- Therefore : two non colinear vectors form a basis
- Any polarization state can be expressed as the sum of two non colinear other states
- Remark : two colinear polarization states are identical

Homework

Find the transformation matrix between between the two following bases :

- Horizontal and Vertical Linear Polarizations
- Right and Left Circular Polarizations

Relationship between Jones and Poynting vectors

Jones vectors also provide information about intensity

Choose an orthonormal basis

(J_1, J_2)

- Hermitian product is null : $\overline{J_1} \cdot J_2 = 0$
- Each vector norm is unity : $\overline{J_1} \cdot J_1 = \overline{J_2} \cdot J_2 = 1$

Hermitian Norm is Intensity

Simple calculations show that :

- If each Jones component is one complex electric field component
- The Hermitian norm is proportional to beam intensity

Polarization as a unique complex number

If the intensity information disappears, polarization is summed up in one complex number

Rule out the intensity

Norm the Jones vector to **unity**

Put 1 as first component

- Multiplying Jones vector by a complex number does not change the polarization state
- Norm the first component to 1 : $\begin{bmatrix} 1 \\ \xi \end{bmatrix}$
- The sole ξ describes the polarization state

Choose between the two

Either you norm the vector, or its first component. **Not both !**

The Stokes parameters

A set of 4 dependent real parameters that can be measured

Sample Jones Vector

$$\begin{bmatrix} A_x \exp(+i\varphi/2) \\ A_y \exp(-i\varphi/2) \end{bmatrix}$$

P_0 Overall Intensity

$$P_0 = A_x^2 + A_y^2 = I$$

P_1 Intensity Différence

$$P_1 = A_x^2 - A_y^2 = I_x - I_y$$

P_2 $\pi/4$ Tilted Basis

$$J_{\pi/4} = \frac{1}{\sqrt{2}} \begin{bmatrix} A_x e^{-i\varphi/2} + A_y e^{+i\varphi/2} \\ A_x e^{-i\varphi/2} - A_y e^{+i\varphi/2} \end{bmatrix}$$

$$P_2 = I_{\pi/4} - I_{-\pi/4} = 2A_x A_y \cos(\varphi)$$

P_3 Circular Basis

$$J_{\text{cir}} = \frac{1}{\sqrt{2}} \begin{bmatrix} A_x e^{-i\varphi/2} - iA_y e^{+i\varphi/2} \\ A_x e^{-i\varphi/2} + iA_y e^{+i\varphi/2} \end{bmatrix}$$

$$P_3 = I_L - I_R = 2A_x A_y \sin(\varphi)$$

4 dependent parameters

$$P_0^2 = P_1^2 + P_2^2 + P_3^2$$

Eigen Polarization states

Polarization states that do not change after propagation in an anisotropic medium

Eigen Polarization states

- Do not change
- Except for Intensity

Hermitian operator

2 eigen polarization states are orthonormal

Linear Anisotropy Eigen Polarizations

- Quarter and half wave plates and Birefringent materials
- Eigen Polarizations are linear along the eigen axes

Circular Anisotropy

- Also called optical activity
- e.g in Faraday rotators and in gyrotory non linear crystals
- Linear polarization is rotated by an angle proportional to propagation distance
- Eigen polarizations are the circular polarizations

Jones Matrices

2D Linear Algebra to compute polarization propagation through devices

Jones matrices in the eigen basis

- Let λ_1 and λ_2 be the two eigenvalues of a given device
- e.g. for linear anisotropy : $\lambda_i = e^{n_i k_0 \Delta z}$
- Jones Matrix is $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

In another basis

use Transformation Matrix

- Let $\vec{J}_1 = \begin{bmatrix} u \\ v \end{bmatrix}$ and $\vec{J}_2 = \begin{bmatrix} -\bar{v} \\ \bar{u} \end{bmatrix}$ be the orthonormal eigen vectors
- $\begin{cases} \mathbf{M} \vec{J}_1 = \lambda_1 \vec{J}_1 \\ \mathbf{M} \vec{J}_2 = \lambda_2 \vec{J}_2 \end{cases} \Rightarrow \mathbf{M} = \begin{bmatrix} \lambda_1 u \bar{u} + \lambda_2 v \bar{v} & (\lambda_1 - \lambda_2) u \bar{v} \\ (\lambda_1 - \lambda_2) v \bar{u} & \lambda_2 v \bar{u} + \lambda_1 v \bar{v} \end{bmatrix}$

The particular case of non absorbing devices

Jones matrix is a unitary operator when $|\lambda_1| = |\lambda_2| = 1$

Nor absorbing neither amplifying devices

- $|\lambda_1| = |\lambda_2| = 1$
- $\mathbf{M} \cdot \overline{\mathbf{M}^t} = \overline{\mathbf{M}^t} \cdot \mathbf{M} = I$
- \mathbf{M} is a unitary operator

Unitary operator properties

- Norm is conserved : Intensity is unchanged after propagation
- Orthogonality is conserved : two initially orthogonal states will remain so after propagation

Jones Matrix of a polarizer

In its eigen basis

- A polarizer is designed for :
 - Full transmission of one linear polarization
 - Zero transmission of its orthogonal counterpart
- Eigen basis Jones matrix : $\mathbf{P}_x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ or $\mathbf{P}_y = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

When transmitted polarization is θ tilted

Change base through $-\theta$ rotation Transformation Matrix

$$\mathbf{R}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

$$\mathbf{P}(\theta) = \mathbf{R}(\theta) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{R}(-\theta) = \begin{bmatrix} \cos^2(\theta) & \sin(\theta)\cos(\theta) \\ \sin(\theta)\cos(\theta) & \sin^2(\theta) \end{bmatrix}$$

Intensity transmitted through a polarizer

From natural or non polarized light

Half the intensity is transmitted

From linearly polarized light

- Transmitted Jones vector in polarizer eigen basis:

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} = \begin{bmatrix} \cos(\theta) \\ 0 \end{bmatrix}$$

- Transmitted Intensity : $\cos^2(\theta)$ **MALUS** law

From circularly polarized light

Show that whatever the polarizer orientation, the transmitted intensity is half the incident intensity.

Linear anisotropy eigen polarization vectors

- Two orthogonal polarization directions
- Two different refraction indexes n_1 and n_2
- Two linear eigen modes along the eigen directions

Jones Matrix in the eigen basis

Express phase delay only

$$\begin{bmatrix} e^{in_1k\Delta z} & 0 \\ 0 & e^{in_2k\Delta z} \end{bmatrix} = e^{i\psi} \begin{bmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{bmatrix} \approx \begin{bmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{bmatrix}$$

Quarter and Half wave plates

Homework

- Find the Jones Matrices of Quarter and Half wave plates
- Find their action on tilted linear polarization (special case for $\pi/4$ tilt)
- Find their action on circular polarization

What is circular anisotropy ?

- Two orthogonal circular eigen polarization states
- Two different refraction indexes n_L and n_R

Jones Matrix in the circular eigen basis Express phase delay only

$$\begin{bmatrix} e^{in_L k \Delta z} & 0 \\ 0 & e^{in_R k \Delta z} \end{bmatrix} = e^{i\psi} \begin{bmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{bmatrix} \approx \begin{bmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{bmatrix}$$

Jones Matrix in a linear polarization basis Transformation matrix

- use $\mathbf{P}_{\text{Lin} \rightarrow \text{Cir}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$ transformation matrix

- $(\mathbf{P}_{\text{Lin} \rightarrow \text{Cir}}) M (\mathbf{P}_{\text{Lin} \rightarrow \text{Cir}})^{-1} = e^{i\psi} \begin{bmatrix} \cos(\phi/2) & \sin(\phi/2) \\ -\sin(\phi/2) & \cos(\phi/2) \end{bmatrix}$

Jones Matrices Composition

The Jones matrices of cascaded optical elements can be composed through Matrix multiplication

Matrix composition

- If a \vec{J}_0 incident light passes through \mathbf{M}_1 and \mathbf{M}_2 in that order
- First transmission: $\mathbf{M}_1 \vec{J}_0$
- Second transmission: $\mathbf{M}_2 \mathbf{M}_1 \vec{J}_0$
- Composed Jones Matrix : $\mathbf{M}_2 \mathbf{M}_1$ Reversed order

Beware of non commutativity

- Matrix product does not commute in general
- Think of the case of a linear anisotropy followed by optical activity
 - in that order
 - in the reverse order

Stokes parameters for partially polarized light

Generalize the coherent definition using the statistical average intensity

Stokes Vector

$$\vec{S} = \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} \langle I_x + I_y \rangle \\ \langle I_x - I_y \rangle \\ \langle I_{\pi/4} - I_{-\pi/4} \rangle \\ \langle I_L - I_R \rangle \end{bmatrix}$$

Polarization degree

$$0 \leq p \leq 1$$

$$p = \frac{\sqrt{P_1^2 + P_2^2 + P_3^2}}{P_0}$$

Stokes decomposition

Polarized and depolarized sum

$$\vec{S} = \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} pP_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} + \begin{bmatrix} (1-p)P_0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \vec{S}_P + \vec{S}_{NP}$$

The Jones Coherence Matrix

Jones Vectors are out

- They describe phase differences
- Meaningless when not monochromatic

Jones Coherence Matrix

- If $\vec{J} = \begin{bmatrix} A_x(t) e^{i\varphi_x(t)} \\ A_y(t) e^{i\varphi_y(t)} \end{bmatrix}$
- $\Gamma_{ij} = \langle \vec{J}_i(t) \overline{\vec{J}_j(t)} \rangle$
- $\Gamma = \langle \overrightarrow{J(t)} \overrightarrow{J(t)}^t \rangle$

Coherence Matrix: explicit formulation

$$\Gamma = \begin{bmatrix} \langle |A_x(t)|^2 \rangle & \langle A_x(t) \overline{A_y(t)} e^{i(\varphi_x - \varphi_y)} \rangle \\ \langle \overline{A_x(t)} A_y(t) e^{-i(\varphi_x - \varphi_y)} \rangle & \langle |A_y(t)|^2 \rangle \end{bmatrix}$$

Jones Coherence Matrix: properties

Trace is Intensity

$$\text{Tr}(\Gamma) = I$$

Base change

Transformation \mathbf{P}

$$\mathbf{P}^{-1}\Gamma\mathbf{P}$$

Relationship with Stokes parameters

from definition

$$\begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -i & i \end{bmatrix} \begin{bmatrix} \Gamma_{xx} \\ \Gamma_{yy} \\ \Gamma_{xy} \\ \Gamma_{yx} \end{bmatrix}$$

Inverse relationship

$$\begin{bmatrix} \Gamma_{xx} \\ \Gamma_{yy} \\ \Gamma_{xy} \\ \Gamma_{yx} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & i \\ 0 & 0 & 1 & -i \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ P_3 \end{bmatrix}$$

Coherence Matrix: further properties

Polarization degree

$$p = \sqrt{\frac{P_1^2 + P_2^2 + P_3^2}{P_0^2}} = \sqrt{1 - \frac{4(\Gamma_{xx}\Gamma_{yy} - \Gamma_{xy}\Gamma_{yx})}{(\Gamma_{xx} + \Gamma_{yy})^2}} = \sqrt{1 - \frac{4\text{Det}(\Gamma)}{\text{Tr}(\Gamma)^2}}$$

Γ Decomposition in polarized and depolarized components

- $\Gamma = \Gamma_P + \Gamma_{NP}$
- Find Γ_P and Γ_{NP} using the relationship with the Stokes parameters

Propagation of the Coherence Matrix

Jones Calculus

- If incoming polarization is $\overrightarrow{J}(t)$
- Output one is $\overrightarrow{J}'(t) = \mathbf{M}\overrightarrow{J}(t)$

Coherence Matrix if \mathbf{M} is unitary

- \mathbf{M} unitary means : linear and/or circular anisotropy only
- $\Gamma' = \langle \overrightarrow{J}'(t) \overrightarrow{J}'(t)^t \rangle$
- $\Gamma' = \mathbf{M} \langle \overrightarrow{J}(t) \overrightarrow{J}(t)^t \rangle \mathbf{M}^{-1}$ Basis change

Polarization degree

- Unaltered for unitary operators Tr and Det are unaltered
- Not the case if a polarizer is present : p becomes **1**

Mueller Calculus

Propagating the Jones coherence matrix is difficult if the operator is not unitary

Jones Calculus raises some difficulties

- Coherence matrix OK for partially polarized light
- Propagation through unitary optical devices
(linear or circular anisotropy only)
- Hard Times if Polarizers are present

The Stokes parameters may be an alternative

- Describing intensity, they can be readily measured
- We will show they can be propagated using 4×4 **real** matrices
- They are the **Mueller** matrices



The projection on a polarization state

Matrix of the polarizer with axis parallel to \vec{V}

Projection on \vec{V} in Jones Basis

P_V

- Orthogonal Linear Polarizations Basis: \vec{X} and \vec{Y}
- Normed Projection Base Vector :
 - $\vec{V} = A_x e^{-i\frac{\varphi}{2}} \vec{X} + A_y e^{i\frac{\varphi}{2}} \vec{Y}$
 - $\vec{V}^t \vec{V} = 1$
- $P_V = \vec{V} \vec{V}^t$ ^a

^aEasy to check in the projection eigen basis

The Pauli Matrices

A base for the 4D 2×2 matrix vector space

$$\sigma_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \sigma_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_3 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

\mathbf{P}_V decomposition

$$\mathbf{P}_V = \frac{1}{2} (p_0 \sigma_0 + p_1 \sigma_1 + p_2 \sigma_2 + p_3 \sigma_3)$$

\mathbf{P}_V composition and Trace property

Trace is the eigen values sum

Projection property

$$\overrightarrow{V}^t \cdot \sigma_j \overrightarrow{V} = \left(\overrightarrow{V}^t \overrightarrow{V} \right) \overrightarrow{V}^t \cdot \sigma_j \overrightarrow{V} = \overrightarrow{V}^t \left(\overrightarrow{V} \overrightarrow{V}^t \right) \sigma_j \overrightarrow{V} = \overrightarrow{V}^t \cdot \mathbf{P}_V \sigma_j \overrightarrow{V}$$

Projection Trace in its eigen basis

- \mathbf{P}_V eigenvalues : 0 & 1 $\text{Tr}(P_V) = 1$
- $\mathbf{P}_V \sigma_j$ eigenvalues : 0 & α $\alpha \leq 1$ $\text{Tr}(P_V \sigma_j) = \alpha$
- $\mathbf{P}_V \sigma_j$ eigenvectors are the same as \mathbf{P}_V :
 - \overrightarrow{V} associated to eigenvalue α
- Project the projection

$$\overrightarrow{V}^t \cdot \mathbf{P}_V \sigma_j \overrightarrow{V} = \alpha = \text{Tr}(\mathbf{P}_V \sigma_j) = \overrightarrow{V}^t \cdot \sigma_j \overrightarrow{V}$$

\mathbf{P}_V Pauli components and physical meaning

Express p_i as a function of \vec{V} and the Pauli matrices, then find their signification

$$\vec{V}^t \cdot \sigma_j \vec{V} = \text{Tr}(\mathbf{P}_V \sigma_j) \qquad \text{Tr}(\sigma_i \sigma_j) = 2\delta_{ij}$$

$$\vec{V}^t \cdot \sigma_j \vec{V} = \text{Tr}(\mathbf{P}_V \sigma_j) = \frac{1}{2} \sum_i \text{Tr}(\sigma_i \sigma_j) p_i = \frac{1}{2} \sum_i 2\delta_{ij} p_i = p_j$$

Project the base vectors on \vec{V}

- Using $\vec{V} = A_x e^{-i\frac{\varphi}{2}} \vec{X} + A_y e^{i\frac{\varphi}{2}} \vec{Y}$
 - $\mathbf{P}_V \vec{X} = A_x^2 \vec{X} + A_x A_y e^{i\varphi} \vec{Y}$
 - $\mathbf{P}_V \vec{Y} = A_y^2 \vec{Y} + A_x A_y e^{-i\varphi} \vec{X}$
- Using the \mathbf{P}_V decomposition on the Pauli Basis
 - $\mathbf{P}_V \vec{X} = \frac{1}{2} (p_0 + p_1) \vec{X} + \frac{1}{2} (p_2 + ip_3) \vec{Y}$
 - $\mathbf{P}_V \vec{Y} = \frac{1}{2} (p_0 - p_1) \vec{Y} + \frac{1}{2} (p_2 - ip_3) \vec{X}$
- Identify

P_V Pauli composition and Stokes parameters

Stokes parameters as P_V decomposition on the Pauli base

- $p_0 = P_0 = A_x^2 - A_y^2 = I_x - I_y$
- $p_1 = P_1 = A_x^2 - A_y^2 = I_x - I_y$
- $p_2 = P_2 = 2A_x A_y \cos(\varphi) = I_{\pi/4} - I_{-\pi/4}$
- $p_3 = P_3 = 2A_x A_y \sin(\varphi) = I_L - I_R$

Propagating through devices: Mueller matrices

$$\vec{V}' = \mathbf{M}_J \vec{V}$$

Projection on \vec{V}'

$$\mathbf{P}_{V'} = \vec{V}' \vec{V}'^t = \mathbf{M}_J \vec{V} \vec{V}^t \mathbf{M}_J^t = \mathbf{M}_J \mathbf{P}_V \mathbf{M}_J^t$$

Trace relationship

$$P'_i = \text{Tr}(\mathbf{P}_{V'} \sigma_i) = \text{Tr}(\mathbf{M}_J \mathbf{P}_V \mathbf{M}_J^t \sigma_i) = \\ \frac{1}{2} \sum_{j=0}^3 \text{Tr}(\mathbf{M}_J \sigma_j \mathbf{M}_J^t \sigma_i) P_j$$

Mueller matrix

$$\vec{S}' = \mathbf{M}_M \vec{S}$$

$$(M_M)_{ij} = \frac{1}{2} \text{Tr}(\mathbf{M}_J \sigma_j \mathbf{M}_J^t \sigma_i)$$

Mueller matrices and partially polarized light

Time average of the previous study

Mueller matrices are time independent

$$\langle \vec{S}' \rangle = \mathbf{M}_M \langle \vec{S} \rangle$$

Mueller calculus can be extended to . . .

- Partially coherent light
- Cascaded optical devices

Final homework

Find the Mueller matrix of each :

- Polarizers along eigen axis or θ tilted
- half and quarter wave plates
- linearly and circularly birefringent crystal