HAL
open science

# Efficient C++ finite element computing with Rheolef <br> Pierre Saramito 

## To cite this version:

Pierre Saramito. Efficient C++ finite element computing with Rheolef. DEA. Grenoble, France, 2012, pp.138. cel-00573970v3

## HAL Id: cel-00573970

https://cel.hal.science/cel-00573970v3
Submitted on 10 Apr 2012 (v3), last revised 2 Jun 2022 (v16)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Efficient C++ finite element computing with Rheolef 

P. Saramito

distributed version 6.0 update 1 April 2012

## Introduction

Rheolef is a programming environment that serves as a convenient laboratory for computations involving finite element methods. Rheolef is both a $\mathbf{C}++$ library and a set of commands for unix shell programming, providing algorithms and data structures.

- Algorithms refer to the most up-to-date ones: preconditioned sparse solvers for linear systems, incompressible elasticity, Stokes and Navier-Stokes flows, characteristic method for convection dominated heat problems, etc. Also nonlinear generic algorithms such as fixed point and damped Newton methods.
- Data structures fit the standard variational formulation concept: spaces, discrete fields, bilinear forms are C++ types for variables, that can be combined in any expressions, as you write it on the paper.

Combined together, as a Lego game, these bricks allows the user to solve most complex nonlinear problems. This Book details, step by step, how some simple and more complex problems can be solved, most of them in less than 20 lines of code. The concision and readability of codes written with Rheolef is certainly a major keypoint of this environment.

## Classical features

- Poisson problems in 1D 2D and 3D with P1 or P2 elements
- Stokes problems in 2D and 3D, with Taylor-Hood P2-P1 or stabilized P1 bubble-P1 elements
- linear elasticity in 2D and 3D, with P1 and P2 elements, including the incompressible and nearly incompressible elasticity
- characteristic method for convection-difusion, time-dependent problems and Navier-Stokes equations.
- input and output in various file format for meshes generators and numerical data visualization systems


## Advanced features

- auto-adaptive mesh based for 2D problems
- axisymmetric problems
- multi-regions and non-constant coefficients
- nonlinear problems with either fixed-point algorithms or a provided generic damped Newton solver
- 3d stereo visualization


## Contacts

```
email Pierre.Saramito@imag.fr
home page http://www-ljk.imag.fr/membres/Pierre.Saramito/rheolef
```

Please send all comments and bug reports by electronic mail to

```
rheolef@grenet.fr
```


## The Rheolef present contributors

from 2008 Ibrahim Cheddadi: dicontinuous Galerkin method for transport problems.
from 2010 Mahamar Dicko: finite element methods for equations on surfaces.
from 2002 Jocelyn Etienne: characteristic method for time-dependent problems.
from 2000 Pierre Saramito: project leader: main developments and code maintainer.
Past contributors
2010 Lara Abouorm: banded level set method for equations on surfaces.
2000 Nicolas Roquet: initial versions of Stokes and Bingham flow solvers.

## Preface to Rheolef version 6.0

## What is new in Rheolef 6.0?

The major main features are:

- support distributed achitectures: the code looks sequential, is easy to read and write but can be run massively parallel and distributed, based on the MPI library.
- high order polynomial approximation: $P_{k}$ basis are introduced in this version, for $k \geq 0$. This feature will be improved in the future developments.
- mesh adaptation and the charateristic method are now available for threedimensional problems.

In order to evoluate in these directions, internal data structures inside the library are completely rewritten in a different way, and thus this version is a completely new library.
Conversely, the library and unix command interfaces was as less as possible modified.
Nevertheless, the user will find some few backward incompatibilities: 5.93 based codes will not directly compile with the 6.0 library version. Let us review how to move a code from 5.93 to 6.0 version.

## Moving to Rheolef 6.0

## 1. Namespace

The namespace rheolef was already introduced in last 5.93 version. Recall that a code usually starts with:

```
#include "rheolef.h"
using namespace rheolef;
```


## 2. Environment

The MPI library requires initialisation and the two command line arguments. This initialisation is performed via the boost::mpi class environment: The code entry point writes:

```
int main (int argc, char** argv) {
    environment rheolef (argc,argv);
    ...
```


## 3. Fields and forms data accessors

The accesses to unknown and blocked data was of a field uh was direct, as uh.u and uh.b. This access is no more possible in a distributed environment, as non-local value requests may be optimized and thus, read and write access may be controled thought accessors. These accessors are named uh.u() and uh.b() for read access, and uh.set_u() and uh.set_b() for write access. Similarly, a form a has accessors as a.uu().
A typical 5.93 code writes:

```
ssk<Float> sa = ldlt(a.uu);
uh.u = sa.solve (lh.u - a.ub*uh.b);
```

and the corresponding 6.0 code is:

```
solver sa (a.uu());
uh.set_u() = sa.solve (lh.u() - a.ub()*uh.b());
```

This major change in the library interface induces the most important work when porting to the 6.0 version.

Notice also that the old ssk<Float> class has been supersetted by the solver class, that manages both direct and iterative solvers in a more effective way. For three-dimensional problems, the iterative solver is the default while direct solvers are used otherwise. In the same spirit, a solver_abtb has been introduced, for Stokes-like mixed problem. These features facilitate the dimension-independent coding style provided by the Rheolef library.

## 4. Distributed input and output streams

Input and output sequential standard streams cin, cout and cerr may now replaced by distributed Rheolef streams din, dout and derr as:

```
din >> omega;
dout << uh;
```

These new streams are available togeher with the idiststream and odiststream classes of the Rheolef library.

## 5. File formats '.geo' and .field have changed

The '.geo' and '.field' file formats have changed. The .mfield is now obsolete: it has been merged into the .field format that supports now multi-component fields. Also, the corresponding mfield unix command is obsolete, as these features are integrated in the field unix command.
At this early stage of the 6.0 version, it is not yet possible to read the old .geo format, but this backward compatibility will be assured soon.

## 6. Space on a domain

A space defined on a domain "boundary" of a mesh omega was defined in the 5.93 version as:

```
space Wh (omega["boundary"], omega, "P1");
```

It writes now:

```
space Wh (omega["boundary"], "P1");
```

as the repetition of omega is no more required.

## Contents

I Getting started with simple problems ..... 9
1 Getting started with Rheolef ..... 13
1.1 The model problem ..... 13
1.2 Approximation ..... 14
1.3 File 'dirichlet.cc' ..... 14
1.4 Comments ..... 14
1.5 How to compile the code ..... 16
1.6 How to run the program ..... 17
1.7 Distributed and parallel runs ..... 17
1.8 Stereo visualization ..... 18
1.9 High-order finite element methods ..... 19
1.10 Tridimensional computations ..... 19
1.11 Quandrangles, prisms and hexahedra ..... 20
2 Standard boundary conditions ..... 23
2.1 Non-homogeneous Dirichlet conditions ..... 23
2.2 Non-homogeneous Neumann boundary conditions for the Helmholtz operator ..... 26
2.3 The Robin boundary conditions ..... 28
3 Non-constant coefficients and multi-regions ..... 31
4 Neumann boundary conditions fot the Laplace operator ..... 37
II Fluids and solids computations ..... 41
5 The linear elasticity and the Stokes problems ..... 43
5.1 The linear elasticity problem ..... 43
5.2 Computing the stress tensor ..... 47
5.3 Mesh adaptation ..... 51
5.4 The Stokes problem ..... 54
5.5 Computing the vorticity ..... 57
5.6 Computing the stream function ..... 60
6 Nearly incompressible elasticity and the stabilized Stokes problems ..... 63
6.1 The incompressible elasticity problem ..... 63
6.2 The $P_{1} b-P_{1}$ element for the Stokes problem ..... 65
6.3 The stabilized Stokes problem ..... 70
6.4 Axisymmetric geometries ..... 70
6.5 The axisymmetric stream function and stress tensor ..... 71
7 Time-dependent problems ..... 75
7.1 The heat equation ..... 75
7.2 The convection-diffusion problem ..... 78
7.3 The Navier-Stokes problem ..... 84
III Advanced and highly nonlinear problems ..... 93
8 The highly nonlinear $p$-laplacian problem ..... 95
8.1 Problem statement ..... 95
8.2 The fixed-point algorithm ..... 96
8.3 The Newton algorithm ..... 100
8.4 The damped Newton algorithm ..... 105
9 Error analysis for $P_{k}$ isoparametric elements ..... 111
9.1 Error analysis ..... 111
IV Technical appendices ..... 121
A How to write a variational formulation? ..... 123
A. 1 The Green formula ..... 123
A. 2 The vectorial Green formula ..... 123
A. 3 The Green formula on a surface ..... 124
B How to prepare a mesh ? ..... 125
B. 1 Bidimensionnal mesh with bamg ..... 125
B. 2 Unidimensionnal mesh with gmsh ..... 126
B. 3 Bidimensionnal mesh with gmsh ..... 127
B. 4 Tridimensionnal mesh with gmsh ..... 128
C GNU Free Documentation License ..... 131
Index of concepts ..... 136
Index of example files ..... 140
Index of programs ..... 141
List of Figures ..... 142

## Part I

## Getting started with simple problems

The first part of this book starts with the Dirichlet problem with homogeneous boundary condition: this example is declined with details in dimension 1,2 and 3 , as a starting point to Rheolef.
Next chapters present various boundary conditions: for completeness, we treat non-homogeneous Dirichlet, Neumann, and Robin boundary conditions for model problems. The last two examples presents some special difficulties that appears in most problems: the first one introduce to problems with non-constant coefficients and the second one, a ill-posed problem where the solution is defined up to a constant.
This first part can be viewed as a pedagogic preparation for more advanced applications, such as Stokes and elasticity, that are treated in the second part of this book. Problem with non-constant coefficients are common as suproblems generated by various algorithms for non-linear problem.

## Chapter 1

## Getting started with Rheolef

For obtaining and installing Rheolef, see the installation instructions on the Rheolef home page:

```
http://www-ljk.imag.fr/membres/Pierre.Saramito/rheolef/
```

All examples presented along the present book are available in the example/ directory of the Rheolef distribution. This directory is given by the following unix command:

```
rheolef-config --exampledir
```

This command returns you a path, something like /usr/share/doc/rheolef-doc/examples/and you should make a copy of these files:

```
cp -a /usr/share/doc/rheolef-doc/examples/ .
cd examples
```

Before to run examples, please check your Rheolef installation with:

```
rheolef-config --check
```


### 1.1 The model problem

Let us consider the classical Poisson problem with homogeneous Dirichlet boundary conditions in a domain bounded $\Omega \subset R^{d}, d=1,2,3$ :
$(P)$ : find $u$, defined in $\Omega$, such that:

$$
\begin{align*}
-\Delta u & =1 \text { in } \Omega  \tag{1.1}\\
u & =0 \text { on } \partial \Omega \tag{1.2}
\end{align*}
$$

where $\Delta$ denotes the Laplace operator. The variational formulation of this problem expresses (see appendix A. 1 for details):
(VF): find $u \in H_{0}^{1}(\Omega)$ such that:

$$
\begin{equation*}
a(u, v)=l(v), \forall v \in H_{0}^{1}(\Omega) \tag{1.3}
\end{equation*}
$$

where the bilinear form $a(.,$.$) and the linear form l($.$) are defined by$

$$
\begin{aligned}
a(u, v) & =\int_{\Omega} \nabla u \cdot \nabla v \mathrm{~d} x, \quad \forall u, v \in H_{0}^{1}(\Omega) \\
l(v) & =\int_{\Omega} v \mathrm{~d} x, \quad \forall v \in L^{2}(\Omega)
\end{aligned}
$$

The bilinear form $a(.,$.$) defines a scalar product in H_{0}^{1}(\Omega)$ and is related to the energy form. This form is associated to the $-\Delta$ operator.

### 1.2 Approximation

Let us introduce a mesh $\mathcal{T}_{h}$ of $\Omega$ and the finite dimensional space $X_{h}$ of continuous piecewise polynomial functions.

$$
X_{h}=\left\{v \in H^{1}(\Omega) ; v_{/ K} \in P_{k}, \forall K \in \mathcal{T}_{h}\right\}
$$

where $k=1$ or 2 . Let $V_{h}=X_{h} \cap H_{0}^{1}(\Omega)$ be the functions of $X_{h}$ that vanishes on the boundary of $\Omega$. The approximate problem expresses:
$(V F)_{h}:$ find $u_{h} \in V_{h}$ such that:

$$
a\left(u_{h}, v_{h}\right)=l\left(v_{h}\right), \forall v_{h} \in V_{h}
$$

By developing $u_{h}$ on a basis of $V_{h}$, this problem reduces to a linear system. The following $\mathrm{C}++$ code implement this problem in the Rheolef environment.

### 1.3 File 'dirichlet.cc'

```
#include "rheolef.h"
using namespace rheolef;
using namespace std;
int main(int argc, char**argv) {
    environment rheolef (argc, argv);
    geo omega (argv[1]);
    space Xh (omega, argv[2]);
    Xh.block ("boundary");
    form a (Xh, Xh, "grad_grad");
    field lh = riesz (Xh, 1);
    field uh (Xh);
    uh ["boundary"] = 0;
    solver sa (a.uu());
    uh.set_u() = sa.solve (lh.u() - a.ub()*uh.b());
    dout << uh;
}
```


### 1.4 Comments

This code applies for both one, two or three dimensional meshes and for both piecewise linear or quadratic finite element approximations. Four major classes are involved, namely: geo, space, form and field.
Let us now comment the code, line by line.

```
#include "rheolef.h"
```

The first line includes the Rheolef header file 'rheolef.h'.

```
using namespace rheolef;
using namespace std;
```

By default, in order to avoid possible name conflicts when using another library, all classe and function names are prefixed by rheolef : : , as in rheolef: :space. This feature is called the name space. Here, since there is no possible conflict, and in order to simplify the syntax, we drop all the rheolef:: prefixes, and do the same with the standard c++ library classes and variables, that are also prefixed by std::.

```
int main(int argc, char**argv) {
```

The entry function of the program is always called main and accepts arguments from the unix command line: argc is the counter of command line arguments and argv is the table of values. The character string $\operatorname{argv}[0]$ is the program name and $\operatorname{argv}[i]$, for $i=1$ to $\operatorname{argc}-1$, are the additional command line arguments.

```
environment rheolef (argc, argv);
```

These two command line parameters are immediatly furnished to the distributed environment initializer of the boost: :mpi library, that is a c++ library based on the usual message passing interface (MPI) library. Notice that this initialization is requiered, even when you run with only one processor.

```
geo omega (argv[1]);
```

This command get the first unix command-line argument argv[1] as a mesh file name and store the corresponding mesh in the variable omega.

```
space Xh (omega, argv[2]);
```

Build the finite element space Xh contains all the piecewise polynomial continuous functions. The polynomial type is the second command-line arguments argv[2], and could be either P1, P2 or any $\mathrm{P} k$, where $k \geq 1$.

```
Xh.block ("boundary");
```

The homogeneous Dirichlet conditions are declared on the boundary.

```
form a (Xh, Xh, "grad_grad");
```

The form $a(.,$.$) is the energy form.$

```
field lh = riesz (Xh, 1);
```

Here $l h($.$) is the Riesz representer of the constant right-hand side f=1$ of the problem.

```
field uh (Xh);
```

The field uh contains the the degrees of freedom.

```
uh ["boundary"] = 0;
```

Some degrees of freedom are prescribed as zero on the boundary.
Let $\left(\varphi_{i}\right)_{0 \leq i<\operatorname{dim}\left(X_{h}\right)}$ be the basis of $X_{h}$ associated to the Lagrange nodes, e.g. the vertices of the mesh for the $P_{1}$ approximation and the vertices and the middle of the edges for the $P_{2}$ approximation. The approximate solution $u_{h}$ expresses as a linear combination of the continuous piecewise polynomial functions $\left(\varphi_{i}\right)$ :

$$
u_{h}=\sum_{i} u_{i} \varphi_{i}
$$

Thus, the field $u_{h}$ is completely represented by its coefficients $\left(u_{i}\right)$. The coefficients $\left(u_{i}\right)$ of this combination are grouped into to sets: some have zero values, from the boundary condition and are related to blocked coefficients, and some others are unknown. Blocked coefficients are stored into the uh.b array while unknown one are stored into uh.u. Thus, the restriction of the bilinear form $a(.,$.$) to X_{h} \times X_{h}$ can be conveniently represented by a block-matrix structure:

$$
a\left(u_{h}, v_{h}\right)=\left(\begin{array}{cc}
\text { vh.u } & \text { vh.b }
\end{array}\right)\left(\begin{array}{cc}
\text { a.uu } & \text { a.ub } \\
\text { a.bu } & \text { a.bb }
\end{array}\right)\binom{\text { uh.u }}{\text { uh.b }}
$$

This representation also applies for the linear form $l($.$) :$

$$
l\left(v_{h}\right)=\left(\begin{array}{ll}
\text { vh.u } & \text { vh.b }
\end{array}\right)\binom{\text { lh.u }}{\text { lh.b }}
$$

Thus, the problem $(V F)_{h}$ writes now:

$$
\left(\begin{array}{cc}
\text { vh.u } & \text { vh.b }
\end{array}\right)\left(\begin{array}{cc}
\text { a.uu } & \text { a.ub } \\
\text { a.bu } & \text { a.bb }
\end{array}\right)\binom{\text { uh.u }}{\text { uh.b }}=\left(\begin{array}{ll}
\text { vh.u } & \text { vh.b }
\end{array}\right)\binom{\text { lh.u }}{\text { lh.b }}
$$

for any vh.u and where vh. $\mathrm{b}=0$. After expansion, the problem reduces to find uh.u such that:

$$
\text { a.uu } * u h . u=1 . \mathrm{u}-\mathrm{a} . \mathrm{ub} * \mathrm{uh} . \mathrm{b}
$$

The resolution of this linear system for the a.uu matrix is then performed. A preliminary step build the $L D L^{T}$ factorization:

```
solver sa (a.uu());
```

Then, the second step solves the unknown part:

```
uh.set_u() = sa.solve (lh.u() - a.ub()*uh.b());
```

When $d>3$, a faster iterative strategy is automatically preferred by the solver class for solving the linear system: in that case, the preliminary step build an incomplete Choleski factorization preconditioner, while the second step runs an iterative method: the preconditioned conjugate gradient algorithm. Finally, the field is printed to standard output:

```
dout << uh;
```

The dout stream is a specific variable defined in the Rheolef library: it is a distributed and parallel extension of the usual cout stream in C++

### 1.5 How to compile the code

First, create a Makefile as follow:

```
include $(shell rheolef-config --libdir)/rheolef/rheolef.mk
CXXFLAGS = $(INCLUDES_RHEOLEF)
LDLIBS = $(LIBS_RHEOLEF)
default: dirichlet
```

Then, enter:

```
make dirichlet
```

Now, your program, linked with Rheolef, is ready to run on a mesh.

### 1.6 How to run the program



Figure 1.1: Solution of the model problem for $d=2$ : (left) $P_{1}$ element; (right) $P_{2}$ element.

Enter the commands:

```
mkgeo_grid -t 10 > square.geo
geo square.geo
```

The first command generates a simple 10 x 10 bidimensional mesh of $\Omega=] 0,1\left[{ }^{2}\right.$ and stores it in the file square.geo. The second command shows the mesh. It uses gnuplot visualization program by default.
The next command performs the computation:

```
./dirichlet square.geo P1 > square.field
field square.field
```


### 1.7 Distributed and parallel runs

Alternatively, a computation in a distributed and parallel environment writes:

```
mpirun -np 4 ./dirichlet square.geo P1 > square.field
```



Figure 1.2: Alternative representations of the solution of the model problem ( $d=2$ and the $P_{1}$ element): (left) in black-and-white; (right) in elevation and stereoscopic anaglyph mode.

### 1.8 Stereo visualization

Also explore some graphic rendering modes (see Fig. 1.2):

```
field square.field -bw
field square.field -gray
field square.field -mayavi
field square.field -elevation -nofill -stereo
```

The last command shows the solution in elevation and in stereoscopic anaglyph mode (see Fig. 1.4, left). The anaglyph mode requires red-cyan glasses: red for the left eye and cyan for the right one, as shown on Fig. 1.3.


Figure 1.3: Red-cyan anaglyph glasses for the stereoscopic visualization.

See
http://en.wikipedia.org/wiki/Anaglyph_image
for more and http://www.alpes-stereo.com/lunettes.html for how to find anaglyph red-cyan glasses.

Please, consults the corresponding unix manual page for more on field, geo and mkgeo_grid:

```
man mkgeo_grid
man geo
man field
```


### 1.9 High-order finite element methods

Turning to the P2 or P3 approximations simply writes:

```
./dirichlet square.geo P2 > square-P2.field
```

field square-P2.field

Fig. 1.1.right shows the result. You can replace the P 2 command-line argument by any $\mathrm{P} k$, where $k \geq 1$.
Now, let us consider a mono-dimensional problem $\Omega=] 0,1[$ :

```
mkgeo_grid -e 10 > line.geo
geo line.geo
./dirichlet line.geo P1 | field -
```

The first command generates a subdivision containing ten edge elements. The last two lines show the mesh and the solution via gnuplot visualization, respectively.
Conversely, the P2 case writes:

```
./dirichlet line.geo P2 | field -
```


### 1.10 Tridimensional computations

Let us consider a three-dimensional problem $\Omega=] 0,1\left[{ }^{3}\right.$. First, let us generate a mesh:

```
mkgeo_grid -T 10 > cube.geo
geo cube.geo
geo cube.geo -stereo -full
geo cube.geo -stereo -cut
```

The previous commands draw the mesh with all internal edges (-full), stereoscopic anaglyph (-stereo) and then with a cut (-cut inside the internal structure: a simple click on the central arrow draws the cut plane normal vector or its origin, while the red square allows a translation.


Figure 1.4: Solution of the model problem for $d=3$ and the $P_{1}$ element : (left) mesh; (right) isovalue, cut planes and stereo anaglyph renderings.

Then, we perform the computation and the visualization:

```
./dirichlet cube.geo P1 > cube.field
field cube.field
```

The visualization presents an isosurface. Also here, you can interact with the cutting plane. Click on IsoSurface in the left menu and change the value of the isosurface. Finally exit from the visualization and explore the stereoscopic anaglyph mode (see Fig. 1.4, right):

```
field cube.field -stereo
```

It is also possible to add a second IsoSurface or ScalarCutPlane module to this scene by using the Visualize menu. After this exploration of the 3D visualisation capacities of our environment, let us go back to the Dirichlet problem and perform the P2 approximation:

```
./dirichlet cube.geo P2 | field -
```


### 1.11 Quandrangles, prisms and hexahedra

Quadrangles and hexahedra are also supported in meshes:

```
mkgeo_grid -q 10 > square.geo
geo square.geo
mkgeo_grid -H 10 > cube.geo
geo cube.geo
```

Notices also that the one-dimensional exact solution writes:

$$
u(x)=\frac{x(1-x)}{2}
$$

while the two-and three dimensional ones support a Fourier expansion (see e.g. [1], annex).

## Chapter 2

## Standard boundary conditions

We show how to deal with various non-homogeneous boundary conditions of Dirichlet, Neuman and Robin type.

### 2.1 Non-homogeneous Dirichlet conditions

## Formulation

We turn now to the case of a non-homogeneous Dirichlet boundary conditions. Let $f \in H^{-1}(\Omega)$ and $g \in H^{\frac{1}{2}}(\partial \Omega)$. The problem writes:
$\left(P_{2}\right)$ find $u$, defined in $\Omega$ such that:

$$
\begin{aligned}
-\Delta u & =f \text { in } \Omega \\
u & =g \text { on } \partial \Omega
\end{aligned}
$$

The variational formulation of this problem expresses:
$\left(V F_{2}\right)$ find $u \in V$ such that:

$$
a(u, v)=l(v), \forall v \in V_{0}
$$

where

$$
\begin{aligned}
a(u, v) & =\int_{\Omega} \nabla u \cdot \nabla v \mathrm{~d} x \\
l(v) & =\int_{\Omega} f v \mathrm{~d} x \\
V & =\left\{v \in H^{1}(\Omega) ; v_{\mid \partial \Omega}=g\right\} \\
V_{0} & =H_{0}^{1}(\Omega)
\end{aligned}
$$

## Approximation

As usual, we introduce a mesh $\mathcal{T}_{h}$ of $\Omega$ and the finite dimensional space $X_{h}$ :

$$
X_{h}=\left\{v \in H^{1}(\Omega) ; v_{/ K} \in P_{k}, \forall K \in \mathcal{T}_{h}\right\}
$$

Then, we introduce:

$$
\begin{aligned}
V_{h} & =\left\{v \in X_{h} ; v_{\mid \partial \Omega}=\pi_{h}(g)\right\} \\
V_{0, h} & =\left\{v \in X_{h} ; v_{\mid \partial \Omega}=0\right\}
\end{aligned}
$$

where $\pi_{h}$ denotes the Lagrange interpolation operator. The approximate problem writes: $\left(V F_{2}\right)_{h}:$ find $u_{h} \in V_{h}$ such that:

$$
a\left(u_{h}, v_{h}\right)=l\left(v_{h}\right), \forall v_{h} \in V_{0, h}
$$

The following $\mathrm{C}++$ code implement this problem in the $\mathbf{R h e o l e f}$ environment.

File 'dirichlet-nh.cc'

```
#include "rheolef.h"
using namespace rheolef;
using namespace std;
#include "cosinusprod_laplace.icc"
int main(int argc, char**argv) {
    environment rheolef(argc, argv);
    geo omega (argv[1]);
    size_t d = omega.dimension();
    space Xh (omega, argv[2]);
    Xh.block ("boundary");
    form a (Xh, Xh, "grad_grad");
    field lh = riesz (Xh, f(d));
    field uh (Xh);
    space Wh (omega["boundary"], argv[2]);
    uh ["boundary"] = interpolate(Wh, g(d));
    solver sa (a.uu());
    uh.set_u() = sa.solve (lh.u() - a.ub()*uh.b());
    dout << uh;
}
```

Let us choose $\Omega \subset R^{d}, d=1,2,3$ with

$$
f(x)=d \pi^{2} \prod_{i=0}^{d-1} \cos \left(\pi x_{i}\right) \quad \text { and } \quad g(x)=\prod_{i=0}^{d-1} \cos \left(\pi x_{i}\right)
$$

Remarks the notation $x=\left(x_{0}, \ldots, x_{d-1}\right)$ for points of $R^{d}$ : since all arrays start at index zero in C++ codes, and in order to avoid any mistakes between the code and the mathematical formulation, we also adopt this convention here. This choice of $f$ and $g$ is convenient, since the exact solution is known:

$$
u(x)=\prod_{i=0}^{d-1} \cos \left(\pi x_{i}\right)
$$

The following $\mathrm{C}++$ code implement this problem by using the concept of function object, also called class-function (see e.g. [2]). A convenint feature is the abblility for function objects to store auxilliary parameters, such as the space dimension $d$ for $f$ here, or some constants, as $\pi$ for $f$ and $g$.

File 'cosinusprod_laplace.icc'

```
struct f : unary_function<point,Float> {
    Float operator() (const point& x) const {
        return d*pi*pi*cos(pi*x[0])*\operatorname{cos(pi*x[1])*\operatorname{cos(pi*x[2]); }}}}=\mp@code{*}
    f(size_t d1) : d(d1), pi(acos(Float(-1))) {}
    size_t d; const Float pi;
};
struct g : unary_function<point,Float> {
    Float operator() (const point& x) const {
        return cos(pi*x[0])*cos(pi*x[1])*cos(pi*x[2]); }
    g(size_t d1) : pi(acos(Float(-1))) {}
    const Float pi;
};
```


## Comments

The class point describes the coordinates of a point $\left(x_{0}, \ldots, x_{d-1}\right) \in R^{d}$ as a d-uplet of Float. The Float type is usually a double. This type depends upon the Rheolef configuration (see [3], installation instructions), and could also represent some high precision floating point class. The dirichlet-nh. cc code looks like the previous one dirichlet. cc related to homogeneous boundary conditions. Let us comments the changes. The dimension $d$ comes from the geometry $\Omega$ :

```
size_t d = omega.dimension();
```

The linear form $l($.$) , defined as the Riesz representer of f$, writes:

```
field lh = riesz (Xh,f(d));
```

Notice that the function object $f$ is build with the dimension $d$ as parameter. The space $W_{h}$ of picewise $P_{k}$ functions defined on the boundary $\partial \Omega$ is defined by:

```
space Wh (omega["boundary"], argv[2]);
```

where $P_{k}$ is defined via the second command line argument argv[2]. This space is suitable for the Lagrange interpolation of $g$ on the boundary:

```
uh ["boundary"] = interpolate(Wh, g(d));
```

The values of the degrees of freedom related to the boundary are stored into the field uh.b, where non-homogeneous Dirichlet conditions applies. The rest of the code is similar to the homogeneous Dirichlet case.

### 2.1.1 How to run the program

First, compile the program:

```
make dirichlet-nh
```

Running the program is obtained from the homogeneous Dirichlet case, by remplacing dirichlet by dirichlet-nh:

```
mkgeo_grid -e 10 > line.geo
./dirichlet-nh line.geo P1 > line.field
field line.field
```

for the bidimensional case:

```
mkgeo_grid -t 10 > square.geo
./dirichlet-nh square.geo P1 > square.field
field square.field
```

and for the tridimensional case:

```
mkgeo_grid -T 10 > box.geo
./dirichlet-nh box.geo P1 > box.field
field -mayavi box.field
```

Here, the P1 approximation can be remplaced by P2, P3, etc, by modifying the command-line argument.

### 2.2 Non-homogeneous Neumann boundary conditions for the Helmholtz operator

## Formulation

Let us show how to insert Neumann boundary conditions. Let $f \in H^{-1}(\Omega)$ and $g \in H^{-\frac{1}{2}}(\partial \Omega)$. The problem writes:
$\left(P_{3}\right)$ : find $u$, defined in $\Omega$ such that:

$$
\begin{aligned}
u-\Delta u & =f \text { in } \Omega \\
\frac{\partial u}{\partial n} & =g \text { on } \partial \Omega
\end{aligned}
$$

The variational formulation of this problem expresses:
$\left(V F_{3}\right):$ find $u \in H^{1}(\Omega)$ such that:

$$
a(u, v)=l(v), \forall v \in H^{1}(\Omega)
$$

where

$$
\begin{aligned}
a(u, v) & =\int_{\Omega} u v \mathrm{~d} x+\int_{\Omega} \nabla u . \nabla v \mathrm{~d} x \\
l(v) & =\int_{\Omega} f v \mathrm{~d} x+\int_{\partial \Omega} g v \mathrm{~d} s
\end{aligned}
$$

## Approximation

As usual, we introduce a mesh $\mathcal{T}_{h}$ of $\Omega$ and the finite dimensional space $X_{h}$ :

$$
X_{h}=\left\{v \in H^{1}(\Omega) ; v_{/ K} \in P_{k}, \forall K \in \mathcal{T}_{h}\right\}
$$

The approximate problem writes:
$\left(V F_{3}\right)_{h}$ : find $u_{h} \in X_{h}$ such that:

$$
a\left(u_{h}, v_{h}\right)=l\left(v_{h}\right), \forall v_{h} \in X_{h}
$$

File 'neumann-nh.cc'

```
#include "rheolef.h"
using namespace rheolef;
using namespace std;
#include "sinusprod_helmholtz.icc"
int main(int argc, char**argv) {
    environment rheolef(argc, argv);
    geo omega (argv[1]);
    size_t d = omega.dimension();
    space Xh (omega, argv[2]);
    form m (Xh, Xh, "mass");
    form a (Xh, Xh, "grad_grad");
    a = m + a;
    field lh = riesz(Xh, f(d)) + riesz(Xh, "boundary", g(d));
    field uh (Xh);
    solver sa (a.uu());
    uh.set_u() = sa.solve (lh.u() - a.ub()*uh.b());
    dout << uh;
}
```

Let us choose $\Omega \subset R^{d}, d=1,2,3$ and

$$
\begin{aligned}
& f(x)=\left(1+d \pi^{2}\right) \prod_{i=0}^{d-1} \sin \left(\pi x_{i}\right) \\
& g(x)= \begin{cases}-\pi & \text { when } d=1 \\
-\pi\left(\sum_{i=0}^{d-1} \sin \left(\pi x_{i}\right)\right) & \text { when } d=2 \\
-\pi\left(\sum _ { i = 0 } ^ { d - 1 } \operatorname { s i n } ( \pi x _ { i } ) \operatorname { s i n } \left(x_{(i+1) \bmod d)}\right.\right. & \text { when } d=3\end{cases}
\end{aligned}
$$

This example is convenient, since the exact solution is known:

$$
u(x)=\prod_{i=0}^{d-1} \sin \left(\pi x_{i}\right)
$$

File 'sinusprod_helmholtz.icc'

```
struct f : unary_function<point,Float> {
    Float operator() (const point& x) const {
        switch (d) {
        case 1: return (1+d*pi*pi)*sin(pi*x[0]);
        case 2: return (1+d*pi*pi)*sin(pi*x[0])*sin(pi*x[1]);
        default: return (1+d*pi*pi)*sin(pi*x[0])*sin(pi*x[1])*sin(pi*x[2]);
    }}
f(size_t d1) : d(d1), pi(acos(Float(-1.0))) {}
size_t d; const Float pi;
};
struct g : unary_function<point,Float> {
    Float operator() (const point& x) const {
        switch (d) {
        case 1: return -pi;
        case 2: return -pi*(sin(pi*x[0]) + sin(pi*x[1]));
        default: return -pi*( sin(pi*x[0])*sin(pi*x[1])
                        + sin(pi*x[1])*sin(pi*x[2])
                        + sin(pi*x[2])*sin(pi*x[0]));
    }}
    g(size_t d1) : d(d1), pi(acos(Float(-1.0))) {}
    size_t d; const Float pi;
};
```


## Comments

The neumann-nh.cc code looks like the previous one dirichlet-nh.cc. Let us comments only the changes.

```
form m (Xh, Xh, "mass");
form a (Xh, Xh, "grad_grad");
```

The bilinear forms $m(.,$.$) and a(.,$.$) , also called the mass and energy in mechanics, are defined as:$

$$
m(u, v)=\int_{\Omega} u v \mathrm{~d} x \quad \text { and } \quad a(u, v)=\int_{\Omega} \nabla u . \nabla v \mathrm{~d} x
$$

Then, these two forms are added and stored in $a$ :

```
a = m + a;
```

The right-hand side is computed as:

```
field lh = riesz(Xh, f(d)) + riesz(Xh, "boundary", g(d));
```


### 2.2.1 How to run the program

First, compile the program:

```
make neumann-nh
```

Running the program is obtained from the homogeneous Dirichlet case, by replacing dirichlet by neumann-nh.

### 2.3 The Robin boundary conditions

## Formulation

Let $f \in H^{-1}(\Omega)$ and Let $g \in H^{\frac{1}{2}}(\partial \Omega)$. The problem writes: $\left(P_{4}\right)$ find $u$, defined in $\Omega$ such that:

$$
\begin{aligned}
-\Delta u & =f \text { in } \Omega \\
\frac{\partial u}{\partial n}+u & =g \text { on } \partial \Omega
\end{aligned}
$$

The variational formulation of this problem expresses:
$\left(V F_{4}\right):$ find $u \in H^{1}(\Omega)$ such that:

$$
a(u, v)=l(v), \forall v \in H^{1}(\Omega)
$$

where

$$
\begin{aligned}
a(u, v) & =\int_{\Omega} \nabla u \cdot \nabla v \mathrm{~d} x+\int_{\partial \Omega} u v \mathrm{~d} s \\
l(v) & =\int_{\Omega} u v \mathrm{~d} x+\int_{\partial \Omega} g v \mathrm{~d} s
\end{aligned}
$$

## Approximation

As usual, let

$$
X_{h}=\left\{v \in H^{1}(\Omega) ; v_{/ K} \in P_{k}, \forall K \in \mathcal{T}_{h}\right\}
$$

The approximate problem writes: $\left(V F_{4}\right)_{h}:$ find $u_{h} \in X_{h}$ such that:

$$
a\left(u_{h}, v_{h}\right)=l\left(v_{h}\right), \forall v_{h} \in X_{h}
$$

File 'robin.cc'

```
#include "rheolef.h"
using namespace rheolef;
using namespace std;
#include "cosinusprod_laplace.icc"
int main(int argc, char**argv) {
    environment rheolef(argc, argv);
    geo omega (argv[1]);
    size_t d = omega.dimension();
    space Xh (omega, argv[2]);
    form a (Xh, Xh, "grad_grad");
    form ab (Xh, Xh, "mass", omega["boundary"]);
    a = a + ab;
    field lh = riesz(Xh, f(d)) + riesz(Xh, "boundary", g(d));
    field uh (Xh);
    solver sa (a.uu());
    uh.set_u() = sa.solve (lh.u() - a.ub()*uh.b());
    dout << uh;
}
```


## Comments

The code robin. cc looks like the previous one neumann-nh.cc. Let us comments the changes.

```
form a (Xh, Xh, "grad_grad");
form ab (Xh, Xh, "mass", omega["boundary"]);
a = a + ab;
```

The bilinear forms $a(.,$.$) and a_{b}(.,$.$) are defined as:$

$$
a(u, v)=\int_{\Omega} \nabla u \cdot \nabla v \mathrm{~d} x \quad \text { and } \quad a_{b}(u, v)=\int_{\partial \Omega} u_{\mid \partial \Omega} v_{\mid \partial \Omega} \mathrm{d} s
$$

where $u_{\mid \partial \Omega}$ denotes the restriction to the boundary of a function $u$ defined in $\Omega$. Then the sum $a+a_{b}$ is stored in $a$. The boundary contribution to the $a(.,$.$) form on X_{h} \times X_{h}$ is introduced. Finally, the implementation of the right-hand sides $f$ and $g$ in file 'cosinusprod_laplace.icc' has already been presented on page 24 .

### 2.3.1 How to run the program

First, compile the program:

```
make robin
```

Running the program is obtained from the homogeneous Dirichlet case, by replacing dirichlet by robin.

## Chapter 3

## Non-constant coefficients and multi-regions

This chapter is related to the so-called transmission problem. We introduce some new concepts: problems with non-constant coefficients, regions in the mesh, weighted forms and discontinuous approximations.

## Formulation

Let us consider a diffusion problem with a non-constant diffusion coefficient $\eta$ in a domain bounded $\Omega \subset R^{d}, d=1,2,3:$
$(P):$ find $u$ defined in $\Omega$ such that:

$$
\begin{align*}
-\operatorname{div}(\eta \boldsymbol{\nabla} u) & =f \text { in } \Omega  \tag{3.1}\\
u & =0 \text { on } \Gamma_{\text {left }} \cup \Gamma_{\text {right }}  \tag{3.2}\\
\frac{\partial u}{\partial n} & =0 \text { on } \Gamma_{\text {top }} \cup \Gamma_{\text {bottom }} \text { when } d \geq 2  \tag{3.3}\\
\frac{\partial u}{\partial n} & =0 \text { on } \Gamma_{\text {front }} \cup \Gamma_{\text {back }} \text { when } d=3 \tag{3.4}
\end{align*}
$$

where $f$ is a given source term.

$\qquad$

Figure 3.1: Transmission problem: the domain $\Omega$ partition: ( $\Omega_{\mathrm{west}}$ and $\left.\Omega_{\mathrm{east}}\right)$.

We consider here the important special case when $\eta$ is picewise constant:

$$
\eta(x)= \begin{cases}\varepsilon & \text { when } x \in \Omega_{\text {west }} \\ 1 & \text { when } x \in \Omega_{\text {east }}\end{cases}
$$

where ( $\Omega_{\text {west }}, \Omega_{\text {east }}$ ) is a partition of $\Omega$ in two parts (see Fig. 3.1). This is the so-called transmission problem: the solution and the flux are continuous on the interface $\Gamma_{0}=\partial \Omega_{\text {east }} \cap \partial \Omega_{\text {west }}$ between the two domains where the problem reduce to a constant diffusion one:

$$
\begin{aligned}
u_{\Omega_{\mathrm{west}}} & =u_{\Omega_{\text {east }}} \text { on } \Gamma_{0} \\
\varepsilon \frac{\partial u_{/ \Omega_{\mathrm{west}}}}{\partial n} & =\frac{\partial u_{\Omega_{\mathrm{east}}}}{\partial n} \text { on } \Gamma_{0}
\end{aligned}
$$

It expresses the transmission of the quantity $u$ and its flux accross the interface $\Gamma_{0}$ between two regions that have different diffusion properties: Notice that the more classical problem, with constant diffusion $\eta$ on $\Omega$ is obtained by simply choosing when $\varepsilon=1$.
The variational formulation of this problem expresses:
(VF): find $u \in V$ such that:

$$
a(u, v)=l(v), \forall v \in V
$$

where the bilinear forms $a(.,$.$) and the linear one l($.$) are defined by$

$$
\begin{aligned}
a(u, v) & =\int_{\Omega} \eta \nabla u \cdot \nabla v d x, \quad \forall u, v \in H^{1}(\Omega) \\
l(v) & =\int_{\Omega} f v d x, \quad \forall v \in L^{2}(\Omega) \\
V & =\left\{v \in H^{1}(\Omega) ; v=0 \text { on } \Gamma_{\text {left }} \cup \Gamma_{\text {right }}\right\}
\end{aligned}
$$

The bilinear form $a(.,$.$) defines a scalar product in V$ and is related to the energy form. This form is associated to the $-\operatorname{div} \eta \boldsymbol{\nabla}$ operator.
The approximation of this problem could performed by a standard Lagrange $P_{k}$ continuous approximation.

## File 'transmission.cc'

```
#include "rheolef.h"
#include "rheolef/catchmark.h"
using namespace rheolef;
using namespace std;
int main(int argc, char**argv) {
    environment rheolef (argc, argv);
    const Float epsilon = 0.01;
    geo omega (argv[1]);
    space Xh (omega, argv[2]);
    Xh.block ("left");
    Xh.block ("right");
    string eta_approx = "P" + itos(Xh.degree()-1) + "d";
    space Qh (omega, eta_approx);
    field eta (Qh);
    eta ["east"] = 1;
    eta ["west"] = epsilon;
    form a (Xh, Xh, "grad_grad", eta);
    field lh = riesz (Xh, 1);
    field uh (Xh);
    uh["left"] = uh["right"] = 0;
    solver sa (a.uu());
    uh.set_u() = sa.solve (lh.u() - a.ub()*uh.b());
    dout << catchmark("epsilon") << epsilon << endl
        << catchmark("u") << uh;
}
```


## Comments

This file is quite similar to those studied in the first chapters of this book. Let us comment only the changes. The Dirichlet boundary condition applies no more on the whole boundary $\partial \Omega$ but on two parts $\Gamma_{\text {left }}$ and $\Gamma_{\text {right }}$. On the other boundary parts, an homogeneous Neumann boundary condition is used: since these conditions does not produce any additional terms in the variational formulation, there are also nothing to write in the C++ code for these boundries. We choose $f=1$ : this leads to a convenient test-problem, since the exact solution is known when $\Omega=] 0,1\left[{ }^{d}\right.$ :

$$
u(x)= \begin{cases}\frac{x_{0}}{2 \varepsilon}\left(\frac{1+3 \varepsilon}{2(1+\varepsilon)}-x_{0}\right) & \text { when } x_{0}<1 / 2 \\ \frac{1-x_{0}}{2}\left(x_{0}+\frac{1-\varepsilon}{2(1+\varepsilon)}\right) & \text { otherwise }\end{cases}
$$

The field $\eta$ belongs to a discontinuous finite element $P_{k-1}$ space denoted by $Q_{h}$ :

```
string eta_approx = "P" + itos(Xh.degree()-1) + "d";
space Qh (omega, eta_approx);
field eta (Qh);
```

For instance, when argv[2] contains "P2", i.e. $k=2$, then the string eta_approx takes value "P1d". Then $\eta$ is initialized by:

```
eta["east"] = 1;
eta["weast"] = epsilon;
```

The energy form $a$ is then constructed with $\eta$ as additional parameter:

```
form a (Xh, Xh, "grad_grad", eta);
```

Such forms with a additional weight function are called weighted forms in Rheolef.

## How to run the program?

Build the program as usual: make transmission. Then, creates a one-dimensional geometry with two regions:

```
mkgeo_grid -e 100 -region > line.geo
geo line.geo
```

The trivial mesh generator mkgeo_grid, defines two regions east and west when used with the -region option. This correspond to the situation:

$$
\Omega=[0,1]^{d}, \quad \Omega_{\mathrm{west}}=\Omega \cap\left\{x_{0}<1 / 2\right\} \quad \text { and } \quad \Omega_{\mathrm{east}}=\Omega \cap\left\{x_{0}>1 / 2\right\} .
$$

In order to avoid mistakes with the $\mathrm{C}++$ style indexation, we denote by $\left(x_{0}, \ldots, x_{d-1}\right)$ the cartesian coordinate system in $\mathbb{R}^{d}$.
Finaly, run the program and look at the solution:

```
make transmission
    ./transmission line.geo P1 > line.field
field line.field
```

Since the exact solution is a piecewise second order polynom and the change in the diffusion coefficient value fits the element boundaries, we obtain the exact solution for all the degrees of freedom of any $P_{k}$ approximation, $k \geq 1$, as shown on Fig. 3.2 when $k=1$. Moreover, when $k \geq 2$ then $u_{h}=u$ since $X_{h}$ contains the exact solution $u$.


Figure 3.2: Transmission problem: $u_{h}=\pi_{h}(u)\left(\varepsilon=10^{-2}, d=1, P_{1}\right.$ approximation $)$.

The two dimensionnal case corresponds to the commands:

```
mkgeo_grid -t 10 -region > square.geo
geo square.geo
./transmission square.geo P1 > square.field
field square.field -elevation
```

while the tridimensional to

```
mkgeo_grid -T 10 -region > cube.geo
    ./transmission cube.geo P1 > cube.mfield
field cube.field
```

As for all the others examples, you can replace P1 by higher-order approximations, change elments shapes, such as $\mathrm{q}, \mathrm{H}$ or P , and run distributed computations computations with mpirun.

## Chapter 4

## Neumann boundary conditions fot the Laplace operator

In this chapter we study how to solve a ill-posed problem with a solution defined up to a constant.

## Formulation

Let $f \in L^{2}(\Omega)$ and $g \in H^{\frac{1}{2}}(\partial \Omega)$ satisfying the following compatibility condition:

$$
\int_{\Omega} f \mathrm{~d} x+\int_{\partial \Omega} g \mathrm{~d} s=0
$$

The problem writes:
$\left(P_{5}\right)_{h}$ : find $u$, defined in $\Omega$ such that:

$$
\begin{aligned}
-\Delta u & =f \text { in } \Omega \\
\frac{\partial u}{\partial n} & =g \text { on } \partial \Omega
\end{aligned}
$$

Since this problem only involves the derivatives of $u$, it is clear that its solution is never unique [4, p. 11]. A discrete version of this problem could be solved iteratively by the conjugate gradient or the MINRES algorithm [5]. In order to solve it by a direct method, we turn the difficulty by seeking $u$ in the following space

$$
V=\left\{v \in H^{1}(\Omega) ; \quad b(v, 1)=0\right\}
$$

where

$$
b(v, \mu)=\int_{\Omega} v \mathrm{~d} x, \quad \forall v \in L^{2}(\Omega), \forall \mu \in \mathbb{R}
$$

The variational formulation of this problem writes: $\left(V F_{5}\right):$ find $u \in V$ such that:

$$
a(u, v)=l(v), \forall v \in V
$$

where

$$
\begin{aligned}
a(u, v) & =\int_{\Omega} \nabla u \cdot \nabla v d x \\
l(v) & =m(f, v)+m_{b}(g, v) \\
m(f, v) & =\int_{\Omega} f v d x \\
m_{b}(g, v) & =\int_{\partial \Omega} g v d s
\end{aligned}
$$

Since the direct discretization of the space $V$ is not an obvious task, the constraint $b(u, 1)=0$ is enforced by a Lagrange multiplier $\lambda \in \mathbb{R}$. Let us introduce the Lagrangian, defined for all $v \in H^{1}(\Omega)$ and $\mu \in \mathbb{R}$ by:

$$
L(v, \mu)=\frac{1}{2} a(v, v)+b(v, \mu)-l(v)
$$

The saddle point $(u, \lambda) \in H^{1}(\Omega) \times \mathbb{R}$ of this Lagrangian is characterized as the unique solution of:

$$
\begin{aligned}
a(u, v)+b(v, \lambda) & =l(v), \quad \forall v \in H^{1}(\Omega) \\
b(u, \mu) & =0, \quad \forall \mu \in \mathbb{R}
\end{aligned}
$$

It is clear that if $(u, \lambda)$ is solution of this problem, then $u \in V$ and $u$ is a solution of $\left(V F_{5}\right)$. Conversely, let $u \in V$ the solution of $\left(V F_{5}\right)$. Choosing $v=v_{0}$ where $v_{0}(x)=1, \forall x \in \Omega$ leads to $\lambda$ meas $(\Omega)=l\left(v_{0}\right)$. From the definition of $l($.$) and the compatibility condition between the data f$ and $g$, we get $\lambda=0$. Notice that the saddle point problem extends to the case when $f$ and $g$ does not satisfies the compatibility condition, and in that case $\lambda=l\left(v_{0}\right) /$ meas $(\Omega)$.

## Approximation

As usual, we introduce a mesh $\mathcal{T}_{h}$ of $\Omega$ and the finite dimensional space $X_{h}$ :

$$
X_{h}=\left\{v \in H^{1}(\Omega) ; v_{/ K} \in P_{k}, \forall K \in \mathcal{T}_{h}\right\}
$$

The approximate problem writes:
$\left(V F_{5}\right)_{h}:$ find $\left(u_{h}, \lambda_{h}\right) \in X_{h} \times \mathbb{R}$ such that:

$$
\begin{aligned}
a\left(u_{h}, v\right)+b\left(v, \lambda_{h}\right) & =l_{h}(v), \quad \forall v \in X_{h} \\
b\left(u_{h}, \mu\right) & =0, \quad \forall \mu \in \mathbb{R}
\end{aligned}
$$

where

$$
l_{h}(v)=m\left(\Pi_{h} f, v_{h}\right)+m_{b}\left(\pi_{h} g, v_{h}\right)
$$

File 'neumann-laplace.cc'

```
#include "rheolef.h"
using namespace rheolef;
using namespace std;
size_t d;
Float f (const point& x) { return 1; }
Float g (const point& x) { return -0.5/d; }
int main(int argc, char**argv) {
    environment rheolef (argc, argv);
    geo omega (argv[1]);
    d = omega.dimension();
    space Xh (omega, argv[2]);
    form m (Xh, Xh, "mass");
    form a (Xh, Xh, "grad_grad");
    field b = m*field(Xh,1);
    field lh = riesz(Xh, f) + riesz(Xh, "boundary", g);
    csr<Float> A = {{ a.uu(), b.u()},
            {trans(b.u()), 0 }};
    vec<Float> B = { lh.u(), 0 };
    solver sa (A);
    vec<Float> U = sa.solve (B);
    field uh(Xh);
    uh.set_u() = U [range(0,uh.u().size())];
    Float lambda = (U.size() == uh.u().size()+1) ? U [uh.u().size()] : 0;
#ifdef _RHEOLEF_HAVE_MPI
    mpi::b
#endif // _ RHEOLEF_HAVE_MPI
    dout << uh
        << "lambda" << lambda << endl;
}
```


## Comments

Let $\Omega \subset R^{d}, d=1,2,3$. We choose $f(x)=1$ and $g(x)=-1 /(2 d)$. This example is convenient, since the exact solution is known:

$$
u(x)=-\frac{1}{12}+\frac{1}{2 d} \sum_{i=1}^{d} x_{i}\left(1-x_{i}\right)
$$

The code looks like the previous ones. Let us comment the changes. The discrete bilinear form $b$ is computed as $b_{h} \in X_{h}$ that interprets as a linear application from $X_{h}$ to $\mathbb{R}: b_{h}\left(v_{h}\right)=m\left(v_{h}, 1\right)$. Thus $b_{h}$ is computed as

```
field b = m*field(Xh,1.0);
```

where the discrete bilinear form $m$ is identified to its matrix and field(Xh,1.0) is the constant vector equal to 1 . Let

$$
\mathcal{A}=\left(\begin{array}{cc}
\text { a.uu } & \operatorname{trans}(\mathrm{b} . \mathrm{u}) \\
\text { b.u } & 0
\end{array}\right), \quad \mathcal{U}=\binom{\text { uh.u }}{\text { lambda }}, \quad \mathcal{B}=\binom{\text { lh.u }}{0}
$$

The problem admits the following matrix form:

$$
\mathcal{A} \mathcal{U}=\mathcal{B}
$$

The matrix and right-hand side of the system are assembled by concatenation:

```
csr<Float> A = {{ a.uu, b.u},
    {trans(b.u), 0 }};
vec<Float> B = { lh.u, 0 };
```

where csr and vec are respectively the matrix and vector classes. The csr is the abbreviation of compressed sparse row, a sparse matrix compression standard format. Notice that the matrix $\mathcal{A}$ is symmetric and non-singular, but indefinite : it admits eigenvalues that are either strictly positive or strictly negative. Then, the $u h . u$ vector is extracted from the $U$ one:

```
uh.u = U [range(0,uh.u.size())];
```

The extraction of lambda from $U$ is more technical in a distributed environment. In a sequential one, since it is stored after the uh.u values, it could be simply written as:

```
Float lambda = U [uh.u.size()];
```

In a distributed environment, lambda is stored in $U$ on the last processor, identified by U.comm().size()-1. Here U.comm() denotes the communicator, from the boost::mpi library and $U . \operatorname{comm}() . \operatorname{size}()$ is the number of processors in use, e.g. as specified by the mpirun command. On this last processor, the array $U$ has size equal to $u h . u . \operatorname{size}()+1$ and lambda is stored in U[uh.u.size()]. On the others processors, the array $U$ has size equal to uh.u.size() and lambda is not available. The following statement extract lambda on the last processor and set it to zero on the others:

```
Float lambda = (U.size() == uh.u.size()+1) ? U [uh.u.size()] : 0;
```

Then, the value of lambda is broadcasted on the others processors:

```
mpi::broadcast (U.comm(), lambda, U.comm().size() - 1);
```

The preprocessing guards \#idef...\#endif assure that this example compile also when the library is not installed with the MPI support. Finally, the statement

```
dout << catchmark("u") << uh
    << catchmark("lambda") << lambda << endl;
```

writes the solution $\left(u_{h}, \lambda\right)$. The catchmark function writes marks together with the solution in the output stream. These marks are suitable for a future reading with the same format, as:

```
din >> catchmark("u") >> uh
    >> catchmark("lambda") >> lambda;
```

This is usefull for post-treatment, visualization and error analysis.

### 4.0.2 How to run the program

As usual, enter:

```
make neumann-laplace
mkgeo_grid -t 10 > square.geo
./neumann-laplace square P1 | field -
```


## Part II

## Fluids and solids computations

## Chapter 5

## The linear elasticity and the Stokes problems

### 5.1 The linear elasticity problem

## Formulation

The total Cauchy stress tensor expresses:

$$
\begin{equation*}
\sigma(\mathbf{u})=\lambda \operatorname{div}(\mathbf{u}) \cdot I+2 \mu D(\mathbf{u}) \tag{5.1}
\end{equation*}
$$

where $\lambda$ and $\mu$ are the Lamé coefficients. Here, $D(\mathbf{u})$ denotes the symmetric part of the gradient operator and div is the divergence operator. Let us consider the elasticity problem for the embankment, in $\Omega=] 0,1\left[{ }^{d}, d=2,3\right.$. The problem writes:
$(P):$ find $\mathbf{u}=\left(u_{0}, \ldots, u_{d-1}\right)$, defined in $\Omega$, such that:

$$
\begin{align*}
-\operatorname{div} \sigma(\mathbf{u}) & =\mathbf{f} \text { in } \Omega \\
\frac{\partial \mathbf{u}}{\partial \mathbf{n}} & =0 \text { on } \Gamma_{\text {top }} \cup \Gamma_{\text {right }}  \tag{5.2}\\
\mathbf{u} & =0 \text { on } \Gamma_{\text {left }} \cup \Gamma_{\text {bottom }}, \\
\mathbf{u} & =0 \text { on } \Gamma_{\text {front }} \cup \Gamma_{\text {back }}, \text { when } d=3
\end{align*}
$$

where $\mathbf{f}=(0,-1)$ when $d=2$ and $\mathbf{f}=(0,0,-1)$ when $d=3$. The Lamé coefficients are assumed to satisfy $\mu>0$ and $\lambda+\mu>0$. Since the problem is linear, we can suppose that $\mu=1$ without any loss of generality.


Figure 5.1: The boundary domains for the square and the cube.
recall that, in order to avoid mistakes with the C++ style indexation, we denote by $\left(x_{0}, \ldots, x_{d-1}\right)$ the cartesian coordinate system in $\mathbb{R}^{d}$.
For $d=2$ we define the boundaries:

$$
\begin{array}{ll}
\left.\Gamma_{\text {left }}=\{0\} \times\right] 0,1[, & \left.\Gamma_{\text {right }}=\{1\} \times\right] 0,1[ \\
\left.\Gamma_{\text {bottom }}=\right] 0,1[\times\{0\}, & \left.\Gamma_{\text {top }}=\right] 0,1[\times\{1\}
\end{array}
$$

and for $d=3$ :

$$
\begin{array}{ll}
\left.\Gamma_{\text {back }}=\{0\} \times\right] 0,1\left[^{2},\right. & \left.\Gamma_{\text {front }}=\{1\} \times\right] 0,1[2 \\
\left.\Gamma_{\text {left }}=\right] 0,1[\times\{0\} \times] 0,1[, & \left.\Gamma_{\text {right }}=\right] 0,1[\times\{1\} \times] 0,1[ \\
\left.\Gamma_{\text {bottom }}=\right] 0,1[2 \times\{0\}, & \left.\Gamma_{\text {top }}=\right] 0,1\left[^{2} \times\{1\}\right.
\end{array}
$$

These boundaries are represented on Fig. 5.1.
The variational formulation of this problem expresses:
$(V F):$ find $\mathbf{u} \in \mathbf{V}$ such that:

$$
\begin{equation*}
a(\mathbf{u}, \mathbf{v})=l(\mathbf{v}), \forall \mathbf{v} \in \mathbf{V} \tag{5.3}
\end{equation*}
$$

where

$$
\begin{aligned}
a(\mathbf{u}, \mathbf{v}) & =\lambda \int_{\Omega} \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v} d x+\int_{\Omega} 2 D(\mathbf{u}): D(\mathbf{v}) d x \\
l(\mathbf{v}) & =\int_{\Omega} \mathbf{f . v} d x \\
\mathbf{V} & =\left\{\mathbf{v} \in\left(H^{1}(\Omega)\right)^{2} ; \mathbf{v}=0 \text { on } \Gamma_{\text {left }} \cup \Gamma_{\text {bottom }}\right\}, \text { when } d=2 \\
\mathbf{V} & =\left\{\mathbf{v} \in\left(H^{1}(\Omega)\right)^{3} ; \mathbf{v}=0 \text { on } \Gamma_{\text {left }} \cup \Gamma_{\text {bottom }} \cup \Gamma_{\text {right }} \cup \Gamma_{\text {back }}\right\}, \text { when } d=3
\end{aligned}
$$

## Approximation

We introduce a mesh $\mathcal{T}_{h}$ of $\Omega$ and for $k \geq 1$, the following finite dimensional spaces:

$$
\begin{aligned}
& \mathbf{X}_{h}=\left\{\mathbf{v}_{h} \in\left(H^{1}(\Omega)\right)^{d} ; \mathbf{v}_{h / K} \in\left(P_{k}\right)^{d}, \forall K \in \mathcal{T}_{h}\right\} \\
& \mathbf{V}_{h}=\mathbf{X}_{h} \cap \mathbf{V}
\end{aligned}
$$

The approximate problem writes:
$(V F)_{h}:$ find $\mathbf{u}_{h} \in \mathbf{V}_{h}$ such that:

$$
a\left(\mathbf{u}_{h}, \mathbf{v}_{h}\right)=l\left(\mathbf{v}_{h}\right), \forall \mathbf{v}_{h} \in \mathbf{V}_{h}
$$

File 'embankment.cc'

```
#include "rheolef.h"
using namespace rheolef;
using namespace std;
#include "embankment.icc"
int main(int argc, char**argv) {
    environment rheolef(argc, argv);
    geo omega (argv[1]);
    space Xh = embankment_space (omega, argv[2]);
    Float lambda = (argc > 3) ? atof(argv[3]) : 1;
    size_t d = omega.dimension();
    point f (0,0,0);
    f[d-1] = -1;
    field lh = riesz (Xh, f);
    form a1 (Xh, Xh, "div_div");
    form a2 (Xh, Xh, "2D_D");
    form a = lambda*a1 + a2;
    field uh (Xh, 0);
    solver sa (a.uu());
    uh.set_u() = sa.solve (lh.u() - a.ub()*uh.b());
    dout << catchmark("inv_lambda") << 1/lambda << endl
        << catchmark("u") << uh;
}
```

File 'embankment.icc'

```
space embankment_space (const geo& omega, string approx) {
    space Xh (omega, approx, "vector");
    Xh.block("left");
    if (omega.dimension() >= 2) {
        Xh.block("bottom");
    }
    if (omega.dimension() == 3) {
        Xh.block("right");
        Xh.block("back");
    }
    return Xh;
}
```


## Comments

The space is defined in a separate file 'embankment.icc', since it will be reused in others examples along this chapter:

```
space Vh (omega, "P2", "vector");
```

Note here the multi-component specification "vector" as a supplementary argument to the space constructor. The boundary condition contain a special cases for bi- and tridimensionnal cases. The right-hand-side $\mathbf{f}_{h}$ represents the dimensionless gravity forces, oriented on the vertical axis: the last component of $\mathbf{f}_{h}$ is set to -1 as:

```
fh [omega.dimension()-1] = -1;
```

Finally, the $1 / \lambda$ parameter and the multi-field result are printed, using mark labels, thanks to the catchmark stream manipulator. Labels are convenient for post-processing purpose, as we will see in the next paragraph.

## How to run the program



Figure 5.2: The linear elasticity for $\lambda=1$ and $d=2$ and $d=3$ : both wireframe and filled surfaces ; stereoscopic anaglyph mode for 3D solutions.

We assume that the previous code is contained in the file 'embankment.cc'. Compile the program as usual (see page 16):

```
make embankment
```

and enter the comands:

```
mkgeo_grid -t 10 > square.geo
geo square.geo
```

The triangular mesh has four boundary domains, named left, right, top and bottom. Then, enter:

```
./embankment square.geo P1 > square-P1.field
```

The previous command solves the problem for the corresponding mesh and writes the multicomponent solution in the '.field' file format. Run the deformation vector field visualization using the default gnuplot render:

```
field square-P1.field
field square-P1.field -nofill
```

Note the graphic options usage ; the unix manual for the field command is available as:

```
man field
```

The view is shown on Fig. 5.2. A specific field component can be also selected for a scalar visualization:

```
field -comp O square-P1.field
field -comp 1 square-P1.field
```

Next, perform a $P_{2}$ approximation of the solution:

```
./embankment square.geo P2 > square-P2.field
field square-P2.field -mayavi -nofill
```

Finally, let us consider the three dimensional case

```
mkgeo_grid -T 10 > cube.geo
./embankment cube.geo P1 > cube-P1.field
field cube-P1.field -stereo
field cube-P1.field -stereo -fill
```

The two last commands show the solution in 3D stereoscopic anaglyph mode. The graphic is represented on Fig. 5.2. The $P_{2}$ approximation writes:

```
./embankment cube.geo P2 > cube-P2.field
field cube-P2.field
```


### 5.2 Computing the stress tensor

## Formulation and approximation

The following code computes the total Cauchy stress tensor, reading the Lamé coefficient $\lambda$ and the deformation field $\mathbf{u}_{h}$ from a .field file. Let us introduce:

$$
T_{h}=\left\{\tau_{h} \in\left(L^{2}(\Omega)\right)^{d \times d} ; \tau_{h}=\tau_{h}^{T} \text { and } \tau_{h ; i j / K} \in P_{k-1}, \forall K \in \mathcal{T}_{h}, 1 \leq i, j \leq d\right\}
$$

This computation expresses:
find $\sigma_{h}$ such that:

$$
m\left(\sigma_{h}, \tau\right)=b\left(\tau, \mathbf{u}_{h}\right), \forall \tau \in T_{h}
$$

where

$$
\begin{aligned}
m(\sigma, \tau) & =\int_{\Omega} \sigma: \tau d x \\
b(\tau, \mathbf{u}) & =\lambda \int_{\Omega} \operatorname{div}(\mathbf{u}) \operatorname{tr}(\tau) d x+\int_{\Omega} 2 D(\mathbf{u}): \tau d x
\end{aligned}
$$

where $\operatorname{tr}(\tau)=\sum_{i=1}^{d} \tau_{i i}$ is the trace of the tensor $\tau$.

File 'stress.cc'

```
#include "rheolef.h"
using namespace rheolef;
using namespace std;
int main(int argc, char** argv) {
    environment rheolef (argc,argv);
    Float inv_lambda;
    field uh;
    din >> catchmark("inv_lambda") >> inv_lambda
        >> catchmark("u") >> uh;
    const geo& omega = uh.get_geo();
    const space& Xh = uh.get_space();
    string grad_approx = "P" + itos(Xh.degree()-1) + "d";
    space Th (omega, grad_approx, "tensor");
    space TOh (omega, grad_approx);
    form two_D (Xh, Th, "2D");
    form div (Xh, TOh, "div");
    form inv_mt (Th, Th, "inv_mass");
    form inv_m (TOh, TOh, "inv_mass");
    field trace_h = inv_m*(div*uh);
    field sigma_h = inv_mt*(two_D*uh);
    if (inv_lambda != 0)
        for (size_t i_comp = 0; i_comp < uh.size(); i_comp++)
            sigma_h(i_comp,i_comp) += (1/inv_lambda)*trace_h;
    dout << catchmark("s") << sigma_h;
}
```


## Comments

First notice that this code applies for any deformation field, and is not rectricted to our embankment problem.
The P0 and P1d stands for the piecewise constant and picewise linear discontinuous approximations, respectively. Since elements of $T_{h}$ are discontinuous accross interelement boundaries, the mass operator is block-diagonal and can be inverted one time for all: this operation results in the inv_mass operator.

## How to run the program



Figure 5.3: The stress tensor visualization (linear elasticity $\lambda=1$ ).

First, compile the program:

```
make stress
```

The visualization for the stress tensor as ellipes writes:

```
./stress < square-P1.field > square-stress-P1.field
field square-stress-P1.field -proj -mayavi
```

Recall that the stress, as a derivative of the deformation, is P0 (resp. P1d) and discontinuous when the deformation is P1 (resp. P2) and continuous. The approximate stress tensor field is projected on a continuous piecewise linear space, using the -proj option. Conversely, the 3D visualization bases on ellipsoides:

```
./stress < cube-P1.field > cube-stress-P1.field
field cube-stress-P1.field -proj -mayavi -stereo
```



Figure 5.4: The $\sigma_{01}$ stress component (linear elasticity $\lambda=1$ ): $d=2$ (top) and $d=3$ (bottom) ; $P_{0}$ (left) and $P_{1}$ discontinuous approximation (right).

You can observe a discontinuous constant or piecewise linear representation of the approximate stress component $\sigma_{01}$ (see Fig. 5.4):

```
field square-stress-P1.field -comp 01
field square-stress-P2.field -comp 01 -elevation
field square-stress-P2.field -comp 01 -elevation -stereo
```

Notice that the -stereo implies the -mayavi one, as this feature is not available with others visualization systems. The approximate stress field can be also projected on a continuous piecewise space:

```
field square-stress-P2.field -comp 01 -elevation -proj
```

The tridimensionnal case writes simply (see Fig. 5.4):

```
./stress < cube-P1.field > cube-stress-P1.field
```

```
./stress < cube-P2.field > cube-stress-P2.field
field cube-stress-P1.field -comp 01 -stereo
field cube-stress-P2.field -comp 01 -stereo -iso
```

and also the P1-projected versions write:

```
field cube-stress-P1.field -comp 01 -stereo -proj -iso
field cube-stress-P2.field -comp 01 -stereo -proj -iso
```

These operations can be repeated for each $\sigma_{i j}$ components and for both P1 and P2 approximation of the deformation field.

### 5.3 Mesh adaptation

The main principle of the auto-adaptive mesh writes [6-11]:

```
cin >> omega;
uh = solve(omega);
for (unsigned int i = 0; i < n; i++) {
    ch = criterion(uh);
    omega = adapt(ch);
    uh = solve(omega);
}
```

The initial mesh is used to compute a first solution. The adaptive loop compute an adaptive criterion, denoted by ch, that depends upon the problem under consideration and the polynomial approximation used. Then, a new mesh is generated, based on this criterion. A second solution on an adapted mesh can be constructed. The adaptation loop converges generaly in roughly 5 to 20 iterations.
Let us apply this principle to the elasticity problem.

File 'embankment_adapt.cc'

```
#include "rheolef.h"
using namespace rheolef;
using namespace std;
#include "elasticity_solve.icc"
#include "elasticity_criterion.icc"
#include "embankment.icc"
int main(int argc, char**argv) {
    environment rheolef (argc, argv);
    const Float lambda = 1;
    geo omega (argv[1]);
    adapt_option_type options;
    string approx = (argc > 2) ? argv[2] : "P1";
    options.err = (argc > 3) ? atof (argv[3]) : 5e-3;
    size_t n_adapt = (argc > 4) ? atoi(argv[4]) : 5;
    options.hmin = 0.004;
    for (size_t i = 0; true; i++) {
        space Xh = embankment_space (omega, approx);
        field uh = elasticity_solve (Xh, lambda);
        odiststream of (omega.name(), "field");
        of << catchmark("lambda") << lambda << endl
            << catchmark("u") << uh;
        if (i == n_adapt) break;
        field ch = elasticity_criterion (lambda,uh);
        omega = adapt(ch, options);
        odiststream og (omega.name(), "geo");
        og << omega;
    }
}
```

File 'elasticity_solve.icc'

```
field elasticity_solve(const space& Xh, Float lambda) {
    size_t d = Xh.get_geo().dimension();
    point f (0,0,0);
    f[d-1] = -1;
    field lh = riesz (Xh, f);
    form m (Xh, Xh, "mass");
    form a1 (Xh, Xh, "div_div");
    form a2 (Xh, Xh, "2D_D");
    form a = lambda*a1 + a2;
    solver sa (a.uu());
    field uh (Xh, 0);
    uh.set_u() = sa.solve (lh.u() - a.ub()*uh.b());
    return uh;
}
```

File 'elasticity_criterion.icc'

```
field elasticity_criterion (Float lambda, const field& uh) {
    string grad_approx = (uh.get_approx() == "P2") ? "P1d" : "P0";
    if (grad_approx == "PO") return norm(uh);
    space Th (uh.get_geo(), grad_approx, "tensor");
    space Xh (uh.get_geo(), grad_approx);
    form two_D (uh.get_space(), Th, "2D");
    form div (uh.get_space(), Xh, "div");
    form mt (Th, Th, "mass");
    form m (Xh, Xh, "mass");
    form inv_mt (Th, Th, "inv_mass");
    form inv_m (Xh, Xh, "inv_mass");
    field qh = inv_m*(div*uh);
    field two_Duh= inv_mt*(two_D*uh);
    return sqrt(lambda*sqr(qh) + norm2(two_Duh));
}
```


## Comments

The criterion is here:

$$
c_{h}= \begin{cases}\left|\mathbf{u}_{h}\right| & \text { when using } P_{1} \\ \left(\sigma\left(\mathbf{u}_{h}\right): D\left(\mathbf{u}_{h}\right)\right)^{1 / 2} & \text { when using } P_{2}\end{cases}
$$

The adapt_option_type declaration is used by Rheolef to send options to the mesh generator. The err parameter controls the error via the edge length of the mesh: the smaller it is, the smaller the edges of the mesh are. In our example, is set by default to one. Conversely, the hmin parameter controls minimal edge length.

## How to run the program



Figure 5.5: Adapted meshes: the deformation visualization for $P_{1}$ and $P_{2}$ approximations.

The compilation command writes:

```
make embankment_adapt
```

The mesh loop adaptation is initiated from a bamg mesh (see also appendix B.1).

```
bamg -g square.bamgcad -o square.bamg
bamg2geo square.bamg square.dmn > square.geo
./embankment_adapt square P1
```

The code performs a loop of five mesh adaptations: the corresponding meshes are stored in files, from square-1.geo.gz to square-5.geo.gz, and the associated solutions in files, from square-1.field.gz to square-5.field.gz. The additional '.gz' suffix expresses that the files are compressed using gzip.

```
geo square-5.geo
field square-5.field -mayavi -nofill
```

Note that the '.gz' suffix is automatically assumed by the geo and the field commands.
For a piecewise quadratic approximation:

```
./embankment_adapt square P2
```

Then, the visualization writes:

```
geo square-5.geo
field square-5.field -mayavi -nofill
```

A one-dimensional mesh adaptive procedure is also possible:

```
gmsh -1 line.mshcad -o line.msh
msh2geo < line.msh | geo -upgrade - > line.geo
geo line.geo
./embankment_adapt line P2
geo line-5.geo
field line-5.field -comp 0 -elevation
```

The three-dimensional extension this mesh adaptive procedure is in development.

### 5.4 The Stokes problem

## Formulation

Let us consider the Stokes problem for the driven cavity in $\Omega=] 0,1\left[{ }^{d}, d=2,3\right.$. The problem writes:
(S) find $\mathbf{u}=\left(u_{0}, \ldots, u_{d-1}\right)$ and $p$ defined in $\Omega$ such that:

$$
\begin{aligned}
-\operatorname{div}(2 D(\mathbf{u}))+\nabla p & =0 \text { in } \Omega \\
-\operatorname{div} \mathbf{u} & =0 \text { in } \Omega \\
\mathbf{u} & =(1,0) \text { on } \Gamma_{\text {top }}, \\
\mathbf{u} & =0 \text { on } \Gamma_{\text {left }} \cup \Gamma_{\text {right }} \cup \Gamma_{\text {bottom }}, \\
\frac{\partial u_{0}}{\partial \mathbf{n}}=\frac{\partial u_{1}}{\partial \mathbf{n}} & =u_{2}=0 \text { on } \Gamma_{\text {back }} \cup \Gamma_{\text {front }} \text { when } d=3
\end{aligned}
$$

where $D(\mathbf{u})=\left(\nabla \mathbf{u}+\nabla \mathbf{u}^{T}\right) / 2$. The boundaries are represented on Fig. 5.1, page 44 . The variational formulation of this problem expresses:
$(V F S)$ find $\mathbf{u} \in \mathbf{V}(1)$ and $p \in L_{0}^{2}(\Omega)$ such that:

$$
\begin{array}{ll}
a(\mathbf{u}, \mathbf{v})+b(\mathbf{v}, p) & =0, \forall \mathbf{v} \in \mathbf{V}(0) \\
b(\mathbf{u}, q) & =0, \forall q \in L_{0}^{2}(\Omega)
\end{array}
$$

where

$$
\begin{aligned}
& a(\mathbf{u}, \mathbf{v})= \int_{\Omega} 2 D(\mathbf{u}): D(\mathbf{v}) d x \\
& b(\mathbf{v}, q)=-\int_{\Omega} \operatorname{div}(\mathbf{v}) q d x . \\
& \mathbf{V}(\alpha)=\left\{\mathbf{v} \in\left(H^{1}(\Omega)\right)^{2} ; \mathbf{v}=0 \text { on } \Gamma_{\text {left }} \cup \Gamma_{\text {right }} \cup \Gamma_{\text {bottom }} \text { and } \mathbf{v}=(\alpha, 0) \text { on } \Gamma_{\text {top }}\right\}, \text { when } d=2, \\
& \mathbf{V}(\alpha)=\left\{\mathbf{v} \in\left(H^{1}(\Omega)\right)^{3} ; \mathbf{v}=0 \text { on } \Gamma_{\text {left }} \cup \Gamma_{\text {right }} \cup \Gamma_{\text {bottom }},\right. \\
&\left.\mathbf{v}=(\alpha, 0,0) \text { on } \Gamma_{\text {top }} \text { and } v_{2}=0 \text { on } \Gamma_{\text {back }} \cup \Gamma_{\text {front }}\right\}, \quad \text { when } d=3, \\
& L_{0}^{2}(\Omega)=\left\{q \in L^{2}(\Omega) ; \int_{\Omega} q d x=0\right\} .
\end{aligned}
$$

## Approximation

The Taylor-Hood [12] finite element approximation of the Stokes problem is considered. We introduce a mesh $\mathcal{T}_{h}$ of $\Omega$ and the following finite dimensional spaces:

$$
\begin{aligned}
\mathbf{X}_{h} & =\left\{\mathbf{v} \in\left(H^{1}(\Omega)\right)^{d} ; \mathbf{v}_{/ K} \in\left(P_{2}\right)^{d}, \forall K \in \mathcal{T}_{h}\right\} \\
\mathbf{V}_{h}(\alpha) & =\mathbf{X}_{h} \cap \mathbf{V}(\alpha) \\
Q_{h} & \left.=\left\{q \in L^{2}(\Omega)\right) \cap C^{0}(\bar{\Omega}) ; q_{/ K} \in P_{1}, \forall K \in \mathcal{T}_{h}\right\}
\end{aligned}
$$

The approximate problem writes:
$(V F S)_{h}$ find $\mathbf{u}_{h} \in \mathbf{V}_{h}(1)$ and $p \in Q_{h}$ such that:

$$
\begin{align*}
a\left(\mathbf{u}_{h}, \mathbf{v}\right)+b\left(\mathbf{v}, p_{h}\right) & =0, \forall \mathbf{v} \in \mathbf{V}_{h}(0)  \tag{5.4}\\
b\left(\mathbf{u}_{h}, q\right) & =0, \forall q \in Q_{h}
\end{align*}
$$

## File 'cavity.icc'

```
space cavity_space (const geo& omega_h, std::string approx) {
    space Xh (omega_h, approx, "vector");
    Xh.block("top"); Xh.block("bottom");
    if (omega_h.dimension() == 3) {
        Xh.block("back"); Xh.block("front");
        Xh[1].block("left"); Xh[1].block("right");
    } else {
        Xh.block("left"); Xh.block("right");
    }
    return Xh;
}
field cavity_field (const space& Xh, Float alpha) {
    field uh (Xh, O.);
    uh[0]["top"] = alpha;
    return uh;
}
```

File 'stokes_cavity.cc'

```
#include "rheolef.h"
using namespace rheolef;
using namespace std;
#include "cavity.icc"
int main(int argc, char**argv) {
    environment rheolef (argc, argv);
    geo omega (argv[1]);
    space Xh = cavity_space (omega, "P2");
    field uh = cavity_field (Xh, 1);
    space Qh (omega, "P1");
    field ph (Qh, 0.);
    form a (Xh, Xh, "2D_D");
    form b = - form(Xh, Qh, "div");
    form mp (Qh, Qh, "mass");
    solver_abtb stokes (a.uu(), b.uu(), mp.uu()) ;
    stokes.solve (-(a.ub()*uh.b()), -(b.ub()*uh.b()),
    uh.set_u(), ph.set_u());
    dout << catchmark("u") << uh
        << catchmark("p") << ph;
}
```


## Comments

The spaces and boundary conditions and grouped in specific functions, defined in file 'cavity.icc'. This file is suitable for a future reusage. Next, forms are defined as usual, in file 'stokes_cavity.cc'.

The problem admits the following matrix form:

$$
\left(\begin{array}{cc}
\text { a.uu } & \operatorname{trans(b.uu)} \\
\text { b.uu } & 0
\end{array}\right)\binom{\text { uh.u }}{\text { ph.u }}=\binom{- \text { a.ub } * \text { uh.b }}{- \text { b.ub } * \text { uh.b }}
$$

An initial value for the pressure field is provided:

```
field ph (Qh, 0);
```

The main Stokes solver call writes:

```
solver_abtb stokes (a.uu(), b.uu(), mp.uu());
stokes.solve (-(a.ub()*uh.b()), -(b.ub()*uh.b()),
    uh.set_u(), ph.set_u());
```

For tridimensional geometries $(d=3)$, this system is solved by the preconditioned conjugate gradient algorithm. the preconditioner is here the mass matrix mp.uu for the pressure: as showed in [13], the number of iterations need by the conjugate gradient algorithm to reach a given precision is then independent of the mesh size. For more details, see the Rheolef reference manual related to mixed solvers, available e.g. via the unix command:

```
man solver_abtbc
```

When $d=2$, it is interessant to turn to direct methods and factorize the whole matrix of the linear system. As the pressure is defined up to a constant, the whole matrix is singular. By adding a Lagrange multiplier that impose a nul average pressure value, the system becomes regular and the modified matrix can be inverted. Such a technic has already been presented in section 4 for the Neumann-Laplace problem. Finaly, he choice between iterative and direct algorithm for the Stokes solver is automatically done, regarding the geometry dimension.

## How to run the program



Figure 5.6: The velocity visualization for $d=2$ and $d=3$ with stereo anaglyph.

We assume that the previous code is contained in the file 'stokes_cavity.cc'. Then, compile the program as usual (see page 16):

```
make stokes_cavity
```

and enter the comands:

```
mkgeo_grid -t 10 > square.geo
./stokes_cavity square > square.field
```

The previous command solves the problem for the corresponding mesh and writes the solution in a'.field' file. Run the velocity vector visualization :

```
field square.field -velocity
```

Run also some scalar visualizations:

```
field square.field -comp 0
field square.field -comp 1
field square.field -catchmark p
```

Note the -catchmark option to the field command: the file reader jumps to the label and then starts to read the selected field. Next, perform another computation on a finer mesh:

```
mkgeo_grid -t 20 > square-20.geo
./stokes_cavity square-20.geo > square-20.field
```

and observe the convergence.
Finally, let us consider the three dimensional case:

```
mkgeo_grid -T 5 > cube.geo
./stokes_cavity cube.geo > cube.field
```

and the corresponding visualization:

```
field cube.field -velocity
field cube.field -comp O
field cube.field -comp 1
field cube.field -comp 2
field cube.field -catchmark p
```


### 5.5 Computing the vorticity

## Formulation and approximation

When $d=2$, we define [4, page 30] for any distributions $\phi$ and $\mathbf{v}$ :

$$
\begin{aligned}
\operatorname{curl} \phi & =\left(\frac{\partial \phi}{\partial x_{1}},-\frac{\partial \phi}{\partial x_{0}}\right) \\
\operatorname{curl} \mathbf{v} & =\frac{\partial v_{1}}{\partial x_{0}}-\frac{\partial v_{0}}{\partial x_{1}},
\end{aligned}
$$

and when $d=3$ :

$$
\mathbf{c u r l} \mathbf{v}=\left(\frac{\partial v_{2}}{\partial x_{1}}-\frac{\partial v_{1}}{\partial x_{2}}, \frac{\partial v_{0}}{\partial x_{2}}-\frac{\partial v_{2}}{\partial x_{0}}, \frac{\partial v_{1}}{\partial x_{0}}-\frac{\partial v_{0}}{\partial x_{1}}\right)
$$

Let $\mathbf{u}$ be the solution of the Stokes problem $(S)$. The vorticity is defined by:

$$
\begin{array}{ll}
\omega=\operatorname{curl} \mathbf{u} & \text { when } d=2 \\
\boldsymbol{\omega}=\operatorname{curl} \mathbf{u} & \text { when } d=3
\end{array}
$$

Since the approximation of the velocity is piecewise quadratic, we are looking for a discontinuous piecewise linear vorticity field that belongs to:

$$
\begin{aligned}
Y_{h} & =\left\{\xi \in L^{2}(\Omega) ; \xi / K \in P_{1}, \forall K \in \mathcal{T}_{h}\right\}, & & \text { when } d=2 \\
\mathbf{Y}_{h} & =\left\{\boldsymbol{\xi} \in\left(L^{2}(\Omega)\right)^{3} ; \xi_{i / K} \in P_{1}, \forall K \in \mathcal{T}_{h}\right\}, & & \text { when } d=3
\end{aligned}
$$

The approximate variational formulation writes:

$$
\begin{aligned}
& \omega_{h} \in Y_{h}, \quad \int_{\Omega} \omega_{h} \xi d x=\int_{\Omega} \operatorname{curl} \mathbf{u}_{h} \xi d x, \forall \xi \in Y_{h} \quad \text { when } d=2 \\
& \boldsymbol{\omega} \in \mathbf{Y}_{h}, \quad \int_{\Omega} \boldsymbol{\omega}_{h} \cdot \boldsymbol{\xi} d x=\int_{\Omega} \operatorname{curl} \mathbf{u}_{h} . \boldsymbol{\xi} d x, \forall \boldsymbol{\xi} \in \mathbf{Y}_{h} \quad \text { when } d=3
\end{aligned}
$$

File 'vorticity.cc'

```
#include "rheolef.h"
using namespace rheolef;
using namespace std;
int main(int argc, char** argv) {
    environment rheolef (argc, argv);
    field uh;
    din >> uh;
    string grad_approx = "P" + itos(uh.get_space().degree()-1) + "d";
    string valued = (uh.size() == 3) ? "vector" : "scalar";
    space Lh (uh.get_geo(), grad_approx, valued);
    form curl (uh.get_space(), Lh, "curl");
    form inv_m (Lh, Lh, "inv_mass");
    dout << catchmark("w") << inv_m*(curl*uh);
}
```


## How to run the program



Figure 5.7: The vorticity: elevation view for $d=2$ and vector representation for $d=3$ (with anaglyph).

For $d=2$, just enter:

```
make vorticity
./vorticity < square.field | field -elevation -stereo -
```

and you observe a discontinuous piecewise linear representation of the approximate vorticity. Also, the vorticity presents two sharp peacks at the upper corners of the driven cavity: the vorticity is unbounded and the peacks will increase with mesh refinements. This singularity of the solution is due to the boundary condition for the first component of the velocity $u_{0}$ that jumps from zero to one at the corners. The approximate vorticity field can also be projected on a continuous piecewise linear space, using the -proj option (See Fig. 5.7 left):

```
./vorticity < square.field | field -elevation -stereo -nofill -
./vorticity < square.field | field -elevation -stereo -proj -
```

For $d=3$, the whole vorticity vector can also be visualized (See Fig. 5.7 right):

```
./vorticity < cube.field | field -proj -velocity -stereo -
```

In the previous command, the -proj option has been used: since the 3D render has no support for discontinuous picewise linear fields, the P1-discontinuous field is transformed into a P1-continuous one, thanks to a $L^{2}$ projection. P1 The following command shows the second component of the vorticity vector, roughtly similar to the bidimensionnal case.

```
./vorticity < cube.field | field -comp 1 -
./vorticity < cube.field | field -comp 1 -proj -
```


### 5.6 Computing the stream function

## Formulation and approximation

When $d=3$, the stream function is a vector-valued field $\boldsymbol{\psi}$ that satisfies [4, page 90 ]: $\mathbf{c u r l} \boldsymbol{\psi}=\mathbf{u}$ and $\operatorname{div} \boldsymbol{\psi}=0$. From the identity:

$$
\operatorname{curl} \operatorname{curl} \psi=-\Delta \psi+\nabla(\operatorname{div} \psi)
$$

we obtain the following caracterization of $\boldsymbol{\psi}$ :

$$
\begin{array}{rlcll}
-\Delta \boldsymbol{\psi} & = & \operatorname{curl} \mathbf{u} & & \text { in } \Omega \\
\boldsymbol{\psi} & =0 & & \text { on } \Gamma_{\text {back }} \cup \Gamma_{\text {front }} \cup \Gamma_{\text {top }} \cup \Gamma_{\text {bottom }}, \\
\frac{\partial \boldsymbol{\psi}}{\partial n} & = & 0 & & \text { on } \Gamma_{\text {left }} \cup \Gamma_{\text {right }} .
\end{array}
$$

When $d=2$, the stream function $\psi$ is a scalar-valued field the solution of the following problem [4, page 88]:

$$
\begin{array}{rlll}
-\Delta \psi & = & \operatorname{curl} \mathbf{u} & \\
\text { in } \Omega \\
\psi & =0 & & \text { on } \partial \Omega
\end{array}
$$

File 'streamf_cavity.cc'

```
#include "rheolef.h"
using namespace rheolef;
using namespace std;
int main (int argc, char** argv) {
    environment rheolef (argc, argv);
    field uh;
    din >> uh;
    string valued = (uh.size() == 3) ? "vector" : "scalar";
    space Ph (uh.get_geo(), "P2", valued);
    Ph.block("top"); Ph.block("bottom");
    if (uh.get_geo().dimension() == 3) {'
        Ph.block("back"); Ph.block("front");
    } else {
        Ph.block("left"); Ph.block("right");
    }
    const space& Xh = uh.get_space();
    form a (Ph, Ph, "grad_grad");
    form b (Xh, Ph, "curl");
    field psi_h (Ph, O.);
    field lh = b*uh;
    solver sa (a.uu());
    psi_h.set_u() = sa.solve (lh.u() - a.ub()*psi_h.b());
    dout << catchmark("psi") << psi_h;
}
```


## How to run the program



Figure 5.8: The stream function visualization: isolines for $d=2$, and combined vectors and isonorm surface for $d=3$.

For $d=2$, just enter (see Fig. 5.8 left):

```
make streamf_cavity
./streamf_cavity < square.field | field -bw -
```

For $d=3$, the whole stream function vector can be visualized:

```
./streamf_cavity < cube.field | field -velocity -
```

The second component of the stream function is showed by:

```
./streamf_cavity < cube.field | field -comp 1
```

The combined representation of Fig. 5.8.right has been obtained in two steps. First, enter:

```
./streamf_cavity < cube.field | field -comp 1 -noclean -noexecute -
mv output.vtk psi1.vtk
./streamf_cavity < cube.field | field -velocity -
```

The -noclean -noexecute options cause the creation of the '.vtk' file for the second componnt, without running the mayavi visualization. Next, in the mayavi window associated to the whole stream function, select the File/Load data/VTK file menu and load 'psi1.vtk'. Finaly, select the Vizualize/Module/IsoSurface menu. Observe that the 3D stream function is mainly represented by its second component.

## Chapter 6

## Nearly incompressible elasticity and the stabilized Stokes problems

### 6.1 The incompressible elasticity problem

## Formulation

Let us go back to the linear elasticity problem.
When $\lambda$ becomes large, this problem is related to the incompressible elasticity and cannot be solved as it was previously done. To overcome this difficulty, the pressure is introduced :

$$
p=-\lambda \operatorname{div} \mathbf{u}
$$

and the problem becomes:
$(E)$ find $\mathbf{u}$ and $p$ defined in $\Omega$ such that:

$$
\begin{array}{cc}
-\operatorname{div}(2 D(\mathbf{u})) & +\nabla p=\mathbf{f} \text { in } \Omega \\
-\operatorname{div} \mathbf{u} & -\frac{1}{\lambda} p=0 \text { in } \Omega \\
+B . C . &
\end{array}
$$

The variational formulation of this problem expresses:
(VFE) find $\mathbf{u} \in V(1)$ and $p \in L^{2}(\Omega)$ such that:

$$
\begin{aligned}
& a(\mathbf{u}, \mathbf{v})+b(\mathbf{v}, p)=m(\mathbf{f}, \mathbf{v}), \forall \mathbf{v} \in V(0) \\
& b(\mathbf{u}, q)-c(p, q)=0, \forall q \in L_{0}^{2}(\Omega)
\end{aligned}
$$

where

$$
\begin{aligned}
m(\mathbf{u}, \mathbf{v}) & =\int_{\Omega} \mathbf{u} \mathbf{v} d x \\
a(\mathbf{u}, \mathbf{v}) & =\int_{\Omega} D(\mathbf{u}): D(\mathbf{v}) d x \\
b(\mathbf{v}, q) & =-\int_{\Omega} \operatorname{div}(\mathbf{v}) q d x \\
c(p, q) & =\frac{1}{\lambda} \int_{\Omega} p q d x \\
V & =\left\{\mathbf{v} \in\left(H^{1}(\Omega)\right)^{2} ; \mathbf{v}=0 \text { on } \Gamma_{\text {left }} \cup \Gamma_{\text {bottom }}\right\}
\end{aligned}
$$

When $\lambda$ becomes large, we obtain the incompressible elasticity problem, that coincides with the Stokes problem.

## Approximation

As for the Stokes problem, the Talor-Hood [12] finite element approximation is considered. We introduce a mesh $\mathcal{T}_{h}$ of $\Omega$ and the following finite dimensional spaces:

$$
\begin{aligned}
X_{h} & =\left\{\mathbf{v} \in\left(H^{1}(\Omega)\right) ; \mathbf{v}_{/ K} \in\left(P_{2}\right)^{2}, \forall K \in \mathcal{T}_{h}\right\}, \\
V_{h}(\alpha) & =X_{h} \cap V \\
Q_{h} & \left.=\left\{q \in L^{2}(\Omega)\right) \cap C^{0}(\bar{\Omega}) ; q_{/ K} \in P_{1}, \forall K \in \mathcal{T}_{h}\right\},
\end{aligned}
$$

The approximate problem writes:
$(V F E)_{h}$ find $\mathbf{u}_{h} \in V_{h}(1)$ and $p \in Q_{h}$ such that:

$$
\begin{aligned}
& a\left(\mathbf{u}_{h}, \mathbf{v}\right)+b\left(\mathbf{v}, p_{h}\right)=0, \forall \mathbf{v} \in V_{h}(0) \\
& b\left(\mathbf{u}_{h}, q\right)-c(p, q)=0, \forall q \in Q_{h}
\end{aligned}
$$

## File 'incompressible-elasticity.cc'

```
#include "rheolef.h"
using namespace rheolef;
using namespace std;
#include "embankment.icc"
int main(int argc, char**argv) {
    environment rheolef (argc, argv);
    geo omega (argv[1]);
    Float inv_lambda = (argc > 2 ? atof(argv[2]) : 0);
    size_t d = omega.dimension();
    space Xh = embankment_space(omega, "P2");
    space Qh (omega, "P1");
    point f (0,0,0);
    f [d-1] = -1;
    field lh = riesz (Xh, f);
    form mp (Qh, Qh, "mass");
    form a (Xh, Xh, "2D_D");
    form b (Xh, Qh, "div"); b = -b;
    form c = inv_lambda*mp;
    field uh (Xh, 0);
    field ph (Qh, 0);
    solver_abtb elasticity (a.uu(), b.uu(), c.uu(), mp.uu());
    elasticity.solve (lh.u() - a.ub()*uh.b(), -(b.ub()*uh.b()),
    uh.set_u(), ph.set_u());
    dout << catchmark("inv_lambda") << inv_lambda << endl
        << catchmark("u") << uh
        << catchmark("p") << ph;
}
```


## Comments

The problem admits the following matrix form:

$$
\left(\begin{array}{cc}
\text { a.uu } & \operatorname{trans(b.uu)} \\
\text { b.uu } & - \text { c.uu }
\end{array}\right)\binom{\text { uh.u }}{\text { ph.u }}=\binom{\text { lh.u }-\mathrm{a} . \mathrm{ub} * \mathrm{uh.} . \mathrm{b}}{-\mathrm{b} . \mathrm{ub} * \text { uh.b }}
$$

The problem is similar to the Stokes one (see page 56 ). This system is solved by:

```
solver_abtb elasticity (a.uu(), b.uu(), c.uu(), mp.uu());
elasticity.solve (lh.u() - a.ub()*uh.b(), -(b.ub()*uh.b()),
    uh.set_u(), ph.set_u());
```

For two-dimensional problems, a direct solver is used by default. In the three-dimensional case, an iterative algorithm is the default: the preconditionned conjugate gradient. The preconditioner is here the mass matrix mp. uu for the pressure. As showed in [13], the number of iterations need by the conjugate gradient algorithm to reach a given precision is then independent of the mesh size and is uniformly bounded when $\lambda$ becomes small, i.e. in the incompressible case.

## How to run the program



Figure 6.1: The incompressible linear elasticity $(\lambda=+\infty)$ for $N=2$ and $N=3$.

We assume that the previous code is contained in the file 'incompressible-elasticity.cc'. Compile the program as usual (see page 16):

```
make incompressible-elasticity
```

and enter the comands:

```
mkgeo_grid -t 10 > square.geo
./incompressible-elasticity square.geo 0 > square.field
field square.field -mayavi -nofill
mkgeo_grid -T 10 > cube.geo
./incompressible-elasticity cube.geo 0 > cube.field
field cube.field -mayavi -fill
```

The visualization is performed as usual: see section 5.1, page 46. Compare the results on Fig. 6.1, obtained for $\lambda=+\infty$ with those of FIg. 5.2, page 46, obtained for $\lambda=1$.
Finally, the stress computation and the mesh adaptation loop is left as an execice to the reader.

### 6.2 The $P_{1} b-P_{1}$ element for the Stokes problem

## Formulation and approximation

Let us go back to the Stokes problem. In section 5.4, page 54, the Taylor-Hood finite element was considered. Here, we turn to the mini-element [14] proposed by Arnold, Brezzi and Fortin, also well-known as the P1-bubble element. This element is generaly less precise than the Taylor-Hood one, but becomes popular, mostly because it is easy to implement in two and three dimensions
and furnishes a $P_{1}$ approximation of the velocity field. Moreover, this problem develops some links with stabilization technics and will presents some new Rheolef features.
We consider a mesh $\mathcal{T}_{h}$ of $\Omega \subset \mathbb{R}^{N}, N=2,3$ composed only of simplicial elements: triangles when $N=2$ and tetraedra when $N=3$. The following finite dimensional spaces are introduced:

$$
\begin{aligned}
\mathbf{X}_{h}^{(1)} & =\left\{\mathbf{v} \in\left(H^{1}(\Omega)\right)^{N} ; \mathbf{v}_{/ K} \in\left(P_{1}\right)^{N}, \forall K \in \mathcal{T}_{h}\right\} \\
\mathbf{B}_{h} & =\left\{\boldsymbol{\beta} \in\left(C^{0}(\bar{\Omega})\right)^{N} ; \boldsymbol{\beta}_{/ K} \in B(K)^{N}, \forall K \in \mathcal{T}_{h}\right\} \\
\mathbf{X}_{h} & =\mathbf{X}_{h}^{(1)} \oplus \mathbf{B}_{h} \\
\mathbf{V}_{h}(\alpha) & =X_{h} \cap \mathbf{V}(\alpha), \\
Q_{h} & \left.=\left\{q \in L^{2}(\Omega)\right) \cap C^{0}(\bar{\Omega}) ; q_{/ K} \in P_{1}, \forall K \in \mathcal{T}_{h}\right\}
\end{aligned}
$$

where $B(K)=\operatorname{vect}\left(\lambda_{1} \times \ldots \times \lambda_{N+1}\right)$ and $\lambda_{i}$ are the barycentric coordinates of the simplex $K$. The $B(K)$ space is related to the bubble local space. The approximate problem is similar to (5.4), page 55 , up to the choice of finite dimensional spaces.
Remark that the velocity field splits in two terms: $\mathbf{u}_{h}=\mathbf{u}_{h}^{(1)}+\mathbf{u}_{h}^{(b)}$, where $\mathbf{u}_{h}^{(1)} \in \mathbf{X}_{h}^{(1)}$ is continuous and piecewise linear, and $\mathbf{u}_{h}^{(b)} \in \mathbf{B}_{h}$ is the bubble term.

File 'stokes_contraction_bubble.cc'

```
#include "rheolef.h"
using namespace rheolef;
using namespace std;
#include "poiseuille.h"
int main(int argc, char**argv) {
    environment rheolef (argc, argv);
    geo omega (argv[1]);
    string sys_coord = omega.coordinate_system_name();
    Float c = omega.xmax() [1];
    space X1h (omega, "P1", "vector");
    space Bh (omega, "bubble", "vector");
    X1h.block ("wall");
    X1h.block ("upstream");
    X1h[1].block ("axis");
    X1h[1].block ("downstream");
    space Xh = X1h * Bh;
    space Qh (omega, "P1");
    space Wh (omega["upstream"], "P1");
    field uh (Xh, 0);
    field ph (Qh, 0);
    uh[0][0]["upstream"] = interpolate (Wh, u(c,sys_coord));
    form a1 (X1h, X1h, "2D_D");
    form ab (Bh, Bh, "2D_D");
    form b1 = - form(X1h, Qh, "div");
    form bb = - form(Bh, Qh, "div");
    form a = {{a1, 0},
            { 0, ab}};
    form b = {b1, bb};
    form mp (Qh, Qh, "mass");
    solver_abtb stokes (a.uu(), b.uu(), mp.uu()) ;
    stokes.solve (-(a.ub()*uh.b()), -(b.ub()*uh.b()),
                                    uh.set_u(), ph.set_u());
    dout << catchmark("inv_lambda") << 0 <<< endl
        << catchmark("u") << uh[0]
        << catchmark("ub") << uh[1]
        << catchmark("p") << ph;
}
```


## Comments

The matrix structure is similar to those of the Taylor-Hood element, and thus the same mixed solver approach applies. We consider the abrupt contraction geometry:

$$
\Omega=]-L_{u}, 0[\times] 0, c\left[\cup \left[0, L_{d}[\times] 0,1[\right.\right.
$$

where $c \geq 1$ stands for the contraction ratio, and $L_{u}, L_{d}>0$, are the upstream and downstream tube lengths. The boundary conditions on $\mathbf{u}=\left(u_{0}, u_{1}\right)$ for this test case are:

$$
\begin{aligned}
u_{0} & =u_{\text {poiseuille }} \text { and } u_{1}=0 \text { on } \Gamma_{\text {upstream }} \\
\mathbf{u} & =0 \text { on } \Gamma_{\text {wall }} \\
\frac{\partial u_{0}}{\partial x_{1}} & =0 \text { and } u_{1}=0 \text { on } \Gamma_{\text {axis }} \\
\frac{\partial \mathbf{u}}{\partial n} & =0 \text { on } \Gamma_{\text {downstream }}
\end{aligned}
$$

where

$$
\begin{aligned}
\Gamma_{\text {upstream }} & \left.=\left\{-L_{u}\right\} \times\right] 0, c[ \\
\Gamma_{\text {downstream }} & \left.=\left\{L_{d}\right\} \times\right] 0,1[ \\
\Gamma_{\text {axis }} & =]-L_{u}, L_{d}[\times\{0\} \\
\Gamma_{\text {wall }} & =]-L_{u}, 0[\times\{c\} \cup\{0\} \times] 1, c[\cup] 0, L_{d}[\times\{1\}
\end{aligned}
$$

Notices the automatic computation of the geometric contraction ratio $c$ from the input mesh, as:

```
Float c = omega.xmax()[1];
```

The global form $a(.,$.$) over \mathbf{X}_{h}$ is obtained by concatenation of the forms $a_{1}(.,$.$) and a_{b}(.,$.$) over$ $\mathbf{X}_{h}^{(1)}$ and $\mathbf{B}_{h}$ repectively, thanks to the form initializer list:

$$
\mathrm{a}=\left(\begin{array}{cc}
\mathrm{a} 1 & 0 \\
0 & \mathrm{ab}
\end{array}\right) \quad \text { and } \quad \mathrm{b}=\left(\begin{array}{ll}
\mathrm{b} 1 & \mathrm{bb}
\end{array}\right)
$$

that writes simply with Rheolef:

```
form a = {{a1, 0},
form b = {0, ab}};
```


## File 'poiseuille.h'

```
struct u : std::unary_function<point,Float> {
    Float operator() (const point& x) const {
        return a*(c+x[1])*(c-x[1]); }
    u (const Float& c1, std::string sc) : c(c1)
    {a=(sc== "cartesian") ? 3/(2*pow(c,3)) : 4/pow(c,4); }
    protected: Float c, a;
};
struct psi : std::unary_function<point,Float> {
    Float operator() (const point& x) const {
        return xy ? a*squ (c-x[1])*(2*c+x[1]) : a*sqr(c-x[1])*sqr (c+x[1]); }
    psi (const Float& c1, std::string sc) : c(c1), xy(sc == "cartesian")
    { a = xy ? -1/(2* pow (c,3)) : -1/pow (c,4); }
    protected: Float c, a; bool xy;
};
```

The file poiseuille.h contains code for the velocity and stream function boundary conditions. The Poiseuille velocity upstream boundary condition $u_{\text {poiseuille }}$ has been scaled such that the total flow rate is equal to one. The stream function is equal to -1 on the axis and to zero on the wall. This file contains also a treatment of the axisymmetric variant of the geometry: this case will be presented in the next paragraphs.

## How to run the program



Figure 6.2: Solution of the Stokes problem in the abrupt contraction: (top) the mesh; (center) the $P_{1}$ stream function associated to the $P_{1} b-P_{1}$ element; (bottom) the $P_{2}$ stream function associated to the $P_{2}-P_{1}$ Taylor-Hood element.

The boundary conditions in this example are related to an abrupt contraction geometry with a free surface. The corresponding mesh 'contraction.geo' can be easily builded from the geometry description file 'contraction.mshcad', which is provided in the example directory of the Rheolef distribution. The building mesh procedure is presented with details in appendix B, page B.

```
gmsh -2 contraction.mshcad -o contraction.msh
msh2geo < contraction.msh | geo -upgrade - > contraction.geo
geo contraction.geo
```

The mesh is represented on Fig. 6.2.top. Then, the computation and the visualization writes:

```
./stokes_contraction_bubble contraction.geo > contraction-P1.field
field contraction-P1.field -mayavi -velocity
```

The visualization of the velocity field brings few informations about the properties of the flow. The stream function is more relevant for stationnary flow visualization.

```
File 'streamf_contraction.cc'
```

```
#include "rheolef.h"
using namespace rheolef;
using namespace std;
#include "poiseuille.h"
int main (int argc, char** argv) {
    environment rheolef (argc, argv);
    field uh;
    din >> uh;
    const geo& omega = uh.get_geo();
    string sys_coord = omega.coordinate_system_name();
    Float c = omega.xmax()[1];
    string approx = "P" + itos(uh.get_space().degree());
    space Ph (omega, approx);
    Ph.block("upstream");
    Ph.block("wall");
    Ph.block("axis");
    space Wh (omega["upstream"], approx);
    const space& Xh = uh.get_space();
    form a (Ph, Ph, "s_grad_grad");
    form b (Ph, Xh, "s_curl");
    field psi_h (Ph, 0);
    psi_h["upstream"] = interpolate (Wh, psi(c,sys_coord));
    psi_h["wall"] = 0;
    psi_h["axis"] = -1;
    field lh = b.trans_mult(uh);
    solver sa (a.uu());
    psi_h.set_u() = sa.solve (lh.u() - a.ub()*psi_h.b());
    dout << catchmark("psi") << psi_h;
}
```

Notice the use of special extensions of the grad_grad and curl forms for the stream function computation:

```
form a (Ph, Ph, "s_grad_grad");
form b (Ph, Xh, "s_curl");
```

These form are suitable for the axisymmetric coordinate system specific definition of the stream function, while they coincide with the usual grad_grad and curl forms on cartesian coordinate systems. The axisymmetric case will be presented in the next section.
The stream function $\psi$ (see also section 5.6) is computed and visualized as:

```
make streamf_contraction
./streamf_contraction < contraction-P1.field > contraction-P1-psi.field
field contraction-P1-psi.field
field contraction-P1-psi.field -n-iso 15 -n-iso-negative 10 -bw
```

The $P_{1}$ stream function is represented on Fig. 6.2.center. The stream function is zero along the wall and the line separating the main flow and the vortex located in the outer corner of the contraction. Thus, the isoline assoiated to the zero value separates the main flow from the vortex. In order to observe this vortex, an extra -n-iso-negative 10 option is added: ten isolines are drawed for negatives values of $\psi$, associated to the main flow, and $\mathrm{n}_{-}$iso-10 for the positives values, associated to the vortex.

A similar computation based on the Taylor-Hood $P_{2}-P_{1}$ element is implemented in stokes_contraction.cc. The code is similar, up to the boundary conditions, to stokes_cavity.cc (see page 55): thus it is not listed here but is available in the Rheolef example directory.

```
make stokes_contraction
    ./stokes_contraction contraction.geo > contraction-P2.field
field contraction-P2.field -mayavi -velocity
./streamf_contraction < contraction-P2.field > contraction-P2-psi.field
field contraction-P2-psi.field -n-iso-negative 10 -bw
```

The associated $P_{2}$ stream function is represented on Fig. 6.2.bottom. Observe that the two solutions are similar and that the vortex activity, defined as $\psi_{\max }$, is accurately computed with the two methods (see also [15], Fig. 5.11.a, page 143).

```
field contraction-P1-psi.field -max
field contraction-P2-psi.field -max
```

Recall that the stream function is negative in the main flow and positive in the vortex located in the outer corner of the contraction. Nevertheless, the Taylor-Hood based solution is more accurate : this is perceptible on the graphic, in the region where the upstream vortex reaches the boundary.

### 6.3 The stabilized Stokes problem

An alternative and popular implementation of this element eliminates the unknowns related to the bubble components (see e.g. [16], page 24). Remark that the system has the following structure:

$$
\left(\begin{array}{ccc}
A_{1} & 0 & B_{1}^{T} \\
0 & A_{b} & B_{b}^{T} \\
B_{1} & B_{b} & 0
\end{array}\right)\left(\begin{array}{c}
U_{1} \\
U_{b} \\
P
\end{array}\right)=\left(\begin{array}{c}
F_{1} \\
F_{b} \\
G
\end{array}\right)
$$

This elimination can be easily performed since the form $a_{b}(.,$.$) over \mathbf{B}_{h}$ is diagonal, due to the fact that the bubble functions vanishes on the boundary of elements. The system reduces to:

$$
\left(\begin{array}{ll}
A_{1} & B_{1}^{T} \\
B_{1} & -C
\end{array}\right)\binom{U_{1}}{P}=\binom{F}{\tilde{G}}
$$

where $\tilde{G}=G-B_{b} A_{b}^{-1} F_{b}$ and $C=B_{b} A_{b}^{-1} B_{b}^{T}$. Recall that $A_{b}$ is diagonal, since each bubble function vanishes at element boundary. Remarks that the matrix structure is similar to those of the nealy incompressible elasticity (see 6.1, page 6.1). A direct matrix formulation for this problem is similar to the direct $P_{1}-P_{1}$ stabilized element, proposed by Brezzi and Pitkäranta [17].

### 6.4 Axisymmetric geometries

Axisymmetric geometries are fully supported in Rheolef: the coordinate system is associated to the geometry description, stored together with the mesh in the '.geo' and this information is propagated in spaces, forms and fields without any change in the code. Thus, a code that works in plane a 2 D plane geometry is abble to support a 3 D axisymmetric one without changes. A simple axisymmetric geometry writes:

```
mkgeo_grid -t 10 -zr > square-zr.geo
more square-zr.geo
```

Remark the additional line in the header:

```
coordinate_system zr
```

The axis of symmetry is denoted as $z$ while the polar coordinates are $(r, \theta)$. By symmetry, the problem is supposed to be independent upon $\theta$ and the computationnal domain is described by $\left(x_{0}, x_{1}\right)=(z, r)$. Conversely, in some cases, it could be convenient to swap the order of the coordinates and use $(r, z)$ : this feature is obtanied by the $-r z$ option:

```
mkgeo_grid -t 10 -rz > square-rz.geo
more square-rz.geo
```

Axisymmetric problems uses $L^{2}$ functional space equipped with the following weighted scalar product

$$
(f, g)=\int_{\Omega} f(z, r) g(z, r) r d r d z
$$

and all usual bilinear forms support this weight. Thus, the coordinate system can be chosen at run time and we can expect an efficient source code reduction.

### 6.5 The axisymmetric stream function and stress tensor

In the axisymmetric case, the velocity field $\mathbf{u}=\left(u_{z}, u_{r}\right)$ can be expressed in terms of the Stokes stream function $\psi$ by (see [18, p.453] and [19]):

$$
\begin{equation*}
\mathbf{u}=\left(u_{z}, u_{r}\right)=\left(\frac{1}{r} \frac{\partial \psi}{\partial r}, \quad-\frac{1}{r} \frac{\partial \psi}{\partial z}\right) \tag{6.1}
\end{equation*}
$$

Recall that in the axisymmetric case:

$$
\operatorname{curl} \psi=\left(\frac{1}{r} \frac{\partial(r \psi)}{\partial r},-\frac{\partial \psi}{\partial z}\right)
$$

Thus, from this definition, in axisymmetric geometries $\mathbf{u} \neq \mathbf{c u r l} \psi$ and the definition of $\psi$ differs from the 2D plane or 3 D cases (see section 5.6 , page 60 ).
Let us turn to a variational formulation in order to compte $\psi$ from $\mathbf{u}$. For any $\xi \in H^{1}(\Omega)$, let us multiply (6.1) by $\mathbf{v}=\left(\partial_{r} \xi,-\partial_{z} \xi\right)$ and then integrate over $\Omega$ with the $r \mathrm{~d} r \mathrm{~d} z$ weight. For any known $\mathbf{u}$ velocity field, the problem writes:
$(P):$ find $\psi \in \Psi\left(\psi_{\Gamma}\right)$ such that

$$
a(\psi, \xi)=b(\xi, \mathbf{u}), \quad \forall \xi \in \Psi(0)
$$

where we have introduced the following bilinear forms:

$$
\begin{aligned}
a(\psi, \xi) & =\int_{\Omega}\left(\frac{\partial \psi}{\partial r} \frac{\partial \xi}{\partial r}+\frac{\partial \psi}{\partial z} \frac{\partial \xi}{\partial z}\right) \mathrm{d} r \mathrm{~d} z \\
b(\xi, \mathbf{u}) & =\int_{\Omega}\left(\frac{\partial \xi}{\partial r} u_{z}-\frac{\partial \xi}{\partial z} u_{r}\right) r \mathrm{~d} r \mathrm{~d} z
\end{aligned}
$$

As an exemple, let us reconsider the contraction geometry (see section 6.2, page 65), extended in the axisymmetric case. In that case, the functional space is defined by:

$$
\Psi\left(\psi_{\Gamma}\right)=\left\{\varphi \in H^{1}(\Omega) ; \varphi=\psi_{\Gamma} \quad \text { on } \Gamma_{\text {upstream }} \cup \Gamma_{\text {wall }} \cup \Gamma_{\text {axis }}\right\}
$$

with

$$
\psi_{\Gamma}=\left\{\begin{array}{cll}
\psi_{\text {poiseuile }} & \text { on } & \Gamma_{\text {upstream }} \\
0 & \text { on } & \Gamma_{\text {wall }} \\
-1 & \text { on } & \Gamma_{\text {axis }}
\end{array}\right.
$$

This space corresponds to the imposition of Dirichlet boundary conditions on $\Gamma_{\text {upstream }}, \Gamma_{\text {wall }}$ and $\Gamma_{\text {axis }}$ and a Neuman boundary condition on $\Gamma_{\text {downstream }}$.
These forms are defined in 'streamf_contraction.cc' as:

```
form a (Ph, Ph, "s_grad_grad");
form b (Ph, Xh, "s_curl");
```

Notice that $a$ is similar to the grad_grad form, but where the usual $r \mathrm{~d} r \mathrm{~d} z$ measure is replaced by $\mathrm{d} r \mathrm{~d} z$, i.e. without the $r$ weight. Conversely, $b$ is related to the s_curl operator:

$$
\mathbf{s \_ c u r l} \xi=\left(\begin{array}{ll}
\frac{\partial \xi}{\partial r}, & -\frac{\partial \xi}{\partial z}
\end{array}\right)
$$

that is closely related to the standard curl operator:

$$
\operatorname{curl} \xi=\left(\frac{1}{r} \frac{\partial(r \xi)}{\partial r}, \quad-\frac{\partial \xi}{\partial z}\right)
$$

Notice the apparition of $r$ and $1 / r$ in the last expression.


Figure 6.3: Solution of the axisymmetric Stokes problem in the abrupt contraction: (top) the $P_{2}$ stream function associated to the $P 2-P_{1}$ element; (bottom) comparison with the 2 D cartesian solution (in red).

The following unix commands generate the axisymmetric geometry:

```
gmsh -2 contraction.mshcad -o contraction.msh
msh2geo -zr < contraction.msh | geo -upgrade - > contraction-zr.geo
```

```
more contraction-zr.geo
geo contraction-zr.geo
```

The previous code stokes_contraction.cc and streamf_contraction.cc are both reused as:

```
./stokes_contraction contraction-zr.geo > contraction-zr-P2.field
./streamf_contraction < contraction-zr-P2.field > contraction-zr-P2-psi.field
field contraction-zr-P2-psi.field -n-iso-negative 10 -bw
```

The solution is represented on Fig. 6.3: it slighly differs from the 2D cartesian solution, as computed in the previous section (see Fig. 6.2). The vortex size is smaller but its intensity $\psi_{\max }=1.84 \times 10^{-3}$ is higher. Despite the stream functions looks like similar, the plane solutions


Figure 6.4: Solution of the plane and axisymmetric Stokes problem in the abrupt contraction: cut along the axis of symmetry: (left): $u_{0}$; (right) $\tau_{\theta \theta}$.
are really different, as we can observe from a cut of the first component of the velocity along the axis (see Fig. 6.4):

```
field contraction-P2.field -comp 0 -cut -normal 0 1 -origin 0 1e-15
field contraction-zr-P2.field -comp 0 -cut -normal O 1 -origin 0 1e-15
```

The 1e-15 argument replace the zero value, as the mesh intersection cannot yet be done exactly on the boundary. Notice that the stokes_contraction_bubble.cc can be also reused in a similar way:

```
./stokes_contraction_bubble contraction-zr.geo > contraction-zr-P1.field
./streamf_contraction < contraction-zr-P1.field > contraction-zr-P1-psi.field
field contraction-zr-P1-psi.field -n-iso-negative 10 -bw
```

There is another major difference with axisymmetric problems: the rate of deformation tensor writes:

$$
\tau=2 D(\mathbf{u})=\left(\begin{array}{ccc}
\tau_{z z} & \tau_{r z} & 0 \\
\tau_{r z} & \tau_{r r} & 0 \\
0 & 0 & \tau_{\theta \theta}
\end{array}\right)
$$

Thus, there is an additional non-zero component $\tau_{\theta \theta}$ that is automatically integrated into the computations in Rheolef. The incompressibility relation leads to $\operatorname{tr}(\tau)=\tau_{z z}+\tau_{r r}+\tau_{\theta \theta}=0$. Here $\sigma_{\mathrm{tot}}=-p . I+\tau$ is the total Cauchy stress tensor (by a dimensionless procedure, the visosity can be taken as one). By reusing the stress.cc code (see page 48) we are abble to compute the tensor components:

```
make stress
    ./stress < contraction-zr-P1.field > contraction-zr-P1-tau.field
```

The visualization along the axis of symmetry for the $\tau_{\theta \theta}$ component is obtained by (see Fig. 6.4):

```
field contraction-zr-P1-tau.field -comp 22 -proj -cut -normal 0 1 -origin 0 1e-15
```

Recall that the $\tau_{z z}$ and $\tau_{r r}$ components are obtained by the -comp 00 and -comp 11 options, respectively. The non-zero values along the axis of symmetry expresses the elongationnal effects in the entry region of the abrupt contraction.

## Chapter 7

## Time-dependent problems

### 7.1 The heat equation

## Formulation

Let $T>0, \Omega \subset \mathbb{R}^{d}, d=1,2,3$ and $f$ defined in $\Omega$. The heat problem writes:
$(P):$ find $u$, defined in $\Omega \times] 0, T[$, such that

$$
\begin{aligned}
\frac{\partial u}{\partial t}-\Delta u & =f \text { in } \Omega \times] 0, T[ \\
u(0) & =0 \text { in } \Omega \\
u(t) & =0 \text { on } \partial \Omega \times] 0, T[
\end{aligned}
$$

where $f$ is a known function.

## Approximation

Let $\Delta t>0$ and $t_{n}=n \Delta t, n \geq 0$. The problem is approximated with respect to time by the following first-order implicit Euler scheme:

$$
\frac{u^{n+1}-u^{n}}{\Delta t}-\Delta u^{n+1}=f\left(t_{n+1}\right) \text { in } \Omega
$$

where $u^{n} \approx u(n \Delta t)$ and $u^{(0)}=0$. We reuse the bilinear forms $a$ and $m$ defined in section 1.1, page 13 for the Dirichlet problem and introduce the bilinear form $c=m+\Delta t a$. The variationnal formulation of the time-discretized problem writes:
$(V F)_{n}$ : Let $u^{n}$ being known, find $u^{n+1} \in H_{0}^{1}(\Omega)$ such that

$$
c\left(u^{n+1}, v\right)=m\left(u^{n}+\Delta t f\left(t_{n+1}\right), v\right), \forall v \in H_{0}^{1}(\Omega)
$$

This is a Poisson-like problem. The discretization with respect to space of this problem is similar to those presened in section 1.1, page 13.

File 'heat.cc'

```
#include "rheolef.h"
using namespace rheolef;
using namespace std;
int main (int argc, char **argv) {
    environment rheolef (argc, argv);
    geo omega (argv[1]);
    size_t n_max = (argc > 2) ? atoi(argv[2]) : 100;
    Float delta_t = 0.5/n_max;
    space Xh (omega, "P1");
    Xh.block ("boundary");
    form m (Xh, Xh, "mass");
    form a (Xh, Xh, "grad_grad");
    form c = m + delta_t*a;
    solver sc = ldlt (c.uu());
    field lh = riesz (Xh, 1);
    field uh (Xh, 0);
    branch event ("t","u");
    dout << event (0, uh);
    for (size_t n = 1; n <= n_max; n++) {
        field kh = m*uh + delta_t*lh;
        uh.set_u() = sc.solve (kh.u() - c.ub()*uh.b());
        dout << event (Float(n)*delta_t, uh);
    }
}
```


## Comments

Notice the use of the branch class:

```
branch event ("t","u");
```

this is a wrapper class that is used here to print the branch of solution $\left(t_{n}, u^{n}\right)_{n \geq 0}$, on the standard output in the '. branch' file format. An instruction as:

```
dout << event (t,uh);
```

is equivalent to the formated output

```
dout << catchmark("t") << t << endl
    << catchmark("u") << uh;
```

How to run the program


Figure 7.1: Animation of the solution of the heat problem.

We assume that the previous code is contained in the file 'heat. cc'. Then, compile the program as usual (see page 16):

```
make heat
```

For a one dimensional problem, enter the comands:

```
mkgeo_grid -e 10 > line.geo
./heat line.geo > line.branch
```

The previous commands solve the problem for the corresponding mesh and write the solution in the field-family file format '. branch'. For a bidimensional one:

```
mkgeo_grid -t 10 > square.geo
./heat square.geo > square.branch
```

For a tridimensional one:

```
mkgeo_grid -T 10 > box.geo
./heat box.geo > box.branch
```


## How to run the animation

```
branch line.branch -gnuplot -umax 0.125
```

A gnuplot window appears. Enter q to exit the window. For a bidimensional case, a more sophisticated procedure is required. Enter the following unix commands:

```
branch square.branch -paraview
paraview &
```

A window appears, that looks like a video player. Then, open the File->open menu and load square-..vtk. The first '.' stands for a wildcard, i.e. the time index family. Then, press the apply green button and, clic a first time on the video play button, at the top of the window. Next, go to the object inspector window, select display and clic on the rescale to data range button. Then clic a second time on the video play button. An elevation view can be also obtained: Select the Filter->alphabetical->wrap(scalar) menu, choose 10 as scale factor and press the apply green button. Then, clic on the graphic window, rotate the view and finally re-play the animation
To generate an animation file ${ }^{1}$, go to the File->save animation menu and enter as file name square and as file type jpeg. A collection of jpeg files are generated by paraview. Then, run the unix command:

```
ffmpeg -r 2 -i 'square.%04d.jpg' square.mov
```

The animation file square.mov can now be started from any video player, such as vlc:

```
vlc --loop square.mov
```

For the tridimensional case, the animation feature is similar.

### 7.2 The convection-diffusion problem

## Formulation

Let $T>0$ and $\nu>0$. The convection-diffusion problem writes:
$(P):$ find $\phi$, defined in $\Omega \times] 0, T[$, such that

$$
\begin{aligned}
\frac{\partial \phi}{\partial t}+\mathbf{u} \cdot \nabla \phi-\nu \Delta \phi+\sigma \phi & =0 \text { in } \Omega \times] 0, T[ \\
\phi(0) & =\phi_{0} \text { in } \Omega \\
\phi(t) & \left.=\phi_{\Gamma}(t) \text { on } \partial \Omega \times\right] 0, T[
\end{aligned}
$$

where $\mathbf{u}, \sigma \geq 0, \phi_{0}$ and $\phi_{\Gamma}$ being known. Notice the aditional $\mathbf{u} . \nabla$ operator.

[^0]
## Time approximation

This problem is approximated by the following first-order implicit Euler scheme:

$$
\frac{\phi^{n+1}-\phi^{n} \circ X^{n}}{\Delta t}-\nu \Delta \phi^{n+1}=0 \text { in } \Omega
$$

where $\Delta t>0, \phi^{n} \approx \phi(n \Delta t)$ and $\phi^{(0)}=\phi_{0}$.
Let $t_{n}=n \Delta t, n \geq 0$. The term $X^{n}(x)$ is the position at $t_{n}$ of the particule that is in $x$ at $t_{n+1}$ and is transported by $\mathbf{u}^{n}$. Thus, $X^{n}(x)=X\left(t_{n}, x\right)$ where $X(t, x)$ is the solution of the differential equation

$$
\left\{\begin{aligned}
\frac{\mathrm{d} X}{\mathrm{~d} t} & =\mathbf{u}(X(t, x), t) \quad \text { p.p. } t \in] t_{n}, t_{n+1}[, \\
X\left(t_{n+1}, x\right) & =x .
\end{aligned}\right.
$$

Then $X^{n}(x)$ is approximated by the first-order Euler approximation

$$
X^{n}(x) \approx x-\Delta t \mathbf{n}^{n}(x)
$$

This algorithm has been introduced by O. Pironneau (see e.g. [20]), and is known as the method of characteristic in the finite difference context and as the Lagrange-Galerkin in the finite element one. The efficient evaluation of $\phi_{h} \circ X^{n}(x)$ in an unstructured mesh involves a hierarchical $d$-tree (quadtree, octree) data structure for the localization of the element $K$ of the mesh that contains $x$. When $d=3$ requieres also sophisticted geometric predicates to test whether $x \in K$ without rounding errors, and avoid to conclude that no elements contains a point $x$ close to $\partial K$ up to rounding errors. This problems is addressed in Rheolef based on the cgal library.
The following code implements the classical rotating Gaussian hill test case (see e.g. [21]).

File 'convect.cc'

```
// mkgeo_grid -e 20-a -2 -b 2 > line2.geo
// ./convect-P2 line2 P1 1e-2 100 > line2-P1.branch
// ./convect-P2 line2 P2 1e-2 100 > line2-P2.branch
// ./convect_error < line2-P1.branch > line2-P1-cmp.branch
// ./convect_error < line2-P2.branch > line2-P2-cmp.branch
#include "rheolef.h"
using namespace rheolef;
using namespace std;
#include "rotating-hill.h"
int main (int argc, char **argv) {
    environment rheolef (argc,argv);
    geo omega (argv[1]);
    string approx = (argc > 2) ? argv[2] : "P1";
    Float nu = (argc > 3) ? atof (argv[3]) : 1e-2;
    size_t n_max = (argc > 4) ? atoi(argv[4]) : 50;
    size_t d = omega.dimension();
    Float delta_t = 2*acos(-1.)/n_max;
    space Vh (omega, approx, "vector");
    field uh = interpolate (Vh, u(d));
    space Xh (omega, approx);
    Xh.block ("boundary");
    field phi_h = interpolate (Xh, phi(d,nu,0));
    characteristic X (-delta_t*uh);
    quadrature_option_type qopt;
    qopt.set_family(quadrature_option_type::gauss_lobatto);
    qopt.set_order(Xh.degree());
    form m (Xh, Xh, "mass", qopt);
    form a (Xh, Xh, "grad_grad");
    branch event ("t","phi");
    dout << catchmark("nu") << nu << endl
        << event (0, phi_h);
    for (size_t n = 1; n <= n_max; n++) {
        Float t = n*delta_t;
        field lh = riesz(Xh, compose(phi_h, X), qopt);
        form c = (1 + delta_t*phi::sigma(d,nu,t))*m + delta_t*nu*a;
        solver sc (c.uu());
        phi_h.set_u() = sc.solve (lh.u() - c.ub()*phi_h.b());
        dout << event (t, phi_h);
    }
}
```


## Comments

The characteristic variable X implements the localizer $X^{n}(x)$ :

```
characteristic X (-delta_t*uh);
```

Combined with the compose function, it perform the composition $\phi_{h} \circ X^{n}$. The right-hand side is then computed as usual by using the riesz function:

```
field lh = riesz(Xh, compose(phih, X));
```

Notice also the use of the lumped mass matrix:

```
form m (Xh, Xh, "lumped_mass");
```

The lumped procedure leads to profitable spectral properties of the mass matrix that are benefict to the stability of the method. As test case is described in [22]: we take $\Omega=]-2,2\left[{ }^{d}\right.$ and $T=2 \pi$. This problem provides an example for a convection-diffusion equation and a known analytical solution:

$$
\phi(t, x)=\exp \left(-\lambda t-r(t)\left|x-x_{0}(t)\right|^{2}\right)
$$

where $\lambda=4 \nu t_{0} \geq 0$ with $t_{0}>0$ and $\nu \geq 0, x_{0}(t)$ is the moving center of the hill and $r(t)=$ $1 /\left(t_{0}+4 \nu t\right)$. The source term is time-dependent: $\sigma(t)=\lambda-2 d \nu r(t)$ and has been adjusted such
that the right-hand side is zero. The moving center of the hill $x_{0}(t)$ is associated to the velocity field $\mathbf{u}(t, x)$ as:

| $d$ | $\mathbf{u}(t, x)$ | $x_{0}(t)$ |
| :--- | :--- | :--- |
| 1 | $1 /(2 \pi)$ | $t /(2 \pi)-1 / 2$ |
| 2 | $(y,-x)$ | $(-\cos (t) / 2, \sin (t) / 2)$ |
| 3 | $(y,-x, 0)$ | $(-\cos (t) / 2, \sin (t) / 2,0)$ |

File 'rotating-hill.h'

```
struct u : std::unary_function<point,point> {
    point operator() (const point & x) {
        return (d == 1) ? point(u0) : point(x[1], -x[0]); }
    u (size_t d1) : d(d1), u0 (0.5/acos(Float(-1))) {}
    protected: size_t d; Float u0;
};
struct phi : std::unary_function<point,Float> {
    static Float sigma(size_t d, Float nu1, Float t) {
        const Float t0 = 0.2;
        return 4*nu1/t0 - 2*d*nu1/(to + 4*nu1*t); }
    Float operator() (const point& x) {
        point x0t;
        if (d == 1) { x0t = point(x0[0] + u0*t); }
        else { x0t = point( x0[0]*cos(t) + x0[1]*sin(t),
                        -x0[0]*sin(t) + x0[1]*\operatorname{cos(t));}
        }
        return exp(-4*nu*(t/t0) - dist2(x,x0t)/(t0+4*nu*t));
    }
    phi (size_t d1, Float nu1, Float t1) : d(d1), nu(nu1), t(t1),
        t0(0.2), u0 (0.5/acos(Float(-1))), x0(-0.5,0) {}
    protected: size_t d; Float nu, t, t0, u0; point x0;
};
```

Notice the use of a class-function phi for the implementation of $\phi(t)$ as a function of $x$. Such programation style has been introduced in the standard template library [23], which is a part of the standard C++ library. By this way, for a given $t, \phi(t)$ can be interpolated as an usual function on a mesh.

## How to run the program

We assume that the previous code is contained in the file 'convect. cc'. Then, compile the program as usual (see page 16):

```
make convect
```

and enter the comands: Running the one-dimensional test case:

```
mkgeo_grid -e 500 -a -2 -b 2 > line2.geo
./convect line2.geo P1 > line2.branch
branch line2.branch -gnuplot
```

Notice the hill that moves from $x=-1 / 2$ to $x=1 / 2$. Since the exact solution is known, it is possible to analyze the error:


Figure 7.2: Animation of the solution of the rotating hill problem.
File 'convect_error.cc'

```
#include "rheolef.h"
using namespace rheolef;
using namespace std;
#include "rotating-hill.h"
int main (int argc, char **argv) {
    environment rheolef (argc,argv);
    Float nu;
    din >> catchmark("nu") >> nu;
    branch get ("t","phi");
    branch put ("t","phi_h","pi_h_phi");
    derr << "# t\terror_l2\terror_linf" << endl;
    field phi_h;
    Float err_l2_l2 = 0;
    Float err_linf_linf = 0;
    for (Float t = 0, t_prec = 0; din >> get (t, phi_h); t_prec = t) {
        const space& Xh = phi_h.get_space();
        size_t d = Xh.get_geo().dimension();
        field pi_h_phi = interpolate (Xh, phi(d,nu,t));
        form m (Xh, Xh, "mass");
        field eh = phi_h - pi_h_phi;
        Float err_l2 = sqrt(m(eh,eh));
        Float err_linf = eh.max_abs();
        err_l2_l2 += sqr(err_l2)*(t - t_prec);
        err_linf_linf = max(err_linf_linf, err_linf);
        dout << put (t, phi_h, pi_h_phi);
        derr << t << "\t" << err_l2 << "\t" << err_linf << endl;
    }
    derr << "# error_l2_l2 = " << sqrt(err_l2_l2) << endl;
    derr << "# error_linf_linf = " << err_linf_linf << endl;
}
```

The numerical error $\phi_{h}-\pi_{h}(\phi)$ is computed as:

```
field eh = phih - interpolate (Xh, phi(t));
```

and its $L^{2}$ norm is printed on the standard error. Observe the use of the branch class as both input and output field stream.

```
make convect_error
./convect_error < line2.branch > line2-cmp.branch
branch line2-cmp.branch -gnuplot
```

The instantaneous $L^{2}(\Omega)$ norm is printed at each time step and the total error in $L^{2}(] 0, T\left[; L^{2}(\Omega)\right)$ is finaly printed at the end of the stream.


Figure 7.3: Diffusion-convection when $d=1$ and $\nu=10^{-2}$ : convergence versus $h$ and $\Delta t$ for $P_{1}$ and $P_{2}$ elements: (left) in $L^{2}\left(L^{2}\right)$ norm; (right) in $L^{\infty}\left(L^{\infty}\right)$ norm.

A P2 approximation can be used as well:

```
./convect line2.geo P2 > line2.branch
branch line2.branch -gnuplot
./convect_error < line2.branch > line2-cmp.branch
```

On Fig. 7.3.left we observe the $L^{2}\left(L^{2}\right)$ convergence versus $h$ for the $P_{1}$ and $P_{2}$ elements when $d=1$ : the errors reaches a plateau that decreases versus $\Delta t$. On Fig. 7.3.right the $L^{\infty}\left(L^{\infty}\right)$ norm of the error presents a similar behavior. Since the plateau are equispaced, the convergence versus $\Delta t$ is of first order.
These computation was performed for a convection-diffusion problem with $\nu=10^{-2}$. The pure transport problem ( $\nu=0$, without diffusion) computation is obtained by:

```
./convect line2.geo P1 O > line2.branch
branch line2.branch -gnuplot
```

Let us turn to the two-dimensional test case:

```
mkgeo_grid -t 80 -a -2 -b 2 -c -2 -d 2 > square2.geo
./convect square2.geo P1 > square2.branch
branch square2.branch -paraview
paraview &
```

The visualization and animation are similar to those of the head problem previously presented in paragraph 7.1. Observe the rotating hill. The result is shown on Fig. 7.2. The error analysis writes:

```
./convect_error < square2.geo > square2-cmp.branch
branch square2-cmp.branch -paraview
```

From the paraview menu, you can visualize simultaneously both the approximate solution and the Lagrange interpolate of the exact one. Finaly, the three-dimensional case:

```
mkgeo_grid -T 15 -a -2 -b 2 -c -2 -d 2 -f -2 -g 2 > cube2.geo
./convect cube2.geo P1 > cube2.branch
```

The visualization is similar to the two-dimensional case.

### 7.3 The Navier-Stokes problem

## Formulation

This longer example combines most fonctionalities presented in the previous examples.
Let us consider the Navier-Stokes problem for the driven cavity in $\Omega=] 0,1\left[{ }^{d}, d=2,3\right.$. Let $R e>0$ be the Reynolds number, and $T>0$ a final time. The problem writes:
$(N S):$ find $\mathbf{u}=\left(u_{0}, \ldots, u_{d-1}\right)$ and $p$ defined in $\left.\Omega \times\right] 0, T[$ such that:

$$
\left.\left.\begin{array}{rl}
\operatorname{Re}\left(\frac{\partial \mathbf{u}}{\partial t}+\mathbf{u} \cdot \nabla \mathbf{u}\right)-\operatorname{div}(2 D(\mathbf{u}))+\nabla p & =0 \text { in } \Omega \times] 0, T[, \\
-\operatorname{div} \mathbf{u} & \\
\mathbf{u}(t=0) & =0 \text { in } \Omega \times] 0, T[, \\
\mathbf{u} & \\
\mathbf{u} & \\
& \left.=0 \text { in } \Omega \times\{0, T\}, 0) \text { on } \Gamma_{\text {top }} \times\right] 0, T[, \\
\frac{\partial u_{0}}{\partial \mathbf{n}} & =\frac{\partial u_{1}}{\partial \mathbf{n}}
\end{array}\right)=u_{2}=0 \text { on }\left(\Gamma_{\text {left }} \cup \Gamma_{\text {right }} \cup \Gamma_{\text {back }} \cup \Gamma_{\text {bront }}\right) \times\right] 0, T[\text { when } d=3,
$$

where $D(\mathbf{u})=\left(\boldsymbol{\nabla} \mathbf{u}+\boldsymbol{\nabla} \mathbf{u}^{T}\right) / 2$. This nonlinear problem is the natural extension of the linear Stokes problem, as presented in paragraph 7.3, page 84. The boundaries are represented on Fig. 5.1, page 44.

## Time approximation

Let $\Delta t>0$. Let us consider the following backward second order scheme, for all $\phi \in C^{2}([0, T])$ :

$$
\frac{\mathrm{d} \phi}{\mathrm{~d} t}(t)=\frac{3 \phi(t)-4 \phi(t-\Delta t)+\phi(t-2 \Delta t)}{2 \Delta t}+\mathcal{O}\left(\Delta t^{2}\right)
$$

The problem is approximated by the following second-order implicit Euler scheme:

$$
\begin{array}{rlrl}
\operatorname{Re} \frac{3 \mathbf{u}^{n+1}-4 \mathbf{u}^{n} \circ X^{n}+\mathbf{u}^{n-1} \circ X^{n-1}}{2 \Delta t}-\operatorname{div}\left(2 D\left(\mathbf{u}^{n+1}\right)\right)+\nabla p^{n+1} & =0 \text { in } \Omega, \\
-\operatorname{div} \mathbf{u}^{n+1} & =0 \text { in } \Omega, \\
\mathbf{u}^{n+1} & & =(1,0) \text { on } \Gamma_{\text {top }}, \\
\mathbf{u}^{n+1} & & 0 \text { on } \Gamma_{\text {left }} \cup \Gamma_{\text {right }} \cup \Gamma_{\text {bottom }}, \\
& \frac{\partial u_{0}^{n+1}}{\partial \mathbf{n}}=\frac{\partial u_{1}^{n+1}}{\partial \mathbf{n}}=u_{2}^{n+1} & =0 \text { on } \Gamma_{\text {back }} \cup \Gamma_{\text {front }} \text { when } d=3,
\end{array}
$$

where, following [24, 25]:

$$
\begin{aligned}
X^{n}(x) & =x-\Delta t \mathbf{u}^{*}(x) \\
X^{n-1}(x) & =x-2 \Delta t \mathbf{u}^{*}(x) \\
\mathbf{u}^{*} & =2 \mathbf{u}^{n}-\mathbf{u}^{n-1}
\end{aligned}
$$

It is a second order extension of the method previously introduced in paragraph 7.2 page 79. The scheme defines a second order recurence for the sequence $\left(\mathbf{u}^{n}\right)_{n \geq-1}$, that starts with $\mathbf{u}^{-1}=\mathbf{u}^{0}=0$.

## Variationnal formulation

The variationnal formulation of this problem expresses:
$(N S)_{\Delta t}:$ find $\mathbf{u}^{n+1} \in \mathbf{V}(1)$ and $p^{n+1} \in L_{0}^{2}(\Omega)$ such that:

$$
\begin{array}{lll}
a\left(\mathbf{u}^{n+1}, \mathbf{v}\right)+b\left(\mathbf{v}, p^{n+1}\right) & =m\left(\mathbf{f}^{n}, \mathbf{v}\right), & \forall \mathbf{v} \in \mathbf{V}(0) \\
b\left(\mathbf{u}^{n+1}, q\right) & =0, & \forall q \in L_{0}^{2}(\Omega)
\end{array}
$$

where

$$
\mathbf{f}^{n}=\frac{R e}{2 \Delta t}\left(4 \mathbf{u}^{n} \circ X^{n}-\mathbf{u}^{n-1} \circ X^{n}\right)
$$

where

$$
a(\mathbf{u}, \mathbf{v})=\frac{3 R e}{2 \Delta t} \int_{\Omega} \mathbf{u} . \mathbf{v} d x+\int_{\Omega} 2 D(\mathbf{u}): D(\mathbf{v}) d x
$$

and $b(.,$.$) and \mathbf{V}(\alpha)$ was already introduced in paragraph 5.4 , page 54 , while studying the Stokes problem.

## Space approximation

The Taylor-Hood [12] finite element approximation of this generalised Stokes problem was also considered in paragraph 5.4, page 54. We introduce a mesh $\mathcal{T}_{h}$ of $\Omega$ and the finite dimensional spaces $\mathbf{X}_{h}, \mathbf{V}_{h}(\alpha)$ and $Q_{h}$. The approximate problem writes:
$(N S)_{\Delta t, h}:$ find $\mathbf{u}_{h}^{n+1} \in \mathbf{V}_{h}(1)$ and $p^{n+1} \in Q_{h}$ such that:

$$
\begin{array}{lll}
a\left(\mathbf{u}_{h}^{n+1}, \mathbf{v}\right)+b\left(\mathbf{v}, p_{h}^{n+1}\right) & =m\left(\mathbf{f}_{h}^{n}, \mathbf{v}\right), &  \tag{7.1}\\
b \mathbf{v} \in \mathbf{V}_{h}(0) \\
b\left(\mathbf{u}_{h}^{n+1}, q\right) & &
\end{array}
$$

where

$$
\mathbf{f}_{h}^{n}=\frac{R e}{2 \Delta t}\left(4 \mathbf{u}_{h}^{n} \circ X^{n}-\mathbf{u}_{h}^{n-1} \circ X^{n}\right)
$$

The problem reduces to a sequence resolution of a generalized Stokes problems.

File 'navier_stokes_solve.icc'

```
using namespace std;
int navier_stokes_solve (
    Float Re, Float delta_t, field lOh, field& uh, field& ph,
        size_t& max_iter, Float& tol, odiststream *p_derr=0) {
    const space& Xh = uh.get_space();
    const space& Qh = ph.get_space();
    string label = "navier-stokes-" + Xh.get_geo().name();
    quadrature_option_type qopt;
    qopt.set_family(quadrature_option_type::gauss_lobatto);
    qopt.set_order(Xh.degree()) ;
    form m (Xhh, Xh, "mass", qopt);
    form a (Xh, Xh, "2D_D");
    form mp(Qh, Qh, "mass");
    a = a + 1.5*(Re/delta_t)*m;
    solver sa (a.uu());
    form b (Xh, Qh, "div"); b = -b;
    solver_abtb stokes (a.uu(), b.uu(), mp.uu());
    if (p_derr != 0) *p_derr << "[" <<'label << "] #n |du/dt|" << endl;
    field uh1 = uh;
    for (size_t n = 0; true; n++) {
        field uh2 = uh1;
        uh1 = uh;
        field uh_star = 2.0*uh1 - uh2;
        characteristic X1 ( -delta_t*uh_star);
        characteristic X2 (-2.0*delta_t*uh_star);
        field l1h = riesz(Xh, compose(uh1,X1), qopt);
        field l2h = riesz(Xh, compose(uh2,X2), qopt);
        field lh = lOh + (Re/delta_t)*(2*l1h - 0.5*l2h);
        stokes.solve (lh.u() - a.ub()*uh.b(), -(b.ub()*uh.b()),
                    uh.set_u(), ph.set_u());
            field duh_dt = (3*uh - 4*uh1 + uh2)/(2*delta_t);
            Float residual = sqrt(m(duh_dt,duh_dt));
            if (p_derr != 0) *p_derr << "[" << label << "] "<< n << " " << residual << endl;
            if (residual < tol) {
            tol = residual;
            max_iter = n;
            return 0;
            }
            if (n == max_iter-1) {
            tol = residual;
            return 1;
        }
    }
}
```


## Comments

The navier_stokes_solve function is similar to the 'stokes_cavity.cc'. It solves here a generalised Stokes problem and manages a right-hand side $\mathbf{f}_{h}$ :

```
characteristic X1 ( -delta_t*uh_star);
characteristic X2 (-2.0*delta_t*uh_star);
field l1h = riesz(Xh, compose(uh1,X1), qopt);
field l2h = riesz(Xh, compose(uh2,X2), qopt);
field lh = lOh + (Re/delta_t)*(2*l1h - 0.5*l2h);
```

This last computation is similar to those done in the 'convect. cc' example. The generalized Stokes problem is solved by the solver_abtb class. The stopping criterion is related to the stationnary solution or the maximal iteration number.

File 'navier_stokes_cavity.cc'

```
#include "rheolef.h"
using namespace rheolef;
using namespace std;
#include "navier_stokes_solve.icc"
#include "navier_stokes_criterion.icc"
#include "cavity.icc"
int main (int argc, char**argv) {
    environment rheolef (argc, argv);
    if (argc < 2) {
        cerr << "usage: " << argv[0] << " <geo> <Re> <err> <hmin> <delta t> <n_qdapt>" << endl;
        exit (1);
    }
    geo omega (argv[1]);
    adapt_option_type options;
    Float Re = (argc > 2) ? atof (argv[2]) : 100;
    options.err = (argc > 3) ? atof(argv[3]) : 1e-2;
    size_t n_adapt = (argc > 4) ? atoi(argv[4]) : 5;
    Float delta_t = 0.05;
    options.hmin = 0.004;
    options.hmax = 0.1;
    space Xh = cavity_space (omega, "P2");
    space Qh (omega, "P1");
    field uh = cavity_field (Xh, 1.0);
    field ph (Qh, 0);
    field fh (Xh, O);
    for (size_t i = 0; true; i++) {
        size_t max_iter = 1000;
        Float tol = 1e-5;
        navier_stokes_solve (Re, delta_t, fh, uh, ph, max_iter, tol, &derr);
        odiststream o (omega.name(), "field");
        o << catchmark("Re") << Re << endl
            << catchmark("delta_t") << delta_t << endl
            << catchmark("u") << uh
            << catchmark("p") << ph;
        o.close();
        if (i >= n_adapt) break;
        field ch = navier_stokes_criterion (Re,uh);
        omega = adapt (ch, options);
        o.open (omega.name(), "geo");
        o << omega;
        o.close();
        Xh = cavity_space (omega, "P2");
        Qh = space (omega, "P1");
        uh = cavity_field (Xh, 1.0);
        ph = field(Qh, 0);
        fh = field (Xh, 0);
    }
}
```

File 'navier_stokes_criterion.icc'

```
field navier_stokes_criterion (Float Re, const field& uh) {
    space Xh (uh.get_geo(), "P1d", "vector");
    form mpt (uh.get_space(), Xh, "mass");
    form inv_m (Xh, Xh, "inv_mass");
    field c1 = norm2(inv_m*(mpt*uh));
    space Th (uh.get_geo(), "P1d", "tensor");
    form two_D (uh.get_space(), Th, "2D");
    form inv_mt (Th, Th, "inv_mass");
    field two_Duh = inv_mt*(two_D*uh);
    field c2 = norm2(two_Duh);
    return sqrt(Re*c1 + c2);
}
```


## Comments

The code performs a computation by using adaptive mesh refinement, in order to capture recirculation zones. The adapt_option_type declaration is used by rheolef to send options to the mesh generator. The code reuse the file 'cavity.icc' introduced page 55 . This file contains two functions that defines boundary conditions associated to the cavity driven problem.
The criteria function computes the adaptive mesh refinement criteria:

$$
c_{h}=\left(R e\left|\mathbf{u}_{h}\right|^{2}+2\left|D\left(\mathbf{u}_{h}\right)\right|^{2}\right)^{1 / 2}
$$

The criteria function is similar to those presented in the 'embankment-adapt-2d.cc' example.

## How to run the program



Figure 7.4: Meshes and stream functions associated to the solution of the Navier-Stokes equations for $R e=100$ (top) and $R e=400$ (bottom).

$$
R e=1000: 5873 \text { elements, } 3106 \text { vertices }
$$

$$
\psi_{\max }=1.64 \times 10^{-3}, \psi_{\min }=-0.117
$$



Figure 7.5: Meshes and stream functions associated to the solution of the Navier-Stokes equations for $R e=1000$.

The mesh loop adaptation is initiated from a bamg mesh (see also appendix B.1).

```
bamg -g square.bamgcad -o square.bamg
bamg2geo square.bamg square.dmn > square.geo
```

Then, compile and run the Navier-Stokes solver for the driven cavity for $R e=100$ :

```
make navier_stokes_cavity
    ./navier_stokes_cavity square.geo 100
```

The program performs a computation with $R e=100$. By default the time step is $\Delta t=0.05$ and the computation loops for five mesh adaptations. At each time step, the program prints an approximation of the time derivative, and stops when a stationary solution is reached. Then, we visualise the 'square-5' adapted mesh and its associated solution:

```
geo square-5.geo
field square-5.field.gz -velocity -scale 4 -mayavi
```

Notice the -scale option that applies a multiplicative factor to the arrow length when plotting. The representation of the stream function writes:

```
make streamf_cavity
zcat square-5.field.gz | ./streamf_cavity | field -bw -n-iso-negative 10 -
```

The programs streamf_cavity, already introduced page 60, is here reused. The last options of the field program draws isocontours of the stream function using lines, as shown on Fig. 7.4. The zero isovalue separates the main flow from recirculations, located in corners at the bottom of the cavity.
For $R e=400$ and 1000 the computation writes:
./navier_stokes_cavity square.geo 400
./navier_stokes_cavity square.geo 1000


Figure 7.6: Navier-Stokes: velocity profiles along lines passing throught the center of the cavity, compared with data from [26]: (a) $u_{0}$ along the vertical line; (b) $u_{1}$ along the horizontal line line.

The visualization of the cut of the horizontal velocity along the vertical median line writes:

```
field square-5.field.gz -comp 0 -cut -normal -1 0 -origin 0.5 0
field square-5.field.gz -comp 1 -cut -normal 0 1 -origin 0 0.5
```

Fig. 7.6 compare the cuts with data from [26], table 1 and 2 (see also [27]). Observe that the solution is in good agreement with these previous computations.

| $R e$ |  | $x_{c}$ | $y_{c}$ | $-\psi_{\min }$ | $\psi_{\max }$ |
| :--- | :--- | :--- | :--- | :--- | :---: |
| 100 | present | 0.613 | 0.738 | 0.103 | $9.5 \times 10^{-6}$ |
|  | Labeur and Wells [28] | 0.608 | 0.737 | 0.104 | - |
|  | Donea and Huerta [29] | 0.62 | 0.74 | 0.103 | - |
| 400 | present | 0.554 | 0.607 | 0.111 | $5.6 \times 10^{-4}$ |
|  | Labeur and Wells [28] | 0.557 | 0.611 | 0.115 | - |
|  | Donea and Huerta [29] | 0.568 | 0.606 | 0.110 | - |
|  | present | 0.532 | 0.569 | 0.117 | $1.6 \times 10^{-3}$ |
|  | Labeur and Wells [28] | 0.524 | 0.560 | 0.121 | - |
|  | Donea and Huerta [29] | 0.540 | 0.573 | 0.110 | - |

Figure 7.7: Cavity flow: primary vortex position and stream function value.

Finaly, table 7.7 compares the primary vortex position and its associated stream function value. Notice also the good agreement with previous simulations. The stream function extremal values are obtained by:

```
zcat square-5.field.gz | ./streamf_cavity | field -min -
zcat square-5.field.gz | ./streamf_cavity | field -max -
```

The maximal value has not yet been communicated to our knoledge and is provided in table 7.7 for cross validation purpose. The small program that computes the primary vortex position is showed below.

```
make vortex_position
zcat square-5.field.gz | ./streamf_cavity | ./vortex_position
```

File 'vortex_position.cc'

```
#include "rheolef.h"
using namespace rheolef;
using namespace std;
int main (int argc, char** argv) {
    environment rheolef (argc, argv);
    check_macro (communicator().size() == 1, "please, use sequentially");
    field psi_h;
    din >> psi_h;
    size_t idof_min = 0;
    Float psi_min = std:: numeric_limits<Float>::max();
    for (size_t idof = 0, ndof = psi_h.ndof(); idof < ndof; idof++) {
        if (psi_h.dof(idof) >= psi_min) continue;
        psi_min = psi_h.dof(idof);
        idof_min = idof;
    }
    const array<point>& xdof = psi_h.get_space().get_xdofs();
    point xmin = xdof [idof_min];
    dout << "xc\t\tyc\t\tpsi" << endl
        << xmin[0] << "\t" << xmin[1] << "\t" << psi_min << endl;
}
```

For higher Reynolds number, Shen [30] showed in 1991 that the flow converges to a stationary state for Reynolds numbers up to 10000 ; for Reynolds numbers larger than a critical value $10000<$ $R e_{1}<10500$ and less than another critical value $15000<R e_{2}<16000$, these authors founded that the flow becomes periodic in time which indicates a Hopf bifurcation; the flow loses time periodicity for $R e \geq R e_{2}$. In 1998, Ould Salihi [31] founded a loss of stationarity between 10000 and 20000 . In 2002 , Auteri et al. [32] estimated the critical value for the apparition of the first instability to $R e_{1} \approx 8018$. In 2005, Erturk et al. [33] computed steady driven cavity solutions up to $R e \leq 21000$. Also in 2005 , this result was infirmed by [34]: these authors estimated $R e_{1}$ close to 8000 , in agreement with [32]. The 3D driven cavity has been investigated in [35] by the method of characteristic (see also [36] for 3D driven cavity computations). In conclusion, the exploration of the driven cavity at large Reynolds number is a fundamental challenge in computational fluid dynamics.

## Part III

## Advanced and highly nonlinear problems

## Chapter 8

## The highly nonlinear $p$-laplacian problem

### 8.1 Problem statement

Let us consider the classical p-laplacian problem with homogeneous Dirichlet boundary conditions in a domain bounded $\Omega \subset R^{d}, d=1,2,3$ :
$(P):$ find $u$, defined in $\Omega$ such that:

$$
\begin{aligned}
-\operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) & =f \text { in } \Omega \\
u & =0 \text { on } \partial \Omega
\end{aligned}
$$

where $f$ is known and $f=1$ in the computational examples. When $p=2$, this problem reduces to the linear Poisson problem with homogeneous Dichlet boundary conditions. Otherwise, for any $p>1$, the nonlinear problem is equivalent to the following minimisation problem:
(MP): find $u \in W_{0}^{1, p}(\Omega)$ such that:

$$
u=\underset{v \in W_{0}^{1, p}(\Omega)}{\arg \min } \frac{1}{p} \int_{\Omega}|\nabla v|^{p} \mathrm{~d} x-\int_{\Omega} f v \mathrm{~d} x
$$

where $W_{0}^{1, p}(\Omega)$ denotes the usual Sobolev spaces of functions in $W^{1, p}(\Omega)$ that vanishes on the boundary [37, p. 118]. The variational formulation of this problem expresses:
(VF): find $u \in W_{0}^{1, p}(\Omega)$ such that:

$$
a(u ; u, v)=m(f, v), \forall v \in W_{0}^{1, p}(\Omega)
$$

where $a(.,$.$) and m(.,$.$) are defined for any u_{0}, u, v \in W^{1, p}(\Omega)$ by

$$
\begin{aligned}
a\left(u_{0} ; u, v\right) & =\int_{\Omega}\left|\nabla u_{0}\right|^{p-2} \nabla u \cdot \nabla v \mathrm{~d} x, \quad \forall u, v \in W_{0}^{1, p}(\Omega) \\
m(u, v) & =\int_{\Omega} u v \mathrm{~d} x, \quad \forall u, v \in L^{2}(\Omega)
\end{aligned}
$$

The $m(.,$.$) is here the classical scalar product on L^{2}(\Omega)$, and is related to the mass form. The quantity $a(u ; u, u)=\|\nabla u\|_{p, \Omega}$ induces a norm in $W_{0}^{1, p}$, equivalent to the standard norm. The form $a(. ; .,$.$) is bilinear with respect to the two last variable and is related to the energy form.$

### 8.2 The fixed-point algorithm

### 8.2.1 Principe of the algorithm

This nonlinear problem is then reduced to a sequence of linear subproblems by using the fixed-point algorithm. The sequence $\left(u^{(n)}\right)_{n \geq 0}$ is defined by recurrence as:

- $n=0$ : let $u^{(0)} \in W_{0}^{1, p}(\Omega)$ be known.
- $n \geq 0$ : suppose that $u^{(n)} \in W_{0}^{1, p}(\Omega)$ is known and find $u^{(n+1)} \in W_{0}^{1, p}(\Omega)$ such that:

$$
a\left(u^{(n)} ; u^{(n+1)}, v\right)=m(f, v), \forall v \in W_{0}^{1, p}(\Omega)
$$

Let $u^{(n+1)}=G\left(u^{(n)}\right)$ denotes the operator that solve the previous linear subproblem for a given $u^{(n)}$. Since the solution $u$ satisfies $u=G(u)$, it is a fixed-point of $G$.
Let us introduce a mesh $\mathcal{T}_{h}$ of $\Omega$ and the finite dimensional space $X_{h}$ of continuous picewise polynomial functions and $V_{h}$, the subspace of $X_{h}$ containing elements that vanishes on the boundary of $\Omega$ :

$$
\begin{aligned}
X_{h} & =\left\{v_{h} \in C_{0}^{0}(\bar{\Omega}) ; v_{h / K} \in P_{k}, \forall K \in \mathcal{T}_{h}\right\} \\
V_{h} & =\left\{v_{h} \in X_{h} ; v_{h}=0 \text { on } \partial \Omega\right\}
\end{aligned}
$$

where $k=1$ or 2 . The approximate problem expresses: suppose that $u_{h}^{(n)} \in V_{h}$ is known and find $u_{h}^{(n+1)} \in V_{h}$ such that:

$$
a\left(u_{h}^{(n)} ; u_{h}^{(n+1)}, v_{h}\right)=m\left(f, v_{h}\right), \forall v_{h} \in V_{h}
$$

By developping $u_{h}$ on a basis of $V_{h}$, this problem reduces to a linear system. The implementation with Rheolef, involving weighted forms, is quite standard: the weight field wh is inserted as the last argument to the form constructor. The following code implement this problem in the Rheolefenvironment.

### 8.2.2 File 'p_laplacian_fixed_point.h'

```
#include "not too small.h"
int p_laplacian_fixed_point (
    Float p, field lh,
        field& uh, Float& r, size_t& n)
{
    Float tol = r;
    Float r0 = 0;
    size_t max_iter = n;
    const geo& omega = uh.get_geo();
    const space& Xh = uh.get_space();
    string grad_approx = "P" + itos(Xh.degree()-1) + "d";
    space Th (omega, grad_approx, "vector");
    form inv_mt (Th, Th, "inv_mass");
    form grad (Xh, Th, "grad");
    derr << "# Fixed-point algorithm" << endl
            << "# p = " << p << endl
            << "# n r v" << endl;
    n = 0;
    do {
        field grad_uh = inv_mt*(grad*uh);
        field nh = norm2(grad_uh);
        if (p/2 -1 <= 0) nh = compose (not_too_small(1e-10), nh);
        field wh = pow(nh, p/2-1);
        form a (Xh, Xh, "grad_grad", wh);
        field mrh = a*uh - lh;
        r = mrh.u().max_abs();
        if (n == 0) r0 = r;
        Float v = (n == 0) ? 0 : log10(r0/r)/n;
        derr << n << " " << r << " " << v << endl;
        if (r <= tol || n++ >= max_iter) break;
        solver sa(a.uu());
        uh.set_u() = sa.solve (lh.u() - a.ub()*uh.b());
    } while (true);
    return (r <= tol) ? 0 : 1;
}
```


### 8.2.3 File 'p_laplacian_fixed_point.cc'

```
#include "rheolef.h"
using namespace rheolef;
using namespace std;
#include "p_laplacian_fixed_point.icc"
#include "dirichlet.icc"
int main(int argc, char**argv) {
    environment rheolef (argc,argv);
    geo omega (argv[1]);
    string approx = (argc > 2) ? argv[2] : "P1";
    Float p = (argc > 3) ? atof (argv[3]) : 1.5;
    Float tol = (argc>4) ? atof(argv[4]) : 1e-10;
    size_t max_iter = 500;
    derr << "# P-Laplacian problem by fixed-point:" << endl
        << "# geo = " << omega.name() << endl
        << "# approx = " << approx << endl
        <<"# p = " << p << endl;
    space Xh (omega, approx);
    Xh.block ("boundary");
    field uh (Xh);
    uh ["boundary"] = 0;
    field lh = riesz (Xh, 1);
    dirichlet (lh, uh);
    int status = p_laplacian_fixed_point (p, lh, uh, tol, max_iter);
    dout << catchmark("p") << p << endl
        << catchmark("u") << uh;
    return status;
}
```


### 8.2.4 File 'dirichlet.icc'

```
void dirichlet (field lh, field& uh) {
    const space& Xh = uh.get_space();
    form a (Xh, Xh, "grad_grad");
    solver sa(a.uu());
    uh.set_u() = sa.solve (lh.u() - a.ub()*uh.b());
}
```


### 8.2.5 Comments

The fixed-point algorithm is amorced with $u^{(0)}$ as the solution of the linear problem associated to $p=2$, i.e. the standard Poisson problem with Dirichlet boundary conditions. The construction of the weighted form $a(. ; .,$.$) writes:$

form a (Vh, Vh, "grad_grad", eta_h);

### 8.2.6 Running the program

We assume that the previous code is contained in the file 'p_laplacian_fixed_point.cc'. Compile the program, as usual:

```
make p_laplacian_fixed_point
```

and enter the comands:

```
mkgeo_grid -t 10 > square.geo
geo square.geo
```

The triangular mesh has a boundary domain named boundary.

```
./p_laplacian_fixed_point square.geo P1 1.2 > square.field
```

The previous command solves the problem for the corresponding mesh and writes the solution in the file format '.field'.



Figure 8.1: The $p$-laplacian for $d=2$ : (a) elevation view for $p=1.2$; (b) cut along the first bissectice $x_{0}-x_{1}=0$.

Run the field visualization:

```
field square.field -elevation
field square.field -cut -origin 0.5 0.5 -normal 1 1
```

The first command shows an elevation view of the solution (see 8.1.a) while the second one shows a cut along the first bissectrice $x_{0}=x_{1}$. (see 8.1.b).

### 8.2.7 Convergence properties of the fixed-point algorithm



Figure 8.2: The fixed-point algorithm on the $p$-laplacian for $d=2$ : (a) convergence when $p<2$; (b) when $p>2$; (c) convergence rate versus $p$; (d) convergence rate versus $p$ in semi-log scale.

The fixed-point algorithm prints also at each iteration $n$, the residual term $r_{n}$ in discrete $H^{-1}(\Omega)$ and the convergence rate $v_{n}=\log 10\left(r_{n} / r_{0}\right) / n$. The residual term is defined by

$$
r_{h}^{(n)}=A_{h}\left(u^{(n)}\right)-M_{h} f_{h}
$$

where $A_{h}$ and $M_{h}$ are the discrete operators induced by the forms $a(. ; .,$.$) and m(.,$.$) on V_{h}$, and defined for all $u_{h}, v_{h} \in V h$ by:

$$
\begin{aligned}
A_{h}\left(u_{h}\right) v_{h}^{T} & =a\left(u_{h} ; u_{h}, v_{h}\right) \\
\left(M_{h} u_{h}\right) v_{h}^{T} & =m\left(u_{h}, v_{h}\right)
\end{aligned}
$$

where the elements of $V_{h}$ are identified to elements of $\mathbb{R}^{\operatorname{dim}\left(V_{h}\right)}$. The $W^{-1, p}(\Omega)$ norm, defined for all $r \in W^{-1, p}(\Omega)$ by duality:

$$
\|r\|_{-1, p, \Omega}=\sup _{\substack{v \in W^{1, p}(\Omega) \\\|v\|_{1, p, \Omega}=1}} m(r, v)
$$

By analogy, the discrete $W^{-1, p}(\Omega)$ norm, denoted as $\|\cdot\|_{-1, h}$, is defined by duality for all $r_{h} \in V_{h}$ by:

$$
\left\|r_{h}\right\|_{-1, h}=\sup _{\substack{v_{h} \in V_{h} \\\left\|v_{h}\right\|_{1, p, \Omega}=1}} m\left(r_{h}, v_{h}\right)=\sup _{\mathbf{x} \in \operatorname{xdof}\left(V_{h}\right)}\left|r_{h}(\mathbf{x})\right|
$$

where $\operatorname{xdof}\left(V_{h}\right)$ denotes the set of nodes associated to the $V_{h}$ degrees of freedom. Since elements of $V_{h}$ vanishes on the boundary, the $\operatorname{xdof}\left(V_{h}\right)$ contains all nodes associated to the degrees of freedoms of $X_{h}$ except nodes located on the boudnary. Fig 8.2.a and 8.2.b show that the residual term decreases exponentially versus $n$, since the slope of the plot in semi-log scale tends to be strait. Thus, the convergence rate $v_{n}=\log 10\left(r_{n} / r_{0}\right) / n$ tends to a constant, denoted by $\bar{v}$. Fig 8.2.c shows the dependence of $\bar{v}: r_{n} \approx r_{0} \times 10^{-\bar{v} n}$. Observe that $\bar{v}$ tends to $+\infty$ when $p=2$, since the system becomes linear and the algorithm converge in one iteration. Observe also that $\bar{v}$ tends to zero in $p=1$ and $p=3$. The singularity in $p=1$ is not surprising, since the problem is defined only when $p>1$. Conversely, the singularity in $p=3$ is not clear and requires more analysis. Fig 8.2.d shows the same plot in semi-log scale and shows that $\bar{v}$ behaves as: $\bar{v} \approx 2 \times \log _{10}(|p-2|)$. Finally, this study shows that the residual term behaves as:

$$
r_{n} \approx 2|p-2| r_{0} 10^{-n}
$$

### 8.3 The Newton algorithm

### 8.3.1 Principe of the algorithm

An alternative to the fixed-point algorithm is to solve the nonlinear problem $(P)$ by using the Newton algorithm. Let us consider the following operator:

$$
\begin{aligned}
F: W_{0}^{1, p}(\Omega) & \longrightarrow W^{-1, p}(\Omega) \\
u & \longmapsto F(u)=-\operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right)-f
\end{aligned}
$$

The $F$ operator computes simply the residual term and the problem expresses now as: find $u \in$ $W_{0}^{1, p}(\Omega)$ such that $F(u)=0$.
The Newton algorithm reduces the nonlinear problem into a sequence of linear subproblems: the sequence $\left(u^{(n)}\right)_{n \geq 0}$ is classically defined by recurrence as:

- $n=0$ : let $u^{(0)} \in W_{0}^{1, p}(\Omega)$ be known.
- $n \geq 0$ : suppose that $u^{(n)}$ is known, find $\delta u^{(n)}$, defined in $\Omega$, such that:

$$
F^{\prime}\left(u^{(n)}\right) \delta u^{(n)}=-F\left(u^{(n)}\right)
$$

and then compute explicitely:

$$
u^{(n+1)}:=u^{(n)}+\delta u^{(n)}
$$

The notation $F^{\prime}(u)$ stands for the Fréchet derivative of $F$, as an operator from $W^{-1, p}(\Omega)$ into $W_{0}^{1, p}(\Omega)$. For any $r \in W^{-1, p}(\Omega)$, the linear tangent problem writes: find $\delta u \in W_{0}^{1, p}(\Omega)$ such that:

$$
F^{\prime}(u) \delta u=-r
$$

After the computation of the Fréchet derivative, we obtain the strong form of this problem: $(L T)$ : find $\delta u$, defined in $\Omega$, such that

$$
\begin{array}{rlll}
-\operatorname{div}\left(|\boldsymbol{\nabla} u|^{p-2} \boldsymbol{\nabla}(\delta u)+(p-2)|\boldsymbol{\nabla} u|^{p-4}\{\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla}(\delta u)\} \boldsymbol{\nabla} u\right) & = & -r & \text { in } \Omega \\
\delta u & =0 & \text { on } \partial \Omega
\end{array}
$$

This is a Poisson-like problem with homogeneous Dirichlet boundary conditions and a non-constant tensorial coefficient. The variational form of the linear tangent problem writes:
$(V L T)$ : find $\delta u \in W_{0}^{1, p}(\Omega)$ such that

$$
a_{1}(u ; \delta u, \delta v)=m(r, v), \quad \forall \delta v \in W_{0}^{1, p}(\Omega)
$$

where the $a_{1}\left(. ;\right.$., .) is defined for any $u, \delta u, \delta v \in W_{0}^{1, p}(\Omega)$ by:

$$
a_{1}(u ; \delta u, \delta v)=\int_{\Omega}\left(|\boldsymbol{\nabla} u|^{p-2} \boldsymbol{\nabla}(\delta u) . \boldsymbol{\nabla}(\delta v)+(p-2)|\boldsymbol{\nabla} u|^{p-4}\{\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla}(\delta u)\}\{\boldsymbol{\nabla} u . \boldsymbol{\nabla}(\delta v)\}\right) \mathrm{d} x
$$

For any $\boldsymbol{\xi} \in \mathbb{R}^{d}$ let us denote by $\eta(\boldsymbol{\xi})$ the following $d \times d$ matrix:

$$
\eta(\boldsymbol{\xi})=|\boldsymbol{\xi}|^{p-2} I+(p-2)|\boldsymbol{\xi}|^{p-4} \boldsymbol{\xi} \otimes \boldsymbol{\xi}
$$

where $I$ stands for the $d$-order identity matrix. Then the $a_{1}$ expresses in a more compact form:

$$
a_{1}(u ; \delta u, \delta v)=\int_{\Omega}(\eta(\boldsymbol{\nabla} u) \boldsymbol{\nabla}(\delta u)) \cdot \boldsymbol{\nabla}(\delta v) \mathrm{d} x
$$

Clearly $a_{1}$ is linear and symmetric with respect to the two last variables.

### 8.3.2 File 'p_laplacian_newton.cc'

```
#include "rheolef.h"
#include "rheolef/newton.h"
using namespace rheolef;
using namespace std;
#include "p_laplacian.h"
int main(int argc, char**argv) {
    environment rheolef (argc, argv);
    geo omega_h (argv[1]);
    string approx = (argc > 2) ? argv[2] : "P1";
    Float p = (argc > 3) ? atof(argv[3]) : 1.5;
    derr << "# P-Laplacian problem by Newton:" << endl
        << "# geo = " << omega_h.name() << endl
        << "# approx = " << approx << endl
        << "# p = " << p << endl;
    p_laplacian F (p, omega_h, approx);
    field uh = F.initial();
    Float tol = 1e6*numeric_limits<Float>: : epsilon();
    size_t max_iter = 500;
    int status = newton (F, uh, tol, max_iter, &derr);
    dout << setprecision(numeric_limits<Float>::digits10)
        << catchmark("p") << p << endl
        << catchmark("u") << uh;
    return status;
}
```


### 8.3.3 File 'p_laplacian.h'

```
class p_laplacian {
public:
    typedef field value_type;
    typedef Float float_type;
    p_laplacian(Float p, const geo& omega_h, string approx = "P1");
    void reset(const geo& omega_h, string approx = "previous");
    field initial () const;
    field residue (const field& uh) const;
    void update_derivative (const field& uh) const;
    field derivative_solve (const field& mrh) const;
    field derivative_trans_mult (const field& mrh) const;
    Float norm (const field& uh) const;
    Float dual_norm (const field& Muh) const;
    Float dot (const field& uh, const field& vh) const;
    Float dual_dot (const field& Muh, const field& Mvh) const;
    field criteria(const field& uh) const;
    Float p;
    space Xh, Kh;
    field fh;
    form m, inv_mt, grad;
    solver sm;
    mutable form a1;
    mutable solver sa1;
};
#include "p_laplacian1.icc"
#include "p_laplacian2.icc"
```


### 8.3.4 File 'p_laplacian.icc'

```
#include "dirichlet.icc"
#include "not_too_small.h"
p_laplacian:: p_laplacian(Float p1, const geo& omega_h, string approx1)
    : p(p1), Xh(), Kh(), fh(),
    m(), inv_mt(), grad(), sm(), a1(), sa1() {
    reset(omega_h, approx1);
}
void p_laplacian::reset(const geo& omega_h1, string approx1) {
    if (approx1 == "previous") approx1 = Xh.get_approx();
    Xh = space(omega_h1, approx1);
    Xh.block ("boundary");
    fh = field(Xh, 1);
    m = form (Xh, Xh, "mass");
    sm = solver(m.uu());
    string grad_approx = "P" + itos(Xh.degree()-1) + "d";
    space Th (fh.get_geo(), grad_approx, "vector");
    inv_mt = form (Th, Th, "inv_mass");
    grad = form (Xh, Th, "grad");
    Kh = space(fh.get_geo(), grad_approx, "tensor");
}
field p_laplacian::initial () const {
    field uh(Xh);
    uh [Xh.get_geo()["boundary"]] = 0;
    dirichlet (fh, uh);
    return uh;
}
void p_laplacian:: update_derivative (const field& uh) const {
    field grad_uh = inv_mt*(grad*uh);
    field norm2_grad_uh = norm2(grad_uh);
    if (p/2 - 4 <= 0) norm2_grad_uh = compose (not_too_small(1e-10), norm2_grad_uh);
    field w0h = pow(norm2_grad_uh, p/2)/norm2_grad_uh;
    field w1h = pow(norm2_grad_uh, p/2)/sqr(norm2_grad_uh);
    field eta_h (Kh);
    eta_h(0,0) = w0h + (p-2)*w1h*sqr(grad_uh [0]);
    size_t d = uh.get_geo().dimension();
    if (d >= 2) {
        eta_h(1,1) = w0h + (p-2)*w1h*sqr(grad_uh[1]);
        eta_h(0,1) = (p-2)*w1h*grad_uh[0]*grad_uh[1];
    }
    if (d == 3) {
        eta_h(2,2) = w0h + (p-2)*w1h*sqr(grad_uh[2]);
        eta_h(1,2) = (p-2)*w1h*grad_uh[1]*grad_uh[2];
        eta_h(0,2)= (p-2)*w1h*grad_uh[0]*grad_uh[2];
    }
    a1 = form (Xh, Xh, "grad_grad", eta_h);
    sa1 = solver(a1.uu());
}
field p_laplacian::residue (const field& uh) const {
    field grad_uh = inv_mt*(grad*uh);
    field norm2_grad_uh = norm2(grad_uh);
    field wOh = pow(norm2_grad_uh, p/2-1);
    form a (Xh, Xh, "grad_grad", wOh);
    field mrh = a*uh - m*fh;
    mrh.set_b() = 0;
    return mrh;
}
field p_laplacian:: derivative_solve (const field& mrh) const {
    field delta_uh (Xh,0);
    delta_uh.set_b() = 0;
    delta_uh.set_u() = sa1.solve(mrh.u());
    return delta_uh;
}
```

```
field p_laplacian:: derivative_trans_mult (const field& mrh) const {
    field rh (Xh);
    rh.set_b() = 0;
    rh.set_u() = sm.solve(mrh.u());
    field mgh;
    mgh = a1*rh;
    mgh.set_b() = 0;
    return mgh;
}
Float p_laplacian::dot (const field& uh, const field& vh) const {
    return m(uh,vh);
}
Float p_laplacian:: norm (const field& uh) const {
    return sqrt(m(uh,uh));
}
Float p_laplacian::dual_dot (const field& mrh, const field& msh) const {
    field sh (Xh);
    sh.set_b() = 0;
    sh.set_u() = sm.solve(msh.u());
    return :: dot(mrh,sh);
}
Float p_laplacian::dual_norm (const field& mrh) const {
    return sqrt(dual_dot(mrh,mrh));
}
field p_laplacian::criteria(const field& uh) const {
    if (uh.get_approx() == "P1") return abs(uh);
    field grad_uh = inv_mt*(grad*uh);
    field norm2_grad_uh = norm2(grad_uh);
    return pow(norm2_grad_uh, p/4);
}
```


### 8.3.5 Comments

The code implements a generic Newton algorithm in the file 'newton.h'. The main program is 'p_laplacian_newton.cc', that uses a class p_laplacian. This class interface is definied in the file 'p_laplacian.h' and its implementation in 'p_laplacian.icc' The residual term $F\left(u_{h}\right)$ is computed by the member function residual while the resolution of $F^{\prime}\left(u_{h}\right) \delta u_{h}=M r_{h}$ is performed by the function derivative_solve. The derivative $F^{\prime}\left(u_{h}\right)$ is computed separately by the function update_derivative. Notice that the $a_{1}(u ; .,$.$) bilinear form is a tensorial weighted form, where$ $\eta(\boldsymbol{\nabla} u)$ is the weight tensor. In Rheolef, the tensorial weight field eta h is inserted as an usual scalar weight, by passing the weight parameter as the last argument to the form constructor. The introduction of the class 'p_laplacian' allows an easiest implementation of several variants of the Newton algorithm.

### 8.3.6 Running the program




Figure 8.3: The Newton algorithm on the $p$-laplacian for $d=2$ : (a) convergence when $p<2$; (b) when $p>2$.

We assume that the previous code is contained in the file ' $p$-laplacian-newton.cc'. As usual, enter:

```
make p_laplacian_newton
mkgeo_grid -t 10 > square.geo
./p_laplacian_newton square.geo P1 1.5 | field -mayavi -elevation -
```

The program prints at each iteration $n$, the residual term $r_{n}$ in discrete $L^{2}(\Omega)$ norm. Fig. 8.3.a and. 8.3.b shows that the residual terms tends very fast to zero. Observe that the slope is no more constant in semi-log scale: the convergence rate accelerates and the slope tends to be vertical, the so-called super-linear convergence. This is the major advantage of the Newton method. Also the algorithm converge when $p \geq 3$, until $p \approx 4$. It was not the case with the fixed point algorithm that diverges in that case. Finally, the Newton algorithm diverges for small values of $p$, e.g. $p<1.5$ and the plot is not showed here. Conversely, when $p>4$, the first iterations increases dramatically the residual terms, before to decrease. In that case, another strategy should be considered: the damped Newton algorithm. This is the subject of the next section.

### 8.4 The damped Newton algorithm

### 8.4.1 Principe of the algorithm

The Newton algorithm diverges when the initial $u^{(0)}$ is too far from a solution. Our aim is to modify the Newton algorithm and to obtain a globaly convergent algorith, i.e to converge to a solution for any initial $u^{(0)}$. The basic idea is to decrease the step length while maintaining the direction of the original Newton algorithm:

$$
u^{(n+1)}:=u^{(n)}+\lambda_{n} \delta u^{(n)}
$$

where $\left.\left.\lambda^{(n)} \in\right] 0,1\right]$ and $\delta u^{(n)}$ is the direction from the Newton algorithm, given by:

$$
F^{\prime}\left(u^{(n)}\right) \delta u^{(n)}=-F\left(u^{(n)}\right)
$$

Let $V$ a Banach space and let $T: V \rightarrow \mathbb{R}$ defined for any $v \in V$ by:

$$
T(v)=\frac{1}{2}\left\|C^{-1} F(v)\right\|_{V}^{2}
$$

where $C$ is some non-singular operator, easy to invert, used as a non-linear preconditioner. The simplest case, without preconditioner, is $C=I$. The $T$ function furnishes a measure of the residual term in $L^{2}$ norm. The convergence is global when for any initial $u^{(0)}$, we have for any $n \geq 0$ :

$$
\begin{equation*}
T\left(u^{(n+1)}\right) \leq T\left(u^{(n)}\right)+\alpha\left\langle T^{\prime}\left(u^{(n)}\right), u^{(n+1)}-u^{(n)}\right\rangle_{V^{\prime}, V} \tag{8.1}
\end{equation*}
$$

where $\langle., .\rangle_{V^{\prime}, V}$ is the duality product between $V$ and its dual $V^{\prime}$, and $\left.\alpha \in\right] 0,1[$ is a small parameter. Notice that

$$
T^{\prime}(u)=\left\{C^{-1} F^{\prime}(u)\right\}^{*} C^{-1} F(u)
$$

where the superscript * denotes the adjoint operator, i.e. the transpose matrix the in finite dimensional case. In practice we consider $\alpha=10^{-4}$ and we also use a minimal step length $\lambda_{\min }=1 / 10$ in order to avoid too small steps. Let us consider a fixed step $n \geq 0$ : for convenience the $n$ superscript is dropped in $u^{(n)}$ and $\delta u^{(n)}$. Let $g: \mathbb{R} \rightarrow \mathbb{R}$ defined for any $\lambda \in \mathbb{R}$ by:

$$
g(\lambda)=T(u+\lambda \delta u)
$$

Then :

$$
\begin{aligned}
g^{\prime}(\lambda) & =\left\langle T^{\prime}(u+\lambda \delta u), \delta u\right\rangle_{V^{\prime}, V} \\
& =\left\langle C^{-1} F(u+\lambda \delta u), F^{\prime}(u+\lambda \delta u) C^{-1} \delta u\right\rangle_{V, V^{\prime}}
\end{aligned}
$$

where the superscript * denotes the adjoint operator, i.e. the transpose matrix the in finite dimensional case. The practical algorithm for obtaining $\lambda$ was introduced first in [38] and is also presented in [39, p. 385]. The step length $\lambda$ that satify (8.1) is computed by using a finite sequence $\lambda_{k}, k=0,1 \ldots$ with a second order recurrence:

- $k=0$ : initialisation $\lambda_{0}=1$. If (8.1) is satified whith $u+\lambda_{0} d$ then let $\lambda:=\lambda_{0}$ and the sequence stop here.
- $k=1$ : first order recursion. The quantities $g(0)=f(u)$ et $g^{\prime}(0)=\left\langle f^{\prime}(u), d\right\rangle$ are already computed at initialisation. Also, we already have computed $g(1)=f(u+d)$ when verifiying whether (8.1) was satified. Thus, we consider the following approximation of $g(\lambda)$ by a second order polynom:

$$
\tilde{g}_{1}(\lambda)=\left\{g(1)-g(0)-g^{\prime}(0)\right\} \lambda^{2}+g^{\prime}(0) \lambda+g(0)
$$

After a short computation, we find that the minimum of this polynom is:

$$
\tilde{\lambda}_{1}=\frac{-g^{\prime}(0)}{2\left\{g(1)-g(0)-g^{\prime}(0)\right\}}
$$

Since the initialisation at $k=0$ does not satisfy (8.1), it is possible to show that, when $\alpha$ is small enougth, we have $\tilde{\lambda}_{1} \leq 1 / 2$ and $\tilde{\lambda}_{1} \approx 1 / 2$. Let $\lambda_{1}:=\max \left(\lambda_{\min }, \tilde{\lambda}_{1}\right)$. If (8.1) is satisfied with $u+\lambda_{1} d$ then let $\lambda:=\lambda_{1}$ and the sequence stop here.

- $k \geq 2$ : second order recurrence. The quantities $g(0)=f(u)$ et $\left.g^{\prime}(0)=\right\rangle f^{\prime}(u), d\langle$ are available, ytogether with $\lambda_{k-1}, g\left(\lambda_{k-1}\right), \lambda_{k-2}$ and $g\left(\lambda_{k-2}\right)$. Then, $g(\lambda)$ is approximed by the following third order polynom:

$$
\tilde{g}_{k}(\lambda)=a \lambda^{3}+b \lambda^{2}+g^{\prime}(0) \lambda+g(0)
$$

where $a$ et $b$ are expressed by:

$$
\binom{a}{b}=\frac{1}{\lambda_{k-1}-\lambda_{k-2}}\left(\begin{array}{cc}
\frac{1}{\lambda_{k-1}^{2}} & -\frac{1}{\lambda_{k-2}^{2}} \\
-\frac{\lambda_{k-2}}{\lambda_{k-1}^{2}} & \frac{\lambda_{k-1}}{\lambda_{k-2}^{2}}
\end{array}\right)\binom{g\left(\lambda_{k-1}\right)-g^{\prime}(0) \lambda_{k-1}-g(0)}{g\left(\lambda_{k-2}\right)-g^{\prime}(0) \lambda_{k-2}-g(0)}
$$

The minimum of $\tilde{g}_{k}(\lambda)$ is

$$
\tilde{\lambda}_{k}=\frac{-b+\sqrt{b^{2}-3 a g^{\prime}(0)}}{3 a}
$$

Let $\lambda_{k}=\min \left(1 / 2 \lambda_{k}, \max \left(\tilde{\lambda}_{k} / 10, \tilde{\lambda}_{k+1}\right)\right.$ in order for $\lambda_{k}$ to be at the same order of magnitude as $\lambda_{k-1}$. If (8.1) is satisfied with $u+\lambda_{k} d$ then let $\lambda:=\lambda_{k}$ and the sequence stop here.

The sequence $\left(\lambda_{k}\right)_{k \geq 0}$ is strictly decreasing: when the stopping criteria is not satified until $\lambda_{k}$ reaches the machine precision $\varepsilon_{\text {mach }}$ then the algorithm stops with an error.

### 8.4.2 File 'p_laplacian_damped_newton.cc'

```
#include "rheolef.h"
#include "rheolef/damped-newton.h"
using namespace rheolef;
using namespace std;
#include "p_laplacian.h"
int main(int argc, char**argv) {
    environment rheolef (argc,argv);
    geo omega_h (argv[1]);
    string approx = (argc > 2) ? argv[2] : "P1";
    Float p = (argc > 3) ? atof(argv[3]) : 1.5;
    derr << "# P-Laplacian problem by damped Newton:" << endl
            << "# geo = " << omega_h.name() << endl
            << "# approx = " << approx << endl
            << "# p = " << p << endl;
    p_laplacian F (p, omega_h, approx);
    field uh = F.initial();
    Float tol = numeric_limits<Float>:: epsilon();
    size_t max_iter = 500;
    int status = damped_newton (F, uh, tol, max_iter, &derr);
    dout << catchmark("p") << p << endl
            << catchmark("u") << uh;
    return status;
}
```


### 8.4.3 Comments

The file damped-newton-generic.h implements the damped Newton algorithm for a generic $T(u)$ function, i.e. a generic nonlinear preconditioner. This algorithms use a backtrack strategy implemented in file 'newton-backtrack.h'. The simplest choice of the identity preconditioner $C=I$ i.e. $T(u)=\|F(u)\|_{V^{\prime}}^{2} / 2$ is showed in file damped-newton. h . The gradient at $\lambda=0$ is

$$
T^{\prime}(u)=F^{\prime}(u)^{*} F(u)
$$

and the slope at $\lambda=0$ is:

$$
\begin{aligned}
g^{\prime}(0) & =\left\langle T^{\prime}(u), \delta u\right\rangle_{V^{\prime}, V} \\
& =\left\langle F(u), F^{\prime}(u) \delta u\right\rangle_{V^{\prime}, V^{\prime}} \\
& =-\|F(u)\|_{V^{\prime}}^{2}
\end{aligned}
$$

The 'p_laplacian_damped_newton.cc' is the application program to the p-Laplacian problem together with the $\|\cdot\|_{L^{2}(\Omega)}$ discrete norm for the function $T$.

### 8.4.4 Running the program



Figure 8.4: The damped Newton algorithm on the $p$-laplacian for $d=2$ : (a) convergence when $p<2$; (b) when $p>2$.

We assume that the previous code is contained in the file 'p_laplacian_damped_newton.cc'. As usual, enter:

```
make p_laplacian_damped_newton
mkgeo_grid -t 10 > square.geo
    ./p_laplacian_damped_newton square.geo P1 1.5 | field -
    ./p_laplacian_damped_newton square.geo P1 5.0 | field -
```

The algorithm is now quite robust: the convergence occurs for a large range of $p>1$ values. and has been pushed until $p=100$. The only limitation is due to machine roundoff when $p=1.1$ : the residual term reaches only $10^{-10}$ instead of $10^{-15}$.

### 8.4.5 Robustness and mesh invariance



Figure 8.5: Convergence versus $n$ for various meshes: (a) Newton algorithm when $p=1.7$ and (b) $p=1.6$; (c) Newton algorithm when $p=1.5$; (d) damped-Newton algorithm when $p=1.5$.

Fig. 8.5.a, 8.5.b and. 8.5.c show the convergence of the Newton method when $p=1.7,1.6$ and 1.5 , respectively. Observe that the convergence is asymptotically invariant of the mesh when the element size decreases. The convergence is more difficult when $p$ decreases to 1.5 and the Newton algorithm is no more mesh invariant. Fig. 8.5.d shows the convergence of the damped Newton method when $p=1.5$ : the convergence is now very fast and also mesh-invariant.

## Chapter 9

## Error analysis for $P_{k}$ isoparametric elements

### 9.1 Error analysis

## Principle

Let us go back to the non-homogeneous Dirichlet problem, as presented in section2.1, page 24. Since the solution $u$ is regular, the following error estimates holds:

$$
\begin{aligned}
\left\|u-u_{h}\right\|_{0,2, \Omega} & \approx \mathcal{O}\left(h^{k+1}\right) \\
\left\|u-u_{h}\right\|_{0, \infty, \Omega} & \approx \mathcal{O}\left(h^{k+1}\right) \\
\left\|u-u_{h}\right\|_{1,2, \Omega} & \approx \mathcal{O}\left(h^{k}\right)
\end{aligned}
$$

providing the approximate solution $u_{h}$ uses $P_{k}$ continuous finite element method, $k \geq 1$. Here, $\|\cdot\|_{0,2, \Omega},\|\cdot\|_{0, \infty, \Omega}$ and $\|\cdot\|_{1,2, \Omega}$ denotes as usual the $L^{2}(\Omega), L^{\infty}(\Omega)$ and $H^{1}(\Omega)$ norms.
By denoting $\pi_{h}$ the Lagrange interpolation operator, the triangular inequality leads to:

$$
\left\|u-u_{h}\right\|_{0,2, \Omega} \leq\left\|\left(I-\pi_{h}\right)(u)\right\|_{0,2, \Omega}+\left\|u_{h}-\pi_{h} u\right\|_{0,2, \Omega}
$$

From the fundamental properties of the Laplace interpolation $\pi_{h}$, and since $u$ is smooth enought, we have $\left\|\left(I-\pi_{h}\right)(u)\right\|_{0,2, \Omega} \approx O\left(h^{k+1}\right)$. Thus, we have just to check the $\left\|u_{h}-\pi_{h} u\right\|_{0,2, \Omega}$ term. The following code implement the computation of the error.

File 'cosinusprod_error.cc'

```
#include "rheolef.h"
using namespace rheolef;
using namespace std;
#include "cosinusprod.icc"
int main(int argc, char**argv) {
    environment rheolef(argc, argv);
    Float error_linf_expected = (argc > 1) ? atof(argv[1]) : 1e+38;
    field uh; din >> uh;
    space Xh = uh.get_space();
    size_t d = Xh.get_geo(). dimension();
    field pi_h_u = interpolate(Xh, u(d));
    field eh = uh - pi_h_u;
    form m(Xh, Xh, "mass");
    form a(Xh, Xh, "grad_grad");
    dout << "error_l2 " << sqrt(m(eh,eh)) << endl
        << "error_linf " << eh.max_abs() << endl
        << "error_h1 " << sqrt(a(eh,eh)) << endl;
    return (eh.max_abs() <= error_linf_expected) ? 0 : 1;
}
```

File 'cosinusprod.icc'

```
struct u : std::unary_function<point,Float> {
    Float operator() (const point& x) const {
        return cos(pi*x[0])*\operatorname{cos(pi*x[1])*cos(pi*x[2]); }}
    u(size_t d1) : d(d1), pi(acos(Float(-1.0))) {}
    size_t d; Float pi;
};
```


## Running the program

```
make dirichlet-nh cosinusprod_error
```

After compilation, run the code by using the command:

```
mkgeo_grid -t 10 > square.geo
./dirichlet-nh square.geo P1 | ./cosinusprod_error
```

The three $L^{2}, L^{\infty}$ and $H^{1}$ errors are printed for a $h=1 / 10$ uniform mesh. Note that an unstructured quasi-uniform mesh can be simply generated by using the mkgeo_ugrid command:

```
mkgeo_ugrid -t 10 > square.geo
geo square.geo
```

Let $n_{\mathrm{el}}$ denotes the number of elements in the mesh. Since the mesh is quasi-uniform, we have $h \approx n_{\mathrm{el}}^{\frac{1}{d}}$ where $d$ is the physical space dimension. Here $d=2$ for our bidimensionnal mesh. Figure 9.1 plots in logarithmic scale the error versus $n_{\text {el }}^{\frac{1}{2}}$ for both $P_{k}$ approximations, $k=1,2,3$ and the various norms. Observe that the error behaves as predicted by the theory.


Figure 9.1: Strait geometry: error analysis in $L^{2}, L^{\infty}$ and $H^{1}$ norms.


Figure 9.2: Curved domains (triangles): error analysis in $L^{2}, L^{\infty}$ and $H^{1}$ norms.


Figure 9.3: Curved domains (quadrangles): error analysis in $L^{2}, L^{\infty}$ and $H^{1}$ norms.

## Curved domains

File 'cosinusrad_laplace.icc'

```
struct f : unary_function<point,Float> {
    Float operator() (const point& x) const {
        Float r = sqrt(sqr(x[0])+sqr(x[1])+sqr(x[2]));
        Float sin_over_ar = (r == 0) ? 1 : sin(a*r)/(a*r);
        return sqr(a)*((d-1)*sin_over_ar + cos(a*r)); }
    f(size_t d1) : d(d1), a(acos(Float(-1.0))) {}
    size_t d; Float a;
};
struct g : std::unary_function<point,Float> {
    Float operator() (const point& x) const {
        return cos(a*sqrt(sqr(x[0])+sqr(x[1])+sqr(x[2]))); }
    g(size_t=0) : a(acos(Float(-1.0))) {}
    Float a;
};
```

File 'cosinusrad.icc'

```
struct u : std::unary_function<point,Float> {
    Float operator() (const point& x) const {
        Float r = sqrt(sqr(x[0])+sqr(x[1])+sqr(x[2]));
        return cos(a*r); }
    u(size_t=0) : a(acos(Float(-1.0))) {}
    Float a;
};
```

    sed -e 's/sinusprod/sinusrad/' < dirichlet-nh.cc > dirichlet_nh_ball.cc
    sed -e 's/sinusprod/sinusrad/' < cosinusprod_error.cc > cosinusrad_error.cc
    make dirichlet_nh_ball cosinusrad_error
    mkgeo_ball -order 1 -t \(10>\) circle-P1.geo
    geo circle-P1
    ./dirichlet_nh_ball circle-P1.geo P1 | ./cosinusrad_error
    mkgeo_ball -order 3 -t \(10>\) circle-P3.geo
    geo circle-P3
    ./dirichlet_nh_ball circle-P3.geo P3 | ./cosinusrad_error
    Observe Fig. 9.2: for meshes based on triangles: the error behave as expected when $k=1,2,3,4$.
A similar result occurs for quadrangles, as shown on Fig. 9.3.

```
mkgeo_ball -order 3 -q 10 > circle-q-P3.geo
geo circle-q-P3
./dirichlet_nh_ball circle-q-P3.geo P3 | ./cosinusrad_error
```

These features are curently in development for arbitrarily $P_{k}$ high order approximations and threedimensional geometries.

## Bibliography

[1] P. Saramito and N. Roquet. An adaptive finite element method for viscoplastic fluid flows in pipes. Comput. Meth. Appl. Mech. Engrg., (to appear), 2000. 21
[2] D. R. Musser and A. Saini. C++ STL tutorial and reference guide. Addison Wesley, Reading, 1996. 24
[3] P. Saramito. Rheolef home page, LMC-IMAG. http://www-lmc.imag.fr/lmc-edp/Pierre.Saramito/rheolef/, 2011. 25
[4] V. Girault and P. A. Raviart. Finite Element Methods for Navier-Stokes Equations-Theory and Algorithms. Springer Verlag, 1986. 37, 57, 60
[5] C. C. Paige and M. A. Saunders. Solution of sparse indefinite systems of linear equations. SIAM J. Numer. Anal., 12(4):617-629, 1975. http://www.stanford.edu/group/SOL/software.html. 37
[6] N. Roquet, R. Michel, and P. Saramito. Estimations d'erreur pour un fluide viscoplastique par éléments finis $P_{k}$ et maillages adaptés. C. R. Acad. Sci. Paris, série I, to appear. 51
[7] N. Roquet. Simulation numérique d'écoulements de fluides viscoplastiques par un algorithme de Lagrangien augmenté et une méthode d'éléments finis incompressibles. Université d'Orsay Paris-Sud, Prébublication 97-21, 1997. 51
[8] F. Hecht. Bidimensional anisotropic mesh generator, version 0.67. INRIA, october 1998. ftp://ftp.inria.fr/INRIA/Projects/Gamma/bamg/. 51, 125
[9] H. Borouchaki, P. L. George, F. Hecht, P. Laug, and E. Saltel. Delaunay mesh generation governed by metric specifications. Part I: Algorithms. Finite Elem. Anal. Des., 25:61-83, 1997. 51
[10] M. J. Castro-Diaz, F. Hecht, B. Mohammadi, and O. Pironneau. Anisotropic unstructured mesh adaption for flow simulations. Int. J. Numer. Methods Fluids, 25:475-491, 1997. 51
[11] M. G. Vallet. Génération de maillages anisotropes adaptés. Application à la capture de couches limites. Rapport de Recherche $\mathrm{n}^{0}$ 1360, INRIA, 1990. 51
[12] P. Hood and C. Taylor. A numerical solution of the Navier-Stokes equations using the finite element technique. Comp. and Fluids, 1:73-100, 1973. 55, 64, 85
[13] A. Klawonn. An optimal preconditioner for a class of saddle point problems with a penalty term. SIAM J. Sci. Comput, 19(2):540-552, 1998. 56, 64
[14] D. N. Arnold, F. Brezzi, and M. Fortin. A stable finite element for the Stokes equations, 1983. preprint, university of Pavia (Calcolo). 65
[15] P. Saramito. Simulation numérique d'écoulements de fluides viscoélastiques par éléments finis incompressibles et une méthode de directions alternées; applications. PhD thesis, Institut National Polytechnique de Grenoble, 1990. 70
[16] E. M. Abdalass. Résolution performante du problème de Stokes par mini-éléments, maillages auto-adaptatifs et méthodes multigrilles - applications. Ths̀e de l'École Centrale de Lyon, 1987. 70
[17] F. Brezzi and J. Pitkäranta. On the stabilization of finite element approximation of the Stokes equations. In Efficient solutions of elliptic systems, Kiel, Notes on numerical fluid mechanics, volume 10, pages 11-19, 1984. 70
[18] G. K. Batchelor. An introduction to fluid dynamics. Cambridge university press, UK, sixth edition, 1967. 71
[19] Wikipedia. The Stokes stream function. Wikipedia, 2012. http://en.wikipedia.org/wiki/ Stokes_stream_function. 71
[20] O. Pironneau. Méthode des éléments finis pour les fluides. Masson, 1988. 79
[21] H. Rui and M. Tabata. A second order characteristic finite element scheme for convection diffusion problems. Numer. Math. (to appear), 2001. 79
[22] O. Pironneau and M. Tabata. Stability and convergence of a galerkin-characteristics finite element scheme of lumped mass type. Int. J. Numer. Meth. Fluids, 64:1240-1253, 2010. 80
[23] D. R. Musser and A. Saini. STL tutorial and reference guide. Addison-Wesley, 1996. 81
[24] K. Boukir, Y. Maday, B. Metivet, and E. Razafindrakoto. A high-order characteristic/finite element method for the incompressible Navier-Stokes equations. Int. J. Numer. Meth. Fluids, 25:1421-1454, 1997. 85
[25] G. Fourestey and S. Piperno. A second-order time-accurate ALE Lagrange-Galerkin method applied to wind engineering and control of bridge profiles. Comput. Methods Appl. Mech. Engrg., 193:4117-4137, 2004. 85
[26] U. Ghia, K. N. Ghia, and C. T. Shin. High Re solutions for incompressible flow using the Navier-Stokes equations and a multigrid method. J. Comput. Phys., 48:387-411, 1982. 91, 144
[27] M. M. Gupta and J. C. Kalita. A new paradigm for solving Navier-Stokes equations: streamfunction-velocity formulation. J. Comput. Phys., 207:52-68, 2005. http://www.cours.polymtl.ca/mec6616/cavite_2.pdf. 91
[28] R. J. Labeur and G. N. Wells. A Galerkin interface stabilisation method for the advectiondiffusion and incompressible Navier-Stokes equations. Comput. Meth. Appl. Mech. Engrg., 196(49-52):4985-5000, 2007. 91
[29] J. Donea and A. Huerta. Finite element methods for flow problems. Wiley, New-York, 2003. 91
[30] J. Shen. Hopf bifurcation of the unsteady regularized driven cavity flow. J. Comp. Phys., 95:228-245, 1991. http://www.math.purdue.edu/ shen/pub/Cavity.pdf. 92
[31] M. L. Ould Salihi. Couplage de méthodes numériques en simulation directe d'écoulements incompressibles. PhD thesis, Université J. Fourier, Grenoble, 1998. 92
[32] F. Auteri, N. Parolini, and L. Quartapelle. Numerical investigation on the stability of singular driven cavity flow. J. Comput. Phys., 183(1):1-25, 2002. 92
[33] E. Erturk, T. C. Corke, and C. Gökçol. Numerical solutions of 2-D steady incompressible driven cavity flow at high Reynolds numbers. Int. J. Numer. Meth. Fluids, 48:747-774, 2005. 92
[34] T. Gelhard, G. Lube, M. A. Olshanskii, and J. H. Starcke. Stabilized finite element schemes with LBB-stable elements for incompressible flows. J. Comput. Appl. Math., 177:243-267, 2005. 92
[35] P. D. Minev and C. R. Ethier. A characteristic/finite element algorithm for the 3-D NavierStokes equations using unstructured grids. Comput. Meth. in Appl. Mech. and Engrg., 178(1-2):39-50, 1998. 92
[36] S. Melchior, V. Legat, P. Van Dooren, and A. J. Wathen. Analysis of preconditioned iterative solvers for incompressible flow problems. Int. J. Numer. Meth. Fluids, 2011. 92
[37] H. Brezis. Analyse fonctionnelle. Théorie et application. Masson, Paris, 1983. 95
[38] Jr. J. E. Dennis and R. B. Schnablel. Numerical methods for unconstraint optimization and nonlinear equations. Prentice Hall, Englewood Cliff, N. J., 1983. 106
[39] W. H. Press, S. A. Teulkolsky, W. T. Vetterling, and B. P. Flannery. Numerical recepies in C. The art of scientific computing. Cambridge University Press, UK, second edition, 1997. Version 2.08. 106
[40] C. Geuzaine and J.-F. Remacle. Gmsh: a three-dimensional finite element mesh generator with built-in pre- and post-processing facilities. Int. J. Numer. Meths Engrg., 79(11):13091331, 2009. 125, 127

## Part IV

## Technical appendices

## Appendix A

## How to write a variational formulation?

The major keypoint for using Rheolef is to put the problem in variational form. Then this variational form can be efficiently translated into C++ language. This appendix is dedicated to readers who are not fluent with variational formulations and some related functionnal analysis tools.

## A. 1 The Green formula

Let us come back to the model problem presented in section 1.1, page 13, equations (1.1)-(1.2) and details how this problem is transformed into (1.3).
Let $H_{0}^{1}(\Omega)$ the space of functions whose gradient square has a finite sum over $\Omega$ and that vanishes on $\partial \Omega$ :

$$
H_{0}^{1}(\Omega)=\left\{v \in L^{2}(\Omega) ; \nabla v \in L^{2}(\Omega)^{d} \text { and } v=0 \text { on } \partial \Omega\right\}
$$

We start by multiplying (1.1) by an arbitrarily test-function $v \in H_{0}^{1}(\Omega)$ and then integrate over $\Omega$ :

$$
-\int_{\Omega} \Delta u v \mathrm{~d} x=\int_{\Omega} f v \mathrm{~d} x, \quad \forall v \in H_{0}^{1}(\Omega)
$$

The next step is to invoque an integration by part, the so-called Green formula:

$$
\int_{\Omega} \Delta u v \mathrm{~d} x+\int_{\Omega} \nabla u \cdot \nabla v \mathrm{~d} x=\int_{\partial \Omega} \frac{\partial u}{\partial n} v \mathrm{~d} s, \quad \forall u, v \in H^{1}(\Omega)
$$

Since our test-function $v$ vanishes on the boundary, the integral over $\partial \Omega$ is zero and the problem becomes:

$$
\int_{\Omega} \nabla u \cdot \nabla v \mathrm{~d} x=\int_{\Omega} f v \mathrm{~d} x, \quad \forall v \in H_{0}^{1}(\Omega)
$$

This is exactly the variational formulation (1.3), page 13.

## A. 2 The vectorial Green formula

In this section, we come back to the linear elasticity problem presented in section 5.1 , page 43 , equations (5.1)-(5.2) and details how this problem is transformed into (5.3).
Let $\Gamma_{d}\left(\right.$ resp. $\left.\Gamma_{n}\right)$ denotes the parts of the boundary $\partial \Omega$ related to the homogeneous Dirichlet boundary condition $\mathbf{u}=0$ (resp. the homogeneous Neumann boundary condition $\sigma(\mathbf{u}) \mathbf{n}=0$ ).

We suppose that $\partial \Omega=\bar{\Gamma}_{d} \cap \bar{\Gamma}_{n}$. Let us introduce the following functional space:

$$
\mathbf{V}=\left\{\mathbf{v} \in H^{1}(\Omega)^{d} ; \mathbf{v}=0 \text { on } \Gamma_{d}\right\}
$$

Then, multiplying the first equation of (5.2) by an arbitrarily test-function $\mathbf{v} \in \mathbf{V}$ and then integrate over $\Omega$ :

$$
-\int_{\Omega} \operatorname{div}(\sigma(\mathbf{u})) \cdot \mathbf{v} \mathrm{d} x=\int_{\Omega} \mathbf{f} . \mathbf{v} \mathrm{d} x, \quad \forall \mathbf{v} \in \mathbf{V}
$$

The next step is to invoque an integration by part:

$$
\int_{\Omega} \operatorname{div} \tau \cdot \mathbf{v} \mathrm{d} x+\int_{\Omega} \tau: D(\mathbf{v}) \mathrm{d} x=\int_{\partial \Omega} \tau:(\mathbf{v} \otimes \mathbf{n}) \mathrm{d} s, \quad \forall \tau \in L^{2}(\Omega)^{d \times d}, \forall \mathbf{v} \in \mathbf{V}
$$

Recall that $\operatorname{div} \tau$ denotes $\left(\sum_{j=0}^{d-1} \partial_{j} \tau_{i, j}\right)_{0 \leq i<d}$, i.e. the vector whose component are the divergence of each row of $\tau$. Also, $\sigma: \tau$ denote the double contracted product $\sum_{i, j=0}^{d-1} \sigma_{i, j} \tau_{i, j}$ for any tensors $\sigma$ and $\tau$, and that $\mathbf{u} \otimes \mathbf{v}$ dotes the $\tau_{i, j}=u_{i} v_{j}$ tensor, vectors $\mathbf{u}$ and $\mathbf{v}$. Remark that $\tau:(\mathbf{u} \otimes \mathbf{v})=(\tau \mathbf{v}) \cdot \mathbf{u}=\sum_{i, j=0}^{d-1} \tau_{i, j} u_{i} v_{j}$. Choosing $\tau=\sigma(\mathbf{u})$ in the previous equation leads to:

$$
\int_{\Omega} \sigma(\mathbf{u}): D(\mathbf{v}) \mathrm{d} x=\int_{\partial \Omega}(\sigma(\mathbf{u}) \mathbf{n}) \cdot \mathbf{v} \mathrm{d} s+\int_{\Omega} \mathbf{f} \cdot \mathbf{v} \mathrm{d} x, \quad \forall \mathbf{v} \in \mathbf{V}
$$

Since our test-function $v$ vanishes on $\Gamma_{d}$ and the solution satisfie the homogeneous Neumann boundary condition $\sigma(\mathbf{u}) \mathbf{n}=0$ on $\Gamma_{n}$, the integral over $\partial \Omega$ is zero and the problem becomes:

$$
\int_{\Omega} \sigma(\mathbf{u}): D(\mathbf{v}) \mathrm{d} x=\int_{\Omega} \mathbf{f} . \mathbf{v} \mathrm{d} x, \quad \forall \mathbf{v} \in \mathbf{V}
$$

From the definition of $\sigma(\mathbf{u})$ in (5.1) page 43 we have:

$$
\begin{aligned}
\sigma(\mathbf{u}): D(\mathbf{v}) & =\lambda \operatorname{div}(\mathbf{u})(I: D(\mathbf{v}))+2 \mu D(\mathbf{u}): D(\mathbf{v}) \\
& =\lambda \operatorname{div}(\mathbf{u}) \operatorname{div}(\mathbf{v})+2 \mu D(\mathbf{u}): D(\mathbf{v})
\end{aligned}
$$

and the previous relation becomes:

$$
\int_{\Omega} \lambda \operatorname{div}(\mathbf{u}) \operatorname{div}(\mathbf{v}) \mathrm{d} x+\int_{\Omega} 2 \mu D(\mathbf{u}): D(\mathbf{v}) \mathrm{d} x=\int_{\Omega} \mathbf{f} . \mathbf{v} \mathrm{d} x, \quad \forall \mathbf{v} \in \mathbf{V}
$$

This is exactly the variational formulation (5.3), page 44.

## A. 3 The Green formula on a surface

TODO

## Appendix B

## How to prepare a mesh ?

Since there is many good mesh generators, Rheolefdoes not provide a built-in mesh generator. There are several ways to prepare a mesh for Rheolef.
We present here several procedures: by using the bamg bidimensional anisotropic mesh generator, written by Fréderic Hecht [8], and the gmsh mesh generator, suitable when $d=1,2$ and 3 , and written by Christophe Geuzaine and Jean-Franois Remacle [40].

## B. 1 Bidimensionnal mesh with bamg

We first create a 'square.bamgcad' file:

```
MeshVersionFormatted
    O
Dimension
    2
Vertices
    4
    0}00\quad
    1 0 2
    1 1 3
    0
Edges
    4
    1 2 101
    2 3 102
    3 4 103
    4 1 104
hVertices
    0.1 0.1 0.1 0.1
```

This is an uniform mesh with element size $h=0.1$. We refer to the bamg documentation [8] for the complete file format description. Next, enter the mesh generator commands:
bamg -g square.bamgcad -o square.bamg

Then, create the file 'square. dmn' that associate names to the four boundary domains of the mesh. Here, there is four boundary domains:

```
EdgeDomainNames
    4
    bottom
    right
    top
    left
```

and enter the translation command:

```
bamg2geo square.bamg square.dmn > square.geo
```

This command creates a 'square.geo' file. Look at the mesh via the command:

```
geo square
```

This presents the mesh it in a graphical form, usually with gnuplot. You can switch to the mayavi renders:

```
geo square -mayavi
```

A finer mesh could be generated by:

```
bamg -coef 0.5 -g square.bamgcad -o square-0.5.bamg
```


## B. 2 Unidimensionnal mesh with gmsh

The simplest unidimensional mesh is a line:

```
h_local = 0.1;
Point(1) = {0, 0, 0, h_local};
Point(2) = {1, 0, 0, h_local};
Line(3) = {1,2};
Physical Point("left") = {1};
Physical Point("right") = {2};
Physical Point("boundary") = {1,2};
Physical Line("interior") = {3};
```

The mesh generation command writes:

```
gmsh -1 line.mshcad -format msh -o line.msh
```

Then, the conversion to '.geo' format and the visualization:

```
msh2geo < line.msh | geo -upgrade - > line.geo
geo line
```


## B. 3 Bidimensionnal mesh with gmsh



Figure B.1: Visualization of the gmsh meshes 'square.geo' and 'cube.geo'.

We first create a 'square.mshcad' file:

```
n = 10.0;
hloc = 1.0/n;
Point(1) = {0, 0, 0, hloc};
Point(2) = {1, 0, 0, hloc};
Point(3) = {1, 1, 0, hloc};
Point(4) = {0, 1, 0, hloc};
Line(1) = {1,2};
Line(2) = {2,3};
Line(3) = {3,4};
Line(4) = {4,1};
Line Loop(5) = {1,2,3,4};
Plane Surface(6) = {5} ;
Physical Point("left_bottom") = {1};
Physical Point("right_bottom") = {2};
Physical Point("right_top") = {3};
Physical Point("left_top") = {4};
Physical Line("boundary") = {1,2,3,4};
Physical Line("bottom") = {1};
Physical Line("right") = {2};
Physical Line("top") = {3};
Physical Line("left") = {4};
Physical Surface("interior") = {6};
```

This is an uniform mesh with element size $h=0.1$. We refer to the gmsh documentation [40] for the complete file format description. Next, enter the mesh generator commands:

```
gmsh -2 square.mshcad -format msh -o square.msh
```

Then, enter the translation command:

```
msh2geo < square.msh | geo -upgrade - > square.geo
```

This command creates a 'square.geo' file. Look at the mesh via the command:

```
geo square
```

Remark that the domain names, defined in the .mshcad file, are included in the gmsh .msh input file and are propagated in the .geo by the format conversion.

## B. 4 Tridimensionnal mesh with gmsh

First, create a 'cube.mshcad' file:

```
Mesh.Algorithm = 7; // bamg
Mesh.Algorithm3D = 7; // mmg3d
a = 0; c = 0; f = 0;
b = 1; d = 1; g = 1;
n = 10;
hloc = 1.0/n;
Point(1) = {a, c, f, hloc};
Point(2) = {b, c, f, hloc};
Point(3) = {b, d, f, hloc};
Point(4) = {a, d, f, hloc};
Point(5) = {a, c, g, hloc};
Point(6) = {b, c, g, hloc};
Point(7) = {b, d, g, hloc};
Point(8) = {a, d, g, hloc};
Line(1) = {1,2};
Line(2) = {2,3};
Line(3) = {3,4};
Line(4) = {4,1};
Line(5) = {5,6};
Line(6) = {6,7};
Line(7) = {7,8};
Line(8) = {8,5};
Line(9) = {1,5};
Line(10) = {2,6};
Line(11) = {3,7};
Line(12) = {4,8};
Line Loop(21) = {-1, -4,-3,-2};
Plane Surface(31) = {21} ;
Line Loop(22) = {5,6,7,8};
Plane Surface(32) = {22} ;
Line Loop(23) = {1,10,-5,-9};
Plane Surface(33) = {23} ;
Line Loop(24) = {12, -7,-11,3};
Plane Surface(34) = {24} ;
```

```
Line Loop(25) = {2,11,-6, -10};
Plane Surface(35) = {25} ;
Line Loop(26) = {9,-8,-12,4};
Plane Surface(36) = {26} ;
Surface Loop(41) = {31,32,33,34,35,36};
Volume(51) = {41};
Physical Surface("bottom") = {31};
Physical Surface("top") = {32};
Physical Surface("left") = {33};
Physical Surface("front") = {35};
Physical Surface("right") = {34};
Physical Surface("back") = {36};
Physical Volume("internal") = {51};
```

Next, enter the mesh generator commands:

```
gmsh -3 cube.mshcad -format msh -o cube.msh
```

Then, enter the translation command:

```
msh2geo < cube.msh | geo -upgrade - > cube.geo
```

This command creates a 'cube.geo' file. Look at the mesh via the command:

```
geo cube
geo cube.geo -mayavi -cut
```

The second command allows to see inside the mesh.

## Appendix C

## GNU Free Documentation License

Version 1.1, March 2000

Copyright © 2000 Free Software Foundation, Inc. 59 Temple Place, Suite 330, Boston, MA 02111-1307 USA<br>Everyone is permitted to copy and distribute verbatim copies of this license document, but changing it is not allowed.

* 

Preamble
The purpose of this License is to make a manual, textbook, or other written document "free" in the sense of freedom: to assure everyone the effective freedom to copy and redistribute it, with or without modifying it, either commercially or noncommercially. Secondarily, this License preserves for the author and publisher a way to get credit for their work, while not being considered responsible for modifications made by others.
This License is a kind of "copyleft", which means that derivative works of the document must themselves be free in the same sense. It complements the GNU General Public License, which is a copyleft license designed for free software.
We have designed this License in order to use it for manuals for free software, because free software needs free documentation: a free program should come with manuals providing the same freedoms that the software does. But this License is not limited to software manuals; it can be used for any textual work, regardless of subject matter or whether it is published as a printed book. We recommend this License principally for works whose purpose is instruction or reference.

## Applicability and Definitions

This License applies to any manual or other work that contains a notice placed by the copyright holder saying it can be distributed under the terms of this License. The "Document", below, refers to any such manual or work. Any member of the public is a licensee, and is addressed as "you".
A "Modified Version" of the Document means any work containing the Document or a portion of it, either copied verbatim, or with modifications and/or translated into another language.
A "Secondary Section" is a named appendix or a front-matter section of the Document that deals exclusively with the relationship of the publishers or authors of the Document to the Document's overall subject (or to related matters) and contains nothing that could fall directly within that overall subject. (For example, if the Document is in part a textbook of mathematics, a Secondary

Section may not explain any mathematics.) The relationship could be a matter of historical connection with the subject or with related matters, or of legal, commercial, philosophical, ethical or political position regarding them.
The "Invariant Sections" are certain Secondary Sections whose titles are designated, as being those of Invariant Sections, in the notice that says that the Document is released under this License.
The "Cover Texts" are certain short passages of text that are listed, as Front-Cover Texts or Back-Cover Texts, in the notice that says that the Document is released under this License.
A "Transparent" copy of the Document means a machine-readable copy, represented in a format whose specification is available to the general public, whose contents can be viewed and edited directly and straightforwardly with generic text editors or (for images composed of pixels) generic paint programs or (for drawings) some widely available drawing editor, and that is suitable for input to text formatters or for automatic translation to a variety of formats suitable for input to text formatters. A copy made in an otherwise Transparent file format whose markup has been designed to thwart or discourage subsequent modification by readers is not Transparent. A copy that is not "Transparent" is called "Opaque".
Examples of suitable formats for Transparent copies include plain ASCII without markup, Texinfo input format, $\mathrm{EAT}_{\mathrm{E}} \mathrm{X}$ input format, SGML or XML using a publicly available DTD, and standardconforming simple HTML designed for human modification. Opaque formats include PostScript, PDF, proprietary formats that can be read and edited only by proprietary word processors, SGML or XML for which the DTD and/or processing tools are not generally available, and the machinegenerated HTML produced by some word processors for output purposes only.
The "Title Page" means, for a printed book, the title page itself, plus such following pages as are needed to hold, legibly, the material this License requires to appear in the title page. For works in formats which do not have any title page as such, "Title Page" means the text near the most prominent appearance of the work's title, preceding the beginning of the body of the text.

## Verbatim Copying

You may copy and distribute the Document in any medium, either commercially or noncommercially, provided that this License, the copyright notices, and the license notice saying this License applies to the Document are reproduced in all copies, and that you add no other conditions whatsoever to those of this License. You may not use technical measures to obstruct or control the reading or further copying of the copies you make or distribute. However, you may accept compensation in exchange for copies. If you distribute a large enough number of copies you must also follow the conditions in section 3 .
You may also lend copies, under the same conditions stated above, and you may publicly display copies.

## Copying in Quantity

If you publish printed copies of the Document numbering more than 100, and the Document's license notice requires Cover Texts, you must enclose the copies in covers that carry, clearly and legibly, all these Cover Texts: Front-Cover Texts on the front cover, and Back-Cover Texts on the back cover. Both covers must also clearly and legibly identify you as the publisher of these copies. The front cover must present the full title with all words of the title equally prominent and visible. You may add other material on the covers in addition. Copying with changes limited to the covers, as long as they preserve the title of the Document and satisfy these conditions, can be treated as verbatim copying in other respects.
If the required texts for either cover are too voluminous to fit legibly, you should put the first ones listed (as many as fit reasonably) on the actual cover, and continue the rest onto adjacent pages.

If you publish or distribute Opaque copies of the Document numbering more than 100, you must either include a machine-readable Transparent copy along with each Opaque copy, or state in or with each Opaque copy a publicly-accessible computer-network location containing a complete Transparent copy of the Document, free of added material, which the general network-using public has access to download anonymously at no charge using public-standard network protocols. If you use the latter option, you must take reasonably prudent steps, when you begin distribution of Opaque copies in quantity, to ensure that this Transparent copy will remain thus accessible at the stated location until at least one year after the last time you distribute an Opaque copy (directly or through your agents or retailers) of that edition to the public.
It is requested, but not required, that you contact the authors of the Document well before redistributing any large number of copies, to give them a chance to provide you with an updated version of the Document.

## Modifications

You may copy and distribute a Modified Version of the Document under the conditions of sections 2 and 3 above, provided that you release the Modified Version under precisely this License, with the Modified Version filling the role of the Document, thus licensing distribution and modification of the Modified Version to whoever possesses a copy of it. In addition, you must do these things in the Modified Version:

- Use in the Title Page (and on the covers, if any) a title distinct from that of the Document, and from those of previous versions (which should, if there were any, be listed in the History section of the Document). You may use the same title as a previous version if the original publisher of that version gives permission.
- List on the Title Page, as authors, one or more persons or entities responsible for authorship of the modifications in the Modified Version, together with at least five of the principal authors of the Document (all of its principal authors, if it has less than five).
- State on the Title page the name of the publisher of the Modified Version, as the publisher.
- Preserve all the copyright notices of the Document.
- Add an appropriate copyright notice for your modifications adjacent to the other copyright notices.
- Include, immediately after the copyright notices, a license notice giving the public permission to use the Modified Version under the terms of this License, in the form shown in the Addendum below.
- Preserve in that license notice the full lists of Invariant Sections and required Cover Texts given in the Document's license notice.
- Include an unaltered copy of this License.
- Preserve the section entitled "History", and its title, and add to it an item stating at least the title, year, new authors, and publisher of the Modified Version as given on the Title Page. If there is no section entitled "History" in the Document, create one stating the title, year, authors, and publisher of the Document as given on its Title Page, then add an item describing the Modified Version as stated in the previous sentence.
- Preserve the network location, if any, given in the Document for public access to a Transparent copy of the Document, and likewise the network locations given in the Document for previous versions it was based on. These may be placed in the "History" section. You may omit a network location for a work that was published at least four years before the Document itself, or if the original publisher of the version it refers to gives permission.
- In any section entitled "Acknowledgements" or "Dedications", preserve the section's title, and preserve in the section all the substance and tone of each of the contributor acknowledgements and/or dedications given therein.
- Preserve all the Invariant Sections of the Document, unaltered in their text and in their titles. Section numbers or the equivalent are not considered part of the section titles.
- Delete any section entitled "Endorsements". Such a section may not be included in the Modified Version.
- Do not retitle any existing section as "Endorsements" or to conflict in title with any Invariant Section.

If the Modified Version includes new front-matter sections or appendices that qualify as Secondary Sections and contain no material copied from the Document, you may at your option designate some or all of these sections as invariant. To do this, add their titles to the list of Invariant Sections in the Modified Version's license notice. These titles must be distinct from any other section titles.
You may add a section entitled "Endorsements", provided it contains nothing but endorsements of your Modified Version by various parties - for example, statements of peer review or that the text has been approved by an organization as the authoritative definition of a standard.
You may add a passage of up to five words as a Front-Cover Text, and a passage of up to 25 words as a Back-Cover Text, to the end of the list of Cover Texts in the Modified Version. Only one passage of Front-Cover Text and one of Back-Cover Text may be added by (or through arrangements made by) any one entity. If the Document already includes a cover text for the same cover, previously added by you or by arrangement made by the same entity you are acting on behalf of, you may not add another; but you may replace the old one, on explicit permission from the previous publisher that added the old one.

The author(s) and publisher(s) of the Document do not by this License give permission to use their names for publicity for or to assert or imply endorsement of any Modified Version.

## Combining Documents

You may combine the Document with other documents released under this License, under the terms defined in section 4 above for modified versions, provided that you include in the combination all of the Invariant Sections of all of the original documents, unmodified, and list them all as Invariant Sections of your combined work in its license notice.
The combined work need only contain one copy of this License, and multiple identical Invariant Sections may be replaced with a single copy. If there are multiple Invariant Sections with the same name but different contents, make the title of each such section unique by adding at the end of it, in parentheses, the name of the original author or publisher of that section if known, or else a unique number. Make the same adjustment to the section titles in the list of Invariant Sections in the license notice of the combined work.
In the combination, you must combine any sections entitled "History" in the various original documents, forming one section entitled "History"; likewise combine any sections entitled "Acknowledgements", and any sections entitled "Dedications". You must delete all sections entitled "Endorsements."

## Collections of Documents

You may make a collection consisting of the Document and other documents released under this License, and replace the individual copies of this License in the various documents with a single
copy that is included in the collection, provided that you follow the rules of this License for verbatim copying of each of the documents in all other respects.
You may extract a single document from such a collection, and distribute it individually under this License, provided you insert a copy of this License into the extracted document, and follow this License in all other respects regarding verbatim copying of that document.

## Aggregation With Independent Works

A compilation of the Document or its derivatives with other separate and independent documents or works, in or on a volume of a storage or distribution medium, does not as a whole count as a Modified Version of the Document, provided no compilation copyright is claimed for the compilation. Such a compilation is called an "aggregate", and this License does not apply to the other self-contained works thus compiled with the Document, on account of their being thus compiled, if they are not themselves derivative works of the Document.

If the Cover Text requirement of section 3 is applicable to these copies of the Document, then if the Document is less than one quarter of the entire aggregate, the Document's Cover Texts may be placed on covers that surround only the Document within the aggregate. Otherwise they must appear on covers around the whole aggregate.

## Translation

Translation is considered a kind of modification, so you may distribute translations of the Document under the terms of section 4. Replacing Invariant Sections with translations requires special permission from their copyright holders, but you may include translations of some or all Invariant Sections in addition to the original versions of these Invariant Sections. You may include a translation of this License provided that you also include the original English version of this License. In case of a disagreement between the translation and the original English version of this License, the original English version will prevail.

## Termination

You may not copy, modify, sublicense, or distribute the Document except as expressly provided for under this License. Any other attempt to copy, modify, sublicense or distribute the Document is void, and will automatically terminate your rights under this License. However, parties who have received copies, or rights, from you under this License will not have their licenses terminated so long as such parties remain in full compliance.

## Future Revisions of This License

The Free Software Foundation may publish new, revised versions of the GNU Free Documentation License from time to time. Such new versions will be similar in spirit to the present version, but may differ in detail to address new problems or concerns. See http://www.gnu.org/copyleft/.
Each version of the License is given a distinguishing version number. If the Document specifies that a particular numbered version of this License " or any later version" applies to it, you have the option of following the terms and conditions either of that specified version or of any later version that has been published (not as a draft) by the Free Software Foundation. If the Document does not specify a version number of this License, you may choose any version ever published (not as a draft) by the Free Software Foundation.
*

ADDENDUM: How to use this License for your documents
To use this License in a document you have written, include a copy of the License in the document and put the following copyright and license notices just after the title page:

Copyright © YEAR YOUR NAME. Permission is granted to copy, distribute and/or modify this document under the terms of the GNU Free Documentation License, Version 1.1 or any later version published by the Free Software Foundation; with the Invariant Sections being LIST THEIR TITLES, with the Front-Cover Texts being LIST, and with the Back-Cover Texts being LIST. A copy of the license is included in the section entitled "GNU Free Documentation License".

If you have no Invariant Sections, write "with no Invariant Sections" instead of saying which ones are invariant. If you have no Front-Cover Texts, write "no Front-Cover Texts" instead of "Front-Cover Texts being LIST"; likewise for Back-Cover Texts.
If your document contains nontrivial examples of program code, we recommend releasing these examples in parallel under your choice of free software license, such as the GNU General Public License, to permit their use in free software.

## Index of concepts

algorithm
conjugate gradient, 16, 37, 55, 64
Euler implicit scheme, 75, 78
fixed-point, 96, 105
minres, 37
Newton, 100
damped, 105
approximation, 14
P0, 48, 49
P1b-P1, 65
P1d, 48, 49
P1, 15, 45, 49, 54, 64, 65, 84
P2-P1, Taylor-Hood, 55, 65, 67, 69
P2, 15, 19, 45, 49, 54, 64, 84
Pk, 3, 15, 19
bubble, 65
discontinuous, 33, 48-50
high-order, 3, 15, 19, 35, 116
isoparametric, 116
mixed, 55
$\operatorname{argc}, \operatorname{argv}$, command line arguments, 3, 14
benchmark
driven cavity flow, 54, 84
embankment, 43
flow in an abrupt contraction, 67, 71
rotating hill, 79
boundary condition
Dirichlet, 13, 23, 33, 43, 54, 63, 84, 95, 101
Neumann, 26, 33, 37, 43, 63
Poiseuille flow, 67
Robin, 28
mixed, 63, 67, 72
characteristic method, 3, 79, 84
class
Float, 25
adapt_option_type, 53
branch, 76, 83
characteristic, 80
communicator, 39
csr<T>, 39
doubledouble, 25
environment, 3
field, 3,14
form, 14
geo, 14
idiststream, 4
odiststream, 4, 51
point, 25
solver_abtb, 4, 86
solver, 4, 16, 56
space, 4,14
vec<T>, 39
compilation, 16
convergence
rate, 99, 100
super-linear, 105
versus mesh, 111
degree of freedom, 100
directory of example files, 13, 68, 70
distributed computation, $3,15,17,39$
element shape, 35
error analysis, 83, 111
file format
'. bamgcad’ bamg geometry, 53, 121
'. bamg' bamg mesh, 53, 121
'. branch' family of fields, 77
'.dmn' domain names, 121
'.field' field, 4, 17
'.field' multi-component field, 4, 47
'.geo' mesh, $4,17,68,70,121,122,124$
'.gz' gzip compressed file, 53
‘.jpg' joint photographic experts group, 78
'.mov' quicktime video file, 78
'.mshcad' gmsh geometry, 54, 68, 72, 122
'.msh' gmsh mesh, 54, 68, 72, 122
'. vtk' vtk file, 61, 78
form
2D_D, $43,54,63,66,84$
2D, 48
curl, 57, 72
div_div, 43, 63
div, $48,54,63,66,84$
grad_grad, 15, 72
inv_mass, 48
lumped_mass, 80
mass, 15,111
s_curl, 69, 72
s_grad_grad, 69, 72
concatenation, 67
energy, $14,32,95$
weighted, 33, 96
tensorial weight, 104
Fréchet derivative, 100
function
adapt, 51
catchmark, $39,45,57,76$
compose, 80,88
norm2, 87
riesz, 80
class-function object, 24,81
geometry
axisymmetric, 67, 69, 70
contraction, 67, 71
cube, 19, 124
line, 19, 122
square, 17,123
Green formula, 119
Lagrange
interpolation, 24-26, 38
multiplier, 38, 56
node, 15
Lamé coefficients, 43
Makefile, 16
matrix
block structure, 15
concatenation, 39
factorization
Choleski, 16
lumped mass, 80
mesh, 14, 121
adaptation, 3
anisotropic, 51, 88
generation, 68, 72, 121
namespace
rheolef, 3,14
std, 14
node, 100
norm
in $W^{-1, p}, 100$
discrete version, 100
in $W^{1, p}, 95$
in $W_{0}^{1, p}, 95$
operator
ajoint, 106
curl, 57
discrete, 99
divergence, 43
gradient, 43
symmetric part, 43
Helmholtz, 26
Laplace, 13
parallel computation, 15, 17, 39
polar coordinate system, 70
preconditioner, 55
Choleski incomplete factorization, 16
for nearly incompressible elasticity, 64
for Stokes problem, 55
problem
Helmholtz, 26
Navier-Stokes, 84
Poisson
non-constant tensorial coefficients, 101
Poisson, 13, 23, 28, 37, 75, 95, 98
Stokes, 54, 65, 84
convection-diffusion, 78
elasticity
incompressible, 63
elasticity, 43
heat, 75
linear tangent, 100
nonlinear, 84
p-laplacian, 95
stabilized Stokes, 70
transmission, 32
projection, 49, 59
region, 32,34
residual term, 99, 100
Riesz representer, 15
singular solution, 59
space
$W^{-1, p}$, dual of $W_{0}^{1, p}, 100$
$W^{1, p}, 95$
$W_{0}^{1, p}, 95$
dual, 100
weighted (axisymmetric), 71
stabilization, 63
stream function, 60, 69, 90
axisymmetric, 71
tensor

Cauchy stress, 43, 73
field, 49
rate of deformation, 73
visualization as ellipsoid, 49
variable
derr, 4
din, 4
dout, 4
visualization
animation, 78
elevation view, 18, 99
stereoscopic anaglyph, 18, 47
vortex, 69, 73
vorticity, 57

## Index of example files

Makefile.demo, 16
cavity.icc, 55, 88
contraction.mshcad, 68,72
convect.cc, 80,86
convect_error.cc, 82
cosinusprod.icc, 112
cosinusprod_error.cc, 112
cosinusprod_laplace.icc, 24
cosinusrad.icc, 116
cosinusrad_error.cc, 116
cosinusrad_laplace.icc, 116
cube.mshcad, 124
dirichlet-nh.cc, 24
dirichlet.cc, 14
dirichlet.icc, 98
dirichlet_nh_ball.cc, 116
elasticity_criterion.icc, 52
elasticity_solve.icc, 52
embankment.cc, 45
embankment.icc, 45
embankment_adapt.cc, 52
heat.cc, 76
incompressible-elasticity.cc, 64
line.mshcad, 54, 122
navier_stokes_cavity.cc, 87
navier_stokes_criterion.icc, 87
navier_stokes_solve.icc, 86
neumann-laplace.cc, 38
neumann-nh.cc, 26
p_laplacian.h, 102
p_laplacian1.icc, 103
p_laplacian2.icc, 104
p_laplacian_damped_newton.cc, 107
p_laplacian_fixed_point.cc, 97
p_laplacian_fixed_point.h, 97
p_laplacian_newton.cc, 101
poiseuille.h, 67
robin.cc, 29
rotating-hill.h, 81
sinusprod_helmholtz.icc, 27
square.bamgcad, $53,90,121$
square.dmn, 53, 90, 121
square.mshcad, 123
stokes_cavity.cc, 55
stokes_contraction.cc, 69
stokes_contraction_bubble.cc, 66
streamf_cavity.cc, 60, 90
streamf_contraction.cc, 69, 72
stress.cc, 48, 74
transmission.cc, 33
vortex_position.cc, 92
vorticity.cc, 58

## Index of programs

bamg, 53, 90, 121
bamg2geo, 121
branch, 81
-gnuplot, 78
-paraview, 78, 83
-umax, 78
convect, 81
ffmpeg, 78
field, 4,17
-, 19
-bw, 18, 69, 73, 90
-catchmark, 57
-comp, 47, 50, 73, 91
-cut, 73, 91, 99
-elevation, 18, 50, 99
-fill, 47
-gray, 18
-iso, 50
-max, 70, 91
-mayavi, 18, 47
-min, 91
-n-iso, 69
-n-iso-negative, 69, 73, 90
-noclean, 61
-noexecute, 61
-nofill, 18, 47
-normal, 73, 91, 99
-origin, 73, 91, 99
-proj, 49
-scale, 90
-stereo, 18, 20, 47, 49
-velocity, 57, 90
geo, 17
-cut, 19
-full, 19
-stereo, 19
gmsh, 54, 68, 72, 122
gnuplot, 17, 19, 34, 47, 78, 81, 122
gzip, 53
library
boost, $3,15,39$
CGAL, computational geometry, 79
STL, standard template library, 81
make, 16
man, 19
mayavi, $18,20,61,122$
mkgeo_ball, 116
mkgeo_grid, 17, 81
-H, 20
-T, 19, 84
-a, 81
-b, 81
-c, 83
-d, 83
-e, 19
-f, 84
-g, 84
-q, 20
-region, 34
-t, 17
-zr, 70
mkgeo_ugrid, 112
MPI, message passing interface, 15
mpirun, 17, 35, 39
msh2geo, 68, 122, 124
-zr, 72
paraview, 78,83
rheolef-config, 13
-check, 13
-example-dir, 13
sed, 116
visualization
mesh, 17
deformed, 47
vlc, 78
zcat, 90

## List of Figures

1.1 Solution of the model problem for $d=2$ : (left) $P_{1}$ element; (right) $P_{2}$ element. ..... 17
1.2 Alternative representations of the solution of the model problem $\left(d=2\right.$ and the $P_{1}$ element): (left) in black-and-white; (right) in elevation and stereoscopic anaglyph mode. ..... 18
1.3 Red-cyan anaglyph glasses for the stereoscopic visualization. ..... 18
1.4 Solution of the model problem for $d=3$ and the $P_{1}$ element: (left) mesh; (right) isovalue, cut planes and stereo anaglyph renderings. ..... 20
3.1 Transmission problem: the domain $\Omega$ partition: $\left(\Omega_{\text {west }}\right.$ and $\left.\Omega_{\text {east }}\right)$. ..... 32
3.2 Transmission problem: $u_{h}=\pi_{h}(u)\left(\varepsilon=10^{-2}, d=1, P_{1}\right.$ approximation). ..... 34
5.1 The boundary domains for the square and the cube. ..... 44
5.2 The linear elasticity for $\lambda=1$ and $d=2$ and $d=3$ : both wireframe and filled surfaces ; stereoscopic anaglyph mode for 3D solutions. ..... 46
5.3 The stress tensor visualization (linear elasticity $\lambda=1$ ). ..... 49
5.4 The $\sigma_{01}$ stress component (linear elasticity $\lambda=1$ ): $d=2$ (top) and $d=3$ (bottom) ; $P_{0}$ (left) and $P_{1}$ discontinuous approximation (right). ..... 50
5.5 Adapted meshes: the deformation visualization for $P_{1}$ and $P_{2}$ approximations. ..... 53
5.6 The velocity visualization for $d=2$ and $d=3$ with stereo anaglyph. ..... 56
5.7 The vorticity: elevation view for $d=2$ and vector representation for $d=3$ (with anaglyph). ..... 59
5.8 The stream function visualization: isolines for $d=2$, and combined vectors and isonorm surface for $d=3$. ..... 61
6.1 The incompressible linear elasticity $(\lambda=+\infty)$ for $N=2$ and $N=3$. ..... 65
6.2 Solution of the Stokes problem in the abrupt contraction: (top) the mesh; (center) the $P_{1}$ stream function associated to the $P_{1} b-P_{1}$ element; (bottom) the $P_{2}$ stream function associated to the $P_{2}-P_{1}$ Taylor-Hood element. ..... 68
6.3 Solution of the axisymmetric Stokes problem in the abrupt contraction: (top) the $P_{2}$ stream function associated to the $P 2-P_{1}$ element; (bottom) comparison with the 2D cartesian solution (in red). ..... 72
6.4 Solution of the plane and axisymmetric Stokes problem in the abrupt contraction: cut along the axis of symmetry: (left): $u_{0}$; (right) $\tau_{\theta \theta}$. ..... 73
7.1 Animation of the solution of the heat problem. ..... 77
7.2 Animation of the solution of the rotating hill problem. ..... 82
7.3 Diffusion-convection when $d=1$ and $\nu=10^{-2}$ : convergence versus $h$ and $\Delta t$ for $P_{1}$ and $P_{2}$ elements: (left) in $L^{2}\left(L^{2}\right)$ norm; (right) in $L^{\infty}\left(L^{\infty}\right)$ norm. ..... 83
7.4 Meshes and stream functions associated to the solution of the Navier-Stokes equa- tions for $R e=100$ (top) and $R e=400$ (bottom). ..... 89
7.5 Meshes and stream functions associated to the solution of the Navier-Stokes equa- tions for $R e=1000$. ..... 90
7.6 Navier-Stokes: velocity profiles along lines passing throught the center of the cavity, compared with data from [26]: (a) $u_{0}$ along the vertical line; (b) $u_{1}$ along the horizontal line line. ..... 91
7.7 Cavity flow: primary vortex position and stream function value ..... 91
8.1 The $p$-laplacian for $d=2$ : (a) elevation view for $p=1.2$; (b) cut along the first bissectice $x_{0}-x_{1}=0$. ..... 98
8.2 The fixed-point algorithm on the $p$-laplacian for $d=2$ : (a) convergence when $p<2$; (b) when $p>2$; (c) convergence rate versus $p$; (d) convergence rate versus $p$ in semi-log scale. ..... 99
8.3 The Newton algorithm on the $p$-laplacian for $d=2$ : (a) convergence when $p<2$; (b) when $p>2$. ..... 105
8.4 The damped Newton algorithm on the $p$-laplacian for $d=2$ : (a) convergence when $p<2$; (b) when $p>2$. ..... 108
8.5 Convergence versus $n$ for various meshes: (a) Newton algorithm when $p=1.7$ and (b) $p=1.6$; (c) Newton algorithm when $p=1.5$; (d) damped-Newton algorithm when $p=1.5$. ..... 109
9.1 Strait geometry: error analysis in $L^{2}, L^{\infty}$ and $H^{1}$ norms. ..... 113
9.2 Curved domains (triangles): error analysis in $L^{2}, L^{\infty}$ and $H^{1}$ norms. ..... 114
9.3 Curved domains (quadrangles): error analysis in $L^{2}, L^{\infty}$ and $H^{1}$ norms. ..... 115
B. 1 Visualization of the gmsh meshes 'square.geo' and 'cube.geo'. ..... 127


[^0]:    ${ }^{1}$ At this time, the avi output feature is broken in paraview, and an alternate mpeg output is here suggested.

