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# FORMAL TREATMENT OF RADIATION FIELD FLUCTUATIONS IN VACUUM

Frédéric Schuller, Renaud Savalle, Observatoire de Paris, Meudon, France

Michael Neumann-Spallart, Groupe d'Etude de la Matière Condensée, CNRS/Université de Versailles, Versailles, France

## Abstract

In this note the theory of electro-magnetic field fluctuations is introduced in a somewhat unconventional manner without the use of the more common creation and annihilation operators. Subsequently the example of radiation fields in vacuum is treated by a specific Green's function method.

## Introduction

We quantize the electro-magnetic field by means of brute field operators instead of the introduction of harmonic oscillator modes with creation and annihilation operators as quantization tools. In this way correlation and Green's functions are expressed as quantum averages of products of field operators. Moreover the Green's functions are determined by linear response theory and specific integration methods. From the expressions obtained the thermal radiation energy is deduced, inclusive the peculiar term known as the zero-point energy.

## Linear response

We consider a radiation field represented by a vector potential operator  $\mathbf{A}(\mathbf{r}_0)$  and act on it with a perturbation produced by a classical current density described by component functions  $j_j(\mathbf{r}_0, t)$ . Introducing the corresponding perturbation Hamiltonian

$$(1) H_1 = -A_j(\mathbf{r}_0)j_j(\mathbf{r}_0, t) \quad (\text{summation over repeated indices})$$

we write for the total Hamiltonian of the system

$$(2) H = H_0 - A_j(\mathbf{r}_0)j_j(\mathbf{r}_0, t) = H_0 + H_1$$

Writing accordingly the density operator as  $\rho = \rho_0 + \rho_1$ , the evolution of the system is governed by the Liouville-von Neumann equation

$$(3) \frac{d}{dt}(\rho_0 + \rho_1) = -\frac{i}{\hbar}[H, (\rho_0 + \rho_1)]$$

To first order in  $\rho_1, H_1$  we then obtain

$$(4) \frac{d}{dt}\rho_1 = -\frac{i}{\hbar}[H_0, \rho_1] + \frac{i}{\hbar}[A_j(\mathbf{r}_0)j_j(\mathbf{r}_0, t), \rho_0]$$

Switching to the interaction picture with

$$(5) \rho_1 = e^{-\frac{i}{\hbar}H_0 t} \sigma e^{\frac{i}{\hbar}H_0 t}$$

We obtain from eq.(4)

$$(6) \frac{d\sigma}{dt} = \frac{i}{\hbar} e^{\frac{i}{\hbar}H_0 t} [A_j(\mathbf{r}_0), \rho_0] j_j(\mathbf{r}_0, t) e^{-\frac{i}{\hbar}H_0 t}$$

and after integrating and switching back to  $\rho_1$  a little algebra yields

$$(7) \rho_1 = \frac{i}{\hbar} \int_{-\infty}^t e^{-\frac{i}{\hbar}H_0(t-t')} [A_j(\mathbf{r}_0), \rho_0] j_j(\mathbf{r}_0, t') e^{\frac{i}{\hbar}H_0(t-t')} dt'$$

We now assume for the current a harmonic oscillation form

$$(8) j_j(\mathbf{r}_0, t) = J(\mathbf{r}_0) e^{-i\omega t} \text{ and after setting } t - t' = \tau$$

we then find

$$(9) \rho_1 = \frac{i}{\hbar} J(\mathbf{r}_0) e^{-i\omega t} \int_0^\infty e^{-\frac{i}{\hbar}H_0 \tau} [A_j(\mathbf{r}_0), \rho_0] e^{\frac{i}{\hbar}H_0 \tau} e^{i\omega \tau} d\tau$$

Let us now consider the quantum average

$$(10) \text{Tr} \rho_1 A_i(\mathbf{r}) = \frac{i}{\hbar} J(\mathbf{r}_0) e^{-i\omega t} \int_0^\infty \text{Tr} \left\{ e^{-\frac{i}{\hbar}H_0 \tau} [A_j(\mathbf{r}_0), \rho_0] e^{\frac{i}{\hbar}H_0 \tau} A_i(\mathbf{r}) \right\} e^{i\omega \tau} d\tau$$

Playing with the invariance of the trace under cyclic permutations, eq.(10) can be written

$$(11) \text{Tr} \rho_1 A_i(\mathbf{r}) = \frac{i}{\hbar} J(\mathbf{r}_0) e^{-i\omega t} \int_0^\infty \text{Tr} \{ \rho_0 [A_i(\mathbf{r}, \tau), A_j(\mathbf{r}_0)] \} e^{i\omega \tau} d\tau$$

where we have introduced the Heisenberg operator

$$(12) A_i(\mathbf{r}, \tau) = e^{\frac{i}{\hbar}H_0 \tau} A_i(\mathbf{r}) e^{-\frac{i}{\hbar}H_0 \tau}$$

Designating the traces by the symbol  $\langle \ \rangle$  we thus find

$$(13) \langle A_i(\mathbf{r}, t) \rangle = \frac{i}{\hbar} J(\mathbf{r}_0) e^{-i\omega t} \int_0^\infty \langle [A_i(\mathbf{r}, \tau), A_j(\mathbf{r}_0)] \rangle e^{i\omega \tau} d\tau$$

At this stage it is customary to define a Green's function, related to the averaged commutator by the expression

$$(14) G_{ij}(\mathbf{r}, \mathbf{r}_0, \tau) = \frac{i}{\hbar} \langle [A_i(\mathbf{r}, \tau), A_j(\mathbf{r}_0)] \rangle \Theta(\tau)$$

with  $\Theta(\tau)$  the Heaviside step function. In this way eq.(13) can be written

$$(15) \langle A_i(\mathbf{r}, t) \rangle = \frac{i}{\hbar} J(\mathbf{r}_0) e^{-i\omega t} \int_{-\infty}^\infty G_{ij}(\mathbf{r}, \mathbf{r}_0, \tau) e^{i\omega \tau} d\tau$$

where the integral represents the Fourier transformed Green's function  $G_{ij}(\mathbf{r}, \mathbf{r}_0, \omega)$  thus leading to the relation

$$(16) \langle A_i(\mathbf{r}, t) \rangle = \frac{i}{\hbar} J(\mathbf{r}_0) e^{-i\omega t} G_{ij}(\mathbf{r}, \mathbf{r}_0, \omega)$$

In the important case, where the average field is zero, the above expression describes the field that exists if a harmonic perturbation of frequency  $\omega$  is applied.

### The fluctuation-dissipation theorem

Let us give the expression (14) of the Green's function a more general significance in terms of an average with the complete density operator  $\rho$ . Then define correlation functions by the relations

$$(17a) C_{ij}(\tau) = \langle A_i(\mathbf{r}, \tau) A_j(\mathbf{r}_0) \rangle$$

$$(17b) D_{ij}(\tau) = \langle A_j(\mathbf{r}_0) A_i(\mathbf{r}, \tau) \rangle$$

so that, according to eq.(14), we have

$$(18) G_{ij}(\tau) = \frac{i}{\hbar} (C_{ij}(\tau) - D_{ij}(\tau)) \Theta(\tau) = \frac{i}{\hbar} \Gamma_{ij}(\tau) \Theta(\tau)$$

where we have introduced a function  $\Gamma_{ij}(\tau)$  replacing the initial correlation functions given by eq.'s (17a,b).

With the density operator given by the expression

$$(19) \rho = \frac{1}{Z} e^{-\beta H} ; \quad \beta = \frac{1}{k_B T} \quad \text{and } Z \text{ the partition function}$$

we write explicitly

$$(20) D_{ij}(\tau) = \frac{1}{Z} \text{Tr} \left( e^{-\beta H} A_j(\mathbf{r}_0) e^{\frac{i}{\hbar} H \tau} A_i(\mathbf{r}) e^{-\frac{i}{\hbar} H \tau} \right)$$

with the Fourier transform

$$(21) D_{ij}(\omega) = \frac{1}{Z} \int_{-\infty}^{\infty} \text{Tr} \left( e^{\frac{i}{\hbar} H \tau} A_i(\mathbf{r}) e^{-\frac{i}{\hbar} H \tau} e^{-\beta H} A_j(\mathbf{r}_0) \right) e^{i\omega \tau} d\tau$$

where again the invariance of the trace for cyclic permutations has been used.

Change now the integration path according to

$$(22) \tau' = \tau - i\beta\hbar \quad \text{see figure}$$



then it is easily seen that one has

$$(23) D_{ij}(\tau) = C_{ij}(\tau')$$

yielding for the Fourier transformed quantities the relations

$$(24) D_{ij}(\omega) = e^{-\beta\hbar\omega} \int_{-\infty}^{\infty} C_{ij}(\tau') e^{i\omega\tau'} d\tau' = e^{-\beta\hbar\omega} C_{ij}(\omega)$$

For the Fourier transform of the function  $\Gamma_{ij}(\tau)$  introduced in eq.18) we then find

$$(25) \Gamma_{ij}(\omega) = C_{ij}(\omega) - D_{ij}(\omega) = (1 - e^{-\beta\hbar\omega}) C_{ij}(\omega)$$

Before proceeding further an important remark has to be made. By means of the time reversal operator one can prove the following symmetry relation:

$$(26) C_{ij}(-\tau) = C_{ij}^*(\tau)$$

As a consequence one then easily shows that  $C_{ij}(\omega)$  is a real quantity and by eq.'s (24) and (25) so are  $D_{ij}(\omega)$  and  $\Gamma_{ij}(\omega)$  .

A complication arises in the case of the Fourier transform of the Green's function since according to eq.(18) the lower limit of integration is now 0 instead of  $-\infty$  as usual. A known formula, which shall not be derived here, yields the following expression

$$(27) G_{ij}(\omega) = \frac{i}{\hbar} \int_0^{\infty} \Gamma_{ij}(\tau) e^{i\omega\tau} d\tau = \frac{i}{\hbar} \frac{1}{2} \Gamma_{ij}(\omega) + \frac{i}{2\pi} \mathcal{P} \int_{-\infty}^{\infty} \frac{\Gamma_{ij}(\omega')}{\omega - \omega'} d\omega'$$

where  $\mathcal{P}$  designates a principal value.

Remembering that  $\Gamma(\omega)$  is a real function, it then immediately follows that we have

$$(28) \text{Im}G_{ij}(\omega) = \frac{1}{2\hbar} \Gamma_{ij}(\omega)$$

and further, with eq.(25)

$$(30) C_{ij}(\omega) = 2\hbar(1 - e^{-\beta\hbar\omega})^{-1} \text{Im}G_{ij}(\omega)$$

This important relation expresses the fluctuation-dissipation theorem. That  $\text{Im}G_{ij}(\omega)$  measures energy dissipation can be seen from the fact that eq.(16) is similar to that of a damped electric circuit with  $\frac{i}{\hbar} G_{ij}$  playing the role of its impedance.

## Thermal radiation

It is generally admitted that Maxwell's equations remain valid if classical fields are replaced by quantum averages of the corresponding field operators.

In the case of an infinite space with no charges present we then have for a given mode  $\omega$  the following differential equation:

$$(31) \nabla^2 A_i(\mathbf{r}) + \frac{\omega^2}{c^2} A_i(\mathbf{r}) = -\mu_0 J_i(\mathbf{r})$$

Applying the Green's function method we set

$$(32) A_i(\mathbf{r}) = \int J_j(\mathbf{r}_0) G_{ij}(\mathbf{r}, \mathbf{r}_0, \omega) d^3 \mathbf{r}_0$$

Yielding for the Green's function the following equation:

$$(33) \left( \nabla^2 + \frac{\omega^2}{c^2} \right) G_{ij}(\mathbf{r}, \mathbf{r}_0, \omega) = -\mu_0 \delta_{ij} \delta(\mathbf{r} - \mathbf{r}_0)$$

For diagonal elements we then obtain with  $\frac{\omega}{c} = \kappa$

$$(34) (\nabla^2 + \kappa^2) G_{ii}(\mathbf{x}, \omega) = -\mu_0 \delta(\mathbf{x})$$

where we have set  $\mathbf{r} - \mathbf{r}_0 = \mathbf{x}$

Switching now from  $\mathbf{x}$  - space to  $\mathbf{k}$  - space according to the relations

$$(35) G_{ii}(\mathbf{x}, \omega) = \int G_{ii}(\mathbf{k}, \omega) e^{i\mathbf{k}\cdot\mathbf{x}} d^3 \mathbf{k}$$

and

$$(36) \delta(\mathbf{x}) = \frac{1}{(2\pi)^3} \int e^{i\mathbf{k}\cdot\mathbf{x}} d^3 \mathbf{k}$$

We obtain eq.(34) in the form

$$(37) (k^2 + \kappa^2) G_{ii}(\mathbf{k}, \omega) = -\frac{\mu_0}{(2\pi)^3}$$

With the solution

$$(38) G_{ii}(\mathbf{k}, \omega) = -\frac{1}{(2\pi)^3} \mu_0 \frac{1}{-k^2 + \kappa^2}$$

Substituting this result into eq.(35) leads to the integral expression

$$(39) G_{ii}(\mathbf{x}, \omega) = \frac{\mu_0}{(2\pi)^3} \int \frac{1}{k^2 - \kappa^2} e^{i\mathbf{k}\cdot\mathbf{x}} d^3 \mathbf{k}$$

Explicitly the integral in this expression can be written

$$(40) \int \frac{1}{k^2 - \kappa^2} e^{i\mathbf{k}\cdot\mathbf{x}} d^3 \mathbf{k} = 2\pi \int_0^\pi \sin\theta d\theta \frac{1}{2} \int_{-\infty}^{\infty} \frac{k^2}{k^2 - \kappa^2} e^{i|k|x\cos\theta} dk \quad x = |\mathbf{x}|$$

The integral over  $k$  has two poles at  $k = \pm\kappa$  and it can be evaluated by the usual residuum method with a residuum equal to  $a_{-1} = \frac{k^2}{k+\kappa} \rightarrow \frac{\kappa}{2}$ .

In this way the integral of eq.(40) reduces to

(41)  $I = 2\pi i \frac{\kappa}{2} 2\pi \int_0^\pi e^{ikxcos\theta} \sin\theta d\theta$  . Now the quantity we want to evaluate is the energy density of the radiation field in vacuum. This is related to the correlation function of the electric field at  $\mathbf{x} = 0$  . Taking this limit in eq.(41) we obtain for the integral

$$(42) I = 4\pi^2 \kappa i$$

Eq.(39) then yields the result

$$(43) \text{Im } G_{ii}(0, \omega) = \frac{4\pi^2}{(2\pi)^3} \mu_0 \kappa = \frac{1}{2\pi} \frac{1}{\epsilon_0} \frac{\omega}{c^3} \quad ; \quad c^2 = \frac{1}{\mu_0 \epsilon_0}$$

From eq.(30) we then obtain for the diagonal elements of the correlation function the expression

$$(44) C_{ii}(\omega) = \frac{1}{\pi} \frac{\hbar\omega}{\epsilon_0 c^3} (1 - e^{-\beta\hbar\omega})^{-1}$$

Now this result applies to the vector potential. In order to obtain the corresponding result, designated as  $C_{ii}^E(\omega)$ , for the  $\mathbf{E}$  field we notice that with  $\mathbf{E} = -\dot{\mathbf{A}}$  we can simply introduce a factor  $\omega^2$  and write

$$(45) C_{ii}^E(\omega) = \omega^2 C_{ii}(\omega)$$

Going back to the definition (13a) of the correlation function we then have in  $\tau$  -space and at  $\mathbf{r} = \mathbf{r}_0$

$$(46) C_{ii}^E(\tau) = \langle E_i(r, \tau) E_i(r) \rangle$$

Starting from this expression one can establish, by considering simple oscillatory modes, the following relationship between the correlation function and the energy density of the radiation field:

$$(47) dW = \frac{2\epsilon_0}{2\pi} C_{ii}^E(\omega) d\omega$$

where an additional factor 2 accounts for two polarization directions.

Combining eq.'s (44,45) and (47) we thus obtain

$$(48) dW = \frac{\hbar\omega^3}{\pi^2 c^3} (1 - e^{-\beta\hbar\omega})^{-1} d\omega$$

With the elementary transformation

$$(1 - e^{-\beta\hbar\omega})^{-1} = 1 + \frac{1}{e^{\beta\hbar\omega} - 1}$$

we recover the familiar expression for the energy density of thermal radiation

$$(49) dW = \frac{\hbar\omega^3}{\pi^2 c^3} \left( 1 + \frac{1}{e^{\beta\hbar\omega} - 1} \right) d\omega$$

containing Planck's formula and in addition a first term representing the famous zero-point energy. This awkward term, which gives rise to an infinite energy if it is summed over all

frequencies, has been commented in textbooks in various ways. Here we mention only that it is the origin of the Casimir force, a force between perfectly conducting bodies at very small distances in vacuum. This effect, predicted by Casimir in 1947, has been widely discussed since (see e.g. M. Neumann-Spallart, R. Savalle and F. Schuller, Galilean Electrodynamics, May, June 2015)

### **Bibliography**

R. Kubo Statistical-Mechanical Theory of Irreversible Processes  
J. Phys. Soc. Japan **12**, 570, 1957

R. Kubo The Fluctuation-Dissipation Theorem  
Rep. Prog. Phys. **29**, 255, 1966

K. A. Milton, The Casimir Effect, World Scientific, Singapore, 2001