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## Efficient C++ finite element computing with Rheolef

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# Efficient C++ finite element computing with Rheolef

P. Saramito

distributed version 6.0 update 1 April 2012



# Introduction

**Rheolef** is a programming environment that serves as a convenient laboratory for computations involving finite element methods. **Rheolef** is both a C++ library and a set of commands for **unix shell** programming, providing **algorithms** and **data structures**.

- **Algorithms** refer to the most up-to-date ones: **preconditioned sparse solvers** for linear systems, incompressible elasticity, Stokes and Navier-Stokes flows, **characteristic method** for convection dominated heat problems, etc. Also nonlinear generic algorithms such as **fixed point** and **damped Newton** methods.
- **Data structures** fit the standard **variational formulation** concept: **spaces**, discrete **fields**, bilinear **forms** are C++ types for variables, that can be combined in any expressions, as you write it on the paper.

Combined together, as a Lego game, these bricks allows the user to solve most complex nonlinear problems. This Book details, step by step, how some simple and more complex problems can be solved, most of them in less than 20 lines of code. The **concision** and **readability** of codes written with **Rheolef** is certainly a major keypoint of this environment.

## Classical features

- Poisson problems in 1D 2D and 3D with P1 or P2 elements
- Stokes problems in 2D and 3D, with Taylor-Hood P2-P1 or stabilized P1 bubble-P1 elements
- linear elasticity in 2D and 3D, with P1 and P2 elements, including the incompressible and nearly incompressible elasticity
- characteristic method for convection-difusion, time-dependent problems and Navier-Stokes equations.
- input and output in various file format for meshes generators and numerical data visualization systems

## Advanced features

- auto-adaptive mesh based for 2D problems
- axisymmetric problems
- multi-regions and non-constant coefficients
- nonlinear problems with either fixed-point algorithms or a provided generic damped Newton solver
- 3d stereo visualization

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### The Rheolef present contributors

from 2008 **Ibrahim Cheddadi**: discontinuous Galerkin method for transport problems.

from 2010 **Mahamar Dicko**: finite element methods for equations on surfaces.

from 2002 **Jocelyn Etienne**: characteristic method for time-dependent problems.

from 2000 **Pierre Saramito**: project leader: main developments and code maintainer.

### Past contributors

2010 **Lara Abouorm**: banded level set method for equations on surfaces.

2000 **Nicolas Roquet**: initial versions of Stokes and Bingham flow solvers.

## Preface to Rheolef version 6.0

### What is new in Rheolef 6.0 ?

The major main features are:

- support **distributed architectures**: the code looks sequential, is easy to read and write but can be run massively parallel and distributed, based on the MPI library.
- **high order polynomial** approximation:  $P_k$  basis are introduced in this version, for  $k \geq 0$ . This feature will be improved in the future developments.
- **mesh adaptation** and the **characteristic method** are now available for **three-dimensional** problems.

In order to evolve in these directions, internal data structures inside the library are completely rewritten in a different way, and thus this version is a completely new library.

Conversely, the library and unix command interfaces was as less as possible modified.

Nevertheless, the user will find some few backward incompatibilities: 5.93 based codes will not directly compile with the 6.0 library version. Let us review how to move a code from 5.93 to 6.0 version.

### Moving to Rheolef 6.0

#### 1. Namespace

The namespace `rheolef` was already introduced in last 5.93 version. Recall that a code usually starts with:

```
#include "rheolef.h"
using namespace rheolef;
```

#### 2. Environment

The MPI library requires initialisation and the two command line arguments. This initialisation is performed via the `boost::mpi` class `environment`: The code entry point writes:

```
int main (int argc, char** argv) {
    environment rheolef (argc,argv);
    ...
}
```

#### 3. Fields and forms data accessors

The accesses to unknown and blocked data was of a field `uh` was direct, as `uh.u` and `uh.b`. This access is no more possible in a distributed environment, as non-local value requests may be optimized and thus, read and write access may be controled thought accessors. These accessors are named `uh.u()` and `uh.b()` for read access, and `uh.set_u()` and `uh.set_b()` for write access. Similarly, a form `a` has accessors as `a.uu()`.

A typical 5.93 code writes:

```
ssk<Float> sa = ldlt(a.uu);
uh.u = sa.solve (lh.u - a.ub*uh.b);
```

and the corresponding 6.0 code is:

```
solver sa (a.uu());
uh.set_u() = sa.solve (lh.u() - a.ub()*uh.b());
```

This major change in the library interface induces the most important work when porting to the 6.0 version.

Notice also that the old `ssk<Float>` class has been superseded by the `solver` class, that manages both direct and iterative solvers in a more effective way. For three-dimensional problems, the iterative solver is the default while direct solvers are used otherwise. In the same spirit, a `solver_abtb` has been introduced, for Stokes-like mixed problem. These features facilitate the dimension-independent coding style provided by the **Rheolef** library.

#### 4. Distributed input and output streams

Input and output *sequential* standard streams `cin`, `cout` and `cerr` may now be replaced by *distributed Rheolef* streams `din`, `dout` and `derr` as:

```
din >> omega;
dout << uh;
```

These new streams are available together with the `idiststream` and `odiststream` classes of the **Rheolef** library.

#### 5. File formats '.geo' and '.field' have changed

The '.geo' and '.field' file formats have changed. The `.mfield` is now obsolete: it has been merged into the `.field` format that supports now multi-component fields. Also, the corresponding `mfield` unix command is obsolete, as these features are integrated in the `field` unix command.

At this early stage of the 6.0 version, it is not yet possible to read the old `.geo` format, but this backward compatibility will be assured soon.

#### 6. Space on a domain

A space defined on a domain "boundary" of a mesh `omega` was defined in the 5.93 version as:

```
space Wh (omega["boundary"], omega, "P1");
```

It writes now:

```
space Wh (omega["boundary"], "P1");
```

as the repetition of `omega` is no more required.







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## Part I

# Getting started with simple problems



The first part of this book starts with the Dirichlet problem with homogeneous boundary condition: this example is declined with details in dimension 1, 2 and 3, as a starting point to **Rheolef**.

Next chapters present various boundary conditions: for completeness, we treat non-homogeneous Dirichlet, Neumann, and Robin boundary conditions for model problems. The last two examples presents some special difficulties that appears in most problems: the first one introduce to problems with non-constant coefficients and the second one, a ill-posed problem where the solution is defined up to a constant.

This first part can be viewed as a pedagogic preparation for more advanced applications, such as Stokes and elasticity, that are treated in the second part of this book. Problem with non-constant coefficients are common as subproblems generated by various algorithms for non-linear problem.



# Chapter 1

## Getting started with Rheolef

For obtaining and installing **Rheolef**, see the installation instructions on the **Rheolef** home page:

```
http://www-ljk.imag.fr/membres/Pierre.Saramito/rheolef/
```

All examples presented along the present book are available in the `example/` directory of the **Rheolef** distribution. This directory is given by the following unix command:

```
rheolef-config --exampledir
```

This command returns you a path, something like `/usr/share/doc/rheolef-doc/examples/` and you should make a copy of these files:

```
cp -a /usr/share/doc/rheolef-doc/examples/ .
cd examples
```

Before to run examples, please check your **Rheolef** installation with:

```
rheolef-config --check
```

### 1.1 The model problem

Let us consider the classical Poisson problem with homogeneous Dirichlet boundary conditions in a domain bounded  $\Omega \subset R^d$ ,  $d = 1, 2, 3$ :

*(P): find  $u$ , defined in  $\Omega$ , such that:*

$$-\Delta u = 1 \text{ in } \Omega \tag{1.1}$$

$$u = 0 \text{ on } \partial\Omega \tag{1.2}$$

where  $\Delta$  denotes the Laplace operator. The variational formulation of this problem expresses (see appendix A.1 for details):

*(VF): find  $u \in H_0^1(\Omega)$  such that:*

$$a(u, v) = l(v), \quad \forall v \in H_0^1(\Omega) \tag{1.3}$$

where the bilinear form  $a(.,.)$  and the linear form  $l(.)$  are defined by

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx, \quad \forall u, v \in H_0^1(\Omega)$$
$$l(v) = \int_{\Omega} v \, dx, \quad \forall v \in L^2(\Omega)$$



The bilinear form  $a(\cdot, \cdot)$  defines a scalar product in  $H_0^1(\Omega)$  and is related to the *energy* form. This form is associated to the  $-\Delta$  operator.

## 1.2 Approximation

Let us introduce a mesh  $\mathcal{T}_h$  of  $\Omega$  and the finite dimensional space  $X_h$  of continuous piecewise polynomial functions.

$$X_h = \{v \in H^1(\Omega); v|_K \in P_k, \forall K \in \mathcal{T}_h\}$$

where  $k = 1$  or  $2$ . Let  $V_h = X_h \cap H_0^1(\Omega)$  be the functions of  $X_h$  that vanishes on the boundary of  $\Omega$ . The approximate problem expresses:

$(VF)_h$ : find  $u_h \in V_h$  such that:

$$a(u_h, v_h) = l(v_h), \forall v_h \in V_h$$

By developing  $u_h$  on a basis of  $V_h$ , this problem reduces to a linear system. The following C++ code implement this problem in the **Rheolef** environment.

## 1.3 File ‘dirichlet.cc’

```

1 #include "rheolef.h"
2 using namespace rheolef;
3 using namespace std;
4 int main(int argc, char**argv) {
5     environment rheolef (argc, argv);
6     geo omega (argv[1]);
7     space Xh (omega, argv[2]);
8     Xh.block ("boundary");
9     form a (Xh, Xh, "grad_grad");
10    field lh = riesz (Xh, 1);
11    field uh (Xh);
12    uh ["boundary"] = 0;
13    solver sa (a.uu());
14    uh.set_u() = sa.solve (lh.u() - a.ub()*uh.b());
15    dout << uh;
16 }

```

## 1.4 Comments

This code applies for both one, two or three dimensional meshes and for both piecewise linear or quadratic finite element approximations. Four major classes are involved, namely: **geo**, **space**, **form** and **field**.

Let us now comment the code, line by line.

```
#include "rheolef.h"
```

The first line includes the **Rheolef** header file ‘rheolef.h’.

```
using namespace rheolef;
using namespace std;
```

By default, in order to avoid possible name conflicts when using another library, all classe and function names are prefixed by `rheolef::`, as in `rheolef::space`. This feature is called the name space. Here, since there is no possible conflict, and in order to simplify the syntax, we drop all the `rheolef::` prefixes, and do the same with the standard `c++` library classes and variables, that are also prefixed by `std::`.

```
int main(int argc, char**argv) {
```

The entry function of the program is always called `main` and accepts arguments from the unix command line: `argc` is the counter of command line arguments and `argv` is the table of values. The character string `argv[0]` is the program name and `argv[i]`, for  $i = 1$  to `argc-1`, are the additional command line arguments.

```
environment rheolef (argc, argv);
```

These two command line parameters are immediatly furnished to the distributed environment initializer of the `boost::mpi` library, that is a `c++` library based on the usual message passing interface (MPI) library. Notice that this initialization is required, even when you run with only one processor.

```
geo omega (argv[1]);
```

This command get the first unix command-line argument `argv[1]` as a mesh file name and store the corresponding mesh in the variable `omega`.

```
space Xh (omega, argv[2]);
```

Build the finite element space `Xh` contains all the piecewise polynomial continuous functions. The polynomial type is the second command-line arguments `argv[2]`, and could be either `P1`, `P2` or any  $P_k$ , where  $k \geq 1$ .

```
Xh.block ("boundary");
```

The homogeneous Dirichlet conditions are declared on the boundary.

```
form a (Xh, Xh, "grad_grad");
```

The form  $a(.,.)$  is the energy form.

```
field lh = riesz (Xh, 1);
```

Here  $lh(.)$  is the Riesz representer of the constant right-hand side  $f = 1$  of the problem.

```
field uh (Xh);
```

The field `uh` contains the the degrees of freedom.

```
uh ["boundary"] = 0;
```

Some degrees of freedom are prescribed as zero on the boundary.

Let  $(\varphi_i)_{0 \leq i < \dim(X_h)}$  be the basis of  $X_h$  associated to the Lagrange nodes, e.g. the vertices of the mesh for the  $P_1$  approximation and the vertices and the middle of the edges for the  $P_2$  approximation. The approximate solution  $u_h$  expresses as a linear combination of the continuous piecewise polynomial functions  $(\varphi_i)$ :

$$u_h = \sum_i u_i \varphi_i$$

Thus, the field  $u_h$  is completely represented by its coefficients  $(u_i)$ . The coefficients  $(u_i)$  of this combination are grouped into to sets: some have zero values, from the boundary condition and are related to *blocked* coefficients, and some others are *unknown*. Blocked coefficients are stored into the `uh.b` array while unknown one are stored into `uh.u`. Thus, the restriction of the bilinear form  $a(.,.)$  to  $X_h \times X_h$  can be conveniently represented by a block-matrix structure:

$$a(u_h, v_h) = \begin{pmatrix} \text{vh.u} & \text{vh.b} \end{pmatrix} \begin{pmatrix} \text{a.uu} & \text{a.ub} \\ \text{a.bu} & \text{a.bb} \end{pmatrix} \begin{pmatrix} \text{uh.u} \\ \text{uh.b} \end{pmatrix}$$

This representation also applies for the linear form  $l(.)$ :

$$l(v_h) = \begin{pmatrix} \text{vh.u} & \text{vh.b} \end{pmatrix} \begin{pmatrix} \text{lh.u} \\ \text{lh.b} \end{pmatrix}$$

Thus, the problem  $(VF)_h$  writes now:

$$\begin{pmatrix} \text{vh.u} & \text{vh.b} \end{pmatrix} \begin{pmatrix} \text{a.uu} & \text{a.ub} \\ \text{a.bu} & \text{a.bb} \end{pmatrix} \begin{pmatrix} \text{uh.u} \\ \text{uh.b} \end{pmatrix} = \begin{pmatrix} \text{vh.u} & \text{vh.b} \end{pmatrix} \begin{pmatrix} \text{lh.u} \\ \text{lh.b} \end{pmatrix}$$

for any  $\text{vh.u}$  and where  $\text{vh.b} = 0$ . After expansion, the problem reduces to *find*  $\text{uh.u}$  such that:

$$\text{a.uu} * \text{uh.u} = \text{l.u} - \text{a.ub} * \text{uh.b}$$

The resolution of this linear system for the  $\text{a.uu}$  matrix is then performed. A preliminary step build the  $LDL^T$  factorization:

```
solver sa (a.uu());
```

Then, the second step solves the *unknown part*:

```
uh.set_u() = sa.solve (lh.u() - a.ub()*uh.b());
```

When  $d > 3$ , a faster iterative strategy is automatically preferred by the `solver` class for solving the linear system: in that case, the preliminary step build an incomplete Choleski factorization preconditioner, while the second step runs an iterative method: the preconditioned conjugate gradient algorithm. Finally, the field is printed to standard output:

```
dout << uh;
```

The `dout` stream is a specific variable defined in the **Rheolef** library: it is a distributed and parallel extension of the usual `cout` stream in C++

## 1.5 How to compile the code

First, create a `Makefile` as follow:

```
include $(shell rheolef-config --libdir)/rheolef/rheolef.mk
CXXFLAGS = $(INCLUDES_RHEOLEF)
LDLIBS = $(LIBS_RHEOLEF)
default: dirichlet
```

Then, enter:

```
make dirichlet
```

Now, your program, linked with **Rheolef**, is ready to run on a mesh.

## 1.6 How to run the program

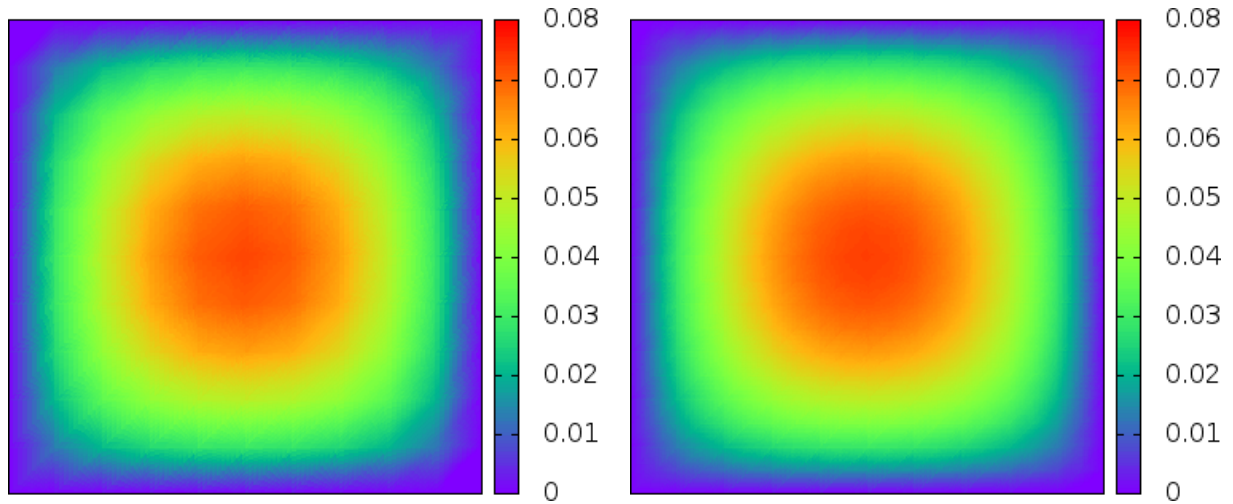


Figure 1.1: Solution of the model problem for  $d = 2$ : (left)  $P_1$  element; (right)  $P_2$  element.

Enter the commands:

```
mkgeo_grid -t 10 > square.geo
geo square.geo
```

The first command generates a simple 10x10 bidimensional mesh of  $\Omega = ]0, 1[^2$  and stores it in the file `square.geo`. The second command shows the mesh. It uses `gnuplot` visualization program by default.

The next command performs the computation:

```
./dirichlet square.geo P1 > square.field
field square.field
```

## 1.7 Distributed and parallel runs

Alternatively, a computation in a distributed and parallel environment writes:

```
mpirun -np 4 ./dirichlet square.geo P1 > square.field
```

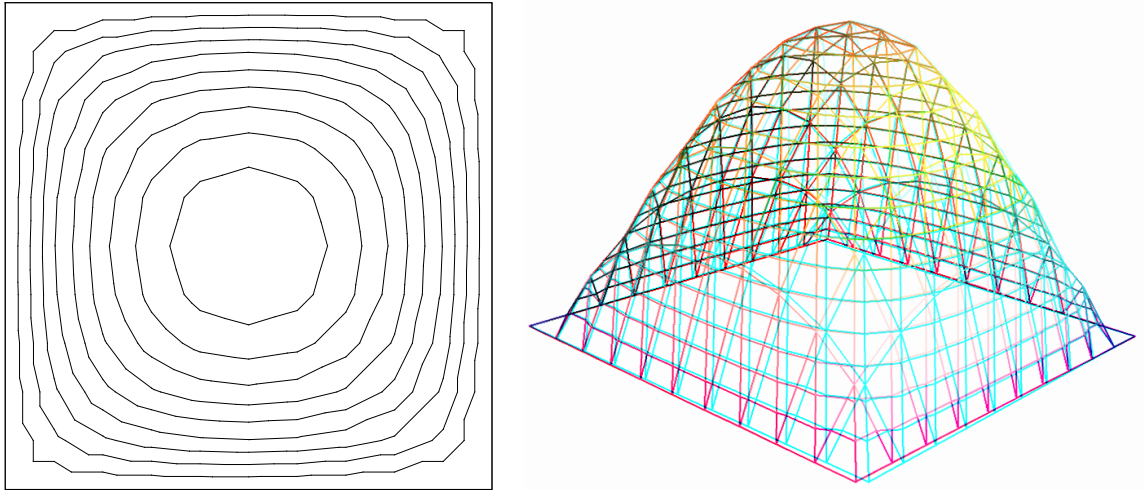


Figure 1.2: Alternative representations of the solution of the model problem ( $d = 2$  and the  $P_1$  element): (left) in black-and-white; (right) in elevation and stereoscopic anaglyph mode.

## 1.8 Stereo visualization

Also explore some graphic rendering modes (see Fig. 1.2):

```
field square.field -bw
field square.field -gray
field square.field -mayavi
field square.field -elevation -nofill -stereo
```

The last command shows the solution in elevation and in stereoscopic anaglyph mode (see Fig. 1.4, left). The anaglyph mode requires red-cyan glasses: red for the left eye and cyan for the right one, as shown on Fig. 1.3.

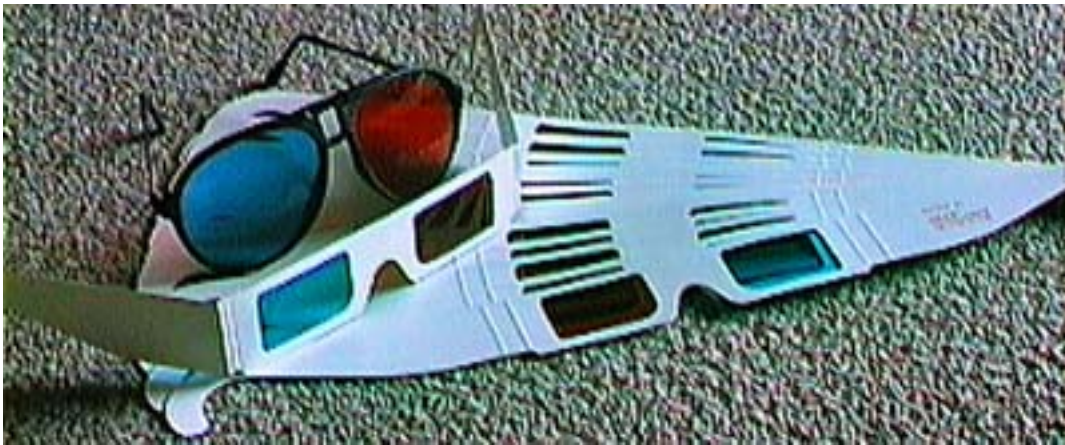


Figure 1.3: Red-cyan anaglyph glasses for the stereoscopic visualization.

See [http://en.wikipedia.org/wiki/Anaglyph\\_image](http://en.wikipedia.org/wiki/Anaglyph_image) for more and <http://www.alpes-stereo.com/lunettes.html> for how to find anaglyph red-cyan glasses.

Please, consults the corresponding unix manual page for more on `field`, `geo` and `mkgeo_grid`:

```
man mkgeo_grid
man geo
man field
```

## 1.9 High-order finite element methods

Turning to the P2 or P3 approximations simply writes:

```
./dirichlet square.geo P2 > square-P2.field
field square-P2.field
```

Fig. 1.1.right shows the result. You can replace the P2 command-line argument by any  $P_k$ , where  $k \geq 1$ .

Now, let us consider a mono-dimensional problem  $\Omega = ]0, 1[$ :

```
mkgeo_grid -e 10 > line.geo
geo line.geo
./dirichlet line.geo P1 | field -
```

The first command generates a subdivision containing ten edge elements. The last two lines show the mesh and the solution via `gnuplot` visualization, respectively.

Conversely, the P2 case writes:

```
./dirichlet line.geo P2 | field -
```

## 1.10 Tridimensional computations

Let us consider a three-dimensional problem  $\Omega = ]0, 1[^3$ . First, let us generate a mesh:

```
mkgeo_grid -T 10 > cube.geo
geo cube.geo
geo cube.geo -stereo -full
geo cube.geo -stereo -cut
```

The previous commands draw the mesh with all internal edges (`-full`), stereoscopic anaglyph (`-stereo`) and then with a cut (`-cut` inside the internal structure: a simple click on the central arrow draws the cut plane normal vector or its origin, while the red square allows a translation).

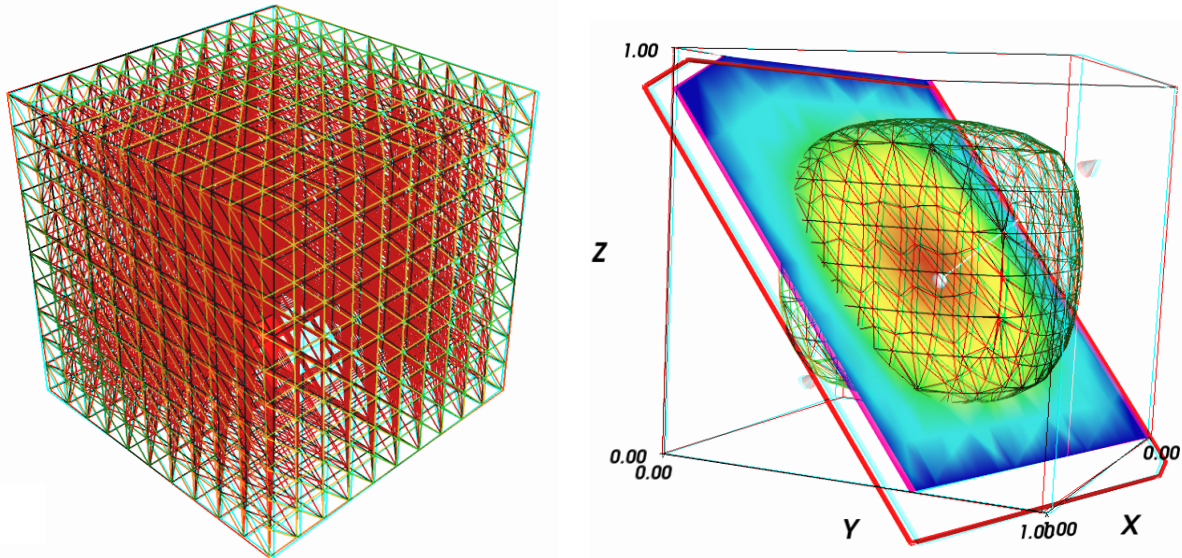


Figure 1.4: Solution of the model problem for  $d = 3$  and the  $P_1$  element : (left) mesh; (right) isovalue, cut planes and stereo anaglyph renderings.

Then, we perform the computation and the visualization:

```
./dirichlet cube.geo P1 > cube.field
field cube.field
```

The visualization presents an isosurface. Also here, you can interact with the cutting plane. Click on `IsoSurface` in the left menu and change the value of the isosurface. Finally exit from the visualization and explore the stereoscopic anaglyph mode (see Fig. 1.4, right):

```
field cube.field -stereo
```

It is also possible to add a second `IsoSurface` or `ScalarCutPlane` module to this scene by using the `Visualize` menu. After this exploration of the 3D visualisation capacities of our environment, let us go back to the Dirichlet problem and perform the  $P_2$  approximation:

```
./dirichlet cube.geo P2 | field -
```

## 1.11 Quadrangles, prisms and hexahedra

Quadrangles and hexahedra are also supported in meshes:

```
mkgeo_grid -q 10 > square.geo
geo square.geo
mkgeo_grid -H 10 > cube.geo
geo cube.geo
```

Notices also that the one-dimensional exact solution writes:

$$u(x) = \frac{x(1-x)}{2}$$

while the two- and three-dimensional ones support a Fourier expansion (see e.g. [1], annex).





## Chapter 2

# Standard boundary conditions

We show how to deal with various non-homogeneous boundary conditions of Dirichlet, Neuman and Robin type.

### 2.1 Non-homogeneous Dirichlet conditions

#### Formulation

We turn now to the case of a non-homogeneous Dirichlet boundary conditions. Let  $f \in H^{-1}(\Omega)$  and  $g \in H^{\frac{1}{2}}(\partial\Omega)$ . The problem writes:

(P<sub>2</sub>) find  $u$ , defined in  $\Omega$  such that:

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega \\ u &= g \text{ on } \partial\Omega \end{aligned}$$

The variational formulation of this problem expresses:

(VF<sub>2</sub>) find  $u \in V$  such that:

$$a(u, v) = l(v), \quad \forall v \in V_0$$

where

$$\begin{aligned} a(u, v) &= \int_{\Omega} \nabla u \cdot \nabla v \, dx \\ l(v) &= \int_{\Omega} f v \, dx \\ V &= \{v \in H^1(\Omega); v|_{\partial\Omega} = g\} \\ V_0 &= H_0^1(\Omega) \end{aligned}$$

#### Approximation

As usual, we introduce a mesh  $\mathcal{T}_h$  of  $\Omega$  and the finite dimensional space  $X_h$ :

$$X_h = \{v \in H^1(\Omega); v|_K \in P_k, \forall K \in \mathcal{T}_h\}$$

Then, we introduce:

$$\begin{aligned} V_h &= \{v \in X_h; v|_{\partial\Omega} = \pi_h(g)\} \\ V_{0,h} &= \{v \in X_h; v|_{\partial\Omega} = 0\} \end{aligned}$$

where  $\pi_h$  denotes the Lagrange interpolation operator. The approximate problem writes:  
 $(VF_2)_h$ : find  $u_h \in V_h$  such that:

$$a(u_h, v_h) = l(v_h), \quad \forall v_h \in V_{0,h}$$

The following C++ code implement this problem in the **Rheolef** environment.

### File ‘dirichlet-nh.cc’

```

1 #include "rheolef.h"
2 using namespace rheolef;
3 using namespace std;
4 #include "cosinusprod_laplace.icc"
5 int main(int argc, char**argv) {
6     environment rheolef(argc, argv);
7     geo omega (argv[1]);
8     size_t d = omega.dimension();
9     space Xh (omega, argv[2]);
10    Xh.block ("boundary");
11    form a (Xh, Xh, "grad_grad");
12    field lh = riesz (Xh, f(d));
13    field uh (Xh);
14    space Wh (omega["boundary"], argv[2]);
15    uh ["boundary"] = interpolate(Wh, g(d));
16    solver sa (a.uu());
17    uh.set_u() = sa.solve (lh.u() - a.ub()*uh.b());
18    dout << uh;
19 }

```

Let us choose  $\Omega \subset R^d$ ,  $d = 1, 2, 3$  with

$$f(x) = d\pi^2 \prod_{i=0}^{d-1} \cos(\pi x_i) \quad \text{and} \quad g(x) = \prod_{i=0}^{d-1} \cos(\pi x_i)$$

Remarks the notation  $x = (x_0, \dots, x_{d-1})$  for points of  $R^d$ : since all arrays start at index zero in C++ codes, and in order to avoid any mistakes between the code and the mathematical formulation, we also adopt this convention here. This choice of  $f$  and  $g$  is convenient, since the exact solution is known:

$$u(x) = \prod_{i=0}^{d-1} \cos(\pi x_i)$$

The following C++ code implement this problem by using the concept of *function object*, also called *class-function* (see e.g. [2]). A convenient feature is the ability for function objects to store auxiliary parameters, such as the space dimension  $d$  for  $f$  here, or some constants, as  $\pi$  for  $f$  and  $g$ .

### File ‘cosinusprod\_laplace.icc’

```

1 struct f : unary_function<point,Float> {
2     Float operator() (const point& x) const {
3         return d*pi*pi*cos(pi*x[0])*cos(pi*x[1])*cos(pi*x[2]); }
4     f(size_t d1) : d(d1), pi(acos(Float(-1))) {}
5     size_t d; const Float pi;
6 };
7 struct g : unary_function<point,Float> {
8     Float operator() (const point& x) const {
9         return cos(pi*x[0])*cos(pi*x[1])*cos(pi*x[2]); }
10    g(size_t d1) : pi(acos(Float(-1))) {}
11    const Float pi;
12 };

```

## Comments

The class `point` describes the coordinates of a point  $(x_0, \dots, x_{d-1}) \in R^d$  as a  $d$ -uplet of `Float`. The `Float` type is usually a `double`. This type depends upon the **Rheolef** configuration (see [3], installation instructions), and could also represent some high precision floating point class. The `dirichlet-nh.cc` code looks like the previous one `dirichlet.cc` related to homogeneous boundary conditions. Let us comments the changes. The dimension  $d$  comes from the geometry  $\Omega$ :

```
size_t d = omega.dimension();
```

The linear form  $l(\cdot)$ , defined as the Riesz representer of  $f$ , writes:

```
field lh = riesz (Xh, f(d));
```

Notice that the function object  $f$  is build with the dimension  $d$  as parameter. The space  $W_h$  of picewise  $P_k$  functions defined on the boundary  $\partial\Omega$  is defined by:

```
space Wh (omega["boundary"], argv[2]);
```

where  $P_k$  is defined via the second command line argument `argv[2]`. This space is suitable for the Lagrange interpolation of  $g$  on the boundary:

```
uh ["boundary"] = interpolate(Wh, g(d));
```

The values of the degrees of freedom related to the boundary are stored into the field `uh.b`, where non-homogeneous Dirichlet conditions applies. The rest of the code is similar to the homogeneous Dirichlet case.

### 2.1.1 How to run the program

First, compile the program:

```
make dirichlet-nh
```

Running the program is obtained from the homogeneous Dirichlet case, by replacing `dirichlet` by `dirichlet-nh`:

```
mkgeo_grid -e 10 > line.geo
./dirichlet-nh line.geo P1 > line.field
field line.field
```

for the bidimensional case:

```
mkgeo_grid -t 10 > square.geo
./dirichlet-nh square.geo P1 > square.field
field square.field
```

and for the tridimensional case:

```
mkgeo_grid -T 10 > box.geo
./dirichlet-nh box.geo P1 > box.field
field -mayavi box.field
```

Here, the P1 approximation can be replaced by P2, P3, etc, by modifying the command-line argument.

## 2.2 Non-homogeneous Neumann boundary conditions for the Helmholtz operator

### Formulation

Let us show how to insert Neumann boundary conditions. Let  $f \in H^{-1}(\Omega)$  and  $g \in H^{-\frac{1}{2}}(\partial\Omega)$ . The problem writes:

$(P_3)$ : find  $u$ , defined in  $\Omega$  such that:

$$\begin{aligned} u - \Delta u &= f \text{ in } \Omega \\ \frac{\partial u}{\partial n} &= g \text{ on } \partial\Omega \end{aligned}$$

The variational formulation of this problem expresses:

$(VF_3)$ : find  $u \in H^1(\Omega)$  such that:

$$a(u, v) = l(v), \quad \forall v \in H^1(\Omega)$$

where

$$\begin{aligned} a(u, v) &= \int_{\Omega} u v \, dx + \int_{\Omega} \nabla u \cdot \nabla v \, dx \\ l(v) &= \int_{\Omega} f v \, dx + \int_{\partial\Omega} g v \, ds \end{aligned}$$

### Approximation

As usual, we introduce a mesh  $\mathcal{T}_h$  of  $\Omega$  and the finite dimensional space  $X_h$ :

$$X_h = \{v \in H^1(\Omega); v|_K \in P_k, \forall K \in \mathcal{T}_h\}$$

The approximate problem writes:

$(VF_3)_h$ : find  $u_h \in X_h$  such that:

$$a(u_h, v_h) = l(v_h), \quad \forall v_h \in X_h$$

### File ‘neumann-nh.cc’

```

1 #include "rheolef.h"
2 using namespace rheolef;
3 using namespace std;
4 #include "sinusprod_helmholtz.icc"
5 int main(int argc, char**argv) {
6     environment rheolef(argc, argv);
7     geo omega (argv[1]);
8     size_t d = omega.dimension();
9     space Xh (omega, argv[2]);
10    form m (Xh, Xh, "mass");
11    form a (Xh, Xh, "grad_grad");
12    a = m + a;
13    field lh = riesz(Xh, f(d)) + riesz(Xh, "boundary", g(d));
14    field uh (Xh);
15    solver sa (a.uu());
16    uh.set_u() = sa.solve (lh.u() - a.ub()*uh.b());
17    dout << uh;
18 }

```

Let us choose  $\Omega \subset R^d$ ,  $d = 1, 2, 3$  and

$$f(x) = (1 + d\pi^2) \prod_{i=0}^{d-1} \sin(\pi x_i)$$

$$g(x) = \begin{cases} -\pi & \text{when } d = 1 \\ -\pi \left( \sum_{i=0}^{d-1} \sin(\pi x_i) \right) & \text{when } d = 2 \\ -\pi \left( \sum_{i=0}^{d-1} \sin(\pi x_i) \sin(x_{(i+1) \bmod d}) \right) & \text{when } d = 3 \end{cases}$$

This example is convenient, since the exact solution is known:

$$u(x) = \prod_{i=0}^{d-1} \sin(\pi x_i)$$

### File ‘sinusprod\_helmholtz.icc’

```

1 struct f : unary_function<point,Float> {
2   Float operator() (const point& x) const {
3     switch (d) {
4       case 1: return (1+d*pi*pi)*sin(pi*x[0]);
5       case 2: return (1+d*pi*pi)*sin(pi*x[0])*sin(pi*x[1]);
6       default: return (1+d*pi*pi)*sin(pi*x[0])*sin(pi*x[1])*sin(pi*x[2]);
7     }
8   f(size_t d1) : d(d1), pi(acos(Float(-1.0))) {}
9   size_t d; const Float pi;
10 };
11 struct g : unary_function<point,Float> {
12   Float operator() (const point& x) const {
13     switch (d) {
14       case 1: return -pi;
15       case 2: return -pi*(sin(pi*x[0]) + sin(pi*x[1]));
16       default: return -pi*( sin(pi*x[0])*sin(pi*x[1])
17                             + sin(pi*x[1])*sin(pi*x[2])
18                             + sin(pi*x[2])*sin(pi*x[0]));
19     }
20   g(size_t d1) : d(d1), pi(acos(Float(-1.0))) {}
21   size_t d; const Float pi;
22 };

```

### Comments

The `neumann-nh.cc` code looks like the previous one `dirichlet-nh.cc`. Let us comments only the changes.

```

form m (Xh, Xh, "mass");
form a (Xh, Xh, "grad_grad");

```

The bilinear forms  $m(.,.)$  and  $a(.,.)$ , also called the *mass* and *energy* in mechanics, are defined as:

$$m(u, v) = \int_{\Omega} u v dx \quad \text{and} \quad a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v dx$$

Then, these two forms are added and stored in  $a$ :

```
a = m + a;
```

The right-hand side is computed as:

```
field lh = riesz(Xh, f(d)) + riesz(Xh, "boundary", g(d));
```

### 2.2.1 How to run the program

First, compile the program:

```
make neumann-nh
```

Running the program is obtained from the homogeneous Dirichlet case, by replacing `dirichlet` by `neumann-nh`.

## 2.3 The Robin boundary conditions

### Formulation

Let  $f \in H^{-1}(\Omega)$  and Let  $g \in H^{\frac{1}{2}}(\partial\Omega)$ . The problem writes:

$(P_4)$  find  $u$ , defined in  $\Omega$  such that:

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega \\ \frac{\partial u}{\partial n} + u &= g \text{ on } \partial\Omega \end{aligned}$$

The variational formulation of this problem expresses:

$(VF_4)$ : find  $u \in H^1(\Omega)$  such that:

$$a(u, v) = l(v), \quad \forall v \in H^1(\Omega)$$

where

$$\begin{aligned} a(u, v) &= \int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} uv \, ds \\ l(v) &= \int_{\Omega} uv \, dx + \int_{\partial\Omega} gv \, ds \end{aligned}$$

### Approximation

As usual, let

$$X_h = \{v \in H^1(\Omega); v|_K \in P_k, \forall K \in \mathcal{T}_h\}$$

The approximate problem writes:

$(VF_4)_h$ : find  $u_h \in X_h$  such that:

$$a(u_h, v_h) = l(v_h), \quad \forall v_h \in X_h$$

## File ‘robin.cc’

```

1 #include "rheolef.h"
2 using namespace rheolef;
3 using namespace std;
4 #include "cosinusprod_laplace.icc"
5 int main(int argc, char**argv) {
6     environment rheolef(argc, argv);
7     geo omega (argv[1]);
8     size_t d = omega.dimension();
9     space Xh (omega, argv[2]);
10    form a (Xh, Xh, "grad_grad");
11    form ab (Xh, Xh, "mass", omega["boundary"]);
12    a = a + ab;
13    field lh = riesz(Xh, f(d)) + riesz(Xh, "boundary", g(d));
14    field uh (Xh);
15    solver sa (a.uu());
16    uh.set_u() = sa.solve (lh.u() - a.ub()*uh.b());
17    dout << uh;
18 }

```

## Comments

The code `robin.cc` looks like the previous one `neumann-nh.cc`. Let us comments the changes.

```

    form a (Xh, Xh, "grad_grad");
    form ab (Xh, Xh, "mass", omega["boundary"]);
    a = a + ab;

```

The bilinear forms  $a(.,.)$  and  $a_b(.,.)$  are defined as:

$$a(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx \quad \text{and} \quad a_b(u, v) = \int_{\partial\Omega} u|_{\partial\Omega} v|_{\partial\Omega} \, ds$$

where  $u|_{\partial\Omega}$  denotes the restriction to the boundary of a function  $u$  defined in  $\Omega$ . Then the sum  $a + a_b$  is stored in  $a$ . The boundary contribution to the  $a(.,.)$  form on  $X_h \times X_h$  is introduced. Finally, the implementation of the right-hand sides  $f$  and  $g$  in file ‘`cosinusprod.laplace.icc`’ has already been presented on page 24.

### 2.3.1 How to run the program

First, compile the program:

```
make robin
```

Running the program is obtained from the homogeneous Dirichlet case, by replacing `dirichlet` by `robin`.





## Chapter 3

# Non-constant coefficients and multi-regions

This chapter is related to the so-called transmission problem. We introduce some new concepts: problems with non-constant coefficients, regions in the mesh, weighted forms and discontinuous approximations.

### Formulation

Let us consider a diffusion problem with a non-constant diffusion coefficient  $\eta$  in a domain bounded  $\Omega \subset R^d$ ,  $d = 1, 2, 3$ :

(P): find  $u$  defined in  $\Omega$  such that:

$$-\operatorname{div}(\eta \nabla u) = f \text{ in } \Omega \quad (3.1)$$

$$u = 0 \text{ on } \Gamma_{\text{left}} \cup \Gamma_{\text{right}} \quad (3.2)$$

$$\frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_{\text{top}} \cup \Gamma_{\text{bottom}} \text{ when } d \geq 2 \quad (3.3)$$

$$\frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_{\text{front}} \cup \Gamma_{\text{back}} \text{ when } d = 3 \quad (3.4)$$

where  $f$  is a given source term.

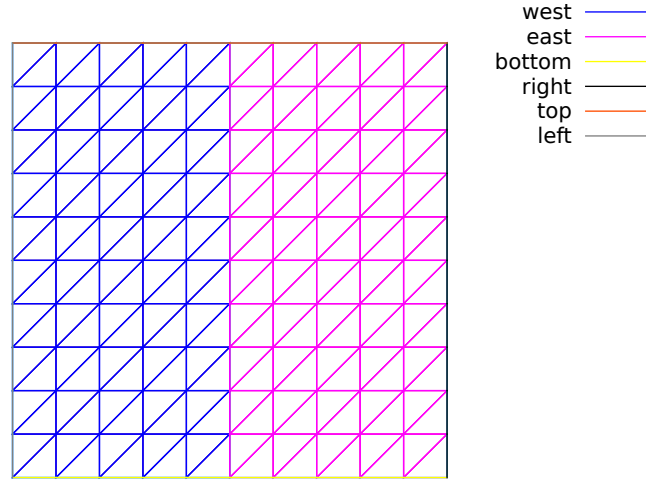


Figure 3.1: Transmission problem: the domain  $\Omega$  partition:  $(\Omega_{\text{west}}$  and  $\Omega_{\text{east}}$ ).

We consider here the important special case when  $\eta$  is picewise constant:

$$\eta(x) = \begin{cases} \varepsilon & \text{when } x \in \Omega_{\text{west}} \\ 1 & \text{when } x \in \Omega_{\text{east}} \end{cases}$$

where  $(\Omega_{\text{west}}, \Omega_{\text{east}})$  is a partition of  $\Omega$  in two parts (see Fig. 3.1). This is the so-called **transmission** problem: the solution and the flux are continuous on the interface  $\Gamma_0 = \partial\Omega_{\text{east}} \cap \partial\Omega_{\text{west}}$  between the two domains where the problem reduce to a constant diffusion one:

$$\begin{aligned} u_{\Omega_{\text{west}}} &= u_{\Omega_{\text{east}}} \text{ on } \Gamma_0 \\ \varepsilon \frac{\partial u_{\Omega_{\text{west}}}}{\partial n} &= \frac{\partial u_{\Omega_{\text{east}}}}{\partial n} \text{ on } \Gamma_0 \end{aligned}$$

It expresses the transmission of the quantity  $u$  and its flux across the interface  $\Gamma_0$  between two regions that have different diffusion properties: Notice that the more classical problem, with constant diffusion  $\eta$  on  $\Omega$  is obtained by simply choosing when  $\varepsilon = 1$ .

The variational formulation of this problem expresses:

(VF): *find*  $u \in V$  *such that*:

$$a(u, v) = l(v), \quad \forall v \in V$$

where the bilinear forms  $a(.,.)$  and the linear one  $l(.)$  are defined by

$$\begin{aligned} a(u, v) &= \int_{\Omega} \eta \nabla u \cdot \nabla v \, dx, \quad \forall u, v \in H^1(\Omega) \\ l(v) &= \int_{\Omega} f v \, dx, \quad \forall v \in L^2(\Omega) \\ V &= \{v \in H^1(\Omega); v = 0 \text{ on } \Gamma_{\text{left}} \cup \Gamma_{\text{right}}\} \end{aligned}$$

The bilinear form  $a(.,.)$  defines a scalar product in  $V$  and is related to the *energy* form. This form is associated to the  $-\text{div } \eta \nabla$  operator.

The approximation of this problem could performed by a standard Lagrange  $P_k$  continuous approximation.

## File ‘transmission.cc’

```

1 #include "rheolef.h"
2 #include "rheolef/catchmark.h"
3 using namespace rheolef;
4 using namespace std;
5 int main(int argc, char**argv) {
6     environment rheolef (argc, argv);
7     const Float epsilon = 0.01;
8     geo omega (argv[1]);
9     space Xh (omega, argv[2]);
10    Xh.block ("left");
11    Xh.block ("right");
12    string eta_approx = "P" + itos(Xh.degree()-1) + "d";
13    space Qh (omega, eta_approx);
14    field eta (Qh);
15    eta ["east"] = 1;
16    eta ["west"] = epsilon;
17    form a (Xh, Xh, "grad_grad", eta);
18    field lh = riesz (Xh, 1);
19    field uh (Xh);
20    uh["left"] = uh["right"] = 0;
21    solver sa (a.uu());
22    uh.set_u() = sa.solve (lh.u() - a.ub()*uh.b());
23    dout << catchmark("epsilon") << epsilon << endl
24         << catchmark("u") << uh;
25 }

```

## Comments

This file is quite similar to those studied in the first chapters of this book. Let us comment only the changes. The Dirichlet boundary condition applies no more on the whole boundary  $\partial\Omega$  but on two parts  $\Gamma_{\text{left}}$  and  $\Gamma_{\text{right}}$ . On the other boundary parts, an homogeneous Neumann boundary condition is used: since these conditions does not produce any additional terms in the variational formulation, there are also nothing to write in the C++ code for these boundaries. We choose  $f = 1$ : this leads to a convenient test-problem, since the exact solution is known when  $\Omega = ]0, 1[^d$ :

$$u(x) = \begin{cases} \frac{x_0}{2\varepsilon} \left( \frac{1+3\varepsilon}{2(1+\varepsilon)} - x_0 \right) & \text{when } x_0 < 1/2 \\ \frac{1-x_0}{2} \left( x_0 + \frac{1-\varepsilon}{2(1+\varepsilon)} \right) & \text{otherwise} \end{cases}$$

The field  $\eta$  belongs to a discontinuous finite element  $P_{k-1}$  space denoted by  $Q_h$ :

```

string eta_approx = "P" + itos(Xh.degree()-1) + "d";
space Qh (omega, eta_approx);
field eta (Qh);

```

For instance, when `argv[2]` contains "P2", i.e.  $k = 2$ , then the string `eta_approx` takes value "P1d". Then  $\eta$  is initialized by:

```

eta["east"] = 1;
eta["west"] = epsilon;

```

The energy form  $a$  is then constructed with  $\eta$  as additional parameter:

```

form a (Xh, Xh, "grad_grad", eta);

```

Such forms with a additional *weight* function are called *weighted forms* in **Rheolef**.

## How to run the program ?

Build the program as usual: `make transmission`. Then, creates a one-dimensional geometry with two regions:

```
mkgeo_grid -e 100 -region > line.geo
geo line.geo
```

The trivial mesh generator `mkgeo_grid`, defines two regions `east` and `west` when used with the `-region` option. This correspond to the situation:

$$\Omega = [0, 1]^d, \quad \Omega_{\text{west}} = \Omega \cap \{x_0 < 1/2\} \quad \text{and} \quad \Omega_{\text{east}} = \Omega \cap \{x_0 > 1/2\}.$$

In order to avoid mistakes with the C++ style indexation, we denote by  $(x_0, \dots, x_{d-1})$  the cartesian coordinate system in  $\mathbb{R}^d$ .

Finally, run the program and look at the solution:

```
make transmission
./transmission line.geo P1 > line.field
field line.field
```

Since the exact solution is a piecewise second order polynom and the change in the diffusion coefficient value fits the element boundaries, we obtain the exact solution for all the degrees of freedom of any  $P_k$  approximation,  $k \geq 1$ , as shown on Fig. 3.2 when  $k = 1$ . Moreover, when  $k \geq 2$  then  $u_h = u$  since  $X_h$  contains the exact solution  $u$ .

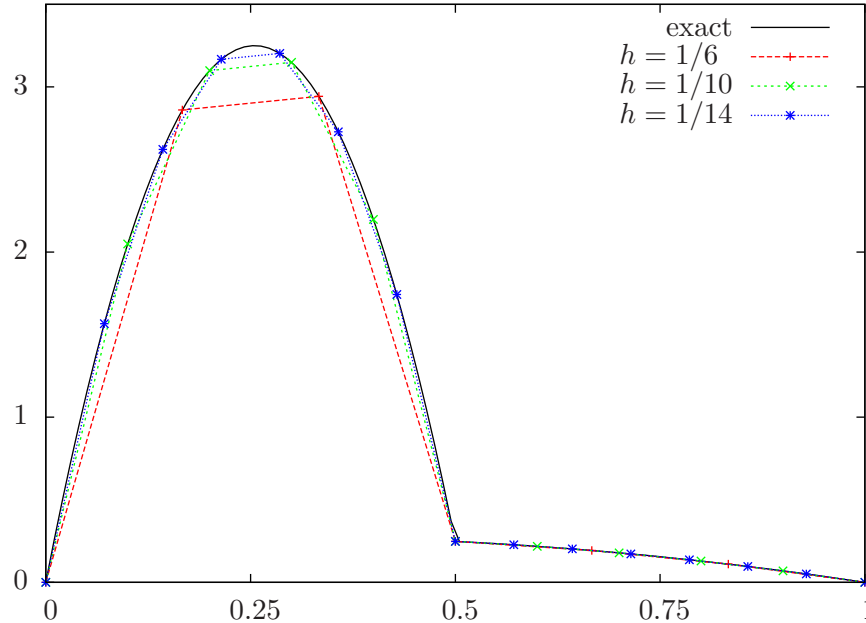


Figure 3.2: Transmission problem:  $u_h = \pi_h(u)$  ( $\varepsilon = 10^{-2}$ ,  $d = 1$ ,  $P_1$  approximation).

The two dimensionnal case corresponds to the commands:

```
mkgeo_grid -t 10 -region > square.geo
geo square.geo
./transmission square.geo P1 > square.field
field square.field -elevation
```

while the tridimensional to

```
mkgeo_grid -T 10 -region > cube.geo
./transmission cube.geo P1 > cube.mfield
field cube.field
```

As for all the others examples, you can replace P1 by higher-order approximations, change elements shapes, such as q, H or P, and run distributed computations with `mpirun`.



## Chapter 4

# Neumann boundary conditions for the Laplace operator

In this chapter we study how to solve a ill-posed problem with a solution defined up to a constant.

### Formulation

Let  $f \in L^2(\Omega)$  and  $g \in H^{\frac{1}{2}}(\partial\Omega)$  satisfying the following compatibility condition:

$$\int_{\Omega} f \, dx + \int_{\partial\Omega} g \, ds = 0$$

The problem writes:

$(P_5)_h$ : find  $u$ , defined in  $\Omega$  such that:

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega \\ \frac{\partial u}{\partial n} &= g \text{ on } \partial\Omega \end{aligned}$$

Since this problem only involves the derivatives of  $u$ , it is clear that its solution is never unique [4, p. 11]. A discrete version of this problem could be solved iteratively by the conjugate gradient or the MINRES algorithm [5]. In order to solve it by a direct method, we turn the difficulty by seeking  $u$  in the following space

$$V = \{v \in H^1(\Omega); \ b(v, 1) = 0\}$$

where

$$b(v, \mu) = \int_{\Omega} v \, dx, \quad \forall v \in L^2(\Omega), \forall \mu \in \mathbb{R}$$

The variational formulation of this problem writes:

$(VF_5)$ : find  $u \in V$  such that:

$$a(u, v) = l(v), \quad \forall v \in V$$

where

$$\begin{aligned} a(u, v) &= \int_{\Omega} \nabla u \cdot \nabla v \, dx \\ l(v) &= m(f, v) + m_b(g, v) \\ m(f, v) &= \int_{\Omega} f v \, dx \\ m_b(g, v) &= \int_{\partial\Omega} g v \, ds \end{aligned}$$



Since the direct discretization of the space  $V$  is not an obvious task, the constraint  $b(u, 1) = 0$  is enforced by a Lagrange multiplier  $\lambda \in \mathbb{R}$ . Let us introduce the Lagrangian, defined for all  $v \in H^1(\Omega)$  and  $\mu \in \mathbb{R}$  by:

$$L(v, \mu) = \frac{1}{2}a(v, v) + b(v, \mu) - l(v)$$

The saddle point  $(u, \lambda) \in H^1(\Omega) \times \mathbb{R}$  of this Lagrangian is characterized as the unique solution of:

$$\begin{aligned} a(u, v) + b(v, \lambda) &= l(v), \quad \forall v \in H^1(\Omega) \\ b(u, \mu) &= 0, \quad \forall \mu \in \mathbb{R} \end{aligned}$$

It is clear that if  $(u, \lambda)$  is solution of this problem, then  $u \in V$  and  $u$  is a solution of  $(VF_5)$ . Conversely, let  $u \in V$  the solution of  $(VF_5)$ . Choosing  $v = v_0$  where  $v_0(x) = 1, \forall x \in \Omega$  leads to  $\lambda \text{meas}(\Omega) = l(v_0)$ . From the definition of  $l(\cdot)$  and the compatibility condition between the data  $f$  and  $g$ , we get  $\lambda = 0$ . Notice that the saddle point problem extends to the case when  $f$  and  $g$  does not satisfies the compatibility condition, and in that case  $\lambda = l(v_0)/\text{meas}(\Omega)$ .

## Approximation

As usual, we introduce a mesh  $\mathcal{T}_h$  of  $\Omega$  and the finite dimensional space  $X_h$ :

$$X_h = \{v \in H^1(\Omega); v|_K \in P_k, \forall K \in \mathcal{T}_h\}$$

The approximate problem writes:

$(VF_5)_h$ : find  $(u_h, \lambda_h) \in X_h \times \mathbb{R}$  such that:

$$\begin{aligned} a(u_h, v) + b(v, \lambda_h) &= l_h(v), \quad \forall v \in X_h \\ b(u_h, \mu) &= 0, \quad \forall \mu \in \mathbb{R} \end{aligned}$$

where

$$l_h(v) = m(\Pi_h f, v_h) + m_b(\pi_h g, v_h)$$

## File ‘neumann-laplace.cc’

```

1 #include "rheolef.h"
2 using namespace rheolef;
3 using namespace std;
4 size_t d;
5 Float f (const point& x) { return 1; }
6 Float g (const point& x) { return -0.5/d; }
7 int main(int argc, char**argv) {
8     environment rheolef (argc, argv);
9     geo omega (argv[1]);
10    d = omega.dimension();
11    space Xh (omega, argv[2]);
12    form m (Xh, Xh, "mass");
13    form a (Xh, Xh, "grad_grad");
14    field b = m*field(Xh,1);
15    field lh = riesz(Xh, f) + riesz(Xh, "boundary", g);
16    csr<Float> A = {{ a.uu(), b.u()},
17                  {trans(b.u()), 0 }};
18    vec<Float> B = { lh.u(), 0 };
19    solver sa (A);
20    vec<Float> U = sa.solve (B);
21    field uh (Xh);
22    uh.set_u() = U [range(0,uh.u().size())];
23    Float lambda = (U.size() == uh.u().size()+1) ? U [uh.u().size()] : 0;
24 #ifdef _RHEOLEF_HAVE_MPI
25    mpi::broadcast (U.comm(), lambda, U.comm().size() - 1);
26 #endif // _RHEOLEF_HAVE_MPI
27    dout << uh
28          << "lambda" << lambda << endl;
29 }

```

## Comments

Let  $\Omega \subset \mathbb{R}^d$ ,  $d = 1, 2, 3$ . We choose  $f(x) = 1$  and  $g(x) = -1/(2d)$ . This example is convenient, since the exact solution is known:

$$u(x) = -\frac{1}{12} + \frac{1}{2d} \sum_{i=1}^d x_i(1 - x_i)$$

The code looks like the previous ones. Let us comment the changes. The discrete bilinear form  $b$  is computed as  $b_h \in X_h$  that interprets as a linear application from  $X_h$  to  $\mathbb{R}$ :  $b_h(v_h) = m(v_h, 1)$ . Thus  $b_h$  is computed as

```
field b = m*field(Xh,1.0);
```

where the discrete bilinear form  $m$  is identified to its matrix and `field(Xh,1.0)` is the constant vector equal to 1. Let

$$\mathcal{A} = \begin{pmatrix} \text{a.uu} & \text{trans(b.u)} \\ \text{b.u} & 0 \end{pmatrix}, \quad \mathcal{U} = \begin{pmatrix} \text{uh.u} \\ \text{lambda} \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} \text{lh.u} \\ 0 \end{pmatrix}$$

The problem admits the following matrix form:

$$\mathcal{A} \mathcal{U} = \mathcal{B}$$

The matrix and right-hand side of the system are assembled by concatenation:

```
csr<Float> A = {{ a.uu,    b.u},
                {trans(b.u), 0 }};
vec<Float> B = { lh.u,    0 };
```

where `csr` and `vec` are respectively the matrix and vector classes. The `csr` is the abbreviation of *compressed sparse row*, a sparse matrix compression standard format. Notice that the matrix  $\mathcal{A}$  is symmetric and non-singular, but indefinite : it admits eigenvalues that are either strictly positive or strictly negative. Then, the `uh.u` vector is extracted from the `U` one:

```
uh.u = U [range(0,uh.u.size())];
```

The extraction of `lambda` from `U` is more technical in a distributed environment. In a sequential one, since it is stored after the `uh.u` values, it could be simply written as:

```
Float lambda = U [uh.u.size()];
```

In a distributed environment, `lambda` is stored in `U` on the last processor, identified by `U.comm().size()-1`. Here `U.comm()` denotes the communicator, from the `boost::mpi` library and `U.comm().size()` is the number of processors in use, e.g. as specified by the `mpirun` command. On this last processor, the array `U` has size equal to `uh.u.size()+1` and `lambda` is stored in `U[uh.u.size()]`. On the others processors, the array `U` has size equal to `uh.u.size()` and `lambda` is not available. The following statement extract `lambda` on the last processor and set it to zero on the others:

```
Float lambda = (U.size() == uh.u.size()+1) ? U [uh.u.size()] : 0;
```

Then, the value of `lambda` is broadcasted on the others processors:

```
mpi::broadcast (U.comm(), lambda, U.comm().size() - 1);
```

The preprocessing guards `#ifdef...#endif` assure that this example compile also when the library is not installed with the MPI support. Finally, the statement

```
dout << catchmark("u") << uh
      << catchmark("lambda") << lambda << endl;
```

writes the solution  $(u_h, \lambda)$ . The `catchmark` function writes marks together with the solution in the output stream. These marks are suitable for a future reading with the same format, as:

```
din >> catchmark("u") >> uh
     >> catchmark("lambda") >> lambda;
```

This is usefull for post-treatment, visualization and error analysis.

## 4.0.2 How to run the program

As usual, enter:

```
make neumann-laplace
mkgeo_grid -t 10 > square.geo
./neumann-laplace square P1 | field -
```

## Part II

# Fluids and solids computations



# Chapter 5

## The linear elasticity and the Stokes problems

### 5.1 The linear elasticity problem

#### Formulation

The total Cauchy stress tensor expresses:

$$\sigma(\mathbf{u}) = \lambda \operatorname{div}(\mathbf{u}) \cdot \mathbf{I} + 2\mu D(\mathbf{u}) \quad (5.1)$$

where  $\lambda$  and  $\mu$  are the Lamé coefficients. Here,  $D(\mathbf{u})$  denotes the symmetric part of the gradient operator and  $\operatorname{div}$  is the divergence operator. Let us consider the elasticity problem for the *embankment*, in  $\Omega = ]0, 1[^d$ ,  $d = 2, 3$ . The problem writes:

(P): find  $\mathbf{u} = (u_0, \dots, u_{d-1})$ , defined in  $\Omega$ , such that:

$$\begin{aligned} -\operatorname{div} \sigma(\mathbf{u}) &= \mathbf{f} \text{ in } \Omega, \\ \frac{\partial \mathbf{u}}{\partial \mathbf{n}} &= 0 \text{ on } \Gamma_{\text{top}} \cup \Gamma_{\text{right}} \\ \mathbf{u} &= 0 \text{ on } \Gamma_{\text{left}} \cup \Gamma_{\text{bottom}}, \\ \mathbf{u} &= 0 \text{ on } \Gamma_{\text{front}} \cup \Gamma_{\text{back}}, \text{ when } d = 3 \end{aligned} \quad (5.2)$$

where  $\mathbf{f} = (0, -1)$  when  $d = 2$  and  $\mathbf{f} = (0, 0, -1)$  when  $d = 3$ . The Lamé coefficients are assumed to satisfy  $\mu > 0$  and  $\lambda + \mu > 0$ . Since the problem is linear, we can suppose that  $\mu = 1$  without any loss of generality.

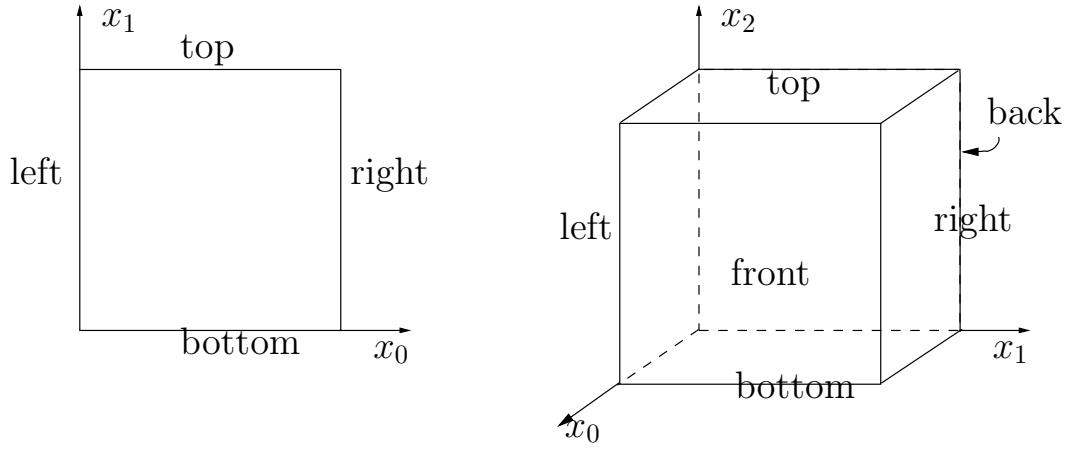


Figure 5.1: The boundary domains for the square and the cube.

recall that, in order to avoid mistakes with the C++ style indexation, we denote by  $(x_0, \dots, x_{d-1})$  the cartesian coordinate system in  $\mathbb{R}^d$ .

For  $d = 2$  we define the boundaries:

$$\begin{aligned} \Gamma_{\text{left}} &= \{0\} \times ]0, 1[, & \Gamma_{\text{right}} &= \{1\} \times ]0, 1[ \\ \Gamma_{\text{bottom}} &= ]0, 1[ \times \{0\}, & \Gamma_{\text{top}} &= ]0, 1[ \times \{1\} \end{aligned}$$

and for  $d = 3$ :

$$\begin{aligned} \Gamma_{\text{back}} &= \{0\} \times ]0, 1[^2, & \Gamma_{\text{front}} &= \{1\} \times ]0, 1[^2 \\ \Gamma_{\text{left}} &= ]0, 1[ \times \{0\} \times ]0, 1[, & \Gamma_{\text{right}} &= ]0, 1[ \times \{1\} \times ]0, 1[ \\ \Gamma_{\text{bottom}} &= ]0, 1[^2 \times \{0\}, & \Gamma_{\text{top}} &= ]0, 1[^2 \times \{1\} \end{aligned}$$

These boundaries are represented on Fig. 5.1.

The variational formulation of this problem expresses:

(VF): find  $\mathbf{u} \in \mathbf{V}$  such that:

$$a(\mathbf{u}, \mathbf{v}) = l(\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}, \quad (5.3)$$

where

$$a(\mathbf{u}, \mathbf{v}) = \lambda \int_{\Omega} \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v} \, dx + \int_{\Omega} 2D(\mathbf{u}) : D(\mathbf{v}) \, dx,$$

$$l(\mathbf{v}) = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx,$$

$$\mathbf{V} = \{\mathbf{v} \in (H^1(\Omega))^2; \mathbf{v} = 0 \text{ on } \Gamma_{\text{left}} \cup \Gamma_{\text{bottom}}\}, \text{ when } d = 2$$

$$\mathbf{V} = \{\mathbf{v} \in (H^1(\Omega))^3; \mathbf{v} = 0 \text{ on } \Gamma_{\text{left}} \cup \Gamma_{\text{bottom}} \cup \Gamma_{\text{right}} \cup \Gamma_{\text{back}}\}, \text{ when } d = 3$$

## Approximation

We introduce a mesh  $\mathcal{T}_h$  of  $\Omega$  and for  $k \geq 1$ , the following finite dimensional spaces:

$$\begin{aligned} \mathbf{X}_h &= \{\mathbf{v}_h \in (H^1(\Omega))^d; \mathbf{v}_{h/K} \in (P_k)^d, \forall K \in \mathcal{T}_h\}, \\ \mathbf{V}_h &= \mathbf{X}_h \cap \mathbf{V} \end{aligned}$$

The approximate problem writes:

(VF)<sub>h</sub>: find  $\mathbf{u}_h \in \mathbf{V}_h$  such that:

$$a(\mathbf{u}_h, \mathbf{v}_h) = l(\mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{V}_h$$

## File ‘embankment.cc’

```

1 #include "rheolef.h"
2 using namespace rheolef;
3 using namespace std;
4 #include "embankment.icc"
5 int main(int argc, char**argv) {
6     environment rheolef(argc, argv);
7     geo omega (argv[1]);
8     space Xh = embankment_space (omega, argv[2]);
9     Float lambda = (argc > 3) ? atof(argv[3]) : 1;
10    size_t d = omega.dimension();
11    point f (0,0,0);
12    f[d-1] = -1;
13    field lh = riesz (Xh, f);
14    form a1 (Xh, Xh, "div_div");
15    form a2 (Xh, Xh, "2D_D");
16    form a = lambda*a1 + a2;
17    field uh (Xh, 0);
18    solver sa (a.uu());
19    uh.set_u() = sa.solve (lh.u() - a.ub()*uh.b());
20    dout << catchmark("inv_lambda") << 1/lambda << endl
21         << catchmark("u") << uh;
22 }

```

## File ‘embankment.icc’

```

1 space embankment_space (const geo& omega, string approx) {
2     space Xh (omega, approx, "vector");
3     Xh.block("left");
4     if (omega.dimension() >= 2) {
5         Xh.block("bottom");
6     }
7     if (omega.dimension() == 3) {
8         Xh.block("right");
9         Xh.block("back");
10    }
11    return Xh;
12 }

```

## Comments

The space is defined in a separate file ‘embankment.icc’, since it will be reused in others examples along this chapter:

```
space Vh (omega, "P2", "vector");
```

Note here the multi-component specification “vector” as a supplementary argument to the `space` constructor. The boundary condition contain a special cases for bi- and tridimensionnal cases. The right-hand-side  $\mathbf{f}_h$  represents the dimensionless gravity forces, oriented on the vertical axis: the last component of  $\mathbf{f}_h$  is set to  $-1$  as:

```
f_h [omega.dimension()-1] = -1;
```

Finally, the  $1/\lambda$  parameter and the multi-field result are printed, using mark labels, thanks to the `catchmark` stream manipulator. Labels are convenient for post-processing purpose, as we will see in the next paragraph.



## How to run the program

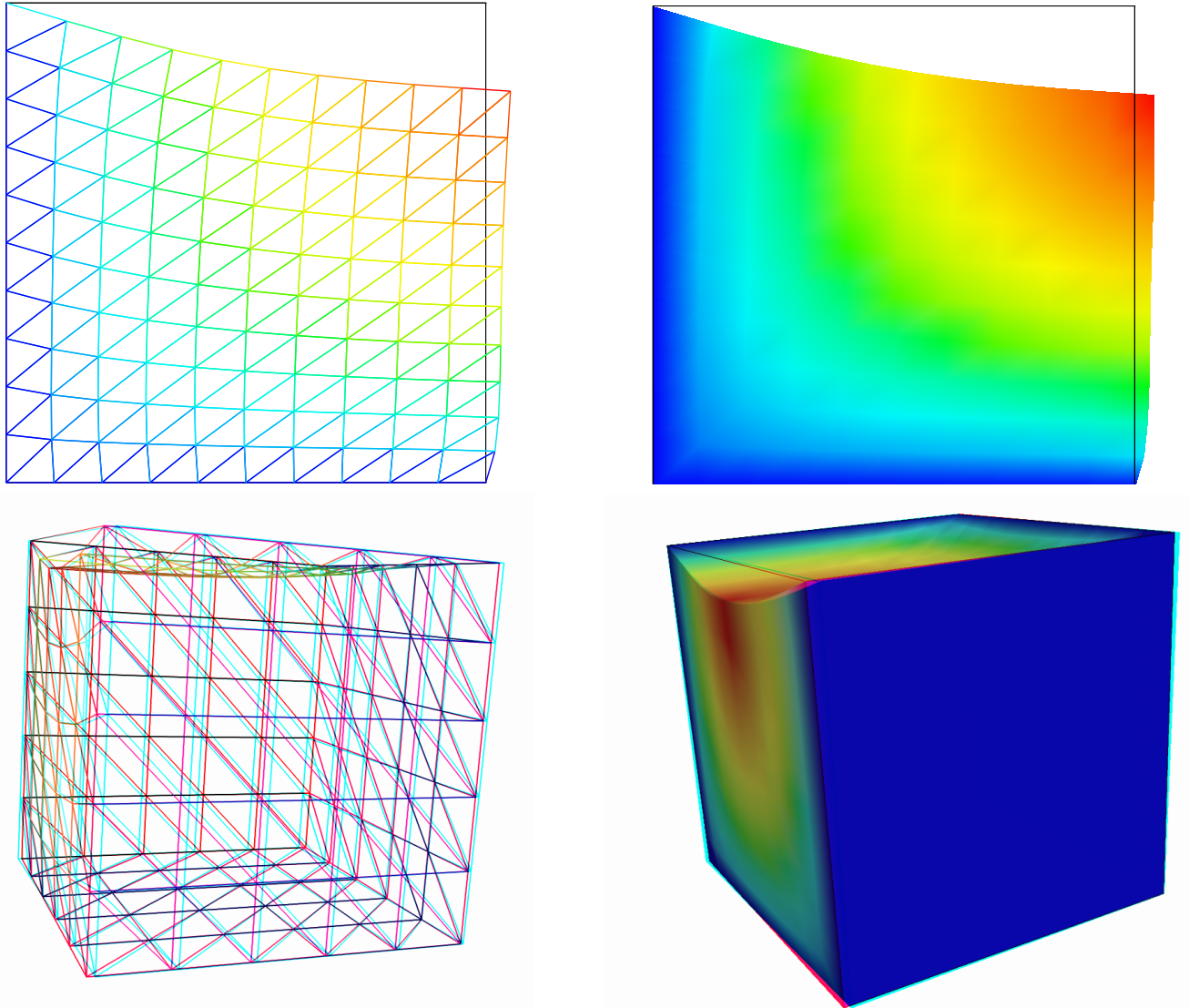


Figure 5.2: The linear elasticity for  $\lambda = 1$  and  $d = 2$  and  $d = 3$ : both wireframe and filled surfaces ; stereoscopic anaglyph mode for 3D solutions.

We assume that the previous code is contained in the file ‘`embankment.cc`’. Compile the program as usual (see page 16):

```
make embankment
```

and enter the comands:

```
mkgeo_grid -t 10 > square.geo
geo square.geo
```

The triangular mesh has four boundary domains, named `left`, `right`, `top` and `bottom`. Then, enter:

```
./embankment square.geo P1 > square-P1.field
```

The previous command solves the problem for the corresponding mesh and writes the multi-component solution in the `.field` file format. Run the deformation vector field visualization using the default `gnuplot` render:

```
field square-P1.field
field square-P1.field -nofill
```

Note the graphic options usage ; the unix manual for the `field` command is available as:

```
man field
```

The view is shown on Fig. 5.2. A specific field component can be also selected for a scalar visualization:

```
field -comp 0 square-P1.field
field -comp 1 square-P1.field
```

Next, perform a  $P_2$  approximation of the solution:

```
./embankment square.geo P2 > square-P2.field
field square-P2.field -mayavi -nofill
```

Finally, let us consider the three dimensional case

```
mkgeo_grid -T 10 > cube.geo
./embankment cube.geo P1 > cube-P1.field
field cube-P1.field -stereo
field cube-P1.field -stereo -fill
```

The two last commands show the solution in 3D stereoscopic anaglyph mode. The graphic is represented on Fig. 5.2. The  $P_2$  approximation writes:

```
./embankment cube.geo P2 > cube-P2.field
field cube-P2.field
```

## 5.2 Computing the stress tensor

### Formulation and approximation

The following code computes the total Cauchy stress tensor, reading the Lamé coefficient  $\lambda$  and the deformation field  $\mathbf{u}_h$  from a `.field` file. Let us introduce:

$$T_h = \{\tau_h \in (L^2(\Omega))^{d \times d}; \tau_h = \tau_h^T \text{ and } \tau_{h;ij/K} \in P_{k-1}, \forall K \in \mathcal{T}_h, 1 \leq i, j \leq d\}$$

This computation expresses:

find  $\sigma_h$  such that:

$$m(\sigma_h, \tau) = b(\tau, \mathbf{u}_h), \forall \tau \in T_h$$

where

$$\begin{aligned} m(\sigma, \tau) &= \int_{\Omega} \sigma : \tau \, dx, \\ b(\tau, \mathbf{u}) &= \lambda \int_{\Omega} \operatorname{div}(\mathbf{u}) \operatorname{tr}(\tau) \, dx + \int_{\Omega} 2D(\mathbf{u}) : \tau \, dx, \end{aligned}$$

where  $\operatorname{tr}(\tau) = \sum_{i=1}^d \tau_{ii}$  is the trace of the tensor  $\tau$ .

### File ‘stress.cc’

```

1 #include "rheolef.h"
2 using namespace rheolef;
3 using namespace std;
4 int main(int argc, char** argv) {
5     environment rheolef (argc, argv);
6     Float inv_lambda;
7     field uh;
8     din >> catchmark("inv_lambda") >> inv_lambda
9         >> catchmark("u") >> uh;
10    const geo& omega = uh.get_geo();
11    const space& Xh = uh.get_space();
12    string grad_approx = "P" + itos(Xh.degree()-1) + "d";
13    space Th (omega, grad_approx, "tensor");
14    space T0h (omega, grad_approx);
15    form two_D (Xh, Th, "2D");
16    form div (Xh, T0h, "div");
17    form inv_mt (Th, Th, "inv_mass");
18    form inv_m (T0h, T0h, "inv_mass");
19    field trace_h = inv_m*(div*uh);
20    field sigma_h = inv_mt*(two_D*uh);
21    if (inv_lambda != 0)
22        for (size_t i_comp = 0; i_comp < uh.size(); i_comp++)
23            sigma_h(i_comp, i_comp) += (1/inv_lambda)*trace_h;
24    dout << catchmark("s") << sigma_h;
25 }

```

### Comments

First notice that this code applies for any deformation field, and is not restricted to our *embankment* problem.

The P0 and P1d stands for the piecewise constant and piecewise linear discontinuous approximations, respectively. Since elements of  $T_h$  are discontinuous across interelement boundaries, the mass operator is block-diagonal and can be inverted one time for all: this operation results in the `inv_mass` operator.

## How to run the program

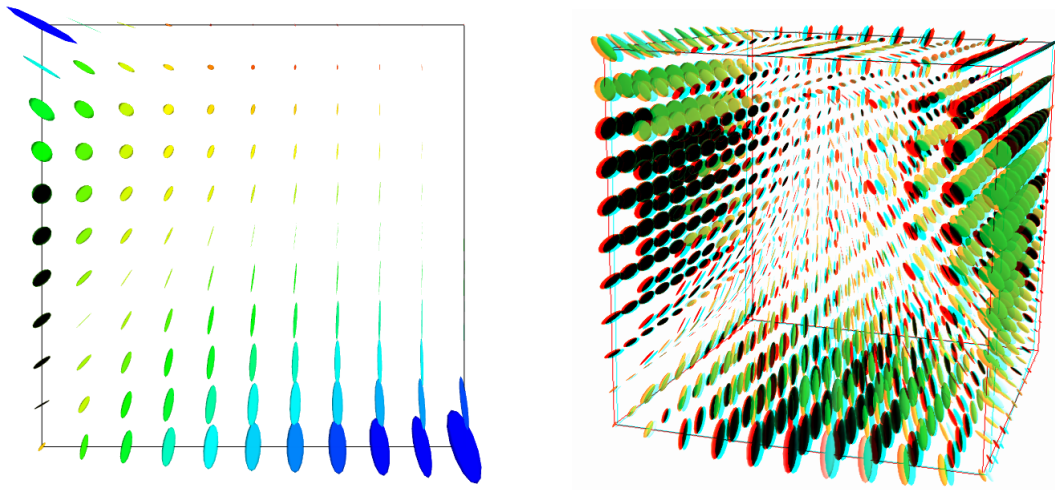


Figure 5.3: The stress tensor visualization (linear elasticity  $\lambda = 1$ ).

First, compile the program:

```
make stress
```

The visualization for the stress tensor as ellipses writes:

```
./stress < square-P1.field > square-stress-P1.field  
field square-stress-P1.field -proj -mayavi
```

Recall that the stress, as a derivative of the deformation, is P0 (resp. P1d) and discontinuous when the deformation is P1 (resp. P2) and continuous. The approximate stress tensor field is projected on a continuous piecewise linear space, using the `-proj` option. Conversely, the 3D visualization bases on ellipsoids:

```
./stress < cube-P1.field > cube-stress-P1.field  
field cube-stress-P1.field -proj -mayavi -stereo
```

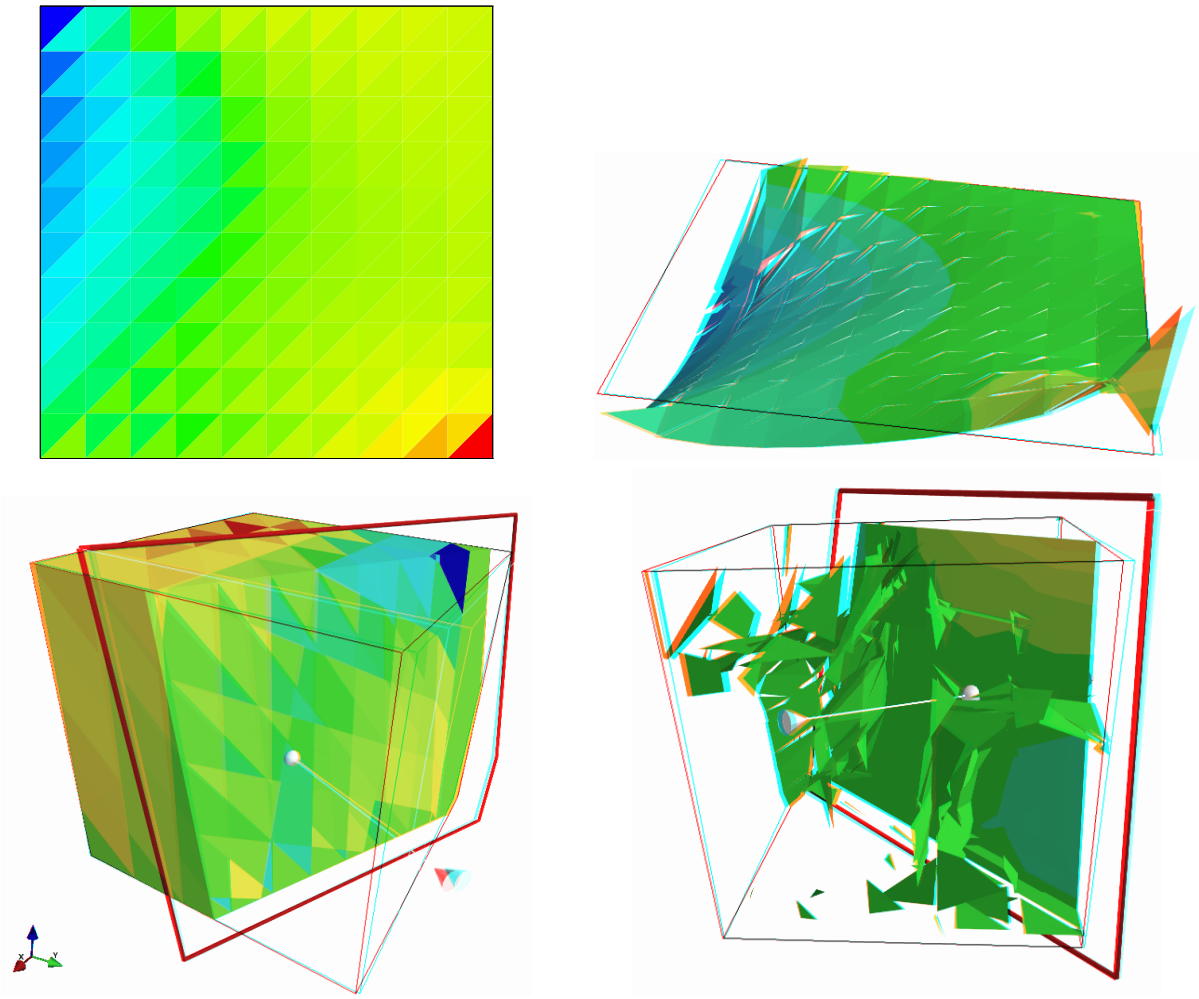


Figure 5.4: The  $\sigma_{01}$  stress component (linear elasticity  $\lambda = 1$ ):  $d = 2$  (top) and  $d = 3$  (bottom) ;  $P_0$  (left) and  $P_1$  discontinuous approximation (right).

You can observe a discontinuous constant or piecewise linear representation of the approximate stress component  $\sigma_{01}$  (see Fig. 5.4):

```
field square-stress-P1.field -comp 01
field square-stress-P2.field -comp 01 -elevation
field square-stress-P2.field -comp 01 -elevation -stereo
```

Notice that the `-stereo` implies the `-mayavi` one, as this feature is not available with others visualization systems. The approximate stress field can be also projected on a continuous piecewise space:

```
field square-stress-P2.field -comp 01 -elevation -proj
```

The tridimensionnall case writes simply (see Fig. 5.4):

```
./stress < cube-P1.field > cube-stress-P1.field
```

```
./stress < cube-P2.field > cube-stress-P2.field  
field cube-stress-P1.field -comp 01 -stereo  
field cube-stress-P2.field -comp 01 -stereo -iso
```

and also the P1-projected versions write:

```
field cube-stress-P1.field -comp 01 -stereo -proj -iso  
field cube-stress-P2.field -comp 01 -stereo -proj -iso
```

These operations can be repeated for each  $\sigma_{ij}$  components and for both P1 and P2 approximation of the deformation field.

### 5.3 Mesh adaptation

The main principle of the auto-adaptive mesh writes [6–11]:

```
cin >> omega;  
uh = solve(omega);  
for (unsigned int i = 0; i < n; i++) {  
    ch = criterion(uh);  
    omega = adapt(ch);  
    uh = solve(omega);  
}
```

The initial mesh is used to compute a first solution. The adaptive loop compute an *adaptive criterion*, denoted by `ch`, that depends upon the problem under consideration and the polynomial approximation used. Then, a new mesh is generated, based on this criterion. A second solution on an adapted mesh can be constructed. The adaptation loop converges generally in roughly 5 to 20 iterations.

Let us apply this principle to the elasticity problem.

## File 'embankment\_adapt.cc'

```

1 #include "rheolef.h"
2 using namespace rheolef;
3 using namespace std;
4 #include "elasticity_solve.icc"
5 #include "elasticity_criterion.icc"
6 #include "embankment.icc"
7 int main(int argc, char**argv) {
8     environment rheolef (argc, argv);
9     const Float lambda = 1;
10    geo omega (argv[1]);
11    adapt_option_type options;
12    string approx = (argc > 2) ? argv[2] : "P1";
13    options.err = (argc > 3) ? atof(argv[3]) : 5e-3;
14    size_t n_adapt = (argc > 4) ? atoi(argv[4]) : 5;
15    options.hmin = 0.004;
16    for (size_t i = 0; true; i++) {
17        space Xh = embankment_space (omega, approx);
18        field uh = elasticity_solve (Xh, lambda);
19        odiststream of (omega.name(), "field");
20        of << catchmark("lambda") << lambda << endl
21           << catchmark("u") << uh;
22        if (i == n_adapt) break;
23        field ch = elasticity_criterion (lambda,uh);
24        omega = adapt(ch, options);
25        odiststream og (omega.name(), "geo");
26        og << omega;
27    }
28 }

```

## File 'elasticity\_solve.icc'

```

1 field elasticity_solve(const space& Xh, Float lambda) {
2     size_t d = Xh.get_geo().dimension();
3     point f (0,0,0);
4     f[d-1] = -1;
5     field lh = riesz (Xh, f);
6     form m (Xh, Xh, "mass");
7     form a1 (Xh, Xh, "div_div");
8     form a2 (Xh, Xh, "2D_D");
9     form a = lambda*a1 + a2;
10    solver sa (a.uu());
11    field uh (Xh, 0);
12    uh.set_u() = sa.solve (lh.u() - a.ub()*uh.b());
13    return uh;
14 }

```

## File 'elasticity\_criterion.icc'

```

1 field elasticity_criterion (Float lambda, const field& uh) {
2     string grad_approx = (uh.get_approx() == "P2") ? "Pid" : "P0";
3     if (grad_approx == "P0") return norm(uh);
4     space Th (uh.get_geo(), grad_approx, "tensor");
5     space Xh (uh.get_geo(), grad_approx);
6     form two_D (uh.get_space(), Th, "2D");
7     form div (uh.get_space(), Xh, "div");
8     form mt (Th, Th, "mass");
9     form m (Xh, Xh, "mass");
10    form inv_mt (Th, Th, "inv_mass");
11    form inv_m (Xh, Xh, "inv_mass");
12    field qh = inv_m*(div*uh);
13    field two_Duh = inv_mt*(two_D*uh);
14    return sqrt(lambda*sqr(qh) + norm2(two_Duh));
15 }

```

## Comments

The criterion is here:

$$c_h = \begin{cases} |\mathbf{u}_h| & \text{when using } P_1 \\ (\sigma(\mathbf{u}_h) : D(\mathbf{u}_h))^{1/2} & \text{when using } P_2 \end{cases}$$

The `adapt_option_type` declaration is used by **Rheolef** to send options to the mesh generator. The `err` parameter controls the error via the edge length of the mesh: the smaller it is, the smaller the edges of the mesh are. In our example, is set by default to one. Conversely, the `hmin` parameter controls minimal edge length.

## How to run the program

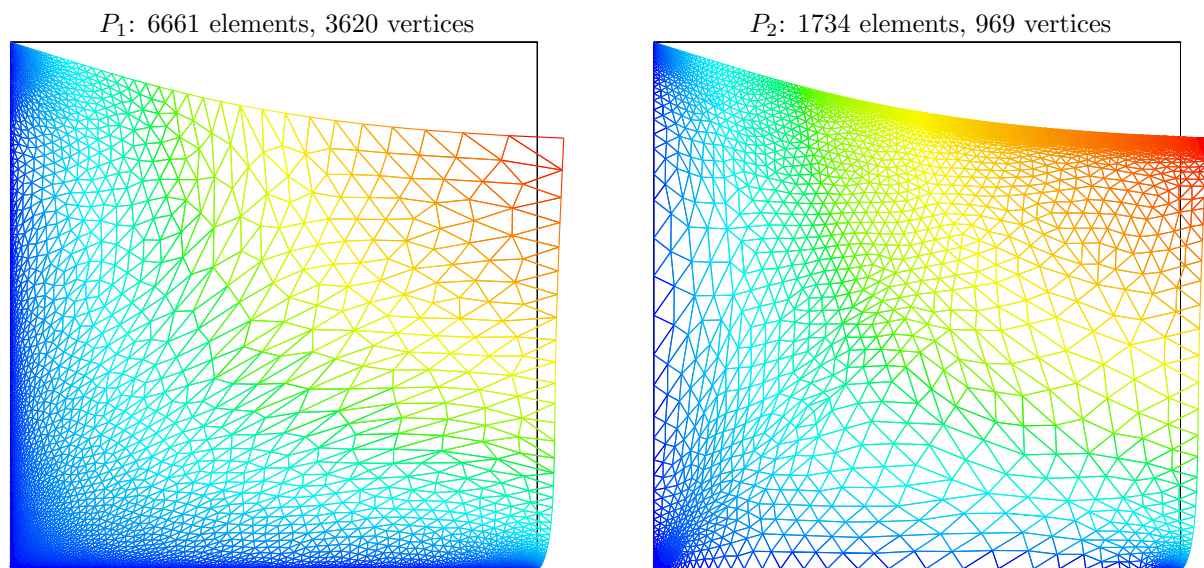


Figure 5.5: Adapted meshes: the deformation visualization for  $P_1$  and  $P_2$  approximations.

The compilation command writes:

```
make embankment_adapt
```

The mesh loop adaptation is initiated from a `bamg` mesh (see also appendix B.1).

```
bamg -g square.bamgcad -o square.bamg
bamg2geo square.bamg square.dmn > square.geo
./embankment_adapt square P1
```

The code performs a loop of five mesh adaptations: the corresponding meshes are stored in files, from `square-1.geo.gz` to `square-5.geo.gz`, and the associated solutions in files, from `square-1.field.gz` to `square-5.field.gz`. The additional `.gz` suffix expresses that the files are compressed using `gzip`.

```
geo square-5.geo
field square-5.field -mayavi -nofill
```

Note that the `.gz` suffix is automatically assumed by the `geo` and the `field` commands.

For a piecewise quadratic approximation:



```
./embankment_adapt square P2
```

Then, the visualization writes:

```
geo square-5.geo
field square-5.field -mayavi -nofill
```

A one-dimensional mesh adaptive procedure is also possible:

```
gmsh -1 line.mshcad -o line.msh
msh2geo < line.msh | geo -upgrade - > line.geo
geo line.geo
./embankment_adapt line P2
geo line-5.geo
field line-5.field -comp 0 -elevation
```

The three-dimensional extension this mesh adaptive procedure is in development.

## 5.4 The Stokes problem

### Formulation

Let us consider the Stokes problem for the driven cavity in  $\Omega = ]0, 1[^d$ ,  $d = 2, 3$ . The problem writes:

(S) find  $\mathbf{u} = (u_0, \dots, u_{d-1})$  and  $p$  defined in  $\Omega$  such that:

$$\begin{aligned} -\operatorname{div}(2D(\mathbf{u})) + \nabla p &= 0 \text{ in } \Omega, \\ -\operatorname{div} \mathbf{u} &= 0 \text{ in } \Omega, \\ \mathbf{u} &= (1, 0) \text{ on } \Gamma_{\text{top}}, \\ \mathbf{u} &= 0 \text{ on } \Gamma_{\text{left}} \cup \Gamma_{\text{right}} \cup \Gamma_{\text{bottom}}, \\ \frac{\partial u_0}{\partial \mathbf{n}} &= \frac{\partial u_1}{\partial \mathbf{n}} = u_2 = 0 \text{ on } \Gamma_{\text{back}} \cup \Gamma_{\text{front}} \text{ when } d = 3, \end{aligned}$$

where  $D(\mathbf{u}) = (\nabla \mathbf{u} + \nabla \mathbf{u}^T)/2$ . The boundaries are represented on Fig. 5.1, page 44.

The variational formulation of this problem expresses:

(VFS) find  $\mathbf{u} \in \mathbf{V}(1)$  and  $p \in L_0^2(\Omega)$  such that:

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= 0, \forall \mathbf{v} \in \mathbf{V}(0), \\ b(\mathbf{u}, q) &= 0, \forall q \in L_0^2(\Omega), \end{aligned}$$

where

$$a(\mathbf{u}, \mathbf{v}) = \int_{\Omega} 2D(\mathbf{u}) : D(\mathbf{v}) dx,$$

$$b(\mathbf{v}, q) = - \int_{\Omega} \operatorname{div}(\mathbf{v}) q dx.$$

$$\mathbf{V}(\alpha) = \{\mathbf{v} \in (H^1(\Omega))^2; \mathbf{v} = 0 \text{ on } \Gamma_{\text{left}} \cup \Gamma_{\text{right}} \cup \Gamma_{\text{bottom}} \text{ and } \mathbf{v} = (\alpha, 0) \text{ on } \Gamma_{\text{top}}\}, \text{ when } d = 2,$$

$$\mathbf{V}(\alpha) = \{\mathbf{v} \in (H^1(\Omega))^3; \mathbf{v} = 0 \text{ on } \Gamma_{\text{left}} \cup \Gamma_{\text{right}} \cup \Gamma_{\text{bottom}}, \\ \mathbf{v} = (\alpha, 0, 0) \text{ on } \Gamma_{\text{top}} \text{ and } v_2 = 0 \text{ on } \Gamma_{\text{back}} \cup \Gamma_{\text{front}}\}, \text{ when } d = 3,$$

$$L_0^2(\Omega) = \{q \in L^2(\Omega); \int_{\Omega} q dx = 0\}.$$

## Approximation

The Taylor-Hood [12] finite element approximation of the Stokes problem is considered. We introduce a mesh  $\mathcal{T}_h$  of  $\Omega$  and the following finite dimensional spaces:

$$\begin{aligned}\mathbf{X}_h &= \{\mathbf{v} \in (H^1(\Omega))^d; \mathbf{v}_{/K} \in (P_2)^d, \forall K \in \mathcal{T}_h\}, \\ \mathbf{V}_h(\alpha) &= \mathbf{X}_h \cap \mathbf{V}(\alpha), \\ Q_h &= \{q \in L^2(\Omega) \cap C^0(\bar{\Omega}); q_{/K} \in P_1, \forall K \in \mathcal{T}_h\},\end{aligned}$$

The approximate problem writes:

$(VFS)_h$  find  $\mathbf{u}_h \in \mathbf{V}_h(1)$  and  $p \in Q_h$  such that:

$$\begin{aligned}a(\mathbf{u}_h, \mathbf{v}) + b(\mathbf{v}, p_h) &= 0, \forall \mathbf{v} \in \mathbf{V}_h(0), \\ b(\mathbf{u}_h, q) &= 0, \forall q \in Q_h.\end{aligned}\tag{5.4}$$

### File ‘cavity.icc’

```

1 space cavity_space (const geo& omega_h, std::string approx) {
2   space Xh (omega_h, approx, "vector");
3   Xh.block("top"); Xh.block("bottom");
4   if (omega_h.dimension() == 3) {
5     Xh.block("back"); Xh.block("front");
6     Xh[1].block("left"); Xh[1].block("right");
7   } else {
8     Xh.block("left"); Xh.block("right");
9   }
10  return Xh;
11 }
12 field cavity_field (const space& Xh, Float alpha) {
13   field uh (Xh, 0.);
14   uh[0]["top"] = alpha;
15   return uh;
16 }

```

### File ‘stokes\_cavity.cc’

```

1 #include "rheolef.h"
2 using namespace rheolef;
3 using namespace std;
4 #include "cavity.icc"
5 int main(int argc, char**argv) {
6   environment rheolef (argc, argv);
7   geo omega (argv[1]);
8   space Xh = cavity_space (omega, "P2");
9   field uh = cavity_field (Xh, 1);
10  space Qh (omega, "P1");
11  field ph (Qh, 0.);
12  form a (Xh, Xh, "2D_D");
13  form b = - form(Xh, Qh, "div");
14  form mp (Qh, Qh, "mass");
15  solver_abtb stokes (a.uu(), b.uu(), mp.uu());
16  stokes.solve (-(a.ub()*uh.b()), -(b.ub()*uh.b()),
17              uh.set_u(), ph.set_u());
18  dout << catchmark("u") << uh
19        << catchmark("p") << ph;
20 }

```

## Comments

The spaces and boundary conditions and grouped in specific functions, defined in file ‘cavity.icc’. This file is suitable for a future reuse. Next, forms are defined as usual, in file ‘stokes\_cavity.cc’.

The problem admits the following matrix form:

$$\begin{pmatrix} \mathbf{a.uu} & \text{trans}(\mathbf{b.uu}) \\ \mathbf{b.uu} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{uh.u} \\ \mathbf{ph.u} \end{pmatrix} = \begin{pmatrix} -\mathbf{a.ub} * \mathbf{uh.b} \\ -\mathbf{b.ub} * \mathbf{uh.b} \end{pmatrix}$$

An initial value for the pressure field is provided:

```
field ph (Qh, 0);
```

The main Stokes solver call writes:

```
solver_abtb stokes (a.uu(), b.uu(), mp.uu());
stokes.solve (-(a.ub()*uh.b()), -(b.ub()*uh.b()),
             uh.set_u(), ph.set_u());
```

For tridimensional geometries ( $d = 3$ ), this system is solved by the preconditioned conjugate gradient algorithm. the preconditioner is here the mass matrix `mp.uu` for the pressure: as showed in [13], the number of iterations need by the conjugate gradient algorithm to reach a given precision is then independent of the mesh size. For more details, see the Rheolef reference manual related to mixed solvers, available e.g. via the unix command:

```
man solver_abtbc
```

When  $d = 2$ , it is interessant to turn to direct methods and factorize the whole matrix of the linear system. As the pressure is defined up to a constant, the whole matrix is singular. By adding a Lagrange multiplier that impose a nul average pressure value, the system becomes regular and the modified matrix can be inverted. Such a technic has already been presented in section 4 for the Neumann-Laplace problem. Finaly, he choice between iterative and direct algorithm for the Stokes solver is automatically done, regarding the geometry dimension.

## How to run the program

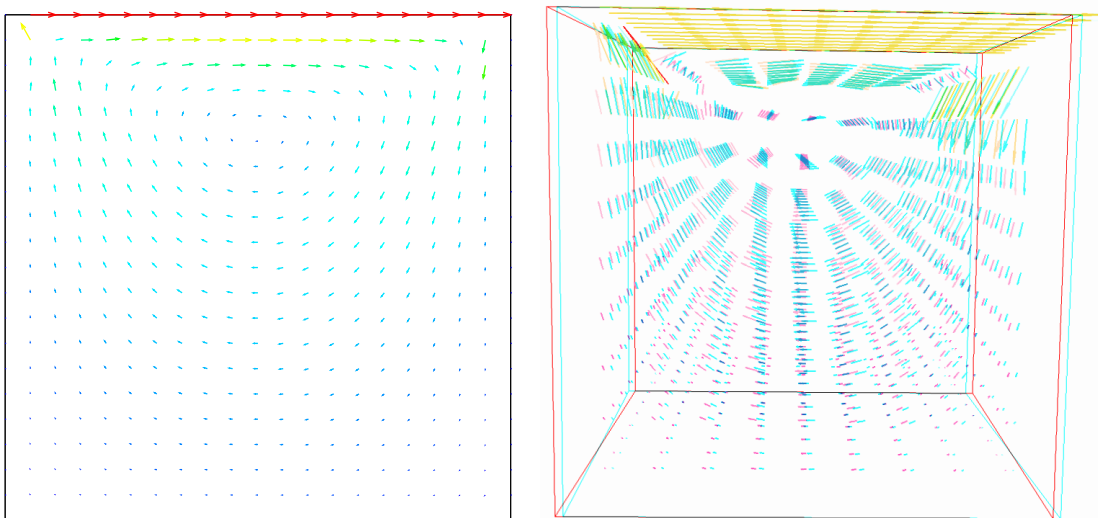


Figure 5.6: The velocity visualization for  $d = 2$  and  $d = 3$  with stereo anaglyph.

We assume that the previous code is contained in the file ‘`stokes_cavity.cc`’. Then, compile the program as usual (see page 16):

```
make stokes_cavity
```

and enter the commands:

```
mkgeo_grid -t 10 > square.geo
./stokes_cavity square > square.field
```

The previous command solves the problem for the corresponding mesh and writes the solution in a '.field' file. Run the velocity vector visualization :

```
field square.field -velocity
```

Run also some scalar visualizations:

```
field square.field -comp 0
field square.field -comp 1
field square.field -catchmark p
```

Note the `-catchmark` option to the `field` command: the file reader jumps to the label and then starts to read the selected field. Next, perform another computation on a finer mesh:

```
mkgeo_grid -t 20 > square-20.geo
./stokes_cavity square-20.geo > square-20.field
```

and observe the convergence.

Finally, let us consider the three dimensional case:

```
mkgeo_grid -T 5 > cube.geo
./stokes_cavity cube.geo > cube.field
```

and the corresponding visualization:

```
field cube.field -velocity
field cube.field -comp 0
field cube.field -comp 1
field cube.field -comp 2
field cube.field -catchmark p
```

## 5.5 Computing the vorticity

### Formulation and approximation

When  $d = 2$ , we define [4, page 30] for any distributions  $\phi$  and  $\mathbf{v}$ :

$$\begin{aligned} \operatorname{curl} \phi &= \left( \frac{\partial \phi}{\partial x_1}, -\frac{\partial \phi}{\partial x_0} \right), \\ \operatorname{curl} \mathbf{v} &= \frac{\partial v_1}{\partial x_0} - \frac{\partial v_0}{\partial x_1}, \end{aligned}$$

and when  $d = 3$ :

$$\mathbf{curl} \mathbf{v} = \left( \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}, \frac{\partial v_0}{\partial x_2} - \frac{\partial v_2}{\partial x_0}, \frac{\partial v_1}{\partial x_0} - \frac{\partial v_0}{\partial x_1} \right)$$

Let  $\mathbf{u}$  be the solution of the Stokes problem ( $S$ ). The vorticity is defined by:

$$\begin{aligned} \omega &= \mathbf{curl} \mathbf{u} && \text{when } d = 2, \\ \boldsymbol{\omega} &= \mathbf{curl} \mathbf{u} && \text{when } d = 3. \end{aligned}$$

Since the approximation of the velocity is piecewise quadratic, we are looking for a discontinuous piecewise linear vorticity field that belongs to:

$$\begin{aligned} Y_h &= \{ \xi \in L^2(\Omega); \xi_{/K} \in P_1, \forall K \in \mathcal{T}_h \}, && \text{when } d = 2 \\ \mathbf{Y}_h &= \{ \boldsymbol{\xi} \in (L^2(\Omega))^3; \xi_{i/K} \in P_1, \forall K \in \mathcal{T}_h \}, && \text{when } d = 3 \end{aligned}$$

The approximate variational formulation writes:

$$\begin{aligned} \omega_h \in Y_h, \quad \int_{\Omega} \omega_h \xi \, dx &= \int_{\Omega} \mathbf{curl} \mathbf{u}_h \xi \, dx, \quad \forall \xi \in Y_h && \text{when } d = 2, \\ \boldsymbol{\omega} \in \mathbf{Y}_h, \quad \int_{\Omega} \boldsymbol{\omega}_h \cdot \boldsymbol{\xi} \, dx &= \int_{\Omega} \mathbf{curl} \mathbf{u}_h \cdot \boldsymbol{\xi} \, dx, \quad \forall \boldsymbol{\xi} \in \mathbf{Y}_h && \text{when } d = 3. \end{aligned}$$

### File ‘vorticity.cc’

```

1 #include "rheolef.h"
2 using namespace rheolef;
3 using namespace std;
4 int main(int argc, char** argv) {
5     environment rheolef (argc, argv);
6     field uh;
7     din >> uh;
8     string grad_approx = "P" + itos(uh.get_space().degree()-1) + "d";
9     string valued = (uh.size() == 3) ? "vector" : "scalar";
10    space Lh (uh.get_geo(), grad_approx, valued);
11    form curl (uh.get_space(), Lh, "curl");
12    form inv_m (Lh, Lh, "inv_mass");
13    dout << catchmark("w") << inv_m*(curl*uh);
14 }

```

## How to run the program

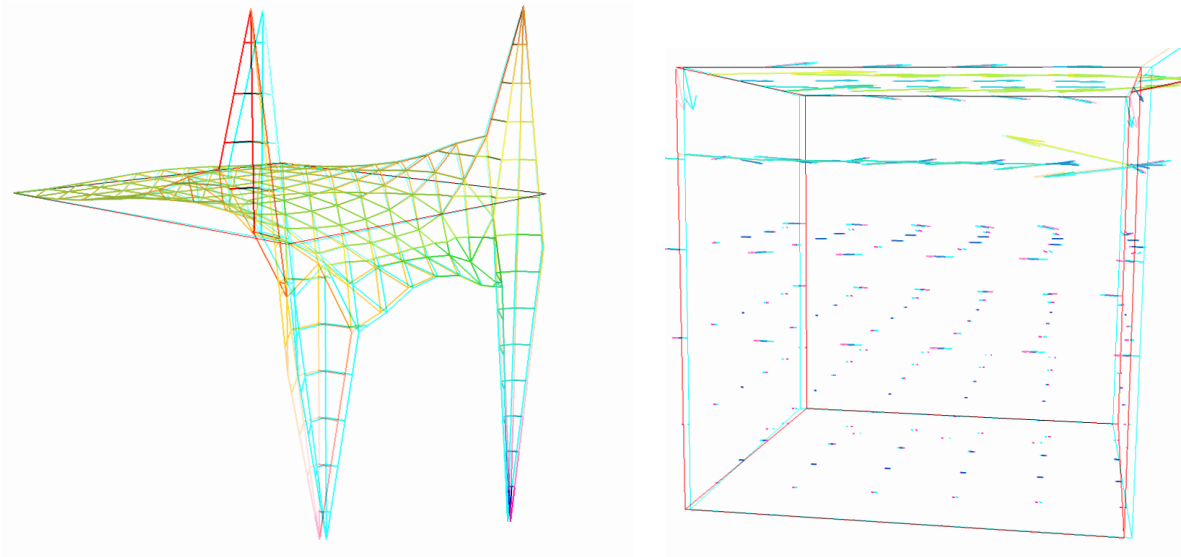


Figure 5.7: The vorticity: elevation view for  $d = 2$  and vector representation for  $d = 3$  (with anaglyph).

For  $d = 2$ , just enter:

```
make vorticity
./vorticity < square.field | field -elevation -stereo -
```

and you observe a discontinuous piecewise linear representation of the approximate vorticity. Also, the vorticity presents two sharp peaks at the upper corners of the driven cavity: the vorticity is unbounded and the peaks will increase with mesh refinements. This singularity of the solution is due to the boundary condition for the first component of the velocity  $u_0$  that jumps from zero to one at the corners. The approximate vorticity field can also be projected on a continuous piecewise linear space, using the `-proj` option (See Fig. 5.7 left):

```
./vorticity < square.field | field -elevation -stereo -nofill -
./vorticity < square.field | field -elevation -stereo -proj -
```

For  $d = 3$ , the whole vorticity vector can also be visualized (See Fig. 5.7 right):

```
./vorticity < cube.field | field -proj -velocity -stereo -
```

In the previous command, the `-proj` option has been used: since the 3D render has no support for discontinuous piecewise linear fields, the P1-discontinuous field is transformed into a P1-continuous one, thanks to a  $L^2$  projection. P1 The following command shows the second component of the vorticity vector, roughly similar to the bidimensionnal case.

```
./vorticity < cube.field | field -comp 1 -
./vorticity < cube.field | field -comp 1 -proj -
```

## 5.6 Computing the stream function

### Formulation and approximation

When  $d = 3$ , the stream function is a vector-valued field  $\psi$  that satisfies [4, page 90]:  $\mathbf{curl} \psi = \mathbf{u}$  and  $\operatorname{div} \psi = 0$ . From the identity:

$$\mathbf{curl} \mathbf{curl} \psi = -\Delta \psi + \nabla(\operatorname{div} \psi)$$

we obtain the following characterization of  $\psi$  :

$$\begin{aligned} -\Delta \psi &= \mathbf{curl} \mathbf{u} && \text{in } \Omega, \\ \psi &= 0 && \text{on } \Gamma_{\text{back}} \cup \Gamma_{\text{front}} \cup \Gamma_{\text{top}} \cup \Gamma_{\text{bottom}}, \\ \frac{\partial \psi}{\partial n} &= 0 && \text{on } \Gamma_{\text{left}} \cup \Gamma_{\text{right}}. \end{aligned}$$

When  $d = 2$ , the stream function  $\psi$  is a scalar-valued field the solution of the following problem [4, page 88]:

$$\begin{aligned} -\Delta \psi &= \operatorname{curl} \mathbf{u} && \text{in } \Omega, \\ \psi &= 0 && \text{on } \partial\Omega. \end{aligned}$$

### File ‘streamf\_cavity.cc’

```

1 #include "rheolef.h"
2 using namespace rheolef;
3 using namespace std;
4 int main (int argc, char** argv) {
5     environment rheolef (argc, argv);
6     field uh;
7     din >> uh;
8     string valued = (uh.size() == 3) ? "vector" : "scalar";
9     space Ph (uh.get_geo(), "P2", valued);
10    Ph.block("top"); Ph.block("bottom");
11    if (uh.get_geo().dimension() == 3) {
12        Ph.block("back"); Ph.block("front");
13    } else {
14        Ph.block("left"); Ph.block("right");
15    }
16    const space& Xh = uh.get_space();
17    form a (Ph, Ph, "grad_grad");
18    form b (Xh, Ph, "curl");
19    field psi_h (Ph, 0.);
20    field lh = b*uh;
21    solver sa (a.uu());
22    psi_h.set_u() = sa.solve (lh.u() - a.ub()*psi_h.b());
23    dout << catchmark("psi") << psi_h;
24 }

```

## How to run the program

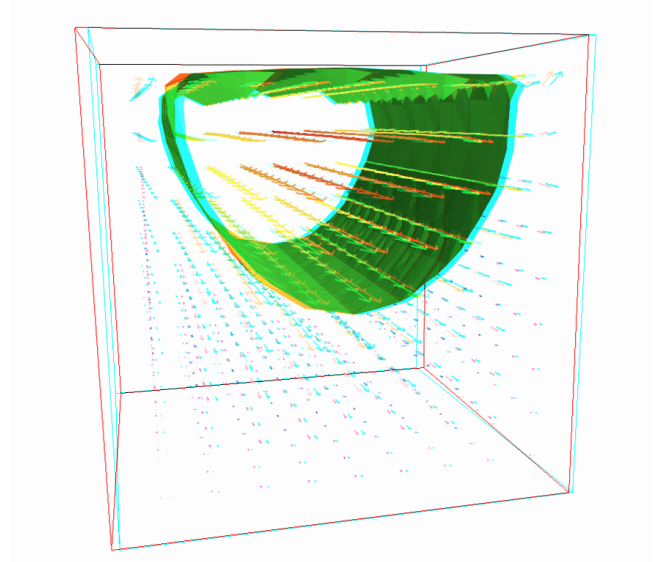
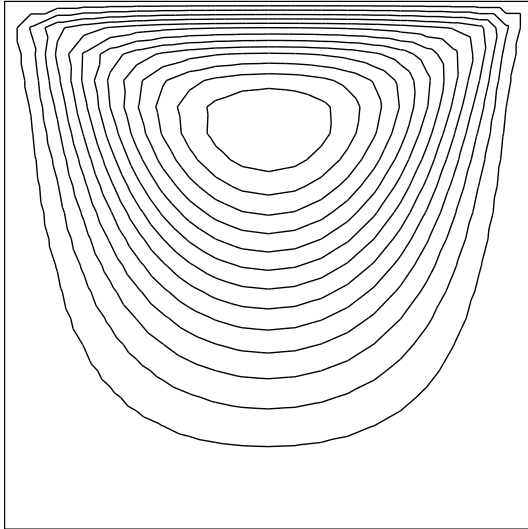


Figure 5.8: The stream function visualization: isolines for  $d = 2$ , and combined vectors and isonorm surface for  $d = 3$ .

For  $d = 2$ , just enter (see Fig. 5.8 left):

```
make streamf_cavity
./streamf_cavity < square.field | field -bw -
```

For  $d = 3$ , the whole stream function vector can be visualized:

```
./streamf_cavity < cube.field | field -velocity -
```

The second component of the stream function is showed by:

```
./streamf_cavity < cube.field | field -comp 1
```

The combined representation of Fig. 5.8.right has been obtained in two steps. First, enter:

```
./streamf_cavity < cube.field | field -comp 1 -noclean -noexecute -
mv output.vtk psi1.vtk
./streamf_cavity < cube.field | field -velocity -
```

The `-noclean -noexecute` options cause the creation of the `.vtk` file for the second component, without running the `mayavi` visualization. Next, in the `mayavi` window associated to the whole stream function, select the `File/Load data/VTK file` menu and load `'psi1.vtk'`. Finally, select the `Vizualize/Module/IsoSurface` menu. Observe that the 3D stream function is mainly represented by its second component.





## Chapter 6

# Nearly incompressible elasticity and the stabilized Stokes problems

### 6.1 The incompressible elasticity problem

#### Formulation

Let us go back to the linear elasticity problem.

When  $\lambda$  becomes large, this problem is related to the *incompressible elasticity* and cannot be solved as it was previously done. To overcome this difficulty, the pressure is introduced :

$$p = -\lambda \operatorname{div} \mathbf{u}$$

and the problem becomes:

(E) find  $\mathbf{u}$  and  $p$  defined in  $\Omega$  such that:

$$\begin{aligned} -\operatorname{div}(2D(\mathbf{u})) + \nabla p &= \mathbf{f} \text{ in } \Omega, \\ -\operatorname{div} \mathbf{u} - \frac{1}{\lambda} p &= 0 \text{ in } \Omega, \\ +B.C. \end{aligned}$$

The variational formulation of this problem expresses:

(VFE) find  $\mathbf{u} \in V(1)$  and  $p \in L^2(\Omega)$  such that:

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= m(\mathbf{f}, \mathbf{v}), \forall \mathbf{v} \in V(0), \\ b(\mathbf{u}, q) - c(p, q) &= 0, \forall q \in L_0^2(\Omega), \end{aligned}$$

where

$$\begin{aligned} m(\mathbf{u}, \mathbf{v}) &= \int_{\Omega} \mathbf{u} \cdot \mathbf{v} \, dx, \\ a(\mathbf{u}, \mathbf{v}) &= \int_{\Omega} D(\mathbf{u}) : D(\mathbf{v}) \, dx, \\ b(\mathbf{v}, q) &= - \int_{\Omega} \operatorname{div}(\mathbf{v}) q \, dx. \\ c(p, q) &= \frac{1}{\lambda} \int_{\Omega} p q \, dx. \\ V &= \{ \mathbf{v} \in (H^1(\Omega))^2; \mathbf{v} = 0 \text{ on } \Gamma_{left} \cup \Gamma_{bottom} \} \end{aligned}$$

When  $\lambda$  becomes large, we obtain the incompressible elasticity problem, that coincides with the Stokes problem.

## Approximation

As for the Stokes problem, the Talar-Hood [12] finite element approximation is considered. We introduce a mesh  $\mathcal{T}_h$  of  $\Omega$  and the following finite dimensional spaces:

$$\begin{aligned} X_h &= \{ \mathbf{v} \in (H^1(\Omega)); \mathbf{v}_{/K} \in (P_2)^2, \forall K \in \mathcal{T}_h \}, \\ V_h(\alpha) &= X_h \cap V, \\ Q_h &= \{ q \in L^2(\Omega) \cap C^0(\bar{\Omega}); q_{/K} \in P_1, \forall K \in \mathcal{T}_h \}, \end{aligned}$$

The approximate problem writes:

$(VFE)_h$  find  $\mathbf{u}_h \in V_h(1)$  and  $p \in Q_h$  such that:

$$\begin{aligned} a(\mathbf{u}_h, \mathbf{v}) + b(\mathbf{v}, p_h) &= 0, \forall \mathbf{v} \in V_h(0), \\ b(\mathbf{u}_h, q) - c(p, q) &= 0, \forall q \in Q_h. \end{aligned}$$

## File ‘incompressible-elasticity.cc’

```

1 #include "rheolef.h"
2 using namespace rheolef;
3 using namespace std;
4 #include "embankment.icc"
5 int main(int argc, char**argv) {
6     environment rheolef (argc, argv);
7     geo omega (argv[1]);
8     Float inv_lambda = (argc > 2 ? atof(argv[2]) : 0);
9     size_t d = omega.dimension();
10    space Xh = embankment_space(omega, "P2");
11    space Qh (omega, "P1");
12    point f (0,0,0);
13    f [d-1] = -1;
14    field lh = riesz (Xh, f);
15    form mp (Qh, Qh, "mass");
16    form a (Xh, Xh, "2D_D");
17    form b (Xh, Qh, "div"); b = -b;
18    form c = inv_lambda*mp;
19    field uh (Xh, 0);
20    field ph (Qh, 0);
21    solver_abtb elasticity (a.uu(), b.uu(), c.uu(), mp.uu());
22    elasticity.solve (lh.u() - a.ub()*uh.b(), -(b.ub()*uh.b()),
23                    uh.set_u(), ph.set_u());
24    dout << catchmark("inv_lambda") << inv_lambda << endl
25          << catchmark("u") << uh
26          << catchmark("p") << ph;
27 }

```

## Comments

The problem admits the following matrix form:

$$\begin{pmatrix} \mathbf{a.uu} & \text{trans}(\mathbf{b.uu}) \\ \mathbf{b.uu} & -\mathbf{c.uu} \end{pmatrix} \begin{pmatrix} \mathbf{uh.u} \\ \mathbf{ph.u} \end{pmatrix} = \begin{pmatrix} \mathbf{lh.u} - \mathbf{a.ub} * \mathbf{uh.b} \\ -\mathbf{b.ub} * \mathbf{uh.b} \end{pmatrix}$$

The problem is similar to the Stokes one (see page 56). This system is solved by:

```

solver_abtb elasticity (a.uu(), b.uu(), c.uu(), mp.uu());
elasticity.solve (lh.u() - a.ub()*uh.b(), -(b.ub()*uh.b()),
                  uh.set_u(), ph.set_u());

```

For two-dimensional problems, a direct solver is used by default. In the three-dimensional case, an iterative algorithm is the default: the preconditioned conjugate gradient. The preconditioner is here the mass matrix `mp.uu` for the pressure. As showed in [13], the number of iterations need by the conjugate gradient algorithm to reach a given precision is then independent of the mesh size and is uniformly bounded when  $\lambda$  becomes small, i.e. in the incompressible case.

## How to run the program

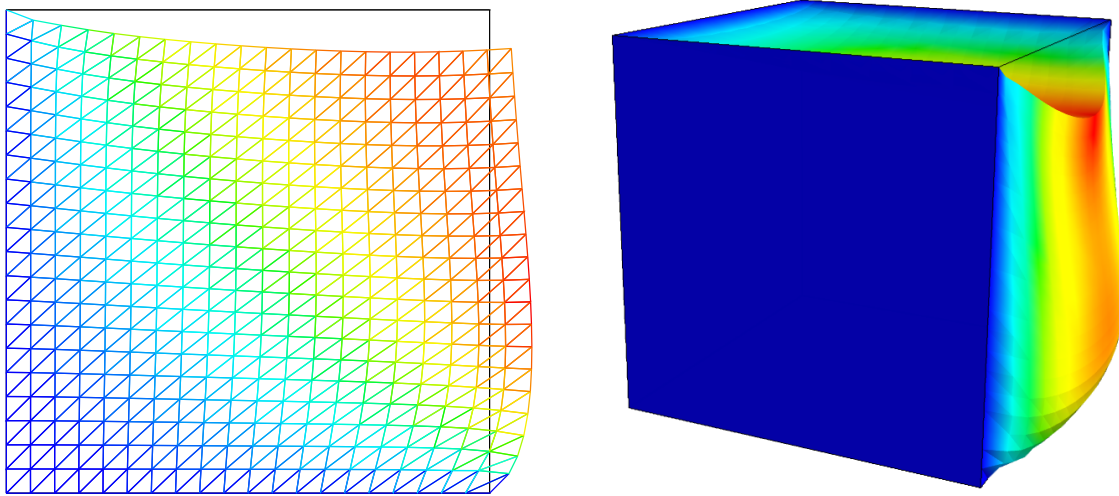


Figure 6.1: The incompressible linear elasticity ( $\lambda = +\infty$ ) for  $N = 2$  and  $N = 3$ .

We assume that the previous code is contained in the file ‘incompressible-elasticity.cc’. Compile the program as usual (see page 16):

```
make incompressible-elasticity
```

and enter the comands:

```
mkgeo_grid -t 10 > square.geo
./incompressible-elasticity square.geo 0 > square.field
field square.field -mayavi -nofill
```

```
mkgeo_grid -T 10 > cube.geo
./incompressible-elasticity cube.geo 0 > cube.field
field cube.field -mayavi -fill
```

The visualization is performed as usual: see section 5.1, page 46. Compare the results on FIG. 6.1, obtained for  $\lambda = +\infty$  with those of FIG. 5.2, page 46, obtained for  $\lambda = 1$ .

Finally, the stress computation and the mesh adaptation loop is left as an exercise to the reader.

## 6.2 The $P_1b - P_1$ element for the Stokes problem

### Formulation and approximation

Let us go back to the Stokes problem. In section 5.4, page 54, the Taylor-Hood finite element was considered. Here, we turn to the mini-element [14] proposed by Arnold, Brezzi and Fortin, also well-known as the  $P_1$ -bubble element. This element is generally less precise than the Taylor-Hood one, but becomes popular, mostly because it is easy to implement in two and three dimensions

and furnishes a  $P_1$  approximation of the velocity field. Moreover, this problem develops some links with stabilization technics and will presents some new **Rheolef** features.

We consider a mesh  $\mathcal{T}_h$  of  $\Omega \subset \mathbb{R}^N$ ,  $N = 2, 3$  composed only of simplicial elements: triangles when  $N = 2$  and tetraedra when  $N = 3$ . The following finite dimensional spaces are introduced:

$$\begin{aligned} \mathbf{X}_h^{(1)} &= \{\mathbf{v} \in (H^1(\Omega))^N; \mathbf{v}_{/K} \in (P_1)^N, \forall K \in \mathcal{T}_h\}, \\ \mathbf{B}_h &= \{\boldsymbol{\beta} \in (C^0(\bar{\Omega}))^N; \boldsymbol{\beta}_{/K} \in B(K)^N, \forall K \in \mathcal{T}_h\} \\ \mathbf{X}_h &= \mathbf{X}_h^{(1)} \oplus \mathbf{B}_h \\ \mathbf{V}_h(\alpha) &= X_h \cap \mathbf{V}(\alpha), \\ Q_h &= \{q \in L^2(\Omega) \cap C^0(\bar{\Omega}); q_{/K} \in P_1, \forall K \in \mathcal{T}_h\}, \end{aligned}$$

where  $B(K) = \text{vect}(\lambda_1 \times \dots \times \lambda_{N+1})$  and  $\lambda_i$  are the barycentric coordinates of the simplex  $K$ . The  $B(K)$  space is related to the *bubble* local space. The approximate problem is similar to (5.4), page 55, up to the choice of finite dimensional spaces.

Remark that the velocity field splits in two terms:  $\mathbf{u}_h = \mathbf{u}_h^{(1)} + \mathbf{u}_h^{(b)}$ , where  $\mathbf{u}_h^{(1)} \in \mathbf{X}_h^{(1)}$  is continuous and piecewise linear, and  $\mathbf{u}_h^{(b)} \in \mathbf{B}_h$  is the bubble term.

### File ‘stokes\_contraction\_bubble.cc’

```

1 #include "rheolef.h"
2 using namespace rheolef;
3 using namespace std;
4 #include "poiseuille.h"
5 int main(int argc, char**argv) {
6     environment rheolef (argc, argv);
7     geo omega (argv[1]);
8     string sys_coord = omega.coordinate_system_name();
9     float c = omega.xmax()[1];
10    space X1h (omega, "P1", "vector");
11    space Bh (omega, "bubble", "vector");
12    X1h.block ("wall");
13    X1h.block ("upstream");
14    X1h[1].block ("axis");
15    X1h[1].block ("downstream");
16    space Xh = X1h * Bh;
17    space Qh (omega, "P1");
18    space Wh (omega["upstream"], "P1");
19    field uh (Xh, 0);
20    field ph (Qh, 0);
21    uh[0][0]["upstream"] = interpolate (Wh, u(c, sys_coord));
22    form a1 (X1h, X1h, "2D_D");
23    form ab (Bh, Bh, "2D_D");
24    form b1 = - form(X1h, Qh, "div");
25    form bb = - form(Bh, Qh, "div");
26    form a = {{a1, 0},
27             {0, ab}};
28    form b = {b1, bb};
29    form mp (Qh, Qh, "mass");
30    solver_abtb stokes (a.uu(), b.uu(), mp.uu());
31    stokes.solve (-(a.ub()*uh.b()), -(b.ub()*uh.b()),
32                uh.set_u(), ph.set_u());
33    dout << catchmark("inv_lambda") << 0 << endl
34          << catchmark("u") << uh[0]
35          << catchmark("ub") << uh[1]
36          << catchmark("p") << ph;
37 }

```

## Comments

The matrix structure is similar to those of the Taylor-Hood element, and thus the same mixed solver approach applies. We consider the abrupt contraction geometry:

$$\Omega = ]-L_u, 0[ \times ]0, c[ \cup [0, L_d[ \times ]0, 1[$$

where  $c \geq 1$  stands for the contraction ratio, and  $L_u, L_d > 0$ , are the upstream and downstream tube lengths. The boundary conditions on  $\mathbf{u} = (u_0, u_1)$  for this test case are:

$$\begin{aligned} u_0 &= u_{\text{poiseuille}} \quad \text{and} \quad u_1 = 0 \quad \text{on } \Gamma_{\text{upstream}} \\ \mathbf{u} &= 0 \quad \text{on } \Gamma_{\text{wall}} \\ \frac{\partial u_0}{\partial x_1} &= 0 \quad \text{and} \quad u_1 = 0 \quad \text{on } \Gamma_{\text{axis}} \\ \frac{\partial \mathbf{u}}{\partial n} &= 0 \quad \text{on } \Gamma_{\text{downstream}} \end{aligned}$$

where

$$\begin{aligned} \Gamma_{\text{upstream}} &= \{-L_u\} \times ]0, c[ \\ \Gamma_{\text{downstream}} &= \{L_d\} \times ]0, 1[ \\ \Gamma_{\text{axis}} &= ]-L_u, L_d[ \times \{0\} \\ \Gamma_{\text{wall}} &= ]-L_u, 0[ \times \{c\} \cup \{0\} \times ]1, c[ \cup [0, L_d[ \times \{1\} \end{aligned}$$

Notices the automatic computation of the geometric contraction ratio  $c$  from the input mesh, as:

```
Float c = omega.xmax()[1];
```

The global form  $a(.,.)$  over  $\mathbf{X}_h$  is obtained by concatenation of the forms  $a_1(.,.)$  and  $a_b(.,.)$  over  $\mathbf{X}_h^{(1)}$  and  $\mathbf{B}_h$  respectively, thanks to the form initializer list:

$$\mathbf{a} = \begin{pmatrix} a_1 & 0 \\ 0 & a_b \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} b_1 & b_b \end{pmatrix}$$

that writes simply with **Rheolef**:

```
form a = {{a1, 0},
          {0, ab}};
form b = {b1, bb};
```

## File ‘poiseuille.h’

```
1 struct u : std::unary_function<point,Float> {
2   Float operator() (const point& x) const {
3     return a*(c+x[1])*(c-x[1]); }
4   u (const Float& c1, std::string sc) : c(c1)
5     { a = (sc == "cartesian") ? 3/(2*pow(c,3)) : 4/pow(c,4); }
6   protected: Float c, a;
7 };
8 struct psi : std::unary_function<point,Float> {
9   Float operator() (const point& x) const {
10    return xy ? a*sqr(c-x[1])*(2*c+x[1]) : a*sqr(c-x[1])*sqr(c+x[1]); }
11   psi (const Float& c1, std::string sc) : c(c1), xy(sc == "cartesian")
12     { a = xy ? -1/(2*pow(c,3)) : -1/pow(c,4); }
13   protected: Float c, a; bool xy;
14 };
```

The file `poiseuille.h` contains code for the velocity and stream function boundary conditions. The Poiseuille velocity upstream boundary condition  $u_{\text{poiseuille}}$  has been scaled such that the total flow rate is equal to one. The stream function is equal to  $-1$  on the axis and to zero on the wall. This file contains also a treatment of the axisymmetric variant of the geometry: this case will be presented in the next paragraphs.

## How to run the program

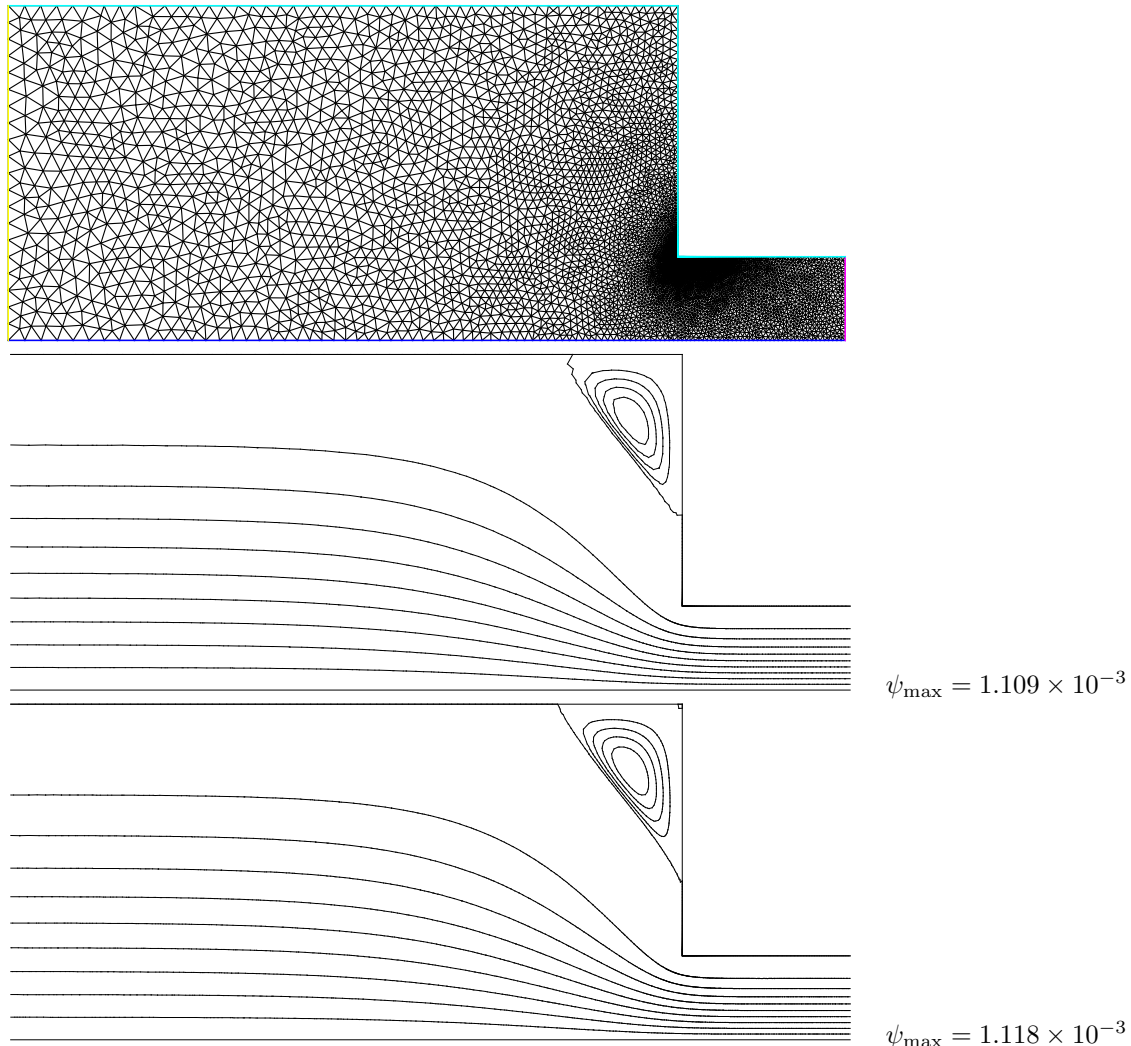


Figure 6.2: Solution of the Stokes problem in the abrupt contraction: (top) the mesh; (center) the  $P_1$  stream function associated to the  $P_1b-P_1$  element; (bottom) the  $P_2$  stream function associated to the  $P_2 - P_1$  Taylor-Hood element.

The boundary conditions in this example are related to an abrupt contraction geometry with a free surface. The corresponding mesh ‘contraction.geo’ can be easily builded from the geometry description file ‘contraction.mshcad’, which is provided in the example directory of the **Rheolef** distribution. The building mesh procedure is presented with details in appendix B, page B.

```
gmsht -2 contraction.mshcad -o contraction.msh
msh2geo < contraction.msh | geo -upgrade - > contraction.geo
geo contraction.geo
```

The mesh is represented on Fig. 6.2.top. Then, the computation and the visualization writes:

```
make stokes_contraction_bubble
```

```
./stokes_contraction_bubble contraction.geo > contraction-P1.field
field contraction-P1.field -mayavi -velocity
```

The visualization of the velocity field brings few informations about the properties of the flow. The stream function is more relevant for stationary flow visualization.

### File ‘streamf\_contraction.cc’

```

1 #include "rheolef.h"
2 using namespace rheolef;
3 using namespace std;
4 #include "poiseuille.h"
5 int main (int argc, char** argv) {
6     environment rheolef (argc, argv);
7     field uh;
8     din >> uh;
9     const geo& omega = uh.get_geo();
10    string sys_coord = omega.coordinate_system_name();
11    Float c = omega.xmax()[1];
12    string approx = "P" + itos(uh.get_space().degree());
13    space Ph (omega, approx);
14    Ph.block("upstream");
15    Ph.block("wall");
16    Ph.block("axis");
17    space Wh (omega["upstream"], approx);
18    const space& Xh = uh.get_space();
19    form a (Ph, Ph, "s_grad_grad");
20    form b (Ph, Xh, "s_curl");
21    field psi_h (Ph, 0);
22    psi_h["upstream"] = interpolate (Wh, psi(c, sys_coord));
23    psi_h["wall"] = 0;
24    psi_h["axis"] = -1;
25    field lh = b.trans_mult(uh);
26    solver sa (a.uu());
27    psi_h.set_u() = sa.solve (lh.u() - a.ub()*psi_h.b());
28    dout << catchmark("psi") << psi_h;
29 }

```

Notice the use of special extensions of the `grad_grad` and `curl` forms for the stream function computation:

```

form a (Ph, Ph, "s_grad_grad");
form b (Ph, Xh, "s_curl");

```

These form are suitable for the axisymmetric coordinate system specific definition of the stream function, while they coincide with the usual `grad_grad` and `curl` forms on cartesian coordinate systems. The axisymmetric case will be presented in the next section.

The stream function  $\psi$  (see also section 5.6) is computed and visualized as:

```

make streamf_contraction
./streamf_contraction < contraction-P1.field > contraction-P1-psi.field
field contraction-P1-psi.field
field contraction-P1-psi.field -n-iso 15 -n-iso-negative 10 -bw

```

The  $P_1$  stream function is represented on Fig. 6.2.center. The stream function is zero along the wall and the line separating the main flow and the vortex located in the outer corner of the contraction. Thus, the isoline associated to the zero value separates the main flow from the vortex. In order to observe this vortex, an extra `-n-iso-negative 10` option is added: ten isolines are drawn for negatives values of  $\psi$ , associated to the main flow, and `n-iso-10` for the positives values, associated to the vortex.



A similar computation based on the Taylor-Hood  $P_2 - P_1$  element is implemented in `stokes_contraction.cc`. The code is similar, up to the boundary conditions, to `stokes_cavity.cc` (see page 55): thus it is not listed here but is available in the **Rheolef** example directory.

```
make stokes_contraction
./stokes_contraction contraction.geo > contraction-P2.field
field contraction-P2.field -mayavi -velocity
./streamf_contraction < contraction-P2.field > contraction-P2-psi.field
field contraction-P2-psi.field -n-iso-negative 10 -bw
```

The associated  $P_2$  stream function is represented on Fig. 6.2.bottom. Observe that the two solutions are similar and that the vortex activity, defined as  $\psi_{\max}$ , is accurately computed with the two methods (see also [15], Fig. 5.11.a, page 143).

```
field contraction-P1-psi.field -max
field contraction-P2-psi.field -max
```

Recall that the stream function is negative in the main flow and positive in the vortex located in the outer corner of the contraction. Nevertheless, the Taylor-Hood based solution is more accurate : this is perceptible on the graphic, in the region where the upstream vortex reaches the boundary.

### 6.3 The stabilized Stokes problem

An alternative and popular implementation of this element eliminates the unknowns related to the bubble components (see e.g. [16], page 24). Remark that the system has the following structure:

$$\begin{pmatrix} A_1 & 0 & B_1^T \\ 0 & A_b & B_b^T \\ B_1 & B_b & 0 \end{pmatrix} \begin{pmatrix} U_1 \\ U_b \\ P \end{pmatrix} = \begin{pmatrix} F_1 \\ F_b \\ G \end{pmatrix}$$

This elimination can be easily performed since the form  $a_b(.,.)$  over  $\mathbf{B}_h$  is diagonal, due to the fact that the bubble functions vanishes on the boundary of elements. The system reduces to:

$$\begin{pmatrix} A_1 & B_1^T \\ B_1 & -C \end{pmatrix} \begin{pmatrix} U_1 \\ P \end{pmatrix} = \begin{pmatrix} F \\ \tilde{G} \end{pmatrix}$$

where  $\tilde{G} = G - B_b A_b^{-1} F_b$  and  $C = B_b A_b^{-1} B_b^T$ . Recall that  $A_b$  is diagonal, since each bubble function vanishes at element boundary. Remarks that the matrix structure is similar to those of the nearly incompressible elasticity (see 6.1, page 6.1). A direct matrix formulation for this problem is similar to the direct  $P_1 - P_1$  stabilized element, proposed by Brezzi and Pitkäranta [17].

### 6.4 Axisymmetric geometries

Axisymmetric geometries are fully supported in **Rheolef**: the coordinate system is associated to the geometry description, stored together with the mesh in the `.geo` and this information is propagated in spaces, forms and fields without any change in the code. Thus, a code that works in plane a 2D plane geometry is able to support a 3D axisymmetric one without changes. A simple axisymmetric geometry writes:

```
mkgeo_grid -t 10 -zr > square-zr.geo
more square-zr.geo
```

Remark the additional line in the header:

```
coordinate_system zr
```

The axis of symmetry is denoted as  $z$  while the polar coordinates are  $(r, \theta)$ . By symmetry, the problem is supposed to be independent upon  $\theta$  and the computational domain is described by  $(x_0, x_1) = (z, r)$ . Conversely, in some cases, it could be convenient to swap the order of the coordinates and use  $(r, z)$ : this feature is obtained by the `-rz` option:

```
mkgeo_grid -t 10 -rz > square-rz.geo
more square-rz.geo
```

Axisymmetric problems uses  $L^2$  functional space equipped with the following weighted scalar product

$$(f, g) = \int_{\Omega} f(z, r) g(z, r) r \, dr \, dz$$

and all usual bilinear forms support this weight. Thus, the **coordinate system can be chosen at run time** and we can expect an efficient source code reduction.

## 6.5 The axisymmetric stream function and stress tensor

In the axisymmetric case, the velocity field  $\mathbf{u} = (u_z, u_r)$  can be expressed in terms of the Stokes stream function  $\psi$  by (see [18, p.453] and [19]):

$$\mathbf{u} = (u_z, u_r) = \left( \frac{1}{r} \frac{\partial \psi}{\partial r}, -\frac{1}{r} \frac{\partial \psi}{\partial z} \right) \quad (6.1)$$

Recall that in the axisymmetric case:

$$\mathbf{curl} \psi = \left( \frac{1}{r} \frac{\partial(r\psi)}{\partial r}, -\frac{\partial \psi}{\partial z} \right)$$

Thus, from this definition, in axisymmetric geometries  $\mathbf{u} \neq \mathbf{curl} \psi$  and the definition of  $\psi$  differs from the 2D plane or 3D cases (see section 5.6, page 60).

Let us turn to a variational formulation in order to compute  $\psi$  from  $\mathbf{u}$ . For any  $\xi \in H^1(\Omega)$ , let us multiply (6.1) by  $\mathbf{v} = (\partial_r \xi, -\partial_z \xi)$  and then integrate over  $\Omega$  with the  $r \, dr \, dz$  weight. For any known  $\mathbf{u}$  velocity field, the problem writes:

(P): find  $\psi \in \Psi(\psi_{\Gamma})$  such that

$$a(\psi, \xi) = b(\xi, \mathbf{u}), \quad \forall \xi \in \Psi(0)$$

where we have introduced the following bilinear forms:

$$\begin{aligned} a(\psi, \xi) &= \int_{\Omega} \left( \frac{\partial \psi}{\partial r} \frac{\partial \xi}{\partial r} + \frac{\partial \psi}{\partial z} \frac{\partial \xi}{\partial z} \right) r \, dr \, dz \\ b(\xi, \mathbf{u}) &= \int_{\Omega} \left( \frac{\partial \xi}{\partial r} u_z - \frac{\partial \xi}{\partial z} u_r \right) r \, dr \, dz \end{aligned}$$

As an exemple, let us reconsider the contraction geometry (see section 6.2, page 65), extended in the axisymmetric case. In that case, the functional space is defined by:

$$\Psi(\psi_{\Gamma}) = \{ \varphi \in H^1(\Omega); \varphi = \psi_{\Gamma} \text{ on } \Gamma_{\text{upstream}} \cup \Gamma_{\text{wall}} \cup \Gamma_{\text{axis}} \}$$

with

$$\psi_{\Gamma} = \begin{cases} \psi_{\text{poiseuille}} & \text{on } \Gamma_{\text{upstream}} \\ 0 & \text{on } \Gamma_{\text{wall}} \\ -1 & \text{on } \Gamma_{\text{axis}} \end{cases}$$

This space corresponds to the imposition of Dirichlet boundary conditions on  $\Gamma_{\text{upstream}}$ ,  $\Gamma_{\text{wall}}$  and  $\Gamma_{\text{axis}}$  and a Neuman boundary condition on  $\Gamma_{\text{downstream}}$ .

These forms are defined in ‘streamf\_contraction.cc’ as:

```
form a (Ph, Ph, "s_grad_grad");
form b (Ph, Xh, "s_curl");
```

Notice that  $a$  is similar to the `grad_grad` form, but where the usual  $r dr dz$  measure is replaced by  $dr dz$ , i.e. without the  $r$  weight. Conversely,  $b$  is related to the `s_curl` operator:

$$\mathbf{s\_curl} \xi = \left( \frac{\partial \xi}{\partial r}, -\frac{\partial \xi}{\partial z} \right)$$

that is closely related to the standard `curl` operator:

$$\mathbf{curl} \xi = \left( \frac{1}{r} \frac{\partial(r\xi)}{\partial r}, -\frac{\partial \xi}{\partial z} \right)$$

Notice the apparition of  $r$  and  $1/r$  in the last expression.

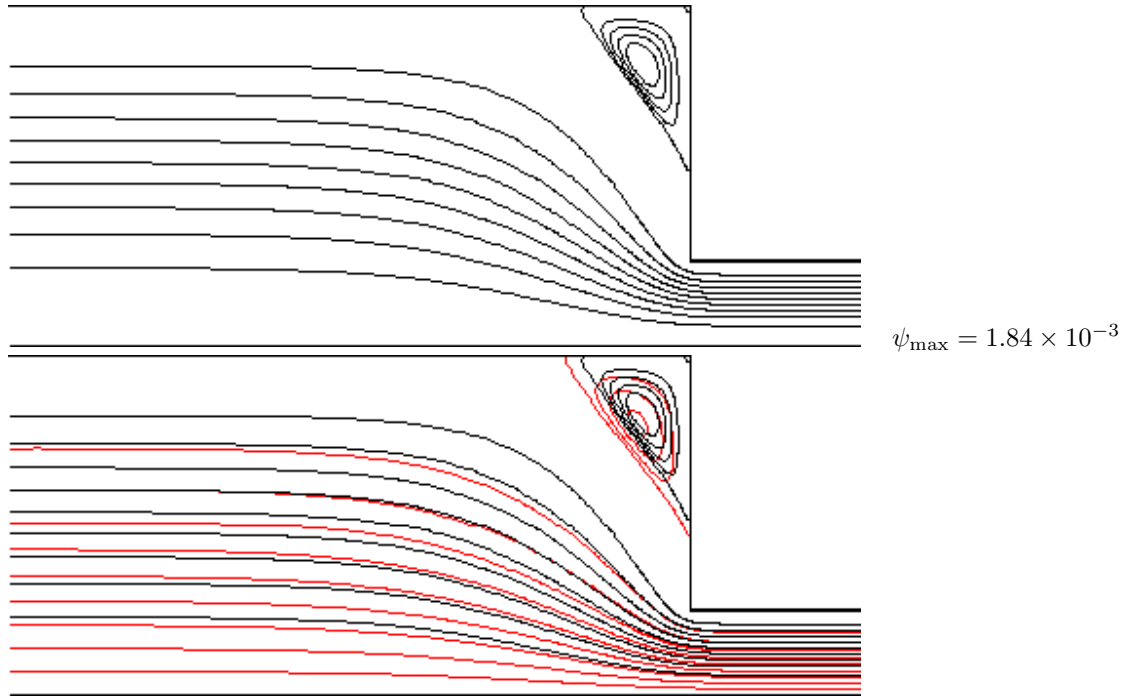


Figure 6.3: Solution of the axisymmetric Stokes problem in the abrupt contraction: (top) the  $P_2$  stream function associated to the  $P_2 - P_1$  element; (bottom) comparison with the 2D cartesian solution (in red).

The following unix commands generate the axisymmetric geometry:

```
gmsh -2 contraction.mshcad -o contraction.msh
msh2geo -zr < contraction.msh | geo -upgrade - > contraction-zr.geo
```

```
more contraction-zr.geo
geo contraction-zr.geo
```

The previous code `stokes_contraction.cc` and `streamf_contraction.cc` are both reused as:

```
./stokes_contraction contraction-zr.geo > contraction-zr-P2.field
./streamf_contraction < contraction-zr-P2.field > contraction-zr-P2-psi.field
field contraction-zr-P2-psi.field -n-iso-negative 10 -bw
```

The solution is represented on Fig. 6.3: it slightly differs from the 2D cartesian solution, as computed in the previous section (see Fig. 6.2). The vortex size is smaller but its intensity  $\psi_{\max} = 1.84 \times 10^{-3}$  is higher. Despite the stream functions looks like similar, the plane solutions

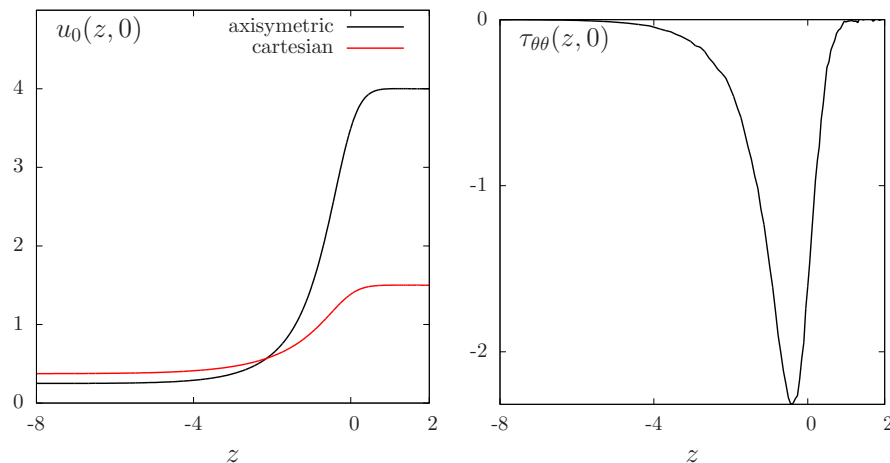


Figure 6.4: Solution of the plane and axisymmetric Stokes problem in the abrupt contraction: cut along the axis of symmetry: (left):  $u_0$ ; (right):  $\tau_{\theta\theta}$ .

are really different, as we can observe from a cut of the first component of the velocity along the axis (see Fig. 6.4):

```
field contraction-P2.field -comp 0 -cut -normal 0 1 -origin 0 1e-15
field contraction-zr-P2.field -comp 0 -cut -normal 0 1 -origin 0 1e-15
```

The `1e-15` argument replace the zero value, as the mesh intersection cannot yet be done exactly on the boundary. Notice that the `stokes_contraction_bubble.cc` can be also reused in a similar way:

```
./stokes_contraction_bubble contraction-zr.geo > contraction-zr-P1.field
./streamf_contraction < contraction-zr-P1.field > contraction-zr-P1-psi.field
field contraction-zr-P1-psi.field -n-iso-negative 10 -bw
```

There is another major difference with axisymmetric problems: the rate of deformation tensor writes:

$$\tau = 2D(\mathbf{u}) = \begin{pmatrix} \tau_{zz} & \tau_{rz} & 0 \\ \tau_{rz} & \tau_{rr} & 0 \\ 0 & 0 & \tau_{\theta\theta} \end{pmatrix}$$

Thus, there is an additional non-zero component  $\tau_{\theta\theta}$  that is automatically integrated into the computations in **Rheolef**. The incompressibility relation leads to  $\text{tr}(\tau) = \tau_{zz} + \tau_{rr} + \tau_{\theta\theta} = 0$ . Here  $\sigma_{\text{tot}} = -p.I + \tau$  is the total Cauchy stress tensor (by a dimensionless procedure, the viscosity can be taken as one). By reusing the `stress.cc` code (see page 48) we are able to compute the tensor components:

```
make stress
./stress < contraction-zr-P1.field > contraction-zr-P1-tau.field
```

The visualization along the axis of symmetry for the  $\tau_{\theta\theta}$  component is obtained by (see Fig. 6.4):

```
field contraction-zr-P1-tau.field -comp 22 -proj -cut -normal 0 1 -origin 0 1e-15
```

Recall that the  $\tau_{zz}$  and  $\tau_{rr}$  components are obtained by the `-comp 00` and `-comp 11` options, respectively. The non-zero values along the axis of symmetry expresses the elongational effects in the entry region of the abrupt contraction.

# Chapter 7

## Time-dependent problems

### 7.1 The heat equation

#### Formulation

Let  $T > 0$ ,  $\Omega \subset \mathbb{R}^d$ ,  $d = 1, 2, 3$  and  $f$  defined in  $\Omega$ . The heat problem writes:

(P): find  $u$ , defined in  $\Omega \times ]0, T[$ , such that

$$\begin{aligned} \frac{\partial u}{\partial t} - \Delta u &= f \text{ in } \Omega \times ]0, T[, \\ u(0) &= 0 \text{ in } \Omega, \\ u(t) &= 0 \text{ on } \partial\Omega \times ]0, T[. \end{aligned}$$

where  $f$  is a known function.

#### Approximation

Let  $\Delta t > 0$  and  $t_n = n\Delta t$ ,  $n \geq 0$ . The problem is approximated with respect to time by the following first-order implicit Euler scheme:

$$\frac{u^{n+1} - u^n}{\Delta t} - \Delta u^{n+1} = f(t_{n+1}) \text{ in } \Omega$$

where  $u^n \approx u(n\Delta t)$  and  $u^{(0)} = 0$ . We reuse the bilinear forms  $a$  and  $m$  defined in section 1.1, page 13 for the Dirichlet problem and introduce the bilinear form  $c = m + \Delta t a$ . The variational formulation of the time-discretized problem writes:

$(VF)_n$ : Let  $u^n$  being known, find  $u^{n+1} \in H_0^1(\Omega)$  such that

$$c(u^{n+1}, v) = m(u^n + \Delta t f(t_{n+1}), v), \forall v \in H_0^1(\Omega).$$

This is a Poisson-like problem. The discretization with respect to space of this problem is similar to those presented in section 1.1, page 13.

## File 'heat.cc'

```

1 #include "rheolef.h"
2 using namespace rheolef;
3 using namespace std;
4 int main (int argc, char **argv) {
5     environment rheolef (argc, argv);
6     geo omega (argv[1]);
7     size_t n_max = (argc > 2) ? atoi(argv[2]) : 100;
8     Float delta_t = 0.5/n_max;
9     space Xh (omega, "Pi");
10    Xh.block ("boundary");
11    form m (Xh, Xh, "mass");
12    form a (Xh, Xh, "grad_grad");
13    form c = m + delta_t*a;
14    solver sc = ldlt (c.uu());
15    field lh = riesz (Xh, 1);
16    field uh (Xh, 0);
17    branch event ("t", "u");
18    dout << event (0, uh);
19    for (size_t n = 1; n <= n_max; n++) {
20        field kh = m*uh + delta_t*lh;
21        uh.set_u() = sc.solve (kh.u() - c.ub()*uh.b());
22        dout << event (Float(n)*delta_t, uh);
23    }
24 }

```

## Comments

Notice the use of the `branch` class:

```
branch event ("t", "u");
```

this is a wrapper class that is used here to print the branch of solution  $(t_n, u^n)_{n \geq 0}$ , on the standard output in the '`.branch`' file format. An instruction as:

```
dout << event (t, uh);
```

is equivalent to the formatted output

```
dout << catchmark("t") << t << endl
      << catchmark("u") << uh;
```

## How to run the program

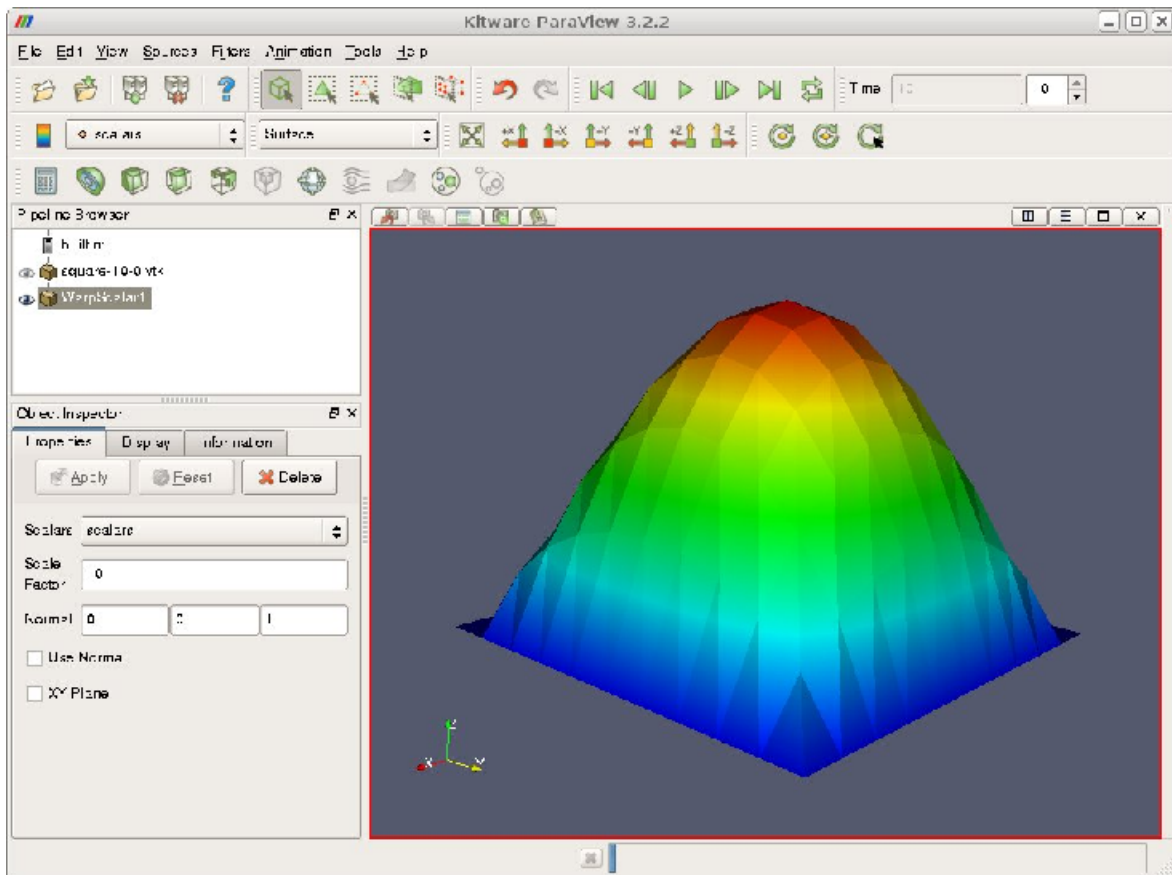


Figure 7.1: Animation of the solution of the heat problem.

We assume that the previous code is contained in the file ‘heat.cc’. Then, compile the program as usual (see page 16):

```
make heat
```

For a one dimensional problem, enter the commands:

```
mkgeo_grid -e 10 > line.geo
./heat line.geo > line.branch
```

The previous commands solve the problem for the corresponding mesh and write the solution in the field-family file format ‘.branch’. For a bidimensional one:

```
mkgeo_grid -t 10 > square.geo
./heat square.geo > square.branch
```

For a tridimensional one:



```
mkgeo_grid -T 10 > box.geo
./heat box.geo > box.branch
```

## How to run the animation

```
branch line.branch -gnuplot -umax 0.125
```

A `gnuplot` window appears. Enter `q` to exit the window. For a bidimensional case, a more sophisticated procedure is required. Enter the following unix commands:

```
branch square.branch -paraview
paraview &
```

A window appears, that looks like a video player. Then, open the `File->open` menu and load `square-*.vtk`. The first `'.'` stands for a wildcard, i.e. the time index family. Then, press the `apply` green button and, clic a first time on the video `play` button, at the top of the window.

Next, go to the `object inspector window`, select `display` and clic on the `rescale to data range` button. Then clic a second time on the video `play` button. An elevation view can be also obtained: Select the `Filter->alphabetical->wrap(scalar)` menu, choose 10 as `scale factor` and press the `apply` green button. Then, clic on the graphic window, rotate the view and finally re-play the animation

To generate an animation file<sup>1</sup>, go to the `File->save animation` menu and enter as file name `square` and as file type `jpeg`. A collection of `jpeg` files are generated by `paraview`. Then, run the unix command:

```
ffmpeg -r 2 -i 'square.%04d.jpg' square.mov
```

The animation file `square.mov` can now be started from any video player, such as `vlc`:

```
vlc --loop square.mov
```

For the tridimensional case, the animation feature is similar.

## 7.2 The convection-diffusion problem

### Formulation

Let  $T > 0$  and  $\nu > 0$ . The convection-diffusion problem writes:

(P): find  $\phi$ , defined in  $\Omega \times ]0, T[$ , such that

$$\begin{aligned} \frac{\partial \phi}{\partial t} + \mathbf{u} \cdot \nabla \phi - \nu \Delta \phi + \sigma \phi &= 0 \text{ in } \Omega \times ]0, T[ \\ \phi(0) &= \phi_0 \text{ in } \Omega \\ \phi(t) &= \phi_\Gamma(t) \text{ on } \partial\Omega \times ]0, T[ \end{aligned}$$

where  $\mathbf{u}$ ,  $\sigma \geq 0$ ,  $\phi_0$  and  $\phi_\Gamma$  being known. Notice the additional  $\mathbf{u} \cdot \nabla$  operator.

<sup>1</sup>At this time, the `avi` output feature is broken in `paraview`, and an alternate `mpeg` output is here suggested.

## Time approximation

This problem is approximated by the following first-order implicit Euler scheme:

$$\frac{\phi^{n+1} - \phi^n \circ X^n}{\Delta t} - \nu \Delta \phi^{n+1} = 0 \quad \text{in } \Omega$$

where  $\Delta t > 0$ ,  $\phi^n \approx \phi(n\Delta t)$  and  $\phi^{(0)} = \phi_0$ .

Let  $t_n = n\Delta t$ ,  $n \geq 0$ . The term  $X^n(x)$  is the position at  $t_n$  of the particule that is in  $x$  at  $t_{n+1}$  and is transported by  $\mathbf{u}^n$ . Thus,  $X^n(x) = X(t_n, x)$  where  $X(t, x)$  is the solution of the differential equation

$$\begin{cases} \frac{dX}{dt} = \mathbf{u}(X(t, x), t) & p.p. \quad t \in ]t_n, t_{n+1}[ \\ X(t_{n+1}, x) = x. \end{cases}$$

Then  $X^n(x)$  is approximated by the first-order Euler approximation

$$X^n(x) \approx x - \Delta t \mathbf{n}^n(x).$$

This algorithm has been introduced by O. Pironneau (see e.g. [20]), and is known as the method of characteristic in the finite difference context and as the Lagrange-Galerkin in the finite element one. The efficient evaluation of  $\phi_h \circ X^n(x)$  in an unstructured mesh involves a hierarchical  $d$ -tree (quadtree, octree) data structure for the localization of the element  $K$  of the mesh that contains  $x$ . When  $d = 3$  requires also sophisticated geometric predicates to test whether  $x \in K$  without rounding errors, and avoid to conclude that no elements contains a point  $x$  close to  $\partial K$  up to rounding errors. This problems is addressed in **Rheolef** based on the **cgal** library.

The following code implements the classical rotating Gaussian hill test case (see e.g. [21]).

## File ‘convect.cc’

```

1 // mkgeo_grid -e 20 -a -2 -b 2 > line2.geo
2 // ./convect-P2 line2 P1 1e-2 100 > line2-P1.branch
3 // ./convect-P2 line2 P2 1e-2 100 > line2-P2.branch
4 // ./convect_error < line2-P1.branch > line2-P1-cmp.branch
5 // ./convect_error < line2-P2.branch > line2-P2-cmp.branch
6 #include "rheolef.h"
7 using namespace rheolef;
8 using namespace std;
9 #include "rotating-hill.h"
10 int main (int argc, char **argv) {
11     environment rheolef (argc,argv);
12     geo omega (argv[1]);
13     string approx = (argc > 2) ? argv[2] : "P1";
14     Float nu = (argc > 3) ? atof(argv[3]) : 1e-2;
15     size_t n_max = (argc > 4) ? atoi(argv[4]) : 50;
16     size_t d = omega.dimension();
17     Float delta_t = 2*acos(-1.)/n_max;
18     space Vh (omega, approx, "vector");
19     field uh = interpolate (Vh, u(d));
20     space Xh (omega, approx);
21     Xh.block ("boundary");
22     field phi_h = interpolate (Xh, phi(d,nu,0));
23     characteristic X (-delta_t*uh);
24     quadrature_option_type qopt;
25     qopt.set_family(quadrature_option_type::gauss_lobatto);
26     qopt.set_order(Xh.degree());
27     form m (Xh, Xh, "mass", qopt);
28     form a (Xh, Xh, "grad_grad");
29     branch event ("t", "phi");
30     dout << catchmark("nu") << nu << endl
31         << event (0, phi_h);
32     for (size_t n = 1; n <= n_max; n++) {
33         Float t = n*delta_t;
34         field lh = riesz(Xh, compose(phi_h, X), qopt);
35         form c = (1 + delta_t*phi::sigma(d,nu,t))*m + delta_t*nu*a;
36         solver sc (c.uu());
37         phi_h.set_u() = sc.solve (lh.u() - c.ub()*phi_h.b());
38         dout << event (t, phi_h);
39     }
40 }

```

## Comments

The characteristic variable  $X$  implements the localizer  $X^n(x)$ :

```
characteristic X (-delta_t*uh);
```

Combined with the `compose` function, it perform the composition  $\phi_h \circ X^n$ . The right-hand side is then computed as usual by using the `riesz` function:

```
field lh = riesz(Xh, compose(phi_h, X));
```

Notice also the use of the *lumped* mass matrix:

```
form m (Xh, Xh, "lumped_mass");
```

The lumped procedure leads to profitable spectral properties of the mass matrix that are benefit to the stability of the method. As test case is described in [22]: we take  $\Omega = ]-2, 2[^d$  and  $T = 2\pi$ . This problem provides an example for a convection-diffusion equation and a known analytical solution:

$$\phi(t, x) = \exp(-\lambda t - r(t)|x - x_0(t)|^2)$$

where  $\lambda = 4\nu t_0 \geq 0$  with  $t_0 > 0$  and  $\nu \geq 0$ ,  $x_0(t)$  is the moving center of the hill and  $r(t) = 1/(t_0 + 4\nu t)$ . The source term is time-dependent:  $\sigma(t) = \lambda - 2d\nu r(t)$  and has been adjusted such

that the right-hand side is zero. The moving center of the hill  $x_0(t)$  is associated to the velocity field  $\mathbf{u}(t, x)$  as:

$d$	$\mathbf{u}(t, x)$	$x_0(t)$
1	$1/(2\pi)$	$t/(2\pi) - 1/2$
2	$(y, -x)$	$(-\cos(t)/2, \sin(t)/2)$
3	$(y, -x, 0)$	$(-\cos(t)/2, \sin(t)/2, 0)$

### File ‘rotating-hill.h’

```

1 struct u : std::unary_function<point,point> {
2   point operator() (const point & x) {
3     return (d == 1) ? point(u0) : point(x[1], -x[0]); }
4   u (size_t d1) : d(d1), u0 (0.5/acos(Float(-1))) {}
5   protected: size_t d; Float u0;
6 };
7 struct phi : std::unary_function<point,Float> {
8   static Float sigma(size_t d, Float nu1, Float t) {
9     const Float t0 = 0.2;
10    return 4*nu1/t0 - 2*d*nu1/(t0 + 4*nu1*t); }
11   Float operator() (const point& x) {
12     point x0t;
13     if (d == 1) { x0t = point(x0[0] + u0*t); }
14     else {
15         x0t = point( x0[0]*cos(t) + x0[1]*sin(t),
16                   -x0[0]*sin(t) + x0[1]*cos(t));
17     }
18     return exp(-4*nu*(t/t0) - dist2(x,x0t)/(t0+4*nu*t));
19   }
20   phi (size_t d1, Float nu1, Float t1) : d(d1), nu(nu1), t(t1),
21     t0(0.2), u0 (0.5/acos(Float(-1))), x0(-0.5,0) {}
22   protected: size_t d; Float nu, t, t0, u0; point x0;
23 };

```

Notice the use of a class-function `phi` for the implementation of  $\phi(t)$  as a function of  $x$ . Such programming style has been introduced in the *standard template library* [23], which is a part of the standard C++ library. By this way, for a given  $t$ ,  $\phi(t)$  can be interpolated as an usual function on a mesh.

### How to run the program

We assume that the previous code is contained in the file ‘`convect.cc`’. Then, compile the program as usual (see page 16):

```
make convect
```

and enter the commands: Running the one-dimensional test case:

```
mkgeo_grid -e 500 -a -2 -b 2 > line2.geo
./convect line2.geo P1 > line2.branch
branch line2.branch -gnuplot
```

Notice the hill that moves from  $x = -1/2$  to  $x = 1/2$ . Since the exact solution is known, it is possible to analyze the error:

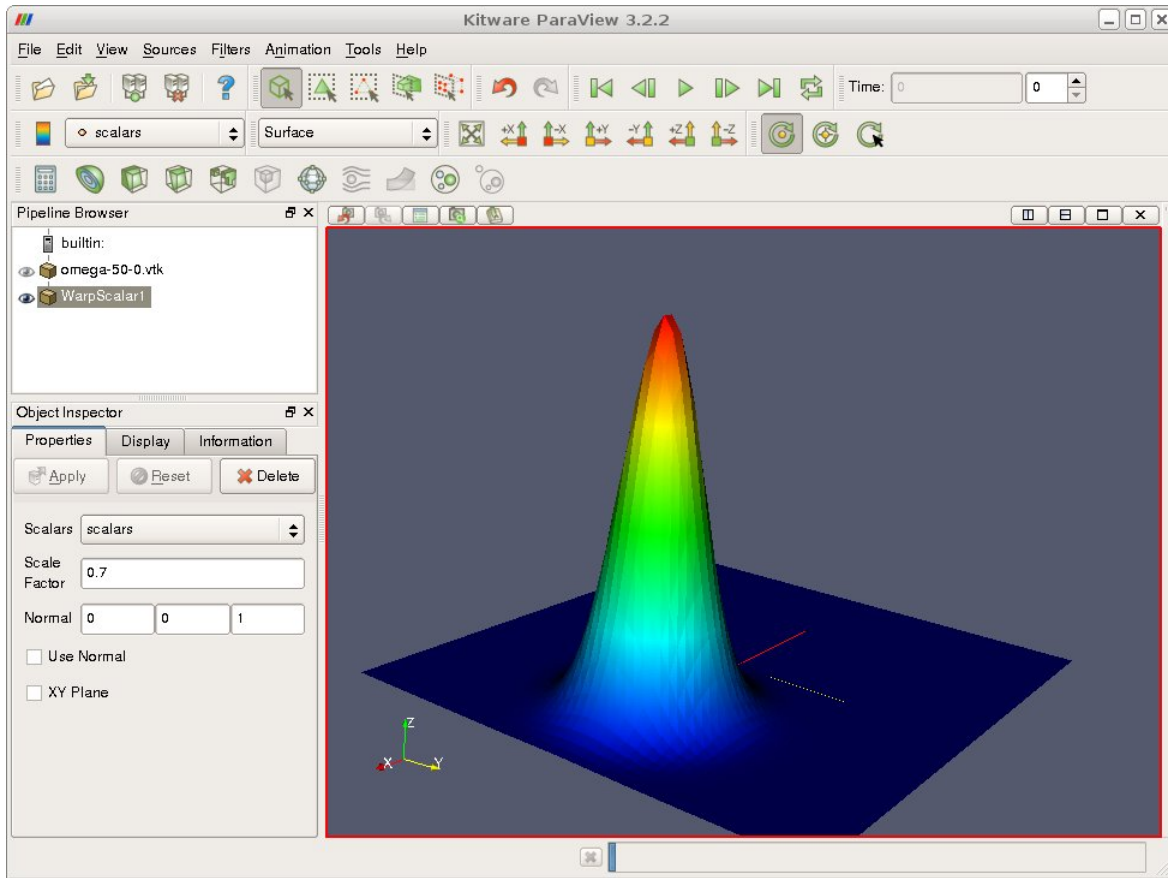


Figure 7.2: Animation of the solution of the rotating hill problem.

## File 'convect\_error.cc'

```

1  #include "rheolef.h"
2  using namespace rheolef;
3  using namespace std;
4  #include "rotating-hill.h"
5  int main (int argc, char **argv) {
6      environment rheolef (argc,argv);
7      Float nu;
8      din >> catchmark("nu") >> nu;
9      branch get ("t","phi");
10     branch put ("t","phi_h","pi_h_phi");
11     derr << "# t\terror_l2\terror_linf" << endl;
12     field phi_h;
13     Float err_l2_l2 = 0;
14     Float err_linf_linf = 0;
15     for (Float t = 0, t_prec = 0; din >> get (t, phi_h); t_prec = t) {
16         const space& Xh = phi_h.get_space();
17         size_t d = Xh.get_geo().dimension();
18         field pi_h_phi = interpolate (Xh, phi(d,nu,t));
19         form m (Xh, Xh, "mass");
20         field eh = phi_h - pi_h_phi;
21         Float err_l2 = sqrt(m(eh,eh));
22         Float err_linf = eh.max_abs();
23         err_l2_l2 += sqr(err_l2)*(t - t_prec);
24         err_linf_linf = max(err_linf_linf, err_linf);
25         dout << put (t, phi_h, pi_h_phi);
26         derr << t << "\t" << err_l2 << "\t" << err_linf << endl;
27     }
28     derr << "# error_l2_l2      = " << sqrt(err_l2_l2) << endl;
29     derr << "# error_linf_linf = " << err_linf_linf << endl;
30 }

```

The numerical error  $\phi_h - \pi_h(\phi)$  is computed as:

```
field eh = phih - interpolate (Xh, phi(t));
```

and its  $L^2$  norm is printed on the standard error. Observe the use of the `branch` class as both input and output field stream.

```
make convect_error
./convect_error < line2.branch > line2-cmp.branch
branch line2-cmp.branch -gnuplot
```

The instantaneous  $L^2(\Omega)$  norm is printed at each time step and the total error in  $L^2([0, T]; L^2(\Omega))$  is finally printed at the end of the stream.

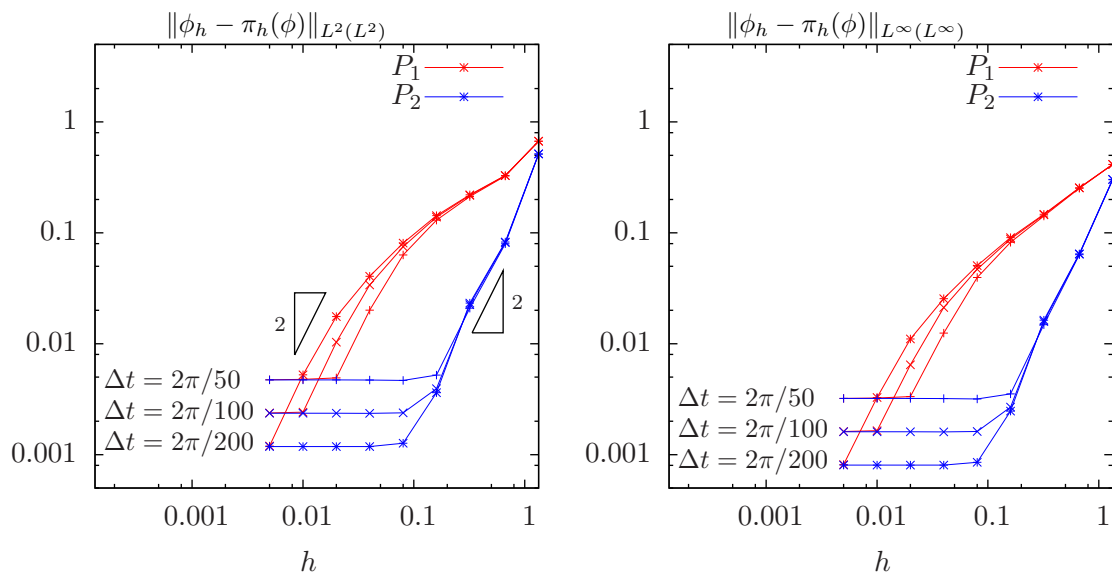


Figure 7.3: Diffusion-convection when  $d = 1$  and  $\nu = 10^{-2}$ : convergence versus  $h$  and  $\Delta t$  for  $P_1$  and  $P_2$  elements: (left) in  $L^2(L^2)$  norm; (right) in  $L^\infty(L^\infty)$  norm.

A  $P_2$  approximation can be used as well:

```
./convect line2.geo P2 > line2.branch
branch line2.branch -gnuplot
./convect_error < line2.branch > line2-cmp.branch
```

On Fig. 7.3.left we observe the  $L^2(L^2)$  convergence versus  $h$  for the  $P_1$  and  $P_2$  elements when  $d = 1$ : the errors reaches a plateau that decreases versus  $\Delta t$ . On Fig. 7.3.right the  $L^\infty(L^\infty)$  norm of the error presents a similar behavior. Since the plateau are equispaced, the convergence versus  $\Delta t$  is of first order.

These computation was performed for a convection-diffusion problem with  $\nu = 10^{-2}$ . The pure transport problem ( $\nu = 0$ , without diffusion) computation is obtained by:

```
./convect line2.geo P1 0 > line2.branch
branch line2.branch -gnuplot
```

Let us turn to the two-dimensional test case:

```

mkgeo_grid -t 80 -a -2 -b 2 -c -2 -d 2 > square2.geo
./convect square2.geo P1 > square2.branch
branch square2.branch -paraview
paraview &

```

The visualization and animation are similar to those of the head problem previously presented in paragraph 7.1. Observe the rotating hill. The result is shown on Fig. 7.2. The error analysis writes:

```

./convect_error < square2.geo > square2-cmp.branch
branch square2-cmp.branch -paraview

```

From the paraview menu, you can visualize simultaneously both the approximate solution and the Lagrange interpolate of the exact one. Finally, the three-dimensional case:

```

mkgeo_grid -T 15 -a -2 -b 2 -c -2 -d 2 -f -2 -g 2 > cube2.geo
./convect cube2.geo P1 > cube2.branch

```

The visualization is similar to the two-dimensional case.

## 7.3 The Navier-Stokes problem

### Formulation

This longer example combines most functionalities presented in the previous examples.

Let us consider the Navier-Stokes problem for the driven cavity in  $\Omega = ]0, 1[^d$ ,  $d = 2, 3$ . Let  $Re > 0$  be the Reynolds number, and  $T > 0$  a final time. The problem writes:

(NS): find  $\mathbf{u} = (u_0, \dots, u_{d-1})$  and  $p$  defined in  $\Omega \times ]0, T[$  such that:

$$\begin{aligned}
Re \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) - \operatorname{div}(2D(\mathbf{u})) + \nabla p &= 0 \text{ in } \Omega \times ]0, T[, \\
-\operatorname{div} \mathbf{u} &= 0 \text{ in } \Omega \times ]0, T[, \\
\mathbf{u}(t=0) &= 0 \text{ in } \Omega \times \{0, T\}, \\
\mathbf{u} &= (1, 0) \text{ on } \Gamma_{\text{top}} \times ]0, T[, \\
\mathbf{u} &= 0 \text{ on } (\Gamma_{\text{left}} \cup \Gamma_{\text{right}} \cup \Gamma_{\text{bottom}}) \times ]0, T[, \\
\frac{\partial u_0}{\partial \mathbf{n}} &= \frac{\partial u_1}{\partial \mathbf{n}} = u_2 = 0 \text{ on } (\Gamma_{\text{back}} \cup \Gamma_{\text{front}}) \times ]0, T[ \text{ when } d = 3,
\end{aligned}$$

where  $D(\mathbf{u}) = (\nabla \mathbf{u} + \nabla \mathbf{u}^T)/2$ . This nonlinear problem is the natural extension of the linear Stokes problem, as presented in paragraph 7.3, page 84. The boundaries are represented on Fig. 5.1, page 44.

### Time approximation

Let  $\Delta t > 0$ . Let us consider the following backward second order scheme, for all  $\phi \in C^2([0, T])$  :

$$\frac{d\phi}{dt}(t) = \frac{3\phi(t) - 4\phi(t - \Delta t) + \phi(t - 2\Delta t)}{2\Delta t} + \mathcal{O}(\Delta t^2)$$

The problem is approximated by the following second-order implicit Euler scheme:

$$\begin{aligned}
Re \frac{3\mathbf{u}^{n+1} - 4\mathbf{u}^n \circ X^n + \mathbf{u}^{n-1} \circ X^{n-1}}{2\Delta t} - \mathbf{div}(2D(\mathbf{u}^{n+1})) + \nabla p^{n+1} &= 0 \text{ in } \Omega, \\
- \mathbf{div} \mathbf{u}^{n+1} &= 0 \text{ in } \Omega, \\
\mathbf{u}^{n+1} &= (1, 0) \text{ on } \Gamma_{\text{top}}, \\
\mathbf{u}^{n+1} &= 0 \text{ on } \Gamma_{\text{left}} \cup \Gamma_{\text{right}} \cup \Gamma_{\text{bottom}}, \\
\frac{\partial u_0^{n+1}}{\partial \mathbf{n}} = \frac{\partial u_1^{n+1}}{\partial \mathbf{n}} = u_2^{n+1} &= 0 \text{ on } \Gamma_{\text{back}} \cup \Gamma_{\text{front}} \text{ when } d = 3,
\end{aligned}$$

where, following [24, 25]:

$$\begin{aligned}
X^n(x) &= x - \Delta t \mathbf{u}^*(x) \\
X^{n-1}(x) &= x - 2\Delta t \mathbf{u}^*(x) \\
\mathbf{u}^* &= 2\mathbf{u}^n - \mathbf{u}^{n-1}
\end{aligned}$$

It is a second order extension of the method previously introduced in paragraph 7.2 page 79. The scheme defines a second order recurrence for the sequence  $(\mathbf{u}^n)_{n \geq -1}$ , that starts with  $\mathbf{u}^{-1} = \mathbf{u}^0 = 0$ .

## Variational formulation

The variational formulation of this problem expresses:

$(NS)_{\Delta t}$ : find  $\mathbf{u}^{n+1} \in \mathbf{V}(1)$  and  $p^{n+1} \in L_0^2(\Omega)$  such that:

$$\begin{aligned}
a(\mathbf{u}^{n+1}, \mathbf{v}) + b(\mathbf{v}, p^{n+1}) &= m(\mathbf{f}^n, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}(0), \\
b(\mathbf{u}^{n+1}, q) &= 0, \quad \forall q \in L_0^2(\Omega),
\end{aligned}$$

where

$$\mathbf{f}^n = \frac{Re}{2\Delta t} (4\mathbf{u}^n \circ X^n - \mathbf{u}^{n-1} \circ X^n)$$

where

$$a(\mathbf{u}, \mathbf{v}) = \frac{3Re}{2\Delta t} \int_{\Omega} \mathbf{u} \cdot \mathbf{v} \, dx + \int_{\Omega} 2D(\mathbf{u}) : D(\mathbf{v}) \, dx$$

and  $b(\cdot, \cdot)$  and  $\mathbf{V}(\alpha)$  was already introduced in paragraph 5.4, page 54, while studying the Stokes problem.

## Space approximation

The Taylor-Hood [12] finite element approximation of this generalised Stokes problem was also considered in paragraph 5.4, page 54. We introduce a mesh  $\mathcal{T}_h$  of  $\Omega$  and the finite dimensional spaces  $\mathbf{X}_h$ ,  $\mathbf{V}_h(\alpha)$  and  $Q_h$ . The approximate problem writes:

$(NS)_{\Delta t, h}$ : find  $\mathbf{u}_h^{n+1} \in \mathbf{V}_h(1)$  and  $p^{n+1} \in Q_h$  such that:

$$\begin{aligned}
a(\mathbf{u}_h^{n+1}, \mathbf{v}) + b(\mathbf{v}, p^{n+1}) &= m(\mathbf{f}_h^n, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}_h(0), \\
b(\mathbf{u}_h^{n+1}, q) &= 0, \quad \forall q \in Q_h.
\end{aligned} \tag{7.1}$$

where

$$\mathbf{f}_h^n = \frac{Re}{2\Delta t} (4\mathbf{u}_h^n \circ X^n - \mathbf{u}_h^{n-1} \circ X^n)$$

The problem reduces to a sequence resolution of a generalized Stokes problems.



## File 'navier\_stokes\_solve.icc'

```

1 using namespace std;
2 int navier_stokes_solve (
3     Float Re, Float delta_t, field l0h, field& uh, field& ph,
4     size_t& max_iter, Float& tol, odistream *p_derr=0) {
5     const space& Xh = uh.get_space();
6     const space& Qh = ph.get_space();
7     string label = "navier-stokes-" + Xh.get_geo().name();
8     quadrature_option_type qopt;
9     qopt.set_family(quadrature_option_type::gauss_lobatto);
10    qopt.set_order(Xh.degree());
11    form m (Xh, Xh, "mass", qopt);
12    form a (Xh, Xh, "2D_D");
13    form mp(Qh, Qh, "mass");
14    a = a + 1.5*(Re/delta_t)*m;
15    solver sa (a.uu());
16    form b (Xh, Qh, "div"); b = -b;
17    solver_abtb stokes (a.uu(), b.uu(), mp.uu());
18    if (p_derr != 0) *p_derr << "[" << label << "]" #n |du/dt| << endl;
19    field uh1 = uh;
20    for (size_t n = 0; true; n++) {
21        field uh2 = uh1;
22        uh1 = uh;
23        field uh_star = 2.0*uh1 - uh2;
24        characteristic X1 (    -delta_t*uh_star);
25        characteristic X2 (-2.0*delta_t*uh_star);
26        field l1h = riesz(Xh, compose(uh1,X1), qopt);
27        field l2h = riesz(Xh, compose(uh2,X2), qopt);
28        field lh = l0h + (Re/delta_t)*(2*l1h - 0.5*l2h);
29        stokes.solve (lh.u() - a.ub()*uh.b(), -(b.ub()*uh.b()),
30                    uh.set_u(), ph.set_u());
31        field duh_dt = (3*uh - 4*uh1 + uh2)/(2*delta_t);
32        Float residual = sqrt(m(duh_dt,duh_dt));
33        if (p_derr != 0) *p_derr << "[" << label << "]" " << n << " " << residual << endl;
34        if (residual < tol) {
35            tol = residual;
36            max_iter = n;
37            return 0;
38        }
39        if (n == max_iter-1) {
40            tol = residual;
41            return 1;
42        }
43    }
44 }

```

## Comments

The `navier_stokes_solve` function is similar to the `'stokes_cavity.cc'`. It solves here a generalised Stokes problem and manages a right-hand side  $f_h$ :

```

characteristic X1 (    -delta_t*uh_star);
characteristic X2 (-2.0*delta_t*uh_star);
field l1h = riesz(Xh, compose(uh1,X1), qopt);
field l2h = riesz(Xh, compose(uh2,X2), qopt);
field lh = l0h + (Re/delta_t)*(2*l1h - 0.5*l2h);

```

This last computation is similar to those done in the `'convect.cc'` example. The generalized Stokes problem is solved by the `solver_abtb` class. The stopping criterion is related to the stationary solution or the maximal iteration number.

## File 'navier\_stokes\_cavity.cc'

```

1 #include "rheolef.h"
2 using namespace rheolef;
3 using namespace std;
4 #include "navier_stokes_solve.icc"
5 #include "navier_stokes_criterion.icc"
6 #include "cavity.icc"
7 int main (int argc, char**argv) {
8     environment rheolef (argc, argv);
9     if (argc < 2) {
10         cerr << "usage: " << argv[0] << " <geo> <Re> <err> <hmin> <delta t> <n_adapt>" << endl;
11         exit (1);
12     }
13     geo omega (argv[1]);
14     adapt_option_type options;
15     Float Re = (argc > 2) ? atof(argv[2]) : 100;
16     options.err = (argc > 3) ? atof(argv[3]) : 1e-2;
17     size_t n_adapt = (argc > 4) ? atoi(argv[4]) : 5;
18     Float delta_t = 0.05;
19     options.hmin = 0.004;
20     options.hmax = 0.1;
21     space Xh = cavity_space (omega, "P2");
22     space Qh (omega, "P1");
23     field uh = cavity_field (Xh, 1.0);
24     field ph (Qh, 0);
25     field fh (Xh, 0);
26     for (size_t i = 0; true; i++) {
27         size_t max_iter = 1000;
28         Float tol = 1e-5;
29         navier_stokes_solve (Re, delta_t, fh, uh, ph, max_iter, tol, &derr);
30         odistream o (omega.name(), "field");
31         o << catchmark("Re") << Re << endl
32           << catchmark("delta_t") << delta_t << endl
33           << catchmark("u") << uh
34           << catchmark("p") << ph;
35         o.close();
36         if (i >= n_adapt) break;
37         field ch = navier_stokes_criterion (Re,uh);
38         omega = adapt (ch, options);
39         o.open (omega.name(), "geo");
40         o << omega;
41         o.close();
42         Xh = cavity_space (omega, "P2");
43         Qh = space (omega, "P1");
44         uh = cavity_field (Xh, 1.0);
45         ph = field (Qh, 0);
46         fh = field (Xh, 0);
47     }
48 }

```

## File 'navier\_stokes\_criterion.icc'

```

1 field navier_stokes_criterion (Float Re, const field& uh) {
2     space Xh (uh.get_geo(), "P1d", "vector");
3     form mpt (uh.get_space(), Xh, "mass");
4     form inv_m (Xh, Xh, "inv_mass");
5     field c1 = norm2(inv_m*(mpt*uh));
6     space Th (uh.get_geo(), "P1d", "tensor");
7     form two_D (uh.get_space(), Th, "2D");
8     form inv_mt (Th, Th, "inv_mass");
9     field two_Duh = inv_mt*(two_D*uh);
10    field c2 = norm2(two_Duh);
11    return sqrt(Re*c1 + c2);
12 }

```

## Comments

The code performs a computation by using adaptive mesh refinement, in order to capture recirculation zones. The `adapt_option_type` declaration is used by `rheolef` to send options to the mesh generator. The code reuse the file `'cavity.icc'` introduced page 55. This file contains two functions that defines boundary conditions associated to the cavity driven problem.

The `criteria` function computes the adaptive mesh refinement criteria:

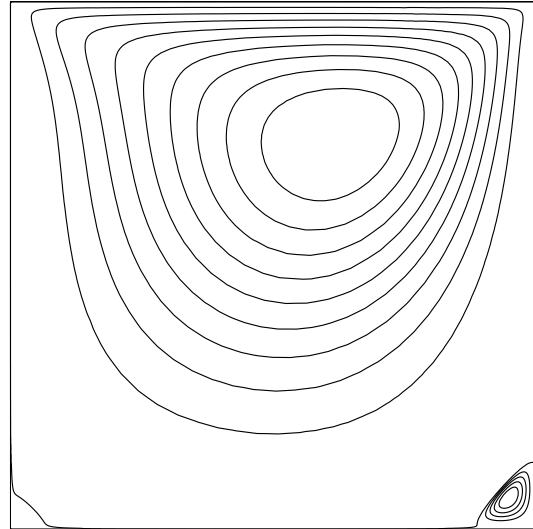
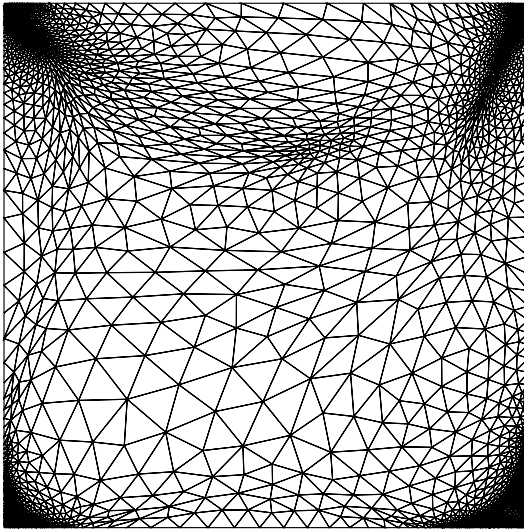
$$c_h = (Re|\mathbf{u}_h|^2 + 2|D(\mathbf{u}_h)|^2)^{1/2}$$

The `criteria` function is similar to those presented in the `'embankment-adapt-2d.cc'` example.

## How to run the program

$Re = 100$ : 4804 elements, 2552 vertices

$\psi_{max} = 9.5 \times 10^{-6}, \psi_{min} = -0.103$



$Re = 400$ : 5233 elements, 2768 vertices

$\psi_{max} = 6.4 \times 10^{-4}, \psi_{min} = -0.111$

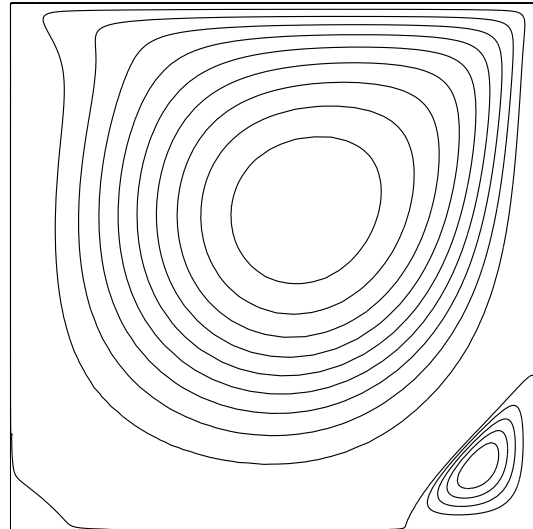
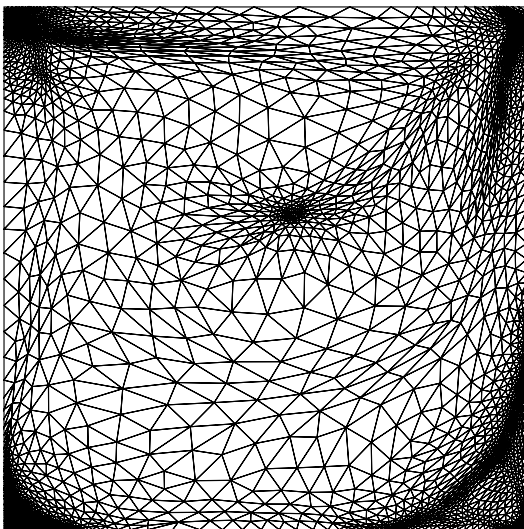


Figure 7.4: Meshes and stream functions associated to the solution of the Navier-Stokes equations for  $Re = 100$  (top) and  $Re = 400$  (bottom).

$Re = 1000$ : 5873 elements, 3106 vertices

$\psi_{max} = 1.64 \times 10^{-3}, \psi_{min} = -0.117$

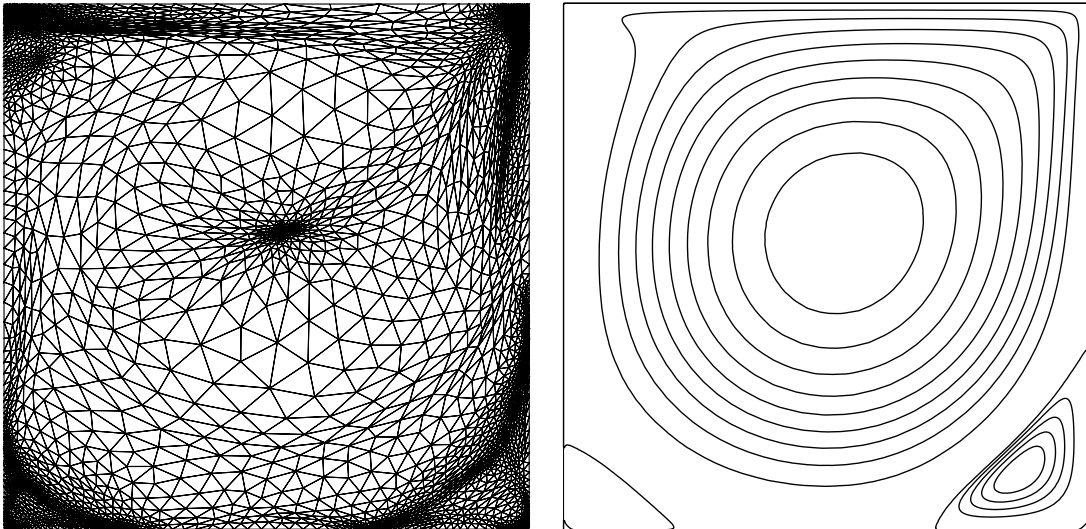


Figure 7.5: Meshes and stream functions associated to the solution of the Navier-Stokes equations for  $Re = 1000$ .

The mesh loop adaptation is initiated from a `bamg` mesh (see also appendix B.1).

```
bamg -g square.bamgcad -o square.bamg
bamg2geo square.bamg square.dmn > square.geo
```

Then, compile and run the Navier-Stokes solver for the driven cavity for  $Re = 100$ :

```
make navier_stokes_cavity
./navier_stokes_cavity square.geo 100
```

The program performs a computation with  $Re = 100$ . By default the time step is  $\Delta t = 0.05$  and the computation loops for five mesh adaptations. At each time step, the program prints an approximation of the time derivative, and stops when a stationary solution is reached. Then, we visualise the ‘square-5’ adapted mesh and its associated solution:

```
geo square-5.geo
field square-5.field.gz -velocity -scale 4 -mayavi
```

Notice the `-scale` option that applies a multiplicative factor to the arrow length when plotting. The representation of the stream function writes:

```
make streamf_cavity
zcat square-5.field.gz | ./streamf_cavity | field -bw -n-iso-negative 10 -
```

The programs `streamf_cavity`, already introduced page 60, is here reused. The last options of the `field` program draws isocontours of the stream function using lines, as shown on Fig. 7.4. The zero isovalue separates the main flow from recirculations, located in corners at the bottom of the cavity.

For  $Re = 400$  and  $1000$  the computation writes:

```
./navier_stokes_cavity square.geo 400
./navier_stokes_cavity square.geo 1000
```

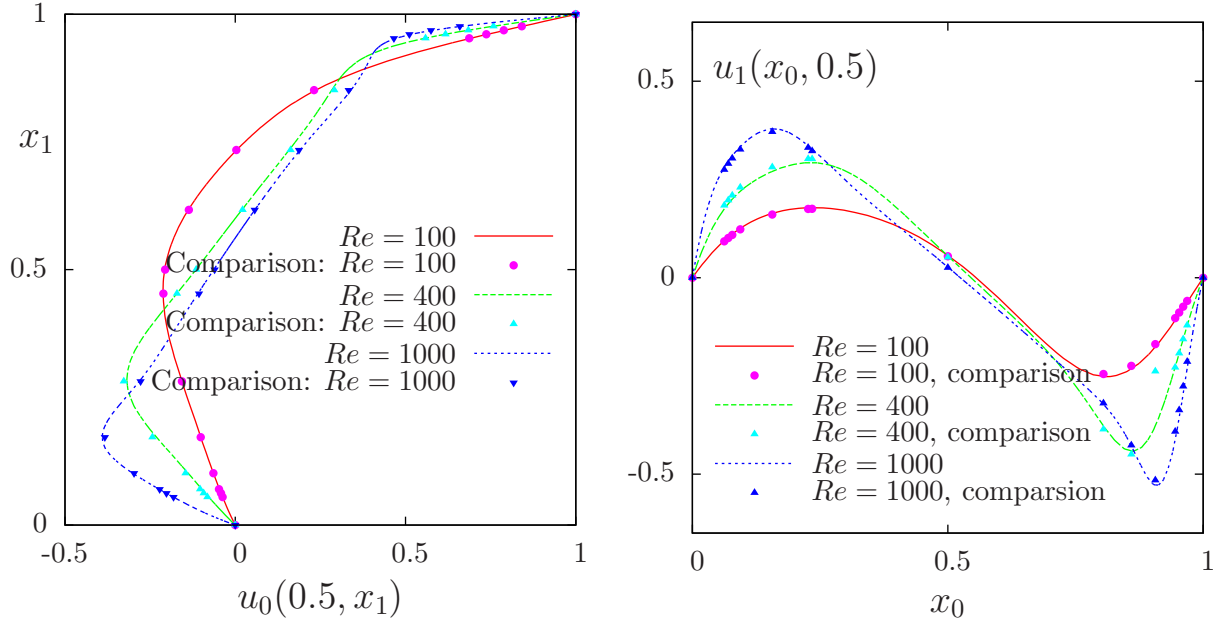


Figure 7.6: Navier-Stokes: velocity profiles along lines passing through the center of the cavity, compared with data from [26]: (a)  $u_0$  along the vertical line; (b)  $u_1$  along the horizontal line.

The visualization of the cut of the horizontal velocity along the vertical median line writes:

```
field square-5.field.gz -comp 0 -cut -normal -1 0 -origin 0.5 0
field square-5.field.gz -comp 1 -cut -normal 0 1 -origin 0 0.5
```

Fig. 7.6 compare the cuts with data from [26], table 1 and 2 (see also [27]). Observe that the solution is in good agreement with these previous computations.

$Re$		$x_c$	$y_c$	$-\psi_{\min}$	$\psi_{\max}$
100	present	0.613	0.738	0.103	$9.5 \times 10^{-6}$
	Labeur and Wells [28]	0.608	0.737	0.104	-
	Donea and Huerta [29]	0.62	0.74	0.103	-
400	present	0.554	0.607	0.111	$5.6 \times 10^{-4}$
	Labeur and Wells [28]	0.557	0.611	0.115	-
	Donea and Huerta [29]	0.568	0.606	0.110	-
1000	present	0.532	0.569	0.117	$1.6 \times 10^{-3}$
	Labeur and Wells [28]	0.524	0.560	0.121	-
	Donea and Huerta [29]	0.540	0.573	0.110	-

Figure 7.7: Cavity flow: primary vortex position and stream function value.

Finally, table 7.7 compares the primary vortex position and its associated stream function value. Notice also the good agreement with previous simulations. The stream function extremal values are obtained by:

```
zcat square-5.field.gz | ./streamf_cavity | field -min -
zcat square-5.field.gz | ./streamf_cavity | field -max -
```

The maximal value has not yet been communicated to our knowledge and is provided in table 7.7 for cross validation purpose. The small program that computes the primary vortex position is showed below.

```
make vortex_position
zcat square-5.field.gz | ./streamf_cavity | ./vortex_position
```

### File ‘vortex\_position.cc’

```

1 #include "rheolef.h"
2 using namespace rheolef;
3 using namespace std;
4 int main (int argc, char** argv) {
5     environment rheolef (argc, argv);
6     check_macro (communicator().size() == 1, "please, use sequentially");
7     field psi_h;
8     din >> psi_h;
9     size_t idof_min = 0;
10    Float psi_min = std::numeric_limits<Float>::max();
11    for (size_t idof = 0, ndof = psi_h.ndof(); idof < ndof; idof++) {
12        if (psi_h.dof(idof) >= psi_min) continue;
13        psi_min = psi_h.dof(idof);
14        idof_min = idof;
15    }
16    const array<point>& xdof = psi_h.get_space().get_xdofs();
17    point xmin = xdof [idof_min];
18    dout << "xc\t\tyc\t\tps" << endl
19         << xmin[0] << "\t" << xmin[1] << "\t" << psi_min << endl;
20 }

```

For higher Reynolds number, Shen [30] showed in 1991 that the flow converges to a stationary state for Reynolds numbers up to 10 000; for Reynolds numbers larger than a critical value  $10\,000 < Re_1 < 10\,500$  and less than another critical value  $15\,000 < Re_2 < 16\,000$ , these authors founded that the flow becomes periodic in time which indicates a Hopf bifurcation; the flow loses time periodicity for  $Re \geq Re_2$ . In 1998, Ould Salihi [31] founded a loss of stationarity between 10 000 and 20 000. In 2002, Auteri et al. [32] estimated the critical value for the apparition of the first instability to  $Re_1 \approx 8018$ . In 2005, Erturk et al. [33] computed steady driven cavity solutions up to  $Re \leq 21\,000$ . Also in 2005, this result was infirmed by [34]: these authors estimated  $Re_1$  close to 8000, in agreement with [32]. The 3D driven cavity has been investigated in [35] by the method of characteristic (see also [36] for 3D driven cavity computations). In conclusion, the exploration of the driven cavity at large Reynolds number is a fundamental challenge in computational fluid dynamics.

## Part III

# Advanced and highly nonlinear problems





## Chapter 8

# The highly nonlinear $p$ -laplacian problem

### 8.1 Problem statement

Let us consider the classical  $p$ -laplacian problem with homogeneous Dirichlet boundary conditions in a domain bounded  $\Omega \subset R^d$ ,  $d = 1, 2, 3$ :

(P): find  $u$ , defined in  $\Omega$  such that:

$$\begin{aligned} -\operatorname{div} (|\nabla u|^{p-2} \nabla u) &= f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{aligned}$$

where  $f$  is known and  $f = 1$  in the computational examples. When  $p = 2$ , this problem reduces to the linear Poisson problem with homogeneous Dirichlet boundary conditions. Otherwise, for any  $p > 1$ , the nonlinear problem is equivalent to the following minimisation problem:

(MP): find  $u \in W_0^{1,p}(\Omega)$  such that:

$$u = \arg \min_{v \in W_0^{1,p}(\Omega)} \frac{1}{p} \int_{\Omega} |\nabla v|^p dx - \int_{\Omega} f v dx,$$

where  $W_0^{1,p}(\Omega)$  denotes the usual Sobolev spaces of functions in  $W^{1,p}(\Omega)$  that vanishes on the boundary [37, p. 118]. The variational formulation of this problem expresses:

(VF): find  $u \in W_0^{1,p}(\Omega)$  such that:

$$a(u; u, v) = m(f, v), \quad \forall v \in W_0^{1,p}(\Omega)$$

where  $a(., .)$  and  $m(., .)$  are defined for any  $u_0, u, v \in W^{1,p}(\Omega)$  by

$$\begin{aligned} a(u_0; u, v) &= \int_{\Omega} |\nabla u_0|^{p-2} \nabla u \cdot \nabla v dx, \quad \forall u, v \in W_0^{1,p}(\Omega) \\ m(u, v) &= \int_{\Omega} u v dx, \quad \forall u, v \in L^2(\Omega) \end{aligned}$$

The  $m(., .)$  is here the classical scalar product on  $L^2(\Omega)$ , and is related to the *mass* form. The quantity  $a(u; u, u) = \|\nabla u\|_{p,\Omega}$  induces a norm in  $W_0^{1,p}$ , equivalent to the standard norm. The form  $a(., ., .)$  is bilinear with respect to the two last variable and is related to the *energy* form.

## 8.2 The fixed-point algorithm

### 8.2.1 Principe of the algorithm

This nonlinear problem is then reduced to a sequence of linear subproblems by using the fixed-point algorithm. The sequence  $(u^{(n)})_{n \geq 0}$  is defined by recurrence as:

- $n = 0$ : let  $u^{(0)} \in W_0^{1,p}(\Omega)$  be known.
- $n \geq 0$ : suppose that  $u^{(n)} \in W_0^{1,p}(\Omega)$  is known and find  $u^{(n+1)} \in W_0^{1,p}(\Omega)$  such that:

$$a(u^{(n)}; u^{(n+1)}, v) = m(f, v), \quad \forall v \in W_0^{1,p}(\Omega)$$

Let  $u^{(n+1)} = G(u^{(n)})$  denotes the operator that solve the previous linear subproblem for a given  $u^{(n)}$ . Since the solution  $u$  satisfies  $u = G(u)$ , it is a fixed-point of  $G$ .

Let us introduce a mesh  $\mathcal{T}_h$  of  $\Omega$  and the finite dimensional space  $X_h$  of continuous piecewise polynomial functions and  $V_h$ , the subspace of  $X_h$  containing elements that vanishes on the boundary of  $\Omega$ :

$$\begin{aligned} X_h &= \{v_h \in C_0^0(\overline{\Omega}); v_{h/K} \in P_k, \forall K \in \mathcal{T}_h\} \\ V_h &= \{v_h \in X_h; v_h = 0 \text{ on } \partial\Omega\} \end{aligned}$$

where  $k = 1$  or  $2$ . The approximate problem expresses: suppose that  $u_h^{(n)} \in V_h$  is known and find  $u_h^{(n+1)} \in V_h$  such that:

$$a(u_h^{(n)}; u_h^{(n+1)}, v_h) = m(f, v_h), \quad \forall v_h \in V_h$$

By developing  $u_h$  on a basis of  $V_h$ , this problem reduces to a linear system. The implementation with **Rheolef**, involving weighted forms, is quite standard: the weight field **wh** is inserted as the last argument to the form constructor. The following code implement this problem in the **Rheolef** environment.

## 8.2.2 File 'p\_laplacian\_fixed\_point.h'

```

1 #include "not_too_small.h"
2 int p_laplacian_fixed_point (
3     Float p,      field lh,
4     field& uh, Float& r, size_t& n)
5 {
6     Float tol = r;
7     Float r0 = 0;
8     size_t max_iter = n;
9     const geo& omega = uh.get_geo();
10    const space& Xh = uh.get_space();
11    string grad_approx = "p" + itos(Xh.degree()-1) + "d";
12    space Th (omega, grad_approx, "vector");
13    form inv_mt (Th, Th, "inv_mass");
14    form grad (Xh, Th, "grad");
15    derr << "# Fixed-point algorithm" << endl
16         << "# p = " << p << endl
17         << "# n r v" << endl;
18    n = 0;
19    do {
20        field grad_uh = inv_mt*(grad*uh);
21        field nh = norm2(grad_uh);
22        if (p/2 -1 <= 0) nh = compose (not_too_small(1e-10), nh);
23        field wh = pow(nh, p/2-1);
24        form a (Xh, Xh, "grad_grad", wh);
25        field mrh = a*uh - lh;
26        r = mrh.u().max_abs();
27        if (n == 0) r0 = r;
28        Float v = (n == 0) ? 0 : log10(r0/r)/n;
29        derr << n << " " << r << " " << v << endl;
30        if (r <= tol || n++ >= max_iter) break;
31        solver sa(a.uu());
32        uh.set_u() = sa.solve (lh.u() - a.ub()*uh.b());
33    } while (true);
34    return (r <= tol) ? 0 : 1;
35 }

```

## 8.2.3 File 'p\_laplacian\_fixed\_point.cc'

```

1 #include "rheolef.h"
2 using namespace rheolef;
3 using namespace std;
4 #include "p_laplacian_fixed_point.icc"
5 #include "dirichlet.icc"
6 int main(int argc, char**argv) {
7     environment rheolef (argc,argv);
8     geo omega (argv[1]);
9     string approx = (argc > 2) ? argv[2] : "P1";
10    Float p = (argc > 3) ? atof(argv[3]) : 1.5;
11    Float tol = (argc > 4) ? atof(argv[4]) : 1e-10;
12    size_t max_iter = 500;
13    derr << "# P-Laplacian problem by fixed-point:" << endl
14         << "# geo = " << omega.name() << endl
15         << "# approx = " << approx << endl
16         << "# p = " << p << endl;
17    space Xh (omega, approx);
18    Xh.block ("boundary");
19    field uh (Xh);
20    uh ["boundary"] = 0;
21    field lh = riesz (Xh, 1);
22    dirichlet (lh, uh);
23    int status = p_laplacian_fixed_point (p, lh, uh, tol, max_iter);
24    dout << catchmark("p") << p << endl
25         << catchmark("u") << uh;
26    return status;
27 }

```

### 8.2.4 File ‘dirichlet.icc’

```

1 void dirichlet (field lh, field& uh) {
2   const space& Xh = uh.get_space();
3   form a (Xh, Xh, "grad_grad");
4   solver sa(a.uu());
5   uh.set_u() = sa.solve (lh.u() - a.ub()*uh.b());
6 }

```

### 8.2.5 Comments

The fixed-point algorithm is amorced with  $u^{(0)}$  as the solution of the linear problem associated to  $p = 2$ , i.e. the standard Poisson problem with Dirichlet boundary conditions. The construction of the weighted form  $a(;\cdot,\cdot)$  writes:

```

field eta_h = pow(sqr(grad_uh[0]) + sqr(grad_uh[1]), p/2.-1);
form a (Vh, Vh, "grad_grad", eta_h);

```

### 8.2.6 Running the program

We assume that the previous code is contained in the file ‘p\_laplacian\_fixed\_point.cc’. Compile the program, as usual:

```
make p_laplacian_fixed_point
```

and enter the comands:

```
mkgeo_grid -t 10 > square.geo
geo square.geo
```

The triangular mesh has a boundary domain named `boundary`.

```
./p_laplacian_fixed_point square.geo P1 1.2 > square.field
```

The previous command solves the problem for the corresponding mesh and writes the solution in the file format ‘.field’.

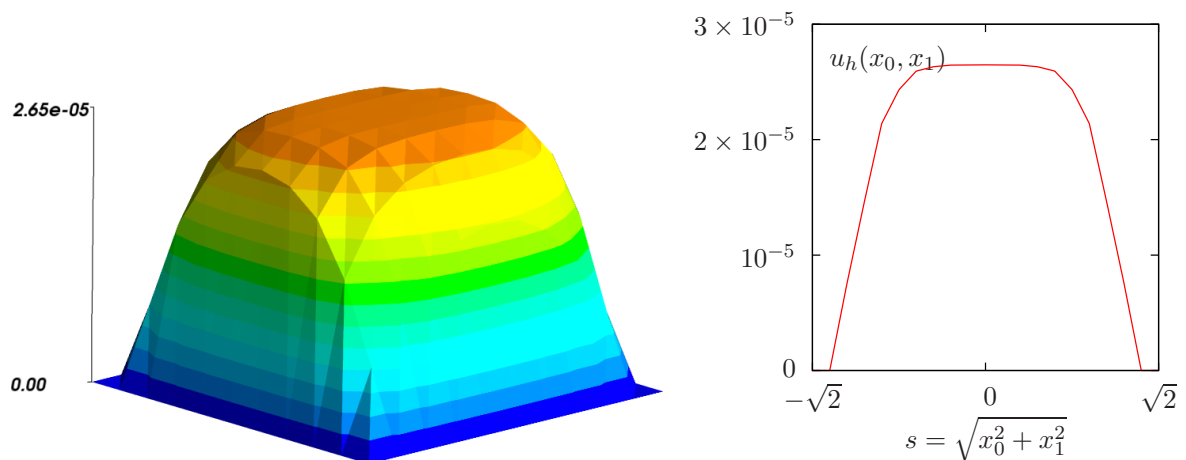


Figure 8.1: The  $p$ -laplacian for  $d = 2$ : (a) elevation view for  $p = 1.2$ ; (b) cut along the first bissectice  $x_0 - x_1 = 0$ .

Run the field visualization:

```
field square.field -elevation
field square.field -cut -origin 0.5 0.5 -normal 1 1
```

The first command shows an elevation view of the solution (see 8.1.a) while the second one shows a cut along the first bissectrice  $x_0 = x_1$ . (see 8.1.b).

### 8.2.7 Convergence properties of the fixed-point algorithm

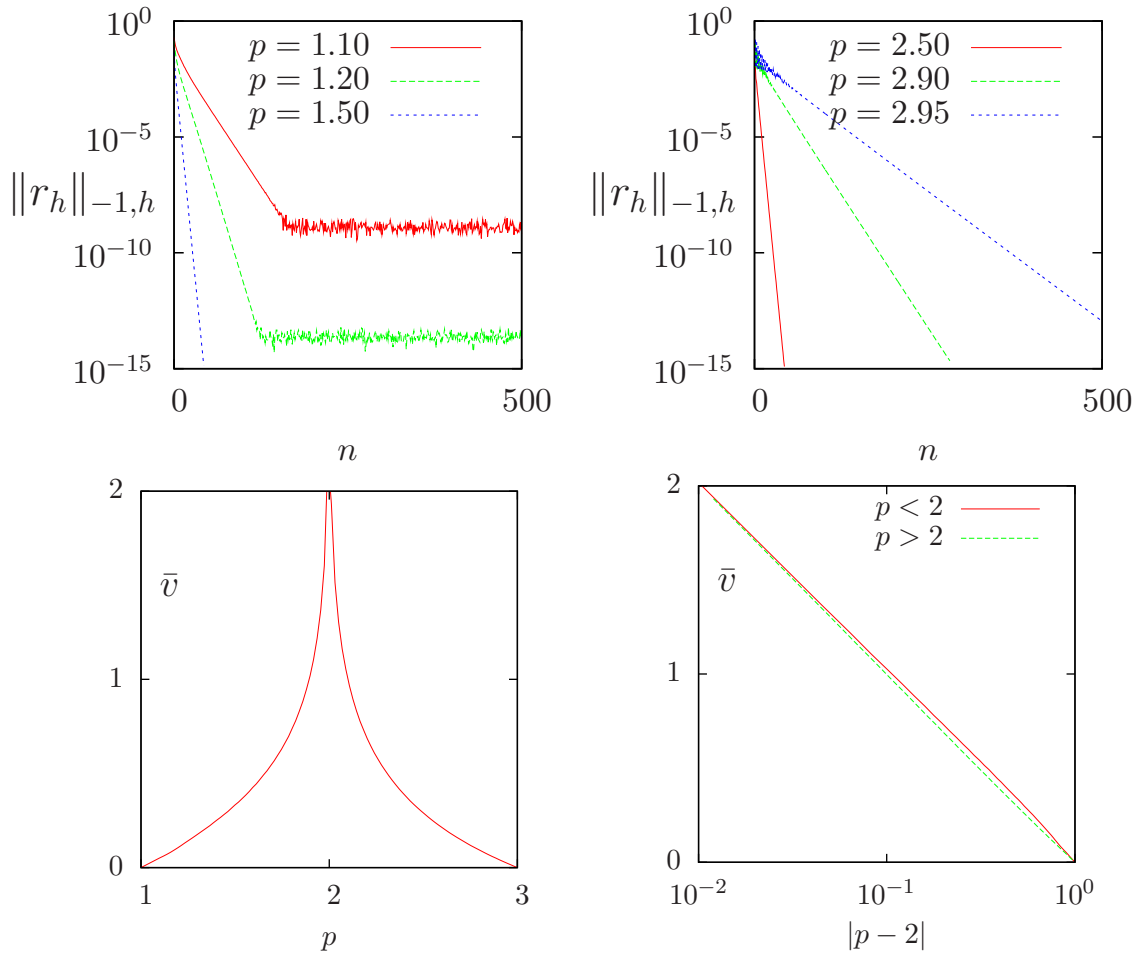


Figure 8.2: The fixed-point algorithm on the  $p$ -laplacian for  $d = 2$ : (a) convergence when  $p < 2$ ; (b) when  $p > 2$ ; (c) convergence rate versus  $p$ ; (d) convergence rate versus  $p$  in semi-log scale.

The fixed-point algorithm prints also at each iteration  $n$ , the residual term  $r_n$  in discrete  $H^{-1}(\Omega)$  and the convergence rate  $v_n = \log_{10}(r_n/r_0)/n$ . The residual term is defined by

$$r_h^{(n)} = A_h(u^{(n)}) - M_h f_h$$

where  $A_h$  and  $M_h$  are the discrete operators induced by the forms  $a(\cdot, \cdot, \cdot)$  and  $m(\cdot, \cdot)$  on  $V_h$ , and defined for all  $u_h, v_h \in V_h$  by:

$$\begin{aligned} A_h(u_h) v_h^T &= a(u_h; u_h, v_h) \\ (M_h u_h) v_h^T &= m(u_h, v_h) \end{aligned}$$

where the elements of  $V_h$  are identified to elements of  $\mathbb{R}^{\dim(V_h)}$ . The  $W^{-1,p}(\Omega)$  norm, defined for all  $r \in W^{-1,p}(\Omega)$  by duality:

$$\|r\|_{-1,p,\Omega} = \sup_{\substack{v \in W^{1,p}(\Omega) \\ \|v\|_{1,p,\Omega}=1}} m(r, v)$$

By analogy, the discrete  $W^{-1,p}(\Omega)$  norm, denoted as  $\|\cdot\|_{-1,h}$ , is defined by duality for all  $r_h \in V_h$  by:

$$\|r_h\|_{-1,h} = \sup_{\substack{v_h \in V_h \\ \|v_h\|_{1,p,\Omega}=1}} m(r_h, v_h) = \sup_{\mathbf{x} \in \text{xdof}(V_h)} |r_h(\mathbf{x})|$$

where  $\text{xdof}(V_h)$  denotes the set of nodes associated to the  $V_h$  degrees of freedom. Since elements of  $V_h$  vanishes on the boundary, the  $\text{xdof}(V_h)$  contains all nodes associated to the degrees of freedoms of  $X_h$  except nodes located on the boundary. Fig 8.2.a and 8.2.b show that the residual term decreases exponentially versus  $n$ , since the slope of the plot in semi-log scale tends to be strait. Thus, the convergence rate  $v_n = \log_{10}(r_n/r_0)/n$  tends to a constant, denoted by  $\bar{v}$ . Fig 8.2.c shows the dependence of  $\bar{v}$ :  $r_n \approx r_0 \times 10^{-\bar{v}n}$ . Observe that  $\bar{v}$  tends to  $+\infty$  when  $p = 2$ , since the system becomes linear and the algorithm converge in one iteration. Observe also that  $\bar{v}$  tends to zero in  $p = 1$  and  $p = 3$ . The singularity in  $p = 1$  is not surprising, since the problem is defined only when  $p > 1$ . Conversely, the singularity in  $p = 3$  is not clear and requires more analysis. Fig 8.2.d shows the same plot in semi-log scale and shows that  $\bar{v}$  behaves as:  $\bar{v} \approx 2 \times \log_{10}(|p - 2|)$ . Finally, this study shows that the residual term behaves as:

$$r_n \approx 2|p - 2| r_0 10^{-n}$$

## 8.3 The Newton algorithm

### 8.3.1 Principe of the algorithm

An alternative to the fixed-point algorithm is to solve the nonlinear problem ( $P$ ) by using the Newton algorithm. Let us consider the following operator:

$$\begin{aligned} F &: W_0^{1,p}(\Omega) &\longrightarrow & W^{-1,p}(\Omega) \\ u &&\longmapsto & F(u) = -\text{div}(|\nabla u|^{p-2} \nabla u) - f \end{aligned}$$

The  $F$  operator computes simply the residual term and the problem expresses now as: find  $u \in W_0^{1,p}(\Omega)$  such that  $F(u) = 0$ .

The Newton algorithm reduces the nonlinear problem into a sequence of linear subproblems: the sequence  $(u^{(n)})_{n \geq 0}$  is classically defined by recurrence as:

- $n = 0$ : let  $u^{(0)} \in W_0^{1,p}(\Omega)$  be known.
- $n \geq 0$ : suppose that  $u^{(n)}$  is known, find  $\delta u^{(n)}$ , defined in  $\Omega$ , such that:

$$F'(u^{(n)}) \delta u^{(n)} = -F(u^{(n)})$$

and then compute explicitly:

$$u^{(n+1)} := u^{(n)} + \delta u^{(n)}$$

The notation  $F'(u)$  stands for the Fréchet derivative of  $F$ , as an operator from  $W^{-1,p}(\Omega)$  into  $W_0^{1,p}(\Omega)$ . For any  $r \in W^{-1,p}(\Omega)$ , the linear tangent problem writes: find  $\delta u \in W_0^{1,p}(\Omega)$  such that:

$$F'(u) \delta u = -r$$

After the computation of the Fréchet derivative, we obtain the strong form of this problem:

(*LT*): find  $\delta u$ , defined in  $\Omega$ , such that

$$\begin{aligned} -\operatorname{div}(|\nabla u|^{p-2}\nabla(\delta u) + (p-2)|\nabla u|^{p-4}\{\nabla u.\nabla(\delta u)\}\nabla u) &= -r \quad \text{in } \Omega \\ \delta u &= 0 \quad \text{on } \partial\Omega \end{aligned}$$

This is a Poisson-like problem with homogeneous Dirichlet boundary conditions and a non-constant tensorial coefficient. The variational form of the linear tangent problem writes:

(*VLT*): find  $\delta u \in W_0^{1,p}(\Omega)$  such that

$$a_1(u; \delta u, \delta v) = m(r, v), \quad \forall \delta v \in W_0^{1,p}(\Omega)$$

where the  $a_1(.,.,.)$  is defined for any  $u, \delta u, \delta v \in W_0^{1,p}(\Omega)$  by:

$$a_1(u; \delta u, \delta v) = \int_{\Omega} (|\nabla u|^{p-2}\nabla(\delta u).\nabla(\delta v) + (p-2)|\nabla u|^{p-4}\{\nabla u.\nabla(\delta u)\}\{\nabla u.\nabla(\delta v)\}) \, dx$$

For any  $\xi \in \mathbb{R}^d$  let us denote by  $\eta(\xi)$  the following  $d \times d$  matrix:

$$\eta(\xi) = |\xi|^{p-2} I + (p-2)|\xi|^{p-4} \xi \otimes \xi$$

where  $I$  stands for the  $d$ -order identity matrix. Then the  $a_1$  expresses in a more compact form:

$$a_1(u; \delta u, \delta v) = \int_{\Omega} (\eta(\nabla u)\nabla(\delta u)).\nabla(\delta v) \, dx$$

Clearly  $a_1$  is linear and symmetric with respect to the two last variables.

### 8.3.2 File ‘p\_laplacian\_newton.cc’

```

1 #include "rheolef.h"
2 #include "rheolef/newton.h"
3 using namespace rheolef;
4 using namespace std;
5 #include "p_laplacian.h"
6 int main(int argc, char**argv) {
7     environment rheolef(argc, argv);
8     geo omega_h(argv[1]);
9     string approx = (argc > 2) ? argv[2] : "P1";
10    Float p = (argc > 3) ? atof(argv[3]) : 1.5;
11    derr << "# P-Laplacian problem by Newton:" << endl
12         << "# geo = " << omega_h.name() << endl
13         << "# approx = " << approx << endl
14         << "# p = " << p << endl;
15    p_laplacian F(p, omega_h, approx);
16    field uh = F.initial();
17    Float tol = 1e6*numeric_limits<Float>::epsilon();
18    size_t max_iter = 500;
19    int status = newton(F, uh, tol, max_iter, &derr);
20    dout << setprecision(numeric_limits<Float>::digits10)
21         << catchmark("p") << p << endl
22         << catchmark("u") << uh;
23    return status;
24 }
```



### 8.3.3 File 'p\_laplacian.h'

```
1 class p_laplacian {
2 public:
3     typedef field value_type;
4     typedef Float float_type;
5     p_laplacian(Float p, const geo& omega_h, string approx = "P1");
6     void reset(const geo& omega_h, string approx = "previous");
7     field initial () const;
8     field residue (const field& uh) const;
9     void update_derivative (const field& uh) const;
10    field derivative_solve (const field& mrh) const;
11    field derivative_trans_mult (const field& mrh) const;
12    Float norm (const field& uh) const;
13    Float dual_norm (const field& Muh) const;
14    Float dot (const field& uh, const field& vh) const;
15    Float dual_dot (const field& Muh, const field& Mvh) const;
16    field criteria(const field& uh) const;
17    Float p;
18    space Xh, Kh;
19    field fh;
20    form m, inv_mt, grad;
21    solver sm;
22    mutable form a1;
23    mutable solver sa1;
24 };
25 #include "p_laplacian1.icc"
26 #include "p_laplacian2.icc"
```

## 8.3.4 File 'p\_laplacian.icc'

```

1 #include "dirichlet.icc"
2 #include "not_too_small.h"
3 p_laplacian::p_laplacian(Float p1, const geo& omega_h, string approx1)
4 : p(p1), Xh(), Kh(), fh(),
5   m(), inv_mt(), grad(), sm(), a1(), sa1() {
6   reset(omega_h, approx1);
7 }
8 void p_laplacian::reset(const geo& omega_h1, string approx1) {
9   if (approx1 == "previous") approx1 = Xh.get_approx();
10  Xh = space(omega_h1, approx1);
11  Xh.block("boundary");
12  fh = field(Xh, 1);
13  m = form(Xh, Xh, "mass");
14  sm = solver(m.uu());
15  string grad_approx = "p" + itos(Xh.degree()-1) + "d";
16  space Th(fh.get_geo(), grad_approx, "vector");
17  inv_mt = form(Th, Th, "inv_mass");
18  grad = form(Xh, Th, "grad");
19  Kh = space(fh.get_geo(), grad_approx, "tensor");
20 }
21 field p_laplacian::initial () const {
22   field uh(Xh);
23   uh[Xh.get_geo()["boundary"]] = 0;
24   dirichlet(fh, uh);
25   return uh;
26 }
27 void p_laplacian::update_derivative (const field& uh) const {
28   field grad_uh = inv_mt*(grad*uh);
29   field norm2_grad_uh = norm2(grad_uh);
30   if (p/2 - 4 <= 0) norm2_grad_uh = compose(not_too_small(1e-10), norm2_grad_uh);
31   field w0h = pow(norm2_grad_uh, p/2)/norm2_grad_uh;
32   field w1h = pow(norm2_grad_uh, p/2)/sqr(norm2_grad_uh);
33   field eta_h(Kh);
34   eta_h(0,0) = w0h + (p-2)*w1h*sqr(grad_uh[0]);
35   size_t d = uh.get_geo().dimension();
36   if (d >= 2) {
37     eta_h(1,1) = w0h + (p-2)*w1h*sqr(grad_uh[1]);
38     eta_h(0,1) = (p-2)*w1h*grad_uh[0]*grad_uh[1];
39   }
40   if (d == 3) {
41     eta_h(2,2) = w0h + (p-2)*w1h*sqr(grad_uh[2]);
42     eta_h(1,2) = (p-2)*w1h*grad_uh[1]*grad_uh[2];
43     eta_h(0,2) = (p-2)*w1h*grad_uh[0]*grad_uh[2];
44   }
45   a1 = form(Xh, Xh, "grad_grad", eta_h);
46   sa1 = solver(a1.uu());
47 }
48 field p_laplacian::residue (const field& uh) const {
49   field grad_uh = inv_mt*(grad*uh);
50   field norm2_grad_uh = norm2(grad_uh);
51   field w0h = pow(norm2_grad_uh, p/2-1);
52   form a(Xh, Xh, "grad_grad", w0h);
53   field mrh = a*uh - m*fh;
54   mrh.set_b() = 0;
55   return mrh;
56 }
57 field p_laplacian::derivative_solve (const field& mrh) const {
58   field delta_uh(Xh,0);
59   delta_uh.set_b() = 0;
60   delta_uh.set_u() = sa1.solve(mrh.u());
61   return delta_uh;
62 }

```

```

1  field p_laplacian::derivative_trans_mult (const field& mrh) const {
2      field rh (Xh);
3      rh.set_b() = 0;
4      rh.set_u() = sm.solve(mrh.u());
5      field mgh;
6      mgh = a1*rh;
7      mgh.set_b() = 0;
8      return mgh;
9  }
10 Float p_laplacian::dot (const field& uh, const field& vh) const {
11     return m(uh,vh);
12 }
13 Float p_laplacian::norm (const field& uh) const {
14     return sqrt(m(uh,uh));
15 }
16 Float p_laplacian::dual_dot (const field& mrh, const field& msh) const {
17     field sh (Xh);
18     sh.set_b() = 0;
19     sh.set_u() = sm.solve(msh.u());
20     return ::dot(mrh,sh);
21 }
22 Float p_laplacian::dual_norm (const field& mrh) const {
23     return sqrt(dual_dot(mrh,mrh));
24 }
25 field p_laplacian::criteria(const field& uh) const {
26     if (uh.get_approx() == "P1") return abs(uh);
27     field grad_uh = inv_mt*(grad*uh);
28     field norm2_grad_uh = norm2(grad_uh);
29     return pow(norm2_grad_uh, p/4);
30 }

```

### 8.3.5 Comments

The code implements a generic Newton algorithm in the file ‘newton.h’. The main program is ‘p\_laplacian\_newton.cc’, that uses a class `p_laplacian`. This class interface is defined in the file ‘p\_laplacian.h’ and its implementation in ‘p\_laplacian.icc’. The residual term  $F(u_h)$  is computed by the member function `residual` while the resolution of  $F'(u_h)\delta u_h = Mr_h$  is performed by the function `derivative_solve`. The derivative  $F'(u_h)$  is computed separately by the function `update_derivative`. Notice that the  $a_1(u; \cdot, \cdot)$  bilinear form is a tensorial weighted form, where  $\eta(\nabla u)$  is the weight tensor. In **Rheolef**, the tensorial weight field `eta_h` is inserted as an usual scalar weight, by passing the weight parameter as the last argument to the form constructor. The introduction of the class ‘p\_laplacian’ allows an easiest implementation of several variants of the Newton algorithm.

### 8.3.6 Running the program

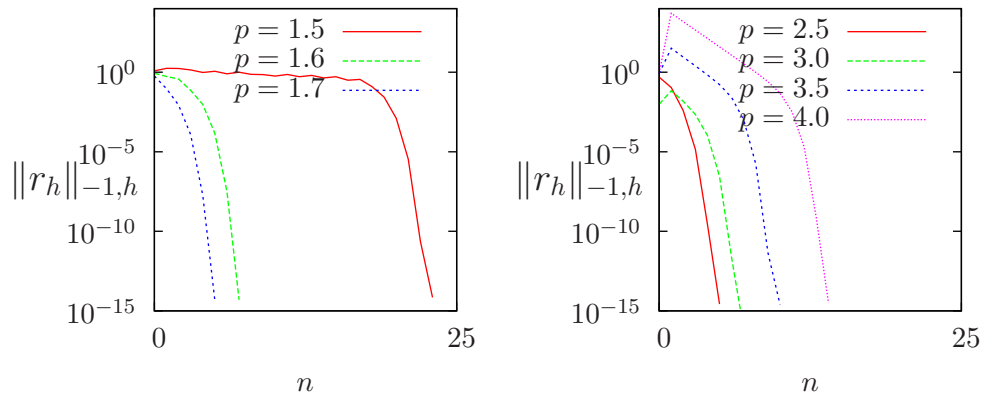


Figure 8.3: The Newton algorithm on the  $p$ -laplacian for  $d = 2$ : (a) convergence when  $p < 2$ ; (b) when  $p > 2$ .

We assume that the previous code is contained in the file ‘p-laplacian-newton.cc’. As usual, enter:

```
make p_laplacian_newton
mkgeo_grid -t 10 > square.geo
./p_laplacian_newton square.geo P1 1.5 | field -mayavi -elevation -
```

The program prints at each iteration  $n$ , the residual term  $r_n$  in discrete  $L^2(\Omega)$  norm. Fig. 8.3.a and. 8.3.b shows that the residual terms tends very fast to zero. Observe that the slope is no more constant in semi-log scale: the convergence rate accelerates and the slope tends to be vertical, the so-called super-linear convergence. This is the major advantage of the Newton method. Also the algorithm converge when  $p \geq 3$ , until  $p \approx 4$ . It was not the case with the fixed point algorithm that diverges in that case. Finally, the Newton algorithm diverges for small values of  $p$ , e.g.  $p < 1.5$  and the plot is not showed here. Conversely, when  $p > 4$ , the first iterations increases dramatically the residual terms, before to decrease. In that case, another strategy should be considered: the damped Newton algorithm. This is the subject of the next section.

## 8.4 The damped Newton algorithm

### 8.4.1 Principle of the algorithm

The Newton algorithm diverges when the initial  $u^{(0)}$  is too far from a solution. Our aim is to modify the Newton algorithm and to obtain a *globally convergent algorithm*, i.e to converge to a solution for any initial  $u^{(0)}$ . The basic idea is to decrease the step length while maintaining the direction of the original Newton algorithm:

$$u^{(n+1)} := u^{(n)} + \lambda_n \delta u^{(n)}$$

where  $\lambda^{(n)} \in ]0, 1]$  and  $\delta u^{(n)}$  is the direction from the Newton algorithm, given by:

$$F' \left( u^{(n)} \right) \delta u^{(n)} = -F \left( u^{(n)} \right)$$

Let  $V$  a Banach space and let  $T : V \rightarrow \mathbb{R}$  defined for any  $v \in V$  by:

$$T(v) = \frac{1}{2} \|C^{-1}F(v)\|_V^2,$$

where  $C$  is some non-singular operator, easy to invert, used as a non-linear preconditioner. The simplest case, without preconditioner, is  $C = I$ . The  $T$  function furnishes a measure of the residual term in  $L^2$  norm. The convergence is global when for any initial  $u^{(0)}$ , we have for any  $n \geq 0$ :

$$T\left(u^{(n+1)}\right) \leq T\left(u^{(n)}\right) + \alpha \left\langle T'\left(u^{(n)}\right), u^{(n+1)} - u^{(n)} \right\rangle_{V',V} \quad (8.1)$$

where  $\langle \cdot, \cdot \rangle_{V',V}$  is the duality product between  $V$  and its dual  $V'$ , and  $\alpha \in ]0, 1[$  is a small parameter. Notice that

$$T'(u) = \{C^{-1}F'(u)\}^* C^{-1}F(u)$$

where the superscript  $*$  denotes the adjoint operator, i.e. the transpose matrix the in finite dimensional case. In practice we consider  $\alpha = 10^{-4}$  and we also use a minimal step length  $\lambda_{\min} = 1/10$  in order to avoid too small steps. Let us consider a fixed step  $n \geq 0$ : for convenience the  $n$  superscript is dropped in  $u^{(n)}$  and  $\delta u^{(n)}$ . Let  $g : \mathbb{R} \rightarrow \mathbb{R}$  defined for any  $\lambda \in \mathbb{R}$  by:

$$g(\lambda) = T(u + \lambda \delta u)$$

Then :

$$\begin{aligned} g'(\lambda) &= \langle T'(u + \lambda \delta u), \delta u \rangle_{V',V} \\ &= \langle C^{-1}F(u + \lambda \delta u), F'(u + \lambda \delta u)C^{-1}\delta u \rangle_{V,V'} \end{aligned}$$

where the superscript  $*$  denotes the adjoint operator, i.e. the transpose matrix the in finite dimensional case. The practical algorithm for obtaining  $\lambda$  was introduced first in [38] and is also presented in [39, p. 385]. The step length  $\lambda$  that satisfy (8.1) is computed by using a finite sequence  $\lambda_k$ ,  $k = 0, 1 \dots$  with a second order recurrence:

- $k = 0$  : initialisation  $\lambda_0 = 1$ . If (8.1) is satisfied with  $u + \lambda_0 d$  then let  $\lambda := \lambda_0$  and the sequence stop here.
- $k = 1$  : first order recursion. The quantities  $g(0) = f(u)$  et  $g'(0) = \langle f'(u), d \rangle$  are already computed at initialisation. Also, we already have computed  $g(1) = f(u + d)$  when verifying whether (8.1) was satisfied. Thus, we consider the following approximation of  $g(\lambda)$  by a second order polynomial:

$$\tilde{g}_1(\lambda) = \{g(1) - g(0) - g'(0)\}\lambda^2 + g'(0)\lambda + g(0)$$

After a short computation, we find that the minimum of this polynomial is:

$$\tilde{\lambda}_1 = \frac{-g'(0)}{2\{g(1) - g(0) - g'(0)\}}$$

Since the initialisation at  $k = 0$  does not satisfy (8.1), it is possible to show that, when  $\alpha$  is small enough, we have  $\tilde{\lambda}_1 \leq 1/2$  and  $\tilde{\lambda}_1 \approx 1/2$ . Let  $\lambda_1 := \max(\lambda_{\min}, \tilde{\lambda}_1)$ . If (8.1) is satisfied with  $u + \lambda_1 d$  then let  $\lambda := \lambda_1$  and the sequence stop here.

- $k \geq 2$  : second order recurrence. The quantities  $g(0) = f(u)$  et  $g'(0) = \langle f'(u), d \rangle$  are available, together with  $\lambda_{k-1}$ ,  $g(\lambda_{k-1})$ ,  $\lambda_{k-2}$  and  $g(\lambda_{k-2})$ . Then,  $g(\lambda)$  is approximated by the following third order polynomial:

$$\tilde{g}_k(\lambda) = a\lambda^3 + b\lambda^2 + g'(0)\lambda + g(0)$$

where  $a$  et  $b$  are expressed by:

$$\begin{pmatrix} a \\ b \end{pmatrix} = \frac{1}{\lambda_{k-1} - \lambda_{k-2}} \begin{pmatrix} \frac{1}{\lambda_{k-1}^2} & -\frac{1}{\lambda_{k-2}^2} \\ -\frac{\lambda_{k-2}}{\lambda_{k-1}^2} & \frac{\lambda_{k-1}}{\lambda_{k-2}^2} \end{pmatrix} \begin{pmatrix} g(\lambda_{k-1}) - g'(0)\lambda_{k-1} - g(0) \\ g(\lambda_{k-2}) - g'(0)\lambda_{k-2} - g(0) \end{pmatrix}$$

The minimum of  $\tilde{g}_k(\lambda)$  is

$$\tilde{\lambda}_k = \frac{-b + \sqrt{b^2 - 3ag'(0)}}{3a}$$

Let  $\lambda_k = \min(1/2 \lambda_k, \max(\tilde{\lambda}_k/10, \tilde{\lambda}_{k+1}))$  in order for  $\lambda_k$  to be at the same order of magnitude as  $\lambda_{k-1}$ . If (8.1) is satisfied with  $u + \lambda_k d$  then let  $\lambda := \lambda_k$  and the sequence stop here.

The sequence  $(\lambda_k)_{k \geq 0}$  is strictly decreasing: when the stopping criteria is not satisfied until  $\lambda_k$  reaches the machine precision  $\varepsilon_{\text{mach}}$  then the algorithm stops with an error.

### 8.4.2 File ‘p\_laplacian\_damped\_newton.cc’

```

1 #include "rheolef.h"
2 #include "rheolef/damped-newton.h"
3 using namespace rheolef;
4 using namespace std;
5 #include "p_laplacian.h"
6 int main(int argc, char**argv) {
7     environment rheolef (argc,argv);
8     geo omega_h (argv[1]);
9     string approx = (argc > 2) ? argv[2] : "P1";
10    Float p = (argc > 3) ? atof(argv[3]) : 1.5;
11    derr << "# P-Laplacian problem by damped Newton:" << endl
12         << "# geo = " << omega_h.name() << endl
13         << "# approx = " << approx << endl
14         << "# p = " << p << endl;
15    p_laplacian F (p, omega_h, approx);
16    field uh = F.initial();
17    Float tol = numeric_limits<Float>::epsilon();
18    size_t max_iter = 500;
19    int status = damped_newton (F, uh, tol, max_iter, &derr);
20    dout << catchmark("p") << p << endl
21         << catchmark("u") << uh;
22    return status;
23 }

```

### 8.4.3 Comments

The file `damped-newton-generic.h` implements the damped Newton algorithm for a generic  $T(u)$  function, i.e. a generic nonlinear preconditioner. This algorithms use a backtrack strategy implemented in file ‘`newton-backtrack.h`’. The simplest choice of the identity preconditioner  $C = I$  i.e.  $T(u) = \|F(u)\|_{V'}^2/2$  is showed in file `damped-newton.h`. The gradient at  $\lambda = 0$  is

$$T'(u) = F'(u)^* F(u)$$

and the slope at  $\lambda = 0$  is:

$$\begin{aligned} g'(0) &= \langle T'(u), \delta u \rangle_{V',V} \\ &= \langle F(u), F'(u)\delta u \rangle_{V',V'} \\ &= -\|F(u)\|_{V'}^2, \end{aligned}$$

The ‘`p_laplacian_damped_newton.cc`’ is the application program to the  $p$ -Laplacian problem together with the  $\|\cdot\|_{L^2(\Omega)}$  discrete norm for the function  $T$ .

### 8.4.4 Running the program

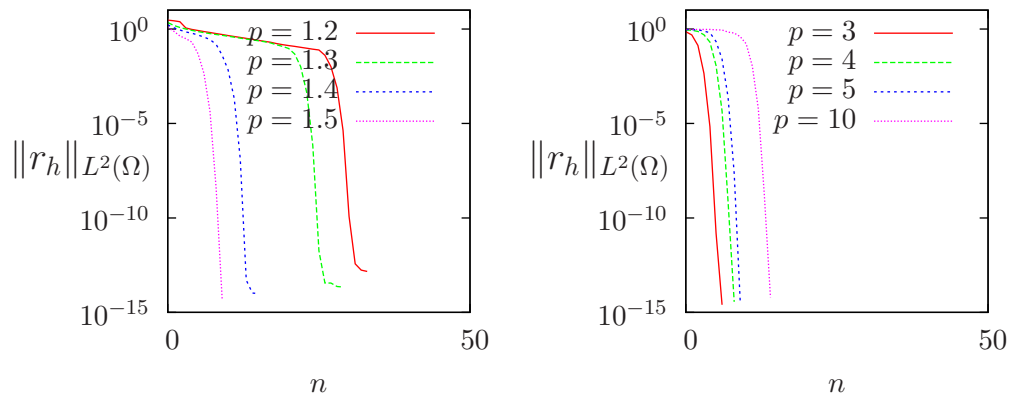


Figure 8.4: The damped Newton algorithm on the  $p$ -laplacian for  $d = 2$ : (a) convergence when  $p < 2$ ; (b) when  $p > 2$ .

We assume that the previous code is contained in the file ‘`p_laplacian_damped_newton.cc`’. As usual, enter:

```
make p_laplacian_damped_newton
mkgeo_grid -t 10 > square.geo
./p_laplacian_damped_newton square.geo P1 1.5 | field -
./p_laplacian_damped_newton square.geo P1 5.0 | field -
```

The algorithm is now quite robust: the convergence occurs for a large range of  $p > 1$  values, and has been pushed until  $p = 100$ . The only limitation is due to machine roundoff when  $p = 1.1$ : the residual term reaches only  $10^{-10}$  instead of  $10^{-15}$ .

## 8.4.5 Robustness and mesh invariance

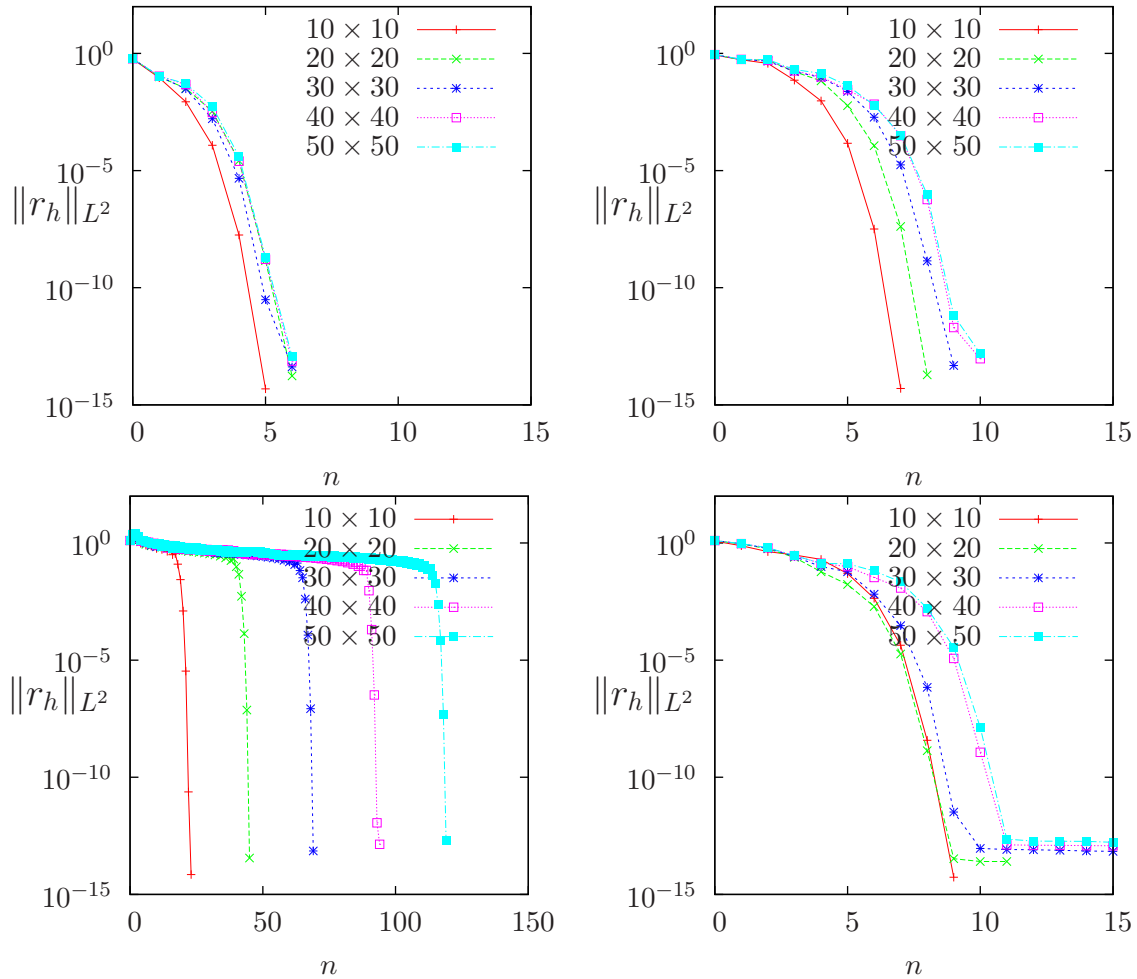


Figure 8.5: Convergence versus  $n$  for various meshes: (a) Newton algorithm when  $p = 1.7$  and (b)  $p = 1.6$ ; (c) Newton algorithm when  $p = 1.5$ ; (d) damped-Newton algorithm when  $p = 1.5$ .

Fig. 8.5.a, 8.5.b and 8.5.c show the convergence of the Newton method when  $p = 1.7$ , 1.6 and 1.5, respectively. Observe that the convergence is asymptotically invariant of the mesh when the element size decreases. The convergence is more difficult when  $p$  decreases to 1.5 and the Newton algorithm is no more mesh invariant. Fig. 8.5.d shows the convergence of the damped Newton method when  $p = 1.5$ : the convergence is now very fast and also mesh-invariant.





## Chapter 9

# Error analysis for $P_k$ isoparametric elements

### 9.1 Error analysis

#### Principle

Let us go back to the non-homogeneous Dirichlet problem, as presented in section 2.1, page 24. Since the solution  $u$  is regular, the following error estimates holds:

$$\begin{aligned}\|u - u_h\|_{0,2,\Omega} &\approx \mathcal{O}(h^{k+1}) \\ \|u - u_h\|_{0,\infty,\Omega} &\approx \mathcal{O}(h^{k+1}) \\ \|u - u_h\|_{1,2,\Omega} &\approx \mathcal{O}(h^k)\end{aligned}$$

providing the approximate solution  $u_h$  uses  $P_k$  continuous finite element method,  $k \geq 1$ . Here,  $\|\cdot\|_{0,2,\Omega}$ ,  $\|\cdot\|_{0,\infty,\Omega}$  and  $\|\cdot\|_{1,2,\Omega}$  denotes as usual the  $L^2(\Omega)$ ,  $L^\infty(\Omega)$  and  $H^1(\Omega)$  norms.

By denoting  $\pi_h$  the Lagrange interpolation operator, the triangular inequality leads to:

$$\|u - u_h\|_{0,2,\Omega} \leq \|(I - \pi_h)(u)\|_{0,2,\Omega} + \|u_h - \pi_h u\|_{0,2,\Omega}$$

From the fundamental properties of the Laplace interpolation  $\pi_h$ , and since  $u$  is smooth enough, we have  $\|(I - \pi_h)(u)\|_{0,2,\Omega} \approx \mathcal{O}(h^{k+1})$ . Thus, we have just to check the  $\|u_h - \pi_h u\|_{0,2,\Omega}$  term. The following code implement the computation of the error.

## File ‘cosinusprod\_error.cc’

```

1 #include "rheolef.h"
2 using namespace rheolef;
3 using namespace std;
4 #include "cosinusprod.icc"
5 int main(int argc, char**argv) {
6     environment rheolef(argc, argv);
7     Float error_linf_expected = (argc > 1) ? atof(argv[1]) : 1e+38;
8     field uh; din >> uh;
9     space Xh = uh.get_space();
10    size_t d = Xh.get_geo().dimension();
11    field pi_h_u = interpolate(Xh, u(d));
12    field eh = uh - pi_h_u;
13    form m(Xh, Xh, "mass");
14    form a(Xh, Xh, "grad_grad");
15    dout << "error_l2" << sqrt(m(eh,eh)) << endl
16         << "error_linf" << eh.max_abs() << endl
17         << "error_h1" << sqrt(a(eh,eh)) << endl;
18    return (eh.max_abs() <= error_linf_expected) ? 0 : 1;
19 }

```

## File ‘cosinusprod.icc’

```

1 struct u : std::unary_function<point,Float> {
2     Float operator() (const point& x) const {
3         return cos(pi*x[0])*cos(pi*x[1])*cos(pi*x[2]); }
4     u(size_t d1) : d(d1), pi(acos(Float(-1.0))) {}
5     size_t d; Float pi;
6 };

```

## Running the program

```
make dirichlet-nh cosinusprod_error
```

After compilation, run the code by using the command:

```
mkgeo_grid -t 10 > square.geo
./dirichlet-nh square.geo P1 | ./cosinusprod_error
```

The three  $L^2$ ,  $L^\infty$  and  $H^1$  errors are printed for a  $h = 1/10$  uniform mesh. Note that an unstructured quasi-uniform mesh can be simply generated by using the `mkgeo_ugrid` command:

```
mkgeo_ugrid -t 10 > square.geo
geo square.geo
```

Let  $n_{\text{el}}$  denotes the number of elements in the mesh. Since the mesh is quasi-uniform, we have  $h \approx n_{\text{el}}^{-\frac{1}{d}}$  where  $d$  is the physical space dimension. Here  $d = 2$  for our bidimensionnal mesh. Figure 9.1 plots in logarithmic scale the error versus  $n_{\text{el}}^{-\frac{1}{2}}$  for both  $P_k$  approximations,  $k = 1, 2, 3$  and the various norms. Observe that the error behaves as predicted by the theory.

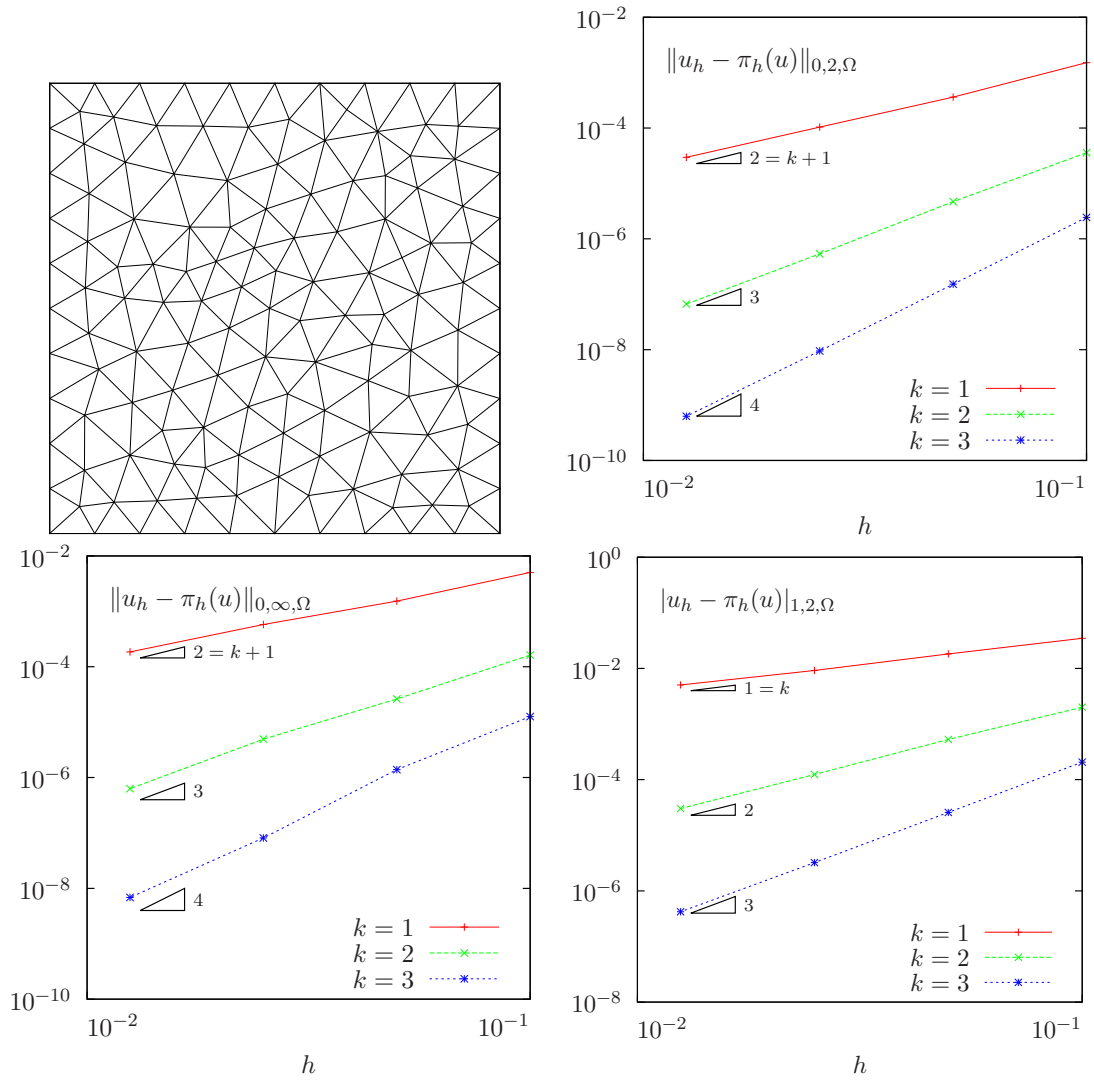
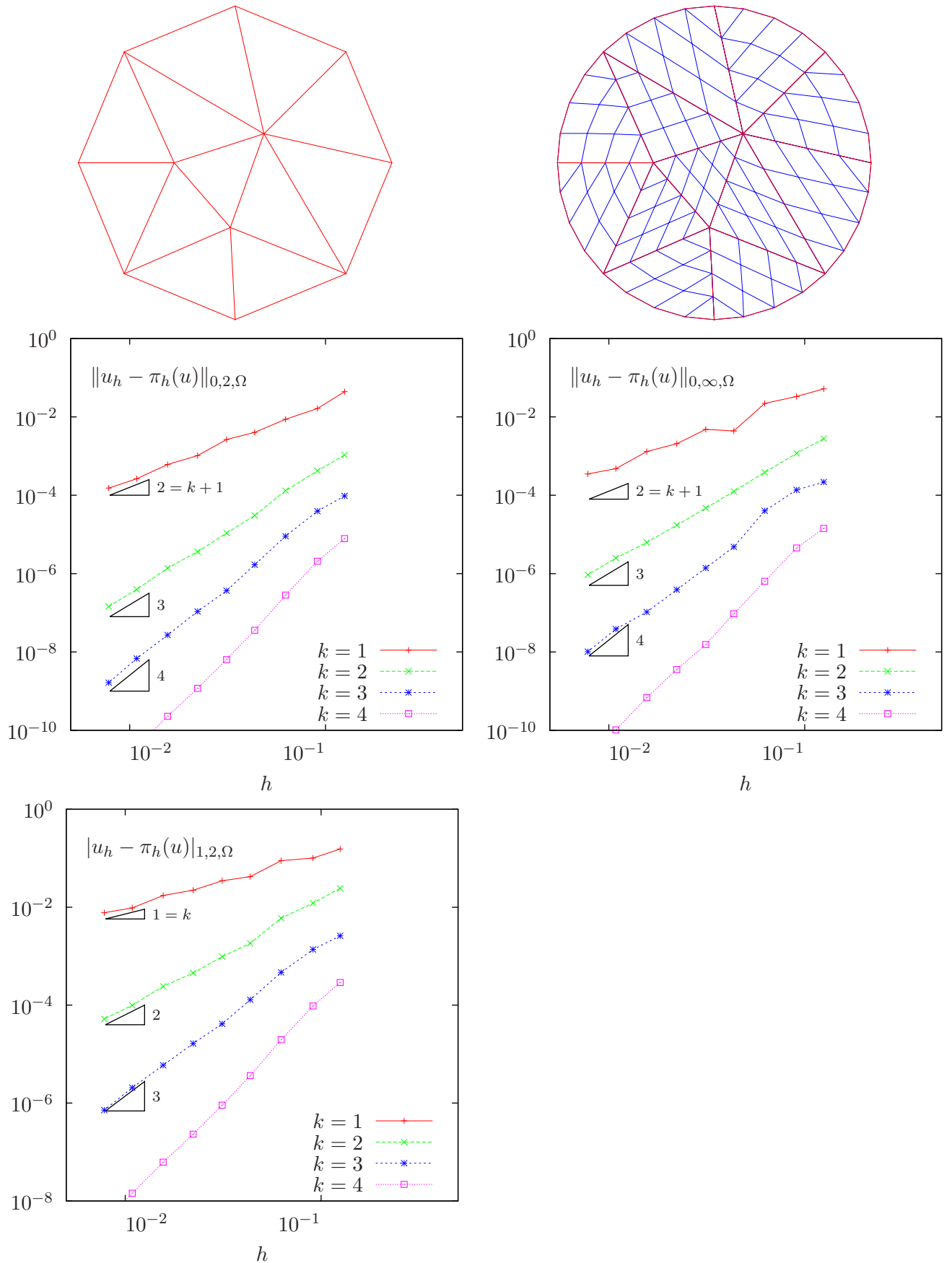


Figure 9.1: Strait geometry: error analysis in  $L^2$ ,  $L^\infty$  and  $H^1$  norms.

Figure 9.2: Curved domains (triangles): error analysis in  $L^2$ ,  $L^\infty$  and  $H^1$  norms.

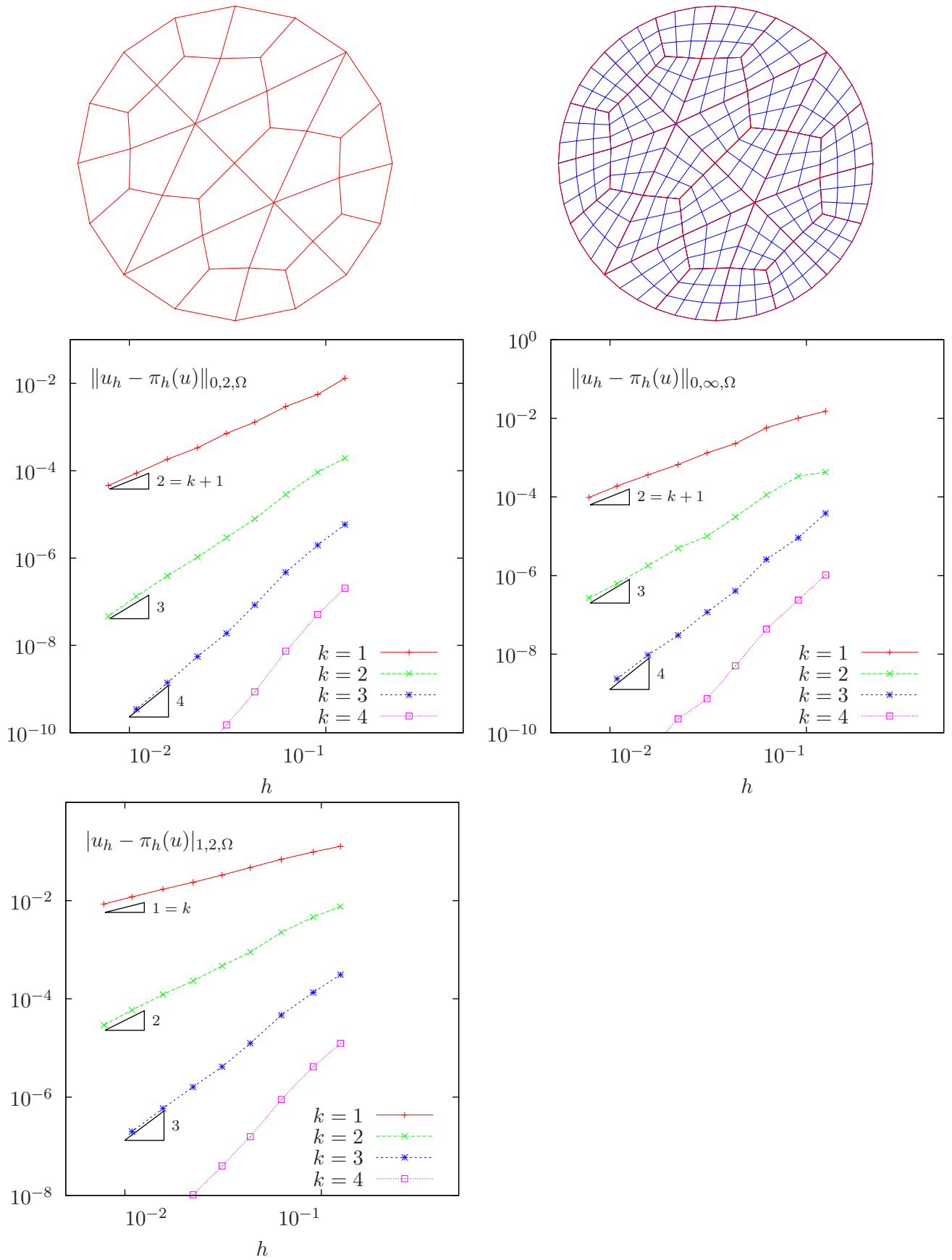


Figure 9.3: Curved domains (quadrangles): error analysis in  $L^2$ ,  $L^\infty$  and  $H^1$  norms.

## Curved domains

### File ‘cosinusrad\_laplace.icc’

```

1 struct f : unary_function<point,Float> {
2   Float operator() (const point& x) const {
3     Float r = sqrt(sqr(x[0])+sqr(x[1])+sqr(x[2]));
4     Float sin_over_ar = (r == 0) ? 1 : sin(a*r)/(a*r);
5     return sqr(a)*((d-1)*sin_over_ar + cos(a*r)); }
6   f(size_t d1) : d(d1), a(acos(Float(-1.0))) {}
7   size_t d; Float a;
8 };
9 struct g : std::unary_function<point,Float> {
10  Float operator() (const point& x) const {
11    return cos(a*sqrt(sqr(x[0])+sqr(x[1])+sqr(x[2]))); }
12  g(size_t=0) : a(acos(Float(-1.0))) {}
13  Float a;
14 };

```

### File ‘cosinusrad.icc’

```

1 struct u : std::unary_function<point,Float> {
2   Float operator() (const point& x) const {
3     Float r = sqrt(sqr(x[0])+sqr(x[1])+sqr(x[2]));
4     return cos(a*r); }
5   u(size_t=0) : a(acos(Float(-1.0))) {}
6   Float a;
7 };

```

```

sed -e 's/sinusprod/sinusrad/' < dirichlet-nh.cc > dirichlet_nh_ball.cc
sed -e 's/sinusprod/sinusrad/' < cosinusprod_error.cc > cosinusrad_error.cc
make dirichlet_nh_ball cosinusrad_error
mkgeo_ball -order 1 -t 10 > circle-P1.geo
geo circle-P1
./dirichlet_nh_ball circle-P1.geo P1 | ./cosinusrad_error
mkgeo_ball -order 3 -t 10 > circle-P3.geo
geo circle-P3
./dirichlet_nh_ball circle-P3.geo P3 | ./cosinusrad_error

```

Observe Fig. 9.2: for meshes based on triangles: the error behave as expected when  $k = 1, 2, 3, 4$ . A similar result occurs for quadrangles, as shown on Fig. 9.3.

```

mkgeo_ball -order 3 -q 10 > circle-q-P3.geo
geo circle-q-P3
./dirichlet_nh_ball circle-q-P3.geo P3 | ./cosinusrad_error

```

These features are curenly in development for arbitrarily  $P_k$  high order approximations and three-dimensional geometries.

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## Part IV

# Technical appendices



# Appendix A

## How to write a variational formulation ?

The major keypoint for using **Rheolef** is to put the problem in variational form. Then this variational form can be efficiently translated into **C++** language. This appendix is dedicated to readers who are not fluent with variational formulations and some related functional analysis tools.

### A.1 The Green formula

Let us come back to the model problem presented in section 1.1, page 13, equations (1.1)-(1.2) and details how this problem is transformed into (1.3).

Let  $H_0^1(\Omega)$  the space of functions whose gradient square has a finite sum over  $\Omega$  and that vanishes on  $\partial\Omega$ :

$$H_0^1(\Omega) = \{v \in L^2(\Omega); \nabla v \in L^2(\Omega)^d \text{ and } v = 0 \text{ on } \partial\Omega\}$$

We start by multiplying (1.1) by an arbitrarily test-function  $v \in H_0^1(\Omega)$  and then integrate over  $\Omega$  :

$$-\int_{\Omega} \Delta u v \, dx = \int_{\Omega} f v \, dx, \quad \forall v \in H_0^1(\Omega)$$

The next step is to invoke an integration by part, the so-called Green formula:

$$\int_{\Omega} \Delta u v \, dx + \int_{\Omega} \nabla u \cdot \nabla v \, dx = \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, ds, \quad \forall u, v \in H^1(\Omega)$$

Since our test-function  $v$  vanishes on the boundary, the integral over  $\partial\Omega$  is zero and the problem becomes:

$$\int_{\Omega} \nabla u \cdot \nabla v \, dx = \int_{\Omega} f v \, dx, \quad \forall v \in H_0^1(\Omega)$$

This is exactly the variational formulation (1.3), page 13.

### A.2 The vectorial Green formula

In this section, we come back to the linear elasticity problem presented in section 5.1, page 43, equations (5.1)-(5.2) and details how this problem is transformed into (5.3).

Let  $\Gamma_d$  (resp.  $\Gamma_n$ ) denotes the parts of the boundary  $\partial\Omega$  related to the homogeneous Dirichlet boundary condition  $\mathbf{u} = 0$  (resp. the homogeneous Neumann boundary condition  $\sigma(\mathbf{u}) \mathbf{n} = 0$ ).

We suppose that  $\partial\Omega = \bar{\Gamma}_d \cap \bar{\Gamma}_n$ . Let us introduce the following functional space:

$$\mathbf{V} = \{\mathbf{v} \in H^1(\Omega)^d; \mathbf{v} = 0 \text{ on } \Gamma_d\}$$

Then, multiplying the first equation of (5.2) by an arbitrarily test-function  $\mathbf{v} \in \mathbf{V}$  and then integrate over  $\Omega$  :

$$-\int_{\Omega} \mathbf{div}(\sigma(\mathbf{u})) \cdot \mathbf{v} \, dx = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx, \quad \forall \mathbf{v} \in \mathbf{V}$$

The next step is to invoke an integration by part:

$$\int_{\Omega} \mathbf{div} \tau \cdot \mathbf{v} \, dx + \int_{\Omega} \tau : D(\mathbf{v}) \, dx = \int_{\partial\Omega} \tau : (\mathbf{v} \otimes \mathbf{n}) \, ds, \quad \forall \tau \in L^2(\Omega)^{d \times d}, \quad \forall \mathbf{v} \in \mathbf{V}$$

Recall that  $\mathbf{div} \tau$  denotes  $\left(\sum_{j=0}^{d-1} \partial_j \tau_{i,j}\right)_{0 \leq i < d}$ , i.e. the vector whose component are the divergence of each row of  $\tau$ . Also,  $\sigma : \tau$  denote the double contracted product  $\sum_{i,j=0}^{d-1} \sigma_{i,j} \tau_{i,j}$  for any tensors  $\sigma$  and  $\tau$ , and that  $\mathbf{u} \otimes \mathbf{v}$  dotes the  $\tau_{i,j} = u_i v_j$  tensor, vectors  $\mathbf{u}$  and  $\mathbf{v}$ . Remark that  $\tau : (\mathbf{u} \otimes \mathbf{v}) = (\tau \mathbf{v}) \cdot \mathbf{u} = \sum_{i,j=0}^{d-1} \tau_{i,j} u_i v_j$ . Choosing  $\tau = \sigma(\mathbf{u})$  in the previous equation leads to:

$$\int_{\Omega} \sigma(\mathbf{u}) : D(\mathbf{v}) \, dx = \int_{\partial\Omega} (\sigma(\mathbf{u}) \mathbf{n}) \cdot \mathbf{v} \, ds + \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx, \quad \forall \mathbf{v} \in \mathbf{V}$$

Since our test-function  $v$  vanishes on  $\Gamma_d$  and the solution satisfy the homogeneous Neumann boundary condition  $\sigma(\mathbf{u}) \mathbf{n} = 0$  on  $\Gamma_n$ , the integral over  $\partial\Omega$  is zero and the problem becomes:

$$\int_{\Omega} \sigma(\mathbf{u}) : D(\mathbf{v}) \, dx = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx, \quad \forall \mathbf{v} \in \mathbf{V}$$

From the definition of  $\sigma(\mathbf{u})$  in (5.1) page 43 we have:

$$\begin{aligned} \sigma(\mathbf{u}) : D(\mathbf{v}) &= \lambda \operatorname{div}(\mathbf{u}) (I : D(\mathbf{v})) + 2\mu D(\mathbf{u}) : D(\mathbf{v}) \\ &= \lambda \operatorname{div}(\mathbf{u}) \operatorname{div}(\mathbf{v}) + 2\mu D(\mathbf{u}) : D(\mathbf{v}) \end{aligned}$$

and the previous relation becomes:

$$\int_{\Omega} \lambda \operatorname{div}(\mathbf{u}) \operatorname{div}(\mathbf{v}) \, dx + \int_{\Omega} 2\mu D(\mathbf{u}) : D(\mathbf{v}) \, dx = \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, dx, \quad \forall \mathbf{v} \in \mathbf{V}$$

This is exactly the variational formulation (5.3), page 44.

### A.3 The Green formula on a surface

TODO

## Appendix B

# How to prepare a mesh ?

Since there is many good mesh generators, **Rheolef** does not provide a built-in mesh generator. There are several ways to prepare a mesh for **Rheolef**.

We present here several procedures: by using the **bamg** bidimensional anisotropic mesh generator, written by Frédéric Hecht [8], and the **gmsh** mesh generator, suitable when  $d = 1, 2$  and  $3$ , and written by Christophe Geuzaine and Jean-Francois Remacle [40].

### B.1 Bidimensionnal mesh with bamg

We first create a ‘square.bamgcad’ file:

```
MeshVersionFormatted
0
Dimension
2
Vertices
4
0 0 1
1 0 2
1 1 3
0 1 4
Edges
4
1 2 101
2 3 102
3 4 103
4 1 104
hVertices
0.1 0.1 0.1 0.1
```

This is an uniform mesh with element size  $h = 0.1$ . We refer to the **bamg** documentation [8] for the complete file format description. Next, enter the mesh generator commands:

```
bamg -g square.bamgcad -o square.bamg
```

Then, create the file ‘square.dmn’ that associate names to the four boundary domains of the mesh. Here, there is four boundary domains:



```
EdgeDomainNames
4
bottom
right
top
left
```

and enter the translation command:

```
bamg2geo square.bamg square.dmn > square.geo
```

This command creates a ‘square.geo’ file. Look at the mesh via the command:

```
geo square
```

This presents the mesh in a graphical form, usually with `gnuplot`. You can switch to the `mayavi` renders:

```
geo square -mayavi
```

A finer mesh could be generated by:

```
bamg -coef 0.5 -g square.bamgcad -o square-0.5.bamg
```

## B.2 Unidimensionnal mesh with gmsh

The simplest unidimensional mesh is a line:

```
h_local = 0.1;
Point(1) = {0, 0, 0, h_local};
Point(2) = {1, 0, 0, h_local};
Line(3) = {1,2};
Physical Point("left") = {1};
Physical Point("right") = {2};
Physical Point("boundary") = {1,2};
Physical Line("interior") = {3};
```

The mesh generation command writes:

```
gmsh -1 line.mshcad -format msh -o line.msh
```

Then, the conversion to ‘.geo’ format and the visualization:

```
msh2geo < line.msh | geo -upgrade - > line.geo
geo line
```

### B.3 Bidimensionnal mesh with gmsh

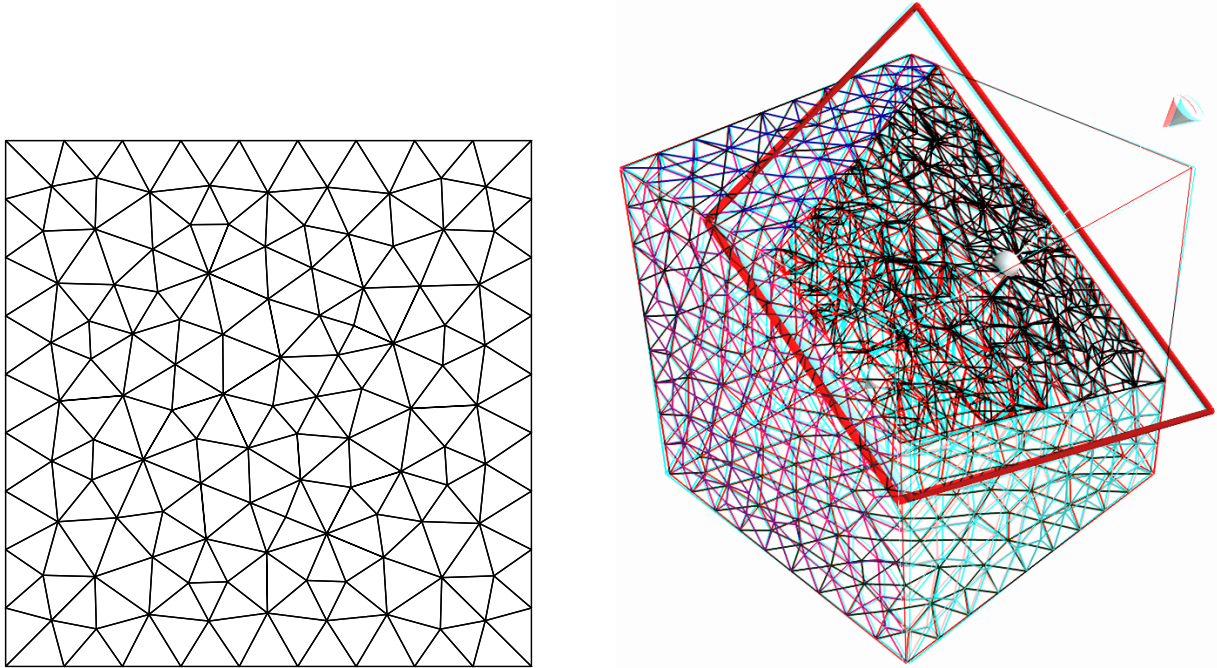


Figure B.1: Visualization of the gmsh meshes 'square.geo' and 'cube.geo'.

We first create a 'square.mshcad' file:

```

n = 10.0;
hloc = 1.0/n;
Point(1) = {0, 0, 0, hloc};
Point(2) = {1, 0, 0, hloc};
Point(3) = {1, 1, 0, hloc};
Point(4) = {0, 1, 0, hloc};
Line(1) = {1,2};
Line(2) = {2,3};
Line(3) = {3,4};
Line(4) = {4,1};
Line Loop(5) = {1,2,3,4};
Plane Surface(6) = {5} ;
Physical Point("left_bottom") = {1};
Physical Point("right_bottom") = {2};
Physical Point("right_top") = {3};
Physical Point("left_top") = {4};
Physical Line("boundary") = {1,2,3,4};
Physical Line("bottom") = {1};
Physical Line("right") = {2};
Physical Line("top") = {3};
Physical Line("left") = {4};
Physical Surface("interior") = {6};

```

This is an uniform mesh with element size  $h = 0.1$ . We refer to the gmsh documentation [40] for the complete file format description. Next, enter the mesh generator commands:

```
gmsht -2 square.mshcad -format msh -o square.msh
```

Then, enter the translation command:

```
msh2geo < square.msh | geo -upgrade - > square.geo
```

This command creates a ‘square.geo’ file. Look at the mesh via the command:

```
geo square
```

Remark that the domain names, defined in the .mshcad file, are included in the gmsht .msh input file and are propagated in the .geo file by the format conversion.

## B.4 Tridimensionnal mesh with gmsht

First, create a ‘cube.mshcad’ file:

```
Mesh.Algorithm = 7; // bamg
Mesh.Algorithm3D = 7; // mmg3d
a = 0; c = 0; f = 0;
b = 1; d = 1; g = 1;
n = 10;
hloc = 1.0/n;
Point(1) = {a, c, f, hloc};
Point(2) = {b, c, f, hloc};
Point(3) = {b, d, f, hloc};
Point(4) = {a, d, f, hloc};
Point(5) = {a, c, g, hloc};
Point(6) = {b, c, g, hloc};
Point(7) = {b, d, g, hloc};
Point(8) = {a, d, g, hloc};
Line(1) = {1,2};
Line(2) = {2,3};
Line(3) = {3,4};
Line(4) = {4,1};
Line(5) = {5,6};
Line(6) = {6,7};
Line(7) = {7,8};
Line(8) = {8,5};
Line(9) = {1,5};
Line(10) = {2,6};
Line(11) = {3,7};
Line(12) = {4,8};
Line Loop(21) = {-1,-4,-3,-2};
Plane Surface(31) = {21} ;
Line Loop(22) = {5,6,7,8};
Plane Surface(32) = {22} ;
Line Loop(23) = {1,10,-5,-9};
Plane Surface(33) = {23} ;
Line Loop(24) = {12,-7,-11,3};
Plane Surface(34) = {24} ;
```

```
Line Loop(25) = {2,11,-6,-10};
Plane Surface(35) = {25} ;
Line Loop(26) = {9,-8,-12,4};
Plane Surface(36) = {26} ;
Surface Loop(41) = {31,32,33,34,35,36};
Volume(51) = {41};
Physical Surface("bottom") = {31};
Physical Surface("top") = {32};
Physical Surface("left") = {33};
Physical Surface("front") = {35};
Physical Surface("right") = {34};
Physical Surface("back") = {36};
Physical Volume("internal") = {51};
```

Next, enter the mesh generator commands:

```
gmsh -3 cube.mshcad -format msh -o cube.msh
```

Then, enter the translation command:

```
msh2geo < cube.msh | geo -upgrade - > cube.geo
```

This command creates a 'cube.geo' file. Look at the mesh via the command:

```
geo cube
geo cube.geo -mayavi -cut
```

The second command allows to see inside the mesh.



# Appendix C

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