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Nicolas Fressengeas

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UE SPM-PHY-S07-101
Gaussian Beams

N. Fressengeas

Laboratoire Matériaux Optiques, Photonique et Systèmes
Unité de Recherche commune à l'Université Paul Verlaine Metz et à Supélec

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Further reading

[KL66, GB94]

A. Gerrard and J.M. Burch.
Introduction to matrix methods in optics.
Dover, 1994.

H. KOGELNIK and T. LI.
Laser beams and resonators.
Appl. Opt., 5(10):1550–1567, Oct 1966.

Course Outline

1. Fundamentals of Gaussian beam propagation
   - Gaussian beams vs. plane waves
   - The fundamental mode
   - Higher order modes
2. Matrix methods for geometrical and Gaussian optics
   - Linear algebra for geometrical optics
   - A few simple matrices
   - Matrix method for Gaussian beams

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Plane waves do not exist

Waves carrying an infinite amount of energy cannot come into existence

Planes waves

- Plane wave have a homogeneous transversal electric field
- Ponting's vector norm, and power density, are also homogeneous
- Total carried power is infinite

Practical use of plane wave theory: usual unsaid approximation

- Plane waves of finite extent are often used
- Strictly speaking, they are not plane waves

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Plane waves, Gaussian beams. . . what else ?

Solutions of the wave equations: one finds only those he was searching for

Solving the wave equation

$$\overrightarrow{\triangle E} = \frac{1}{c^2}\frac{\partial^2 \overrightarrow{E}}{\partial t^2}$$

- Vectorial Partial Derivatives Equations
- Solutions are numerous
- An ansatz[1] is needed to seek solutions

Gaussian beams as an *ansatz*

- We will find another family of solutions
- We never pretend to get them all

[1] An ansatz is an *a priori* hypothesis on the form of the sought solution.

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Gaussian *ansatz*

Plugging the ansatz into the wave equation builds the envelope equation

Introducing a space dependent envelope

- Plane wave: $\overrightarrow{E_0} \times e^{-i\overrightarrow{k}\cdot\overrightarrow{r}}$
- Gaussian ansatz : $u(x,y,z)\,\overrightarrow{e_x} \times e^{-ikz}$
  - $u(x,y,z)$: complex beam envelope
  - $\overrightarrow{e_x}$ unit vector
- The envelope $u(x,y,z)$ is our new unknown

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- Wave equation: $\overrightarrow{\Delta E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$
- Scalar harmonic wave equation: $\Delta E + k^2 E = 0$
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$$\Delta u(x,y,z)e^{-ikz} + k^2 u(x,y,z)e^{-ikz} = 0$$

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- Scalar harmonic wave equation: $\triangle E + \frac{\omega^2}{c^2} E = 0$
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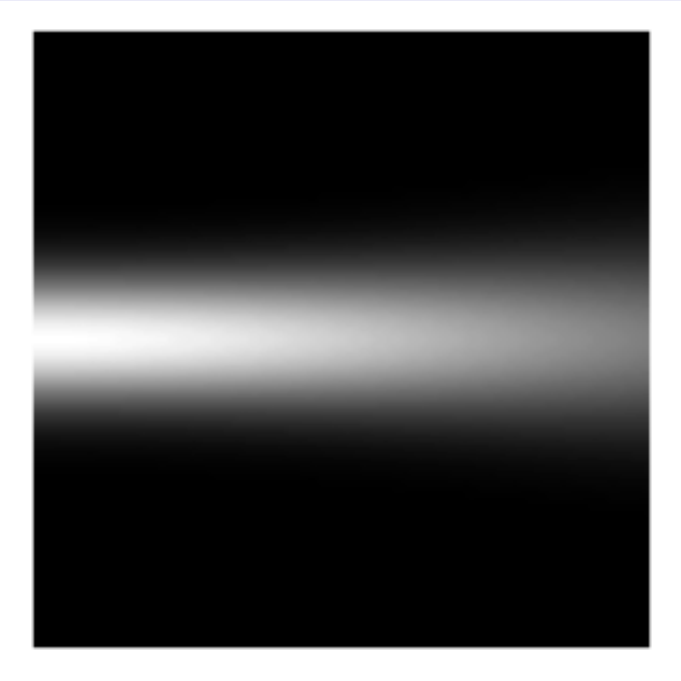
The paraxial approximation

Also known as Gauss conditions, Slow Varying Envelope. . .

Non Paraxial Beam

Paraxial Beam

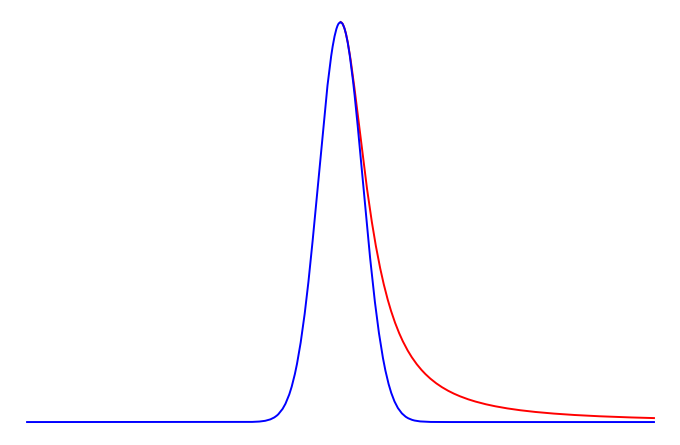


Paraxial approximation and partial derivatives

Assuming small angles is equivalent to neglecting z derivatives

Transversal variation *vs.* longitudinal variation

Non Paraxial Beam



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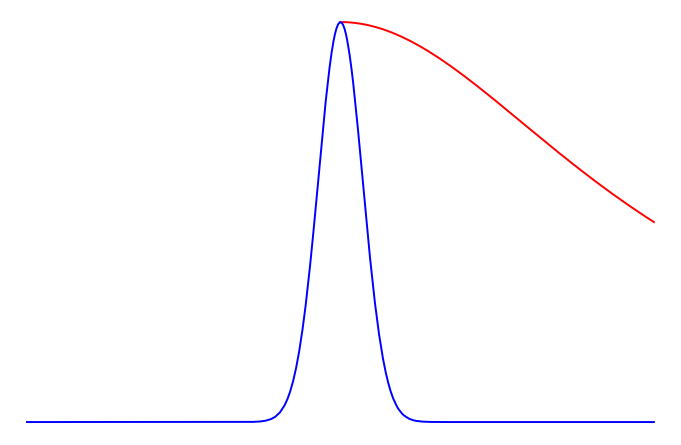

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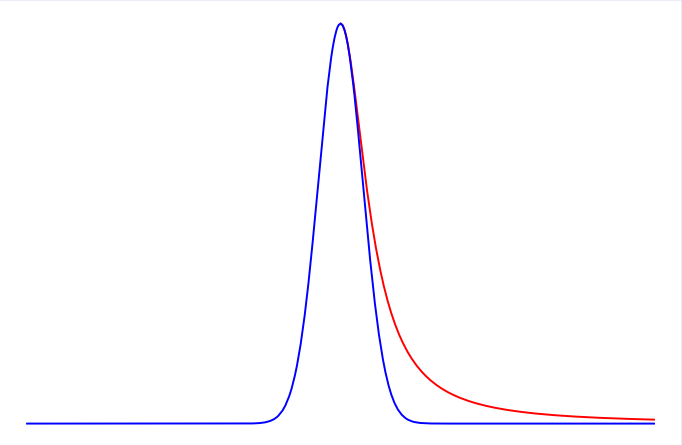

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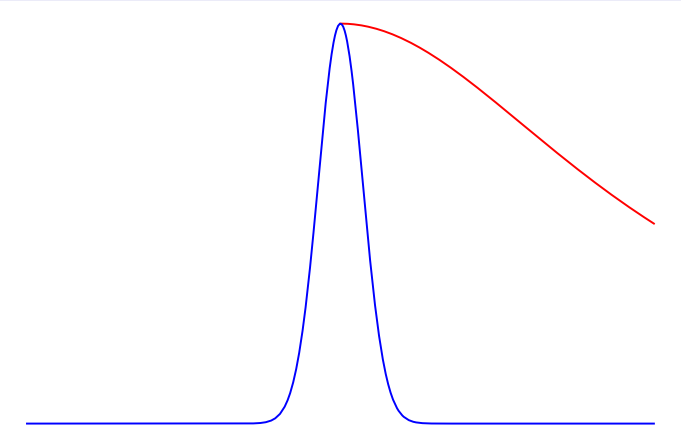

Transversal Laplacian

$$\frac{\partial^2}{\partial z^2} \ll \frac{\partial^2}{\partial x^2} \qquad \triangle \approx \triangle_\perp$$

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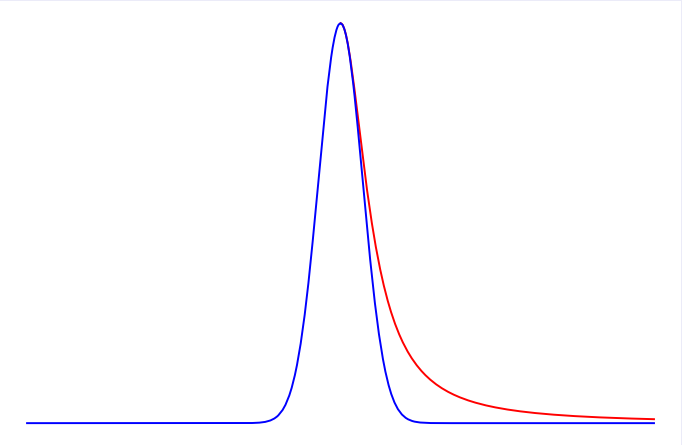

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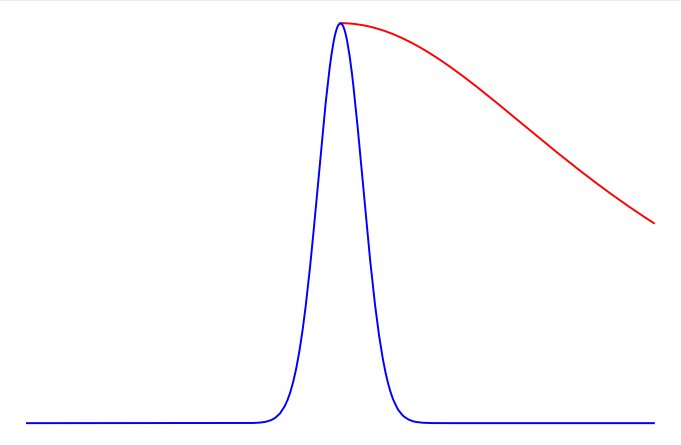

Paraxial wave equation

$$\triangle_{\perp} u - 2ik\frac{\partial u}{\partial z} = 0$$

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A first solution to the paraxial wave equation

The simplest one, though probably the more important

Wave Propagation Equation

$$\triangle_{\perp} u - 2ik\frac{\partial u}{\partial z} = 0$$

A simple *ansatz*

$$u = e^{-i\left(P(z)+\frac{k}{2q(z)}r^2\right)}$$

Complex beam radius $q(z)$

- Real part: phase variations
- Imaginary part: intensity variations
- Plugging ansatz: $q' = 1$
- Integration: $q(z) = q(0) + z$

Phase shift $P(z)$

- Phase shift with respect to the plane wave
- $qP' + i = 0$

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The complex beam radius $q(z)$

A closer look to the signification on a complex parameter

$$u = e^{-i\left(P(z)+\frac{k}{2q(z)}r^2\right)}$$

A complex parameter is linked to two real ones

$$\frac{1}{q} = \frac{1}{R} - i\frac{\lambda}{\pi W^2}$$

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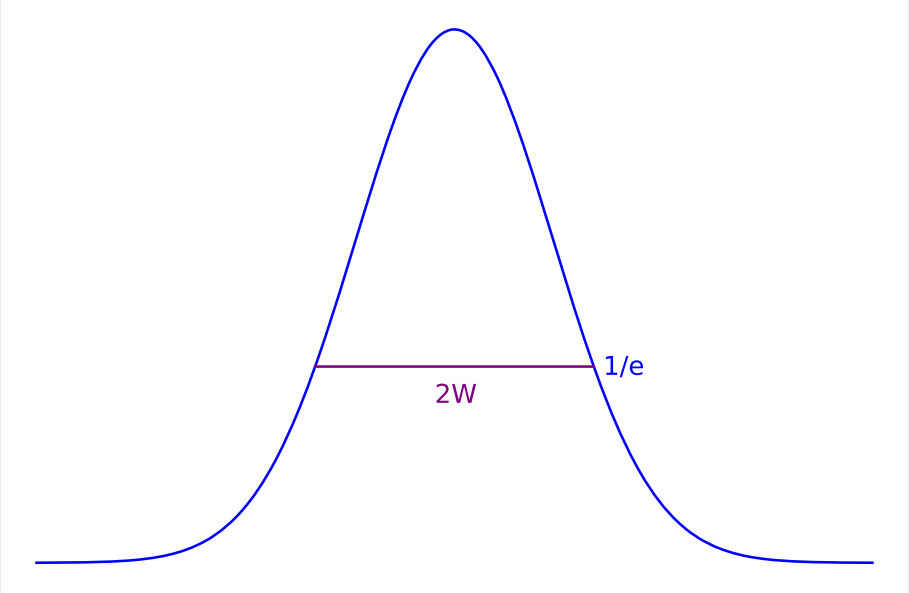

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What is R ?

Spherical wavefront of radius R at abscissa z

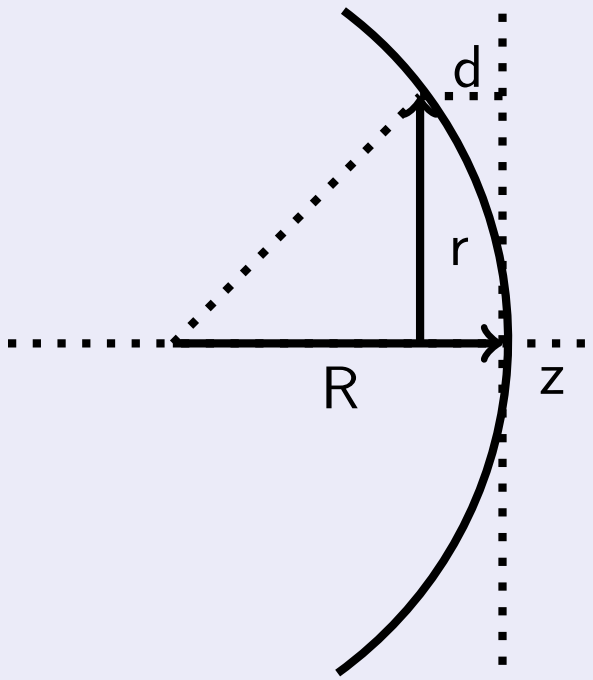
Phase at abscissa z

- Constant phase on sphere
- Phase $\propto d, \quad r \ll R$

Gaussian ansatz

- $u = e^{-i\left(P(z)+k\frac{r^2}{2R(z)}\right)} e^{-\frac{r^2}{W(z)^2}}$

R radius spherical wavefront



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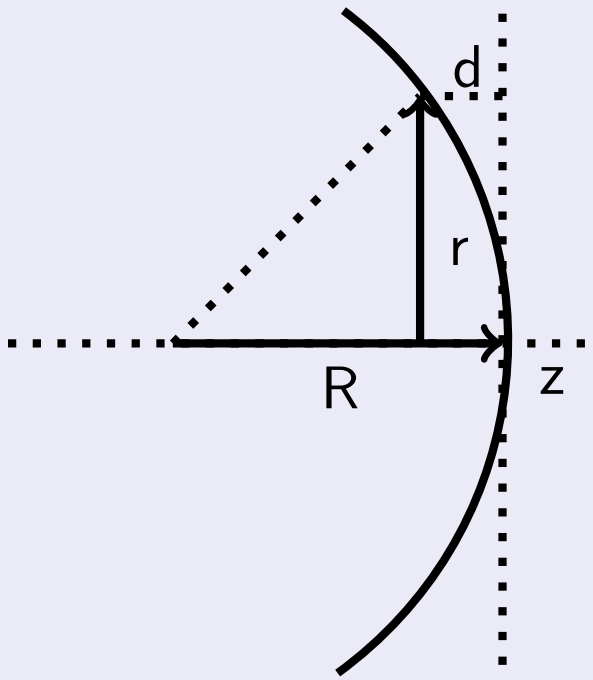

Spherical wavefront of radius R at abscissa z

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- Phase $\propto d, \quad r \ll R$
- $(R - d)^2 + r^2 = R^2$

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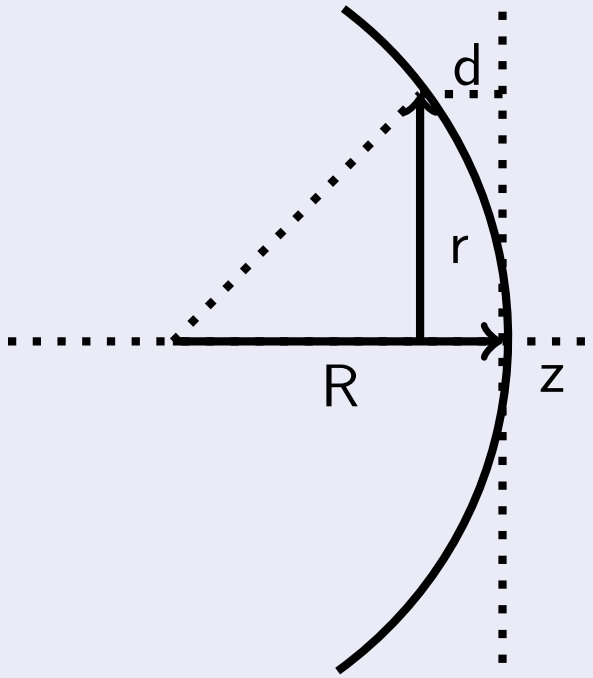
R radius spherical wavefront



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- $d = R - \sqrt{R^2 - r^2}$

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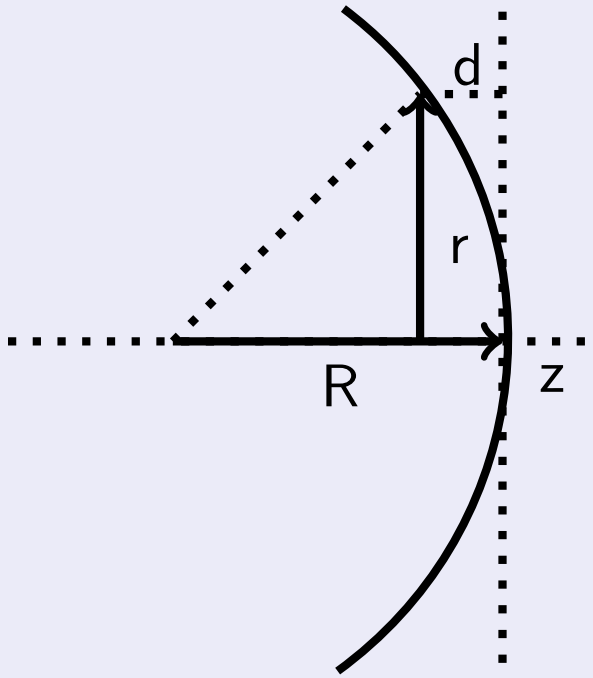

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Phase at abscissa z

- Constant phase on sphere
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- $d = R - \sqrt{R^2 - r^2}$
- $d = R\left(1 - \sqrt{1 - \frac{r^2}{R^2}}\right)$

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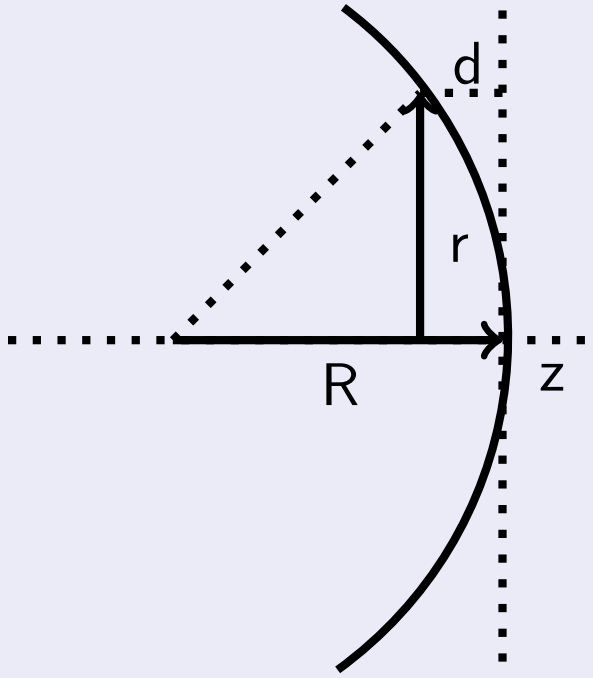

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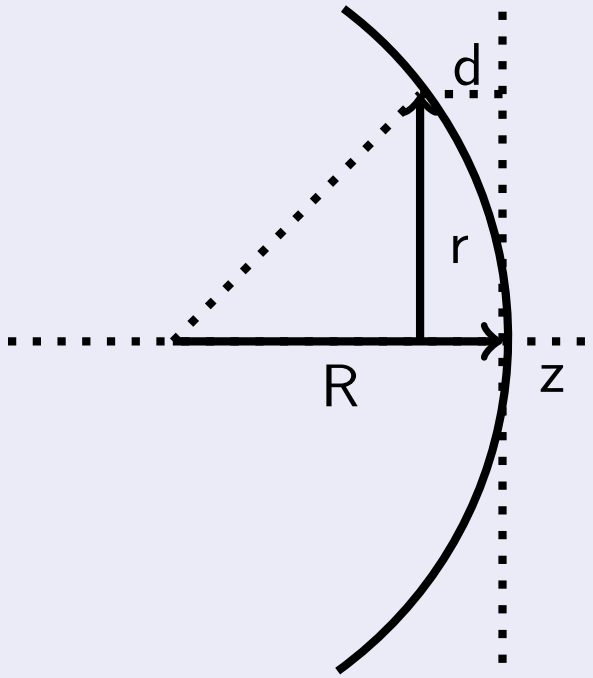
- $d = R - \sqrt{R^2 - r^2}$
- $d \approx \frac{r^2}{2R}$

Gaussian ansatz

$$u = e^{-i\left(P(z)+k\frac{r^2}{2R(z)}\right)} e^{-\frac{r^2}{W(z)^2}}$$

R radius spherical wavefront



Gaussian Beam Complex Amplitude

Where the Gaussian beam amplitude is derived from the ansatz and $q' = 1$

A quick summary

- Ansatz : $u = e^{-i\left(P(z)+\frac{k}{2q(z)}r^2\right)}$
- Complex beam radius : $\frac{1}{q} = \frac{1}{R} - i\frac{\lambda}{\pi W^2}$
- Beam radius equation : $q' = 1$

Assuming a plane wavefront for $z = 0$

- $q(0) = i\frac{\pi W_0^2}{\lambda}$
- $W^2(z) = W_0^2\left[1+\left(\frac{\lambda z}{\pi W_0^2}\right)^2\right]$
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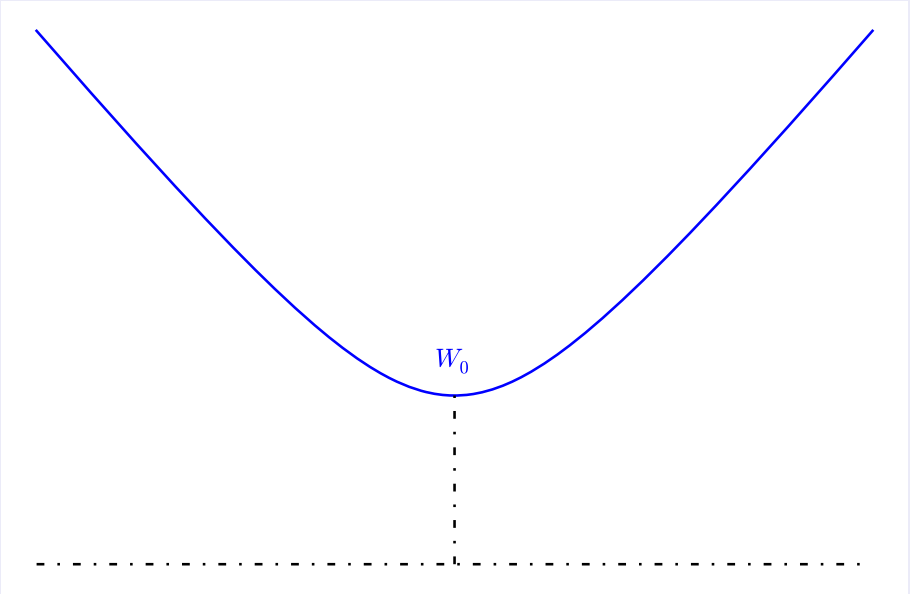

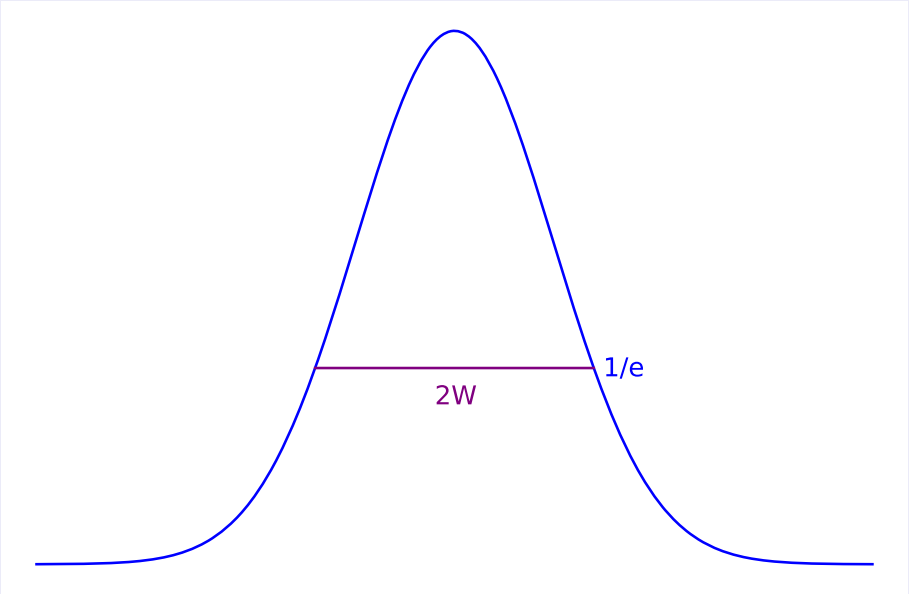

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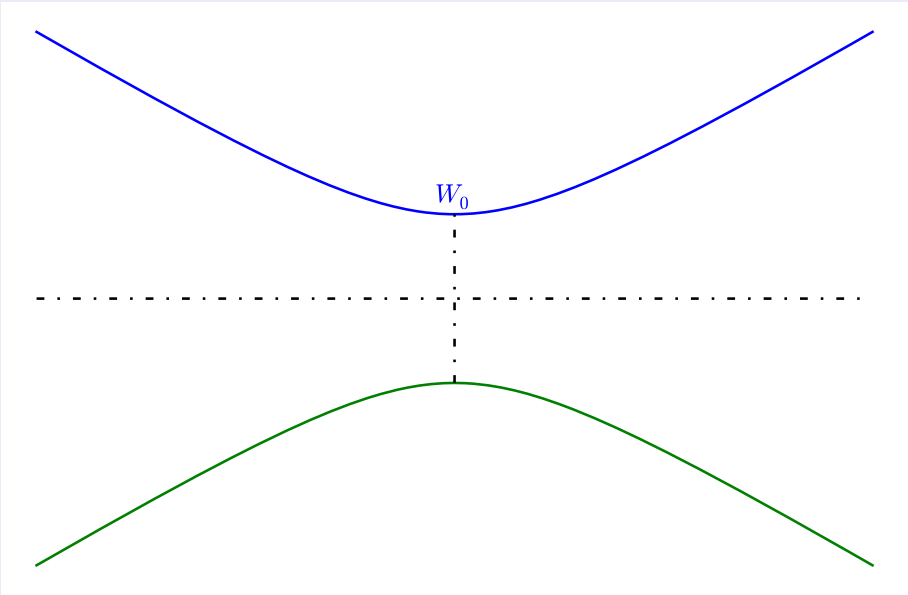

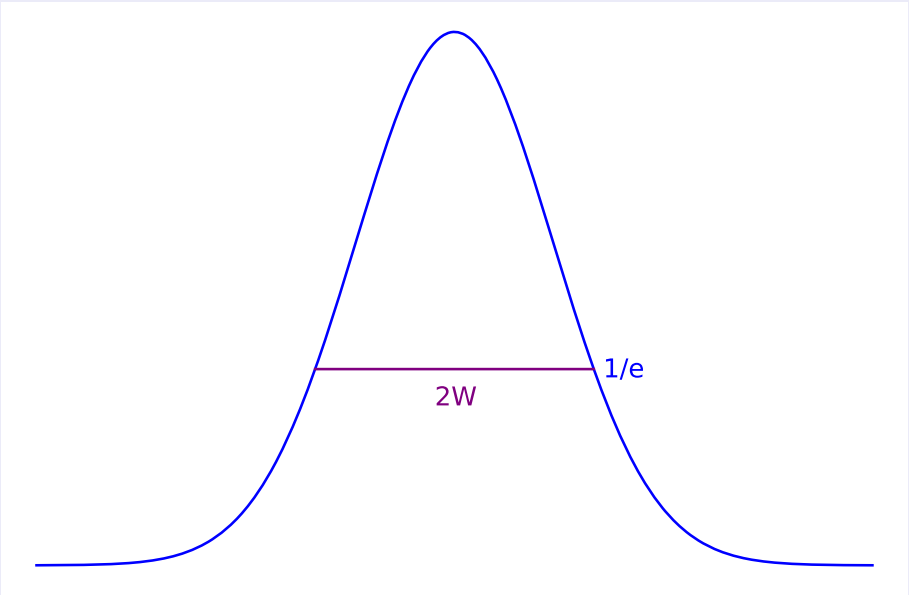

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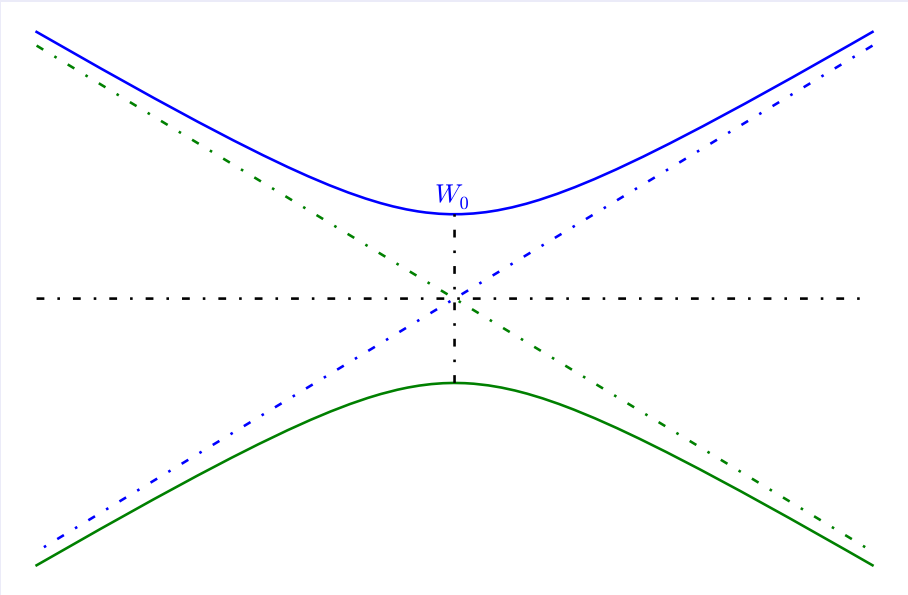

Asymptotes

- $\frac{\lambda z}{\pi W_0^2} \gg 1 \quad \Rightarrow \quad W(z) \approx \frac{\lambda z}{\pi W_0}$
- $\gamma = \frac{\lambda}{\pi W_0}$

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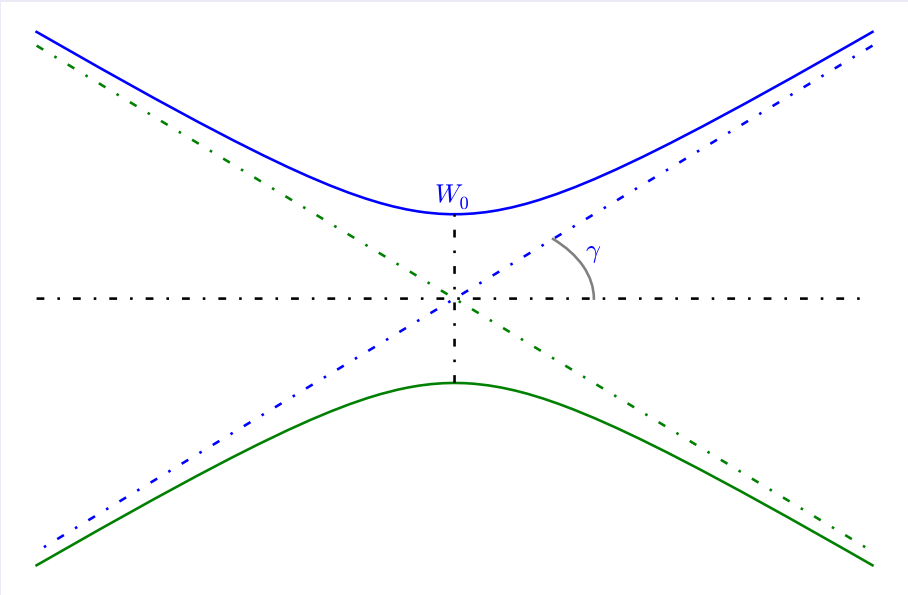

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Gaussian wavefront curvature

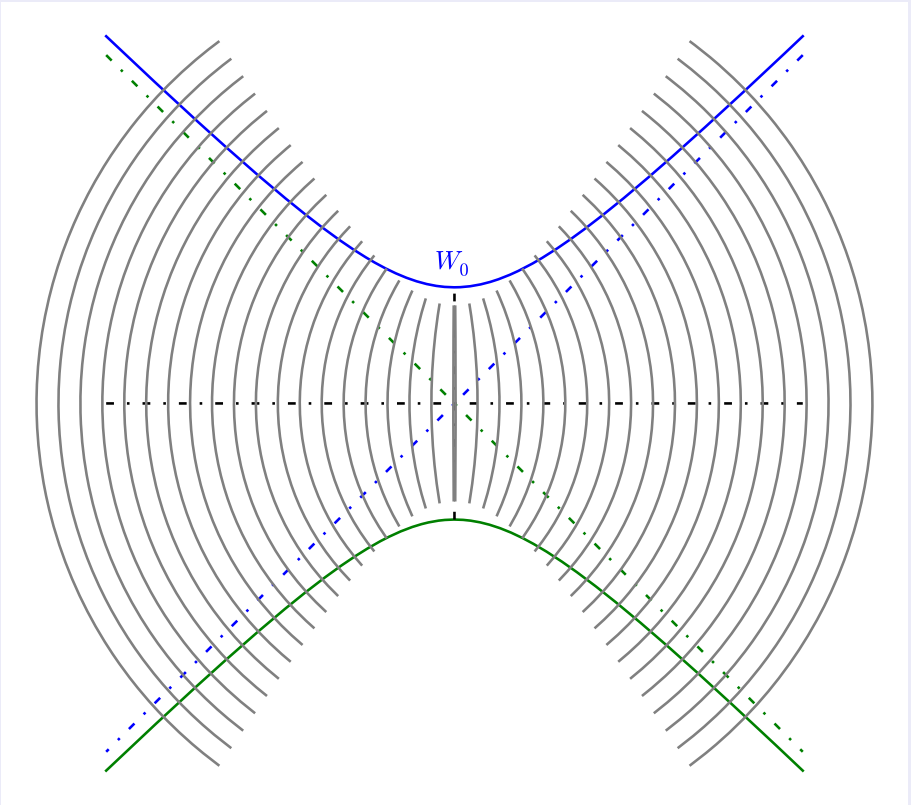
$$R(z) = z\left[1 + \left(\frac{\pi W_0^2}{\lambda z}\right)^2\right]$$

Plane and spherical limits

- For small z : $R = \infty$

  plane wavefront
- For high z : $R \approx z$

  spherical wavefront



The Rayleigh length

A quantitative criterion to decide whether a Gaussian beam is plane or spherical

Plane for small z $\quad \frac{\lambda z}{\pi W_0^2} \ll 1$

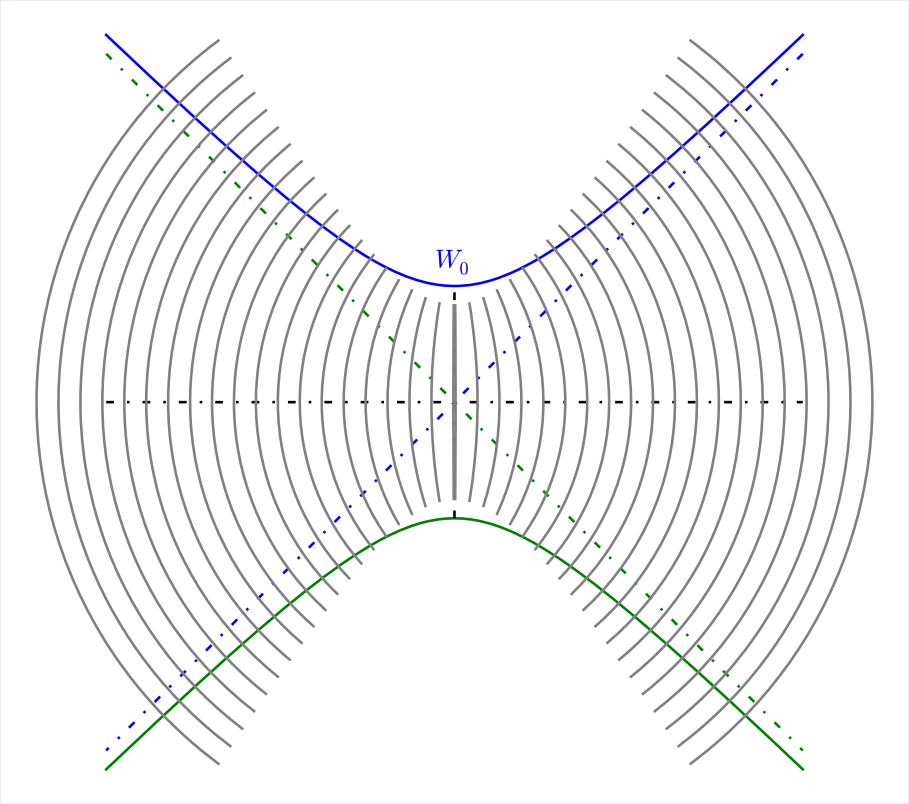
- $W(z) \approx W_0$
- $\lim_{z \to 0} R(z) = \infty$

Spherical for high z $\quad \frac{\lambda z}{\pi W_0^2} \gg 1$

- $W(z) \approx \frac{\lambda z}{\pi W_0}$
- $R(z) \approx z$

The Rayleigh length is the limit

$$L_R = \frac{\pi W_0^2}{\lambda}$$



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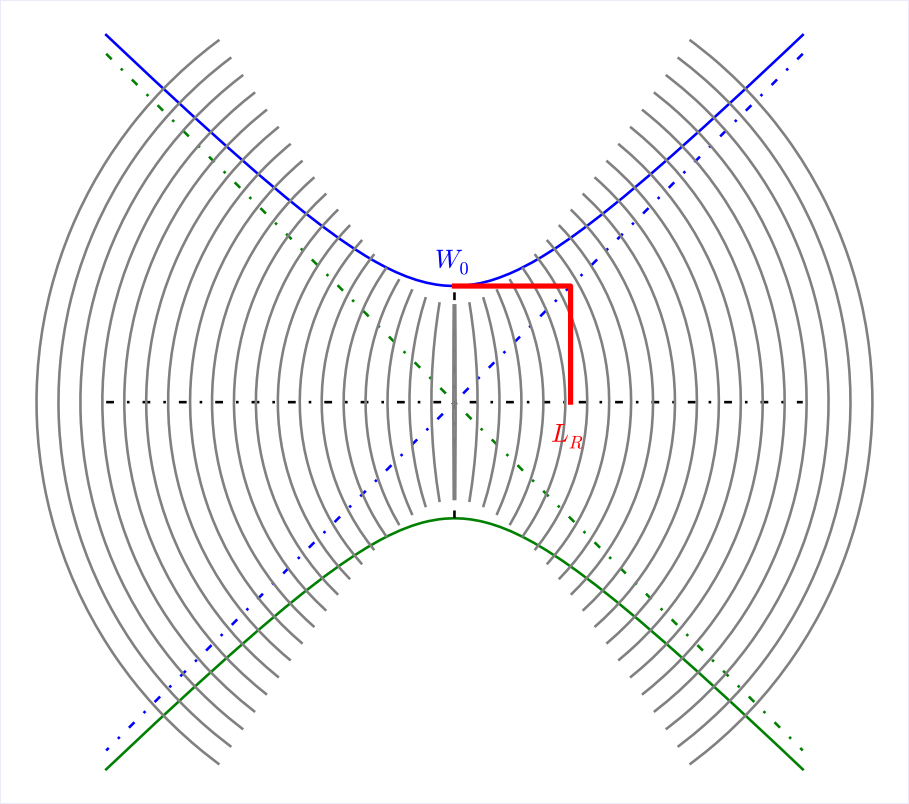

The homogeneous phase shift $P(z)$

$u = e^{-i\left(P(z)+\frac{k}{2q(z)}r^2\right)}$

Recall the equation

$$qP' + i = 0 \quad \Leftrightarrow \quad P'(z) = -\frac{i}{z + iL_R}$$

Integrate it

$$iP(z) = \ln\left(1 - i\frac{z}{L_R}\right)$$

Complex phase meaning

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Complex phase meaning

- Real part : Phase shift with respect to plane wave
- Imaginary part: $\frac{W_0}{W(z)}$ factor to ensure energy conservation

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The fundamental Gaussian mode

General expression

$$E(r,z) = \frac{W_0}{W(z)} e^{-i[kz+P(z)]-r^2\left(\frac{1}{W(z)^2}+i\frac{k}{2R(z)}\right)}$$

With, in (nearly) the order of appearance on the screen

- $W^2(z) = W_0^2\left[1+\left(\frac{z}{L_R}\right)^2\right]$
- $R(z) = z\left[1+\left(\frac{L_R}{z}\right)^2\right]$
- $P(z) = -\tan^{-1}\left(\frac{z}{L_R}\right)$
- Rayleigh length $L_R = \frac{\pi W_0^2}{\lambda}$
- Diffraction half angle: $\gamma \approx \frac{\lambda}{\pi W_0} = \frac{W_0}{L_R}$

1. Fundamentals of Gaussian beam propagation
   - Gaussian beams vs. plane waves
   - The fundamental mode
   - Higher order modes
2. Matrix methods for geometrical and Gaussian optics
   - Linear algebra for geometrical optics
   - A few simple matrices
   - Matrix method for Gaussian beams

High order Hermite-Gaussian modes

A Cartesian family of higher order modes

Ansatz

$$u(x,y,z) = g\left(\frac{x}{W(z)}\right) h\left(\frac{y}{W(z)}\right) e^{-i\left(P(z)+\frac{k}{2q(z)}\left(x^2+y^2\right)\right)}$$

Plugged into the wave equation

- $q' = 1$
- $\exists m \in \mathbb{N},\ \frac{\partial g}{\partial x^2} - 2x\frac{\partial g}{x} + 2mg = 0$
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Behavior of Hermite-Gaussian modes

Each mode is a mere space modulation of the fundamental

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- Same equation for q as in the fundamental mode
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- Solutions are, by definition, the orthogonal Hermite polynomials
- $\mathcal{H}_0 = 1, \quad \mathcal{H}_1 = x, \quad \mathcal{H}_2 = 4x^2 - 1, \quad \mathcal{H}_3 = 8x^3 - 12x \ldots$
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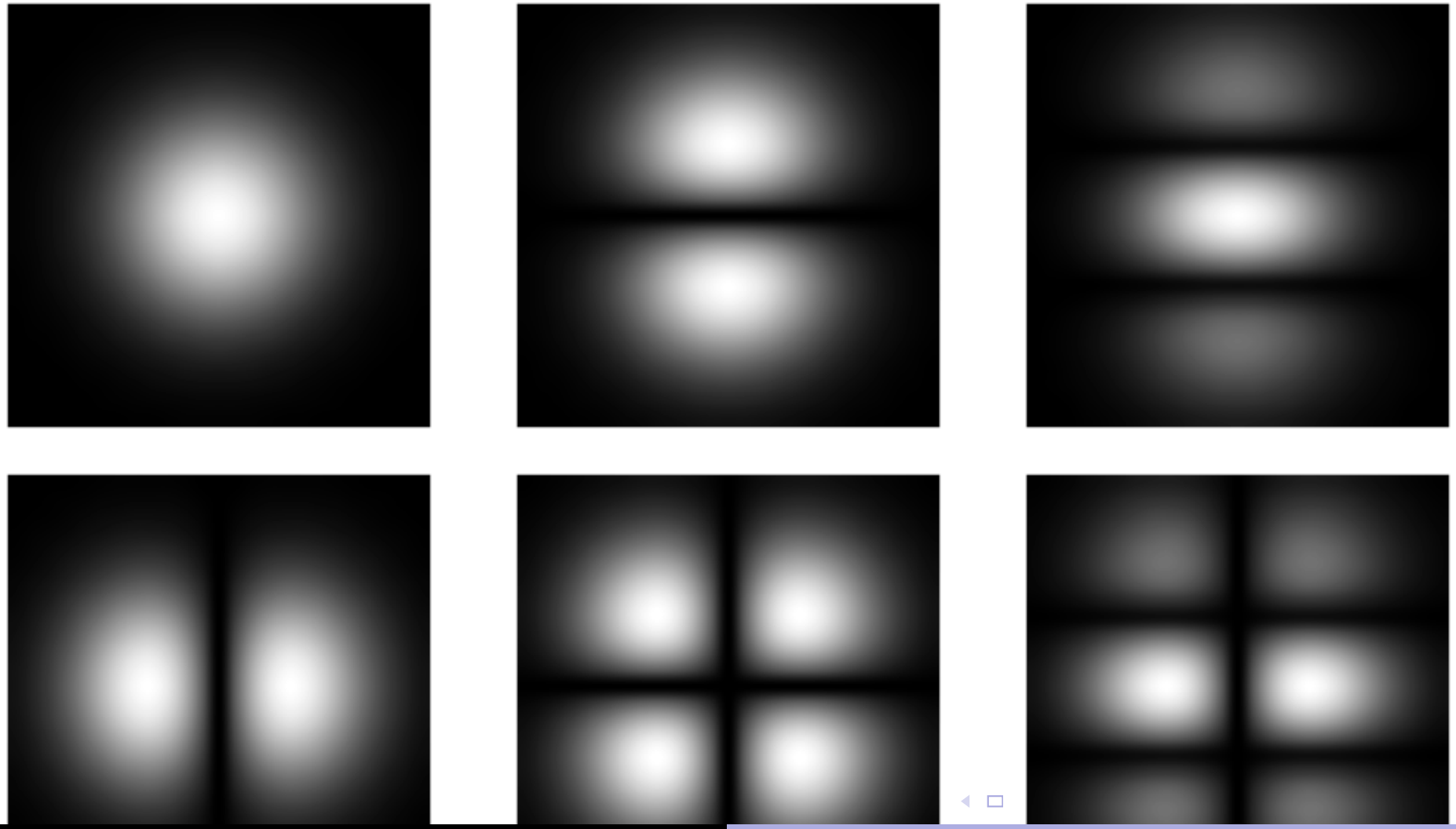
Intensity profiles of Hermite Gaussian (HG) modes

The intensity if proportional to the squared envelope



High order Laguerre-Gaussian modes

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Ansatz

$$u(r,\phi,z) = g\left(\frac{r}{W(z)}\right) e^{-i\left(P(z)+\frac{k}{2q(z)}r^2+l\phi\right)}$$

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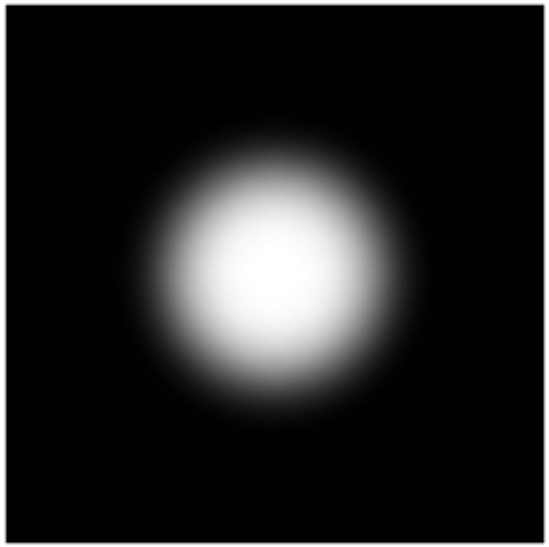
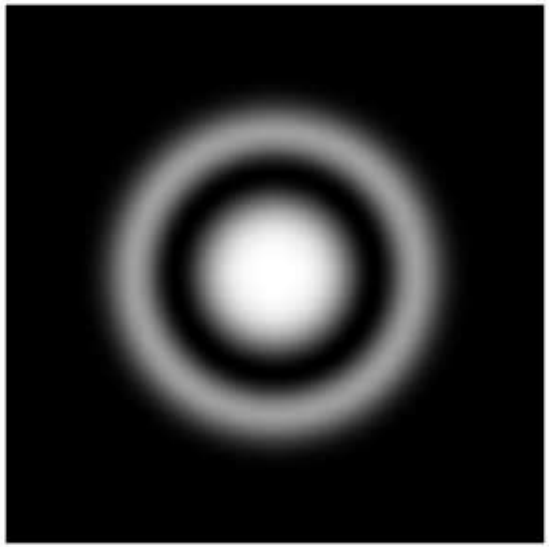


$LG(0,0)$

$LG(0,1)$: vortex

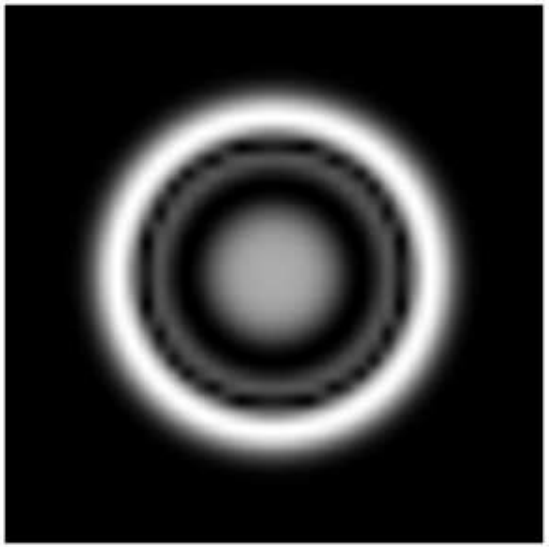


$LG(0,2)$

Intensity profiles of other Laguerre Gaussian (LG) modes

The intensity if proportional to the squared envelope

$LG(1,2)$



$LG(1,3)$

$LG(2,3)$

Homogeneous phase shift is different for high order modes

$qP' + (1 + m + n)j = 0$ $\qquad$ $qP' + (1 + 2p + l)j = 0$

A small phase difference between modes around the beam waist

- Slightly different optical paths for different orders
- Slightly different oscillating frequencies in lasers
- Usually forgotten

Homogeneous phase shift is different for high order modes

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- Slightly different oscillating frequencies in lasers
- Usually forgotten

Homogeneous phase shift is different for high order modes

$qP' + (1 + m + n)j = 0$ $\qquad$ $qP' + (1 + 2p + l)j = 0$

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- Linear algebra for geometrical optics
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Geometrical optics framework

Where is is shown that rays are not so thin as you may think

Geometrical optics *do not* deal with thin rays

- A thin ray has a thin waist: it should diffract $\gamma = \frac{\lambda}{\pi W_0}$
- Thin rays are seldom alone: their meaning is collective
- A ray is a Poynting vector curve
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Do geometrical optics deal with plane and spherical waves ?

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Geometrical optic is Gaussian optics

Transversely limited plane waves — **Parallel rays**

Gaussian Beams within their Rayleigh zone

Transversely limited spherical waves — **Con(Di)verging rays**

Gaussian Beams far from their Rayleigh zone

Orders of magnitude

- He-Ne laser: $W_0 \approx 1\mathrm{mm}, \lambda = 633\mathrm{nm}, L_R \approx 5\mathrm{m}$
- GSM Antenna: $W_0 \approx 1\mathrm{m}, \lambda \approx 33\mathrm{cm}, L_R \approx 10\mathrm{m}$

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Geometrical optics is linear

Geometrical optics stems entirely from Descartes law $n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$

Descartes made paraxial

Paraxial approximation : $\theta \ll 1$ $n_1\theta_1 \approx n_2\theta_2$

Geometrical optics is linear algebra

- Paraxial Descartes is linear
- Straight line propagation is linear
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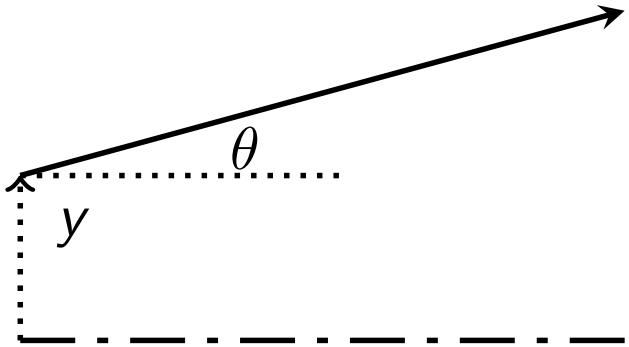
Matrix geometrical optics

A 2 dimensional linear algebra framework

The ray vector $v = \begin{pmatrix} y \\ \theta \end{pmatrix}$

- y : distance from the axis
- θ : angle to the axis



An optical system $v' = Mv$

- M is a 2×2 real matrix
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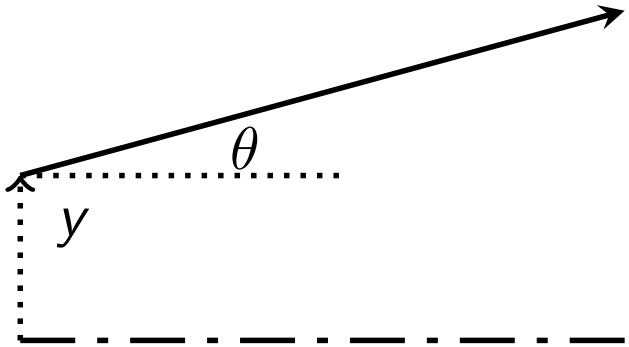

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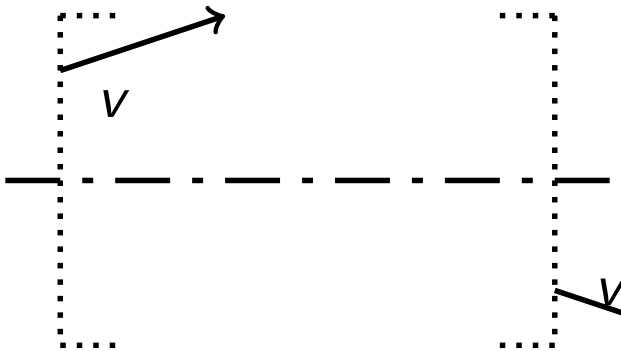
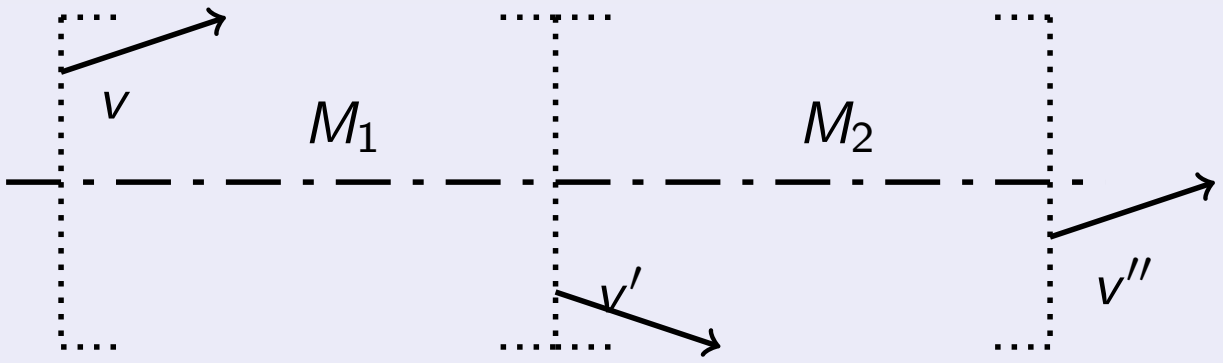


Optical system composition

Optical system composition reduced to matrix product

Optical System Composition



Matrix Composition

- $v' = M_1 \cdot v$
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Complex systems

Compose simple systems

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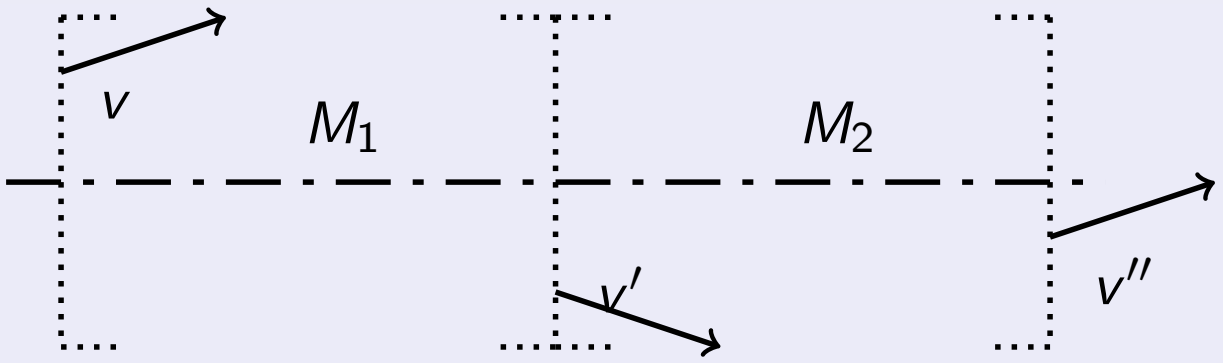

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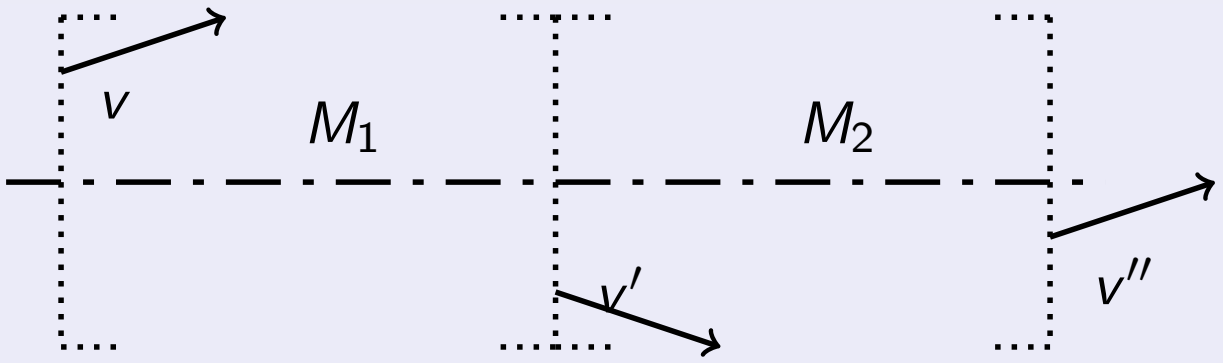

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Propagation in a homogeneous medium

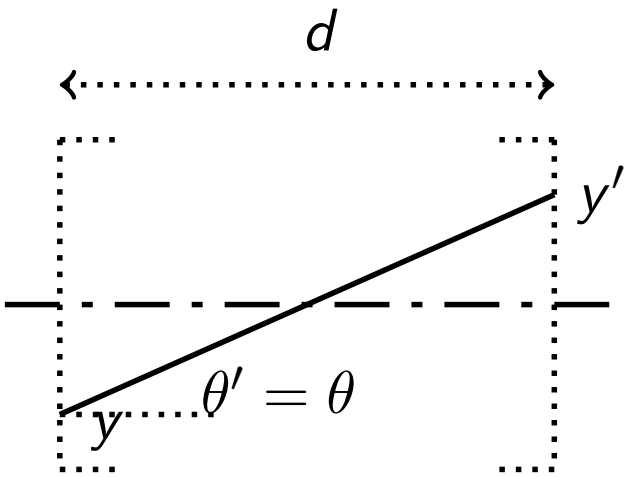
$$\begin{pmatrix} y' \\ \theta' \end{pmatrix} = M_d \begin{pmatrix} y \\ \theta \end{pmatrix}$$

Light propagates in straight line

- No direction change: $\theta' = \theta$
- $y' = y + d \sin(\theta)$

M_d

$$\begin{pmatrix} y' \\ \theta' \end{pmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{pmatrix} y \\ \theta \end{pmatrix}$$



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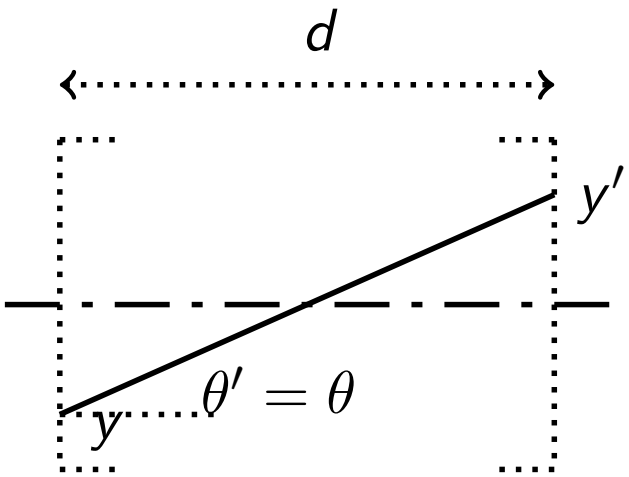

Passing through a plane interface

$$\begin{pmatrix} y' \\ \theta' \end{pmatrix} = M_p \begin{pmatrix} y \\ \theta \end{pmatrix}$$

Descartes

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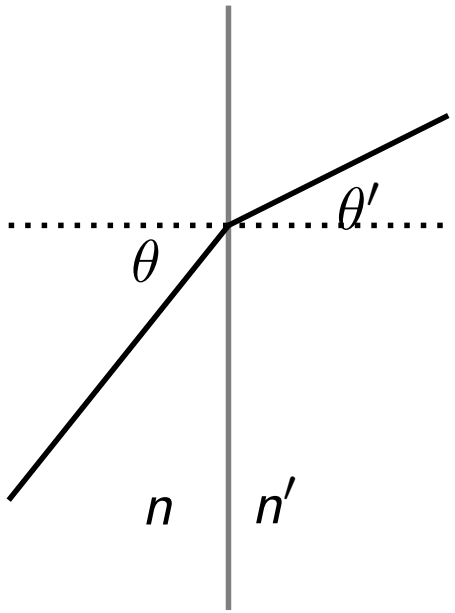

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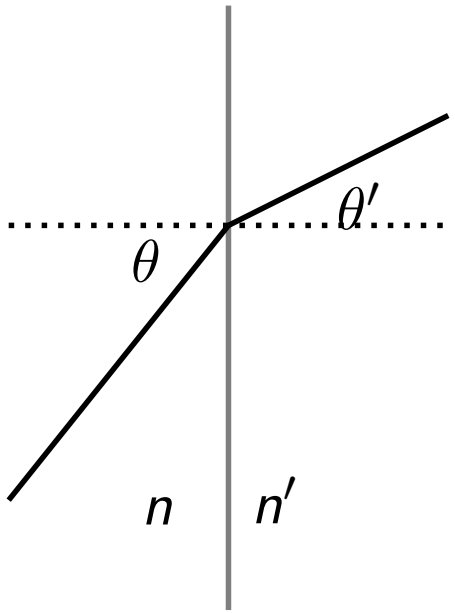

Passing through a thin lens

$$\begin{pmatrix} y' \\ \theta' \end{pmatrix} = M_l \begin{pmatrix} y \\ \theta \end{pmatrix}$$

Two characteristic rays

- No propagation: $y' = y$
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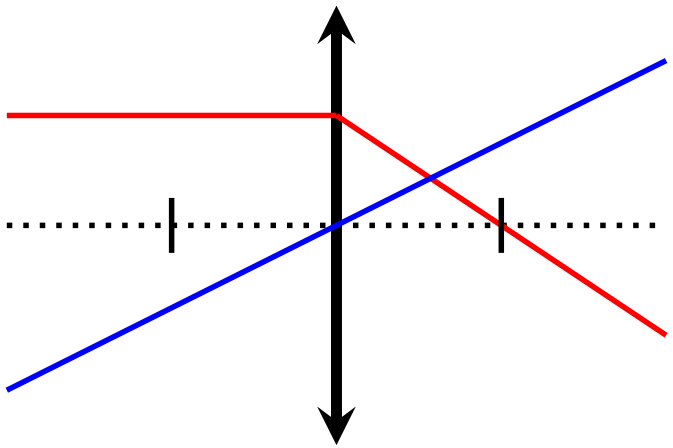

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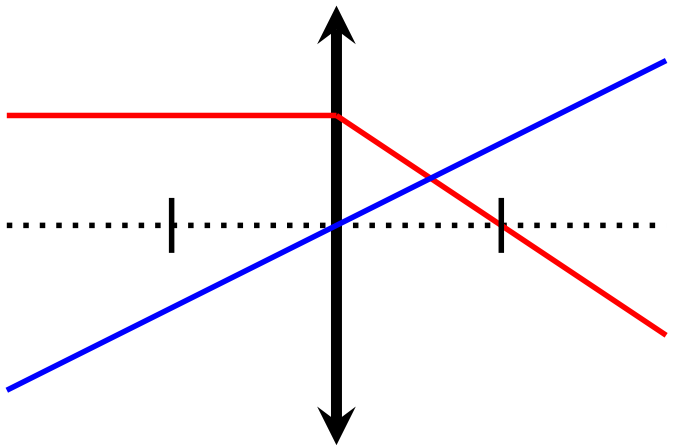

Passing through a (thin) spherical interface

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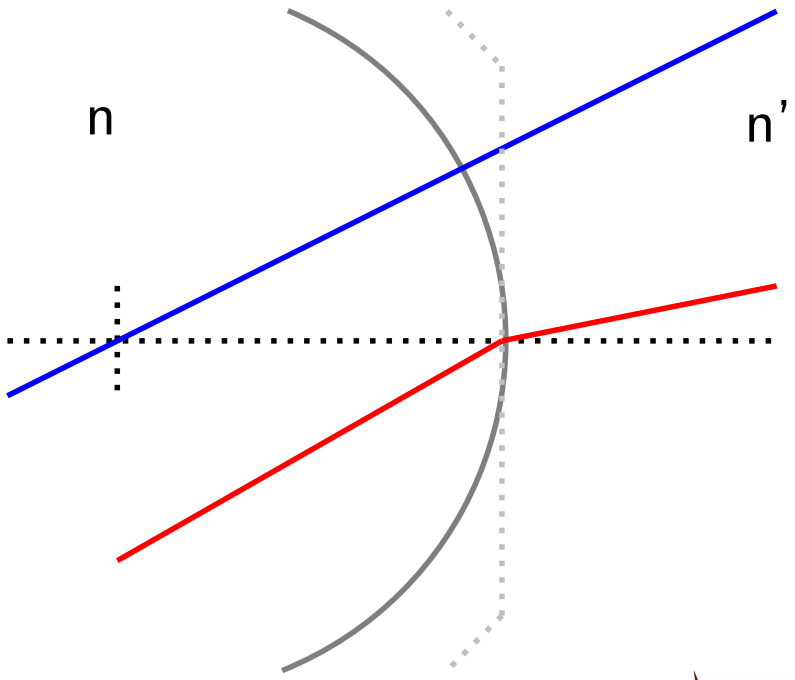

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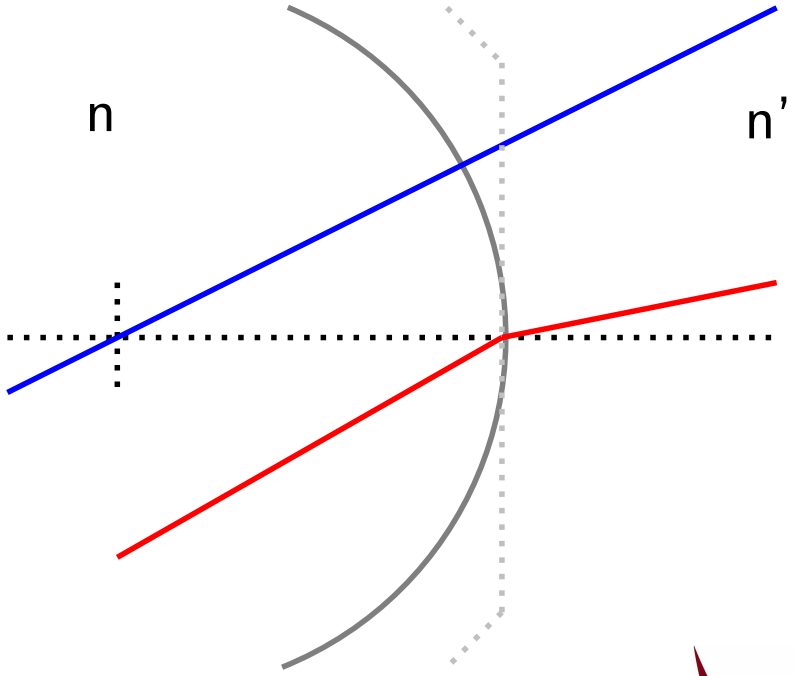
$\theta' = ay + b\theta$

Solve for (a, b) using

- $(y, \theta') = (R\theta, \theta)$
- $(y, \theta') = \left(0, \frac{n}{n'}\theta\right)$



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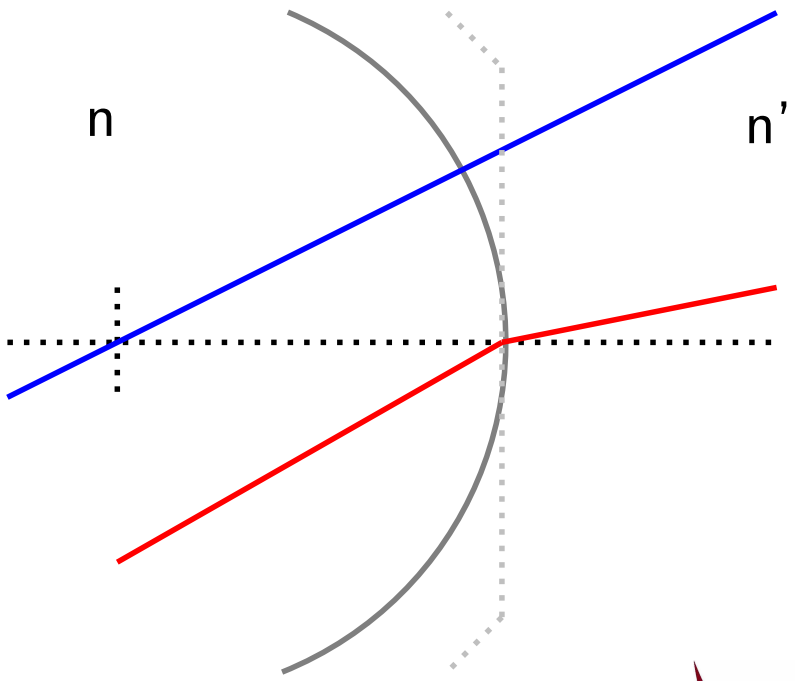

Mirrors

Unfolding the light

Plane mirrors as if they did not exist

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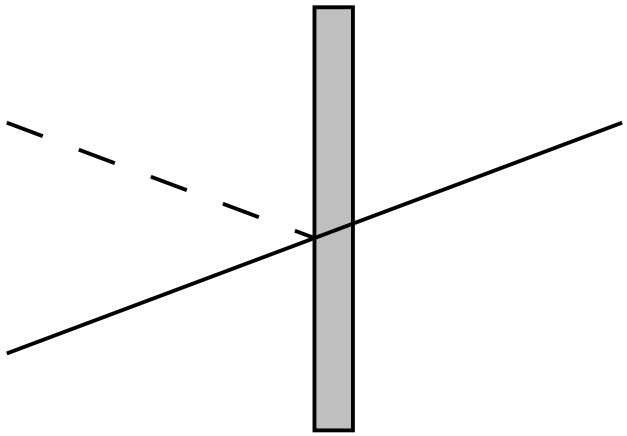

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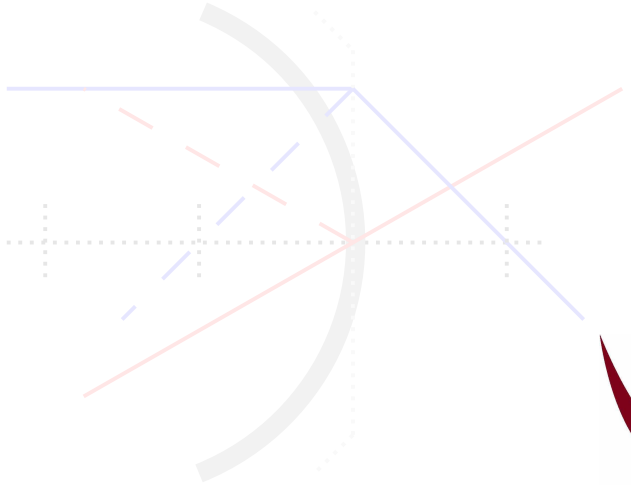

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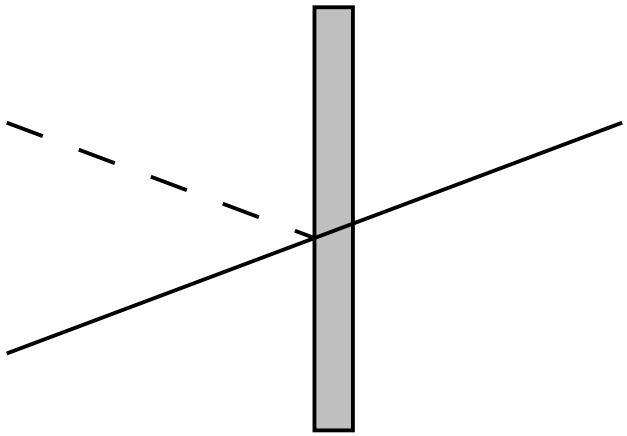

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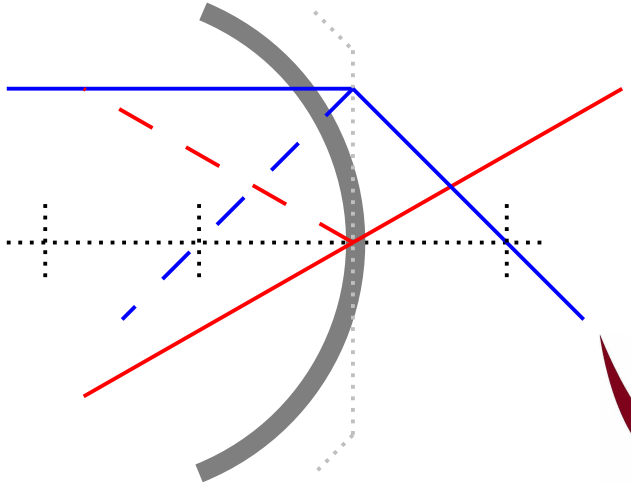

Matrix property

A determinant property stemming from all the simple matrices determinants

n: start index $\qquad$ n': stop index

$$\forall M,\ \det(M) = \frac{n}{n'}$$

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Gaussian modes, propagation and lenses

A Gaussian mode does not change upon propagation of by passing through thin interfaces or lenses

$$g\left(\frac{x}{W(z)}\right) h\left(\frac{y}{W(z)}\right) e^{-i\left(P(z)+\frac{k}{2q(z)}\left(x^2+y^2\right)\right)}$$

- z independent modulation of the fundamental mode
- Free space $q' = 1$ common property
- Thin lens does not change mode profile

Common $R(z)$ and $W(z)$ behavior

All the modes share the same laws on $q(z)$, $R(z)$ and $W(z)$

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Gaussian beam propagation: the ABCD law

The transformation of the complex radius q for simple optical systems

Free space $q' = 1$

- $q' = q + d$
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- $\frac{q'}{q} = \frac{n'}{n} \Rightarrow q' = \frac{1 \times q}{\frac{n}{n'}}$
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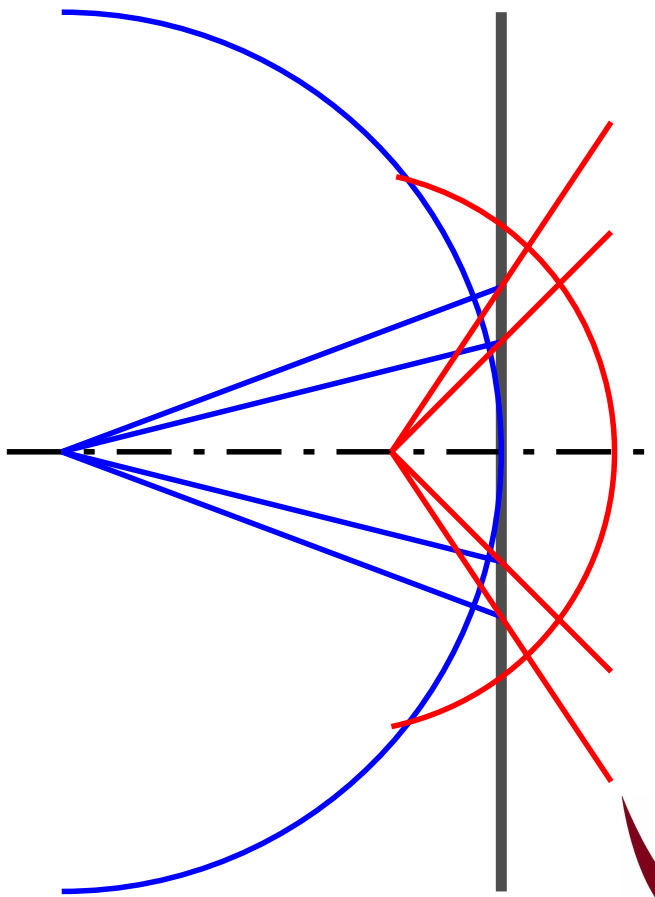

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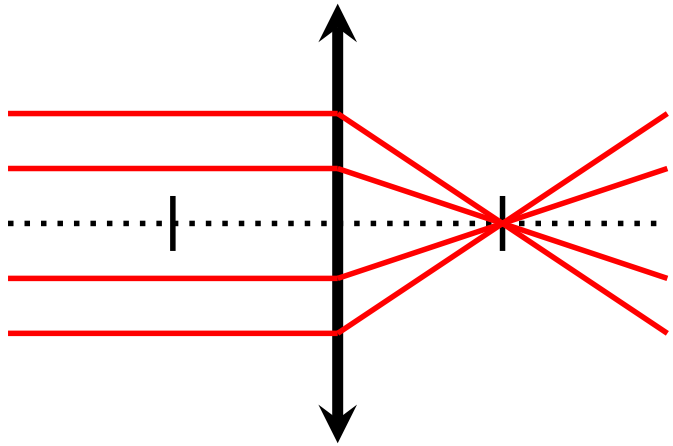

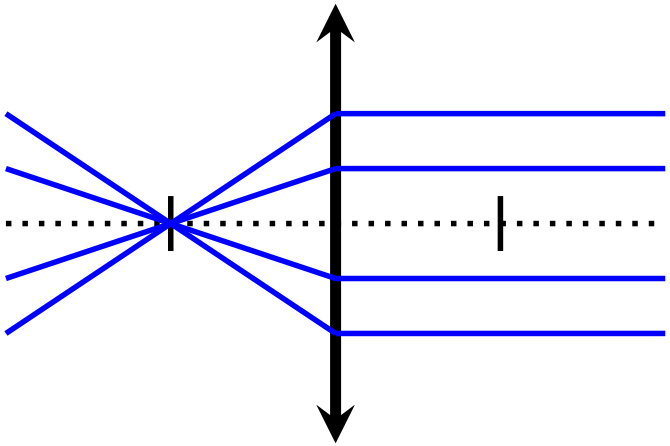

Thin lens $\frac{1}{R'} = \frac{1}{R} - \frac{1}{f}$

- $\frac{1}{q'} = \frac{1}{q} - \frac{1}{f} \Rightarrow q' = \frac{1}{-\frac{1}{f}q+1}$
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Thin lens $\frac{1}{R'} = \frac{1}{R} - \frac{1}{f}$

- $\frac{1}{q'} = \frac{1}{q} - \frac{1}{f} \Rightarrow q' = \frac{1}{-\frac{1}{f}q+1}$
- $M_l = \begin{pmatrix} 1 \times q & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$

Kogelnik's ABCD law

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \Rightarrow q' = \frac{Aq + B}{Cq + D}$$

Gaussian beam propagation: the ABCD law

The transformation of the complex radius q for simple optical systems

Free space $q' = 1$

- $q' = q + d$
- $M_d = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$

Plane interface $n/n' = R/R'$

- $\frac{q'}{q} = \frac{n'}{n} \Rightarrow q' = \frac{1 \times q}{\frac{n}{n'}}$
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Geometrical and Gaussian optics are linked through paraxial approx.

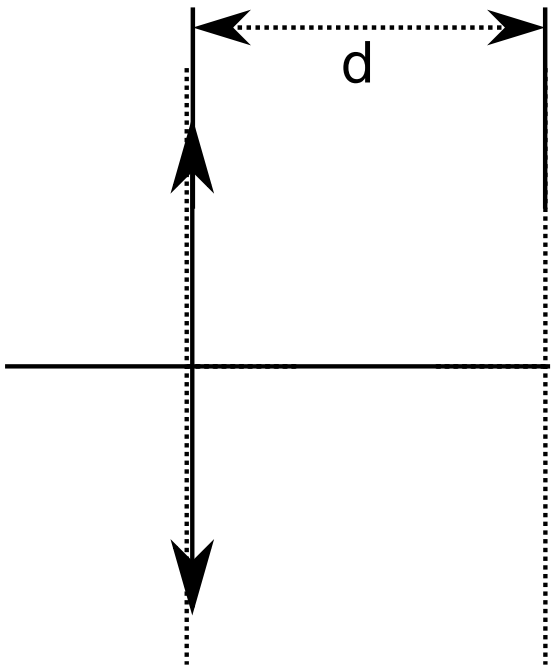
Gaussian beam propagation can be evaluated, for any mode, using simple matrix geometrical optics

Focal lens example

Using the ABCD law to verify that parallel rays do converge on the focal plane

Parallel input beam

- Input plane : just before lens
- Output plane : after length d
- Input beam at waist:
  $q_0 = i\frac{\pi W_0^2}{\lambda}$



Focal lens example

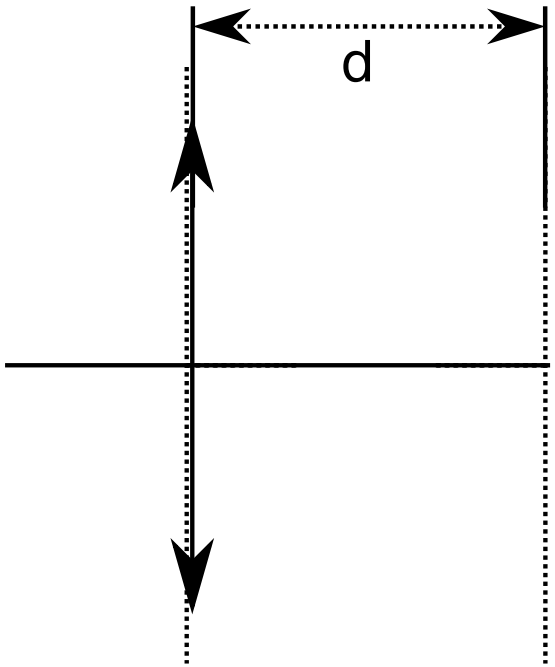
Using the ABCD law to verify that parallel rays do converge on the focal plane

Parallel input beam

- Input plane : just before lens
- Output plane : after length d
- Input beam at waist: $q_0 = iL_{R_0}$

Propagation matrix

- lens: $M_f = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$
- distance d : $M_d = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$



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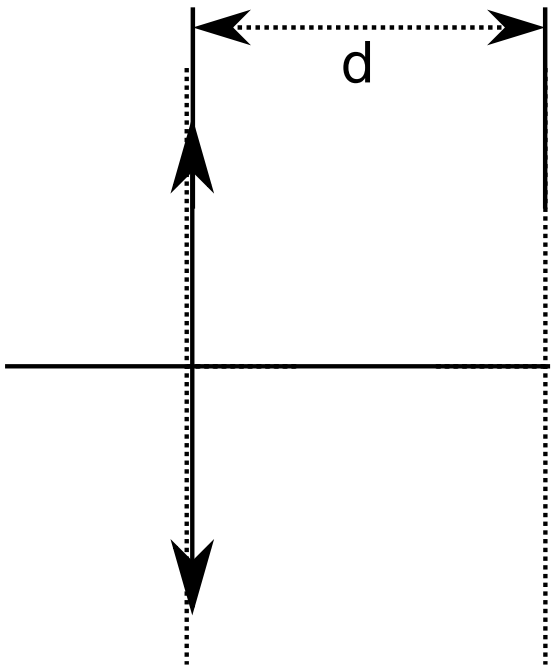

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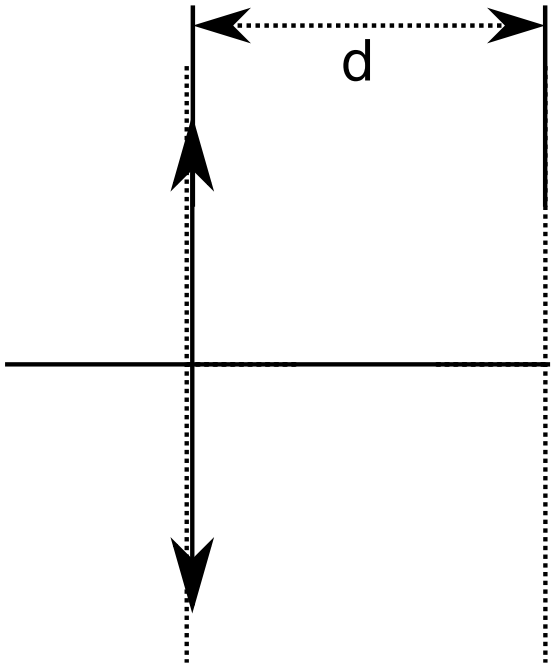
ABCD law

$$q_1 = \frac{df + i(f-d)L_{R_0}}{f - iL_{R_0}}$$



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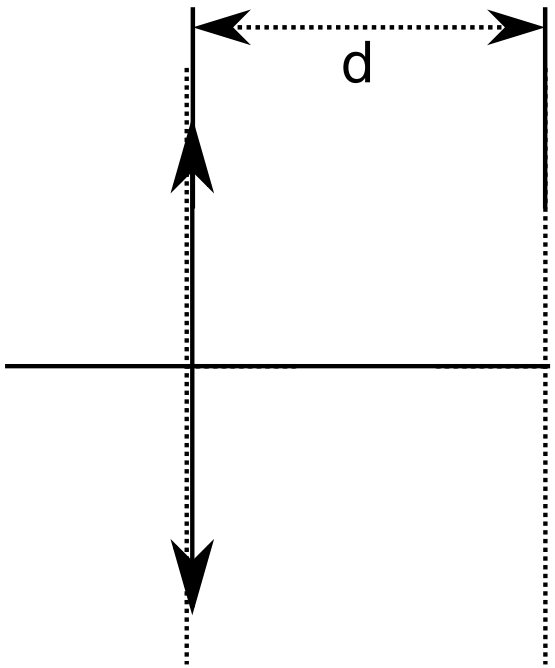

d for plane wavefront: imaginary q_1

$$d = \frac{f}{1 + \left(\frac{f}{L_{R_0}}\right)^2}$$