Algebraic approaches to nuclear structure

K. Heyde

ALGEBRAIC APPROACHES TO NUCLEAR STRUCTURE

K.Heyde
Institute for Theoretical Physics,
Proeftuinstraat, 86 - B 9000 Gent (Belgium)

In the present lecture notes, I have tried to present a short overview of how recent developments in nuclear structure calculations, concentrating mainly on low-lying collective excitations, can be understood in the light of the nuclear shell-model. The particular approach where the independent-particle motion in an average nuclear potential and the residual interactions amongst nucleons are assumed to form the basic ingredients for studying nuclear collective motion is of course a personal view. Also, many aspects of recent developments in nuclear structure studies starting from self-consistent methods (eg using Skyrme type forces), the interesting studies on nuclear deformation within the Nilsson model (Crancked shell-model, etc.), the observation in a systematic way of coexistence of different types of excitations near closed shells (0^+ levels in Pb, Sn region) nor the extended shell-model calculations (MONSTER, ...) could be discussed in this lectures although sometimes very exciting results have been obtained.

1.1. Introduction

When studying the atomic nucleus, in the early stages, use has been made of nuclear models to approximate the nuclear many-body problem that can be described by a Hamiltonian of the form

$$H = \sum_{i=1}^{A} t_i + \frac{1}{2} \sum_{i,j=1}^{A} v_{i,j} + \cdots \quad , \tag{1.1}$$

for the A-body system. Starting from the nuclear shell-model as outlined by M.G.Mayer[1] and H.Jensen[2], which emphasises the independent particle motion in an average potential U(r), the structure of many properties (excitation energies, electromagnetic properties, ...) is largely understood. In the average potential, the one-body Schrödinger equation can be solved for the single-particle wave functions and single-particle energies as

$$h_o(i)\varphi_a(\vec{r}_i) = \varepsilon_a \varphi_a(\vec{r}_i) \quad , \tag{1.2}$$

and the original Hamiltonian can be rewritten as

$$H = H_o + H_{res} \quad , \tag{1.3}$$

with

$$H_o = \sum_{i=1}^{A} h_o(i) \quad , \tag{1.4}$$

where H_o describes the independent-particle model and H_{res} the residual interactions that have to be treated in the basis spanned by the independent-particle wave functions. Even in its simplest form, the independent-particle model describes the ground-state spin of most odd-A nuclei. However, for nuclei with many nucleons outside of the closed shells, determined via the shell-model itself, the residual interactions play a major role in determining the precise nuclear structure. Here, the

independent-particle shell-model wave functions $\varphi_a(\vec{r}_i)$ are used to construct a basis for expressing the total nuclear wave function as

$$\bar{\psi} = \sum_i a_i \psi_i \quad , \tag{1.5}$$

where ψ_i is a short-hand notation for a particular wave function conform to a given nucleus A(Z,N) i.e.

$$\psi_i \equiv \psi(((j_1)^{n_1}_{\alpha_1 J_1} \ (j_2)^{n_2}_{\alpha_2 J_2})_{J_{12}} \ \ldots) JM \quad , \tag{1.6}$$

where n_i nucleons are moving in the single-particle orbital characterized by the quantum number $j_i (j_i \equiv n_i, l_i, j_i)$. Here, J_i is the angular momentum, α_i denotes extra quantum numbers needed to specify states uniquely and J is the total angular momentum ($\sum_i n_i = n$) where n is the number of valence nucleons outside of the closed shells).
The eigenvalue equation, using the Hamiltonian of eq. (1.3), now leads to a matrix eigenvalue equation

$$H\bar{\psi} = E\bar{\psi} \quad , \tag{1.7}$$

or substituting the expression (1.5), one gets

$$\sum_{l,k=1}^{n} a_{lp'} H_{lk} a_{kp} = E_p \delta_{pp'} \quad , \tag{1.8}$$

where

$$H_{lk} = \langle \psi_l | H_o + H_{res} | \psi_k \rangle \quad . \tag{1.9}$$

In the next paragrahs, I will shortly discuss, as a reminder, some properties of the two-particle system relating to the pairing properties of the nuclear interaction and sketch the extension to n particles in many shells.

1.2. The nuclear shell-model

1.2.1. Two-particle system

Using the notation $j_a(\equiv n_a,l_a,j_a)$, two-particle angular momentum and antisymmetrized wave functions can be constructed using the wave functions of the independent-particle shell-model as

$$\psi(j_1j_2,JM) = \frac{1}{\sqrt{2}} \sum_{m_1m_2} \langle j_1m_1j_2m_2|j_1j_2,JM\rangle$$
$$[\varphi_{j_1m_1}(1)\varphi_{j_2m_2}(2) - \varphi_{j_1m_1}(2)\,\varphi_{j_2m_2}(1)] \quad , \qquad (1.10)$$

or

$$\psi(j^2,JM) = \sum_{m} \langle jm\ jm'|JM\rangle \varphi_{jm}(1)\ \varphi_{jm'}(2) \quad , \qquad (1.11)$$

where in (1.10) $|j_1-j_2| \leq J \leq j_1+j_2$ and in (1.11) $J = 0,2,4, \ldots, 2j-1$.

When considering non-identical nucleons, the isospin quantum number has also to be taken into account. When calculating the nuclear two-particle interaction matrix elements, (using Dirac bra-ket notation) one gets the binding energy that are

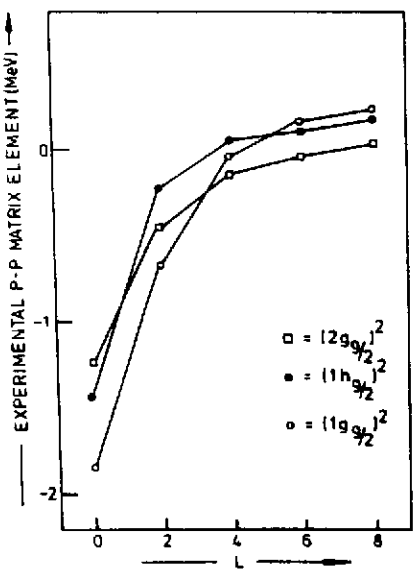

fig.1.1.

The experimental particle-particle matrix elements for three different shell-model orbitals

$$\left.\begin{aligned} &\Delta E_J = \langle j_1 j_2, JM | H_{res} | j_1 j_2, JM \rangle \\ \text{or} \\ &\Delta E_J = \langle j^2, JM | H_{res} | j^2, JM \rangle \end{aligned}\right\} , \qquad (1.12)$$

depicted in fig.1.1 for some typical j-values. All two-particle energy spectra for nuclei having just two-particles (or holes) outside closed shells are of this form eg ^{18}O, ^{42}Ca, ^{134}Te, ^{206}Pb, ... and indicate the pairing aspect of the residual two-body interaction. Most effective interactions (that are constructed explicitely for describing energy spectra in a restricted model space) have this pairing property[3,4]. Also, more realistic interactions, constructed for describing free nucleon-nucleon scattering outside the nucleus, are shown to exhibit this pairing property when putting nucleons in the nuclear medium in a given model space[5]. In the remaining part of this lecture, I will simplify the residual interaction to the pairing component only and discuss some consequences for the energy spectra when considering many more valence nucleons.

1.2.2. Particles in degenerate shells

Starting from the above pairing aspects, it is tempting to use an interaction that has this pairing property of acting only in $J^\pi = 0^+$ states. Knowing that a given two-body interaction, given in coordinate space can be written in a second quantized form

$$V_{12} \rightarrow \frac{1}{4} \sum_{\alpha\beta\gamma\delta} \langle \alpha\beta | V | \gamma\delta \rangle a^+_\alpha a^+_\beta a_\delta a_\gamma \quad , \qquad (1.13)$$

we can define an interaction Hamiltonian, immediately in second quantized form as[4,6]

$$H = - G \sum_{m,m'>0} a^+_{jm} a^+_{j-m} a_{j-m'} a_{jm'} (-1)^{2j+m+m'} \quad . \qquad (1.14)$$

In the two-particle subspace (m-states), H has the matrix respresentation

$$H = - G \begin{bmatrix} 1 & 1 & 1 & \cdot & \cdot \\ 1 & 1 & 1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \end{bmatrix} \quad . \tag{1.15}$$

It can be shown that the state

$$\frac{1}{\sqrt{\Omega}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ \cdot \end{bmatrix} = \frac{1}{\sqrt{\Omega}} \sum_{m>o} (-1)^{j+m} a^{+}_{jm} a^{+}_{j-m} \, |0\rangle \tag{1.16}$$

$$= (j)^2 J=0 \; M=0\rangle \quad ,$$

is the lowest energy eigenstate of (1.15). Here $|0\rangle$ denotes the closed-shell wave function and Ω is the shell degeneracy or $\Omega \equiv j + 1/2$. The eigenenergy of the state (1.16) is $E_o = - G.\Omega$.

To solve now for the n-particle system the pairing interaction problem, we define the pair creation operator

$$S^{+}_{j} = \frac{1}{\sqrt{\Omega}} \sum_{m>o} (-1)^{j+m} \, a^{+}_{jm} a^{+}_{j-m} \quad , \tag{1.17}$$

which creates the two-particle $J^{\pi} = 0^{+}$ state. Using this operator S^{+}_{j}, the Hamiltonian (1.14) can be rewritten as

$$H = - G \, \Omega \, S^{+}_{j} S_{j} . \tag{1.18}$$

The commutation relations are

$$[S_j, S^{+}_{j}] = 1 - \frac{\hat{n}}{\Omega} \quad , \tag{1.19}$$

where $\hat{n}$ is the number operator

$$\hat{n} = \Sigma_m a^{+}{}_{jm} a_{jm} = \sum_{m>o} (a^{+}_{jm} a_{jm} + a^{+}_{j-m} a_{j-m}) \quad . \tag{1.20}$$

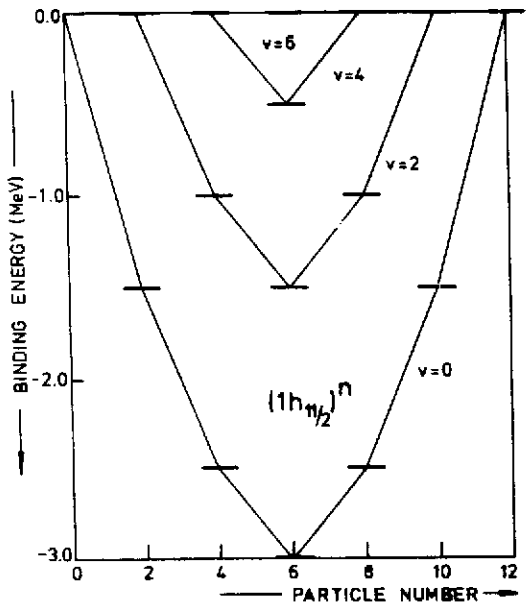

fig.1.2.

The spectrum for a pairing interaction within the $1h_{11/2}$ orbital $(1h_{11/2})^n_v$ for v= 0,2, 4 and 6 as a function of n. The pairing strength is G=0.25 MeV.

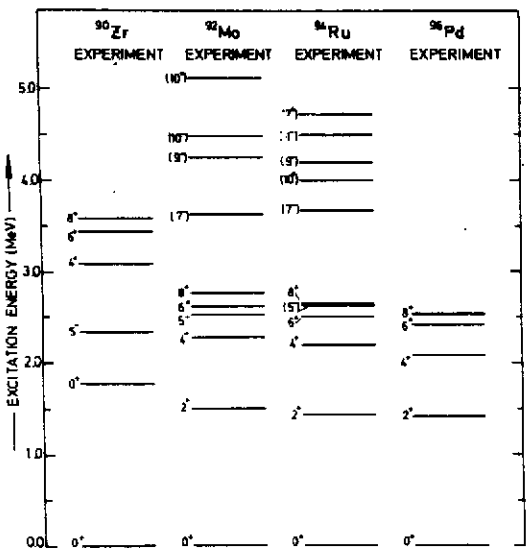

fig.1.3.
The experimental spectra for the N=50 single-closed shell nuclei ^{90}Zr, ^{92}Mo, ^{94}Ru, ^{96}Pd. The seniority v=2 states are clearly observed $(1g9/2)^n_{v=2}$.

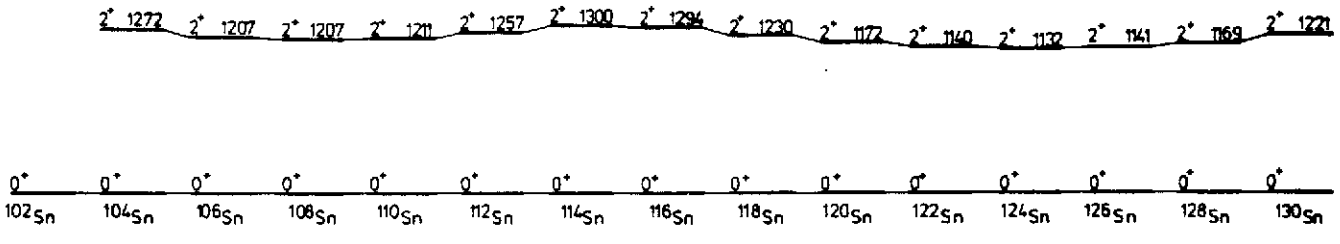

fig.1.4. Systematics of the first 2^+ level in the even-even Sn nuclei ($102 \leq A \leq 130$).

A class of eigenstates of H (eq. 1.14) are now given by

$$[H,S_j^+] = - G\, S_j^+(\Omega-\hat{n}) = - G(\Omega-\hat{n}+2)S_j^+ \quad . \tag{1.21}$$

Starting from zero valence particles, one gets

$$\left.\begin{aligned} H\, S_j^+|0\rangle &= - G\, \Omega\, S_j^+|0\rangle \quad , \\ H(S_j^+)^2|0\rangle &= - 2G\, (\Omega-1)(S_j^+)^2|0\rangle \quad , \\ &\vdots \qquad\qquad\qquad \vdots \\ H\, (S_j^+)^{n/2}|0\rangle &= - \frac{G}{4}\, n(2\Omega-n+2)(S_j^+)^{n/2}|0\rangle \quad . \end{aligned}\right\} \tag{1.22}$$

Here all particles are coupled pairwise to $J^\pi = 0^+$. They are described as seniority v = 0 states, where the seniority quantum number v denotes number of unpaired particles. In shorthand we can write

$$|n,v=0\rangle = (S_j^+)^{n/2}|0\rangle \quad , \tag{1.23}$$

and

$$E_{v=0}(n) = - \frac{G}{4}\, n(2\Omega-n+2) \quad . \tag{1.24}$$

One can generalize this procedure in adding to the S_j^+ operator, the $\Omega-1$ operators B_J^+ which create pairs coupled to angular momentum $J(J \neq 0)$ defined as

$$B_J^+ = \sum_{m>0} (-1)^{j-m} \langle jmj-m|J0\rangle\, a_{jm}^+ a_{j-m}^+ \quad , \tag{1.25}$$

and obey the relation

$$[H,B_J^+]|0\rangle = 0 \quad . \tag{1.26}$$

So we can construct a set of seniority v = 2 states

$$
\begin{aligned}
&H\, B_J^+|0\rangle \equiv H|n=2,v=2\ J\rangle = 0 \quad , \\
&H\, S_J^+\, B_J^+|0\rangle \equiv H|4,2,J\rangle = -\, G(\Omega-2)\,|4,2,J\rangle \quad , \\
&\quad\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
&H(S_J^+)^{\frac{n}{2}-1}\, B_J^+|0\rangle \equiv H|n,2,J\rangle = -\,\frac{G}{4}\,(n-2)(2\Omega-n)\,|n,2,J\rangle \quad .
\end{aligned}
\tag{1.27}
$$

For v > 2 we cannot continue in this way since an overcomplete set of states $B_J^+B_J^+|0\rangle$, ... arises. Still we can calculate the energy since a state with maximum seniority v = n has energy zero or $H|n,v=n\rangle = 0$. We can now calculate succesively

$$
\left.
\begin{aligned}
&H|n,n-2\rangle = H(S_J^+)\,|n-2,n-2\rangle = -\,G(\Omega-n+2)\,|n,n-2\rangle \quad . \\
&H|n,n-4\rangle = H(S_J^+)^2\,|n-4,n-4\rangle = -\,G(2\Omega-2n+6)\,|n,n-4\rangle \quad . \\
&\quad\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
&H|n,v\rangle = H(S_J^+)^{\left(\frac{n-v}{2}\right)}\,|v,v\rangle = -\,\frac{G}{4}\,(n-v)(2\Omega-n-v+2)\,|n,v\rangle \quad .
\end{aligned}
\right\}
\tag{1.28}
$$

giving the general expression for the spectrum

$$
E_v(n) - E_0(n) = \frac{G}{4}\,v(2\Omega-v+2) \quad . \tag{1.29}
$$

(see fig.1.2 and 1.3 for some illustrations).

1.2.3. Particles in non-degenerate shells

If we now have two or n particles in non-degenerate shells, much of the simplicity of the above discussion is lost but the major ingredients of classification of energy spectra in families of different seniority v = 0 , v = 2, v = 4, ..., remains and the above approximation of treating the many body

system of n identical nucleons outside closed shells using a product wave function of pairs is a good approximation[4,6]. Many calculations using this pair approximation and different types of residual interactions have been carried out under the name of BCS (Bardeen-Cooper-Schrieffer theory), broken-pair approximation, generalized seniority, (see ref.4 for a more detailed discussion and references to the literature) and all result in a good description of single-closed shell nuclei (see fig.1.4 for an example).

When however, both proton and neutron valence numbers are present this seniority classification breaks down rapidly and completely different types of energy spectra, reminiscent of coherent collective motion start to appear[7]. In the second lecture we will come back to this point and try to develop a microscopic basis for describing nuclear collective motion. Now, we mention a number of striking features relating to the presence of both active protons and neutrons.

1.3. Collective modes of the nucleus

Parallel to the study of the nuclear shell-model to a high degree of sophistication, the study of nuclear collective motion as a description of a liquid drop has been developed by A.Bohr, B.Mottelson[7-9], S.G.Nilsson [10] and many others in particular in the Copenhagen school[8,9].

Here, one starts of with a nuclear shape, for which the radius vector R is written as

$$R = R_0(1 + \sum_{\lambda\mu} \alpha_{\lambda\mu} Y_{\lambda\mu}(\theta,\varphi)), \qquad (1.30)$$

where λ describes the multipolarity of the shape ($\lambda = 2$: quadrupole, $\lambda = 3$: octupole, ... see fig.1.5). The dynamics for such an object in the small amplitude limit of oscillations around a spherical equilibrium shape is described starting form the Hamiltonian (for quadrupole vibrations)

$$H = \frac{1}{2} B \sum_{\mu} |\dot{\alpha}_\mu|^2 + \frac{C}{2} \sum_{\mu} |\alpha_\mu|^2 \quad . \qquad (1.31)$$

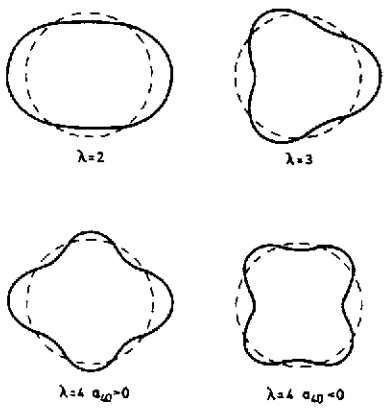

fig.1.5.
Nuclear shapes with quadrupole (λ=2), octupole (λ=3) and hexadecapole (λ=4) deformations.

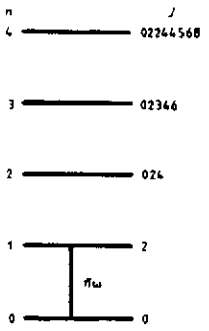

fig.1.6.
The quadrupole phonon spectrum for $n \leq 4$. The angular momenta are indicated on the right hand side.

Defining the momentum $\pi_\mu = B\,\alpha_\mu^*$ or $-i\hbar(\partial/\partial\alpha_\mu)$, one can quantize the Hamiltonian in eq (1.31). It is convenient to define creation and annihilation operators for oscillator quanta of a given multipolarity via the relation

$$b_\mu^+ = -\frac{1}{2a_0}\,\alpha_\mu + (-1)^\mu\, a_0\,\frac{\partial}{\partial\alpha_{-\mu}} \quad (a_0 = \sqrt{\hbar^2/4BC}) \ . \tag{1.32}$$

Using these boson operators, satisfying boson commutation relations

$$[b_\mu', b_\mu^+] = \delta_{\mu\mu'} \tag{1.33}$$

the Hamiltonian takes the form

$$H = \hbar\omega \sum_\mu (b_\mu^+ b_\mu + \frac{1}{2}), \tag{1.34}$$

with frequency $\omega = \sqrt{C/B}$. The energy spectrum consists of equidistant sets of levels with angular momenta determined by the multipolarity λ=2 (see fig. 1.6). In more realistic cases, the

degeneracies are broken through anharmonicities in the vibrational Hamiltonian and many such posibilities have been studied over the years[8].

Whenever the nucleus is described by a non-spherical shape, rotations can result besides vibrational motion. Here, it is convenient to transform from the coordinates α_μ to three Euler angles and two intrinsic variables (for $\lambda = 2$) using

$$\alpha_\mu = \sum_\nu D^2_{\mu\nu}(\Omega)\,\alpha'_\nu \quad , \tag{1.35}$$

With $D^2_{\mu\nu}$ the Wigner D or rotation matrix. In the new system we require

$$\alpha'_1 = \alpha'_{-1} = 0 \quad , \quad \alpha'_2 = \alpha'_{-2} \quad , \tag{1.36}$$

if the new "intrinsic" axis are to be the principal axis of the nuclear shape. So, we can parametrize

$$\left.\begin{aligned} \alpha'_0 &= \beta\cos\gamma \\ \alpha'_2 &= \frac{\beta}{\sqrt{2}}\sin\gamma \end{aligned}\right\} \quad . \tag{1.37}$$

Now, since the original potential energy was written in terms of rotationally invariant expressions of the coordinates, the following two forms will frequently occur i.e.

$$\left.\begin{aligned} (\alpha \times \alpha)^{(0)}_0 &= \frac{\beta^2}{\sqrt{5}} \\ (\alpha \times \alpha \times \alpha)^{(0)}_0 &= -\sqrt{\frac{2}{35}}\,\beta^3\cos 3\gamma \end{aligned}\right\} \quad . \tag{1.38}$$

Here, γ measures the departure from axial symmetry with $\gamma = 0°$ giving a prolate, $\gamma = 60°$ an oblate shape and thereby all shapes can be obtained within the sector $(0°, 60°)$. Also, the potential

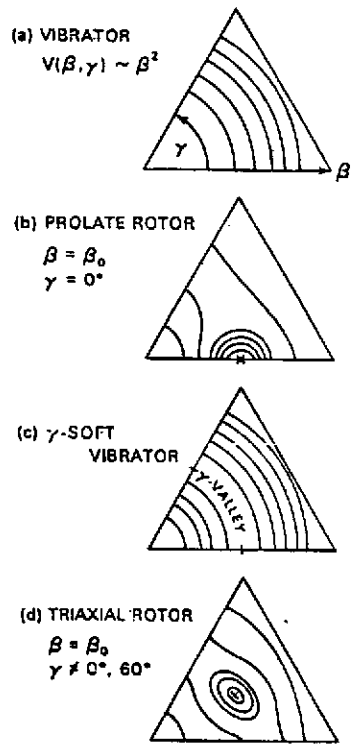

fig.1.7.

Different shapes $V(\beta,\gamma)$ in the β, (γ=0°-γ=60°) sector corresponding to a spherical vibrator, a prolate rotor, a γ-soft vibrator and a triaxial rotor.

energy $V(\beta,\gamma)$ will determine the nuclear dynamics in a major way (fig.1.7). The energy for rigid, rotation for an axially symmetric nucleus around an axis perpendicular to the symmetry axis then becomes

$$E_J = \frac{\hbar^2}{2\mathcal{J}} J(J+1) \quad , \tag{1.39}$$

with as a wave function

$$\psi_{JM} = \frac{2J+1}{8\pi^2} D^J_{MO}(\Omega) \quad . \tag{1.40}$$

In the more general case, when a general (β,γ) shape exists, vibrations in the β and γ direction can be formed and classified according to the number of quanta for each type (β-vibrations : n_β, γ-vibrations : n_γ). This more general wave function becomes

$$\psi_{JM} = \sqrt{\frac{2J+1}{16\pi^2}} \left(D^J_{MK}(\Omega) + (-1)^{J-K} D^J_{M-K}(\Omega)\right) \varphi_{n_\beta}(\beta) \varphi_{n_\gamma}(\gamma) \quad , \tag{1.41}$$