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FORMAL THEORY OF ELECTRO-MAGNETIC FIELD FLUCTUATIONS

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In this lecture the theory of electro-magnetic field fluctuations is introduced in a somewhat unconventional manner. Subsequently the specific example of electro-magnetic fields in infinite media is treated according to the usual Green's function method.

Electric field correlation functions

We define correlation functions by the following relations:

$$(1a) \quad C_{ij}(t) = \langle E_i(\mathbf{r}', t) E_j(\mathbf{r}, 0) \rangle = Tr \left(\rho e^{\frac{i}{\hbar} H t} E_i(\mathbf{r}', 0) e^{-\frac{i}{\hbar} H t} E_j(\mathbf{r}, 0) \right)$$

$$(1b) \quad D_{ij}(t) = \langle E_j(\mathbf{r}, 0) E_i(\mathbf{r}', t) \rangle = Tr \left(\rho E_j(\mathbf{r}, 0) e^{\frac{i}{\hbar} H t} E_i(\mathbf{r}', 0) e^{-\frac{i}{\hbar} H t} \right)$$

The indices i and j stand for cartesian field components conceived as operators. The brackets then designate statistical ensemble averages represented by traces taken with the corresponding density matrix ρ .

In order to establish a relationship between the two correlation functions, we use the identity

$$(2) \quad (Tr Q)^* = Tr Q^+$$

which is obvious given the fact that trace elements are not affected by permutation of indices.

For hermitian operators such that $\rho = \rho^\dagger, A = A^\dagger, B = B^\dagger$ we also have

$(\rho AB)^\dagger = BA\rho$ and further, due to the invariance of the trace under cyclic permutations,

$TrBA\rho = Tr\rho BA$. This immediately yields

$$(3) \quad C_{ij}^*(t) = Tr(\rho E_j(\mathbf{r}, 0) E_i(\mathbf{r}', t)) = D_{ij}(t)$$

Introducing real functions $a(t)$, $b(t)$, we can therefore write

$$(4a) \quad C_{ij}(t) = a(t) + i b(t)$$

$$(4b) \quad D_{ij}(t) = a(t) - i b(t)$$

Symmetry properties with respect to time reversal can be approached by means of the time reversal operator T . This operator replaces wave functions in coordinate representation by their complex conjugates. T is antilinear and $TT = 1$.

Using the fact that the property $TAT = A^*$ yields the identity

$$Tr(TQT) = TrQ^* = (TrQ)^*$$

we write

(5) $C_{ij}^*(t) = Tr(T\rho E_i(\mathbf{r}', t) E_j(\mathbf{r}, 0) T)$. Furthermore, with time reversal acting on the electric field and on the evolution operator according to the relations

$$TET = E, \quad T e^{\frac{i}{\hbar} Ht} T = e^{-\frac{i}{\hbar} Ht}$$

we have

$$(6) \quad T E_i(\mathbf{r}', t) T = T e^{\frac{i}{\hbar} Ht} E_i(\mathbf{r}', 0) e^{-\frac{i}{\hbar} Ht} T = e^{-\frac{i}{\hbar} Ht} E_i(\mathbf{r}', 0) e^{\frac{i}{\hbar} Ht}$$

and therefore from eq.'s (5) and (1a)

$$(7) \quad C_{ij}^*(t) = \text{Tr} \left(\rho e^{-\frac{i}{\hbar} H t} E_i(\mathbf{r}', 0) e^{\frac{i}{\hbar} H t} E_j(\mathbf{r}, 0) \right) = C_{ij}(-t).$$

Finally by combining eq.'s (3) and (7), we arrive at the following symmetry relations:

$$(8) \quad C_{ij}^*(t) = C_{ij}(-t) = D_{ij}(t) = D_{ij}^*(-t)$$

This implies that $a(t)$ and $b(t)$ are respectively even and odd functions of t .

FOURIER TRANSFORMS

Let us define Fourier transforms of our correlation functions by the following relations:

$$(9) \quad C_{ij}(\omega) = \int dt e^{i\omega t} C_{ij}(t) \quad ; \quad D_{ij}(\omega) = \int dt e^{i\omega t} D_{ij}(t)$$

As has been stated above, the real parts of $C_{ij}(t)$, $D_{ij}(t)$ are even functions of time whereas the imaginary parts are odd. Therefore, introducing Fourier transforms

$$a(\omega) = \int dt e^{i\omega t} a(t) \quad ; \quad b(\omega) = \int dt e^{i\omega t} b(t)$$

and writing

$$(10a) \quad C_{ij}(\omega) = a(\omega) + ib(\omega)$$

$$(10b) \quad D_{ij}(\omega) = a(\omega) - ib(\omega)$$

it follows that $a(\omega)$ must be real and $b(\omega)$ imaginary. As a consequence it is clear that both $C_{ij}(\omega)$ and $D_{ij}(\omega)$ must be real.

Furthermore, replacing ω by $-\omega$ we write

$$C_{ij}(-\omega) = \int dt e^{-i\omega t} C_{ij}(t)$$

and hence with eq.(6)

$$C_{ij}^*(-\omega) = \int dt e^{i\omega t} C_{ij}^*(t) = \int dt e^{i\omega t} D_{ij}(t)$$

yielding as a result the symmetry relation

$$(11) \quad C_{ij}^*(-\omega) = D_{ij}(\omega)$$

In terms of $a(\omega)$, $b(\omega)$ this relation takes the form

$$(12) \quad a(-\omega) + ib(-\omega) = a(\omega) - ib(\omega)$$

given the fact that

$$(13) \quad b^*(-\omega) = -b(\omega).$$

Eq.(13) shows that $a(\omega)$ and $b(\omega)$ are respectively even and odd functions of ω .

We derive now another very important relationship between the functions $C_{ij}(\omega)$ and $D_{ij}(\omega)$.

Consider the following integral :

$$(14) \quad I(\omega) = \int dt e^{i\omega t} \langle B(0)A(t) \rangle$$

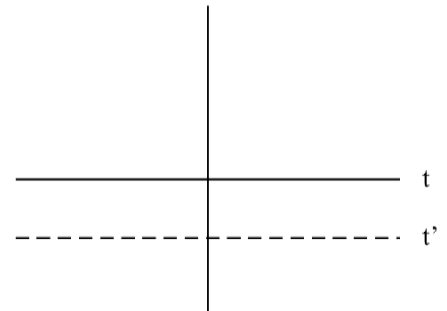
With the average obtained by taking the trace with the density operator represented by the Boltzmann factor $(1/Z)e^{-\beta H}$ this integral is explicitly given by the expression

$$(15) \quad I(\omega) = (1/Z) \int dt e^{i\omega t} \text{Tr} e^{-\beta H} B e^{iHt/\hbar} A e^{-iHt/\hbar} = (1/Z) \int dt e^{i\omega t} \text{Tr} e^{-iH(t-i\hbar\beta)/\hbar} B e^{iHt/\hbar} A$$

Now, setting $t - i\hbar\beta = t'$, integration along the real t axis can be replaced by integration over t' i.e. over a line parallel to this axis at distance $-i\hbar\beta$. (figure).

Hence with $\int dt \rightarrow \int dt'$ we obtain

$$(16) \quad \begin{aligned} I(\omega) &= (1/Z) \int dt' e^{i\omega(t'+i\hbar\beta)} \text{Tr} e^{-iHt'/\hbar} B e^{iH(t'+i\hbar\beta)/\hbar} A = \\ &e^{-\beta\hbar\omega} \int dt' e^{i\omega t'} \text{Tr} e^{-\beta H} e^{iHt'/\hbar} A e^{-iHt'/\hbar} B = \\ &e^{-\beta\hbar\omega} \int dt e^{i\omega t} \langle A(t)B(0) \rangle \end{aligned}$$



where the invariance of the trace with respect to cyclic permutation has been used.

Applying this result to the functions defined by eq.'s (1a,b) we thus arrive at the following relation

$$(17) \quad D_{ij} = e^{-\beta\hbar\omega} C_{ij}(\omega)$$

The Green's function

The Green's function involves a commutator between field variables according to the defining relation

$$(18) \quad G_{ij}(t) = \frac{i}{\hbar} \langle [E_i(\mathbf{r}, t), E_j(\mathbf{r}, 0)] \rangle \Theta(t)$$

with $\Theta(t)$ the Heaviside step function.

In terms of correlation functions we thus have the expression

$$(19) \quad G_{ij}(t) = \frac{i}{\hbar} (C_{ij}(t) - D_{ij}(t)) \Theta(t)$$

Taking the Fourier transform of this expression, by applying the general formula (A6) derived in the appendix, leads to the result

$$(20) \quad G_{ij}(\omega) = \frac{i}{\hbar} \left\{ \frac{1}{2} (C_{ij}(\omega) - D_{ij}(\omega)) + \frac{i}{2\pi} P \int d\omega' (C_{ij}(\omega') - D_{ij}(\omega')) / (\omega - \omega') \right\}$$

We now turn to the special form of the correlation functions as defined by eq.'s (7),(8) and the corresponding Fourier transforms of eq.'s (10a,b). Then eq.(20) takes the form

$$(21) \quad G_{ij}(\omega) = -\frac{1}{\hbar} b(\omega) - \frac{i}{\pi\hbar} P \int d\omega' \frac{b(\omega')}{\omega - \omega'}$$

Now, as stated above (cf. line after (10a,b)) $b(\omega)$ is purely imaginary so that on the r.h.s. the first term is also imaginary whereas the second one is real. We can therefore write

$$(22) \quad b(\omega) = -i\hbar \text{Im} G_{ij}(\omega)$$

On the other hand we have from eq.'s (10a,b)

$$2ib(\omega) = C_{ij}(\omega) - D_{ij}(\omega).$$

With the symmetry relation (17) this yields

$$(23) \quad 2ib(\omega) = C_{ij}(\omega) (1 - e^{-\beta\hbar\omega})$$

We thus arrive at the following relationship between Green's and correlation function:

$$(24) \quad \text{Im} G_{ij}(\omega) = \frac{1}{2\hbar} (1 - e^{-\beta\hbar\omega}) C_{ij}(\omega)$$

with the inverted relation

$$(25) \quad C_{ij}(\omega) = 2\hbar (1 - e^{-\beta\hbar\omega})^{-1} \text{Im} G_{ij}(\omega)$$

Symmetrized relations

Let us now consider a symmetrized correlation function defined by the expression

$$(26) \quad \Phi_{ij}^E(\omega) = \frac{1}{2} (C_{ij}(\omega) + D_{ij}(\omega)) = \frac{1}{2} (1 + e^{-\beta\hbar\omega}) C_{ij}(\omega).$$

where the relation (17) has been used.

With $C_{ij}(\omega)$ given by eq.(25) this expression takes the form

$$(27) \quad \Phi_{ij}^E = \hbar \frac{1 + e^{-\beta\hbar\omega}}{1 - e^{-\beta\hbar\omega}} \text{Im} G_{ij}(\omega) = \hbar \coth\left(e^{\frac{\beta\hbar\omega}{2}}\right) \text{Im} G_{ij}(\omega)$$

As will be shown below, this equation represents a law known as the *fluctuation-dissipation theorem*.

At this stage it is convenient to introduce the vector potential \mathbf{A} . From its definition

$$(28) \quad \mathbf{B} = \nabla \times \mathbf{A}$$

together with Maxwell's equation, it follows

$$(29) \quad \mathbf{E} = -\dot{\mathbf{A}}$$

For the corresponding correlation functions we thus have $\Phi_{ij}^E(t) = -\frac{\partial^2}{\partial t^2} \Phi_{ij}^A(t)$ yielding Fourier components obeying the relation

$$(30) \quad \Phi_{ij}^E(\omega) = \omega^2 \Phi_{ij}^A$$

Defining a Green's function G_{ij}^A with in eq.(18) E-components replaced by A-components, we write instead of eq.(27)

$$(31) \quad \Phi_{ij}^A(\omega) = \hbar \coth\left(e^{\frac{\beta\hbar\omega}{2}}\right) \text{Im} G_{ij}^A(\omega)$$

Φ_{ij}^E is thus obtained by multiplying this quantity with ω^2 according to eq.(30).

Linear response

Let us consider a quantum system driven by an external perturbation. Let's write the Hamiltonian in the form

$$(32) \quad H = H_0 - BS(t)$$

with H_0 the Hamiltonian of the unperturbed system and $-BS(t)$ the perturbation Hamiltonian with B a certain operator and $S(t)$ a given function of time.

As an example suppose that $S(t)$ represents a classical current density component $j_i(t)$ at position \mathbf{r}_0 and $A_i(\mathbf{r}_0)$ a vector potential operator at that same position. $BS(t)$ then stands for the interaction potential $A_i(\mathbf{r}_0)j_i(t)$.

Note that from now on we apply Einstein's summation prescription i.e. automatic summation over repeated indices.

We start from the Liouville equation

$$(33) \quad \frac{d}{dt}(\rho_0 + \rho_1) = -\frac{i}{\hbar}[H, (\rho_0 + \rho_1)]$$

where we have split the density operator ρ into an unperturbed part ρ_0 , corresponding to H_0 and a part ρ_1 due to the application of the external perturbation. Substituting for H the expression of eq.(32) and expanding to lowest order in the perturbation, we then derive for ρ_1 the equation

$$(34) \quad \frac{d}{dt}\rho_1 = -\frac{i}{\hbar}[H_0, \rho_1] + \frac{i}{\hbar}[B, \rho_0]S(t)$$

Introducing an interaction representation defined in the usual way by performing the transformation

$$(35) \quad \rho_1 = e^{-\frac{i}{\hbar}H_0 t} \sigma e^{\frac{i}{\hbar}H_0 t}$$

we obtain for the reduced density operator σ the equation

$$(36) \quad \frac{d}{dt}\sigma = e^{\frac{i}{\hbar}H_0t} [B, \rho_0] S(t) e^{-\frac{i}{\hbar}H_0t}$$

for which an exact solution is given by the expression

$$(37) \quad \sigma = \frac{i}{\hbar} \int_{-\infty}^t dt' e^{\frac{i}{\hbar}H_0t'} [B, \rho_0] S(t') e^{-\frac{i}{\hbar}H_0t'}$$

Transforming back to ρ_1 by inverting eq.(35) we arrive at the result

$$(38) \quad \rho_1(t) = \frac{i}{\hbar} \int_{-\infty}^t dt' e^{-\frac{i}{\hbar}H_0(t-t')} [B, \rho_0] S(t') e^{\frac{i}{\hbar}H_0(t-t')}$$

By means of this expression we can now calculate the change in the average value of some operator Q produced by applying the external perturbation. We therefore consider the expression

$$(39) \quad \delta \langle Q \rangle = Tr \{ \rho_1 Q \} = \frac{i}{\hbar} Tr \int_{-\infty}^t dt [B, \rho_0] Q(t-t') S(t')$$

where we have substituted eq.(38) for ρ_1 and where we have used the defining relation

$$(40) \quad Q(t-t') = e^{\frac{i}{\hbar}H_0(t-t')} Q e^{-\frac{i}{\hbar}H_0(t-t')}$$

after having shifted the factor $e^{-\frac{i}{\hbar}H_0(t-t')}$ in the right position. Writing explicitly

$$[B, \rho_0] Q = B \rho_0 Q - \rho_0 B Q \quad ; \quad Tr B \rho_0 Q = Tr Q B \rho_0 = Tr \rho_0 Q B$$

and hence

$$Tr \{ [B, \rho_0] Q \} = Tr \{ \rho_0 [Q, B] \}$$

we arrive at the expression

$$(41) \quad \delta \langle Q \rangle = \frac{i}{\hbar} \int_{-\infty}^t dt' \langle [Q(t-t'), B] \rangle S(t')$$

Specializing to the case where Q represents the vector potential operator at position \mathbf{r} and where $BS(t')$ is the interaction energy of the external current with the field at position \mathbf{r}_0 , such that

$$BS(t') = A_j(\mathbf{r}_0)j_j(t') ; \quad Q = \mathbf{A}(\mathbf{r}),$$

eq.(41) yields the expression, using $A_j(\mathbf{r}_0) = A_j(\mathbf{r}_0, 0)$,

$$(42) \quad \langle \mathbf{A}(\mathbf{r}, t) \rangle = \frac{i}{\hbar} \int_{-\infty}^t dt' \langle [\mathbf{A}(\mathbf{r}, t-t'), A_j(\mathbf{r}_0, 0)] \rangle j_j(t')$$

in the case however that without the external perturbation the average \mathbf{A} value is zero .

Assuming that the external perturbation oscillates at frequency ω , according to the expression $j_j(t') = j_j e^{-i\omega t'}$ and making the substitution $t' = t - \tau$, we find

$$(43) \quad \langle \mathbf{A}(\mathbf{r}, t) \rangle = \frac{i}{\hbar} e^{-i\omega t} \int_0^{\infty} d\tau \langle [\mathbf{A}(\mathbf{r}, \tau), A_j(\mathbf{r}_0, 0)] \rangle e^{i\omega\tau} j_j$$

Introducing the Green's function

$$(44) \quad G_{ij}(\mathbf{r}, \mathbf{r}_0; \tau) = \frac{i}{\hbar} \langle [A_i(\mathbf{r}, \tau), A_j(\mathbf{r}_0, 0)] \rangle \Theta(\tau)$$

we can write eq.(43) in the form

$$(45) \quad \langle A_i(\mathbf{r}, t) \rangle = e^{-i\omega t} \int_{-\infty}^{\infty} d\tau G_{ij}(\mathbf{r}, \mathbf{r}_0; \tau) e^{i\omega\tau} j_j$$

With the Fourier transform of the Green's function defined by the expression

$$G_{ij}(\mathbf{r}, \mathbf{r}_0; \omega) = \int_{-\infty}^{\infty} d\tau G_{ij}(\mathbf{r}, \mathbf{r}_0; \tau) e^{i\omega\tau}$$

we obtain for the response of the system to a monochromatic perturbation the following result:

$$(46) \quad \langle A_i(\mathbf{r}, t) \rangle = e^{-i\omega t} G_{ij}(\mathbf{r}, \mathbf{r}_0; \omega) j_j$$

Setting further

$$(47) \quad \langle A_i(\mathbf{r}, t) \rangle \equiv \bar{A}_i(\mathbf{r}, \omega) e^{-i\omega t}.$$

we finally arrive at the expression

$$(48) \quad \bar{A}_i(\mathbf{r}, \omega) = G_{ij}(\mathbf{r}, \mathbf{r}_0; \omega) j_j(\mathbf{r}_0)$$

Note that so far in this section we have dropped the superscription on the Green's function.

Energy dissipation

According to the definition $BS(t) = A_i(\mathbf{r}_0) j_i(t)$ the mean value of the energy density of the system considered above is given by the expression

$$(49) \quad E = -\bar{A}_i j_i$$

For the change in energy density we therefore have

$$(50) \quad W = \dot{E} = -\dot{\bar{A}}_i \dot{j}_i$$

Having assumed for j_i a monochromatic oscillation, we write $\dot{j}_i = \frac{1}{-i\omega} j_i$. This relation together with eq.(48) thus yields for W the value

$$(51) \quad W = -G_{ij}^A j_j \left(\frac{1}{-i\omega} j_i \right) = -\frac{i}{\omega} G_{ij}^A j_j j_i$$

Setting $j_j j_i = j^2 \delta_{ij}$ and taking then the real part we obtain for the change in energy density the result

$$(52) \quad W = \frac{1}{\omega} \text{Im} G_{ij}^A j^2 = \omega \text{Im} G_{ij}^E j^2$$

Comparing with the energy dissipation per unit volume $W = \rho j^2$ in a classical resistor with resistivity ρ , we can consider $\text{Im} G_{ij}^E$ as a measure for energy dissipation in the present case.

Moreover we can interpret eq.(27) as a general relation between fluctuation and dissipation in a given system.

The differential equations

Generally speaking, Green's functions are a familiar tool for solving differential equations. In the present case the differential equation that has to be solved is deduced from Maxwell's equation

$$(53) \quad \nabla \times \mathbf{B} = \frac{1}{c^2} \left(\varepsilon \frac{\partial \mathbf{E}}{\partial t} + \frac{\mathbf{j}}{\varepsilon_0} \right)$$

with ε the dielectric constant.

Considering Fourier components we replace $\frac{\partial \mathbf{E}}{\partial t}$ by $-i\omega \mathbf{E}$ so that with $\mathbf{E} = i\omega \mathbf{A}$ and $\mathbf{B} = \nabla \times \mathbf{A}$ we have

$$(54) \quad \nabla \times \nabla \times \mathbf{A} = \frac{\omega^2}{c^2} \varepsilon \mathbf{A} + \frac{1}{c^2 \varepsilon_0} \mathbf{j}$$

Since in Maxwell's equations field variables enter only with their average values, we can now use the expression of eq.(48) for the vector potential. Integrating this expression over the entire space where $\mathbf{j}(\mathbf{r}_0)$ is present, we write

$$(55) \quad \mathbf{A}_i(\mathbf{r}, \omega) = \int G_{ij}(\omega; \mathbf{r}, \mathbf{r}_0) j_j(\mathbf{r}_0) d^3 \mathbf{r}_0$$

Using the relation $\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \Delta \mathbf{A}$ we further write

$$(56) \quad \nabla(\nabla \cdot \mathbf{A}) - \Delta \mathbf{A} - \frac{\omega^2}{c^2} \varepsilon \mathbf{A} - \frac{1}{c^2 \varepsilon_0} \mathbf{j}$$

Introducing components x_k of \mathbf{r} and j_j of \mathbf{j} (recalling that throughout this treatise we apply Einstein's summation prescription) we have

$$\nabla \cdot \mathbf{A} = \frac{\partial}{\partial x_k} A_k$$

and eq.(56) becomes

$$(57) \quad \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_k} A_k - \Delta A_i - \frac{\omega^2}{c^2} \varepsilon A_i = -\frac{1}{c^2 \varepsilon_0} j_i$$

Setting $A_i = \delta_{ki} A_k$ this equation takes the form

$$(58) \left\{ \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_k} - \Delta \delta_{ki} - \frac{\omega^2}{c^2} \varepsilon \delta_{ki} \right\} A_k = -\frac{1}{c^2 \varepsilon_0} j_i$$

Substituting now for A_k the result of eq.(55) we obtain

$$(59) \int \left\{ \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_k} - \Delta \delta_{ki} - \frac{\omega^2}{c^2} \varepsilon \delta_{ki} \right\} G_{kj}(\omega; \mathbf{r}, \mathbf{r}_0) j_j(\mathbf{r}_0) d^3 \mathbf{r}_0 = -\frac{1}{c^2 \varepsilon_0} j_i(\mathbf{r})$$

This equation is satisfied if the following relation is fulfilled:

$$(60) \left\{ \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_k} - \Delta \delta_{ki} - \frac{\omega^2}{c^2} \varepsilon \delta_{ki} \right\} G_{kj}(\omega; \mathbf{r}, \mathbf{r}_0) = -\frac{1}{c^2 \varepsilon_0} \delta_{ij} \delta(\mathbf{r} - \mathbf{r}_0)$$

Note that this can immediately verified by introducing the latter expression into eq.(59).

With eq. (60) we recover the usual form of a Green's function's differential equation characterized by the fact that the external perturbation is no longer contained in it.

We shall now proceed to solving tis differential equation. In order to simplify matters we shall restrict ourselves to the case of an infinite medium. Clearly in this case the Green's function will only depend on the difference $\mathbf{x} = \mathbf{r} - \mathbf{r}_0$ and eq.(60) can be written as follows:

$$(61) \left\{ \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_k} - \Delta \delta_{ki} - \frac{\omega^2}{c^2} \varepsilon \delta_{ki} \right\} G_{kj}(\omega; \mathbf{x}) = -\frac{1}{c^2 \varepsilon_0} \delta_{ij} \delta(\mathbf{x})$$

As a first step we introduce Fourier transformed quantities defined by the relations

$$(62a) G_{kj}(\omega; \mathbf{x}) = \int G_{kj}(\omega; \mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}} d^3 k$$

$$(62b) \delta(\mathbf{x}) = \frac{1}{(2\pi)^3} \int e^{i\mathbf{k} \cdot \mathbf{x}} d^3 k$$

Substituting these expressions into eq.(61) transforms this differential equation into the following set of algebraic equations:

$$(63) \quad (k_i k_k - k^2 \delta_{ki} - \kappa^2 \delta_{ki}) G_{kj}(\omega; \mathbf{k}) = \frac{1}{(2\nu)^2} \frac{1}{c^2 \epsilon_0} \delta_{ij}$$

where we have set

$$(64) \quad \frac{\omega^2}{c^2} \epsilon = -\kappa^2.$$

The solution of these equations can be found by means of the trial expression

$$G_{kj} = A \delta_{kj} + B k_k k_j.$$

Substituting this relation into eq.(63) and summing over the index k yields for the unknown coefficients the equation

$$(A - B\kappa^2) k_i k_j - A(k^2 + \kappa^2) \delta_{ij} = \frac{1}{(2\pi)^3} \frac{1}{c^2 \epsilon_0} \delta_{ij}$$

with the result

$$A = -\frac{1}{(2\pi)^3} \frac{1}{c^2 \epsilon_0} \frac{1}{k^2 + \kappa^2} ; \quad B = -\frac{1}{(2\pi)^3} \frac{1}{\kappa^2} \left(\frac{1}{c^2 \epsilon_0} \frac{1}{k^2 + \kappa^2} \right)$$

Hence the solution of the set of eq.'s(63) is given by the expression

$$(65) \quad G_{kj}(\omega; \mathbf{k}) = -\frac{1}{(2\pi)^3} \frac{1}{c^2 \epsilon_0} \frac{1}{k^2 + \kappa^2} \left(\delta_{kj} + \frac{k_k k_j}{\kappa^2} \right)$$

Naturally, in order to obtain the electromagnetic field variables from this intermediate solution, its inverse Fourier transform has first to be taken.

Thermal radiation

As an example of application of the above concepts we shall now derive the frequency distribution spectrum of thermal radiation. In that case the quantity of interest is the trace of the field correlation matrix of eq.(30). According to eq.(31) it involves the trace of the Green's function, with the additional condition that their values are taken at $\mathbf{r} \rightarrow \mathbf{r}_0$ i.e. at $\mathbf{x} \rightarrow 0$. The easiest way to reach this goal is to take first the trace in \mathbf{k} - space and afterwards perform a transformation to \mathbf{x} - space. The first step therefore consists in operating a contraction in eq.(65) with the result

$$(66) \quad G_{kk}(\omega; \mathbf{k}) = -\frac{1}{(2\pi)^3} \frac{1}{c^2 \epsilon_0} \frac{1}{k^2 + \kappa^2} \left(3 + \frac{k^2}{\kappa^2} \right)$$

Rearranging this expression we write

$$(67) \quad G_{kk}(\omega; \mathbf{k}) = -\frac{1}{(2\pi)^3} \frac{1}{c^2 \epsilon_0} \left(\frac{1}{k^2 + \kappa^2} + \frac{1}{\kappa^2} \right)$$

Transforming into \mathbf{x} - space by means of the defining relation

$$(68) \quad G_{kk}(\omega; \mathbf{x}) = \int G_{kk}(\omega; \mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}} d^3k$$

we obtain

$$(69) \quad G_{kk}(\omega; \mathbf{x}) = -\frac{1}{c^2 \epsilon_0} \left(\frac{1}{(2\pi)^3} \int \frac{1}{k^2 + \kappa^2} e^{i\mathbf{k} \cdot \mathbf{x}} d^3k + \frac{1}{\kappa^2} \delta(\mathbf{x}) \right)$$

where the rule $\int e^{i\mathbf{k} \cdot \mathbf{x}} d^3k = (2\pi)^3 \delta(\mathbf{x})$ has been used.

In order to evaluate the integral inside the parenthesis we introduce polar coordinates through the relations $\mathbf{k} \cdot \mathbf{x} = kx \cos \vartheta$; $d^3k = 2\pi k \sin \vartheta k d\vartheta dk$ and write

$$\int \frac{1}{k^2 + \kappa^2} e^{i\mathbf{k} \cdot \mathbf{x}} d^3k = 2\pi \int_0^\infty \frac{k^2}{k^2 + \kappa^2} dk \int_0^\pi e^{ikx \cos \vartheta} \sin \vartheta d\vartheta$$

Performing the integration over ϑ we thus arrive at the expression

$$(70) \int \frac{1}{k^2 + \kappa^2} e^{i\mathbf{k}\cdot\mathbf{x}} d^3k = \frac{4\pi}{x} \int_0^\infty \frac{k}{k^2 + \kappa^2} \sin(kx) dk$$

The integral that is left has the value

$$(71) \int_0^\infty \frac{k}{k^2 + \kappa^2} \sin(kx) dk = \frac{\pi}{2} e^{-\kappa x} \quad ; \quad \text{Re } \kappa > 0$$

At this point a difficulty arises if the medium considered is the vacuum. As indicated above, the value given by eq.(71) implies the additional condition

$\text{Re } \kappa > 0$. Now from eq.(64) it follows that in vacuum with $\varepsilon = 1$ we have $\kappa = i \frac{\omega}{c}$

and hence $\text{Re } \kappa = 0$. This problem can however be circumvented by assuming that a positive infinitesimal quantity has been added to κ inside the integrand but does not appear on the r.h.s. of eq.(71). We thus finally obtain

$$(72) \int \frac{1}{k^2 + \kappa^2} e^{i\mathbf{k}\cdot\mathbf{x}} d^3k = \frac{2\pi^2}{x} e^{-i\frac{\omega}{c}x}$$

Substituting this result into eq.(69) and taking the imaginary part we arrive at the expression

$$(73) \text{Im } G_{kk}^A(\omega; \mathbf{x}) = \frac{1}{2\pi} \frac{1}{c^2 \varepsilon_0} \frac{1}{x} \sin\left(\frac{\omega}{c}x\right)$$

where we have added the superscript A to recall that so far the vector potential is the relevant parameter. The frequency distribution of the thermal radiation is found from the fluctuation-dissipation theorem, (31) and the subsequent remark, which has to be applied in the form

$$(74) \Phi_{kk}^E = \hbar \omega^2 \coth\left(e^{\frac{\beta \hbar \omega}{2}}\right) \text{Im } G_{kk}(\omega; \mathbf{r}, \mathbf{r})$$

This notation indicates that in eq.(73) the limit $x \rightarrow 0$ has to be taken. Thus we find in the end

$$(75) \Phi_{kk}^E = \frac{1}{2\pi} \frac{\omega^3}{c^3 \varepsilon_0} \coth\left(e^{\frac{\beta \hbar \omega}{2}}\right).$$

Using further the identity

$$\coth \frac{\beta \hbar \omega}{2} = \frac{1 + e^{-\beta \hbar \omega}}{1 - e^{-\beta \hbar \omega}} = 2 \left(\frac{1}{2} + \frac{1}{e^{\beta \hbar \omega} - 1} \right)$$

we recover Planck's radiation law with in addition a term corresponding to the famous zero-point energy. This awkward term, which gives rise to an infinite energy if it is summed over all frequencies, has been commented in textbooks in various ways. Here we mention only that it is the origin of e.g. the Casimir force, a force between parallel and perfectly conducting plates in vacuum. This effect, predicted by Casimir in 1947 has been widely discussed since (Schuller and Savalle 2011).

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APPENDIX

A useful integral formula

We consider a function of the type

$$(A1) \quad G(t) = F(t)\Theta(t)$$

with $F(t)$ a well behaved function and $\Theta(t)$ the Heaviside step function. We define the Fourier transform of $G(t)$ by the relation

$$(A2) \quad G(\omega) = \int_{-\infty}^{+\infty} dt G(t) e^{i\omega t} = \int_0^{\infty} dt F(t) e^{i\omega t}$$

Introducing the Fourier transform

$$(A3) \quad F(\omega) = \int_{-\infty}^{+\infty} dt F(t) e^{i\omega t}$$

we invert this relation and express $F(t)$ in the form

$$(A4) \quad F(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega' F(\omega') e^{-i\omega' t}$$

Substituting this expression into the r.h.s. of eq.(A2) and inserting a convergence factor $e^{-\eta t}$ we find

$$(A5) \quad G(\omega) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega' F(\omega') \int_0^{\infty} dt e^{[i(\omega-\omega')-\eta]t} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} d\omega' \frac{1}{-i(\omega-\omega')+\eta} F(\omega')$$

Using for the limit $\eta \rightarrow 0^+$ the well-known expression involving a delta function and a principal value

$$\frac{1}{-i(\omega-\omega')+\eta} = \frac{i}{\omega-\omega'+i\eta} \rightarrow \pi\delta(\omega-\omega') + iP \frac{1}{\omega-\omega'}$$

we arrive at the following final result:

$$(A6) \quad G(\omega) = \frac{1}{2}F(\omega) + \frac{i}{2\pi}P \int_{-\infty}^{+\infty} d\omega' \frac{F(\omega')}{\omega - \omega'}$$