

# UE SPM-PHY-S07-101

## Gaussian Beams

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## Further reading

[KL66, GB94]



A. Gerrard and J.M. Burch.

*Introduction to matrix methods in optics.*

Dover, 1994.



H. KOGELNIK and T. LI.

Laser beams and resonators.

*Appl. Opt.*, 5(10):1550–1567, Oct 1966.

# Course Outline

- 1 Fundamentals of Gaussian beam propagation
  - Gaussian beams vs. plane waves
  - The fundamental mode
  - Higher order modes
- 2 Matrix methods for geometrical and Gaussian optics
  - Linear algebra for geometrical optics
  - A few simple matrices
  - Matrix method for Gaussian beams

# Plane waves do not exist

Waves carrying an infinite amount of energy cannot come into existence

## Planes waves

- Plane wave have a **homogeneous** transversal electric field
- Poynting's vector norm, and **power density**, are also homogeneous
- Total carried power is infinite

## Practical use of plane wave theory: usual unsaid approximation

- Plane waves of **finite extent** are often used
- Strictly speaking, they are **not** plane waves
- To what extent can we assume they are plane waves ?

## Plane waves, Gaussian beams. . . what else ?

Solutions of the wave equations: one finds only those he was searching for

### Solving the wave equation

$$\overline{\Delta} \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$

- Vectorial Partial Derivatives Equations
- Solutions are numerous
- An **ansatz**<sup>1</sup> is needed to seek solutions

### Gaussian beams as an *ansatz*

- We will find another **family of solutions**
- We never pretend to get them all

---

<sup>1</sup>An ansatz is an *a priori* hypothesis on the form of the sought solution.

## Gaussian *ansatz*

Plugging the ansatz into the wave equation builds the envelope equation

### Introducing a space dependent envelope

- Plane wave:  $\vec{E}_0 \times e^{-i\vec{k} \cdot \vec{r}}$
- Gaussian ansatz :  $u(x, y, z) \vec{e}_x \times e^{-ikz}$ 
  - $u(x, y, z)$  : **complex** beam envelope
  - $\vec{e}_x$  unit vector
- The envelope  $u(x, y, z)$  is our **new unknown**

### Envelope equation

- Scalar harmonic wave equation:  $\Delta E + k^2 E = 0$
- Envelope equation:  $\Delta u - 2ik \frac{\partial u}{\partial z} = 0$

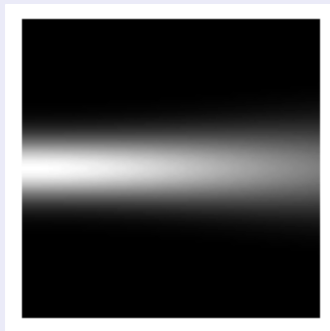
# The paraxial approximation

Also known as Gauss conditions, Slow Varying Envelope. . .

Non Paraxial Beam



Paraxial Beam

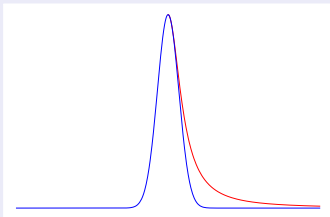


## Paraxial approximation and partial derivatives

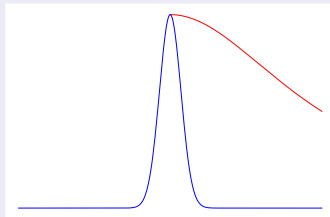
Assuming small angles is equivalent to neglecting  $z$  derivatives

Transversal variation vs. longitudinal variation

Non Paraxial Beam



Paraxial Beam



Transversal Laplacian

$$\frac{\partial^2}{\partial z^2} \ll \frac{\partial^2}{\partial x^2}$$

$$\Delta \approx \Delta_{\perp}$$



# A first solution to the paraxial wave equation

The simplest one, though probably the more important

## Wave Propagation Equation

$$\Delta_{\perp} u - 2ik \frac{\partial u}{\partial z} = 0$$

## Complex beam radius $q(z)$

- Real part: phase variations
- Imaginary part: intensity variations
- Plugging ansatz:  $q' = 1$
- Integration:  $q(z) = q(0) + z$

## A simple *ansatz*

$$u = e^{-i\left(P(z) + \frac{k}{2q(z)} r^2\right)}$$

## Phase shift $P(z)$

- Phase shift **with respect** to the plane wave
- $qP' + i = 0$

## The complex beam radius

$$q(z)$$

A closer look to the signification on a complex parameter

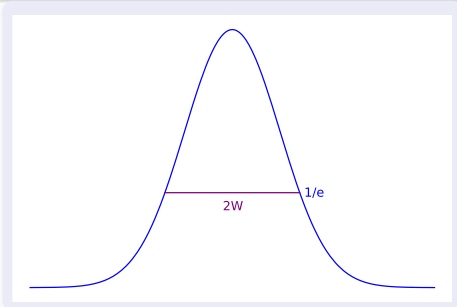
$$u = e^{-i\left(P(z) + \frac{k}{2q(z)}r^2\right)}$$

A **complex** parameter is linked to two **real** ones

$$\frac{1}{q} = \frac{1}{R} - i\frac{\lambda}{\pi W^2}$$

The *ansatz* re-written

$$u = e^{-i\left(P(z) + k\frac{r^2}{2R(z)}\right)} e^{-\frac{r^2}{W(z)^2}}$$



## Spherical wavefront of radius $R$ at abscissa $z$

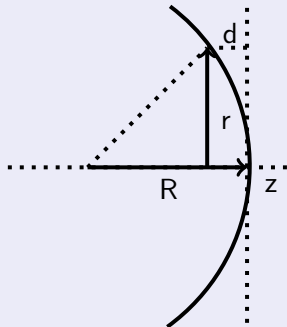
### Phase at abscissa $z$

- Constant phase on sphere
- Phase  $\propto d$ ,  $r \ll R$
- $d = R - \sqrt{R^2 - r^2}$
- $d \approx \frac{r^2}{2R}$

### Gaussian ansatz

$$u = e^{-i\left(P(z) + k \frac{r^2}{2R(z)}\right)} e^{-\frac{r^2}{W(z)^2}}$$

### $R$ radius spherical wavefront



## Gaussian Beam Complex Amplitude

Where the Gaussian beam amplitude is derived from the ansatz and  $q' = 1$

### A quick summary

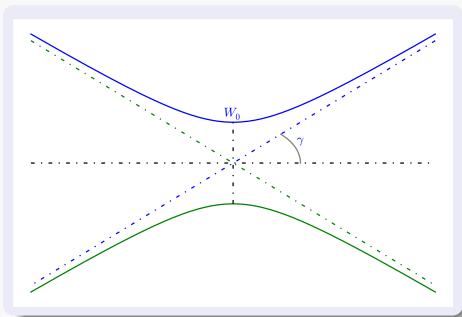
- Ansatz :  $u = e^{-i\left(P(z) + \frac{k}{2q(z)}r^2\right)}$
- Complex beam radius :  $\frac{1}{q} = \frac{1}{R} - i\frac{\lambda}{\pi W^2}$
- Beam radius equation :  $q' = 1 \quad q(z) = q(0) + z$

### Assuming a plane wavefront for $z = 0$

- $q(0) = i\frac{\pi W_0^2}{\lambda}$
- $W^2(z) = W_0^2 \left[ 1 + \left( \frac{\lambda z}{\pi W_0^2} \right)^2 \right]$
- $R(z) = z \left[ 1 + \left( \frac{\pi W_0^2}{\lambda z} \right)^2 \right]$

## Gaussian Beam Intensity

$$W^2(z) = W_0^2 \left[ 1 + \left( \frac{\lambda z}{\pi W_0^2} \right)^2 \right]$$



### Asymptotes

- $\frac{\lambda z}{\pi W_0^2} \gg 1 \Rightarrow W(z) \approx \frac{\lambda z}{\pi W_0}$
- $\gamma = \frac{\lambda}{\pi W_0}$

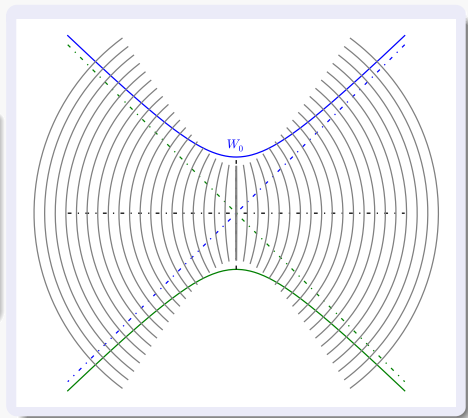
$W_0$  : Beam Waist

## Gaussian wavefront curvature

$$R(z) = z \left[ 1 + \left( \frac{\pi W_0^2}{\lambda z} \right)^2 \right]$$

### Plane and spherical limits

- For small  $z$  :  $R = \infty$   
plane wavefront
- For high  $z$  :  $R \approx z$   
spherical wavefront



## The Rayleigh length

A quantitative criterion to decide whether a Gaussian beam is plane or spherical

Plane for small  $z$

$$\frac{\lambda z}{\pi W_0^2} \ll 1$$

- $W(z) \approx W_0$
- $\lim_{z \rightarrow 0} R(z) = \infty$

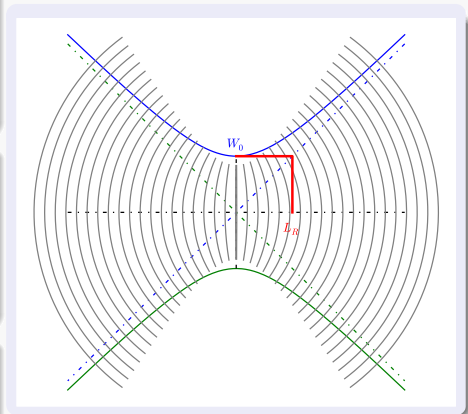
Spherical for high  $z$

$$\frac{\lambda z}{\pi W_0^2} \gg 1$$

- $W(z) \approx \frac{\lambda z}{\pi W_0}$
- $R(z) \approx z$

The Rayleigh length is the limit

$$L_R = \frac{\pi W_0^2}{\lambda}$$



## The homogeneous phase shift

$$P(z)$$

$$u = e^{-i\left(P(z) + \frac{k}{2q(z)}r^2\right)}$$

Recall the equation

$$qP' + i = 0$$

$\Leftrightarrow$

$$P'(z) = -\frac{i}{z + iL_R}$$

Integrate it

$$P(z) = i \ln\left(\frac{W_0}{W(z)}\right) - \tan^{-1}\left(\frac{z}{L_R}\right)$$

Complex phase meaning

- Real part : Phase shift with respect to plane wave
- Imaginary part:  $\frac{W_0}{W(z)}$  factor to ensure energy conservation



## The fundamental Gaussian mode

### General expression

$$E(r, z) = \frac{W_0}{W(z)} e^{-i[kz + P(z)] - r^2 \left( \frac{1}{W(z)^2} + i \frac{k}{2R(z)} \right)}$$

With, in (nearly) the order of appearance on the screen

- $W^2(z) = W_0^2 \left[ 1 + \left( \frac{z}{L_R} \right)^2 \right]$
- $R(z) = z \left[ 1 + \left( \frac{L_R}{z} \right)^2 \right]$
- $P(z) = -\tan^{-1} \left( \frac{z}{L_R} \right)$
- Rayleigh length  $L_R = \frac{\pi W_0^2}{\lambda}$
- Diffraction half angle:  $\gamma \approx \frac{\lambda}{\pi W_0} = \frac{W_0}{L_R}$

# High order Hermite-Gaussian modes

A Cartesian family of higher order modes

## Ansatz

$$u(x, y, z) = g\left(\frac{x}{W(z)}\right) h\left(\frac{y}{W(z)}\right) e^{-i\left(P(z) + \frac{k}{2q(z)}(x^2 + y^2)\right)}$$

## Plugged into the wave equation

- $q' = 1$
- $\exists m \in \mathbb{N}, \frac{\partial g}{\partial x^2} - 2x \frac{\partial g}{\partial x} + 2mg = 0$
- $\exists n \in \mathbb{N}, \frac{\partial h}{\partial y^2} - 2y \frac{\partial h}{\partial y} + 2nh = 0$
- $qP' + (1 + m + n)j = 0$

## Behavior of Hermite-Gaussian modes

Each mode is a mere space modulation of the fundamental

$$q' = 1$$

- Same equation for  $q$  as in the fundamental mode
- $W(z)$  and  $R(z)$  retain their meanings and properties
- Rayleigh length and diffraction angle are unchanged

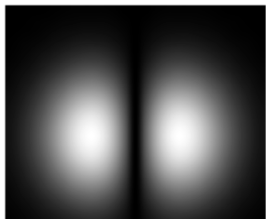
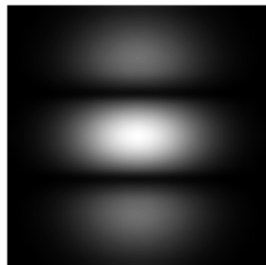
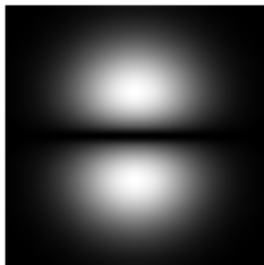
$$\frac{\partial^2 g}{\partial x^2} - 2x \frac{\partial g}{\partial x} + 2mg = 0$$

$$\frac{\partial h}{\partial y^2} - 2y \frac{\partial h}{\partial y} + 2nh = 0$$

- Solutions are, by definition, the orthogonal Hermite polynomials
- $\mathcal{H}_0 = 1$ ,  $\mathcal{H}_1 = x$ ,  $\mathcal{H}_2 = 4x^2 - 1$ ,  $\mathcal{H}_3 = 8x^3 - 12x \dots$
- $\mathcal{H}_n$  has degree  $n$
- $g\left(\frac{x}{W(z)}\right) h\left(\frac{y}{W(z)}\right) = \mathcal{H}_m\left(\sqrt{2}\frac{x}{W(z)}\right) \mathcal{H}_n\left(\sqrt{2}\frac{y}{W(z)}\right)$

## Intensity profiles of Hermite Gaussian (HG) modes

The intensity is proportional to the squared envelope



# High order Laguerre-Gaussian modes

A cylindrical family of higher order modes

## Ansatz

$$u(r, \phi, z) = g\left(\frac{r}{W(z)}\right) e^{-i\left(P(z) + \frac{k}{2q(z)}r^2 + l\phi\right)}$$

## Plugged into the wave equation

- $q' = 1$
- $\exists (l, p) \in \mathbb{N}^2, r \frac{\partial^2 g}{\partial r^2} - (l + 1 - x) \frac{\partial g}{\partial x} + pg = 0$
- $qP' + (1 + 2p + l)j = 0$

## Behavior of Laguerre-Gaussian modes

Each mode is a mere space modulation of the fundamental

$$q' = 1$$

same as HG modes

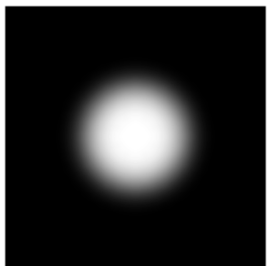
- Same equation for  $q$  as in the fundamental mode
- $W(z)$  and  $R(z)$  retain their meanings and properties
- Rayleigh length and diffraction angle are unchanged

$$r \frac{\partial^2 g}{\partial r^2} - (l + 1 - x) \frac{\partial g}{\partial x} + pg = 0$$

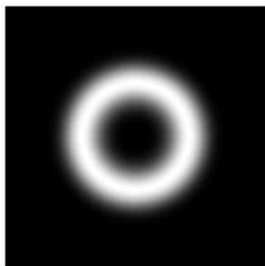
- Solutions are, by definition, the orthogonal generalized Laguerre polynomials
- $\mathcal{L}_0^{(l)} = 1$ ,  $\mathcal{L}_1^{(l)} = -x + l + 1$ ,  $\mathcal{L}_2^{(l)} = \frac{x^2}{2} - (l + 2)x + \frac{(l+1)(l+2)}{2}$
- $\mathcal{L}_m^{(l)}$  has degree  $m$
- $g\left(\frac{r}{W(z)}\right) = \left(\sqrt{2} \frac{r}{W(z)}\right)^l \mathcal{L}_p^{(l)}\left(2 \frac{r^2}{W^2(z)}\right)$

## Intensity profiles of Laguerre Gaussian (LG) modes

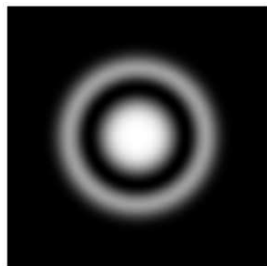
The intensity is proportional to the squared envelope



$LG(0,0)$



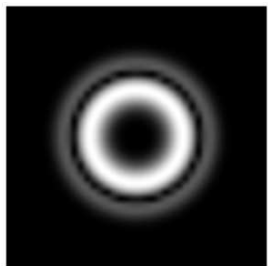
$LG(0,1)$  : vortex



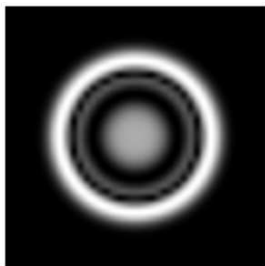
$LG(0,2)$

## Intensity profiles of other Laguerre Gaussian (LG) modes

The intensity is proportional to the squared envelope



$LG(1,2)$



$LG(1,3)$



$LG(2,3)$



## Homogeneous phase shift is different for high order modes

$$qP' + (1 + m + n)j = 0$$

$$qP' + (1 + 2p + l)j = 0$$

A small phase difference between modes around the beam waist

- Slightly different optical paths for different orders
- Slightly different oscillating frequencies in lasers
- Usually forgotten

## Geometrical optics framework

Where is it shown that rays are not so thin as you may think

Geometrical optics *do not* deal with thin rays

- A thin ray has a thin **waist**: it should diffract
- Thin rays are **seldom alone**: their meaning is collective
- A ray is a Poynting vector curve
- A **bunch of rays** describes a **wavefront**

$$\gamma = \frac{\lambda}{\pi W_0}$$

Do geometrical optics deal with plane and spherical waves ?

- **Parallel** rays imply a **plane wavefront**
- **Converging** or diverging rays imply a **spherical wavefront**
- But neither of them has an infinite extension !

## Geometrical optic is Gaussian optics

Transversely limited plane waves

Parallel rays

Gaussian Beams **within their** Rayleigh zone

Transversely limited spherical waves

Con(Di)verging rays

Gaussian Beams **far from** their Rayleigh zone

Orders of magnitude

- He-Ne laser:  $W_0 \approx 1\text{mm}$ ,  $\lambda = 633\text{nm}$ ,  $L_R \approx 5\text{m}$
- GSM Antenna:  $W_0 \approx 1\text{m}$ ,  $\lambda \approx 33\text{cm}$ ,  $L_R \approx 10\text{m}$

## Geometrical optics is linear

Geometrical optics stems entirely from Descartes law

$$n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$$

### Descartes made paraxial

Paraxial approximation :  $\theta \ll 1$

$$n_1 \theta_1 \approx n_2 \theta_2$$

### Geometrical optics is linear algebra

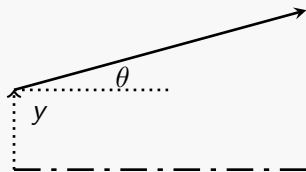
- Paraxial Descartes is linear
- Straight line propagation is linear
- The behavior of a ray through any optical system can be described linearly

## Matrix geometrical optics

A 2 dimensional linear algebra framework

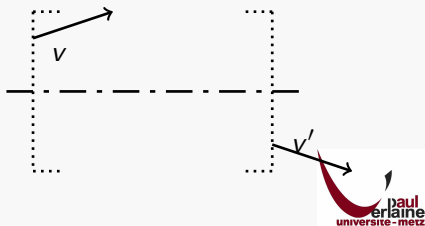
The ray vector  $v = \begin{pmatrix} y \\ \theta \end{pmatrix}$

- $y$  : distance from the axis
- $\theta$  : angle to the axis



An optical system  $v' = Mv$

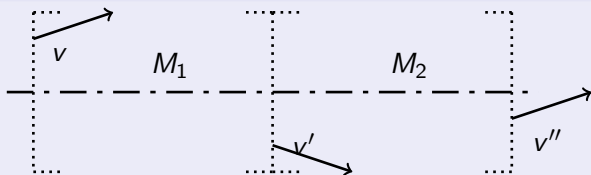
- $M$  is a  $2 \times 2$  real matrix
- It can describe any centered paraxial optical system



## Optical system composition

Optical system composition reduced to matrix product

### Optical System Composition



### Matrix Composition

- $v' = M_1 \cdot v$
- $v'' = M_2 \cdot v'$
- $v'' = M_2 M_1 \cdot v$

### Complex systems

Compose simple systems

## Propagation in a homogeneous medium

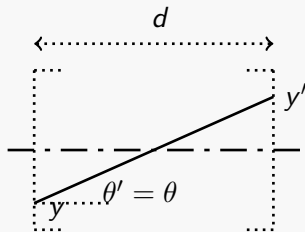
$$\begin{pmatrix} y' \\ \theta' \end{pmatrix} = M_d \begin{pmatrix} y \\ \theta \end{pmatrix}$$

### Light propagates in straight line

- No direction change:  $\theta' = \theta$
- $y' = y + d \sin(\theta)$

$M_d$

$$\begin{pmatrix} y' \\ \theta' \end{pmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{pmatrix} y \\ \theta \end{pmatrix}$$



## Passing through a plane interface

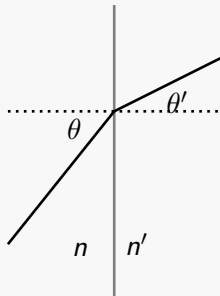
$$\begin{pmatrix} y' \\ \theta' \end{pmatrix} = M_p \begin{pmatrix} y \\ \theta \end{pmatrix}$$

### Descartes

- No propagation:  $y' = y$
- $n \sin(\theta) = n' \sin(\theta')$
- $\theta' \approx \frac{n}{n'} \theta$

### $M_p$

$$\begin{pmatrix} y' \\ \theta' \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{n}{n'} \end{bmatrix} \begin{pmatrix} y \\ \theta \end{pmatrix}$$





## Passing through a thin lens

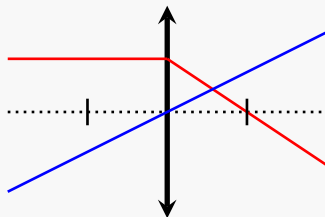
$$\begin{pmatrix} y' \\ \theta' \end{pmatrix} = M_l \begin{pmatrix} y \\ \theta \end{pmatrix}$$

### Two characteristic rays

- No propagation:  $y' = y$
- Blue ray:  $y = 0 \Rightarrow \theta' = \theta$
- Red ray:  $\theta = 0 \Rightarrow \theta' = -\frac{1}{f}y$

$M_l$

$$\begin{pmatrix} y' \\ \theta' \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{pmatrix} y \\ \theta \end{pmatrix}$$



## Passing through a (thin) spherical interface

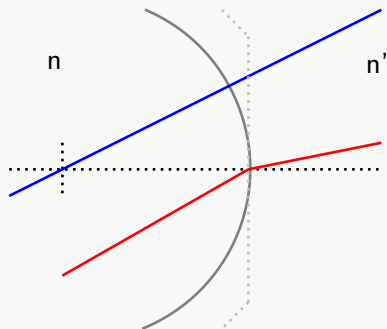
$$\begin{pmatrix} y' \\ \theta' \end{pmatrix} = M_s \begin{pmatrix} y \\ \theta \end{pmatrix}$$

### Descartes

- Thin interface  
No propagation:  $y' = y$
- Blue ray:  $y \approx R\theta \Rightarrow \theta' = \theta$
- Red ray:  $y = 0 \Rightarrow \theta' \approx \frac{n}{n'}\theta$

### $M_s$

$$\begin{pmatrix} y' \\ \theta' \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{n'-n}{n'R} & \frac{n}{n'} \end{bmatrix} \begin{pmatrix} y \\ \theta \end{pmatrix}$$



## Mirrors

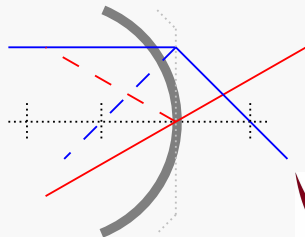
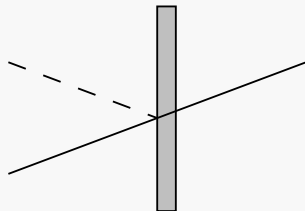
### Unfolding the light

Plane mirrors as if they did not exist

$$\begin{pmatrix} y' \\ \theta' \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} y \\ \theta \end{pmatrix}$$

Spherical Mirrors are thin lenses

$$\begin{pmatrix} y' \\ \theta' \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{bmatrix} \begin{pmatrix} y \\ \theta \end{pmatrix}$$



## Matrix property

A determinant property stemming from all the simple matrices determinants

$n$ : start index

$n'$ : stop index

$$\forall M, \det(M) = \frac{n}{n'}$$

## Gaussian modes, propagation and lenses

A Gaussian mode does not change upon propagation of by passing through thin interfaces or lenses

$$g\left(\frac{x}{W(z)}\right) h\left(\frac{y}{W(z)}\right) e^{-i\left(P(z) + \frac{k}{2q(z)}(x^2 + y^2)\right)}$$

- $z$  independent modulation of the fundamental mode
- Free space  $q' = 1$  common property
- Thin lens does not change mode profile

### Common $R(z)$ and $W(z)$ behavior

All the modes share the same laws on  $q(z)$ ,  $R(z)$  and  $W(z)$

## Gaussian beam propagation: the ABCD law

The transformation of the complex radius  $q$  for simple optical systems

Free space  $q' = 1$

- $q' = q + d$
- $M_d = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$

Thin lens  $\frac{1}{R'} = \frac{1}{R} - \frac{1}{f}$

- $\frac{1}{q'} = \frac{1}{q} - \frac{1}{f} \Rightarrow q' = \frac{1}{-\frac{1}{f}q + 1}$
- $M_l = \begin{pmatrix} 1 \times q & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$

Plane interface  $n/n' = R/R'$

- $\frac{q'}{q} = \frac{n'}{n} \Rightarrow q' = \frac{1 \times q}{\frac{n'}{n}}$
- $M_p = \begin{pmatrix} 1 & 0 \\ 0 & \frac{n}{n'} \end{pmatrix}$

Kogelnik's ABCD law

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \Rightarrow q' = \frac{Aq + B}{Cq + D}$$

Geometrical and Gaussian optics are linked through paraxial approx.

Gaussian beam propagation can be evaluated, for **any** mode, using simple matrix geometrical optics

## Focal lens example

Using the ABCD law to verify that parallel rays do converge on the focal plane

### Parallel input beam

- Input plane : just before lens
- Output plane : after length  $d$
- Input beam at waist:  $q_0 = iL_{R0}$

### Propagation matrix

$$M_d \cdot M_f = \begin{pmatrix} -\frac{d}{f} + 1 & d \\ -\frac{1}{f} & 1 \end{pmatrix}$$

### ABCD law

$$q_1 = \frac{df + i(f - d)L_{R0}}{f - iL_{R0}}$$

$d$  for plane wavefront: imaginary  $q_1$

$$d = \frac{f}{1 + \left(\frac{f}{L_{R0}}\right)^2}$$

