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## Gaussian Beams

Nicolas Fressengeas

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# Gaussian Beams

N. Fressengeas

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Unité de Recherche commune à l'Université de Lorraine et à Supélec

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## Further reading

[KL66, GB94]



A. Gerrard and J.M. Burch.

*Introduction to matrix methods in optics.*

Dover, 1994.



H. KOGELNIK and T. LI.

Laser beams and resonators.

*Appl. Opt.*, 5(10):1550–1567, Oct 1966.



# Course Outline

- 1 Fundamentals of Gaussian beam propagation
  - Gaussian beams vs. plane waves
  - The fundamental mode
  - Higher order modes
- 2 Matrix methods for geometrical and Gaussian optics
  - Linear algebra for geometrical optics
  - A few simple matrices
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# Plane waves do not exist

Waves carrying an infinite amount of energy cannot come into existence

## Planes waves

- Plane wave have a **homogeneous** transversal electric field
- Poynting's vector norm, and power density, are also homogeneous
- Total carried power is infinite

Practical use of plane wave theory: usual unsaid approximation

- Plane waves of **finite extent** are often used
- Strictly speaking, they are **not** plane waves



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## Plane waves, Gaussian beams. . . what else ?

Solutions of the wave equations: one finds only those he was searching for

### Solving the wave equation

$$\Delta \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$

- Vectorial Partial Derivatives Equations
- Solutions are numerous
- An **ansatz**<sup>1</sup> is needed to seek solutions

### Gaussian beams as an *ansatz*

- We will find another family of solutions
- We never pretend to get them all

<sup>1</sup>An ansatz is an *a priori* hypothesis on the form of the sought solution.



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## Gaussian *ansatz*

Plugging the ansatz into the wave equation builds the envelope equation

### Introducing a space dependent envelope

- Plane wave:  $\vec{E}_0 \times e^{-i\vec{k} \cdot \vec{r}}$
- Gaussian ansatz :  $u(x, y, z) \vec{e}_x \times e^{-ikz}$ 
  - $u(x, y, z)$  : complex beam envelope
  - $\vec{e}_x$  unit vector
- The envelope  $u(x, y, z)$  is our new unknown

Envelope equation



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### Envelope equation

- Wave equation:  $\Delta \vec{E} = -\frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$
- Scalar harmonic wave equation:  $\Delta E + E = 0$
- Plane wave:  $\Delta e^{-i\vec{k}\cdot\vec{r}} = -k^2 e^{-i\vec{k}\cdot\vec{r}}$



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 $\Delta u(x, y, z) e^{-ikz} + k^2 u(x, y, z) e^{-ikz} = 0$



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- Scalar harmonic wave equation:  $\Delta E + k^2 E = 0$
- Envelope equation:  $\frac{\partial^2(u e^{-ikz})}{\partial z^2} + \Delta_{\perp} u e^{-ikz} + k^2 u e^{-ikz} = 0$



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### Envelope equation

- Scalar harmonic wave equation:  $\Delta E + k^2 E = 0$
- Envelope equation:  $e^{-ikz} (\Delta u - 2ik \frac{\partial u}{\partial z}) = 0$



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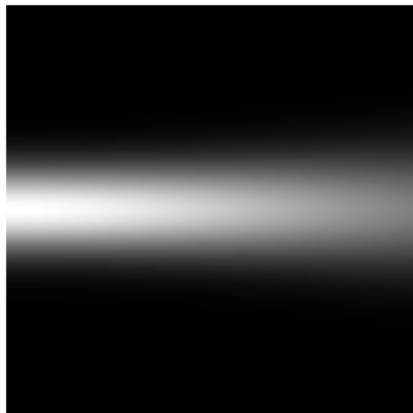
# The paraxial approximation

Also known as Gauss conditions, Slow Varying Envelope. . .

Non Paraxial Beam



Paraxial Beam

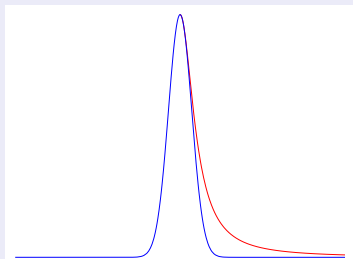


## Paraxial approximation and partial derivatives

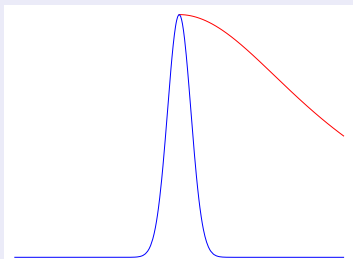
Assuming small angles is equivalent to neglecting  $z$  derivatives

Transversal variation vs. longitudinal variation

Non Paraxial Beam



Paraxial Beam

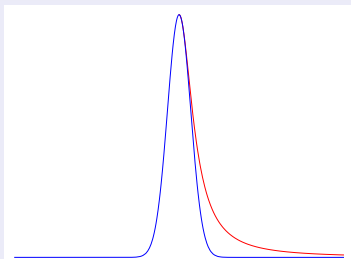


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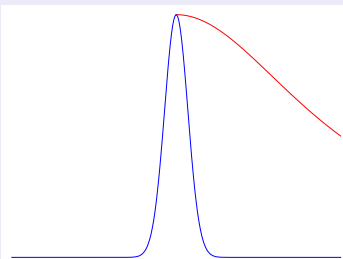
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Paraxial Beam



Transversal Laplacian

$$\frac{\partial^2}{\partial z^2} \ll \frac{\partial^2}{\partial x^2}$$

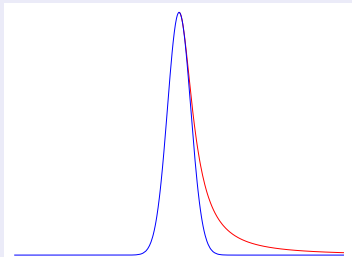
$$\Delta \approx \Delta_{\perp}$$

## Paraxial approximation and partial derivatives

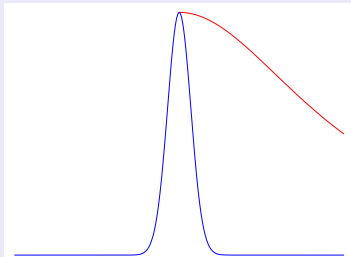
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Paraxial wave equation

$$\Delta_{\perp} u - 2ik \frac{\partial u}{\partial z} = 0$$





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# A first solution to the paraxial wave equation

The simplest one, though probably the more important

## Wave Propagation Equation

$$\Delta_{\perp} u - 2ik \frac{\partial u}{\partial z} = 0$$

### Complex beam radius $q(z)$

- Real part: phase variations
- Imaginary part: intensity variations

• Choosing  $q = iR$  yields

•  $\Delta_{\perp} e^{-ikR} = -k e^{-ikR}$

### A simple *ansatz*

$$u = e^{-i\left(P(z) + \frac{k}{2q(z)} r^2\right)}$$

### Phase shift $P(z)$

- Phase shift with respect to the plane wave
- $qP' + i = 0$



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- Integration:  $q(z) = q(0) + z$

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## The complex beam radius

A closer look to the signification on a complex parameter

$$q(z)$$

$$u = e^{-i\left(P(z) + \frac{k}{2q(z)}r^2\right)}$$

A **complex** parameter is linked to two **real** ones

$$\frac{1}{q} = \frac{1}{R} - i\frac{\lambda}{\pi W^2}$$



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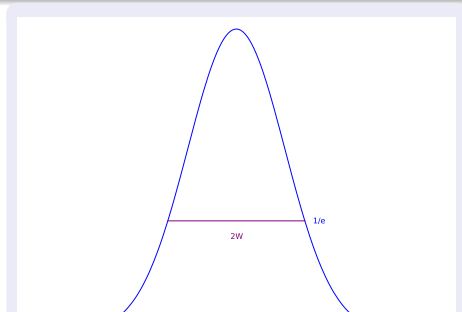
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What is  $R$  ?



## Spherical wavefront of radius $R$ at abscissa $z$

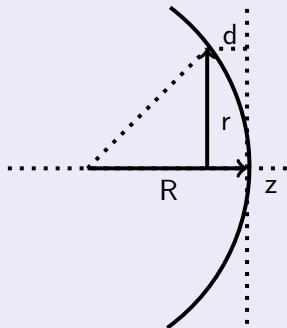
### Phase at abscissa $z$

- Constant phase on sphere
- Phase  $\propto d$ ,  $r \ll R$

### Gaussian ansatz

- $u = e^{-i\left(P(z) + k\frac{r^2}{2R(z)}\right)} e^{-\frac{r^2}{W(z)^2}}$

### $R$ radius spherical wavefront



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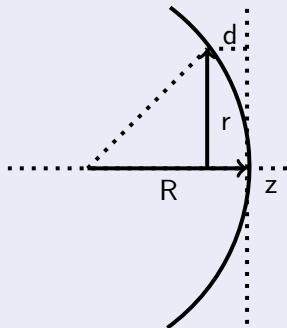
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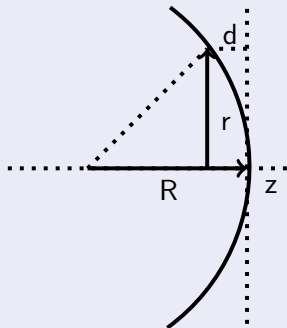
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- $(R - d)^2 + r^2 = R^2$

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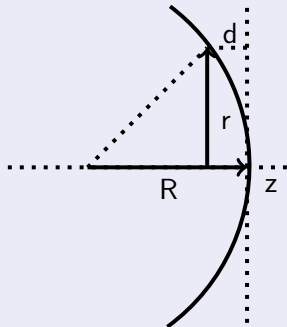
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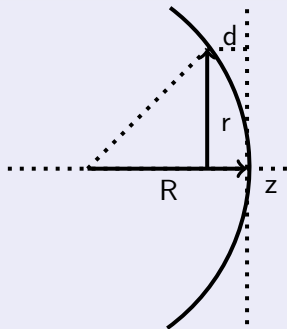
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- $d = R \left( 1 - \sqrt{1 - \frac{r^2}{R^2}} \right)$

### Gaussian ansatz

$$u = e^{-i\left(P(z) + k \frac{r^2}{2R(z)}\right)} e^{-\frac{r^2}{w(z)^2}}$$

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## Spherical wavefront of radius $R$ at abscissa $z$

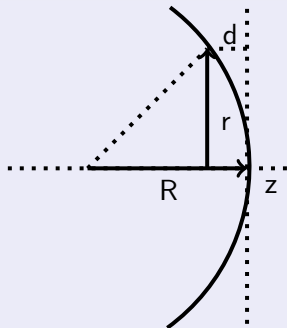
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- Constant phase on sphere
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- $d \approx \frac{r^2}{2R}$

### Gaussian ansatz

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## Gaussian Beam Complex Amplitude

Where the Gaussian beam amplitude is derived from the ansatz and  $q' = 1$

### A quick summary

- Ansatz :  $u = e^{-i\left(P(z) + \frac{k}{2q(z)}r^2\right)}$
- Complex beam radius :  $\frac{1}{q} = \frac{1}{R} - i\frac{\lambda}{\pi W^2}$
- Beam radius equation :  $q' = 1$

Assuming a plane wavefront for  $z = 0$

- $q(0) = i\frac{\pi W_0^2}{\lambda}$



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Assuming a plane wavefront for  $z = 0$

- $q(0) = i\frac{\pi W_0^2}{\lambda}$

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## Gaussian Beam Complex Amplitude

Where the Gaussian beam amplitude is derived from the ansatz and  $q' = 1$

### A quick summary

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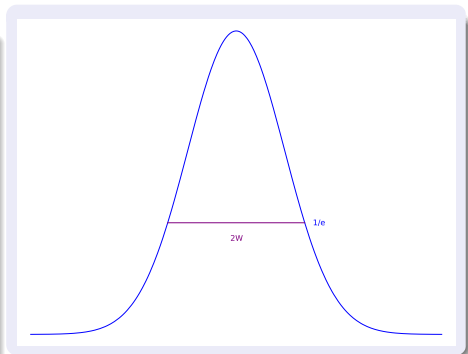
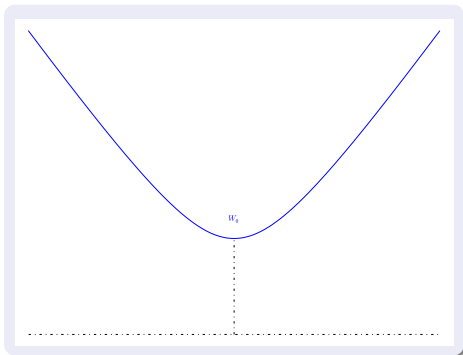
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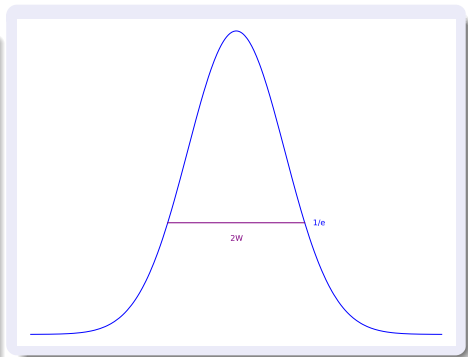
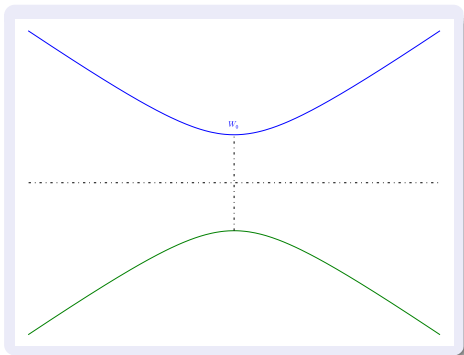
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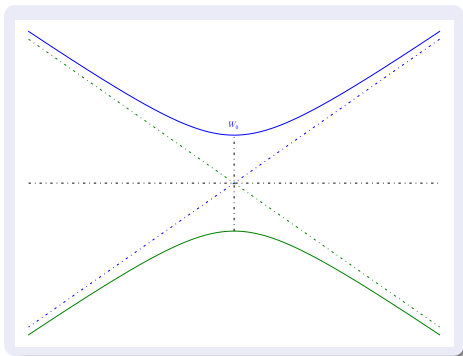
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### Asymptotes

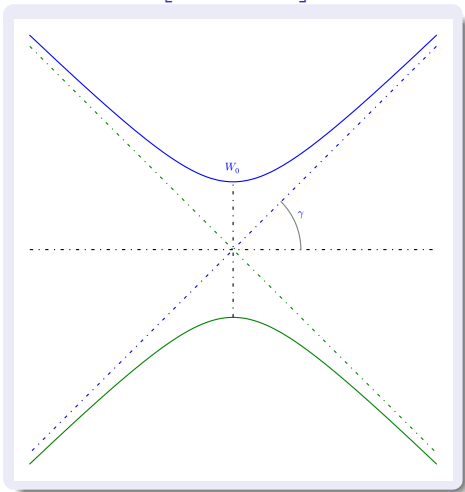
- $\frac{\lambda z}{\pi W_0^2} \gg 1 \Rightarrow W(z) \approx \frac{\lambda z}{\pi W_0}$
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$W_0$  : Beam Waist



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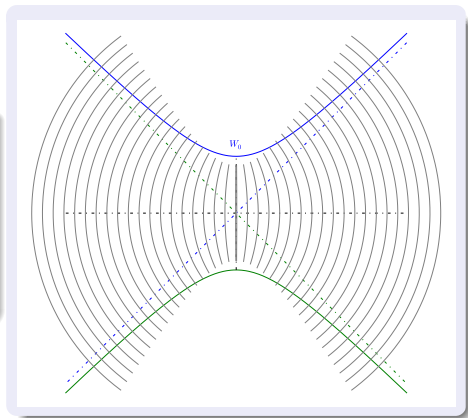


## Gaussian wavefront curvature

$$R(z) = z \left[ 1 + \left( \frac{\pi W_0^2}{\lambda z} \right)^2 \right]$$

### Plane and spherical limits

- For small  $z$  :  $R = \infty$   
plane wavefront
- For high  $z$  :  $R \approx z$   
spherical wavefront



## The Rayleigh length

A quantitative criterion to decide whether a Gaussian beam is plane or spherical

Plane for small  $z$

$$\frac{\lambda z}{\pi W_0^2} \ll 1$$

- $W(z) \approx W_0$
- $\lim_{z \rightarrow 0} R(z) = \infty$

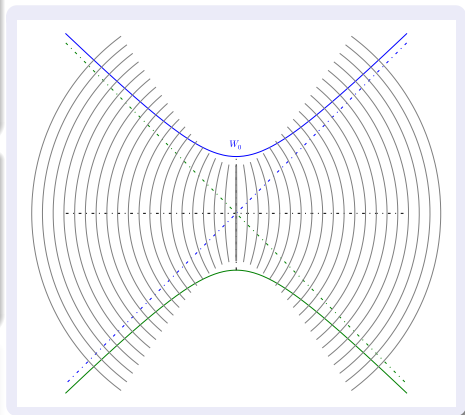
Spherical for high  $z$

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$$L_R = \frac{\pi W_0^2}{\lambda}$$



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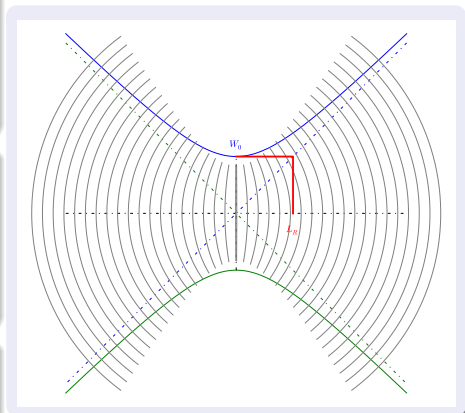
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## The homogeneous phase shift

$$P(z)$$

$$u = e^{-i\left(P(z) + \frac{k}{2q(z)}r^2\right)}$$

Recall the equation

$$qP' + i = 0$$

$\Leftrightarrow$

$$P'(z) = -\frac{i}{z + iL_R}$$

Integrate it

$$iP(z) = \ln\left(1 - i\frac{z}{L_R}\right)$$

Complex phase meaning





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• Real part: Phase shift with respect to plane wave

• Imaginary part: Phase shift with respect to Gaussian beam



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## The fundamental Gaussian mode

### General expression

$$E(r, z) = \frac{W_0}{W(z)} e^{-i[kz + P(z)] - r^2 \left( \frac{1}{W(z)^2} + i \frac{k}{2R(z)} \right)}$$

With, in (nearly) the order of appearance on the screen

- $W^2(z) = W_0^2 \left[ 1 + \left( \frac{z}{L_R} \right)^2 \right]$
- $R(z) = z \left[ 1 + \left( \frac{L_R}{z} \right)^2 \right]$
- $P(z) = -\tan^{-1} \left( \frac{z}{L_R} \right)$
- Rayleigh length  $L_R = \frac{\pi W_0^2}{\lambda}$
- Diffraction half angle:  $\gamma \approx \frac{\lambda}{\pi W_0} = \frac{W_0}{L_R}$



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# High order Hermite-Gaussian modes

A Cartesian family of higher order modes

Ansatz

$$u(x, y, z) = g\left(\frac{x}{W(z)}\right) h\left(\frac{y}{W(z)}\right) e^{-i\left(P(z) + \frac{k}{2q(z)}(x^2 + y^2)\right)}$$

Plugged into the wave equation

$$\bullet \quad q' = 1$$

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## Behavior of Hermite-Gaussian modes

Each mode is a mere space modulation of the fundamental

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- $\mathcal{H}_n$  has degree  $n$
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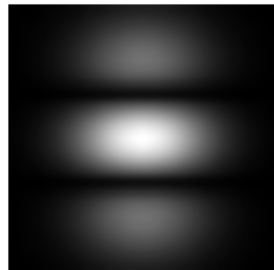
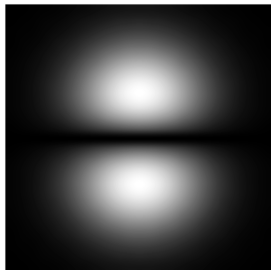
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A cylindrical family of higher order modes

Ansatz

$$u(r, \phi, z) = g\left(\frac{r}{W(z)}\right) e^{-i\left(P(z) + \frac{k}{2q(z)}r^2 + l\phi\right)}$$

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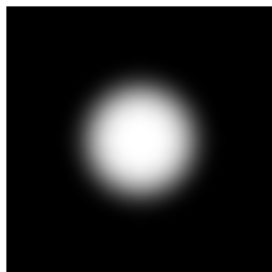
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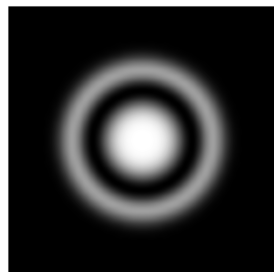
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$LG(0,0)$



$LG(0,1)$  : vortex



$LG(0,2)$

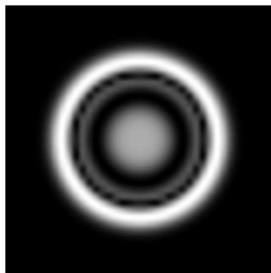


## Intensity profiles of other Laguerre Gaussian (LG) modes

The intensity is proportional to the squared envelope



$LG(1,2)$



$LG(1,3)$



$LG(2,3)$



## Homogeneous phase shift is different for high order modes

$$qP' + (1 + m + n)j = 0$$

$$qP' + (1 + 2p + l)j = 0$$

A small phase difference between modes around the beam waist

- Slightly different optical paths for different orders
- Slightly different oscillating frequencies in lasers
- Usually forgotten





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## Geometrical optics framework

Where is it shown that rays are not so thin as you may think

Geometrical optics *do not* deal with thin rays

- A thin ray has a thin **waist**: it should diffract
- Thin rays are seldom alone: their meaning is collective
- A ray is a Poynting vector curve
- A bunch of rays describes a wavefront

$$\gamma = \frac{\lambda}{\pi W_0}$$

Do geometrical optics deal with plane and spherical waves ?



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- But neither of them has an infinite extension !





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- A bunch of rays describes a wavefront

$$\gamma = \frac{\lambda}{\pi W_0}$$

Do geometrical optics deal with plane and spherical waves ?

- Parallel rays imply a plane wavefront
- **Converging** or diverging rays imply a **spherical wavefront**
- But neither of them has an infinite extension !



## Geometrical optics framework

Where is it shown that rays are not so thin as you may think

Geometrical optics *do not* deal with thin rays

- A thin ray has a thin waist: it should diffract
- Thin rays are seldom alone: their meaning is collective
- A ray is a Poynting vector curve
- A bunch of rays describes a wavefront

$$\gamma = \frac{\lambda}{\pi W_0}$$

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## Geometrical optic is Gaussian optics

Transversely limited plane waves

Parallel rays

Gaussian Beams **within their** Rayleigh zone

Transversely limited spherical waves

Con(Di)verging rays

Gaussian Beams **far from** their Rayleigh zone

Orders of magnitude

- He-Ne laser:  $W_0 \approx 1\text{mm}$ ,  $\lambda = 633\text{nm}$ ,  $L_R \approx 5\text{m}$
- GSM Antenna:  $W_0 \approx 1\text{m}$ ,  $\lambda \approx 33\text{cm}$ ,  $L_R \approx 10\text{m}$



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## Geometrical optics is linear

Geometrical optics stems entirely from Descartes law

$$n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$$

### Descartes made paraxial

Paraxial approximation :  $\theta \ll 1$

$$n_1 \theta_1 \approx n_2 \theta_2$$

Geometrical optics is linear algebra

- Paraxial Descartes is linear
- Straight line propagation is linear
- The behavior of a ray through any optical system can be described linearly



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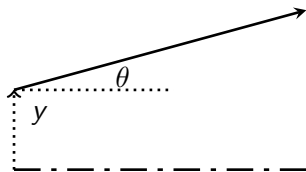


## Matrix geometrical optics

A 2 dimensional linear algebra framework

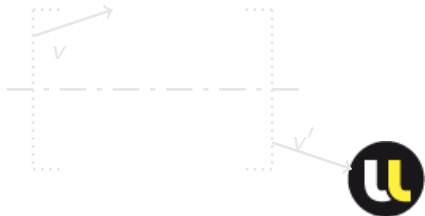
The ray vector  $v = \begin{pmatrix} y \\ \theta \end{pmatrix}$

- $y$  : distance from the axis
- $\theta$  : angle to the axis



An optical system  $v' = Mv$

- $M$  is a  $2 \times 2$  real matrix
- It can describe any centered paraxial optical system





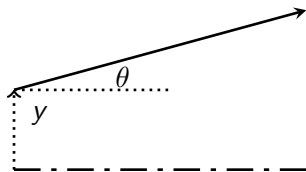
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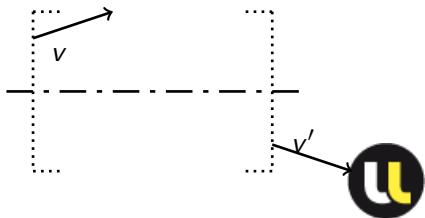
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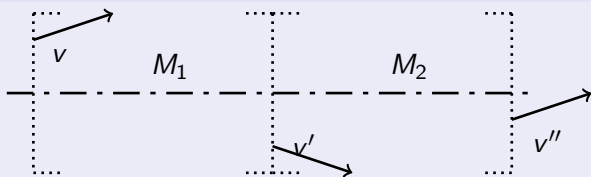
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# Optical system composition

Optical system composition reduced to matrix product

## Optical System Composition



## Matrix Composition

- $v' = M_1 \cdot v$
- $v'' = M_2 \cdot v'$
- $v'' = M_2 M_1 \cdot v$

## Complex systems

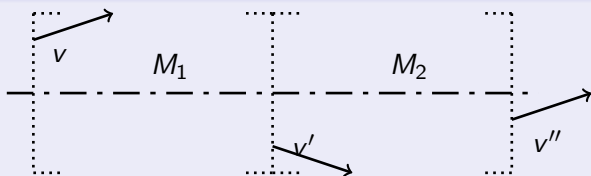
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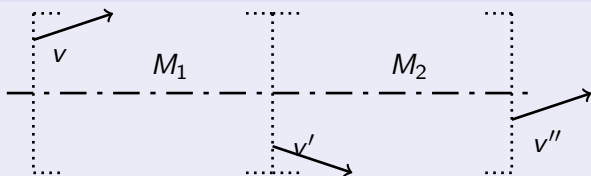
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## Propagation in a homogeneous medium

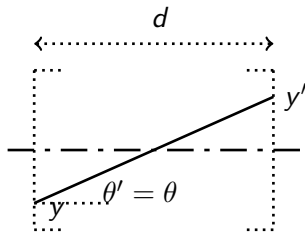
$$\begin{pmatrix} y' \\ \theta' \end{pmatrix} = M_d \begin{pmatrix} y \\ \theta \end{pmatrix}$$

### Light propagates in straight line

- No direction change:  $\theta' = \theta$
- $y' = y + d \sin(\theta)$

$M_d$

$$\begin{pmatrix} y' \\ \theta' \end{pmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{pmatrix} y \\ \theta \end{pmatrix}$$



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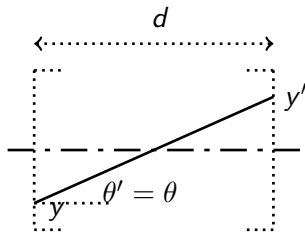
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## Passing through a plane interface

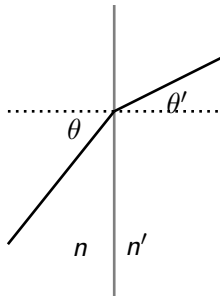
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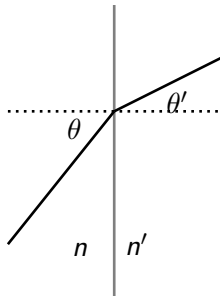
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## Passing through a thin lens

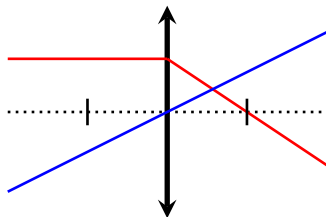
$$\begin{pmatrix} y' \\ \theta' \end{pmatrix} = M_l \begin{pmatrix} y \\ \theta \end{pmatrix}$$

### Two characteristic rays

- No propagation:  $y' = y$
- Blue ray:  $y = 0 \Rightarrow \theta' = \theta$
- Red ray:  $\theta = 0 \Rightarrow \theta' = -\frac{1}{f}y$

$M_l$

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## Passing through a thin lens

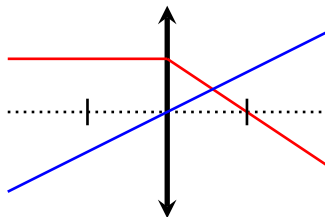
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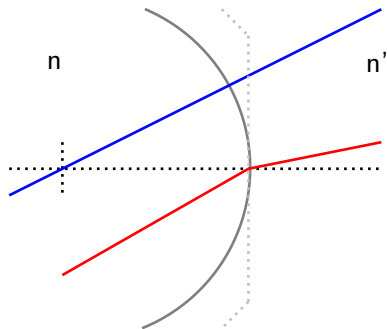


## Passing through a (thin) spherical interface

$$\begin{pmatrix} y' \\ \theta' \end{pmatrix} = M_s \begin{pmatrix} y \\ \theta \end{pmatrix}$$

### Descartes

- Thin interface  
No propagation:  $y' = y$
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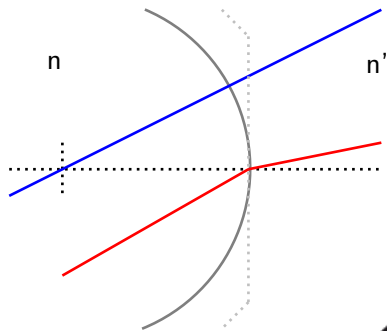
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$$\theta' = ay + b\theta$$

Solve for  $(a, b)$  using

- $(y, \theta') = (R\theta, \theta)$
- $(y, \theta') = (0, \frac{n}{n'}\theta)$



## Passing through a (thin) spherical interface

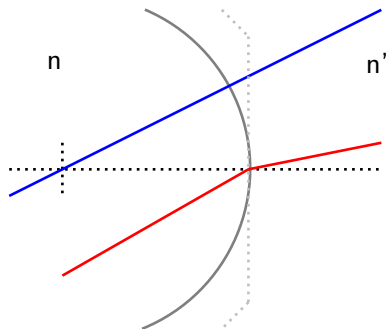
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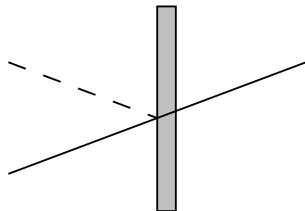


## Mirrors

### Unfolding the light

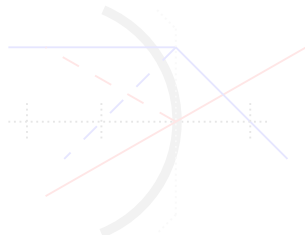
Plane mirrors as if they did not exist

$$\begin{pmatrix} y' \\ \theta' \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} y \\ \theta \end{pmatrix}$$



Spherical Mirrors are thin lenses

$$\begin{pmatrix} y' \\ \theta' \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{bmatrix} \begin{pmatrix} y \\ \theta \end{pmatrix}$$



## Mirrors

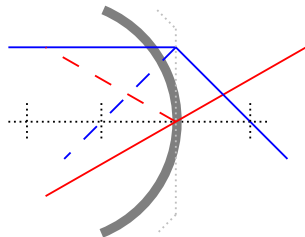
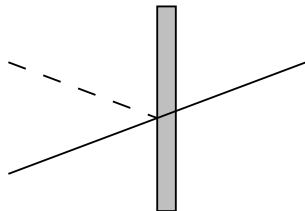
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## Matrix property

A determinant property stemming from all the simple matrices determinants

$n$ : start index

$n'$ : stop index

$$\forall M, \det(M) = \frac{n}{n'}$$



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## Gaussian modes, propagation and lenses

A Gaussian mode does not change upon propagation of by passing through thin interfaces or lenses

$$g\left(\frac{x}{W(z)}\right) h\left(\frac{y}{W(z)}\right) e^{-i\left(P(z) + \frac{k}{2q(z)}(x^2 + y^2)\right)}$$

- $z$  independent modulation of the fundamental mode
- Free space  $q' = 1$  common property
- Thin lens does not change mode profile

Common  $R(z)$  and  $W(z)$  behavior

All the modes share the same laws on  $q(z)$ ,  $R(z)$  and  $W(z)$



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## Gaussian beam propagation: the ABCD law

The transformation of the complex radius  $q$  for simple optical systems

Free space

$$q' = 1$$

- $q_1 = q_0 + d$
- $M_d = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$



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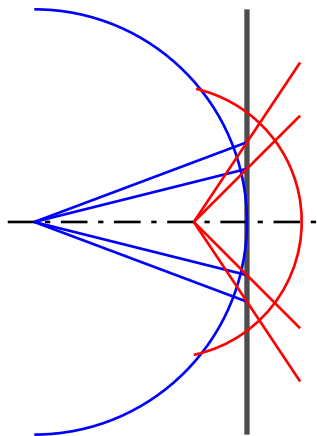
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Plane interface

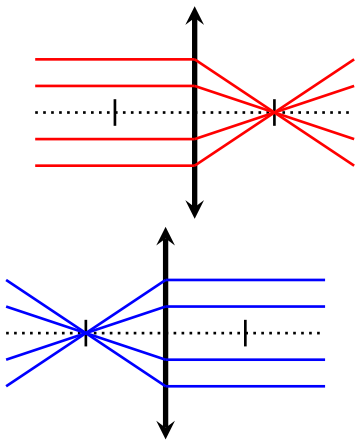
$$n_0/n_1 = R_0/R_1$$

- $\frac{q_1}{q_0} = \frac{n_1}{n_0} \Rightarrow q_1 = \frac{1 \times q_0}{\frac{n_1}{n_0}}$
- $M_p = \begin{pmatrix} 1 & 0 \\ 0 & \frac{n_0}{n_1} \end{pmatrix}$



## Gaussian beam propagation: the ABCD law

The transformation of the complex radius  $q$  for simple optical systems



Thin lens

$$\frac{1}{R_1} = \frac{1}{R_0} - \frac{1}{f}$$

$$\bullet \frac{1}{q_1} = \frac{1}{q_0} - \frac{1}{f} \Rightarrow q_1 = \frac{1}{-\frac{1}{f}q_0 + 1}$$

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Kogelnik's ABCD law

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \Rightarrow q_1 = \frac{Aq_0 + B}{Cq_0 + D}$$





## Gaussian beam propagation: the ABCD law

The transformation of the complex radius  $q$  for simple optical systems

Free space  $q' = 1$

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Plane interface  $n_0/n_1 = R_0/R_1$

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Geometrical and Gaussian optics are linked through paraxial approx.

Gaussian beam propagation can be evaluated, for **any** mode, using simple matrix geometrical optics



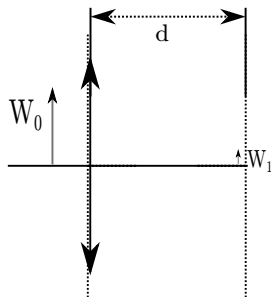
## Focal lens example

Using the ABCD law to verify that parallel rays do converge on the focal plane

### Parallel input beam

- Input plane : just before lens
- Output plane : after length  $d$
- Input beam at waist:

$$q_0 = i \frac{\pi W_0^2}{\lambda}$$



## Focal lens example

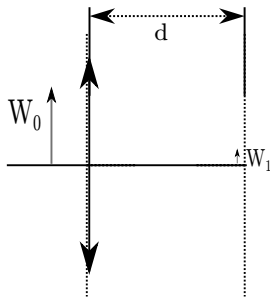
Using the ABCD law to verify that parallel rays do converge on the focal plane

### Parallel input beam

- Input plane : just before lens
- Output plane : after length  $d$
- Input beam at waist:  $q_0 = iL_{R_0}$

### Propagation matrix

- lens:  $M_f = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$
- distance  $d$  :  $M_d = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$



## Focal lens example

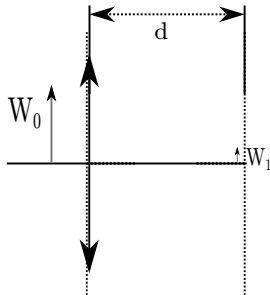
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$$M_d \cdot M_f = \begin{pmatrix} -\frac{d}{f} + 1 & d \\ -\frac{1}{f} & 1 \end{pmatrix}$$



## Focal lens example

Using the ABCD law to verify that parallel rays do converge on the focal plane

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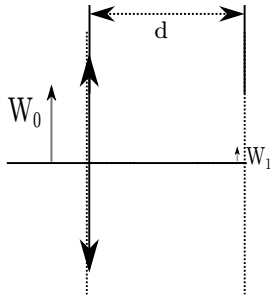
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### ABCD law

$$q_1 = \frac{df + i(f - d)L_{R_0}}{f - iL_{R_0}}$$



## Focal lens example

Using the ABCD law to verify that parallel rays do converge on the focal plane

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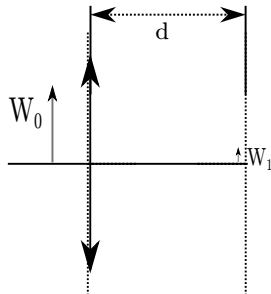
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### ABCD law

$$q_1 = \frac{df + i(f - d)L_{R_0}}{f - iL_{R_0}}$$



$d$  for plane wavefront: imaginary  $q_1$

$$d = \frac{f}{1 + \left(\frac{f}{L_{R_0}}\right)^2}$$

# Transformation of a parallel beam by a lens

The basics of geometrical optics

As we just saw

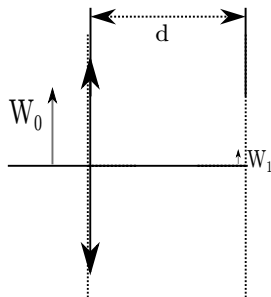
$$L_{R_0} \ll d$$

$$q_1 = \frac{df + i(f - d)L_{R_0}}{f - iL_{R_0}} \quad \& \quad d \approx f$$

Identifying  $W_1$  in  $q_1$

$$L_{R_0} \ll d$$

$$W_1 = \frac{\lambda|f|}{\pi W_0}$$



## Transformation of a parallel beam by a lens

The basics of geometrical optics

As we just saw

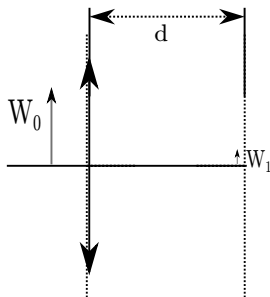
$$L_{R_0} \ll d$$

$$q_1 = \frac{df + i(f - d)L_{R_0}}{f - iL_{R_0}} \quad \& \quad d \approx f$$

Identifying  $W_1$  in  $q_1$

$$L_{R_0} \ll d$$

$$W_1 = \frac{\lambda|f|}{\pi W_0}$$

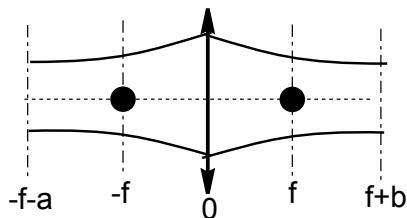




## Transformation of a diffracting Gaussian beam by a lens

From  $-f - a$  to  $f + b$

- Input  $q_0 = iL_{R_0}$  at  $-f - a$
- Output  $q_1$  at  $f + b$
- Assume  $L_{R_0} \ll a$  and  $L_{R_1} \ll b$



Propagation Matrix

$$\begin{pmatrix} 1 & f+b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & f+a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{b}{f} & -\frac{ab-f^2}{f} \\ -\frac{1}{f} & -\frac{a}{f} \end{pmatrix}$$

From waist to waist

assuming  $q_1$  imaginary

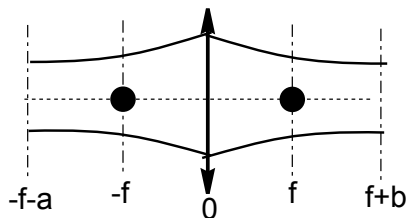
$$ab = f^2$$

$$\left(\frac{W_1}{W_0}\right)^2 = \frac{b}{a}$$

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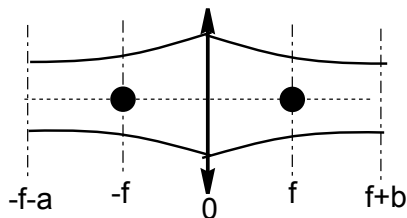
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