

Gaussian Beams

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Further reading

[KL66, GB94]



A. Gerrard and J.M. Burch.

Introduction to matrix methods in optics.

Dover, 1994.



H. KOGELNIK and T. LI.

Laser beams and resonators.

Appl. Opt., 5(10):1550–1567, Oct 1966.



Course Outline

- 1 Fundamentals of Gaussian beam propagation
 - Gaussian beams vs. plane waves
 - The fundamental mode
 - Higher order modes
- 2 Matrix methods for geometrical and Gaussian optics
 - Linear algebra for geometrical optics
 - A few simple matrices
 - Matrix method for Gaussian beams



Plane waves do not exist

Waves carrying an infinite amount of energy cannot come into existence

Planes waves

- Plane wave have a **homogeneous** transversal electric field
- Poynting's vector norm, and **power density**, are also homogeneous
- Total carried power is infinite

Practical use of plane wave theory: usual unsaid approximation

- Plane waves of **finite extent** are often used
- Strictly speaking, they are **not** plane waves
- To what extent can we assume they are plane waves ?



Plane waves, Gaussian beams. . . what else ?

Solutions of the wave equations: one finds only those he was searching for

Solving the wave equation

$$\Delta \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$

- Vectorial Partial Derivatives Equations
- Solutions are numerous
- An **ansatz**¹ is needed to seek solutions

Gaussian beams as an *ansatz*

- We will find another **family of solutions**
- We never pretend to get them all

¹An ansatz is an *a priori* hypothesis on the form of the sought solution.



Gaussian *ansatz*

Plugging the ansatz into the wave equation builds the envelope equation

Introducing a space dependent envelope

- Plane wave: $\vec{E}_0 \times e^{-i\vec{k} \cdot \vec{r}}$
- Gaussian ansatz : $u(x, y, z) \vec{e}_x \times e^{-ikz}$
 - $u(x, y, z)$: **complex** beam envelope
 - \vec{e}_x unit vector
- The envelope $u(x, y, z)$ is our **new unknown**

Envelope equation

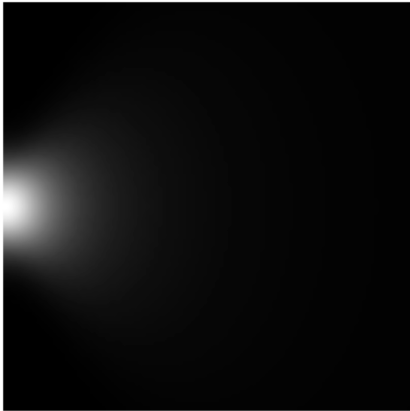
- Scalar harmonic wave equation: $\Delta E + k^2 E = 0$
- Envelope equation: $\Delta u - 2ik \frac{\partial u}{\partial z} = 0$



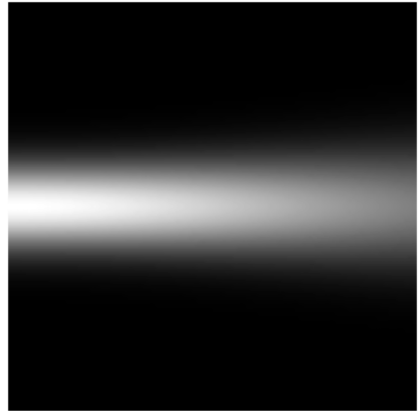
The paraxial approximation

Also known as Gauss conditions, Slow Varying Envelope. . .

Non Paraxial Beam



Paraxial Beam

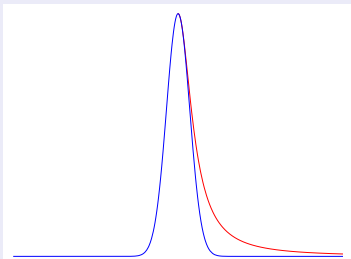


Paraxial approximation and partial derivatives

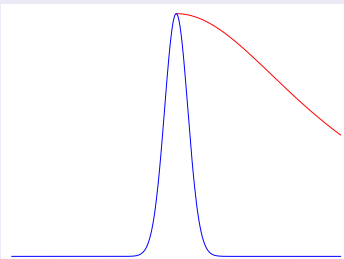
Assuming small angles is equivalent to neglecting z derivatives

Transversal variation vs. longitudinal variation

Non Paraxial Beam



Paraxial Beam



Transversal Laplacian

$$\frac{\partial^2}{\partial z^2} \ll \frac{\partial^2}{\partial x^2}$$

$$\Delta \approx \Delta_{\perp}$$

A first solution to the paraxial wave equation

The simplest one, though probably the more important

Wave Propagation Equation

$$\Delta_{\perp} u - 2ik \frac{\partial u}{\partial z} = 0$$

A simple *ansatz*

$$u = e^{-i\left(P(z) + \frac{k}{2q(z)} r^2\right)}$$

Complex beam radius $q(z)$

- Real part: phase variations
- Imaginary part: intensity variations
- Plugging ansatz: $q' = 1$
- Integration: $q(z) = q(0) + z$

Phase shift $P(z)$

- Phase shift **with respect** to the plane wave
- $qP' + i = 0$



The complex beam radius

$$q(z)$$

A closer look to the signification on a complex parameter

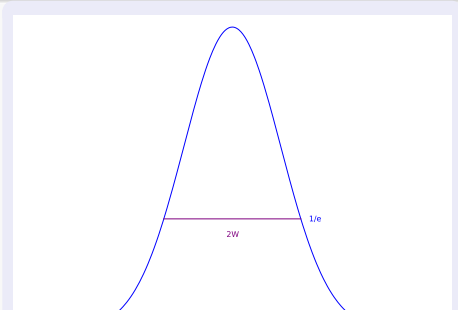
$$u = e^{-i\left(P(z) + \frac{k}{2q(z)}r^2\right)}$$

A **complex** parameter is linked to two **real** ones

$$\frac{1}{q} = \frac{1}{R} - i\frac{\lambda}{\pi W^2}$$

The *ansatz* re-written

$$u = e^{-i\left(P(z) + k\frac{r^2}{2R(z)}\right)} e^{-\frac{r^2}{W(z)^2}}$$



Spherical wavefront of radius R at abscissa z

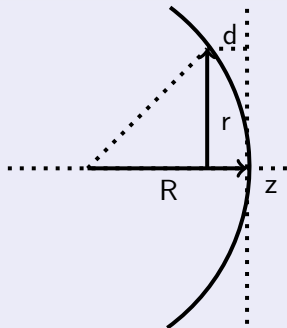
Phase at abscissa z

- Constant phase on sphere
- Phase $\propto d$, $r \ll R$
- $d = R - \sqrt{R^2 - r^2}$
- $d \approx \frac{r^2}{2R}$

Gaussian ansatz

$$u = e^{-i\left(P(z) + k \frac{r^2}{2R(z)}\right)} e^{-\frac{r^2}{W(z)^2}}$$

R radius spherical wavefront



Gaussian Beam Complex Amplitude

Where the Gaussian beam amplitude is derived from the ansatz and $q' = 1$

A quick summary

- Ansatz : $u = e^{-i\left(P(z) + \frac{k}{2q(z)}r^2\right)}$
- Complex beam radius : $\frac{1}{q} = \frac{1}{R} - i\frac{\lambda}{\pi W^2}$
- Beam radius equation : $q' = 1 \quad q(z) = q(0) + z$

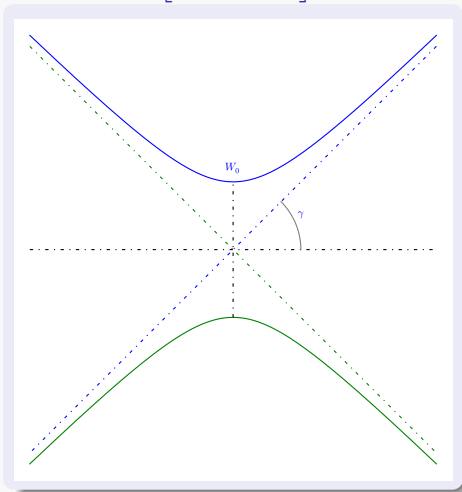
Assuming a plane wavefront for $z = 0$

- $q(0) = i\frac{\pi W_0^2}{\lambda}$
- $W^2(z) = W_0^2 \left[1 + \left(\frac{\lambda z}{\pi W_0^2} \right)^2 \right]$
- $R(z) = z \left[1 + \left(\frac{\pi W_0^2}{\lambda z} \right)^2 \right]$



Gaussian Beam Intensity

$$W^2(z) = W_0^2 \left[1 + \left(\frac{\lambda z}{\pi W_0^2} \right)^2 \right]$$



Asymptotes

- $\frac{\lambda z}{\pi W_0^2} \gg 1 \Rightarrow W(z) \approx \frac{\lambda z}{\pi W_0}$
- $\gamma = \frac{\lambda}{\pi W_0}$

W_0 : Beam Waist

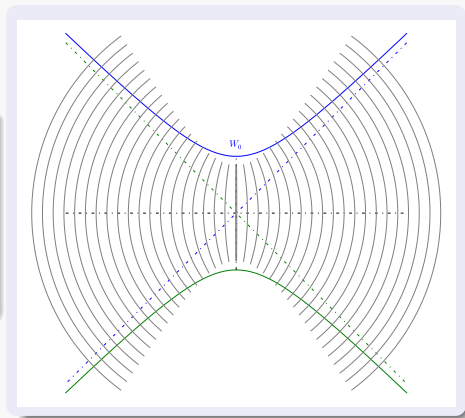


Gaussian wavefront curvature

$$R(z) = z \left[1 + \left(\frac{\pi W_0^2}{\lambda z} \right)^2 \right]$$

Plane and spherical limits

- For small z : $R = \infty$
plane wavefront
- For high z : $R \approx z$
spherical wavefront



The Rayleigh length

A quantitative criterion to decide whether a Gaussian beam is plane or spherical

Plane for small z

$$\frac{\lambda z}{\pi W_0^2} \ll 1$$

- $W(z) \approx W_0$
- $\lim_{z \rightarrow 0} R(z) = \infty$

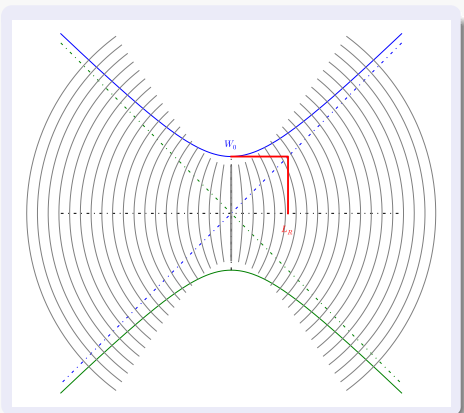
Spherical for high z

$$\frac{\lambda z}{\pi W_0^2} \gg 1$$

- $W(z) \approx \frac{\lambda z}{\pi W_0}$
- $R(z) \approx z$

The Rayleigh length is the limit

$$L_R = \frac{\pi W_0^2}{\lambda}$$



The homogeneous phase shift

$$P(z)$$

$$u = e^{-i\left(P(z) + \frac{k}{2q(z)}r^2\right)}$$

Recall the equation

$$qP' + i = 0$$

\Leftrightarrow

$$P'(z) = -\frac{i}{z + iL_R}$$

Integrate it

$$P(z) = i \ln\left(\frac{W_0}{W(z)}\right) - \tan^{-1}\left(\frac{z}{L_R}\right)$$

Complex phase meaning

- Real part : Phase shift with respect to plane wave
- Imaginary part: $\frac{W_0}{W(z)}$ factor to ensure energy conservation



The fundamental Gaussian mode

General expression

$$E(r, z) = \frac{W_0}{W(z)} e^{-i[kz + P(z)] - r^2 \left(\frac{1}{W(z)^2} + i \frac{k}{2R(z)} \right)}$$

With, in (nearly) the order of appearance on the screen

- $W^2(z) = W_0^2 \left[1 + \left(\frac{z}{L_R} \right)^2 \right]$
- $R(z) = z \left[1 + \left(\frac{L_R}{z} \right)^2 \right]$
- $P(z) = -\tan^{-1} \left(\frac{z}{L_R} \right)$
- Rayleigh length $L_R = \frac{\pi W_0^2}{\lambda}$
- Diffraction half angle: $\gamma \approx \frac{\lambda}{\pi W_0} = \frac{W_0}{L_R}$



High order Hermite-Gaussian modes

A Cartesian family of higher order modes

Ansatz

$$u(x, y, z) = g\left(\frac{x}{W(z)}\right) h\left(\frac{y}{W(z)}\right) e^{-i\left(P(z) + \frac{k}{2q(z)}(x^2 + y^2)\right)}$$

Plugged into the wave equation

- $q' = 1$
- $\exists m \in \mathbb{N}, \frac{\partial g}{\partial x^2} - 2x \frac{\partial g}{\partial x} + 2mg = 0$
- $\exists n \in \mathbb{N}, \frac{\partial h}{\partial y^2} - 2y \frac{\partial h}{\partial y} + 2nh = 0$
- $qP' + (1 + m + n)j = 0$



Behavior of Hermite-Gaussian modes

Each mode is a mere space modulation of the fundamental

$$q' = 1$$

- Same equation for q as in the fundamental mode
- $W(z)$ and $R(z)$ retain their meanings and properties
- Rayleigh length and diffraction angle are unchanged

$$\frac{\partial^2 g}{\partial x^2} - 2x \frac{\partial g}{\partial x} + 2mg = 0$$

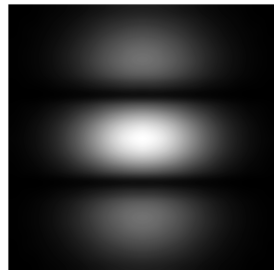
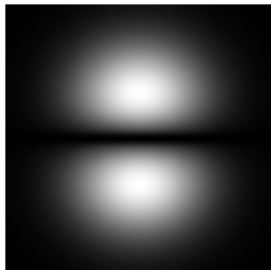
$$\frac{\partial h}{\partial y^2} - 2y \frac{\partial h}{\partial y} + 2nh = 0$$

- Solutions are, by definition, the orthogonal Hermite polynomials
- $\mathcal{H}_0 = 1$, $\mathcal{H}_1 = x$, $\mathcal{H}_2 = 4x^2 - 1$, $\mathcal{H}_3 = 8x^3 - 12x \dots$
- \mathcal{H}_n has degree n
- $g\left(\frac{x}{W(z)}\right) h\left(\frac{y}{W(z)}\right) = \mathcal{H}_m\left(\sqrt{2}\frac{x}{W(z)}\right) \mathcal{H}_n\left(\sqrt{2}\frac{y}{W(z)}\right)$



Intensity profiles of Hermite Gaussian (HG) modes

The intensity is proportional to the squared envelope



High order Laguerre-Gaussian modes

A cylindrical family of higher order modes

Ansatz

$$u(r, \phi, z) = g\left(\frac{r}{W(z)}\right) e^{-i\left(P(z) + \frac{k}{2q(z)}r^2 + l\phi\right)}$$

Plugged into the wave equation

- $q' = 1$
- $\exists (l, p) \in \mathbb{N}^2, r \frac{\partial^2 g}{\partial r^2} - (l + 1 - x) \frac{\partial g}{\partial x} + pg = 0$
- $qP' + (1 + 2p + l)j = 0$



Behavior of Laguerre-Gaussian modes

Each mode is a mere space modulation of the fundamental

$$q' = 1$$

same as HG modes

- Same equation for q as in the fundamental mode
- $W(z)$ and $R(z)$ retain their meanings and properties
- Rayleigh length and diffraction angle are unchanged

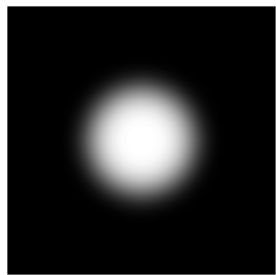
$$r \frac{\partial^2 g}{\partial r^2} - (l + 1 - x) \frac{\partial g}{\partial x} + pg = 0$$

- Solutions are, by definition, the orthogonal generalized Laguerre polynomials
- $\mathcal{L}_0^{(l)} = 1$, $\mathcal{L}_1^{(l)} = -x + l + 1$, $\mathcal{L}_2^{(l)} = \frac{x^2}{2} - (l + 2)x + \frac{(l+1)(l+2)}{2}$
- $\mathcal{L}_m^{(l)}$ has degree m
- $g\left(\frac{r}{W(z)}\right) = \left(\sqrt{2} \frac{r}{W(z)}\right)^l \mathcal{L}_p^{(l)}\left(2 \frac{r^2}{W^2(z)}\right)$



Intensity profiles of Laguerre Gaussian (LG) modes

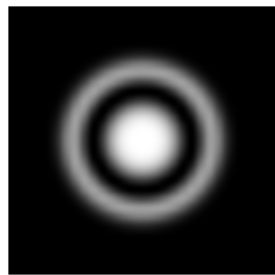
The intensity is proportional to the squared envelope



$LG(0,0)$



$LG(0,1)$: vortex



$LG(0,2)$



Intensity profiles of other Laguerre Gaussian (LG) modes

The intensity is proportional to the squared envelope



$LG(1,2)$



$LG(1,3)$



$LG(2,3)$



Homogeneous phase shift is different for high order modes

$$qP' + (1 + m + n)j = 0$$

$$qP' + (1 + 2p + l)j = 0$$

A small phase difference between modes around the beam waist

- Slightly different optical paths for different orders
- Slightly different oscillating frequencies in lasers
- Usually forgotten



Geometrical optics framework

Where is it shown that rays are not so thin as you may think

Geometrical optics *do not* deal with thin rays

- A thin ray has a thin **waist**: it should diffract $\gamma = \frac{\lambda}{\pi W_0}$
- Thin rays are **seldom alone**: their meaning is collective
- A ray is a Poynting vector curve
- A **bunch of rays** describes a **wavefront**

Do geometrical optics deal with plane and spherical waves ?

- **Parallel** rays imply a **plane wavefront**
- **Converging** or diverging rays imply a **spherical wavefront**
- But neither of them has an infinite extension !



Geometrical optic is Gaussian optics

Transversely limited plane waves

Parallel rays

Gaussian Beams **within their** Rayleigh zone

Transversely limited spherical waves

Con(Di)verging rays

Gaussian Beams **far from** their Rayleigh zone

Orders of magnitude

- He-Ne laser: $W_0 \approx 1\text{mm}$, $\lambda = 633\text{nm}$, $L_R \approx 5\text{m}$
- GSM Antenna: $W_0 \approx 1\text{m}$, $\lambda \approx 33\text{cm}$, $L_R \approx 10\text{m}$



Geometrical optics is linear

Geometrical optics stems entirely from Descartes law

$$n_1 \sin(\theta_1) = n_2 \sin(\theta_2)$$

Descartes made paraxial

Paraxial approximation : $\theta \ll 1$

$$n_1 \theta_1 \approx n_2 \theta_2$$

Geometrical optics is linear algebra

- Paraxial Descartes is linear
- Straight line propagation is linear
- The behavior of a ray through any optical system can be described linearly

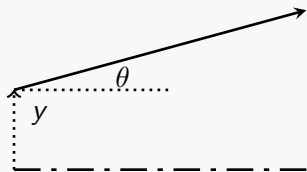


Matrix geometrical optics

A 2 dimensional linear algebra framework

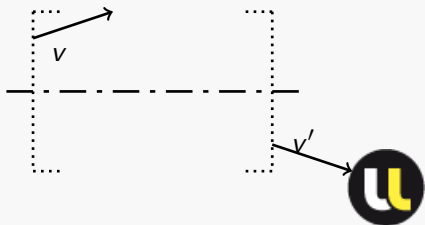
The ray vector $v = \begin{pmatrix} y \\ \theta \end{pmatrix}$

- y : distance from the axis
- θ : angle to the axis



An optical system $v' = Mv$

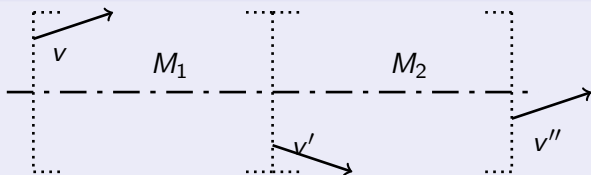
- M is a 2×2 real matrix
- It can describe any centered paraxial optical system



Optical system composition

Optical system composition reduced to matrix product

Optical System Composition



Matrix Composition

- $v' = M_1 \cdot v$
- $v'' = M_2 \cdot v'$
- $v'' = M_2 M_1 \cdot v$

Complex systems

Compose simple systems



Propagation in a homogeneous medium

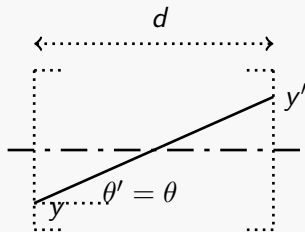
$$\begin{pmatrix} y' \\ \theta' \end{pmatrix} = M_d \begin{pmatrix} y \\ \theta \end{pmatrix}$$

Light propagates in straight line

- No direction change: $\theta' = \theta$
- $y' = y + d \sin(\theta)$

M_d

$$\begin{pmatrix} y' \\ \theta' \end{pmatrix} = \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{pmatrix} y \\ \theta \end{pmatrix}$$



Passing through a plane interface

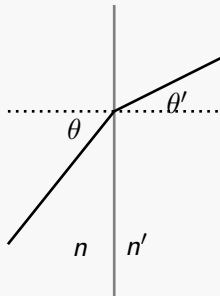
$$\begin{pmatrix} y' \\ \theta' \end{pmatrix} = M_p \begin{pmatrix} y \\ \theta \end{pmatrix}$$

Descartes

- No propagation: $y' = y$
- $n \sin(\theta) = n' \sin(\theta')$
- $\theta' \approx \frac{n}{n'} \theta$

M_p

$$\begin{pmatrix} y' \\ \theta' \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{n}{n'} \end{bmatrix} \begin{pmatrix} y \\ \theta \end{pmatrix}$$



Passing through a thin lens

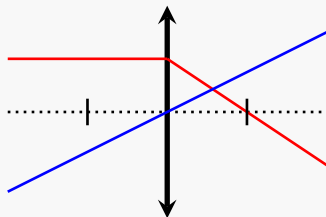
$$\begin{pmatrix} y' \\ \theta' \end{pmatrix} = M_l \begin{pmatrix} y \\ \theta \end{pmatrix}$$

Two characteristic rays

- No propagation: $y' = y$
- Blue ray: $y = 0 \Rightarrow \theta' = \theta$
- Red ray: $\theta = 0 \Rightarrow \theta' = -\frac{1}{f}y$

M_l

$$\begin{pmatrix} y' \\ \theta' \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{pmatrix} y \\ \theta \end{pmatrix}$$



Passing through a (thin) spherical interface

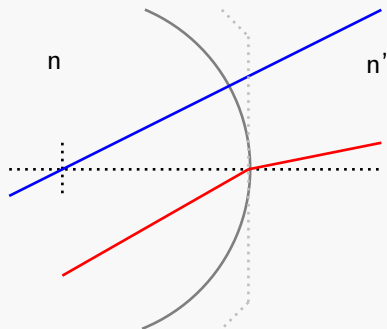
$$\begin{pmatrix} y' \\ \theta' \end{pmatrix} = M_s \begin{pmatrix} y \\ \theta \end{pmatrix}$$

Descartes

- Thin interface
No propagation: $y' = y$
- Blue ray: $y \approx R\theta \Rightarrow \theta' = \theta$
- Red ray: $y = 0 \Rightarrow \theta' \approx \frac{n}{n'}\theta$

M_s

$$\begin{pmatrix} y' \\ \theta' \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{n'-n}{n'R} & \frac{n}{n'} \end{bmatrix} \begin{pmatrix} y \\ \theta \end{pmatrix}$$

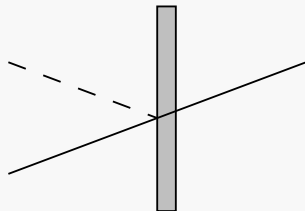


Mirrors

Unfolding the light

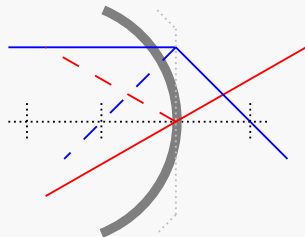
Plane mirrors as if they did not exist

$$\begin{pmatrix} y' \\ \theta' \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} y \\ \theta \end{pmatrix}$$



Spherical Mirrors are thin lenses

$$\begin{pmatrix} y' \\ \theta' \end{pmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{2}{R} & 1 \end{bmatrix} \begin{pmatrix} y \\ \theta \end{pmatrix}$$



Matrix property

A determinant property stemming from all the simple matrices determinants

n : start index

n' : stop index

$$\forall M, \det(M) = \frac{n}{n'}$$



Gaussian modes, propagation and lenses

A Gaussian mode does not change upon propagation or by passing through thin interfaces or lenses

$$g\left(\frac{x}{W(z)}\right) h\left(\frac{y}{W(z)}\right) e^{-i\left(P(z) + \frac{k}{2q(z)}(x^2 + y^2)\right)}$$

- z independent modulation of the fundamental mode
- Free space $q' = 1$ common property
- Thin lens does not change mode profile

Common $R(z)$ and $W(z)$ behavior

All the modes share the same laws on $q(z)$, $R(z)$ and $W(z)$



Gaussian beam propagation: the ABCD law

The transformation of the complex radius q for simple optical systems

Free space

$$q' = 1$$

- $q_1 = q_0 + d$
- $M_d = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$

Plane interface

$$n_0/n_1 = R_0/R_1$$

- $\frac{q_1}{q_0} = \frac{n_1}{n_0} \Rightarrow q_1 = \frac{1 \times q_0}{\frac{n_1}{n_0}}$
- $M_p = \begin{pmatrix} 1 & 0 \\ 0 & \frac{n_0}{n_1} \end{pmatrix}$

Thin lens

$$\frac{1}{R_1} = \frac{1}{R_0} - \frac{1}{f}$$

- $\frac{1}{q_1} = \frac{1}{q_0} - \frac{1}{f} \Rightarrow q_1 = \frac{1}{-\frac{1}{f}q_0 + 1}$
- $M_l = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$

Kogelnik's ABCD law

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \Rightarrow q_1 = \frac{Aq_0 + B}{Cq_0 + D}$$

Geometrical and Gaussian optics are linked through paraxial approx.

Gaussian beam propagation can be evaluated, for **any** mode, using simple matrix geometrical optics

Focal lens example

Using the ABCD law to verify that parallel rays do converge on the focal plane

Parallel input beam

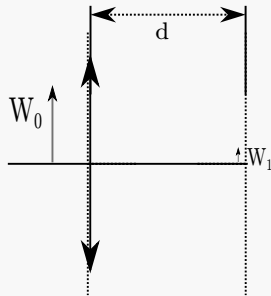
- Input plane : just before lens
- Output plane : after length d
- Input beam at waist: $q_0 = iL_{R0}$

Propagation matrix

$$M_d \cdot M_f = \begin{pmatrix} -\frac{d}{f} + 1 & d \\ -\frac{1}{f} & 1 \end{pmatrix}$$

ABCD law

$$q_1 = \frac{df + i(f - d)L_{R0}}{f - iL_{R0}}$$



d for plane wavefront: imaginary q_1

$$d = \frac{f}{1 + \left(\frac{f}{L_{R0}}\right)^2}$$

Transformation of a parallel beam by a lens

The basics of geometrical optics

As we just saw

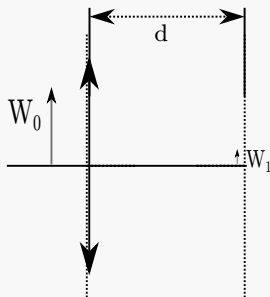
$$L_{R_0} \ll d$$

$$q_1 = \frac{df + i(f - d)L_{R_0}}{f - iL_{R_0}} \quad \& \quad d \approx f$$

Identifying W_1 in q_1

$$L_{R_0} \ll d$$

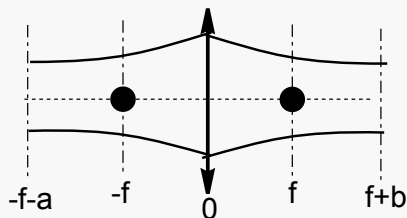
$$W_1 = \frac{\lambda|f|}{\pi W_0}$$



Transformation of a diffracting Gaussian beam by a lens

From $-f - a$ to $f + b$

- Input $q_0 = iL_{R_0}$ at $-f - a$
- Output q_1 at $f + b$
- Assume $L_{R_0} \ll a$ and $L_{R_1} \ll b$



Propagation Matrix

$$\begin{pmatrix} 1 & f+b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & f+a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -\frac{b}{f} & -\frac{ab-f^2}{f} \\ -\frac{1}{f} & -\frac{a}{f} \end{pmatrix}$$

From waist to waist

assuming q_1 imaginary

$$ab = f^2$$

$$\left(\frac{W_1}{W_0}\right)^2 = \frac{b}{a}$$