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# Tropical geometry and amoebas

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ABSTRACT. The objective of this course (ALGANT course, M2, Bordeaux, January-April 2012) was originally to present various facets of the so-called “tropical calculus”, as it appeared in various mathematical domains, including theoretical as applied ones (algebraic geometry, optimization and control theory, dynamical systems, *etc.*, see [Gaub] for a wide panorama in such applied directions). The original idea was to address students starting with a M1 background as prerequisite ; a short appendix about distribution and current theory needed then to be added since students were not familiar to such concepts. Since it was of course necessary to make a choice (for this 30 hours course !), it revealed necessary to focus on the interplay between *tropical* (hence intrinsically *ultrametric*) mathematics and the recent theory of *amæbas* and *coamæbas* in *complex* analytic geometry (toric varieties, current theory towards intersection or complex dynamics, study of differential systems such as hypergeometric systems of the Horn or Gelfand-Krapanov-Zelevinsky type), hence in the *archimedean* context. “*Deformation*” from complex to tropical became then one of the main leitmotives of this course. These notes own indeed a lot to all what I learned, as the naive non-specialist I was, from my friends and collaborators Mikael Passare (who unfortunately disappeared so brutally last September [Pas]) and August Tsikh. Both of them made me really enjoy such fascinating aspects of mathematics. These notes come as a modest tribute to Mikael’s memory. The course intended to be as self-contained as it could be. Nevertheless, it also planned to focus on a wide range of open questions ; which I indeed liked here was that most of them could be formulated in most elementary terms, that is without too much preliminary background. Many of the references I quoted here (as well as others) may often be found as preprints on the site <http://front.math.ucdavis.edu/>

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## Tropical algebraic calculus

*Tropical calculus* (also called “*max-plus*” or “*min-plus*” calculus) was originally introduced by computer scientists. The terminology “tropical” stands as a tribute to the memory of the Brazilian computer scientist and logician Imre Simon (1943-2009), who was one among the pioneers of such a theory. Tropical calculus then developed extensively in applied mathematics (control theory, optimization, more recently dynamical systems), in relation with convex analysis. In the past ten years, such algebraic calculus also incarnated in a geometric setting, and *tropical geometry* became quite an active field, in between complex or real algebraic theory (in general quite hard to deal with, either from the geometric or even topological point of view) and (much more simple) geometry and combinatorics on weighted graphs. In this chapter, one introduces various approaches towards algebraic aspects of *tropical calculus*. It will give us the opportunity to introduce the various frames in which tropical geometry as degeneration of real or complex algebraic geometry will be later on presented.

### 1.1. Various frames for tropical calculus

In this section, one introduces various frames within which *tropical calculus* can be developed.

- The first frame (subsection 1.1.1) is the “historic” one. It leads to the construction of a “classical” semiring structure on  $[-\infty, \infty[$ , that is achieved as the final state ( $h = 0$ ) in the so-called *dequantization process* of Litvinov-Maslov, deforming (from  $h > 0$ ) the (quantum) semiring structure associated to  $(\mathbb{R}_{\geq 0}, +, \times)$ , then transposing it to  $[-\infty, \infty[$  via the logarithm map  $\log$ . It leads to the “classical” semiring structure of the *tropical semiring*  $(\text{Trop}, \boxplus, \boxtimes)$  (note that  $\text{Trop} = [-\infty, \infty[$  pointsetwise).
- The second frame (subsection 1.1.2), still in the real setting, is a modification of the “classical” semiring structure on  $[-\infty, \infty[$  introduced above, in such a way that the operation of “subtraction” (which does not make sense there) can now be carried through. It is based on an idea introduced by Z. Izhakian (see for example [Izh1]). It leads to the construction of the *supertropical semiring*  $(\text{Trop}^*, \boxplus, \boxtimes)$  ( $\text{Trop}^*$  is pointsetwise the disjoint union of  $\text{Trop} = [-\infty, \infty[$  with a “ghost” copy of  $\mathbb{R}$ ). Such a frame will in fact lead, geometrically speaking, to an algebraically closed structure (on the model  $(\mathbb{C} = \mathbb{R} + i\mathbb{R}, +, \times)$ ) fits for the realization of the integral closure of  $(\mathbb{R}, +, \times)$ , where basic results from classical complex algebraic geometry over  $\mathbb{C}$  (such as Hilbert’s nullstellensatz) will later on be transposed.

- The third frame (subsection 1.1.3), which stands also in the real setting, is based on the construction on  $[-\infty, \infty[$  ( $[0, \infty[$  transposed *via* the logarithmic map) of an *hyperfield structure*, that is, roughly, a field structure in which the addition  $\boxplus$  is multivalued instead of being single valued. Such a structure will be that of the *tropical hyperfield*  $(\text{Trop}, \boxplus, \boxtimes)$ . Here again, as in the first frame presented here,  $\text{Trop} = [-\infty, \infty[$  pointsetwise.
- The two last frames introduced in subsection 1.1.4 (first in the complex setting, then in the real one after restriction to the real line) are that of the *hyperfield of complex tropical numbers*  $(\text{Trop}_{\mathbb{C}}, \boxplus_{\mathbb{C}}, \boxtimes_{\mathbb{C}})$  and of the *hyperfield of real tropical numbers*  $(\text{Trop}_{\mathbb{R}}, \boxplus_{\mathbb{R}}, \boxtimes_{\mathbb{R}})$ ; here again, the addition  $\boxplus_{\mathbb{C}}$  or  $\boxplus_{\mathbb{R}}$  is multivalued. They provide (in the complex setting) tropical companions for the complex affine or projective schemes

$$\begin{aligned} \mathbb{C}^n &= \text{Spec}(\mathbb{C}[X_1, \dots, X_n]) \\ \mathbb{T}^n &= (\mathbb{C}^*)^n = \text{Spec}(\mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]) \\ \mathbb{P}^n(\mathbb{C}) &= \text{Spec}(\text{Proj}(\mathbb{C}[X_0, \dots, X_n])), \dots, \end{aligned}$$

which are the standard frames where to do complex algebraic geometry. Such constructions are also based on a *dequantization process*, starting from  $\mathbb{C}$  or  $\mathbb{R}$  equipped with their usual operations. They were introduced by O. Viro (see [Vir]). In the complex setting (the hyperfield of complex tropical numbers  $(\text{Trop}_{\mathbb{C}}, \boxplus_{\mathbb{C}}, \boxtimes_{\mathbb{C}})$ ), this approach stands for a straightforward “tropical degeneration” of the algebraically closed field  $(\mathbb{C}, +, \times)$ ; it will be used later on to deform objects from classical complex algebraic geometry (in  $\mathbb{C}^n, \mathbb{T}^n, \mathbb{P}^n(\mathbb{C}), \dots$ ) into “degenerated” tropical objects (usually, as we will see, simpler to deal with) in  $(\text{Trop}_{\mathbb{C}})^n, (\text{Trop}_{\mathbb{C}} \setminus \{0\})^n$ , etc.

**1.1.1. The semiring**  $(\text{Trop}, \boxplus, \boxtimes)$ . On  $[-\infty, \infty[$ , let us define two internal operations, namely the *tropical addition* and the *tropical multiplication* :

$$(1.1) \quad \begin{aligned} \forall a, b \in [-\infty, \infty[, \quad a \boxplus b &:= \max(a, b) \\ \forall a, b \in [-\infty, \infty[, \quad a \boxtimes b &:= a + b. \end{aligned}$$

The role of zero is now played by  $-\infty$  since  $a \boxplus (-\infty) = (-\infty) \boxplus a = a$  for any  $a \in [-\infty, \infty[$ , while the role of unit element is played by 0 since

$$a \boxtimes 0 = 0 \boxtimes a = a + 0 = 0 + a = a \quad \forall a \in [-\infty, \infty[.$$

Tropical addition is *idempotent*, that is

$$(1.2) \quad a \boxplus a = \max(a, a) = a \quad \forall a \in [-\infty, \infty[.$$

When  $a \in \mathbb{R}$ , there cannot be then any  $b \in [-\infty, \infty[$  such that  $a \boxplus b = -\infty$ , which means that no element (besides  $-\infty$ ) admits an “inverse” respect to tropical addition. It does not either make sense to speak about “*tropical subtraction*” : the idempotency relation (1.2) can be viewed as which replaces the lack of subtraction.

Nevertheless, note that one can define *tropical division*, namely  $a \boxminus b := a - b$  when  $a \in [-\infty, \infty[$  and  $b \in \mathbb{R}$ . The tropical semiring<sup>1</sup> thus constructed is denoted as  $(\text{Trop}, \boxplus, \boxtimes)$ .

The tropical semiring  $(\text{Trop}, \boxplus, \boxtimes)$  can be understood as the result of a continuous deformation  $h \rightarrow T_h$  of the semiring  $(\mathbb{R}_{\geq 0}, +, \times)$ . More precisely, for any  $h > 0$ , let the operations  $\boxplus_h$  and  $\boxtimes_h$  on  $\mathbb{R} \cup \{-\infty\} = \text{Trop}_h$  be defined as

$$a \boxplus_h b := h \log(e^{a/h} + e^{b/h}), \quad a \boxtimes_h b := a + b.$$

This leads to the realization of the semi-ring  $(\text{Trop}_h, \boxplus_h, \boxtimes_h)$ , which is isomorphic to  $(\mathbb{R}_{\geq 0}, +, \times)$  through  $t \in \mathbb{R}_{\geq 0} \mapsto h \log t$ . When  $h > 0$  tends to 0, such a semiring  $(\text{Trop}_h, \boxplus_h, \boxtimes_h)$  degenerates into the tropical semiring  $(\text{Trop}, \boxplus, \boxtimes)$ : such a degeneracy (in the reverse way) is known in physics as the *Litvinov-Maslov quantization* of the “classical” semi-ring  $(\text{Trop}, \boxplus, \boxtimes)$  (of which  $(\text{Trop}_h, \boxplus_h, \boxtimes_h)$  for  $h > 0$  sufficiently small is a *quantum approximation*<sup>2</sup>). The degeneracy process itself is then known as the *Litvinov-Maslov dequantization* process.

**1.1.2. The supertropical semiring  $(\text{Trop}^*, \boxplus, \boxtimes)$ .** The idempotency relation (1.2) prevents indeed from defining a “subtraction” in  $(\text{Trop}, \boxplus, \boxtimes)$ . In order to overcome such a difficulty (which will be important later on to define and manipulate *algebraic tropical sets*), Z. Izhakian proposed in [Izh1] to “duplicate”  $\mathbb{R} \cup \{-\infty\}$  with a “ghost” copy  $\mathbb{R}' \cup \{-\infty\}$ , the two copies  $\mathbb{R} \cup \{-\infty\}$  and  $\mathbb{R}' \cup \{-\infty\}$  sharing the same element  $-\infty$ . The union  $\text{Trop}^*$  of such two copies is equipped with an order according to the following rules :

$$(1.3) \quad \begin{aligned} & \forall a \in \text{Trop}^*, \quad -\infty \prec a \\ & \forall a, b \in \mathbb{R}, \quad a \leq b \implies (a \prec b, a \prec b', a' \prec b, a' \prec b') \\ & \forall a \in \mathbb{R}, \quad a \prec a'. \end{aligned}$$

By convention, we extend  $a \mapsto a'$  to  $\mathbb{R}'$  setting  $(a')' = a'$  for any  $a \in \mathbb{R}$ . One defines then the *supertropical* semiring

$$(\text{Trop}^*, \boxplus, \boxtimes)$$

as the set  $(\mathbb{R} \cup \{-\infty\}) \cup (\mathbb{R}' \cup \{-\infty\})$  equipped with the “extended” tropical addition and multiplication :

$$(1.4) \quad \begin{cases} a \boxplus b := \begin{cases} \max_{\prec}(a, b) & \text{if } a \neq b \\ a' & \text{if } a = b \neq -\infty \\ -\infty & \text{if } a = b = -\infty \end{cases} & \forall a, b \in \text{Trop}^* \\ \begin{cases} (-\infty) \boxtimes a = a \boxtimes (-\infty) & \forall a \in \text{Trop}^* \\ a \boxtimes b := a + b, \quad a' \boxtimes b = a \boxtimes b' = a' \boxtimes b' := (a + b)' & \forall a, b \in \mathbb{R}. \end{cases} \end{cases}$$

The key point is that the relation  $a \boxplus a = a'$  (for any  $a \in \mathbb{R}$ ) replaces now the idempotency relation.

<sup>1</sup>One may say also sometimes “semifield” since any non-zero element  $a$  admits an inverse  $(-a)$  for the tropical multiplication. Note anyway that 0 is different from the neutral element  $(-\infty)$  with respect to tropical addition  $\boxplus$ .

<sup>2</sup>We follow here the presentation by O.A. Viro in [Vir], section 1.3.



Let also  $\pi : (\text{Trop}^*, \boxplus, \boxtimes) \rightarrow (\text{Trop}, \boxplus, \boxtimes)$  be the surjective homomorphism of semirings defined by

$$(1.5) \quad \pi(a) = \pi(a') = a \quad \forall a \in \mathbb{R}, \quad \pi(-\infty) = -\infty.$$

In  $\text{Trop}^*$ , one needs to make the distinction between two copies of  $\mathbb{R}$ : the “visible” or “tangible” one (it includes  $-\infty$ ) and the image of  $\mathbb{R}$  by  $a \mapsto a'$  (the “ghost” part of  $\text{Trop}^*$ ). The visible and ghost part realize a partition of  $\text{Trop}^*$  into two disjoint subsets. Elements of the form  $a'$  (for  $a \in \mathbb{R}$ ), which form the ghost copy, have vocation to be those which will be “ignored”.

**1.1.3. The tropical hyperfield**  $(\text{Trop}, \tilde{\boxplus}, \boxtimes)$ . On  $\mathbb{R}_{\geq 0}$ , one can define an addition  $\tilde{\boxplus}$  as a multivalued (instead of single valued) operation. This can be done in two ways, in accordance with the *triangle inequality* formulated in  $\mathbb{R}$  either in the *archimedean* ( $|a - b| \leq |a \pm b| \leq |a| + |b|$ ), or in the *ultrametric* context ( $|a + b| \leq \max(|a|, |b|)$ ). In the first case, one defines

$$t \tilde{\boxplus}_{\text{arch}} s := [|t - s|, t + s].$$

In the second case, one defines

$$t \tilde{\boxplus}_{\text{ultra}} s := \begin{cases} \max(t, s) & \text{if } t \neq s \\ [0, t] & \text{if } t = s. \end{cases}$$

Both multivalued additions  $\tilde{\boxplus}_{\text{arch}}$  and  $\tilde{\boxplus}_{\text{ultra}}$  are commutative. They also are associative in the following sense :

$$(t \tilde{\boxplus}_{s_1}) \tilde{\boxplus}_{s_2} := \bigcup_{\tau \in t \tilde{\boxplus}_{s_1}} \{\tau \tilde{\boxplus}_{s_2}\} = \bigcup_{\sigma \in s_1 \tilde{\boxplus}_{s_2}} \{t \tilde{\boxplus}_{\sigma}\} := t \tilde{\boxplus}_{(s_1 \tilde{\boxplus}_{s_2})}.$$

They both admit 0 as a neutral element, that is

$$0 \tilde{\boxplus} t = t \tilde{\boxplus} 0 = s \quad \forall t \in \mathbb{R}_{\geq 0},$$

and both are such that, for any  $t \in \mathbb{R}_{\geq 0}$ ,  $t$  is the unique element in  $\mathbb{R}_{\geq 0}$  such that

$$0 \in t \tilde{\boxplus} t \quad \forall t \in \mathbb{R}_{\geq 0}.$$

Moreover, the multiplication  $\times : (t, s) \mapsto ts$  is commutative, in both cases distributive with respect to the addition  $\tilde{\boxplus}$ . Moreover, in both cases  $(\mathbb{R}_{>0}, \times) = (\mathbb{R}_{\geq 0} \setminus \{0\}, \times)$  is a multiplicative group. Such a structure  $(\mathbb{R}_{\geq 0}, \tilde{\boxplus}, \times)$  is called an *hyperfield* structure<sup>3</sup>. When transposed to  $[-\infty, \infty[$  thanks to the logarithmic map  $\log$ , the structure  $(\mathbb{R}_{\geq 0}, \tilde{\boxplus}_{\text{ultra}}, \times)$  becomes the *tropical hyperfield structure*  $(\text{Trop}, \tilde{\boxplus}, \boxtimes)$  on  $[-\infty, \infty[$ . The price to pay to get an hyperfield structure (that is, in some “weak” sense, to be able to handle a “subtraction”) instead of the semiring structure  $(\text{Trop}, \boxplus, \boxtimes)$  as in subsection 1.1.1, is that the tropical addition is now multivalued instead of being single valued : when  $a = b$ ,  $a \tilde{\boxplus} b = [-\infty, a]$ . Note that it remains  $a = \max(a, a) = a \boxplus a \in a \tilde{\boxplus} a = [-\infty, a]$ .

<sup>3</sup>The notions of hyperring and hyperfield were introduced around 1956 by Marc Krasner [Kras]. One can find an up-to-date presentation in the abstract of course by A. Connes in Collège de France, 2009-2010 [Connes].

**1.1.4. Hyperfields of tropical complex or real numbers.** Tropical calculus cannot be straightforwardly transposed to the complex setting ( $\mathbb{C}$  instead of  $[-\infty, \infty]$ ) since there is no order in  $\mathbb{C}$ . Nevertheless, one can take into account the fact that the complex plane is equipped with an orientation, therefore carries a notion of positivity, in order to extend tropical calculus to the complex setting. The key new concept we introduce here, following O. Viro [Vir], as we already did in subsection 1.1.3 above, is that of *multivalued tropical addition* and related “hypergroup” structure : such will be the case for the tropical complex addition of two complex numbers.

As for the Litminov-Maslov dequantization process, the idea is to transform  $(\mathbb{C}, +, \times)$  into a degenerated structure. Namely, for any  $h > 0$ , denote the “addition”  $\boxplus_{\mathbb{C}, h}$  on  $\mathbb{C}$  as

$$(z, w) \mapsto z \boxplus_{\mathbb{C}, h} w := S_h^{-1}(S_h(z) + S_h(w)),$$

where  $S_h$  denotes the invertible map  $S_h : \mathbb{C} \leftrightarrow \mathbb{C}$  defined as

$$S_h(z) = \begin{cases} |z|^{1/h} \frac{z}{|z|} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0. \end{cases}$$

Keep the multiplication as  $z \boxtimes_{\mathbb{C}, h} w := z + w$  for any  $h > 0$ . When  $h > 0$  tends to  $0_+$  and  $|z| \neq |w|$ , one can check that  $\lim_{h \rightarrow 0_+} (z \boxplus_{\mathbb{C}, h} w) = z$  when  $|z| > |w|$  or  $w$  when  $|z| < |w|$ ; when  $|z| = |w|$  and  $z + w \neq 0$ , then

$$\lim_{h \rightarrow 0_+} (z \boxplus_{\mathbb{C}, h} w) = |z| \frac{z + w}{|z + w|};$$

when  $z + w = 0$ , then  $\lim_{h \rightarrow 0_+} (z \boxplus_{\mathbb{C}, h} w) = 0$ .

The limit addition

$$(z, w) \mapsto z \boxplus_{\mathbb{C}, 0} w := \lim_{h \rightarrow 0_+} (z \boxplus_{\mathbb{C}, h} w)$$

does not define a nice addition : it is not continuous (with respect to the usual topology on  $\mathbb{C}$ ) and it is not associative : for example

$$((-1) \boxplus_{\mathbb{C}, 0} i) \boxplus_{\mathbb{C}, 0} 1 = \exp(3i\pi/8) \neq (-1) \boxplus_{\mathbb{C}, 0} (i \boxplus_{\mathbb{C}, 0} 1) = \exp(5i\pi/8).$$

In order to overcome such a difficulty, one introduces again a concept of multivalued addition. The idea is that the addition  $\boxtimes_{\mathbb{C}, 0} = \boxtimes_{\mathbb{C}}$  is multivalued and that its graph  $\Gamma_{\boxtimes_{\mathbb{C}}} \subset \mathbb{C}^2 \times \mathbb{C} = \mathbb{C}^3$  is achieved as the “limit set”:

$$(1.6) \quad \left\{ (z, w, u) \in \mathbb{C}^3; \exists (h_k)_{k \geq 0}, \lim_{k \rightarrow +\infty} h_k = 0_+, \right. \\ \left. \exists (z_k, w_k, u_k) \in \Gamma_{\boxplus_{\mathbb{C}, h_k}}, \lim_{k \rightarrow +\infty} (z_k, w_k, u_k) = (z, w, u) \right\}.$$

In order to do so, one introduces the following definition.

**DEFINITION 1.1** (tropical multivalued addition of two complex numbers). Let  $z$  and  $w$  be two complex numbers. If  $|z| \neq |w|$ , the tropical sum  $z \tilde{\boxplus}_{\mathbb{C}} w$  is defined as  $z$  when  $|z| > |w|$  and  $w$  when  $|z| < |w|$ . When  $|z| = |w| = r$  and  $z + w \neq 0$ , so that  $z = re^{i\alpha}$ ,  $w = re^{i\beta}$ , with  $|\beta - \alpha| < \pi$ , one defines

$$(1.7) \quad z \tilde{\boxplus}_{\mathbb{C}} w := \left\{ ze^{i\theta}; \theta \in [\alpha, \beta] \right\}.$$

Finally

$$(1.8) \quad z \tilde{\boxplus}_{\mathbb{C}} (-z) := \overline{D(0, |z|)} \quad \forall z \in \mathbb{C}.$$

Tropical addition over the set of complex numbers thus defines a multivalued map

$$\tilde{\boxplus}_{\mathbb{C}} : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C}$$

which is commutative, associative, *i.e.*

$$(z \tilde{\boxplus}_{\mathbb{C}} w_1) \boxplus w_2 := \bigcup_{\xi \in z \tilde{\boxplus}_{\mathbb{C}} w_1} \{\xi \tilde{\boxplus}_{\mathbb{C}} w_2\} = \bigcup_{\eta \in w_1 \tilde{\boxplus}_{\mathbb{C}} w_2} \{z \tilde{\boxplus}_{\mathbb{C}} \eta\} := z \tilde{\boxplus}_{\mathbb{C}} (w_1 \tilde{\boxplus}_{\mathbb{C}} w_2),$$

admits as neutral element 0, *i.e.*

$$0 \tilde{\boxplus}_{\mathbb{C}} w = w \tilde{\boxplus}_{\mathbb{C}} 0 = w \quad \forall w \in \mathbb{C},$$

and is such that, for any  $z \in \mathbb{C}$ ,  $-z$  is the unique complex number such that

$$0 \in (z \tilde{\boxplus}_{\mathbb{C}} (-z)) = (-z) \tilde{\boxplus}_{\mathbb{C}} z.$$

The most difficult point to prove is associativity, which is just technical, so that we do not reproduce it here (see Lemma 2.B in [Vir]).

Such a multivalued operation confers to  $(\text{Trop}_{\mathbb{C}}, \tilde{\boxplus}_{\mathbb{C}})$  a structure of *commutative hypergroup* ( $\text{Trop}_{\mathbb{C}} = \mathbb{C}$  pointsetwise). Note that this operation can be restricted to the real line  $\mathbb{R}$  and defines then a structure  $(\text{Trop}_{\mathbb{R}}, \tilde{\boxplus}_{\mathbb{R}})$  of commutative hypergroup ( $\text{Trop}_{\mathbb{R}} = \mathbb{R}$  pointsetwise).

Topologically speaking, the tropical addition  $\tilde{\boxplus}_{\mathbb{C}}$  is made continuous thanks to the choice of a topology on the set  $\{0, 1\}^{\mathbb{C}} = \mathcal{P}(\mathbb{C})$  of all subsets of  $\mathbb{C}$ . One chooses here the *upper Vietoris topology*: a basis of open neighborhood of  $A \in \mathcal{P}(\mathbb{C})$  consists of all subsets  $\{0, 1\}^W \simeq \mathcal{P}(W) \subset \{0, 1\}^{\mathbb{C}} \simeq \mathcal{P}(\mathbb{C})$ ,  $W$  being any open neighborhood of  $A$  in  $\mathbb{C}$ . We leave here to the reader the proof of the continuity of  $\tilde{\boxplus}_{\mathbb{C}}$  respect to choice of the usual topology on  $\mathbb{C}^2$  and the upper Vietoris topology on  $\{0, 1\}^{\mathbb{C}} \simeq \mathcal{P}(\mathbb{C})$  (see Theorem 4.B in [Vir]). Similarly, the multivalued addition  $\tilde{\boxplus}_{\mathbb{R}}$  is continuous with respect to the choice of the usual topology on  $\mathbb{R}^2$  and the upper Vietoris topology on  $\{0, 1\}^{\mathbb{R}} \simeq \mathcal{P}(\mathbb{R})$ .

The tropical multiplication  $\boxtimes_{\mathbb{C}}$  between complex numbers is the singlevalued operation defined as

$$z \boxtimes_{\mathbb{C}} w := z + w.$$

It restricts to  $\text{Trop}_{\mathbb{R}} = \mathbb{R}$  as the tropical multiplication  $\boxtimes_{\mathbb{R}}$ . The tropical multiplication  $\boxtimes_{\mathbb{C}}$  (*resp.*  $\boxtimes_{\mathbb{R}}$ ) is commutative, associative, distributive with respect to the tropical multivalued addition  $\tilde{\boxplus}_{\mathbb{C}}$  (*resp.*  $\tilde{\boxplus}_{\mathbb{R}}$ ). Since the tropical multiplication induces a group structure on  $(\text{Trop}_{\mathbb{C}} \setminus \{0\}, \boxtimes_{\mathbb{C}})$  or  $(\text{Trop}_{\mathbb{R}} \setminus \{0\}, \boxtimes_{\mathbb{R}})$ , it confers, together with the tropical addition  $\tilde{\boxplus}_{\mathbb{C}}$  (*resp.*  $\tilde{\boxplus}_{\mathbb{R}}$ ), an *hyperfield* structure  $(\text{Trop}_{\mathbb{C}}, \tilde{\boxplus}_{\mathbb{C}}, \boxtimes_{\mathbb{C}})$  (*resp.*  $(\text{Trop}_{\mathbb{R}}, \tilde{\boxplus}_{\mathbb{R}}, \boxtimes_{\mathbb{R}})$ ) on  $\mathbb{C} = \text{Trop}_{\mathbb{C}}$  (*resp.* on  $\mathbb{R} = \text{Trop}_{\mathbb{R}}$ ).

The hyperfield  $(\text{Trop}_{\mathbb{C}}, \tilde{\boxplus}_{\mathbb{C}}, \boxtimes_{\mathbb{C}})$  is called *hyperfield of tropical complex numbers*, while the hyperfield  $(\text{Trop}_{\mathbb{R}}, \tilde{\boxplus}_{\mathbb{R}}, \boxtimes_{\mathbb{R}})$  is called *hyperfield of tropical real numbers*.

The complex conjugaison  $z \rightarrow \bar{z}$  realizes an automorphism  $\varphi$  of hyperfields from  $(\text{Trop}_{\mathbb{C}}, \tilde{\boxplus}_{\mathbb{C}}, \boxtimes_{\mathbb{C}})$  into itself (*i.e.*  $\varphi(z \boxtimes_{\mathbb{C}} w) \subset \varphi(z) \boxtimes_{\mathbb{C}} \varphi(w)$  and  $\varphi(z \tilde{\boxplus}_{\mathbb{C}} w) = \varphi(z) \tilde{\boxplus}_{\mathbb{C}} \varphi(w)$ ).

## 1.2. $(\text{Trop}, \boxplus, \boxtimes)$ -polynomial objects in $n$ variables

**1.2.1.  $(\text{Trop}, \boxplus, \boxtimes)$ -polynomials and polynomial functions.** One introduces in this subsection polynomial objects within the  $(\text{Trop}, \boxplus, \boxtimes)$  frame introduced in subsection 1.1.1. A  $(\text{Trop}, \boxplus, \boxtimes)$ -polynomial function  $p$  in  $n$  real variables

is (by definition) a sum of tropical monomial functions on  $\text{Trop}^n$ , of the form

$$(1.9) \quad p : x = (x_1, \dots, x_n) \in \text{Trop}^n = [-\infty, \infty[^n \mapsto \boxplus_{\alpha \in \mathbb{N}^n} (a_\alpha \boxtimes x_1^{\boxtimes \alpha_1} \boxtimes \dots \boxtimes x_n^{\boxtimes \alpha_n}),$$

where the  $a_\alpha$ ,  $\alpha \in \mathbb{N}^n$  are elements in  $\text{Trop}$  such that all  $a_\alpha$ , but a finite number of them, equal  $-\infty$ .

One should be careful here not to make a confusion between such a “functional” notion and that of (Trop,  $\boxplus$ ,  $\boxtimes$ )-*polynomial* : a (Trop,  $\boxplus$ ,  $\boxtimes$ )-*polynomial*  $\mathfrak{p}$  in  $n$  variables is a formal sum

$$(1.10) \quad \mathfrak{p} := \boxplus_{\alpha \in \mathbb{N}^n} (a_\alpha \boxtimes x_1^{\boxtimes \alpha_1} \boxtimes \dots \boxtimes x_n^{\boxtimes \alpha_n}),$$

where  $(a_\alpha)_{\alpha \in \mathbb{N}^n}$  denotes a collection of elements in  $[-\infty, \infty[$  such that all  $a_\alpha$ , but a finite number, equal  $-\infty$ . As we will see in subsection 1.2.3, the expression (1.9) for a (Trop,  $\boxplus$ ,  $\boxtimes$ )-polynomial function in terms of the  $\alpha \in \mathbb{N}^n$  and  $a_\alpha \in [-\infty, \infty[$  is not unique (which makes a crucial difference with the situation for polynomial functions respect to the standard operations  $(+, \times)$  on  $\mathbb{R}^n$ ), though there exists (in some sense) some “maximal” choice ( $a_\alpha$  being maximal respect to the usual order on  $[-\infty, \infty[$  for each index  $\alpha$ ). Therefore, it is not possible to retrieve the coefficients  $a_\alpha$  from the values of the (Trop,  $\boxplus$ ,  $\boxtimes$ )-polynomial function  $p$ . For example, if  $n = 1$  and  $a, b, c$  are three real numbers such that  $a \leq (b + c)/2$ , then, for any  $k \in \mathbb{N}^*$ ,

$$a + kx \leq \frac{1}{2}((b + c + (k - 1)x + (k + 1)x) \leq \max(b + (k - 1)x, c + (k + 1)x),$$

so that, for any  $x \in \mathbb{R}$ ,

$$\max(b + (k - 1)x, a + kx, c + (k + 1)x) = \max(b + (k - 1)x, (-\infty) + kx, c + (k + 1)x),$$

which proves that the two (Trop,  $\boxplus$ ,  $\boxtimes$ )-polynomials

$$(b \boxtimes x^{\boxtimes k-1}) \boxplus ((-\infty) \boxtimes x^{\boxtimes k}) \boxplus (c \boxtimes x^{\boxtimes k+1}) \quad \& \quad a \boxtimes x^{\boxtimes k}$$

are evaluated as the same (Trop,  $\boxplus$ ,  $\boxtimes$ )-polynomial function on (Trop,  $\boxplus$ ,  $\boxtimes$ ). This indeed emphasizes the fact that the notions of (Trop,  $\boxplus$ ,  $\boxtimes$ )-*polynomial function* and (Trop,  $\boxplus$ ,  $\boxtimes$ )-*polynomial* are of different nature. The (Trop,  $\boxplus$ ,  $\boxtimes$ )-polynomial function  $p$  in (1.9) appears as the “evaluation” of the (Trop,  $\boxplus$ ,  $\boxtimes$ )-polynomial  $\mathfrak{p}$  in (1.10) on  $\text{Trop}^n$ . The evaluation  $p$  (as in (1.9)) of the (Trop,  $\boxplus$ ,  $\boxtimes$ )-polynomial (1.10) on  $\text{Trop}^n$  does not provide enough information to be able to retrieve from it the list of the coefficients  $a_\alpha$ ,  $\alpha \in \mathbb{N}^n$ , of  $\mathfrak{p}$ , which is not really a surprise : for example, the evaluation<sup>4</sup> of  $\mathfrak{p} = X^3 - 3X^2 + 2X$  (considered as a formal expression with real coefficients) on  $\mathbb{F}_3$  gives zero, thus does not provide any information on the whole list of coefficients  $[1 \ -3 \ 2 \ 0]$  of  $\mathfrak{p}$ .

If one expresses the tropical operations  $\boxplus$  and  $\boxtimes$  in terms of their definitions (1.1), one can see immediately that the (Trop,  $\boxplus$ ,  $\boxtimes$ )-polynomial function defined as (1.9) is the convex function

$$(1.11) \quad x = (x_1, \dots, x_n) \in [-\infty, \infty[^n \mapsto p(x) = \max_{\alpha \in \mathbb{N}^n} (a_\alpha + \langle \alpha, x \rangle).$$

Such a representation allows to extend the notion of (Trop,  $\boxplus$ ,  $\boxtimes$ )-polynomial function to that of *Laurent* (Trop,  $\boxplus$ ,  $\boxtimes$ )-*polynomial function*, allowing the “multi-exponents”  $\alpha$  in (1.9) to belong to  $\mathbb{Z}^n$  instead of  $\mathbb{N}^n$ . Of course, one needs now to

<sup>4</sup>I quoted such a illuminating example from the presentation of C. Kisman in [Kis].

restrict the evaluation to  $\mathbb{R}^n = (\text{Trop} \setminus \{-\infty\})^n$  instead of  $\text{Trop}^n$ . One will from now on consider  $(\text{Trop}, \boxplus, \boxtimes)$ -polynomial functions, when they are restricted to  $(\text{Trop} \setminus \{-\infty\})^n = \mathbb{R}^n$ , as Laurent  $(\text{Trop}, \boxplus, \boxtimes)$ -polynomial functions

$$(1.12) \quad x = (x_1, \dots, x_n) \in (\text{Trop} \setminus \{-\infty\})^n = \mathbb{R}^n \mapsto p(x) = \max_{\alpha \in \mathbb{Z}^n} (a_\alpha + \langle \alpha, x \rangle),$$

all  $a_\alpha$ , but a finite number, being equal to  $-\infty$ . As for tropical polynomial functions and tropical polynomials, one should be careful here to distinguish between Laurent polynomial tropical functions and *Laurent*  $(\text{Trop}, \boxplus, \boxtimes)$ -polynomials. A *Laurent*  $(\text{Trop}, \boxplus, \boxtimes)$ -polynomial is (by definition) a formal sum

$$(1.13) \quad \mathbf{p} := \boxplus_{\alpha \in \mathbb{Z}^n} (a_\alpha \boxtimes x_1^{\boxtimes \alpha_1} \boxtimes \dots \boxtimes x_n^{\boxtimes \alpha_n}),$$

where  $(a_\alpha)_{\alpha \in \mathbb{Z}^n}$  denotes a collection of elements in  $[-\infty, \infty[$  such that all  $a_\alpha$ , but a finite number, equal  $-\infty$ . Here again, the “evaluation” of the Laurent  $(\text{Trop}, \boxplus, \boxtimes)$ -polynomial  $\mathbf{p}$  on  $(\text{Trop} \setminus \{-\infty\})^n = \mathbb{R}^n$  does not allow in general the possibility to retrieve the coefficients  $a_\alpha$  involved in the expression (1.13) of  $\mathbf{p}$ .

**1.2.2. Tropical hypersurface defined by a Laurent  $(\text{Trop}, \boxplus, \boxtimes)$ -polynomial.** In complex algebraic geometry, the algebraic hypersurface defined by a polynomial function  $P \in \mathbb{C}[X_1, \dots, X_n]$  in the affine scheme

$$\mathbb{A}^n := \text{Spec}(\mathbb{C}[X_1, \dots, X_n])$$

is (from the pointset point of view) the zero set

$$P^{-1}(0) = V_{\mathbb{C}}(P) = \{\zeta \in \mathbb{C}^n ; P(\zeta) = 0\}.$$

If  $P \in \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$  is a Laurent polynomial (thus defining a Laurent polynomial function on  $\mathbb{T}^n = (\mathbb{C}^*)^n = \text{Spec}(\mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}])$ ), the algebraic hypersurface defined by  $P$  is the zero set  $V_{\mathbb{T}}(P) = \{\zeta \in \mathbb{T}^n ; P(\zeta) = 0\}$  in the affine scheme  $\mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ .

Such a point of view cannot be carried to the tropical real setting  $(\text{Trop}, \boxplus, \boxtimes)$ . Let indeed  $p$  be a Laurent  $(\text{Trop}, \boxplus, \boxtimes)$ -polynomial function on  $(\text{Trop} \setminus \{-\infty\})^n$ ,  $\boxplus, \boxtimes$  such as (1.12). There is no  $x \in \mathbb{R}^n$  such that  $p(x) = -\infty$ , except if all  $a_\alpha$  in (1.12) equal  $-\infty$ , in which case  $p \equiv -\infty$ , thus  $V_{\text{trop}}(p) := \{p = -\infty\} = \mathbb{R}^n$ . So, one needs to propose an alternate definition for the *tropical hypersurface*  $V_{\text{trop}}(p) \subset \mathbb{R}^n = (\text{Trop} \setminus \{-\infty\})^n$  defined by a Laurent  $(\text{Trop}, \boxplus, \boxtimes)$ -polynomial function.

**DEFINITION 1.2** (tropical hypersurface attached to a Laurent tropical polynomial function). Let  $p$  be a Laurent tropical polynomial function such as (1.12). The *tropical hypersurface*  $V_{\text{trop}}(p)$  defined by  $p$  is set of critical points (in  $\mathbb{R}^n$ ) of the convex function  $x \in \mathbb{R}^n \mapsto p(x)$  in (1.12), that is the subset of points in  $\mathbb{R}^n$  where at least two different affine functions  $a_{\alpha_1} + \langle \alpha_1, \cdot \rangle$  and  $a_{\alpha_2} + \langle \alpha_2, \cdot \rangle$  involved in the right-hand side of (1.12) (among those that are not identically equal to  $-\infty$ ) coincide. The definition of  $V_{\text{trop}}(p)$  does not of course depend on the representation of  $p$  as (1.9). If  $\mathbf{p}$  is a  $(\text{Trop}, \boxplus, \boxtimes)$ -Laurent polynomial, one defines  $V_{\text{trop}}(\mathbf{p})$  as  $V_{\text{trop}}(p)$ ,  $p$  being the evaluation of  $\mathbf{p}$  on  $(\text{Trop} \setminus \{-\infty\})^n$ .

**REMARK 1.3.** Note that, if  $p$  is the Laurent  $(\text{Trop}, \boxplus, \boxtimes)$ -monomial function

$$x \in (\text{Trop} \setminus \{-\infty\})^n = \mathbb{R}^n \mapsto a + \langle \alpha, x \rangle,$$

then  $V_{\text{trop}}(p) = \emptyset$  (following definition 1.2). This can be viewed in accordance with the fact that  $V_{\mathbb{T}}(P) = \{z \in \mathbb{T}^n ; P(z) = 0\} = \emptyset$  when  $P$  is a Laurent monomial in

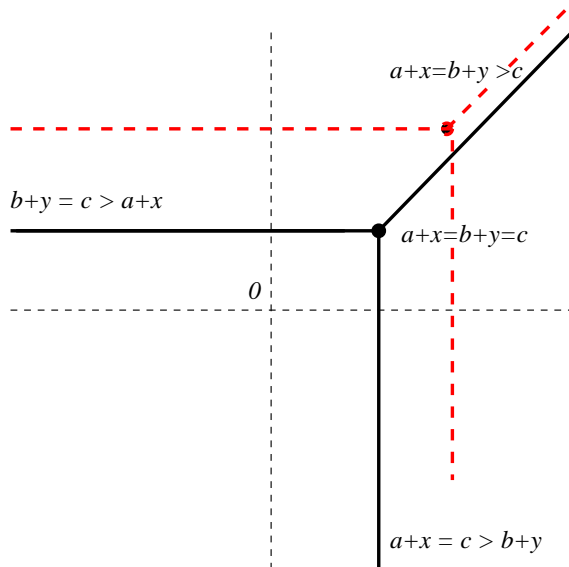


FIGURE 1.1. Tropical lines in  $\text{Trop}^2$

$X_1^{\pm 1}, \dots, X_n^{\pm n}$ . A more pertinent justification for Definition 1.2 will be given later on (Remark 1.13, section 1.3.2).

EXAMPLE 1.4 (tropical lines in  $(\text{Trop} \setminus \{-\infty\})^2$ ). Let  $a, b, c$  three real numbers (considered tropically, that is in  $\text{Trop} \setminus \{-\infty\} = \mathbb{R}$ ) and  $\mathfrak{l}_{a,b,c}$  the  $(\text{Trop}, \boxplus, \boxtimes)$ -polynomial

$$\mathfrak{l}_{a,b,c} = (a \boxtimes x) \boxplus (b \boxtimes y) \boxplus c$$

(considered here as a Laurent  $(\text{Trop}, \boxplus, \boxtimes)$ -polynomial). One has

$$p(x, y) = \max(a + x, b + y, c).$$

Therefore, one can figure  $V_{\text{trop}}(\mathfrak{l}_{a,b,c})$  as on figure 1.1. Note that two generic tropical lines intersect at exactly one point in  $\text{Trop}^2$  (this is a particular case of the tropical version of Bézout theorem that will be stated later in this course) ; the non generic situation reduces to the generic one modulo a perturbation argument (move slightly one of the two tropical lines in this case).

**1.2.3. The Legendre-Fenchel transform : a “tropical” pendant of the Fourier transform.** Let  $f$  be an arbitrary function from  $\mathbb{R}^n = (\text{Trop} \setminus \{-\infty\})^n$  to  $[-\infty, \infty]$  ; its *Legendre-Fenchel transform* is the function defined in another copy of  $\mathbb{R}^n$  (a “dual” copy) as

$$\xi \in \mathbb{R}^n \mapsto \check{f}(\xi) := \sup_{x \in \mathbb{R}^n} (\langle \xi, x \rangle - f(x)) \in [-\infty, \infty].$$

When  $f \leq g$  on  $\mathbb{R}^n$ , one has  $\check{f} \geq \check{g}$ , thus  $\check{\check{f}} \leq \check{\check{g}}$ . If  $g$  is convex, lower semicontinuous<sup>5</sup> and never takes the value  $-\infty$  (or is identically  $-\infty$ ), then  $\check{\check{g}} = g$ . This implies that, for any function  $f$  from  $\mathbb{R}^n$  to  $[-\infty, \infty]$ ,  $\check{\check{f}} \leq f$  on  $\mathbb{R}^n$ , the equality holding if and only if  $f$  is convex, lower semicontinuous, never takes the value  $-\infty$  or equals

<sup>5</sup>*i.e.* the set  $\{f > \alpha\}$  is open for any  $\alpha$ .

identically  $-\infty$ . For basic facts related to the Legendre-Fenchel transform and its application to optimization (*e.g.* Hamilton-Jacobi equations), see for example [YW], pp. 253–258. It will play a very important role all through this course, as a key companion to the concept of convexity, as well as an efficient tool towards an “analytic” incarnation of duality (as the Fourier transform is also).

The Legendre-Fenchel transform plays an important role respect to the “maximal” representation of a tropical polynomial. Namely, one has the following Proposition.

**PROPOSITION 1.1** (“maximal” representation of Laurent (Trop,  $\boxplus$ ,  $\boxtimes$ )-polynomial functions). *Let  $p : \mathbb{R}^n \mapsto \mathbb{R}$  be a non identically  $-\infty$  Laurent (Trop,  $\boxplus$ ,  $\boxtimes$ )-polynomial function, represented as (1.12). Then  $p$  admits also the representation*

$$(1.14) \quad p(x) = \sup_{\alpha \in \mathbb{Z}^n} (-\check{p}(\alpha) + \langle \alpha, x \rangle).$$

Moreover, the representation (1.14) stands for the representation

$$p(x) = \sup_{\alpha \in \mathbb{Z}^n} (b_{\alpha, \max} + \langle \alpha, x \rangle),$$

where, for each  $\alpha_0 \in \mathbb{Z}^n$ ,  $b_{\alpha_0, \max}$  is maximal among all  $b_{\alpha_0}$  involved in any representation

$$p(x) = \sup_{\alpha \in \mathbb{Z}^n} (b_{\alpha} + \langle \alpha, x \rangle)$$

of  $p$  in  $\mathbb{R}^n$  as a Laurent (Trop,  $\boxplus$ ,  $\boxtimes$ )-polynomial function.

**REMARK 1.5.** The representation formula (1.14) can be expressed in terms of tropical operations  $\boxplus$  and  $\boxtimes$  as

$$\forall x \in \mathbb{R}^n, \quad p(x) = \boxplus_{\alpha \in \mathbb{Z}^n} (-\check{p}(\alpha)) \boxtimes x^{\boxtimes \alpha},$$

where  $x \mapsto x^{\boxtimes \alpha}$  stands for the linear map  $x \mapsto \langle \alpha, x \rangle$ . Thus  $-\check{p}(\alpha)$ ,  $\alpha \in \mathbb{Z}^n$ , plays (respect to tropical operations) the role played by the Fourier coefficient  $\hat{f}(\alpha)$  when  $f(\theta) = P(e^{i\theta_1}, \dots, e^{i\theta_n}) = \sum_{\alpha \in \mathbb{Z}^n} \hat{f}(\alpha) e^{i\langle \alpha, \theta \rangle}$ ,  $P$  being a Laurent polynomial in  $\mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ . Formula  $\check{p} = p$  (which is valid in this case since  $p$  is convex and never takes the value  $-\infty$ ) stands thus for an analog of Fourier inversion formula. The Legendre-Fenchel transform thus appears in the “tropical” world  $(\text{Trop} \setminus \{-\infty\})^n$  as the tropical pendant of the Fourier transform in the “real” worlds  $\mathbb{R}^n$  or (better)  $\mathbb{T}^n$ .

**PROOF.** The proof we reproduce here was suggested by A. Rashkovskii and presented in this form in [Kis]. Let

$$(1.15) \quad p(x) = \sup_{\alpha \in \mathbb{Z}^n} (b_{\alpha} + \langle \alpha, x \rangle)$$

be a representation for  $p$ . Let  $f$  be the function from  $\mathbb{R}^n$  to  $[-\infty, \infty]$  defined by

$$f(x) = \begin{cases} +\infty & \text{if } x \in \mathbb{R}^n \setminus \mathbb{Z}^n, \\ -b_x & \text{if } x \in \mathbb{Z}^n \end{cases}$$

so that  $p$  is the Legendre-Fenchel transform of  $f$  thanks to the representation (1.15). Since the third Legendre-Fenchel transform of an arbitrary function  $f$  from  $\mathbb{R}^n$  to  $[-\infty, \infty]$  always equal the first, one has  $\check{p} = \check{f}$  and  $\check{\check{p}} = \check{\check{f}} = p$ . One has also, for any  $x \in \mathbb{R}^n$ ,

$$p(x) = \sup_{\xi \in \mathbb{R}^n} (\langle x, \xi \rangle - \check{p}(\xi)) = \sup_{\alpha \in \mathbb{Z}^n} (\langle x, \alpha \rangle - \check{p}(\alpha))$$

(it is indeed enough to take the supremum over  $\alpha \in \mathbb{Z}^n$  instead of  $\alpha \in \mathbb{R}^n$  since  $p$  is a Laurent (Trop,  $\boxplus$ ,  $\boxtimes$ )-polynomial function), which implies that  $p$  admits the representation formula (1.14). For any  $\alpha_0 \in \mathbb{Z}^n$ , one has

$$-\check{p}(\alpha_0) = -\check{f}(\alpha_0) \geq -f(\alpha_0) = b_{\alpha_0}.$$

This concludes the proof of Proposition 1.1.  $\square$

### 1.3. Tropicalisation with respect to a valuation

The point of view which is presented in that section is that proposed by M. M. Kapranov [Krap] in 2000 (see also [EKL] or [It1], [It2], section 5). It will give us the opportunity (see Remark 1.13) to justify Definition 1.2.

Let  $\mathbb{K}$  be an algebraically closed commutative field. A *valuation*<sup>6</sup> on  $(\mathbb{K}, +, \times)$  (with rank 1, which will be implicit all along this section) is a map

$$\nu : \mathbb{K} \rightarrow \text{Trop}$$

such that

$$(1.16) \quad \begin{aligned} \forall x, y \in \mathbb{K}, \quad \nu(x \times y) &= \nu(x) \boxtimes \nu(y), \quad \nu(x + y) \leq \nu(x) \boxplus \nu(y) \\ \nu(x) = -\infty &\iff x = 0. \end{aligned}$$

Note that  $\nu(1) = 2\nu(-1) = 0$ , which implies  $\nu(1) = \nu(-1) = 0$ . The valuation is said to be *non trivial* when the valuation group (or *value group*  $\Gamma_\nu := \nu(\mathbb{K}^*)$ ) is a dense subgroup of  $\mathbb{R}$  (such a subgroup is known to be either dense or discrete). Thanks to a normalization, one assumes here that  $1 \in \Gamma_\nu$ , so that there exists  $t \in \mathbb{K}^*$  with  $-\nu(t) = 1$  ( $t$  has “order” equal to one). One denotes  $-\nu : \mathbb{K} \rightarrow ]-\infty, \infty]$  as the *order*. The local ring  $R_{\mathbb{K}, -\nu}$  defined as

$$R_{\mathbb{K}, -\nu} := \{x \in \mathbb{K}; -\nu(x) \geq 0\}$$

is called the *valuation ring* associated with the order  $-\nu$ . Its maximal ideal is

$$\mathfrak{M}_{\mathbb{K}, -\nu} := \{x \in \mathbb{K}; -\nu(x) > 0\}$$

and the residue field  $k_{-\nu} = R_{\mathbb{K}, -\nu}/\mathfrak{M}_{\mathbb{K}, -\nu}$  is also algebraically closed. When we speak about a valuation  $\nu$  all over this section, it will always be assumed to be non trivial.

**EXAMPLE 1.6** (ultrametric absolute value). If  $|\cdot|_u$  is an *ultrametric* (or *non-archimedean*) non trivial absolute value on  $\mathbb{K}$ , then the map

$$x \mapsto \log |x|_u \in [-\infty, \infty[ = \text{Trop}$$

defines a (non trivial) valuation, since the absolute value obeys the two rules  $|xy|_u = |x|_u \times |y|_u$  and  $|x + y|_u \leq \max(|x|_u, |y|_u)$ . The example of  $\mathbb{K} = \mathbb{C}_p$  (the completion of the integral closure of  $\mathbb{Q}_p$  with respect to the metric induced by the  $p$ -adic norm  $|x|_p := p^{-\nu_p(x)}$ ) provides an illustration, the valuation being in this case  $-\nu_p$  (note the presence of the factor  $(-1)$ , see also Example 1.7 below). In fact, there is a bijective correspondence between the set of valuations on  $\mathbb{K}$  and that of ultrametric absolute values on  $\mathbb{K}$  (to an ultrametric absolute value  $|\cdot|_u$ , one associates the

<sup>6</sup>The definition one proposes here differs up to a sign from the usual one :  $-\nu$  is a valuation in the classical sense when  $\nu$  is a valuation in our sense.



valuation  $\log | \cdot |_u$ ). Non trivial valuations correspond to non trivial ultrametric absolute values. Note that, if  $| \cdot |_u$  is an ultrametric absolute value on  $\mathbb{K}$ , then

$$| \cdot |_u : (\mathbb{K}, +) \rightarrow (\text{Trop}_{\mathbb{R}}, \tilde{\boxplus}_{\mathbb{R}})$$

defines an homomorphism of hypergroups, that is

$$|x + y|_u \in |x|_u \tilde{\boxplus}_{\mathbb{R}} |y|_u \quad \forall x, y \in \mathbb{K}.$$

EXAMPLE 1.7 (Puiseux theory, *minus order* valuation). Let  $t$  be a transcendental parameter over  $\mathbb{C}$  and  $\mathbb{K} := \overline{\mathbb{C}((t))}$  be the algebraic closure of the quotient field  $\mathbb{C}((t))$  of the ring  $\mathbb{C}[[t]]$  of formal power series in  $t$  with complex coefficients. By Puiseux theory, any element  $x$  in  $\mathbb{K}$  has a Puiseux expansion

$$x = c_1(x)t^{q_1(x)} + c_2(x)t^{q_2(x)} + \dots,$$

where  $q_1(x) < q_2(x) < \dots$  are rational numbers and  $c_1(x), c_2(x), \dots$  are complex coefficients. The field of Puiseux series over  $\mathbb{C}$ , which elements are formal sums  $\sum_{k > k_0} c_k t^{k/p}$ ,  $p \in \mathbb{N}^*$ ,  $k_0 \in \mathbb{Z}$ ,  $c_k \in \mathbb{C}$ , is the algebraic closure of  $\mathbb{C}((t))$  (see for example Corollary 3.15 in [Eis]). The map

$$\nu : x \in \mathbb{K} \rightarrow -q_1(x) \in \text{Trop} \quad (\nu(x) = -\infty \iff x = 0)$$

satisfies  $\nu(x \times y) = \nu(x) \boxtimes \nu(y)$  and  $\nu(x + y) \leq \max(\nu(x), \nu(y))$  since one has  $\min(q_1(x), q_1(y)) \leq q_1(x + y)$  (taking  $q_1(0) = +\infty$ ) for any  $x, y \in \mathbb{K}$ . Such a map  $\nu$  defines a non trivial valuation on  $\mathbb{K}$ , which is called the *minus order map* (its opposite being called the *order map*).

Given such an algebraically commutative field  $\mathbb{K}$  equipped with a (non trivial) valuation (think about  $\mathbb{K} = \overline{\mathbb{C}(t)}$  or  $\mathbb{K} = \overline{\mathbb{C}((t))}$  and  $\nu$  equal minus the *order map* as in example 1.7), one can tropicalise regular functions on  $\text{Spec}(\mathbb{K}[X_1^{\pm 1}, \dots, X_n^{\pm 1}])$  (that is Laurent polynomials in  $n$  variables with coefficients in  $\mathbb{K}$ ) and therefore attach to any closed algebraic subset in  $(\mathbb{K}^*)^n$  a “tropical” companion in  $(\text{Trop} \setminus \{-\infty\})^n$ . Such “tropicalisation” with respect to a given valuation is presented in sections 1.3.1 to 1.3.3 below.

### 1.3.1. Tropicalisation of a Laurent polynomial with respect to a valuation.

DEFINITION 1.8 (tropicalisation of a Laurent polynomial with respect to a valuation). Consider an algebraically closed field  $(\mathbb{K}, +, \times)$  equipped with a non trivial valuation  $\nu$  and

$$(1.17) \quad P = \sum_{\alpha \in \text{Supp}(P)} c_{\alpha} X^{\alpha} \in \mathbb{K}[X_1^{\pm 1}, \dots, X_n^{\pm 1}].$$

The *tropicalisation of the Laurent polynomial  $P$  with respect to the valuation  $\nu$*  is the Laurent  $(\mathbb{T}, \boxplus, \boxtimes)$ -polynomial

$$\mathfrak{p} = \text{trop}_{\nu}(P) = \boxplus_{\alpha \in \mathbb{Z}^n} (a_{\alpha}(P) \boxtimes x_1^{\boxtimes \alpha_1} \boxtimes \dots \boxtimes x_n^{\boxtimes \alpha_n}),$$

where  $a_{\alpha}(P) = -\infty$  when  $\alpha$  does not belong to the support<sup>7</sup> of the Laurent polynomial  $P$  and  $a_{\alpha}(P) = \nu(c_{\alpha})$  for any  $\alpha \in \text{Supp}(P)$ . The *tropicalisation of the Laurent polynomial map  $P : (\mathbb{K}^*)^n \rightarrow \mathbb{K}$  with respect to the valuation  $\nu$*  is the

<sup>7</sup>The *support* of a Laurent polynomial  $\sum_{\alpha} c_{\alpha} X^{\alpha}$  denotes the finite subset  $\{\alpha \in \mathbb{Z}^n ; c_{\alpha} \neq 0\}$  of the lattice  $\mathbb{Z}^n$ .

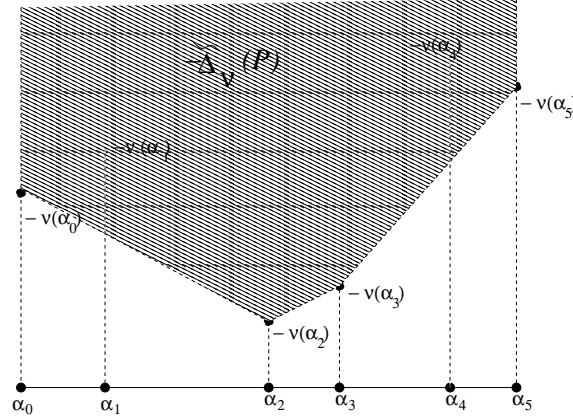


FIGURE 1.2. Newton-Puiseux diagram and symmetric (with respect to the horizontal axis) of  $\tilde{\Delta}_\nu(P)$  (when  $n = 1$ ).

Laurent (Trop,  $\boxplus$ ,  $\boxtimes$ )-polynomial function  $p$  corresponding to the “evaluation” of  $\mathfrak{p} = \text{trop}_\nu(P)$  on  $(\text{Trop} \setminus \{-\infty\})^n$ .

Let  $P \in \mathbb{K}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$  as in (1.17). Two objects of combinatoric nature are related, one to  $P$ , one to its tropicalisation  $\mathfrak{p} = \text{trop}_\nu(P)$  with respect to a given valuation  $\nu$ .

- The *Newton polyhedron*  $\Delta(P) \subset \mathbb{R}^n$  is the compact polyhedron defined in  $\mathbb{R}^n$  as

$$(1.18) \quad \Delta(P) := \overline{\text{conv}(\text{Supp}(P))} \subset \mathbb{R}^n,$$

that is the closed convex envelope of the support of the Laurent polynomial  $P$ . It is only related to the Laurent polynomial  $P$  itself.

- The *roof* of  $P$  (with respect to a valuation  $\nu$  on  $\mathbb{K}$ ) is a closed unbounded polyhedron in  $\mathbb{R}^{n+1}$  related in fact to the tropicalisation  $\mathfrak{p} = \text{trop}_\nu(P)$  of  $P$  with respect to the valuation  $\nu$ . It is sometimes also called the *extended Newton polyhedron* of  $P$  with respect to the valuation  $\nu$  and defined as :

$$(1.19) \quad \tilde{\Delta}_\nu(P) := \overline{\text{conv}\{(\alpha, b); \alpha \in \text{Supp}(P), b \in \mathbb{R}, b \leq \nu(c_\alpha)\}} \subset \mathbb{R}^{n+1}.$$

EXAMPLE 1.9 (Newton polyhedron and roof in the one dimensional case). Let  $P \in \mathbb{K}[X, X^{-1}]$  be a Laurent polynomial that is not a monomial, which can be expressed as

$$P(X) = \sum_{k=0}^m c_{\alpha_k} X^{\alpha_k} = c_{\alpha_0} X^{\alpha_0} \left( 1 + \frac{c_{\alpha_1}}{c_{\alpha_0}} X^{\alpha_1 - \alpha_0} + \dots + \frac{c_{\alpha_m}}{c_{\alpha_0}} X^{\alpha_m - \alpha_0} \right),$$

where  $\alpha_0 < \alpha_1 < \dots < \alpha_m$  denotes some strictly increasing sequence of integers ( $m \geq 1$ ). Then one has  $\Delta(P) = [\alpha_0, \alpha_m]$ . Let  $d = \alpha_m - \alpha_0$ . The symmetric of  $\tilde{\Delta}_\nu(P)$  with respect to the real axis is the Newton-Puiseux diagram of  $P$ . Its upper bound is the Newton-Puiseux diagram of  $P$  (with respect to  $-\nu : \mathbb{K} \rightarrow ]-\infty, \infty]$ , which is a valuation in the usual sense), see figure 1.2. The slopes  $\rho_1, \dots, \rho_r$  of the different segments in the Newton-Puiseux diagram correspond to the distinct values of  $-\nu(\xi_j)$ , when  $\{\xi_1, \dots, \xi_d\}$  stands for the list of roots (counted with multiplicities)

of  $P$  in  $\mathbb{K}^*$ ; for each  $k = 1, \dots, r$ , the length of the interval of  $\Delta(P) = [\alpha_0, \alpha_m]$  over which the slope of the Newton-Puiseux polygon equals  $\rho_k$  figures the number of roots  $\xi_j$  such that  $-\nu(\xi_j) = \rho_j$ . For a proof of such result (in the case where  $-\nu = \nu_p$  is the  $p$ -adic valuation on  $\mathbb{K} = \mathbb{C}_p$ , which extends in a straightforward way to the general case), see section 6, more precisely Theorem 6.4.7 in [Gouv]. The slopes of the extended Newton polyhedron  $\tilde{\Delta}_\nu(P)$  correspond thus to the distinct values of  $\nu(\xi_j)$ ,  $j = 1, \dots, d$ . The extended Newton polyhedron  $\tilde{\Delta}_\nu(P)$  is therefore directly related (in the case  $n = 1$ ) to the zero set  $V_{\mathbb{K}^*}(P)$ .

**1.3.2. Tropicalisation of an algebraic hypersurface with respect to a valuation.** Let  $\mathbb{K}$  be a commutative algebraically closed field. Geometrically speaking, any Laurent polynomial  $P \in \mathbb{K}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$  defines an algebraic hypersurface  $V_{\mathbb{K}^*}(P) = P^{-1}(0)$  in  $(\mathbb{K}^*)^n$ . Such an object is relevant to complex algebraic geometry in the algebraic affine scheme  $\text{Spec}(\mathbb{K}[X_1^{\pm 1}, \dots, X_n^{\pm 1}])$ . Of course, when looking at  $V_{\mathbb{K}^*}(P)$  as a geometric object, one forgets about any algebraic information which is carried by the Laurent polynomial itself<sup>8</sup> (here, multiplicities attached to irreducible components), or better by the principal ideal  $(P) \subset \mathbb{K}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ . Nevertheless, when  $\mathbb{K}$  is equipped with a (non trivial) valuation  $\nu$ , then, following Definition 1.2, one can derive from the complex geometric object  $V_{\mathbb{K}^*}(P)$  a “geometric” object in the tropical world, namely the “tropicalisation” of the hypersurface

$$V_{\mathbb{K}^*}(P) = \{z \in (\mathbb{K}^*)^n; P(z) = 0\} = \{z \in (\mathbb{K}^*)^n; Q(z) = 0 \quad \forall Q \in (P)\}$$

with respect to the given valuation  $\nu$  (think again about the case  $\mathbb{K} = \overline{\mathbb{C}(t)}$  or  $\mathbb{K} = \overline{\mathbb{C}((t))}$ ,  $\nu$  being minus the order map as in (1.7)). Such an object (that will be relevant to tropical, hence real, geometry), will of course be of much more simple nature than the original one  $V_{\mathbb{K}^*}(P)$ . Though its definition does not take into account any algebraic information carried by the principal ideal  $(P)$ , such a tropicalisation will be called *tropical variety of the principal ideal  $(P) \subset \mathbb{K}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$*  with respect to the valuation  $\nu$ .

DEFINITION 1.10 (tropical variety of a principal ideal with respect to a valuation on some algebraically closed field  $\mathbb{K}$ ). Let  $(\mathbb{K}, \nu)$  be an algebraically closed commutative field equipped with a valuation, and  $P$  be a Laurent polynomial in  $\mathbb{K}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ . The *tropical variety of the principal ideal  $(P) \subset \mathbb{K}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$*  with respect to the valuation  $\nu$  is the tropical hypersurface attached to the Laurent tropical polynomial  $\text{trop}_\nu(P)$  introduced in Definition 1.8. It is denoted as  $V_{\text{Trop}, \nu}((P))$ .

In order to extend (in the next subsection 1.3.3) such a definition to arbitrary ideals in  $\mathbb{K}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ , we need to propose two re-interpretations of  $V_{\text{Trop}, \nu}((P))$ . In order to settle them, let us start with the following definition.

DEFINITION 1.11 (lobsided lists in  $\text{Trop} \setminus \{-\infty\}$ ). A finite list  $\{a_1, \dots, a_r\}$  of elements in  $\text{Trop} \setminus \{-\infty\} = \mathbb{R}$  (equipped with the tropical operations  $\boxplus$  and  $\boxtimes$ ) is *tropically lobsided* (or *lobsided in the ultrametric sense*) if and only if one the  $a_j$ ,  $j = 1, \dots, r$ , is strictly greater than the tropical sum of all the others, that is the maximum (respect to the usual order on  $\mathbb{R}$ ) of all the others. This is equivalent to say that there the maximum cannot be achieved twice among the list  $\{a_1, \dots, a_r\}$ .

<sup>8</sup>On the other hand, when  $\mathbb{K} = \mathbb{C}$ , such an information is for example contained in the integration current  $[V] = dd^c \log |P|^2$ , this Lelong-Poincaré formula being understood in the sense of currents,  $dd^c = (i/2\pi)\partial\bar{\partial}$ . We will come back to those aspects in a forthcoming chapter.

REMARK 1.12 (lopsided lists of elements  $(\mathbb{R}_{>0}, +, \times)$ ). The notion of tropically lopsided list introduced in Definition 1.11 should be compared to that of lopsided list in the classical semi-ring  $(\mathbb{R}_{>0}, +, \times)$  (that is, in the archimedean sense) : a list of strictly positive numbers  $\{e^{y_1}, \dots, e^{y_r}\}$  is *lopsided (in the archimedean sense)* if and only if one of the  $e^{y_j}$ ,  $j = 1, \dots, r$ , is strictly greater than the (ordinary) sum of all others. It is then equivalent to say (thanks to the triangle inequality for the usual archimedean absolute value on  $\mathbb{R}_{>0}$ ) that the list  $\{e^{y_1}, \dots, e^{y_r}\}$  is not lopsided (in the archimedean sense) if and only if there exist  $\theta_1, \dots, \theta_r$  in  $\mathbb{R}$  such that

$$\sum_{j=1}^r e^{\theta_j + iy_j} = 0.$$

Let  $P \in \mathbb{K}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$  be a Laurent polynomial such as (1.17). For any multi-exponent  $\alpha \in \text{Supp}(P)$  and any  $a = (a_1, \dots, a_n) \in (\text{Trop} \setminus \{-\infty\})^n = \mathbb{R}^n$ , let

$$\text{trop}_\nu(c_\alpha X^\alpha)(a) := \nu(c_\alpha) \boxtimes a_1^{\boxtimes \alpha_1} \boxtimes \dots \boxtimes a_n^{\boxtimes \alpha_n}.$$

It follows immediately from the observation contained in Definition 1.11 about the characterization of tropical lopsidedness that

$$(1.20) \quad \begin{aligned} V_{\text{Trop}, \nu}((P)) = \\ = \left\{ a \in (\text{Trop} \setminus \{-\infty\})^n ; \left\{ \text{trop}_\nu(c_\alpha X^\alpha)(a) \right\}_{\alpha \in \text{Supp}(P)} \text{ is not trop. lopsided} \right\}. \end{aligned}$$

This provides the first re-interpretation for  $V_{\text{Trop}, \nu}((P))$  we are looking for.

Consider now the map  $\nu : (\mathbb{K}^*)^n \rightarrow (\text{Trop} \setminus \{-\infty\})^n = \mathbb{R}^n$  defined as

$$(1.21) \quad \nu(x_1, \dots, x_n) = (\nu(x_1), \dots, \nu(x_n)).$$

When  $\nu$  corresponds to an ultrametric absolute value  $|\cdot|_u$  on  $\mathbb{K}$ , see Example 1.6, this is the map

$$(1.22) \quad \text{Log}|\cdot|_u : (x_1, \dots, x_n) \in (\mathbb{K}^*)^n \mapsto (\log|x_1|_u, \dots, \log|x_n|_u).$$

One can state the following important proposition ([**Krap**]), that provides the second interpretation for  $V_{\text{Trop}, \nu}(P)$  we are looking for.

PROPOSITION 1.2. *Let  $\mathbb{K}$  be an algebraically closed field equipped with a non trivial valuation  $\nu$  and  $P \in \mathbb{K}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ . The tropical variety of the principal ideal  $((P))$  with respect to the valuation  $\nu$  can be re-interpreted as*

$$(1.23) \quad V_{\text{Trop}, \nu}((P)) = \overline{\nu(V_{\mathbb{K}^*}(P))} \subset \mathbb{R}^n = (\text{Trop} \setminus \{-\infty\})^n.$$

Moreover, when  $\nu(\mathbb{K}^*) = \mathbb{R}$ , then

$$(1.24) \quad V_{\text{Trop}, \nu}((P)) = \nu(V_{\mathbb{K}^*}(P)) \subset \mathbb{R}^n = (\text{Trop} \setminus \{-\infty\})^n.$$

PROOF. See [**EKL**], theorem 2.1.1. Let us first prove the easiest part, which is the inclusion  $\overline{\nu(V_{\mathbb{K}^*}(P))} \subset V_{\text{Trop}, \nu}(P)$ . Let  $x \in (\mathbb{K}^*)^n$ , such that  $P(x) = 0$ . Let  $a = \nu(x)$ . Let  $\alpha_0 \in \text{Supp}(P)$  be such that  $\text{trop}_\nu(c_{\alpha_0} X^{\alpha_0})(a)$  is maximal among all real numbers involved in the list  $\left\{ \text{trop}_\nu(c_\alpha X^\alpha)(a) \right\}_{\alpha \in \text{Supp}(P)}$ . One has

$$(1.25) \quad c_{\alpha_0} x^{\alpha_0} = - \sum_{\substack{\alpha \in \text{Supp}(P) \\ \alpha \neq \alpha_0}} c_\alpha x^\alpha$$

since  $P(x) = 0$ . Taking the images by  $\nu$  of both sides of (1.25), then using the rules (1.16) to which a valuation such as  $\nu$  obeys, it follows that

$$\text{trop}_\nu(c_{\alpha_0} X^{\alpha_0})(a) \leq \max_{\substack{\alpha \in \text{Supp}(P) \\ \alpha \neq \alpha_0}} (\text{trop}_\nu(c_\alpha X^\alpha)(a)).$$

As a consequence, the list  $\{\text{trop}_\nu(c_\alpha X^\alpha)(a)\}_{\alpha \in \text{Supp}(P)}$  fails to be tropically lob-sided, which implies (in view of (1.20)) that  $a \in V_{\mathbb{T}, \nu}((P))$ . Since  $V_{\text{Trop}, \nu}((P))$  is clearly closed in  $\mathbb{R}^n$ , one has the first inclusion  $\overline{\nu(V_{\mathbb{K}^*}(P))} \subset V_{\text{Trop}, \nu}((P))$ .

Proving the reverse inclusion amounts to prove that  $V_{\text{Trop}, \nu}(P) \cap \Gamma_\nu^n \subset \overline{\nu(V_{\mathbb{K}^*}(P))}$  since the valuation  $\nu$  is assumed to be non trivial, thus  $\overline{\Gamma_\nu^n} = \mathbb{R}^n$ . Let  $a = (a_1, \dots, a_n) \in V_{\text{Trop}, \nu}((P)) \cap \Gamma_\nu^n$ , so that  $a_j = \nu(\xi_{a,j})$ ,  $\xi_{a,j} \in \mathbb{K}^*$ , for any  $j = 1, \dots, n$ . If one performs the change of variables  $x_j \longleftrightarrow \xi_{a,j} x_j$  in  $(\mathbb{K}^*)^n$ , proving that  $a$  belongs to  $\overline{\nu(V_{\mathbb{K}^*}(P))}$  amounts to prove

$$(1.26) \quad (0, \dots, 0) \in V_{\text{Trop}, \nu}((P)) \implies (0, \dots, 0) \in \overline{\nu(V_{\mathbb{K}^*}(P))}.$$

The hypothesis about  $(0, \dots, 0)$  in (1.26) means that the extended Newton polyedron  $\tilde{\Delta}_\nu(P)$  has an horizontal face with positive dimension and maximal altitude  $b_{\max} \in \mathbb{R}$ . Let  $F$  be the maximal face (with respect to the inclusion order) among those that share such property, and  $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$  such that, whenever  $[(a_1, b_{\max}), (a_2, b_{\max})]$  is an edge of  $F$ , then  $\langle k, a_1 - a_2 \rangle \neq 0$ . Let  $P_k$  be the Laurent polynomial in  $\mathbb{K}[t, t^{-1}]$  defined as

$$P_k(t) = P(t^{k_1}, \dots, t^{k_n}) = \sum_{\alpha \in \text{Supp}(P)} c_\alpha t^{\langle k, \alpha \rangle}.$$

The extended Newton polyedron  $\tilde{\Delta}_\nu(P_k)$  (in  $\mathbb{R}^2$ ) admits an horizontal edge with maximal altitude  $b_{\max}$ , which implies (see Example (1.9)) that  $P_k$  admits a root with  $t_0$  such that  $\nu(t_0) = 0$ , therefore that  $P$  admits a root  $z_0 = (t_0^{k_1}, \dots, t_0^{k_n})$  such that  $\nu(z_0) = (0, \dots, 0)$ .  $\square$

**REMARK 1.13.** If  $\mathfrak{p}$  is a Laurent (Trop,  $\boxplus$ ,  $\boxtimes$ )-polynomial such as (1.13), which coefficients  $a_\alpha$  can be interpreted, when  $a_\alpha \neq -\infty$ , as  $a_\alpha = \nu(c_\alpha)$ , where  $\nu$  is a non trivial valuation on some algebraically closed commutative field  $\mathbb{K}$  and coefficients  $c_\alpha$  belong to  $\mathbb{K}^*$ , then  $V_{\text{trop}}(\mathfrak{p})$  (in Definition 1.2) can be interpreted as  $\overline{\nu(V_{\mathbb{T}}(P))}$ , where  $P$  is the Laurent polynomial in  $\mathbb{K}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$  obtained from  $P$  just replacing the ‘‘tropical’’ coefficient  $a_\alpha$  by the coefficient  $c_\alpha \in \mathbb{K}^*$ .

**1.3.3. Tropicalisation of the algebraic variety of an ideal with respect to a valuation.** Proposition 1.2 suggests that Definition 1.10 can be extended to the case of general ideals in  $\mathbb{K}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ . Given a valuation  $\nu$  on  $\mathbb{K}$ , one can associate to each such ideal  $I \subset \mathbb{K}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$  a tropical geometric object, the *tropical variety of the ideal  $I$*  (with respect to the valuation  $\nu$ ).

**DEFINITION 1.14** (tropical variety of an ideal with respect to a valuation on some algebraically closed field  $\mathbb{K}$ ). Let  $(\mathbb{K}, \nu)$  be an algebraically closed commutative field equipped with a valuation, and  $I$  be an ideal in  $\mathbb{K}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ . The *tropical variety of the ideal  $I \subset \mathbb{K}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$*  is the closed set in  $\mathbb{R}^n = (\text{Trop} \setminus \{-\infty\})^n$  defined as

$$(1.27) \quad V_{\text{Trop}, \nu}(I) := \overline{\nu(V_{\mathbb{K}^*}(I))} \subset (\text{Trop} \setminus \{-\infty\})^n.$$

It is also called the *ultrametric amoeba* of  $I$  with respect to the (ultrametric) *logarithmic map*

$$\text{Log}_\nu | : (x_1, \dots, x_n) \in (\mathbb{K}^*)^n \mapsto (\log |x_1|_\nu, \dots, \log |x_n|_\nu) = \nu(x).$$

PROPOSITION 1.3. *For any ideal  $I \subset \mathbb{K}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ , one has*

$$(1.28) \quad V_{\text{Trop}, \nu}(I) = \bigcap_{P \in I} V_{\text{Trop}, \nu}((P)).$$

Thanks to (1.20), such a tropical variety  $V_{\text{Trop}, \nu}(I)$  is thus defined as the set of  $a \in \mathbb{R}^n$  such that, for any

$$P = \sum_{\alpha \in \text{Supp}(P)} c_\alpha X^\alpha \in I,$$

the finite list

$$\{\text{trop}_\nu(c_\alpha X^\alpha)(a)\}_{\alpha \in \text{Supp}(P)}$$

is not tropically lobsided.

PROOF. One has<sup>9</sup>

$$V_{\mathbb{K}^*}(I) = \bigcap_{P \in I} V_{\mathbb{K}^*}((P)).$$

It follows from (1.23) that, for any  $x \in V_{\mathbb{K}^*}(I) = \bigcap_{P \in I} V_{\mathbb{K}^*}((P))$ , one has that  $\nu(x) \in V_{\text{Trop}, \nu}((P))$  for any  $P \in I$ , which implies the inclusion

$$V_{\text{Trop}, \nu}(I) \subset \bigcap_{P \in I} V_{\text{Trop}, \nu}((P)).$$

In order to prove the reverse inclusion (the argument here being quoted from [SpSt], section 2), we need to use an auxiliary assertion, which arises from Gröbner basis theory<sup>10</sup>. It follows from (1.20) that

$$(1.29) \quad a \in \bigcap_{P \in I} V_{\mathbb{K}^*}((P)) \iff \left( \forall P = \sum_{\alpha \in \text{Supp}(P)} c_\alpha X^\alpha \in I, \{ \text{trop}_\nu(c_\alpha X^\alpha)(a) \}_{\alpha \in \text{Supp}(P)} \text{ is not trop. lobsided} \right).$$

Such condition (1.29) can be rephrased in different terms : for any monomial  $c_\alpha X^\alpha$  and every  $a \in \mathbb{R}^n$ , one defines the  $(-\nu, a)$ -weight of  $c_\alpha X^\alpha$  as  $-\nu(c_\alpha) + \langle a, \alpha \rangle$ . The *initial form* of a Laurent polynomial  $P = \sum_\alpha c_\alpha X^\alpha$  is then the element in  $k_{-\nu}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$  defined as follows : let<sup>11</sup>

$$\tilde{P}_a(X) := P(t^{a_1} X_1, \dots, t^{a_n} X_n)$$

(remember  $t \in \mathbb{K}^*$  is an element with order 1, that is such that  $\nu(t) = -1$ ) and  $-\nu_P(a)$  be the smallest  $(-\nu, a)$ -weight among those of all monomial involved in  $P$  and  $\text{init}_{-\nu, a}(P)$  be the image of  $t^{\nu_P(a)} P_a$  in  $k_{-\nu}[X_1, \dots, X_n]$ . The ideal  $\text{init}_{-\nu, a}(I)$

<sup>9</sup>In fact, since  $\mathbb{K}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$  is noetherian, there should be indeed a finite collection of polynomials  $P_1, \dots, P_r$  in  $I$  such that  $V_{\mathbb{K}^*}(I) = V_{\mathbb{K}^*}((P_1)) \cap \dots \cap V_{\mathbb{K}^*}((P_r))$ . Unfortunately, as it will appear in the proof, it does not seem to be possible to keep track only of that finite set of Laurent polynomials  $\{P_1, \dots, P_r\}$  which are enough to describe  $V_{\mathbb{K}^*}(I)$ .

<sup>10</sup>See e.g. [CLO], chapter 2.

<sup>11</sup>This is somehow only formal since  $a_1, \dots, a_n$  are here real numbers ; nevertheless, such a formal reasoning is sufficient here since our objective is just to define the initial ideal  $\text{init}_{-\nu, a}(P)$ .

is the ideal generated by all  $\text{init}_{-\nu,a}(P)$  for all  $P \in I$ . One can then rephrase (1.29) as

$$(1.30) \quad a \in \bigcap_{P \in I} V_{K^*}((P)) \iff \left( \text{init}_{-\nu,-a}(I) \text{ does not contain any monomial} \right).$$

Let  $a \in \bigcap_{P \in I} V_{K^*}((P)) \cap \nu((\mathbb{K}^*)^n)$ . One has  $a = \nu(x_a)$ ,  $x_a \in (\mathbb{K}^*)^n$ . In view of the equivalence (1.30), in order to prove the inclusion

$$\bigcap_{P \in I} V_{\text{Trop},\nu}((P)) \cap \nu((\mathbb{K}^*)^n) \subset V_{\text{Trop},\nu}(I),$$

it is enough, if one makes the change of variables  $x_j \longleftarrow x_{a,j}x_j$ , to prove that

$$(1.31) \quad \left( \text{init}_{-\nu,0}(I) \text{ does not contain any monomial} \right) \implies 0 \in V_{\text{Trop},\nu}(I).$$

Suppose then that  $\text{init}_{-\nu,0}(I)$  does not contain any monomial. This implies (thanks to Hilbert's nullstellensatz) that the ideal  $\text{init}_{-\nu,0}(I) \subset k_{-\nu}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$  admits a zero  $\bar{\xi}$  in  $(k_{-\nu}^*)^n$ . The key point is to show that such a zero  $\bar{\xi}$  can be "lifted" to a zero  $\xi$  of  $I$  in  $(\mathbb{K}^*)^n$ , which is indeed a bit technical. Such a zero  $\bar{\xi}$  corresponds to a maximal ideal  $\bar{\mathfrak{m}}_{\bar{\xi}} = (X_1 - \bar{\xi}_1, X_n - \bar{\xi}_n)$  in  $k_{-\nu}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ . Let  $S_{\bar{\mathfrak{m}}_{\bar{\xi}}}$  be the multiplicative subset<sup>12</sup> in  $R_{\mathbb{K},-\nu}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$  which consists of elements whose reduction modulo  $\mathfrak{M}_{\mathbb{K},-\nu}$  does not belong to the maximal ideal  $\bar{\mathfrak{m}}_{\bar{\xi}}$ . One has that  $S_{\bar{\mathfrak{m}}_{\bar{\xi}}} \cap I = \emptyset$  since  $\bar{\xi}$  is a zero of  $\text{init}_{-\nu,0}(I)$ . Moreover, consider the map

$$\varphi_{\bar{\mathfrak{m}}_{\bar{\xi}}} : R_{\mathbb{K},-\nu} \longmapsto \frac{S_{\bar{\mathfrak{m}}_{\bar{\xi}}}^{-1} R_{\mathbb{K},-\nu}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]}{S_{\bar{\mathfrak{m}}_{\bar{\xi}}}^{-1} \cdot (I \cap R_{\mathbb{K},-\nu}[X_1^{\pm 1}, \dots, X_n^{\pm 1}])}.$$

It is injective (easy to check<sup>13</sup>). Take then a minimal prime ideal

$$\mathfrak{P} \subset \frac{S_{\bar{\mathfrak{m}}_{\bar{\xi}}}^{-1} R_{\mathbb{K},-\nu}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]}{S_{\bar{\mathfrak{m}}_{\bar{\xi}}}^{-1} (I \cap R_{\mathbb{K},-\nu}[X_1^{\pm 1}, \dots, X_n^{\pm 1}])}.$$

such that  $\mathfrak{P} \otimes_{R_{\mathbb{K},-\nu}} \mathbb{K}$  is a proper ideal in  $\mathbb{K}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]/I$ ; such a proper ideal is contained in a maximal ideal  $(X_1 - \xi_1, \dots, X_n - \xi_n)$ ,  $\xi = (\xi_1, \dots, \xi_n) \in V_{\mathbb{K}^*}(I)$ . It remains to show that  $\nu(\xi) = (0, \dots, 0)$ . Clearing denominators, one can rewrite  $X_j - \xi_j \in I$  as  $\alpha_j X_j - \beta_j \in I \cap R_{\mathbb{K},-\nu}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ ,  $\alpha_j$  and  $\beta_j$  not lying both in  $\mathfrak{M}_{\mathbb{K},-\nu}$  (since  $\xi \in (k_{-\nu}^*)^n$ ). If  $\alpha_j \in \mathfrak{M}_{\mathbb{K},-\nu}$  for some  $j$ , then  $\text{init}_{-\nu,0}(I)$  contains  $\text{int}_{-\nu,0}(\beta_j)$ , which is a unit in  $\mathbb{K}^*$ ; this contradicts the fact that  $\text{init}_{-\nu,0}(I)$  does not contain a monomial. If  $\alpha_j \notin \mathfrak{M}_{\mathbb{K},-\nu}$  and  $-\beta_j/\alpha_j \not\equiv \bar{\xi}_j$ , then the reduction modulo  $\mathfrak{M}_{\mathbb{K},-\nu}$  of  $\alpha_j X_j - \beta_j$  does not belong to  $\bar{\mathfrak{m}}_{\bar{\xi}}$ , which means that  $\alpha_j X_j - \beta_j$  belongs to the multiplicative subset  $S_{\bar{\mathfrak{m}}_{\bar{\xi}}}$  and is a unit of  $S_{\bar{\mathfrak{m}}_{\bar{\xi}}}^{-1} R_{\mathbb{K},-\nu}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ ; thus  $\mathfrak{P}$  is the unit ideal, which is not prime, which leads also to a contradiction. Therefore  $-\alpha_j/\beta_j \equiv \bar{\xi}_j$  modulo  $\mathfrak{M}_{\mathbb{K},-\nu}$ , which means that  $\nu(-\alpha_j/\beta_j) = \nu(\xi_j) = 0$  for  $j = 1, \dots, n$ , that is  $0 \in \nu(\mathbb{K}^*)^n$ .  $\square$

<sup>12</sup>A *multiplicative subset* in a commutative unitary ring is a subset which does not contain 0, contains 1, and is closed under multiplication.

<sup>13</sup>If  $x$  in  $R_{\mathbb{K},-\nu}$  is in the kernel of  $\varphi_{\bar{\mathfrak{m}}_{\bar{\xi}}}$ , there exists an element  $s \in S_{\bar{\mathfrak{m}}_{\bar{\xi}}}$  such that  $sx \in I$ , therefore  $s \in I$  since  $x$  is invertible in  $\mathbb{K}$ , which is indeed a contradiction because  $S_{\bar{\mathfrak{m}}_{\bar{\xi}}} \cap I = \emptyset$ .

## Archimedean amoebas and coamoebas

The archimedean triangle inequality  $|a - b| \leq |a \pm b| \leq |a| + |b|$ , is known to be far more difficult to handle in geometric problems than the ultrametric triangle inequality  $|a + b| \leq \max(|a|, |b|)$ . On the other hand, classical questions in complex algebraic or analytic geometry arise in the archimedean context, not in the ultrametric one as in section 1.3. Therefore, it is important to extend Definitions 1.10 (completed by Proposition 1.2) or 1.14 to the archimedean context. We will also profit in such a context from the fact that the logarithmic map  $\text{Log} | \cdot |_{\text{arch}}$  is paired with an “argument” multivalued map  $\arg$ .

### 2.1. Amoeba and coamoeba in the codimension one algebraic case

**2.1.1. Amoeba of a Laurent polynomial.** Let  $F \in \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$  be a true<sup>1</sup> Laurent polynomial in  $n$  variables, that is a regular function on the affine algebraic variety  $\text{Spec}(\mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}])$ . It is well known that developing the meromorphic function (in  $n$  variables)  $1/F$  as a (absolutely) convergent Laurent power series in  $z_1^{\pm 1}, \dots, z_n^{\pm 1}$  in the Reinhardt<sup>2</sup> open subset

$$\mathbb{T}^n \setminus \{(z_1 e^{i\theta_1}, \dots, z_n e^{i\theta_n}); z \in V_{\mathbb{T}}(F), \theta \in \mathbb{R}^n\},$$

that is as

$$(2.1) \quad \frac{1}{F(z)} = \sum_{k=0}^{\infty} a_k z^{-\nu_k}, \quad a_k \in \mathbb{C}, \quad \nu_k \in \mathbb{Z}^n,$$

or (if one adopts the discrete Fourier point of view and set  $x_j = \log |z_j|$ ,  $j = 1, \dots, n$ )

$$(2.2) \quad \frac{1}{F(e^{x_1+i\theta_1}, \dots, r_n e^{x_n+i\theta_n})} = \sum_{k=0}^{\infty} a_k e^{-\langle \nu_k, x \rangle} e^{-i\langle \nu_k, \theta \rangle}, \quad a_k \in \mathbb{C}, \quad \nu_k \in \mathbb{Z}^n, \quad \forall \theta \in \mathbb{R}^n,$$

is indeed an ambiguous question. For example, when  $n = 1$ , the moduli of the roots  $\alpha_1, \dots, \alpha_p$  of  $\zeta \in \mathbb{C}^* \mapsto F(\zeta)$  determine a subdivision

$$r = 0 < r_1 < \dots < r_q < r_{q+1} = +\infty$$

such that  $1/F$  admits a convergent development such as (2.1) in each radial domain  $\{r_j < |z| < r_{j+1}\}$ ,  $j = 0, \dots, q$ , all such developments being distincts. Note that the development “at infinity”, that is in the domain  $\{|z| > r_q\}$ , is the most exploited

<sup>1</sup>That is, the Newton polyhedron  $\Delta(F)$  has a non-empty interior in  $\mathbb{R}^n$ ; if it is not the case, the situation reduces to that where  $F$  can be considered as a Laurent polynomial in strictly less than  $n$  variables.

<sup>2</sup>A Reinhardt open set  $R$  in  $\mathbb{T}^n$  is a connected open subset  $R$  in  $\mathbb{T}^n$  such that, whenever  $z \in R$ ,  $(e^{i\theta_1} z_1, \dots, e^{i\theta_n} z_n) \in R$  for any  $\theta \in \mathbb{R}^n$ . It is called a Reinhardt domain when it is additionally connected.



in digital signal processing or information theory since one can associate to it the action of a causal digital filter (see *e.g.* section II.2.2 in [YW], chapter 10).

In order to clarify such a question, let us forget for the moment the *functional* nature of the problem (involving indeed a Laurent polynomial  $F$ ) and define, following [GKZ], the *geometric* concept of *amœba* (which is attached only to the algebraic hypersurface  $V_{\mathbb{T}}(F) = \{z \in \mathbb{T}^n; F(z) = 0\}$ ). Let us introduce, as in section 1.3 (but now in the Archimedean context instead of the ultrametric one, compare to (1.22)) the archimedean logarithmic map  $\text{Log} |_{\text{arch}} = \text{Log}$  from  $\mathbb{T}^n = (\mathbb{C}^*)^n$  in  $\mathbb{R}^n$  defined as:

$$(2.3) \quad \text{Log}(z_1, \dots, z_n) := (\log |z_1|, \dots, \log |z_n|).$$

An important property of the logarithmic map on  $]0, +\infty[$  is that  $|\log t| = o(t^k)$  for any  $k \in \mathbb{N}^*$  when  $t$  tends to  $+\infty$  and  $|\log t| = o(t^{-k})$  for any  $k \in \mathbb{N}^*$  when  $t$  tends to  $0_+$ . It is a well known fact then that changing the usual scale on  $]0, \infty[$  into the logarithmic scale  $t \longleftrightarrow u = \log t$  allows to make closer to the “accessible” world the description of objects in  $]0, +\infty[$  when they escape toward  $+\infty$  (or toward  $0_+$ ).

DEFINITION 2.1 (amœba of an algebraic hypersurface in  $\mathbb{T}^n$ ). The (archimedean) *amœba* of the algebraic hypersurface  $V_{\mathbb{T}}(F) \subset \mathbb{T}^n$  is the subset  $\mathcal{A}_{V_{\mathbb{T}}(F)} \subset \mathbb{R}^n$  defined as

$$(2.4) \quad \mathcal{A}_{V_{\mathbb{T}}(F)} := \text{Log}(V_{\mathbb{T}}(F)).$$

It is also called (archimedean) *amœba of the Laurent polynomial  $F$*  (though it depends only on  $V_{\mathbb{T}}(F)$ ).

REMARK 2.2 (closedness of the archimedean amœba). Since  $\text{Log}$  is a proper continuous map from  $\mathbb{T}^n$  to  $\mathbb{R}^n$ , the archimedean amœba  $\mathcal{A}_{V_{\mathbb{T}}(F)}$  is a closed subset in  $\mathbb{R}^n$ . Note also that the open set  $\text{Log}^{-1}(\mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(F)})$  (in  $\mathbb{T}^n$ ) coincides with the open subset of  $\mathbb{T}^n$  defined as  $\mathbb{T}^n \setminus \{(z_1 e^{i\theta_1}, \dots, z_n e^{i\theta_n}); z \in V_{\mathbb{T}}(F), \theta \in \mathbb{R}^n\}$ .

EXAMPLE 2.3 (amœba of a affine line in  $\mathbb{T}^2$ ). Consider the affine line  $L_0 := \{z + w - 1 = 0\}$  in  $\mathbb{T}^2$ . A point  $(z, w)$  in  $\mathbb{T}^2$  belongs to  $L_0$  if and only if  $|z| \neq 0$  and  $|w| \neq 0$  satisfy the three conditions:

$$(2.5) \quad |z| + |w| \geq 1, \quad |z| + 1 \geq |w|, \quad |w| + 1 \geq |z|.$$

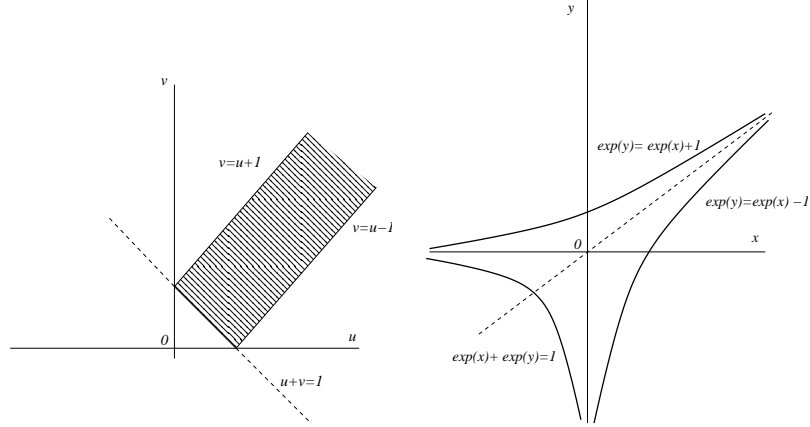
This is equivalent to say that the family  $\{1, |z|, |w|\}$  is not lobsided in the archimedean sense (see Remark 1.12). Such conditions (2.5) stand for the necessary and sufficient conditions ensuring that the positive numbers  $1, |z|, |w|$  can be interpreted as the lengths of the 1-dimensional faces (facets) of a triangle. Then, the amœba of the complex line  $\{z + w = 1\}$  is the image of the domain

$$\{(u, v) \in ]0, \infty[^2; u + v \leq 1, u + 1 \leq v, v + 1 \leq u\}$$

by the map  $(u, v) \mapsto (\log u, \log v)$  (see figure 2.1 below).

Example 2.3 reveals indeed some weak point inherent to the definition of amœba : namely, such a definition does not preserve (in any kind of form) some important information (for example of topological nature) that could be contained in the geometric data  $V(F)$ . For example (see *e.g.* [Mir], Problem II.4, G), the *genus*<sup>3</sup> (as

<sup>3</sup>The *genus* of a compact Riemann surface  $\mathcal{X}$  is the number of “holes” in this surface, that is the rank of the cohomology group  $H^1(\mathcal{X})$  (or, thanks to de Rham’s theorem, that of the homology group  $H_1(\mathcal{X}, \mathbb{Z})$ ). The genus of any smooth projective curve with degree  $d$  equals  $(d-1)(d-2)/2$ .


 FIGURE 2.1. Amœba of the line  $L_0 : z + w - 1 = 0$ 

a Riemann surface) of the smooth *Fermat curve*  $\{[z_0 : z_1 : z_2] \in \mathbb{P}^2(\mathbb{C}) ; z_1^d + z_2^d = z_0^d\}$  equals  $(d-1)(d-2)/2$ , that is it varies with  $d \in \mathbb{N}^*$  (in particular, the Fermat curve is an elliptic curve if and only if  $d = 3$ ). On the other side, the amœba of  $\{(z, w) \in \mathbb{T}^2 ; z^d + w^d = 1\}$  is nothing but just an homothetic of the amœba of the tropical line  $L_0$ , which shows that the topological information provided by the genus is lost when taking archimedean amœbas.

Nevertheless, given a Laurent polynomial  $F \in \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ , the geometric information carried by  $\mathcal{A}_{V_{\mathbb{T}}(F)}$  answers the indeterminacy problem that occurs when developing the meromorphic function  $1/F : \mathbb{T}^n \rightarrow \mathbb{C}$  as a convergent power series such as (2.1). Namely, one has the important proposition.

**PROPOSITION 2.1 ([GKZ]).** *Let  $F \in \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$  be a true Laurent polynomial in  $n$  variables. There is a one-to-one correspondence between the connected components  $C$  of  $\mathbb{R}^n \setminus \mathcal{A}_V(F)$  and the maximal Reinhardt domains  $E$  of convergence (in  $\mathbb{T}^n \setminus V_{\mathbb{T}}(F)$ ) for possible developments (2.1) of  $1/F$  as a convergent Laurent power series in  $z_1^{\pm 1}, \dots, z_n^{\pm 1}$ . Such a one-to-one correspondence associates to the maximal Reinhardt domain of convergence  $E \subset \mathbb{T}^n$  (for some Laurent development of the form (2.1) for  $1/F$ ) its image  $\text{Log } E \subset \mathbb{R}^n$ . As a consequence, any connected component  $C = \text{Log } E$  of  $\mathbb{R}^n \setminus \mathcal{A}_V(F)$  is an open convex subset in  $\mathbb{R}^n$ .*

**PROOF.** Let  $C$  be a connected (open<sup>4</sup>) component of  $\mathbb{R}^n \setminus \mathcal{A}_V(F)$  and  $z_C$  be any point in  $\text{Log}^{-1}(C) \subset \mathbb{T}^n$ . In a neighborhood of  $z_C$  in  $\mathbb{T}^n$ , there exists a (unique) Laurent development  $\mathcal{D}_C$  for  $1/F$  (of the form (2.1)). Such a development converges in the Reinhardt domain  $\text{Log}^{-1}(C)$ , therefore (thanks to a well known result by N. Abel<sup>5</sup>), also in  $\text{Log}^{-1}(\widehat{C})$ , where  $\widehat{C}$  denotes the convex envelope of  $C$  in  $\mathbb{R}^n$ . Let  $E_C$  be the maximal Reinhardt domain of convergence for the development  $\mathcal{D}_C$  which is valid about  $z_C$ . One has  $\text{Log}^{-1}(\widehat{C}) \subset E_C$ , therefore  $\widehat{C} \subset \text{Log } E_C$ . Since  $E_C$  is a Reinhardt domain of convergence for  $1/F$ , one has  $\text{Log } E_C \subset \mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(F)}$ .

<sup>4</sup>The connected components of an open subset in  $\mathbb{R}^n$  are open subsets in  $\mathbb{R}^n$ .

<sup>5</sup>If a Laurent power series of the form (2.1) converges in a neighborhood of two points  $\zeta$  and  $\eta$  in  $\mathbb{T}^n$ , it converges normally on  $\{(z_1, \dots, z_n) ; |\zeta_j| \leq |z_j| \leq |\eta_j|\}$ .

Therefore, one has  $C = \widehat{C} = \text{Log } E_C$ . The map which associates to any maximal Reinhardt domain  $E$  of convergence for some Laurent development of the form (2.1) its image by  $\text{Log}$  is therefore surjective. On the other hand, such a map is injective since, for any connected component of  $\mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(F)}$ , for any  $z_C$  in  $\text{Log}^{-1}(C)$ , there is a unique Laurent development of the form (2.1) for  $1/F$  which is valid in a neighborhood of  $z_C$ . The map  $E \mapsto C = \text{Log } E$  realizes thus a bijection between the set of maximal Reinhardt domains of convergence for some Laurent development of the form (2.1) and the set of connected components of  $\mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(F)}$ .  $\square$

REMARK 2.4. Note that one may reconstruct the Laurent development  $\mathcal{D}_C$  (or the form (2.1)) for  $1/F$  that corresponds to a given connected component  $C$  of  $\mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(F)}$ . Namely, take an arbitrary point  $x_C$  in  $C$  and compute, for each  $\nu \in \mathbb{Z}^n$ , the Fourier coefficient

$$(2.6) \quad \gamma_{C,x,\nu}[1/F] := \frac{1}{(2i\pi)^n} \int_{[0,2\pi]^n} \frac{1}{F(e^{x_1+i\theta_1}, \dots, e^{x_n+i\theta_n})} e^{i\langle \nu, \theta \rangle} d\theta_1 \dots d\theta_n.$$

Keep only those  $\nu$  such that  $\gamma_{C,x,\nu}[1/F] \neq 0$  and organize them as an ordered sequence  $(\nu_{C,x,k})_{k \geq 0}$ . Then, take

$$a_{C,x,k} := \gamma_{C,x,\nu}[1/F] \times e^{\langle \nu_{C,x,k}, x \rangle} \quad \forall k \in \mathbb{N}.$$

Neither the  $\nu_{C,x,k}$  nor the corresponding  $a_{C,x,k}$  depend in fact on the arbitrary choice of  $x$  in  $C$ . The coefficients  $a_{C,k}$  and the  $\nu_{C,k}$  are precisely those involved in the Laurent development for  $1/F$  of the form (2.1) which is valid in the maximal domain of convergence  $E_C$  which corresponds to  $C$  *via* the correspondence described in Proposition 2.1.

**2.1.2. Coamœba of a Laurent polynomial.** As we pointed it out in the introduction, the usual (archimedean) logarithmic map is paired with a multivalued argument function  $\arg$ . Given a true Laurent polynomial  $F \in \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ , it is therefore natural to introduce also, instead of the image of  $V_{\mathbb{T}}(F)$  through the logarithmic map  $\text{Log}$ , the image  $\mathcal{A}'_{V_{\mathbb{T}}(F)}$  of  $V_{\mathbb{T}}(F)$  through the argument map  $\arg : \mathbb{T}^n \mapsto (\mathbb{S}^1)^n$  ( $\mathbb{S}^1 = \mathbb{R}/(2\pi\mathbb{Z})$ ). In the same vein than the description of  $\mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(F)}$  clarifies the ambiguity inherent to the development of  $1/F$  as a Laurent power series in  $z_1^{\pm 1}, \dots, z_n^{\pm 1}$  in  $\text{Log}^{-1}(\mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(F)})$  (see Proposition 2.1), the description of  $(\mathbb{S}^1)^n \setminus \overline{\mathcal{A}'_{V_{\mathbb{T}}(F)}}$  will clarify (see Proposition 2.3 below) the ambiguity inherent to the representation in the conic open set  $\arg^{-1}((\mathbb{S}^1)^n \setminus \overline{\mathcal{A}'_{V_{\mathbb{T}}(F)})$  of  $z \mapsto 1/F(z)$  as an (absolutely) convergent Mellin integral transform, namely

$$(2.7) \quad \frac{1}{F(z)} = \int_{\xi^* + i\mathbb{R}^n} \mathcal{M}(\lambda_1, \dots, \lambda_n) \left( \prod_{j=1}^n z_j^{-\lambda_j} \right) d\lambda_1 \dots d\lambda_n$$

$$\mathcal{M} \in H(\text{int}(\Delta(F)) + i\mathbb{R}^n), \quad \xi^* \in \text{int}(\Delta(F))$$

(the representation (2.7) being independent on the choice of  $\xi^*$  in the interior of  $\Delta(F)$ ), or (if one adopts the continuous Fourier point of view and set  $z_j = e^{x_j + i\theta_j}$  for  $j = 1, \dots, n$ ),

$$(2.8) \quad \frac{e^{\langle \xi^*, x + i\theta \rangle}}{F(e^{x_1 + i\theta_1}, \dots, e^{x_n + i\theta_n})} = \int_{\mathbb{R}^n} \mathcal{M}(\xi^* + i\omega) e^{\langle \omega, \theta \rangle} e^{-i\langle \omega, x \rangle} d\omega,$$

$$g \in H(\text{int}(\Delta(F)) + i\mathbb{R}^n), \quad \xi^* \in \text{int}(\Delta(F)).$$

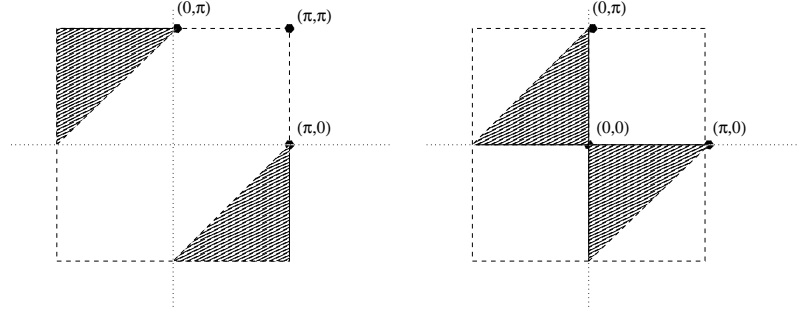


FIGURE 2.2. Coamœbas (represented here in  $] -\pi, \pi]^2$ ) of the complex lines  $\{z + w + 1 = 0\}$  and  $\{z + w - 1 = 0\}$  in  $\mathbb{T}^2$

We therefore introduce the following companion definition to Definition 2.1.

DEFINITION 2.5 (coamœba of an algebraic hypersurface in  $\mathbb{T}^n$ ). The *coamœba* of the algebraic hypersurface  $V_{\mathbb{T}}(F) := \{z \in \mathbb{T}^n; F(z) = 0\}$  is the subset  $\mathcal{A}'_{V_{\mathbb{T}}(F)}$  of  $(\mathbb{S}^1)^n = (\mathbb{R}/2\pi\mathbb{Z})^n$  defined as

$$(2.9) \quad \mathcal{A}'_{V_{\mathbb{T}}(F)} := \arg(V_{\mathbb{T}}(F)).$$

It is also called *coamœba of the Laurent polynomial  $F$*  (though it depends only on  $V_{\mathbb{T}}(F)$ ).

EXAMPLE 2.6 (coamœba of a affine line in  $\mathbb{T}^2$ ). Consider the complex line  $\{z + w + 1 = 0\}$  in  $\mathbb{T}^2$ . Let  $(\theta_1, \theta_2) \in ] -\pi, \pi]^2$ . A necessary and sufficient condition for the existence of  $r_1 > 0$  and  $r_2 > 0$  such that  $r_1 e^{i\theta_1} + r_2 e^{i\theta_2} + 1 = 0$  is that  $|\theta_1 - \theta_2| > \pi$  or  $(\theta_1, \theta_2) = (\pi, 0), (0, \pi), (\pi, \pi)$ . One can thus represent the coamœba of the line  $\{z + w + 1 = 0\}$  in  $] -\pi, \pi]^2$  on figure 2.2 (left). Note that this coamœba, when considered as a subset of  $(\mathbb{R}/(2\pi\mathbb{Z}))^2 \simeq (\mathbb{S}^1)^2$ , is not a closed subset. The coamœba of the line  $\{z + w - 1 = 0\}$  in  $\mathbb{T}^2$  is represented in  $] -\pi, \pi]^2$  on figure 2.2 (right). Note that taking coamœbas makes a distinction between the two lines  $\{z + w + 1 = 0\}$  and  $\{z + w - 1 = 0\}$ , while of course their amœbas are the same. The same phenomenon occurs for the Fermat curve  $\{z^d + w^d - 1 = 0\} = V_{\mathbb{T}}(L_{d-1})$ ,  $d \in \mathbb{N}^*$ : one has  $\theta \in \mathcal{A}'_{V_{\mathbb{T}}(L_{d-1})}$  if and only if  $d \times \theta \in \mathcal{A}'_{V_{\mathbb{T}}(L_0)}$ , which now makes a significative difference between the coamœbas of the Fermat curves when  $d$  varies.

The first observation that occurs from Example 2.6 is that the coamœba  $\mathcal{A}'_{V_{\mathbb{T}}(F)}$  fails in general to be closed in  $(\mathbb{S}^1)^n$ . Therefore, the interesting open subset to study in  $(\mathbb{S}^1)^n$  is  $(\mathbb{S}^1)^n \setminus \overline{\mathcal{A}'_{V_{\mathbb{T}}(F)}}$ , which corresponds to the  $(2\pi\mathbb{Z})^n$ -periodic open subset of  $\mathbb{R}^n$  denoted as  $\mathbb{R}^n \setminus \pi^{-1}(\overline{\mathcal{A}'_{V_{\mathbb{T}}(F)})$ , where  $\pi : \mathbb{R}^n \rightarrow (\mathbb{R}/(2\pi\mathbb{Z})^n = (\mathbb{S}^1)^n$  denotes the canonical projection.

Let  $\mathfrak{C}$  be a connected component of the open set  $\mathbb{R}^n \setminus \pi^{-1}(\overline{\mathcal{A}'_{V_{\mathbb{T}}(F)})$  and  $\mathfrak{C}$  be the conic domain in  $\mathbb{T}^n$  defined as

$$(2.10) \quad \mathfrak{C}_{\mathfrak{C}} := \{(r_1 e^{i\theta_1}, \dots, r_n e^{i\theta_n}); \theta \in \mathfrak{C}\}$$

(note that  $\mathfrak{C}_{\mathfrak{C}} = \mathfrak{C}_{\mathfrak{C}'}$  if  $\mathfrak{C}$  and  $\mathfrak{C}'$  are translated modulo  $(2\pi\mathbb{Z})^n$ ). The relation between the connected components of  $\mathbb{R}^n \setminus \pi^{-1}(\overline{\mathcal{A}'_{V_{\mathbb{T}}(F)})$  and the Mellin transform is given by the following Proposition ([NP], [Ant]). This is the pendant of the

relation that exists between connected components of  $\mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(F)}$  and the Fourier transform (c.f. Proposition 2.1 and Remark 2.4).

**PROPOSITION 2.2** (directional Mellin Transform). *Let  $F \in \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$  be a true Laurent polynomial in  $n$  variables and  $\mathcal{A}'_{V_{\mathbb{T}}(F)}$  be its coamæba in  $(\mathbb{S}^1)^n$ . Let  $\mathfrak{C}$  be a connected component of  $\mathbb{R}^n \setminus \pi^{-1}(\overline{\mathcal{A}'_{V_{\mathbb{T}}(F)})}$ . Then, for any  $s$  in the “tube” domain  $\text{int}(\Delta(F)) + i\mathbb{R}^n$ , for any  $\theta \in \mathfrak{C}$ , the poly-contour integral*

$$\int_{]0, \infty[ \times (e^{i\theta_1}, \dots, e^{i\theta_n})} \frac{1}{F(\zeta)} \prod_{j=1}^n \zeta_j^{s_j} \bigwedge_{j=1}^n \frac{d\zeta_j}{\zeta_j} = \int_{\mathbb{R}^n} \frac{e^{\langle s, x+i\theta \rangle}}{F(e^{x+i\theta})} dx_1 \dots dx_n$$

corresponds to an absolutely convergent integral (in the Lebesgue sense), and is independent on the choice of  $\theta \in \mathfrak{C}$ . Moreover, the map

$$(2.11) \quad \mathcal{M}_{\mathfrak{C}}[1/F] : s \in \text{int}(\Delta(F)) + i\mathbb{R}^n \mapsto \frac{1}{(2i\pi)^n} \int_{\mathbb{R}^n} \frac{e^{\langle s, x+i\theta \rangle}}{F(e^{x+i\theta})} dx_1 \dots dx_n$$

is holomorphic in the tube domain  $\text{int}(\Delta(F)) + i\mathbb{R}^n$ . It is called the  $\mathfrak{C}$ -directional Mellin transform of the rational function  $1/F$ . Moreover, if  $u^*$  denotes an arbitrary interior point in  $\Delta(F)$ , one has the Mellin inversion formula :

$$(2.12) \quad \forall z \in \mathfrak{C}, \quad \frac{1}{F(z)} = \int_{u^* + i\mathbb{R}^n} \mathcal{M}_{\mathfrak{C}}[1/F](u^* + i\omega) \prod_{j=1}^n z_j^{-u_j^* - i\omega_j} d\omega_1 \dots d\omega_n,$$

where the determination of the multi-argument of  $z$  which is chosen to express each  $z_j^{-u_j^* - i\omega_j}$  in the integrant of (2.12) is that which belongs precisely to  $\mathfrak{C}$ .

Let us postpone for the moment the proof of this technical proposition. As a consequence of Proposition 2.2, one can state here the following companion to Proposition 2.1 (see [NP], Theorem 4 or also [Ant]) :

**PROPOSITION 2.3.** *Let  $F \in \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$  be a true Laurent polynomial in  $n$  variables and  $\mathcal{A}'_{V_{\mathbb{T}}(F)}$  be its coamæba. The connected components  $\mathfrak{C}$  of the open set  $\mathbb{R}^n \setminus \pi^{-1}(\overline{\mathcal{A}'_{V_{\mathbb{T}}(F)})}$  are convex. Moreover, for each such component  $\mathfrak{C}$ , the conic open subset  $\mathfrak{C}_{\mathfrak{C}}$  defined as (2.10) is the maximal open subset of  $\mathbb{T}^n$  in which the Mellin integral representation formula (2.7) holds, where  $\mathcal{M} = \mathcal{M}_{\mathfrak{C}}[1/F]$  denotes the  $\mathfrak{C}$ -directional Mellin transform of the rational function  $1/F$ .*

**PROOF OF PROPOSITION 2.3.** Let  $\mathfrak{C}$  be a connected component of the open set  $\mathbb{R}^n \setminus \pi^{-1}(\overline{\mathcal{A}'_{V_{\mathbb{T}}(F)})}$ . Consider an arbitrary  $u$  in the interior of  $\Delta(F)$  and the holomorphic maps defined in the tubular domain  $\mathfrak{C} + i\mathbb{R}^n$  as

$$(2.13) \quad \begin{aligned} \theta - ix &\mapsto \frac{1}{F(e^{i(\theta-ix)})} = \frac{1}{F(e^{x+i\theta})} \\ \theta - ix &\mapsto \int_{u^* + i\mathbb{R}^n} \mathcal{M}_{\mathfrak{C}}[1/F](u^* + i\omega) \prod_{j=1}^n e^{-\langle x+i\theta, u^* + i\omega \rangle} d\omega_1 \dots d\omega_n. \end{aligned}$$

It follows from Bochner’s tube theorem [Boch] (this theorem plays here the role which is played by N. Abel’s result in the proof of Proposition 2.1) that both holomorphic functions (2.13) in the tubular domain  $\mathfrak{C} + i\mathbb{R}^n$  extend holomorphically to the wider tubular convex domain  $\widehat{\mathfrak{C}} + i\mathbb{R}^n$ , where  $\widehat{\mathfrak{C}}$  denotes the convex hull of  $\mathfrak{C}$  in  $\mathbb{R}^n$ . It then follows from the analytic continuation principle (which implies here

that equality (2.12) between these two functions in the tubular domain  $\mathfrak{C} + i\mathbb{R}^n$  extends analytically to the wider one  $\widehat{\mathfrak{C}} + i\mathbb{R}^n$  that one has necessarily  $\mathfrak{C} = \widehat{\mathfrak{C}}$ , which proves that  $\mathfrak{C}$  is convex. The remaining assertions in Proposition 2.3 follow from Proposition 2.2 as in Proposition 2.1.  $\square$

REMARK 2.7 (digital *versus* analogic AR filters). The integral Mellin representation formula (2.12) in  $\mathfrak{C}_{\mathfrak{C}}$  can be interpreted as a continuous analog of the developed expression (2.1) for  $1/F$  in  $\text{Log}^{-1}(C)$  (where  $C$  is now a connected component of the complement of the amoeba  $\mathcal{A}_{V_{\mathbb{T}}(F)}$ ). Such a representation formula (2.12) (as well as the description of all of them, such as given by the correspondence  $(\mathfrak{C}_{\mathfrak{C}}, \mathcal{M}_{\mathfrak{C}}[1/F]) \longleftrightarrow \mathfrak{C}$  settled in Proposition 2.3) is an important tool in analogic (instead of digital) signal processing. Note that the expression (2.11) for the directional Mellin transform with respect to  $\mathfrak{C}$  plays (in the analogic instead of digital setting) the same role than that the set of relations (2.6) (see Remark 2.4), which precisely allow to retrieve the Laurent development (2.1) (that is the digital AR filter<sup>6</sup> corresponding to  $1/F$ ) from the rational function  $1/F$ .

It remains to sketch briefly the proof of the technical Proposition 2.2.

PROOF OF PROPOSITION 2.12. Let  $\theta \in \mathbb{R}^n \setminus \pi^{-1}(\overline{\mathcal{A}'_{V_{\mathbb{T}}(F)}})$ . The first thing to prove is that, for any  $s$  in the tube domain  $\text{int}(\Delta(F)) + i\mathbb{R}^n$ , the integral

$$(2.14) \quad \int_{\mathbb{R}^n} \frac{e^{\langle s, x + i\theta \rangle}}{F(e^{x+i\theta})} dx_1 \dots dx_n$$

is absolutely convergent and independent of  $\theta$ . For each face<sup>7</sup>  $\delta$  of  $\Delta(F)$ , denote as  $F_{\delta}$  be the Laurent polynomial obtained from  $F$  just keeping the monomials  $X_1^{\alpha_1} \dots X_n^{\alpha_n}$  such that  $\alpha = (\alpha_1, \dots, \alpha_n)$  belongs to the face  $\delta$  (note that  $F_{\Delta(F)} = F$ ). It follows from Rouché's theorem (see Theorem 3 in [NP]) that the condition  $\theta \in \mathbb{R}^n \setminus \pi^{-1}(\overline{\mathcal{A}'_{V_{\mathbb{T}}(F)}})$  implies that, for any face  $\delta$  of  $\Delta(F)$ ,  $F_{\delta}$  has no zero on the multidirectional half-line  $\{z \in \mathbb{T}^n; \arg(z) = \pi(\theta)\}$ . As a consequence, for any  $s$  in the tubular domain  $\text{int}(\Delta(F)) + i\mathbb{R}^n$ , the integral (2.14) is absolutely convergent as a Lebesgue integral. Provided  $\theta$  is been kept fixed in  $\mathbb{R}^n \setminus \pi^{-1}(\overline{\mathcal{A}'_{V_{\mathbb{T}}(F)}})$ , the integral (2.14) depends holomorphically on  $s$  when  $s \in \text{int}(\Delta(F)) + i\mathbb{R}^n$ . On the other hand, it follows from the classical one variable residue formula applied to integrals of holomorphic forms (in one complex variables) on paths such as pictured on figure 2.3 that, for any  $s \in \text{int}(\Delta(F)) + i\mathbb{R}^n$ , the integral (2.14) does not depend in  $\theta$ , provided  $\theta$  remains in a fixed connected component  $\mathfrak{C}$  of  $\mathbb{R}^n \setminus \pi^{-1}(\overline{\mathcal{A}'_{V_{\mathbb{T}}(F)}})$ . The directional Mellin integral  $\mathcal{M}_{\mathfrak{C}}[1/F]$  defined in (2.11) is thus well defined for each connected component  $\mathfrak{C}$  of  $\mathbb{R}^n \setminus \pi^{-1}(\overline{\mathcal{A}'_{V_{\mathbb{T}}(F)}})$  as an holomorphic function in the tubular domain  $\text{int}(\Delta(F)) + i\mathbb{R}^n$ . Formula (2.12) (when  $z = (e^{x_1+i\theta_1}, \dots, e^{x_n+i\theta_n})$ ) belongs to the multi-conic sector  $\mathfrak{C}_{\mathfrak{C}}$  follows from Fourier inversion formula for

<sup>6</sup>“AR” stands here for “*all recursive*”, which is the terminology used in information theory.

<sup>7</sup>A “face” of a convex compact polyedron  $\Delta \subset \mathbb{R}^n$  with non-empty interior in  $\mathbb{R}^n$  (which is the case here for  $\Delta = \Delta(F)$  since  $F$  is supposed to be a “true” Laurent polynomial in  $n$  variables) is the intersection of  $\Delta$  with some affine subspace  $\{x; \langle a, x \rangle + b = 0\}$  for some  $a, b \in \mathbb{R}^n$ . Faces of  $\Delta$  are also convex compact polyedrons with (relative) dimension between 0 (0-dimensional faces beings “*vertices*”) and  $n$  ( $\Delta(F)$  itself being the unique  $n$ -dimensional face, take  $a = b = 0$ ). Faces with dimension 1 are called “*edges*” of  $\Delta(F)$ , faces with dimension  $n - 1$  are called “*facets*” of  $\Delta(F)$ .

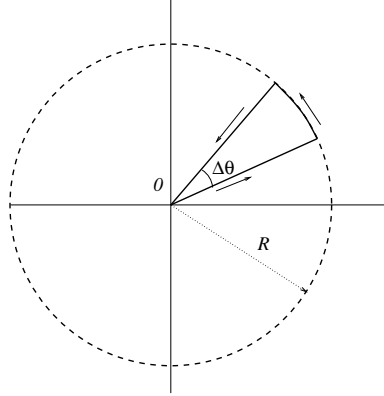


FIGURE 2.3. The directional Mellin transform is independent on the direction  $\theta \in \mathfrak{C}$ .

elements in the space  $\mathcal{S}(\mathbb{R}^n)$  of smooth functions which are rapidly decreasing at infinity as well as all their derivatives, namely

$$\forall \varphi \in \mathcal{S}(\mathbb{R}^n), \quad \widehat{\varphi} \in \mathcal{S}(\mathbb{R}^n), \quad \text{and} \quad \varphi(x) = \frac{1}{(2i\pi)^n} \int_{\mathbb{R}^n} \widehat{\varphi}(\omega) e^{i\langle \omega, x \rangle} d\omega \quad \forall x \in \mathbb{R}^n,$$

applied to the fonction  $x \mapsto e^{\langle u^*, x + i\theta \rangle} / F(e^{x+i\theta})$ ,  $\theta \in \mathfrak{C}$ ,  $u^* \in \text{int}(\Delta(F)) + i\mathbb{R}^n$ .  $\square$

## 2.2. Archimedean amœba and coamœba of a an algebraic subset $V \subset \mathbb{T}^n$

It is natural to extend Definitions 2.1 and 2.5 to the case where the algebraic hypersurface  $V(F)$  is any closed algebraic subset  $V$  in  $\mathbb{T}^n = \text{Spec}(\mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}])$ , that is the zero set of any proper ideal  $I \subset \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ .

DEFINITION 2.8. Let  $V = V(I) \subset \mathbb{T}^n$  be the zero set of a proper polynomial ideal  $I$  in  $\mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ . The *archimedean amœba* of  $V$  is defined as

$$(2.15) \quad \mathcal{A}_V := \text{Log}(V) \subset \mathbb{R}^n.$$

The *archimedean coamœba* of  $V$  is defined as

$$(2.16) \quad \mathcal{A}'_V := \arg(V) \subset (\mathbb{S}^1)^n.$$

This definition 2.15 is just the pendant in the archimedean (instead of ultrametric) context of that of  $V_{\text{Trop}, \nu}(I)$  (see (1.27)),  $I$  being an ideal in  $\mathbb{K}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ , where  $\mathbb{K}$  denotes some algebraically closed commutative field equipped with a non trivial valuation  $\nu$ . The possibility to profit from the intrinsic relation between complex conjugaison and positivity in the  $(\mathbb{C}, +, \times)$  usual calculus<sup>8</sup> makes indeed quite easy to obtain in the archimedean context the analog of the first assertion in Proposition 1.3. The same reasoning leads to an analog statement concerning coamœbas. Namely, one has ([Purb], with respect to the assertion about amœbas):

PROPOSITION 2.4. *Let  $V$  be any closed algebraic subset in  $\mathbb{T}^n$  and  $I$  be any proper polynomial ideal in  $\mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$  such that*

$$V = V_{\mathbb{T}}(I) = \{z \in \mathbb{T}^n; F(z) = 0 \quad \forall F \in I\}.$$

<sup>8</sup> $Re(P(z)\overline{P(z)}) \geq 0$  for  $z \in \mathbb{T}^n$  and any  $P \in \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ .

Then,

$$(2.17) \quad \mathcal{A}_V = \bigcap_{F \in I} \mathcal{A}_{V_{\mathbb{T}}(F)}, \quad \mathcal{A}'_V = \bigcap_{F \in I} \mathcal{A}'_{V_{\mathbb{T}}(F)}.$$

REMARK 2.9. As a consequence of Proposition 2.4, one could call (as in [Purb])  $\mathcal{A}_{V(I)}$  the *archimedean amoeba of the polynomial ideal*  $I \subset \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$  (as  $\mathcal{A}_{V_{\mathbb{T}}(F)}$  is called the *amoeba of the Laurent polynomial*  $F$ ). The same holds with respect to coamoebas.

PROOF. Since  $V = V_{\mathbb{T}}(I) \subset V_{\mathbb{T}}(F)$  for any  $F \in I$ , one has the inclusions  $\mathcal{A}_V \subset \mathcal{A}_{V_{\mathbb{T}}(F)}$  and  $\mathcal{A}'_V \subset \mathcal{A}'_{V_{\mathbb{T}}(F)}$  for any  $F \in I$ , that is  $\mathcal{A}_{V_{\mathbb{T}}(I)} \subset \bigcap_{F \in I} \mathcal{A}_{V_{\mathbb{T}}(F)}$  and  $\mathcal{A}'_{V_{\mathbb{T}}(I)} \subset \bigcap_{F \in I} \mathcal{A}'_{V_{\mathbb{T}}(F)}$ . Let now  $\{F_1, \dots, F_M\}$  be a finite set of generators for the ideal  $I$  and  $x \in \mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(I)}$ . Consider the Laurent polynomial

$$\mathcal{F}_x(X) := \sum_{j=1}^M F_j(X) \overline{F_j}(e^{2x_1} X_1^{-1}, \dots, e^{2x_n} X_n^{-1}).$$

If  $z = (e^{x_1 + i\theta_1}, \dots, e^{x_n + i\theta_n}) \in \text{Log}^{-1}(x)$ , one has

$$\mathcal{F}_x(z) = \sum_{j=1}^M F_j(z) \overline{F_j}(\bar{z}) = \sum_{j=1}^M |F_j(z)|^2 > 0.$$

Therefore, one has

$$\left( x \in \mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(I)} \right) \implies \left( \exists \mathcal{F} = \mathcal{F}_x \in I, \quad x \in \mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(F)} \right),$$

that is

$$\mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(I)} \subset \bigcup_{F \in I} (\mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(F)}).$$

Taking complements in  $\mathbb{R}^n$  leads to  $\bigcap_{F \in I} \mathcal{A}_{V_{\mathbb{T}}(F)} \subset \mathcal{A}_{V_{\mathbb{T}}(I)}$ .

Let now  $\theta \in (\mathbb{S}^1)^n \setminus \mathcal{A}'_{V_{\mathbb{T}}(I)}$ . Consider this time the Laurent polynomial

$$\mathfrak{F}_\theta(X) := \sum_{j=1}^M F_j(X) \overline{F_j}(e^{-2i\theta_1} X_1, \dots, e^{-2i\theta_n} X_n).$$

If  $z = (e^{x_1 + i\theta_1}, \dots, e^{x_n + i\theta_n}) \in \text{arg}^{-1}(\theta)$ , one has

$$\mathfrak{F}_\theta(z) = \sum_{j=1}^M F_j(z) \overline{F_j}(\bar{z}) = \sum_{j=1}^M |F_j(z)|^2 > 0.$$

Therefore, one has

$$\left( \theta \in (\mathbb{S}^1)^n \setminus \mathcal{A}'_{V_{\mathbb{T}}(I)} \right) \implies \left( \exists \mathfrak{F} = \mathfrak{F}_\theta \in I, \quad \theta \in (\mathbb{S}^1)^n \setminus \mathcal{A}'_{V_{\mathbb{T}}(F)} \right),$$

that is

$$(\mathbb{S}^1)^n \setminus \mathcal{A}'_{V_{\mathbb{T}}(I)} \subset \bigcup_{F \in I} ((\mathbb{S}^1)^n \setminus \mathcal{A}'_{V_{\mathbb{T}}(F)}).$$

Taking complements in  $(\mathbb{S}^1)^n$  leads to  $\bigcap_{F \in I} \mathcal{A}'_{V_{\mathbb{T}}(F)} \subset \mathcal{A}'_{V_{\mathbb{T}}(I)}$ .  $\square$



Recall (see Remark 1.12) that a list of strictly positive numbers  $\{e^{y_1}, \dots, e^{y_r}\}$  is *lobsided* (in the archimedean sense) if and only if one of the  $e^{y_j}$ ,  $j = 1, \dots, r$ , is strictly greater than the (ordinary) sum of all others. This is the exact pendant in the archimedean setting of the notion of *tropical lobsidedness* in the ultrametric setting (see Definition 1.11).

Wen  $c_0, \dots, c_n$  are  $n + 1$  non zero complex numbers and

$$L_c(X) = c_0 + c_1 X_1 + \dots + c_n X_n,$$

it is equivalent to say (see Remark 1.12) that  $x$  belongs to  $\mathcal{A}_{V_{\mathbb{T}}(L_c)}$  or that the list  $\{|c_0|, |c_1|e^{x_1}, \dots, |c_n|e^{x_n}\}$  is not lobsided in the archimedean sense. Unfortunately, things are more involved when  $F$  is replaced by an arbitrary Laurent polynomial in  $n$  variables and the companion archimedean assertion to (1.20) does not quite hold for a general Laurent polynomial  $F$ . One needs to replace (1.20) by the following lemma [**Purb**].

LEMMA 2.10 (archimedean amoeba of a Laurent polynomial and lobsidedness). *Let  $F \in \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$  be a Laurent polynomial in  $n$  variables and, for  $k \in \mathbb{N}^*$ ,*

$$(2.18) \quad F_k(X) = \prod_{(l_1, \dots, l_n) \in \mathbb{F}_k^n} F(e^{2i\pi l_1/k} X_1, \dots, e^{2i\pi l_n/k} X_n), \quad \text{where } \mathbb{F}_k := \mathbb{Z}/k\mathbb{Z}.$$

*Then, one has  $\mathcal{A}_{V_{\mathbb{T}}(F)} = \mathcal{A}_{V_{\mathbb{T}}(F_k)}$ . Moreover, for any  $x \in \mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(F)}$ , one can find  $k(x) \in \mathbb{N}^*$  such that, if  $k \geq k(x)$  and*

$$F_k(X) = \sum_{\alpha \in \text{Supp}(F_k)} c_{k,\alpha} X^\alpha,$$

*the list  $\{|c_{k,\alpha}| e^{\langle \alpha, x \rangle}; \alpha \in \text{Supp}(F_k)\}$  is lobsided in the archimedean sense.*

As a consequence of this Lemma 2.10 (let us postpone its proof for the moment), combined with Proposition 2.4, the following archimedean companion to the second assertion in Proposition 1.3 also holds [**Purb**], namely :

PROPOSITION 2.5. *Let  $V$  be any closed algebraic subset in  $\mathbb{T}^n$  and  $I$  be any proper polynomial ideal in  $\mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$  such that*

$$V = V_{\mathbb{T}}(I) = \{z \in \mathbb{T}^n; F(z) = 0 \quad \forall F \in I\}.$$

*Then, the archimedean amoeba  $\mathcal{A}_{V_{\mathbb{T}}(I)}$  coincides with the subset of  $x \in \mathbb{R}^n$  such that, for any*

$$F = \sum_{\alpha \in \text{Supp}(F)} c_\alpha X^\alpha \in I,$$

*the finite list*

$$\{|c_\alpha| e^{\langle \alpha, x \rangle}; \alpha \in \text{Supp}(F)\}$$

*is not tropically lobsided in the archimedean sense.*

PROOF. If  $x \in \mathcal{A}_{V_{\mathbb{T}}(I)}$ , there exists  $\theta \in \mathbb{R}^n$  such that  $F(e^{x_1+i\theta_1}, \dots, e^{x_n+i\theta_n}) = 0$  for any  $F \in I$ . For any such  $F$ , the list  $\{|c_\alpha| e^{\langle \alpha, x \rangle}; \alpha \in \text{Supp}(F)\}$  cannot then be lobsided in the archimedean sense since this would contradict the archimedean triangle inequality. This proves one of the needed inclusions. In order to prove the second one, one takes first complements in the equality  $\mathcal{A}_{V_{\mathbb{T}}(I)} = \bigcap_{F \in I} \mathcal{A}_{V_{\mathbb{T}}(F)}$ , then uses Lemma 2.10 for each  $F \in I$  (note that  $F_{k(x)} \in I$  as well).  $\square$

PROOF OF LEMMA 2.10. Assume that the Newton polyhedron  $\Delta(F) \subset \mathbb{R}_{\xi_1, \dots, \xi_n}^n$  has non empty interior in  $\mathbb{R}^n$  (this is not restrictive, otherwise, one needs to consider the Laurent polynomial  $F$  being truly in strictly less than  $n$  variables). Let

$$c_F := \max_{1 \leq j \leq n} \left( \max_{\Delta(F)} \xi_j - \min_{\Delta(F)} \xi_j \right) > 0.$$

Let  $k \in \mathbb{N}^*$ . For any  $(l_1, \dots, l_n) \in \mathbb{F}_k^n$  and any  $z \in \mathbb{T}^n$ , both points  $z = (z_1, \dots, z_n)$  and  $(e^{2i\pi l_1/k} z_1, \dots, e^{2i\pi l_n/k} z_n)$  are carried to the same point in  $\mathbb{R}^n$  via the logarithmic map  $\text{Log}$ . It follows that the amoebas  $\mathcal{A}_{V_{\mathbb{T}}(F)}$  and  $\mathcal{A}_{V_{\mathbb{T}}(F_k)}$  do indeed coincide. Since the Laurent polynomial  $F_k$  is invariant under the action of  $\mathbb{F}_k^n$  on  $\mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$  defined as as

$$(e^{2i\pi l_1/k}, \dots, e^{2i\pi l_n/k}) \cdot (X_1, \dots, X_n) = (e^{2i\pi l_1/k} X_1, \dots, e^{2i\pi l_n/k} X_n),$$

and its Newton polyhedron equals  $\Delta(F_k) = \Delta(F) + \dots + \Delta(F)$  ( $k^n$  times), that is  $\Delta(F_k) = k^n \Delta(F)$ ,  $F_k$  is of the form

$$F_k(X) = \sum_{\alpha \in k\mathbb{Z}^n \cap k^n \Delta(F)} c_{k,\alpha} X^\alpha.$$

The cardinal of the support of  $F_k$  can be estimated in terms of the Ehrhart polynomial of the Newton polyhedron  $\Delta(F)$ : the *Ehrhart polynomial*  $E_{\Delta(F)}$  of  $\Delta(F)$  is (see [Fult], section 4.4, or also [Beck]) the polynomial in  $t$  (with degree  $n$ ) defined as

$$\begin{aligned} \text{card}(t\Delta(F) \cap \mathbb{Z}^n) &= \frac{n! \text{Vol}_n(\Delta(F))}{n!} t^n + \frac{(n-1)! \text{Vol}_{n-1, \text{norm}}(\partial\Delta(F))}{2(n-1)!} t^{n-1} \\ &+ (?) t^{n-2} + \dots + (?) t + 1 \quad \forall t \in \mathbb{N}; \end{aligned}$$

here  $\text{Vol}_{n-1, \text{norm}}(\partial\Delta(F))$  means the  $(n-1)$ -dimensional volume on the facets of  $\Delta(F)$ , normalized<sup>9</sup> with respect to the lattice induced by  $\mathbb{Z}^n$  on the vectorial linear  $(n-1)$ -dimensional subspace corresponding to each facet. Note also that the Ehrhart polynomial  $E_{\Delta(F)}$  can also be interpreted as the restriction to  $\mathbb{N}$  of the Hilbert polynomial

$$t \in \mathbb{Z} \mapsto E_{\Delta(F)}(t) = \chi\left(\mathcal{X}(\Sigma(\Delta(F))), \mathcal{O}(t\Delta(F))\right),$$

where  $\mathcal{X}(\Sigma(\Delta(F)))$  denotes the toric variety attached to the normal fan  $\Sigma(\Delta(F))$  of  $\Delta(F)$ ,  $\mathcal{O}(t\Delta(F))$  the line bundle corresponding to the Cartier ample divisor  $\Delta(F)$  on  $\mathcal{X}(\Delta(F))$ ,  $\chi$  being the Euler-Poincaré characteristic<sup>10</sup>. One has also (see [Fult], section 4.4)

$$(-1)^n E_{\Delta(F)}(-t) = \text{card}(\text{int}(t\Delta(F)) \cap \mathbb{Z}^n) \quad \forall t \in \mathbb{N}^*,$$

which implies that

$$\chi(\Delta(F), \mathcal{O}(\Delta(F))) = 1 - \text{card}(\text{int}(\Delta(F)) \cap \mathbb{Z}^n),$$

<sup>9</sup>As usual, “normalisation” means here that the  $(n-1)$ -volume of the simplex built with the the basis of the induced lattice on the linear subspace corresponding to the facet equals  $1/(n-1)!$ . The same normalisation holds also for the definition of  $\text{Vol}_n$  (this time with  $\mathbb{Z}^n$  as the lattice and  $n$  instead of  $n-1$ ).

<sup>10</sup>We recall here that the Euler characteristic of an invertible sheaf  $\mathcal{F}$  on a projective algebraic variety  $\mathcal{X}$  is defined as

$$\chi(\mathcal{X}, \mathcal{F}) = \text{rank}(H^0(\mathcal{X}, \mathcal{F})) - \text{rank}(H^1(\mathcal{X}, \mathcal{F})) + \text{rank}(H^2(\mathcal{X}, \mathcal{F})) - \dots$$

(all Čech cohomology groups involved here being finitely dimensional). For a line bundle, take as invertible sheaf the sheaf of holomorphic sections.

that is  $\text{card}(\text{int}(\Delta(F)) \cap \mathbb{Z}^n)$  stands for the *arithmetic genus* of the line bundle  $\Delta(F)$ . An estimate  $d_F$  for  $\sup_{t \in \mathbb{N}^*} (E_{\Delta(F)}(t)/t^n)$  is provided in [BtMc]. Since one has  $\Delta(F_k) = k^n \Delta(F)$ , one has

$$(2.19) \quad \text{card}(\text{Supp}(F_k)) \leq \text{card}(k^{n-1} \Delta(F) \cap \mathbb{Z}^n) = E_{\Delta(F)}(k^{n-1}) \leq k^{n^2-n} d_F.$$

Suppose that  $x \in \mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(F)} = \mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(F_k)}$  is such that its distance to the amoeba  $\mathcal{A}_{V_{\mathbb{T}}(F)} = \mathcal{A}_{V_{\mathbb{T}}(F_k)}$  equals at least  $\epsilon = \epsilon(x) > 0$ . We claim that, provided

$$(2.20) \quad k \geq \frac{1}{\epsilon(x)} \left( (n^2 - 1) \log k + \log \left( \frac{16}{3} c_F d_F \right) \right),$$

there exists at least one  $\alpha = \alpha_{k,x} \in \text{Supp}(F_k)$ , such that

$$|c_{k,\alpha}| e^{\langle \alpha, x \rangle} > d_F k^{n^2-n} |c_{k,\alpha'}| e^{\langle \alpha', x \rangle} \quad \forall \alpha' \in \text{Supp}(F_k) \setminus \{\alpha\}.$$

Since  $\text{card}(\text{Supp}(F_k)) \leq d_F k^{n^2-n}$  (see (2.19)), it will follow then that

$$(2.21) \quad |c_{k,\alpha}| e^{\langle \alpha, x \rangle} > \sum_{\substack{\alpha' \in \text{Supp}(F_k) \\ \alpha' \neq \alpha}} |c_{k,\alpha'}| e^{\langle \alpha', x \rangle},$$

which means precisely that the finite list

$$\{|c_{k,\alpha}| e^{\langle \alpha, x \rangle}; \alpha \in \text{Supp}(F_k)\}$$

is lobsided in the archimedean sense. It remains to prove such claim.

Fix  $x = (x_1, \dots, x_n)$  as before, at least  $\epsilon = \epsilon(x)$ -distant from  $\mathcal{A}_{V_{\mathbb{T}}(F)}$ . For any  $j = 1, \dots, n$ , let  $\Gamma_{x,j}$  be the annulus in the complex plane

$$\Gamma_{x,j} = \{\zeta \in \mathbb{C}; x_j - \epsilon < \log |\zeta| < x_j + \epsilon\}.$$

Let  $z_x \in \text{Log}^{-1}(x)$ . For any  $k \in \mathbb{N}^*$ , the Laurent polynomial in one variable

$$f_{z_x,j,k} := F_k(z_{x,1}, \dots, z_{x,j-1}, Y, z_{x,j+1}, \dots, z_{x,n})$$

has no root within the annulus  $\Gamma_{x,j}$  (otherwise, this would contradict the fact that the distance from  $x$  to the amoeba  $\mathcal{A}_{V_{\mathbb{T}}(F_k)} = \mathcal{A}_{V_{\mathbb{T}}(F)}$  is at least  $\epsilon$ ). The length of its (1-dimensional) Newton polyhedron is bounded by  $c_F k^n$  and all points in its support belong to  $k\mathbb{Z}$ . A technical lemma<sup>11</sup> in one variable shows (Lemma 2.1 and 2.2 in [Purb]) that, if condition (2.20) is fulfilled, there is indeed one “distinguished” monomial among those involved in  $f_{z_x,j,k}$  such that, when evaluated at  $e^{x_j}$ , its modulus is strictly bigger than  $2 d_F k^{n^2-n}$  times the modulus of any other one. Thanks to a connexity argument, this “distinguished” monomial remains the same for all  $f_{z_x,j,k}$ , when  $j$  and  $k$  are kept fixed and  $z_x$  varies in the connected set  $\text{Log}^{-1}(x)$ . Moreover, it corresponds to some  $X^{k\nu_j}$ -term, where  $\nu_j = \nu_j(k) \in \mathbb{Z}$

<sup>11</sup>In order to guess such an argument, take a polynomial  $f(Y) = \prod_{j=1}^d (Y - \alpha_j)$  in one variable such that all roots are non zero and have distinct moduli  $|\alpha_1| > \dots > |\alpha_d| > 0$ . The polynomial  $f_k(Y) = \prod_{j=1}^d (Y^k - \alpha_j^k)$  has as a “leading” expression

$$Y^{kd} - \alpha_1^k Y^{k(d-1)} + (\alpha_1 \alpha_2)^k Y^{k(d-2)} + \dots + (-1)^d (\alpha_1 \dots \alpha_d)^k.$$

If  $e^x$  belongs to the annulus  $|\alpha_{l+1}| < e^x < |\alpha_l|$  (for  $1 \leq l \leq d$ , with the convention that  $\alpha_{d+1} = 0$ ), then, when  $k$  becomes large enough, the monomial term  $(\alpha_1 \dots \alpha_l)^k Y^{n-l}$  is, when evaluated at  $Y = e^x$ , the “leading” monomial (when taking its modulus) among those involved in  $f_k$  and evaluated at  $e^x$ .

remains independent of  $z_x$  provided  $k$  fulfills (2.20). Let  $\nu = (\nu_1, \dots, \nu_n)$ . One can decompose thus  $F_k$  as :

$$F_k(X) = c_{k,k\nu} X^{k\nu} + \sum_{\substack{\alpha \in \text{Supp}(F_k) \\ \alpha \neq k\nu}} c_{k,\alpha} X^\alpha.$$

Let  $M_{k,x}$  be the evaluation of the modulus of  $|c_{k,k\nu} X^{k\nu}|$  at an arbitrary point  $z_x$  in  $\text{Log}^{-1}(x)$  (it is of course independent on the point  $z_x$ ) and  $\mu_{k,x}$  be the maximum of all evaluations at  $z_x$  of all  $|c_{k,\alpha} X^\alpha|$  for  $\alpha \in \text{Supp}(F_k)$ ,  $\alpha \neq k\nu$ . The property that is shared by all  $f_{z_x,j,k}$  when  $z_x \in \text{Log}^{-1}(x)$  (the evaluation of a the modulus of the  $Y^{k\nu_j}$  term at  $e^{x_j}$  is strictly bigger than  $2 d_F k^{n^2-n}$  times the evaluation of modulus of all other monomials), combined with the fact that  $\text{card}(\text{Supp}(F_k)) \leq d_F k^{n^2-n}$ , implies that

$$\mu_{k,x} < \frac{M_{k,x} + \mu_{k,x} d_F k^{n^2-n}}{2 d_F k^{n^2-n}},$$

from which it follows that  $\mu_{k,x} < d_F k^{n^2-n} M_{k,x}$ , which concludes the proof of our claim and thus the proof of Lemma 2.10.  $\square$

REMARK 2.11. Let  $\eta \in ]0, 1[$ . Note that, refining the choice of  $k = k(\epsilon)$ , one could as well ensure that, for any  $x$  at a distance at least equal to  $\epsilon$  from the amoeba  $\mathcal{A}_{V_{\mathbb{T}}(F)}$ , for any  $k \geq k(\eta, \epsilon) \gg 1$ , there exists at least one  $\alpha = \alpha_{k,x}$  in  $\text{Supp}(F_k)$  such that the inequality

$$(2.22) \quad \sum_{\substack{\alpha' \in \text{Supp}(F_k) \\ \alpha' \neq \alpha}} |c_{k,\alpha'}| e^{\langle \alpha', x \rangle} < \eta |c_{k,\alpha}| e^{\langle \alpha, x \rangle}$$

holds (instead of (2.21)). We will use this remark later on (in the proof of Proposition 3.5).



## From complex algebraic geometry to real tropical geometry

### 3.1. Deformation of the amoeba of an algebraic hypersurface

We will show in this section how one can “deform” in a continuous way the archimedean amoeba  $\mathcal{A}_{V_{\mathbb{T}}(F)}$  of a Laurent polynomial  $F \in \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$  (better considered now in the tropical world  $((\text{Trop} \setminus \{-\infty\})^n, \boxplus, \boxtimes)$  instead of  $(\mathbb{R}^n, +, \times)$ ) into a tropical hypersurface in  $(\text{Trop} \setminus \{-\infty\})^n$ . Such a tropical hypersurface will thus be “interpreted” in a natural way as the “*spine*” of the archimedean amoeba  $\mathcal{A}_{V_{\mathbb{T}}(F)}$  (which will confort the parallel one could make with amoebas in biology, besides the fact that such animals are known to be invertebrate !). In this section, the algebraic hypersurface  $V_{\mathbb{T}}(F)$  is considered in  $\mathbb{T}^n$  ( $\mathbb{C}$  being equipped here with its usual operations  $+$  and  $\times$ ). Later on, one will compare such a tropical deformation of  $V_{\mathbb{T}}(F)$  with the image by  $\text{Log}$  of the “zero set” of  $F$ , considered this time in the hyperfield of tropical numbers, namely in  $((\text{Trop}_{\mathbb{C}} \setminus \{0\})^n, \boxplus_{\mathbb{C}}, \boxtimes_{\mathbb{C}})$  (see subsection 1.1.4).

**3.1.1. Multiplicities attached to  $F$  with respect to connected components of  $\mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(F)}$ .** Let  $F \in \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$  be a Laurent polynomial and  $\mathcal{A}_{V_{\mathbb{T}}(F)}$  its archimedean amoeba. Let  $C$  be a connected component of  $\mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(F)}$ ,  $x$  be a point in  $C$ , and  $z_x$  be an arbitrary point in  $\text{Log}^{-1}(x)$ . Consider, for each  $j = 1, \dots, n$ , the continuous loop

$$(3.1) \quad \theta_j \in [0, 2\pi] \longmapsto F(z_{x,1}, \dots, z_{x,j-1}, e^{x_j + i\theta_j}, z_{x,j+1}, \dots, z_n) \in \mathbb{C}^*,$$

that is the image by  $F$  of the continuous loop

$$\gamma_{z_x, j} : \theta_j \in [0, 2\pi] \longmapsto (z_{x,1}, \dots, z_{x,j-1}, e^{x_j + i\theta_j}, z_{x,j+1}, \dots, z_n) \in \text{Log}^{-1}(x).$$

The degree  $\nu_{F, x, z_x, j}$  of the loop  $F \circ \gamma_{z_x, j}$  remains constant while  $x \in C$  and  $z_x \in \text{Log}^{-1}(x)$  since this integer valued degree varies continuously on the connected set

$$\{(x, z_x); x \in C, z_x \in \text{Log}^{-1}(x)\} \subset \mathbb{R}^n \times \mathbb{T}^n.$$

Therefore, there exists  $(\nu_{F, C, 1}, \dots, \nu_{F, C, n}) \in \mathbb{Z}^n$  such that

$$\forall x \in C, \forall z_x \in \text{Log}^{-1}(x), \forall j = 1, \dots, n, \quad \nu_{F, x, z_x, j} = \nu_{F, C, j}.$$

**DEFINITION 3.1** (multiplicities attached to a Laurent polynomial with respect to its amoeba). Let  $F \in \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$  be a Laurent polynomial in  $n$  variables and  $C$  be a connected component of  $\mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(F)}$ . The  $n$ -uplet  $\nu_{F, C} := (\nu_{F, C, 1}, \dots, \nu_{F, C, n}) \in \mathbb{Z}^n$  is called the *multiplicity attached to  $F$  with respect to the connected component  $C$  of  $\mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(F)}$* .

REMARK 3.2. It is important to observe that such multiplicities depend indeed on the Laurent polynomial  $F$  itself, before depending on its amoeba. They therefore carry some algebraic information, not only a geometric one (as the amoeba only does). For example, if  $F$  is replaced by a power  $F^k$  or even some  $F_k$ ,  $k \geq 2$ , as in Lemma 2.10, the amoeba does not change, while of course multiplicities are drastically modified. When  $F$  is replaced by  $F \times X^\alpha$ , where  $X^\alpha$  denotes a monomial, multiplicities  $\nu_{F,C}$  are transformed into  $\nu_{F,C} + \alpha$ . Multiplicities depend indeed on  $F$ , that is on the principal ideal  $(F) \subset \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ , not on the hypersurface  $V_{\mathbb{T}}(F) \subset \mathbb{T}^n$ .

EXAMPLE 3.3 (the one-dimensional case). Let  $f$  be a polynomial in one variable

$$f(X) = c_0 Y^d + c_1 Y^{d-1} + \dots + c_d,$$

with Newton polyhedron  $[0, d]$ ,  $d \in \mathbb{N}^*$  (that is  $c_d \neq 0$ ). Thanks to Lebesgue's theorem about continuity of integrals depending on a parameter  $x$ , the function

$$(3.2) \quad G_f : x = x + iy \in \mathbb{C} = \mathbb{R} + i\mathbb{R} \longmapsto - \int_0^{2\pi} \log |f(e^{x+i\theta})| d\theta$$

( $G$  stands here for ‘‘Green’’ since one will later on interpret such a function as a Green function) is well defined and is continuous, despite the fact that  $\log |f|$  admits singularities at all points in  $V_{\mathbb{T}}(f) = \{\alpha \in \mathbb{T}; f = 0\} = f^{-1}(0)$  (such singularities being in fact of logarithmic nature, thus integrable). If  $f$  has no zero in the open annulus  $\{\zeta \in \mathbb{T}; \gamma < \log |\zeta| < \gamma'\}$ , then  $G_f$  is  $C^\infty$  in  $] \gamma, \gamma' [$ . Moreover, since

$$d_x(\log |f(e^{x+i\theta})|) = e^x \operatorname{Re} \left( \frac{f'(e^{x+i\theta})}{f(e^{x+i\theta})} e^{i\theta} \right) dx, \quad \forall x \in ] \gamma, \gamma' [, \quad \forall \theta \in [0, 2\pi],$$

one has

$$\begin{aligned} \forall x \in ] \gamma, \gamma' [, \quad G'_f(x) &= -\operatorname{Re} \left( \frac{1}{2\pi} \int_0^{2\pi} \left( \frac{f'(e^{x+i\theta})}{f(e^{x+i\theta})} \right) e^{x+i\theta} d\theta \right) \\ &= -\frac{1}{2i\pi} \int_{\theta \mapsto e^{x+i\theta}} \frac{f'(\zeta)}{f(\zeta)} d\zeta = -\operatorname{card} (f^{-1}(0) \cap \overline{D(0, e^\gamma)}). \end{aligned}$$

If  $C$  is a connected component of  $\mathbb{R} \setminus \mathcal{A}_{V_{\mathbb{T}}(f)}$ , one has then

$$(3.3) \quad \nu_{f,C} = \operatorname{card} (f^{-1}(0) \cap \overline{D(0, e^{\inf C})}).$$

In particular,  $\nu_{f,C}$  is an integer that belongs to the Newton polyhedron (here  $[0, d]$ ) of  $f$ . If the amoeba of  $f$  is  $\{a_1, \dots, a_{d'}\}$ , with  $a_1 < a_2 < \dots < a_{d'}$  ( $d' \leq d$ ), then the function  $G_f$  is the piecewise linear continuous function with slope  $\nu_{f,C}$  (as in 3.3) on each open interval  $C = ]a_l, a_{l+1}[$ ,  $l = 0, \dots, d'$  (with  $a_0 = -\infty$  and  $a_{d'+1} = +\infty$ ). This is just an alternative way to phrase the well known Jensen's formula, see *e.g.* [BG] or [Yac], Theorem 4.4. In particular, one has  $G_f(x) \equiv -\log |c_0|$  on  $] -\infty, a_1[ + i\mathbb{R}$  and  $G_f(x) \equiv -\log |c_d| - dx$  on  $] a_{d'}, +\infty[ + i\mathbb{R}$ . Note that, apart from these two ‘‘extremal’’ cases, the constant coefficient  $\tau_C$  such that  $G_f(x) \equiv -\tau_C - \nu_C x$  on  $C + i\mathbb{R}$  cannot be recovered easily in terms of the  $\log |c_\alpha|$ , where  $\alpha \in \operatorname{Supp}(f)$  (it can be expressed inductively in terms of  $\log |c_0|$ , the  $a_l$  and the  $\nu_{f,]a_l, a_{l+1}[}$  thanks to Jensen's formula, see [BG] or [Yac], Theorem 4.4).

It follows from the argument principle that one has, for each connected component  $C$  of  $\mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(F)}$ , for any  $x \in \mathbb{C}$ , for any  $j = 1, \dots, n$ ,

$$(3.4) \quad \nu_{F,C,j} = \frac{1}{(2i\pi)^n} \int_{\text{Log}^{-1}(x)} \frac{\partial F}{\partial \zeta_j}(\zeta) \frac{1}{F(\zeta)} \bigwedge_{j=1}^n \frac{d\zeta_j}{\zeta_j},$$

where the  $n$ -cycle

$$\text{Log}^{-1}(x) = \{(e^{x_1+i\theta_1}, \dots, e^{x_n+i\theta_n}); \theta_j \in [0, 2\pi], j = 1, \dots, n\}$$

is oriented as in the multidimensional Cauchy representation formula.

As in example 3.3, one can introduce a distribution  $G_F$  in  $\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n$  as follows : consider the locally integrable function  $z = x + iy \mapsto \log |F(e^{x+iy})|$  as a distribution<sup>1</sup> in  $\mathbb{C}^n$  that is  $(2\pi, \dots, 2\pi)$ -periodic in  $y = \text{Im } z$ , that is as a distribution on  $\mathbb{R}^n + i(\mathbb{S}^1)^n$ , then take  $G_F$  such that :

$$(3.5) \quad \langle G_F, \varphi \rangle := - \left\langle \log |F(e^{x+iy})|, \int_{(\mathbb{S}^1)^n} \varphi(x + i\theta) d\sigma_{(\mathbb{S}^1)^n}(\theta) \right\rangle \quad \forall \varphi \in \mathcal{D}(\mathbb{C}^n),$$

where  $d\sigma_{(\mathbb{S}^1)^n}$  denotes the normalized Haar measure on the group  $(\mathbb{S}^1)^n$ . This distribution  $G_F$  is of course the distribution-function represented by the function

$$(3.6) \quad G_F : z = x + iy \in \mathbb{R}^n + i\mathbb{R}^n \mapsto - \int_{(\mathbb{S}^1)^n} \log |F(e^{x+i\theta})| d\sigma_{(\mathbb{S}^1)^n}(\theta).$$

Nevertheless, since one will later transpose this construction to the case where the Laurent polynomial  $F$  is replaced by an ideal  $I \subset \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ , we preferred to present the “distribution” point of view (3.5) (which is in accordance with the presentation in [Rash]) instead of the “functional” presentation (3.6).

The computations which were done in example 3.3 can be repeated in the  $n$ -dimensional case. One has then the following proposition.

**PROPOSITION 3.1 ([Ronk]).** *The function  $G_F$  defined in (3.6) is  $C^\infty$  in each  $C + i\mathbb{R}^n$ ,  $C$  being a connected component of  $\mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(F)}$ . Moreover*

$$(3.7) \quad dG_F + \sum_{j=1}^n \nu_{F,C,j} dx_j \equiv 0 \quad \text{in } C + i\mathbb{R}^n,$$

that is, the function  $G_F$  is affine in  $C + i\mathbb{R}^n$ , of the form

$$(3.8) \quad G_F(x + iy) + \tau_{F,C} + \langle \nu_{F,C}, x \rangle = 0 \quad \forall x \in \mathbb{C}, \forall y \in \mathbb{R}^n.$$

for some real constant  $\tau_{F,C}$ .

**DEFINITION 3.4 (Ronkin function).** The function  $-G_F$  has been introduced by L. Ronkin in [Ronk] and is thus called the *Ronkin function* corresponding to the Laurent polynomial  $F$ .

As in the one variable case, one has the following result about multiplicities.

**PROPOSITION 3.2 ([FPT]).** *Let  $F \in \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$  and  $\mathcal{A}_{V_{\mathbb{T}}(F)}$  its archimedean amoeba. For each component  $C$  of  $\mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(F)}$ , the multiplicity  $\nu_{F,C}$  is a point in  $\mathbb{Z}^n \cap \Delta(F)$ . Moreover, the map which assigns to each connected component  $C$  of  $\mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(F)}$  the multiplicity  $\nu_{F,C} \in \mathbb{Z}^n \cap \Delta(F)$  is injective. As a consequence,*

<sup>1</sup>See Appendix A for a brief presentation of distribution theory.



the number of connected components of  $\mathbb{R}^n \setminus \mathcal{A}_{V(\mathbb{T})}(F)$  is bounded from above by  $\text{card}(\Delta(P) \cap \mathbb{Z}^n) = E_{\Delta(F)}(1)$ .

PROOF. Let  $x \in C$  and  $a = (a_1, \dots, a_n) \in \mathbb{Z}^n \setminus \{0\}$ . Let  $z_x \in \text{Log}^{-1}(x)$ . The loop

$$\gamma_{z_x, a} : \theta \in [0, 2\pi] \mapsto (z_{x,1}e^{ia_1\theta}, \dots, z_{x,n}e^{ia_n\theta}) \in \text{Log}^{-1}(x)$$

is homologous (in  $\text{Log}^{-1}(x)$ ) to the 1-cycle

$$\sum_{j=1}^n a_j \left( \theta_j \in [0, 2\pi] \mapsto (z_{x,1}, \dots, z_{x,j-1}, z_{x,j}e^{i\theta_j}, z_{x,j+1}, \dots, z_{x,n}) \right) = \sum_{j=1}^n a_j \gamma_{z_x, j}.$$

Therefore, the number of zero-poles of the Laurent polynomial in one variable  $f_{z_x, a}(Y) = F(z_{x,1}Y^{a_1}, \dots, z_{x,n}Y^{a_n})$  inside the unit disc can be expressed thanks to the argument principle as

$$\frac{1}{2i\pi} \int_{F \circ \gamma_{z_x, a}} \frac{d\zeta}{\zeta} = \frac{1}{2i\pi} \int_{\gamma_{z_x, a}} \frac{dF}{F} = \sum_{j=1}^n a_j \deg(F \circ \gamma_{z_x, j}) = \langle a, \nu_{F, C} \rangle.$$

Since the top degree of the Laurent polynomial  $f_{z_x, a}$  is  $\max_{\xi^* \in \Delta(F)} \langle a, \xi^* \rangle$ , one has

$$\langle a, \nu_{F, C} \rangle \leq \max_{\xi^* \in \Delta(F)} \langle a, \xi^* \rangle \quad \forall a \in \mathbb{Z}^n \setminus \{0\}.$$

This implies  $\nu_{F, C} \in \Delta(F)$ .

Consider now two points  $x \in C \cap \mathbb{Q}^n$  and  $x' \in C' \cap \mathbb{Q}^n$  in two different connected components of  $\mathbb{R}^n \setminus \mathcal{A}_{V(\mathbb{T})}(F)$ . Then the segment  $[x, x']$  necessarily hits  $\mathcal{A}_{V(\mathbb{T})}(F)$ . Let  $x' = x + ta$  with  $t \in \mathbb{Q}_{>0}$  and  $a \in \mathbb{Z}^n \setminus \{0\}$ . Let us prove that  $\langle a, \nu_{F, C'} \rangle > \langle a, \nu_{F, C} \rangle$ , interpreting (see above) these two numbers as the numbers of zero-poles respectively of the Laurent polynomials (in one variable)  $f_{z_{x'}, a}$  and  $f_{z_x, a}$  when  $z_{x'} \in \text{Log}^{-1}(x')$  and  $z_x \in \text{Log}^{-1}(x)$ . Since  $|z_{x', j}|/|z_{x, j}| = e^{ta_j}$  for  $j = 1, \dots, n$ , one may interpret also  $\langle \nu_{F, C'}, a \rangle$  as the number of zero-poles of  $f_{z_x, a}$  (pick up if necessary another  $z_x \in \text{Log}^{-1}(x)$  and replace the old one) inside a disc with radius  $e^t$  instead of 1. It is impossible for  $f_{z_x, a}$  to be zero-free in the annulus  $\{\zeta \in \mathbb{C}; 1 < |\zeta| < e^t\}$ ; if it was the case,  $[x, x']$  would not intersect  $\mathcal{A}_{V(\mathbb{T})}(F)$ , which means that  $x$  and  $x'$  would be in the same connected component of  $\mathbb{R}^n \setminus \mathcal{A}_{V(\mathbb{T})}(F)$ . Thus  $\nu_{F, C'}$  certainly differs from  $\nu_{F, C}$  whenever  $C$  and  $C'$  are distinct.  $\square$

Let  $\tau \prec \Delta(F)$  be a face of the Newton polyedron  $\Delta(F)$ . Let  $\Gamma_\tau \subset \mathbb{R}^n$  be the  $((n - \dim \tau)$ -dimensional) *polar cone* of the face  $\tau \prec \Delta(F)$  with respect to the Newton polyedron  $\Delta(F)$ , that is

$$\Gamma_\tau := \{a \in \mathbb{R}^n; \{u \in \Delta(F); \langle a, u \rangle = \max_{v \in \Delta(F)} \langle a, v \rangle\} = \tau\}.$$

( $-\Gamma_\tau$  is called the *dual cone*<sup>2</sup> of the face  $\tau \prec \Delta(F)$  with respect to  $\Delta(F)$ ). Any point  $\nu$  in  $\mathbb{Z}^n \cap \Delta(F)$  which is not a vertex of  $\Delta(F)$  belongs to the relative interior of a unique face  $\tau(\nu) \prec \Delta(F)$ . One can therefore attach to such a point  $\nu \in \mathbb{Z}^n \cap \Delta(F)$  a polar cone  $\Gamma_\nu$  in  $\mathbb{R}^n$ :  $\Gamma_\nu = \Gamma_{\tau(\nu)}$  if  $\nu$  is not a vertex of  $\Delta(F)$ ,  $\Gamma_\nu = \Gamma_{\{\nu\}}$  if  $\nu = \sigma$  is a vertex of  $\Delta(F)$ . Note that, if  $\nu$  belongs to the relative interior of  $\Delta(F)$ ,

<sup>2</sup>When  $F$  is a true Laurent polynomial in  $n$  variables, that is the interior of  $\Delta(F)$  is not empty, the closures of all dual cones corresponding to all faces of  $\Delta(F)$  are all strict cones in  $\mathbb{R}^n$  which form the fan  $\Sigma(\Delta(F))$ , from which one can construct the toric complete projective variety  $\mathcal{X}(\Sigma(\Delta(F)))$ , see subsection 4.2.1 in the next chapter for a short recap about complete toric varieties.

$\Gamma_\nu = \Gamma_{\Delta(F)} = (\text{vec}(\Delta(F)))^\perp$ ; in particular  $\Gamma_\nu = \{0\}$  for such  $\nu$  in the relative interior of  $\Delta(F)$  when  $F$  is a true Laurent polynomial in  $n$  variables (that is, the interior of  $\Delta(F)$  in  $\mathbb{R}^n$  is nonempty).

PROPOSITION 3.3 (existence of recession cones, [FPT]). *Each vertex  $\sigma$  of  $\Delta(F)$  corresponds to a unique open connected component  $C_\sigma$  of  $\mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(F)}$ . One has*

$$(3.9) \quad G_F(x + iy) + \log |c_\sigma| + \langle \sigma, x \rangle = 0 \quad \forall x \in C_\sigma, \forall y \in \mathbb{R}^n.$$

*If  $C_\nu$  is the unique open connected component of  $\mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(F)}$  corresponding to some  $\nu \in \mathbb{Z}^n \cap \Delta(F)$ , then*

$$C_\nu + \Gamma_\nu \subset C_\nu$$

*and no strictly larger open cone  $\Gamma \subset \mathbb{R}^n$  than  $\Sigma_\nu$  fulfills  $C_\nu + \Gamma \subset C_\nu$ . The cone  $\Gamma_\nu$  is thus called the recession cone of the open connected component  $C_\nu$ .*

PROOF. Let  $\sigma \in \mathbb{Z}^n$  be a vertex of  $\Delta(F)$ . Clearly, one can find points  $z$  in  $\mathbb{T}^n$  such that

$$(3.10) \quad |c_\sigma z^\sigma| > \sum_{\alpha \neq \sigma \in \text{Supp}(F)} |c_\alpha z^\alpha|,$$

that is points where the family  $\{|c_\alpha| e^{\langle \alpha, x \rangle}, \alpha \in \text{Supp}(F)\}$  (where  $x = \text{Log } z$ ) is lopsided in the archimedean sense, with leading term  $|c_\sigma| e^{\langle \sigma, x \rangle}$ . In the connected component  $C_\sigma$  containing such points, the multiplicity  $\nu_{F, C_\sigma}$  equals  $\sigma$  since, for  $x \in C_\sigma$  such that (3.10) holds for any  $z_x \in \text{Log}^{-1}(x)$ , the loop (3.1) is clearly homologous (in  $\mathbb{C}^*$ ) to the loop

$$\theta_j \in [0, 2\pi] \mapsto c_\sigma z_{x,1}^{\sigma_1} \dots z_{x,j-1}^{\sigma_{j-1}} e^{\sigma_j(x_j + i\theta_j)} z_{x,j+1}^{\sigma_{j+1}} \dots z_{x,n}^{\sigma_n}.$$

Any vertex  $\sigma$  of  $\Delta(F)$  stands for the multiplicity of a unique connected open component  $C_\sigma$  of  $\mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(F)}$ . For any  $x$  in  $C_\sigma$  such that (3.10) holds for any point  $z = z_x \in \text{Log}^{-1}(x)$ , for any  $y \in \mathbb{R}^n$ , one has

$$\begin{aligned} G_F(x + iy) &= \\ &= -\frac{1}{(2i\pi)^n} \int_{[0, 2\pi]^n} \left( \log(|c_\sigma| e^{\langle \sigma, x \rangle}) + \log \left| 1 + \frac{\sum_{\alpha \neq \sigma \in \text{Supp}(F)} c_\alpha e^{\langle \alpha, x + i\theta \rangle}}{|c_\sigma| e^{\langle \sigma, x \rangle}} \right| \right) d\theta_1 \dots d\theta_n \\ &= -\frac{1}{(2i\pi)^n} \int_{[0, 2\pi]^n} \log(|c_\sigma| e^{\langle \sigma, x \rangle}) d\theta_1 \dots d\theta_n \\ &= -\log |c_\sigma| - \langle \sigma, x \rangle, \end{aligned}$$

which proves (3.9).

It remains to prove the last assertion. One needs to prove that, given  $x \in C_\nu$  (for some  $\nu \in \mathbb{Z}^n \cap \Delta(F)$ ) and  $a \in \mathbb{R}^n \setminus \{0\}$ , the half-line  $x + \mathbb{R}_{\geq 0} a$  remains disjoint from  $\mathcal{A}_{V_{\mathbb{T}}(F)}$  if and only if  $a \in \Gamma_\nu \setminus \{0\}$  (that is  $\langle a, \nu \rangle = \max_{v \in \Delta(F)} \langle a, v \rangle$ ). It is even sufficient to prove this for  $a \in \mathbb{Z}^n \setminus \{0\}$  since  $\mathbb{Q}^n \setminus \{0\}$  is dense in  $\mathbb{R}^n \setminus \{0\}$ . The fact that the half-line  $x + \mathbb{R}_{\geq 0} a$  remains disjoint from  $\mathcal{A}_{V_{\mathbb{T}}(F)}$  (which means it remains in  $C_\nu$ ) is equivalent to say (see the proof of Proposition 3.2 above) that  $\langle a, \nu \rangle$  represents the number of zero-poles of the one variable Laurent polynomial  $f_{z_{x'}, a} := F(z_{x',1} Y^{a_1}, \dots, z_{x',n} Y^{a_n})$ , when  $z_{x'} \in \text{Log}^{-1}(x')$ ,  $x'$  being any point in  $x + \mathbb{R}_{\geq 0} a$ . Since  $\max_{v \in \Delta(F)} \langle a, v \rangle$  stands for the top degree of this Laurent polynomial  $f_{z_{x'}, a}$ , the condition  $\langle a, \nu \rangle = \max_{v \in \Delta(F)} \langle a, v \rangle$  is fulfilled. If it was not, our interpretation of  $\langle a, \nu \rangle$  as the number of zeroes of the Laurent polynomial  $f_{z_{x'}, a}$ ,

when  $z_{x'} \in \text{Log}^{-1}(x')$ ,  $x'$  being any point in  $x + \mathbb{R}_{\geq 0} a$ , would fail. This proves that  $\Gamma_\nu$  is indeed the largest open cone  $\Gamma \subset \mathbb{R}^n$  such that  $C_\nu + \Gamma_\nu \subset C_\nu$ . The last assertion is thus proved.  $\square$

**3.1.2. The positive current  $[\mathcal{A}_F]$ , the Ronkin function  $R_F$  and its tropical deformation.** We will take profit from this section to introduce basic rudiments about the theory of *currents* (more specifically of *positive currents*) in  $\mathbb{C}^n$ . See first Appendix A for a brief overview of distribution theory. Since the “currential” point of view plays today an important role in complex geometry (intersection theory, potential theory, complex dynamics, *etc.*), it seems important to take here the opportunity of presenting archimedean amoebas in order to sketch a brief overview of such theory. For more details, we mention here basic references such as [Lel], [De0], [YNiam] (Chapter 2) or [De1]. Let  $n \in \mathbb{N}^*$  and  $p, q$  be positive integers less than  $n$ . A  $(p, q)$ -*current* (or a *current with bidegree  $(p, q)$*  or *bidimension  $(n-p, n-q)$* ) in  $\mathbb{C}^n$  is by definition an element of the dual of the space  $\mathcal{D}^{n-p, n-q}(\mathbb{C}^n)$  of smooth  $(n-p, n-q)$ -forms differential forms with compact support (so-called  $(n-p, n-q)$  *test-forms*)  $\varphi$  in  $\mathbb{C}^n$  such as

$$(3.11) \quad \varphi(z) = \sum_{\substack{I=\{i_1, \dots, i_{n-p}\} \subset \{1, \dots, n\}, \#I=n-p \\ K=\{k_1, \dots, k_{n-q}\} \subset \{1, \dots, n\}, \#K=n-q}} \varphi_{I,K}(z) dz_I \wedge d\bar{z}_K, \quad \varphi_{I,K} \in \mathcal{D}(\mathbb{C}^n),$$

keeping here to the notations

$$dz_I := \bigwedge_{\nu=1}^{n-p} dz_{i_\nu}, \quad d\bar{z}_K := \bigwedge_{\nu=1}^{n-q} d\bar{z}_{k_\nu}.$$

Note that one takes here by convention  $1 \leq i_1 < i_2 < \dots < i_{n-p} \leq n$  and  $1 \leq k_1 < k_2 < \dots < k_{n-q} \leq n$ . One can thus consider a  $(p, q)$  current as a differential  $(p, q)$ -form in  $\mathbb{C}^n$  with coefficients distributions, namely

$$(3.12) \quad T(z) = \sum_{\substack{J=\{j_1, \dots, j_p\} \subset \{1, \dots, n\}, \#J=p \\ L=\{l_1, \dots, l_q\} \subset \{1, \dots, n\}, \#L=q}} T^{J,L}(z) dz_J \wedge d\bar{z}_L, \quad T_{J,L} \in \mathcal{D}'(\mathbb{C}^n),$$

still keeping here to the notations

$$dz_J := \bigwedge_{\nu=1}^p dz_{j_\nu}, \quad d\bar{z}_L := \bigwedge_{\nu=1}^q d\bar{z}_{l_\nu}.$$

Note that one takes here, as before, by convention  $1 \leq j_1 < j_2 < \dots < j_p \leq n$  and  $1 \leq l_1 < l_2 < \dots < l_q \leq n$ . The *duality bracket* between  $T$  (expressed as (3.12)) and  $\varphi$  (expressed as (3.11)) is therefore defined as

$$(3.13) \quad \langle T(z), \varphi(z) \rangle := (-2i)^n \sum_{I,K} \epsilon_{I,K} \langle T^{I',K'}, \varphi_{I,K} \rangle$$

where  $I' := \{1, \dots, n\} \setminus I$ ,  $K' := \{1, \dots, n\} \setminus K$  and  $\epsilon_{I,K} = \pm 1$  is chosen in such a way that

$$i^n dz_{I'} \wedge d\bar{z}_{K'} \wedge dz_I \wedge d\bar{z}_K = 2^n \epsilon_{I,K} \bigwedge_{j=1}^n (dx_j \wedge dy_j)$$

( $z_j = x_j + iy_j$ ,  $j = 1, \dots, n$ ).

The *de Rham d-operator* (acting on differential forms with distribution coefficients, thanks to the differentiation of distributions) splits as  $d = \partial + \bar{\partial}$ , where  $\partial$  differentiates  $(p, q)$  currents into  $(p + 1, q)$  currents (0 when  $p = n$ ), while  $\bar{\partial}$  differentiates  $(p, q)$ -currents into  $(p, q + 1)$  currents (0 when  $q = n$ ). The second-order operator  $(i/\pi) \partial \circ \bar{\partial}$  (transforming  $(p, q)$ -currents into  $(p + 1, q + 1)$  currents, 0 when either  $p$  or  $q$  equals  $n$ ) is denoted<sup>3</sup> as  $dd^c$ .

A  $(m, m)$ -current  $T$  in  $\mathbb{C}^n$  ( $0 \leq m \leq n$ ) is said to be *positive* if and only if, for any  $(1, 0)$  test-forms  $\psi_1 = \sum_1^n \psi_{1,j}(z) dz_j, \dots, \psi_m = \sum_1^n \psi_{m,j}(z) dz_j$ , one has

$$(3.14) \quad \left\langle T(z), \bigwedge_{\nu=1}^m (i\psi_\nu(z) \wedge \overline{\psi_\nu(z)}) \right\rangle \geq 0.$$

This implies indeed that all distribution coefficients  $T^{J,L}$  involved in (3.12) are measures, the “diagonal” coefficients  $T^{J,J}$  ( $\#J = m$ ) being even positive measures which sum  $\sum_{\#J=m} T^{J,J}$  is called (as usual in linear algebra) the *trace measure* of such a positive  $(m, m)$ -current  $T$ . The trace measure governs in fact all other (complex) measure coefficients (which are such that  $T^{J,L} = \overline{T^{L,J}}$  for  $J, L$ , arbitrary subsets of  $\{1, \dots, n\}$  with cardinal  $m$ ), see [De0] or also [YNiam], Chapter 2, exercice 2.9).

Positive  $d$ -closed  $(m, m)$ -currents in  $\mathbb{C}^n$  play an important role in complex geometry, since the class they form includes *integration currents* on purely  $n - m$  dimensional closed analytic subsets of  $\mathbb{C}^n$ . As an example (when  $m = 1$ ), consider a Laurent polynomial  $F(X) = \prod_{\nu=1}^q F_\nu^{q_\nu} \in \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ , together with its primary factorization (the  $F_\nu$  are here prime elements in  $\mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ ). The current

$$dd^c[\log |F(e^{x_1+iy_1}, \dots, e^{x_n+iy_n})|] = \frac{i}{2\pi} \partial \bar{\partial} [\log |F(e^{x_1+iy_1}, \dots, e^{x_n+iy_n})|^2]$$

is the *integration current on the hypersurface*<sup>4</sup>  $\{z = x + iy \in \mathbb{C}^n ; F(\exp(z)) = 0\}$ , where

$$\exp(z) := (e^{z_1}, \dots, e^{z_n}),$$

multiplicities being taken into account : it associates to any  $(n - 1, n - 1)$ -test form  $\varphi$  in  $\mathbb{C}^n$  the quantity

$$(3.15) \quad \langle [(F \circ \exp)^{-1}(0)], \varphi \rangle := \sum_{\nu=1}^q q_\nu \int_{\text{Reg} [(F_\nu \circ \exp)^{-1}(0)]} \varphi.$$

Here  $\text{Reg} [(F_\nu \circ \exp)^{-1}(0)]$  denotes the submanifold of non singular (*i.e.* smooth) points of the closed analytic hypersurface defined as the zero set of the entire function  $F_\nu \circ \exp$  ; integration of  $\varphi$  on such set makes sense since the set of singular points of such an hypersurface is a closed analytic subset with dimension at most  $n - 2$  (that is Lebesgue negligible in the  $(n - 1)$ -hypersurface on which integration

<sup>3</sup>Sometimes  $dd^c$  denotes the operator  $(i/(2\pi)) \partial \circ \bar{\partial}$ . We prefer here the convention that  $dd^c = (i/\pi) \partial \circ \bar{\partial}$ , which occurs most frequently in russian terminology (as for example in [Ronk] or [Rash]).

<sup>4</sup>This extends in higher dimension Lelong-Poincaré equation, see Examples A.7 and A.8 in Appendix A.

is performed), see [Lel]. Due to the  $2i\pi\mathbb{Z}^n$  periodicity of  $\exp$ , one can write

$$[(F \circ \exp)^{-1}(0)](z) = \sum_{j=1}^n \sum_{l=1}^n T^{j,l}(x+iy) dz_j \wedge d\bar{z}_l,$$

where all distributions  $(x, y) \mapsto T^{j,l}(x+iy)$  are  $(2\pi, \dots, 2\pi)$  periodic in  $y$ , that is can be considered as distributions on  $\mathbb{R}^n \times (\mathbb{S}^1)^n$ . One defines then a  $(1, 1)$   $d$ -closed positive current  $[\mathcal{A}_F]$  as

$$(3.16) \quad [\mathcal{A}_F](z) = \sum_{j=1}^n \sum_{l=1}^n \left( \int_{(\mathbb{S}^1)^n} T^{j,l}(x+i\theta) d\sigma_{(\mathbb{S}^1)^n}(\theta) \right) dz_j \wedge d\bar{z}_l,$$

where  $d\sigma_{(\mathbb{S}^1)^n}$  denotes the normalized Haar measure on the group  $(\mathbb{S}^1)^n$  (averaging distributions  $T^{j,l}(x+iy)$  with respect to  $y \in (\mathbb{S}^1)^n$  is naturally defined here by duality). It is convenient therefore to note in short :

$$(3.17) \quad [\mathcal{A}_F] = \int_{(\mathbb{S}^1)^n} [(F \circ u \exp)^{-1}(0)] d\sigma_{(\mathbb{S}^1)^n}(u),$$

where

$$u \exp : z \in \mathbb{C}^n \mapsto (u_1 e^{z_1}, \dots, u_n e^{z_n}) \quad \text{for } u = (u_1, \dots, u_n) \in (\mathbb{S}^1)^n.$$

It is immediate to check that, in the sense of currents in  $\mathbb{C}_{x+iy}^n$ ,

$$(3.18) \quad dd^c R_F(z) = [\mathcal{A}_F],$$

where  $R_F$  denotes the Ronkin function

$$(3.19) \quad R_F : z = x + iy \in \mathbb{C}^n \mapsto \int_{(\mathbb{S}^1)^n} \log |F(\exp(x+i\theta))| d\sigma_{(\mathbb{S}^1)^n}(\theta)$$

introduced in Definition 3.4 as the opposite of the Green function  $G_F$  defined in (3.6).

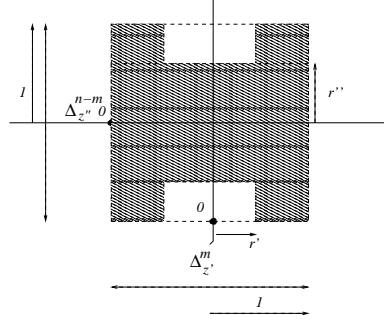
Because of the construction of  $[\mathcal{A}_F]$  by averaging of integration currents and of the definition of the archimedean amoeba  $\mathcal{A}_{V_{\mathbb{T}}(F)}$ , the support<sup>5</sup> of the  $d$ -closed  $(1, 1)$  positive current  $[\mathcal{A}_F]$  is the closed tube  $\mathcal{A}_{V_{\mathbb{T}}(F)} + i\mathbb{R}^n$ , which means that the connected components of the complement of its support are the open tubes  $C + i\mathbb{R}^n$ , where  $C$  denotes any open connected component of  $\mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(F)}$ . Unfortunately, the Lelong number (*i.e.* local mass) of  $[\mathcal{A}_F]$  at an arbitrary point  $z = x + iy$  in its support, which is defined as

$$\nu_z([\mathcal{A}_F]) := \lim_{r \rightarrow 0_+} \left( \frac{1}{r^{2(n-1)}} \int_{\|\zeta - z\| \leq r} [\mathcal{A}_F] \wedge (dd^c \|\zeta - z\|)^{n-1} \right)$$

(see *e.g.* [De0] or [YNiam], Chapter 2), equals 0, since the set of  $\theta \in (\mathbb{S}^1)^n$  such that  $\exp(x+i\theta)$  belongs to  $V_{\mathbb{T}}(F)$  has Lebesgue measure 0 in  $(\mathbb{S}^1)^n$ . The current  $[\mathcal{A}_F]$  presents therefore local mass nowhere in  $\mathbb{C}^n$ .

Nevertheless, geometrically speaking,  $d$ -closed  $(m, m)$ -currents are linked (precisely through the geometry of the connected components of their supports) with a geometric notion of  $(n-m)$ -(pluri)convexity in  $\mathbb{C}^n$ . In order to settle this notion, let us

<sup>5</sup>The support of a  $(p, q)$  current in  $\mathbb{C}^n$  is the complement of the largest open subset of  $\mathbb{C}^n$  to which the restriction of the current equals 0.

FIGURE 3.1. A  $(m, n - m)$ -Hartogs' figure  $\Delta_{r', r''}^{m, n-m}$ 

recall what a  $(m, n - m)$  Hartogs' figure (also called also “Hartogs' marmite” when  $m = 1$ ) in the polydisc

$$\Delta^n := \{(z', z'') \in \mathbb{C}^n = \mathbb{C}^m \times \mathbb{C}^{n-m}; \max_{1 \leq \mu \leq m} |z'_\mu| < 1, \max_{1 \leq \nu \leq n-m} |z''_\nu| < 1\}$$

is. Such a  $(m, n - m)$ -Hartogs' figure in  $\Delta^n$  is an open subset of  $\Delta^n$  of the form

$$\Delta_{r', r''}^{m, n-m} := \{(z', z'') \in \Delta_{z'}^m \times \Delta_{z''}^{n-m}; \max_{1 \leq \mu \leq m} |z'_\mu| > r' \text{ or } \max_{1 \leq \nu \leq n-m} |z''_\nu| < r''\}$$

for some  $r', r'' \in ]0, 1[$ . (see figure 3.1).

**DEFINITION 3.5** ( $(n - m)$ -pseudoconvexity). An open connected subset  $U \subset \mathbb{C}^n$  is said to be  $(n - m)$ -pseudoconvex if and only if, each time there exists an injective holomorphic map  $h : \Delta^n \rightarrow \mathbb{C}^n$  such that  $h(\Delta_{r', r''}^{m, n-m}) \subset U$  for some Hartogs's figure in  $\Delta^n$ , then  $h(\Delta^n) \subset U$ . When  $m = 1$ ,  $(n - 1)$ -pseudoconvex open connected subsets of  $\mathbb{C}^n$  are called *pseudoconvex*.

**REMARK 3.6.** When  $m = 1$ , note that the polydisc  $\Delta^n$  is the envelope of holomorphy of any Hartogs' figure  $\Delta_{r', r''}^{1, n-1}$ . Pseudoconvex domains in  $\mathbb{C}^n$  are exactly domains of holomorphy in  $\mathbb{C}^n$ . A domain of holomorphy in  $\mathbb{C}^n$  is an open connected subset  $U \subset \mathbb{C}^n$  such that, for any compact set  $K \subset\subset U$ , the set

$$\widehat{K}_U := \{z \in U; |h(z)| \leq \sup_K |h| \quad \forall h \in \text{Hol}(U)\}$$

(the *envelope of holomorphy of  $K$  in  $U$* ) remains such that  $\widehat{K} \subset\subset U$ . What makes a significant difference between  $\mathbb{C}$  and  $\mathbb{C}^n$  (when  $n > 1$ ) is that, when  $n > 1$ , not all connected open subsets are domains of holomorphy, but only those which are precisely pseudoconvex, namely *i.e.*  $(n - 1)$ -pseudoconvex.

The result that relates  $d$ -closed  $(m, m)$  positive currents with  $\mathbb{C}^n$  with (pluri)convex geometry is the following result (that one will admit in this course).

**THEOREM 3.7** (see [FS], Section 2, Corollary 2.6). *The connected open components of the complement of the support of any  $(m, m)$   $d$ -closed positive current in  $\mathbb{C}^n$  are  $(n - m)$  pseudoconvex. In particular, the connected components of the complement of the support of any  $(1, 1)$   $d$ -closed positive current in  $\mathbb{C}^n$  are domains of holomorphy.*

As an application of Theorem 3.7, the open tubes  $C + i\mathbb{R}^n$ , when  $C$  is an open connected component of  $\mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(F)}$ , are pseudoconvex open tubes in  $\mathbb{C}^n$ , that is domains of holomorphy. As a consequence, it follows from Bochner's theorem [Boch] that  $C$  is necessarily convex, which provides some way to recover the convexity of open connected components of the complement of the archimedean amoeba  $\mathcal{A}_{V_{\mathbb{T}}(F)}$ . Theorem 3.7 will be later on invoked in order to derive a substitute for the convexity property for the connected components of  $\mathcal{A}_{V_{\mathbb{T}}(I)}$ , when  $I$  is an ideal in  $\mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$  such that  $V_{\mathbb{T}}(I)$  is a  $(n - m)$ -purely dimensional algebraic subvariety of  $\mathbb{T}^n$ , with  $m \in \{2, \dots, n - 1\}$  instead of  $m = 1$ .

Since  $dd^c R_F = [\mathcal{A}_F]$  is a positive current, the continuous Ronkin function  $R_F$  is plurisubharmonic in  $\mathbb{C}^n$ , that is its restriction to each complex line in  $\mathbb{C}^n$  is subharmonic<sup>6</sup>. Since  $R_F$  depends only on the  $x$  coordinates and subharmonicity in the one dimensional real setting corresponds to convexity (see e.g. [Yac], Chapter 4, section 4.1),  $x \mapsto R_F(x)$  is a convex continuous function in  $\mathbb{R}^n$ . Let us introduce its tropical deformation as follows : for any connected component  $C$  of  $\mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(F)}$ , it follows from Proposition 3.1 that

$$\forall z = x + iy \in C + i\mathbb{R}^n, R_F(z) = \tau_{F,C} + \langle \nu_{F,C}, x \rangle$$

for some real constant  $\tau_{F,C}$  (that can be only easily computed in terms of the coefficients of  $F$  when  $\nu_{F,C}$  is a vertex  $\sigma$  of the Newton polyhedron  $\Delta(F)$ , since it equals in this case  $\log |c_{\sigma}|$ ). The convex function

$$(3.20) \quad x \in \mathbb{R}^n \mapsto \max_C (\tau_{F,C} + \langle \nu_{F,C}, x \rangle)$$

is dominated by the function  $x \mapsto R_F(x)$  everywhere in  $\mathbb{R}^n$  since it is affine in each  $C$  and equals  $x \mapsto R_F(x)$  there. Consider the tropical Laurent polynomial

$$(3.21) \quad \mathfrak{p}_{R_F} := \boxplus_C (\tau_{F,C} \boxtimes x_1^{\boxtimes \nu_{F,C,1}} \boxtimes \dots \boxtimes x_n^{\boxtimes \nu_{F,C,n}}).$$

The tropical polynomial  $\mathfrak{p}_{R_F}$  defined in (3.21) will be naturally considered as the tropical deformation of the Ronkin function  $z = x + iy \mapsto R_F(x)$ . The tropical hypersurface  $V_{\text{trop}}(\mathfrak{p}_{R_F})$ , defined in  $\mathbb{R}^n$  as the set of critical points of the convex function (3.20) (see Definition 1.2), lies in the archimedean amoeba  $\mathcal{A}_{V_{\mathbb{T}}(F)}$ . We will show in the next subsection that this tropical hypersurface can be interpreted as the spine of  $\mathcal{A}_{V_{\mathbb{T}}(F)}$ , onto which the archimedean amoeba  $\mathcal{A}_{V_{\mathbb{T}}(F)}$  precisely retracts continuously. A serious stumbling block in this approach remains the fact that the real numbers  $\tau_{F,C}$  (on which the construction of the tropical deformation  $\mathfrak{p}_{R_F}$  relies) cannot be in general expressed in terms of the coefficients of  $F$ . In order to by-pass such a difficulty, we will propose next a “uniform approximating procedure” due to K. Purbhoo [Purb] (in  $(\text{Trop} \setminus \{-\infty\})^n$  equipped with the usual distance in  $\mathbb{R}^n$ ) toward the spine  $V_{\text{trop}}(\mathfrak{p}_{R_F})$ , involving this time instead of the unknown  $\tau_{F,C}$  real numbers that are algebraically explicitly computable in terms of the coefficients of the Laurent polynomials  $F_k$ ,  $k \gg 1$ , introduced in (2.18) (cf. Lemma 2.10).

### 3.1.3. The amoeba $\mathcal{A}_{V_{\mathbb{T}}(F)}$ retracted on its tropical spine. Let

$$F \in \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$$

<sup>6</sup>A typical example of a plurisubharmonic function in an open subset  $U \subset \mathbb{C}^n$  is  $u = \log(|f_1|^2 + \dots + |f_m|^2)$ , where  $f_1, \dots, f_m$  are holomorphic in  $U$  : the current  $dd^c u$  is indeed in this case a positive  $d$ -closed  $(1, 1)$ -current.

be a Laurent polynomial,  $\mathcal{A}_{\mathbb{T}(F)}$  be its archimedean amceba, and

$$(3.22) \quad \mathfrak{p}_{R_F} := \bigsqcup_C (\tau_{F,C} \boxtimes x_1^{\boxtimes \nu_{F,C,1}} \boxtimes \cdots \boxtimes x_n^{\boxtimes \nu_{F,C,n}})$$

be the tropical deformation of its Ronkin function  $R_F : \mathbb{R}^n = (\text{Trop} \setminus \{-\infty\})^n \rightarrow \mathbb{R}$ . The Newton polyedron of the tropical polynomial  $\mathfrak{p}_{R_F}$  (in  $(\mathbb{R}^n)^* \simeq \mathbb{R}^n$ ), as defined in (1.18), coincides with the Newton polyedron  $\Delta(F)$  because of the first assertion in Proposition 3.3.

Consider the corresponding Laurent tropical polynomial function :

$$p_{R_F} : x \in (\text{Trop} \setminus \{-\infty\})^n = \mathbb{R}^n \mapsto \max_C (\tau_{F,C} + \langle \nu_{F,C}, x \rangle) \in (\text{Trop} \setminus \{-\infty\}) = \mathbb{R}$$

and its Legendre-Fenchel transform  $\check{p}_{R_F}$  (see subsection 1.2.3), defined in the dual copy  $(\mathbb{R}^n)^* \simeq \mathbb{R}^n$  of  $\mathbb{R}^n$  in which the Newton polyedron of  $\mathfrak{p}_{R_F}$  (that is in fact  $\Delta(F)$ , as noticed above) precisely lies. Remember (see Proposition 1.1) that the “maximal representation” of the Laurent tropical function  $p_{R_F}$  is indeed

$$(3.23) \quad p_{R_F} : x \in (\text{Trop} \setminus \{-\infty\})^n = \mathbb{R}^n \mapsto \max_C (-\check{p}_{R_F}(\nu_{F,C}) + \langle \nu_{F,C}, x \rangle),$$

which suggests that the real numbers  $-\check{p}_{R_F}(\nu_{F,C})$  are in some sense more “relevant” (with respect to the evaluation of  $\mathfrak{p}_{R_F}$  on  $(\text{Trop} \setminus \{-\infty\})^n$ ) than the (somehow mysterious) constant coefficients  $\tau_{F,C}$ .

Consider, in  $\mathbb{R}^n \times \Delta(F) \subset \mathbb{R}^n \times (\mathbb{R}^n)^*$ , the positive function

$$(x, \xi) \in \mathbb{R}^n \times \Delta(F) \\ \mapsto H_F(\xi, x) := \check{p}_{R_F}(\xi) - (\langle \xi, x \rangle - p_{R_F}(x)) = p_{R_F}(x) + \check{p}_{R_F}(\xi) - \langle \xi, x \rangle.$$

For all  $\xi \in \Delta(F)$ , the closed subset

$$K^\xi := \{x \in \mathbb{R}^n ; H_F(\xi, x) = 0\} \subset \mathbb{R}^n$$

is clearly convex since  $x \mapsto H_F(x, \xi)$  is a convex function. Since  $p_{R_F}$  is the evaluation of a Laurent tropical polynomial function,  $K^\xi$  is even a convex polytope for any  $\xi \in \Delta(F)$ . Moreover, the closed convex polytopes  $K^\xi$ ,  $\xi \in \Delta(F)$ , are such that

$$(3.24) \quad \bigcup_{\xi \in \Delta(F)} K^\xi = (\text{Trop} \setminus \{-\infty\})^n = \mathbb{R}^n.$$

This comes from the fact that, for any  $x \in \mathbb{R}^n$ , there exists always (see (3.23)) at least one  $\xi = \nu_{F,C(x)} \in \Delta(F)$  such that  $p_{R_F}(x) = -\check{p}_{R_F}(\nu_{F,C(x)}) + \langle \nu_{F,C(x)}, x \rangle$ , which implies that  $H_F(\nu_{F,C(x)}, x) = 0$ , *i.e.*  $x \in K^{\nu_{F,C(x)}}$ . In fact, the collection of all  $K^\xi$ ,  $\xi \in \Delta(F)$ , defines a *polytopal convex subdivision* of  $\mathbb{R}^n$ . That is<sup>7</sup>, besides the fact that each  $K^\xi$  is a closed convex polytope of  $\mathbb{R}^n$  for any  $\xi \in \Delta(F)$  and (3.24) is fulfilled, one has :

- whenever  $K^{\xi_1} \cap K^{\xi_2}$  is nonempty for some  $\xi_1, \xi_2 \in \Delta(F)$ , the intersection  $K^{\xi_1} \cap K^{\xi_2}$  equals  $K^\xi$  for some  $\xi \in \Delta(F)$  ;
- whenever  $K^{\xi_1} \subset K^{\xi_2}$  for some  $\xi_1, \xi_2 \in \Delta(F)$ , one has  $K^{\xi_1} \prec K^{\xi_2}$ , *i.e.*  $K^{\xi_1}$  is a *face* of the closed convex subset  $K^{\xi_2}$ , that is the intersection of  $K^{\xi_2}$  with the boundary of one of its supporting half-spaces ;

<sup>7</sup>A fundamental example of polytopal convex subdivision of  $\mathbb{R}^n$  occurs in the construction of complete algebraic toric varieties (see [Fult]) : a *complete fan* in  $N \otimes_{\mathbb{Z}} \mathbb{R}$ , where  $N$  denotes a lattice with rank  $n$  (*e.g.*  $\mathbb{Z}^n$ ), is a *polytopal convex subdivision* of  $N \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^n$ . Cells of this convex subdivision are in that case *strict rational cones* with respect to the lattice  $N$ .



- any face  $\tau \prec K^\xi$  for some  $\xi \in \Delta(F)$  is such that  $\tau = K^{\xi'}$  for some  $\xi' \in \Delta(F)$ .

The closed convex polytopes  $K^\xi$ ,  $\xi \in \Delta(F)$ , are called *cells* of this polytopal convex subdivision  $\mathcal{C}(F)$  of  $\mathbb{R}^n$ . With respect to the first point, take  $\xi = (\xi_1 + \xi_2)/2$ . With respect to the second point, one can check that, if  $K^{\xi_1} \subset K^{\xi_2}$ , then necessarily

$$K^{\xi_1} = \{x \in K^{\xi_2}; \langle \xi^1 - \xi^2, x \rangle = \sup_{x \in K^{\xi_2}} \langle \xi^1 - \xi^2, x \rangle\} \prec K^{\xi_2}.$$

Finally, with respect to the third point (which is the most delicate), let  $\xi \in \Delta(F)$  and

$$(3.25) \quad \tau := \{x \in K^\xi; \langle \eta, x \rangle = \sup_{x \in K^\xi} \langle \eta, x \rangle\}$$

for some  $\eta \in (\mathbb{R}^n)^* \setminus \{0\}$ . One notices that  $t \mapsto H_F(\xi + t\eta, x)$  is affine in  $[0, t_0]$  for  $t_0 \ll 1$  (depending on  $\xi$  and  $\eta$ , but not of  $x$ ). To check this point, take  $x_0 \in K^\xi$  such that  $\langle \eta, x_0 \rangle$  is maximal and make the two following observations :

- on one hand,

$$(3.26) \quad \check{p}_{R_F}(\xi + t\eta) \geq \langle \xi + t\eta, x_0 \rangle - p_{R_F}(x_0) = \check{p}_{R_F}(\xi) + t\langle \eta, x_0 \rangle$$

since  $x_0 \in K^\xi$  ;

- on the other hand, if one separates the family of components  $C$  between those ( $C^{x_0}$ ) such that  $\tau_{F, C^{x_0}} + \langle \nu_{F, C^{x_0}}, x_0 \rangle = p_{R_F}(x_0)$  and the others, one has

$$p_{R_F} \geq \max_{C^{x_0}} (\tau_{F, C^{x_0}} + \langle \nu_{F, C^{x_0}}, \cdot \rangle) := p_{R_F}^{x_0},$$

therefore  $\check{p}_{R_F} \leq \check{p}_{R_F}^{x_0}$ , which implies (since the Legendre-Fenchel transform reverses inequalities, see subsection 1.2.3)

$$(3.27) \quad \begin{aligned} \check{p}_{R_F}(\xi + t\eta) &\leq \check{p}_{R_F}^{x_0}(\xi + t\eta) = \sup_x (\langle \xi, x \rangle - p_{R_F}^{x_0}(x)) + t\langle \eta, x_0 \rangle \\ &= \sup_x (\langle \xi, x \rangle - p_{R_F}(x)) + t\langle \eta, x_0 \rangle = \check{p}_{R_F}(x) + t\langle \eta, x_0 \rangle \end{aligned}$$

for  $0 \leq t \leq t_0$  (for  $t_0 > 0$  small enough, depending on  $x_0$  and  $\eta$ , that is on  $\xi$  and  $\eta$ ) thanks to the choice of  $x_0 \in K^\xi$  so that to maximize  $x \mapsto \langle \eta, x \rangle$ .

Combining (3.26) with (3.27), one gets the fact that  $t \mapsto H_F(\xi + t\eta, x)$  is affine for  $t \in [0, t_0]$  provided  $t_0 = t_0(\xi, \eta)$  is small enough. Taking  $\xi' := \xi + t_0\eta/2$ , one has, for any  $x \in \mathbb{R}^n$ ,

$$H_F(\xi', x) = \frac{H_F(\xi, x) + H_F(\xi + t_0\eta, x)}{2}.$$

When  $x \notin K^\xi$ ,  $H_F(\xi', x)$  is then strictly positive since  $H_F \geq 0$  and  $H_F(\xi, x) > 0$ . When  $x \in K^\xi \setminus \tau$ , one has  $H_F(\xi + t_0\eta, x) = t_0\langle \eta, y - x \rangle$  for all  $y \in \tau$  (since  $H_F(\xi, x) = H_F(\xi, y) = 0$  because both  $x, y$  belong to  $K^\xi$ , and  $t \mapsto H_F(\xi + t\eta, \cdot)$  is linear on  $[0, t_0]$ ), that is  $H_F(\xi + t_0\eta, x) > 0$  since  $\langle \eta, x - y \rangle > 0$  when  $x \in K^\xi \setminus \tau$  and  $y \in \tau$ , therefore  $H_F(\xi', x) > 0$  for such  $x \in K^\xi \setminus \tau$ . Finally, if both  $x$  and  $y$  lie in  $\tau$ , then linearity of  $t \mapsto H_F(\xi + t\eta, \cdot)$  and the definition (3.25) of  $\eta$  imply that

$$H_F(\xi', x) - H_F(\xi', y) = H_F(\xi, x) - H_F(\xi, y) - (t_0/2)\langle \eta, x - y \rangle = 0.$$

The function  $x \mapsto H_F(\xi', x)$  is therefore constant in  $\tau$ . But we claim that there is some  $x$  (necessarily in  $\tau$ ) such that  $H_F(\xi', x) = 0$ . If our claim is true, one has  $\tau = K^{\xi'}$ . The claim follows from the following important lemma, that will be quite useful also later on in this course (so we give here a detailed proof) :

LEMMA 3.8 (see [PaR], Lemma 1 (iv)). *The set of  $\xi \in (\mathbb{R}^n)^* \simeq \mathbb{R}^n$  where the continuous convex function*

$$(3.28) \quad x \in \mathbb{R}^n \longmapsto p_{R_F}(x) - \langle \xi, x \rangle$$

*is bounded from below on  $\mathbb{R}^n$  equals  $\Delta(F)$ . Moreover, for any  $\xi \in \Delta(F)$ , the infimum of this function in  $\mathbb{R}^n$  is achieved at some point  $x_\xi \in \mathbb{R}^n$ .*

PROOF. Let us prove the first assertion. All functions

$$x \in \mathbb{R}^n \longmapsto p_{R_F}(x) - \langle \sigma, x \rangle \geq \tau_{F, C_\sigma},$$

where  $\sigma$  is a vertex of  $\Delta(F)$ , are bounded from below in  $\mathbb{R}^n$ , which implies, since  $\Delta(F)$  is the convex hull of the set of its vertices, that, for any  $\xi \in \Delta(F)$ , the function

$$(3.29) \quad x \in \mathbb{R}^n \longmapsto p_{R_F}(x) - \langle \xi, x \rangle$$

is bounded from below in  $\mathbb{R}^n$ . On the opposite, if  $\xi \notin \Delta(F)$ , there is  $a \in \mathbb{R}^n \setminus \{0\}$ , together with a vertex  $\sigma_a$  of  $\Delta(F)$ , such that

$$\langle \xi, a \rangle > \sup_{\eta \in \Delta(F)} \langle \eta, a \rangle = \langle \sigma_a, a \rangle.$$

If  $x \in C_{\sigma_a}$ , one has also  $x + ta \in C_{\sigma_a}$  for any  $t > 0$  (since  $C_{\sigma_a} + \Gamma_{\sigma_a} \subset C_{\sigma_a}$ , see Proposition 3.3) ; therefore

$p_{R_F}(x+ta) - \langle \xi, x+ta \rangle = p_{R_F}(x+ta) - \langle \xi, x+ta \rangle = \tau_{F, C_{\sigma_a}} - \langle \xi - \sigma_a, x+ta \rangle \rightarrow -\infty$  when  $t$  tends to  $+\infty$ . The function (3.29) is therefore not bounded from below in that case.

Let us prove now the second assertion. If  $\xi$  lies in the relative interior of  $\Delta(F)$ , one can see that the function (3.29) (which is known to be bounded from below in  $\mathbb{R}^n$  thanks to the first assertion) tends to  $+\infty$  when  $\|x\|$  tends to  $+\infty$  ; therefore it achieves its infimum at some  $x_\xi \in \mathbb{R}^n$ . Take now  $\xi$  in the relative interior of the face

$$\delta := \left\{ \xi \in \Delta(F) ; \langle \xi, y \rangle = \sup_{\eta \in \Delta(F)} \langle \eta, y \rangle \right\} \prec \Delta(F),$$

where  $y \in \mathbb{R}^n \setminus \{0\}$ , and

$$p_{R_F, \delta} : x \in \mathbb{R}^n \longmapsto \sup_{\{C ; \nu_{F, C} \in \delta\}} (\tau_{F, C} + \langle \nu_{F, C}, x \rangle)$$

the corresponding “ $\delta$ -truncated” tropical Laurent polynomial function (one keeps only tropical monomials with multi-exponents lying in the face  $\delta$ ). Clearly one has  $p_{R_F} \geq p_{R_F, \delta}$  in  $\mathbb{R}^n$ . The function

$$x \longmapsto p_{R_F, \delta}(x) - \langle \xi, x \rangle$$

is bounded from below in  $\mathbb{R}^n$  and achieves its infimum at some point  $x_\delta \in \mathbb{R}^n$  (see the first case studied), thus at all points  $x_\delta + ty$ . There is at most a finite number of  $\nu_{F, C} \notin \delta$  such that  $\tau_{F, C} + \langle \nu_{F, C}, x_\delta \rangle > p_{R_F, \delta}(x_\delta)$ . On the other hand, for any  $C$  such that  $\nu_{F, C} \notin \delta$ , the function

$$t \in \mathbb{R}_{\geq 0} \longmapsto p_{R_F, \delta}(x_\delta + ty) - \langle \nu_{F, C}, x_\delta + ty \rangle$$

is an increasing function of  $t$  which tends to  $+\infty$  when  $t$  tends to  $+\infty$ . One has therefore, for  $t$  large enough,  $p_{R_F}(x_\delta + ty) = p_{R_F, \delta}(x_\delta + ty)$ . It follows that the function

$$x \longmapsto p_{R_F}(x) - \langle \xi, x \rangle \geq p_{R_F, \delta}(x) - \langle \xi, x \rangle$$

(the two functions compared here being equal for  $x = x_\delta + ty$  when  $t$  is large) achieves certainly its infimum at some  $x = x_\delta + ty$  for some  $t$  large enough. This concludes the proof of the second assertion.  $\square$

The claim follows from the following fact : for any  $\xi \in \Delta(F)$ , there is at least one  $x_\xi \in \mathbb{R}^n$  such that  $H_F(\xi, x_\xi) = 0$  : if  $\xi \in \Delta(F)$ , Lemma 3.8 implies indeed the existence of  $x_\xi \in \mathbb{R}^n$  such that the function

$$x \mapsto p_{R_F}(x) - \langle \xi, x \rangle$$

achieves its infimum at  $x_\xi$  ; since  $\check{p}_{R_F} = p_{R_F}$ , one has  $\check{p}_{R_F}(x_\xi) = \langle \xi, x_\xi \rangle - p_{R_F}(x_\xi)$ , that is  $H_F(\xi, x_\xi) = 0$ .

We are thus done with the justification that  $\mathcal{C}(F)$  was a polytopal convex subdivision of  $\mathbb{R}^n$ . Let us now transpose the construction of the polytopal convex subdivision  $\mathcal{C}(F)$  in the “dual” setting. For all  $x \in \mathbb{R}^n$ , the closed subset

$$K_x^* := \{\xi \in \Delta(F) ; H_F(\xi, x) = 0\} \subset \Delta(F)$$

is clearly a compact convex polytope since  $\xi \mapsto H_F(\xi, x)$  is again a convex function and  $K_x^*$  (which is compact, since included in  $\Delta(F)$ ) may be defined as the intersection of only a finite number of closed half-spaces. Moreover, the compact convex polytopes  $K_x^*$ ,  $x \in \mathbb{R}^n$ , are such that

$$(3.30) \quad \bigcup_{x \in \mathbb{R}^n} K_x^* = \Delta(F).$$

(since, as seen before as a consequence of Lemma 3.8, for any  $\xi \in \Delta(F)$ , there is at least one  $x_\xi \in \mathbb{R}^n$  such that  $H_F(\xi, x_\xi) = 0$ ). This proves (3.30). In fact, the collection of all  $K_x^*$ ,  $x \in \mathbb{R}^n$ , defines also a *polytopal convex subdivision*, this time of the convex polyhedron  $\Delta(F) \subset (\mathbb{R}^n)^* \simeq \mathbb{R}^n$ . That is, besides the fact that each  $K_x^*$  is a compact convex polytope contained in  $\Delta(F)$  for any  $x \in \mathbb{R}^n$  and (3.30) is fulfilled, one has :

- whenever  $K_{x_1}^* \cap K_{x_2}^*$  is nonempty for some  $x_1, x_2 \in \mathbb{R}^n$ , the intersection  $K_{x_1}^* \cap K_{x_2}^*$  equals  $K_x^*$  for some  $x \in \mathbb{R}^n$  ;
- whenever  $K_{x_1}^* \subset K_{x_2}^*$  for some  $x_1, x_2 \in \mathbb{R}^n$ , one has  $K_{x_1}^* \prec K_{x_2}^*$  ;
- any face  $\delta^* \prec K_x^*$  for some  $x \in \mathbb{R}^n$  is such that  $\delta^* = K_{x'}^*$  for some  $x' \in \mathbb{R}^n$ .

The compact convex polytopes  $K_x^*$ ,  $x \in \mathbb{R}^n$ , are called *cells* of this polytopal convex subdivision  $\mathcal{C}^*(F)$  of  $\Delta(F)$ . The proof  $\mathcal{C}^*(F)$  is a polytopal convex subdivision of  $\Delta(F)$  is based on the same arguments than those used to prove that  $\mathcal{C}(F)$  is a polytopal convex subdivision of  $\mathbb{R}^n$ .

Let us indicate here a quite convenient way to “visualize” the polytopal convex subdivision  $\mathcal{C}^*(F)$ . If

$$\mathfrak{p} := \boxplus_{\alpha \in \mathbb{Z}^n} (a_\alpha \boxtimes x_1^{\boxtimes \alpha_1} \boxtimes \cdots \boxtimes x_n^{\boxtimes \alpha_n})$$

is a Laurent tropical polynomial (such as the tropical deformation  $\mathfrak{p}_{R_F}$  of the Ronkin function  $R_F$ ), one may define its Newton polyhedron  $\Delta(\mathfrak{p})$  as in (1.18) and its *roof*<sup>8</sup>

<sup>8</sup>In the same vein one introduced, in the ultrametric context, the *roof* of a polynomial in  $\mathbb{K}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ , where  $\mathbb{K}$  denotes an algebraically closed field equipped with a valuation, see (1.19).

as follows : take the evaluation  $p$  of  $\mathfrak{p}$  in  $(\text{Trop} \setminus \{-\infty\})^n$  and its ‘‘maximal’’ representation

$$p(x) = \max_{\alpha \in \text{Supp}(\mathfrak{p})} (-\check{p}(\alpha) + \langle \alpha, x \rangle);$$

then set

$$(3.31) \quad \tilde{\Delta}(\mathfrak{p}) := \overline{\text{conv} \{(\alpha, b); \alpha \in \text{Supp}(\mathfrak{p}), b \in \mathbb{R}, b \leq -\check{p}(\alpha)\}} \subset \mathbb{R}^{n+1}.$$

Bounded faces of this unbounded convex polytope  $\tilde{\Delta}(\mathfrak{p})$  (called also the *extended Newton polyhedron* of the Laurent tropical polynomial  $\mathfrak{p}$ ) are mapped *via* the projection  $(x, b) \in \mathbb{R}^{n+1} \rightarrow x \in \mathbb{R}^n$  on compact convex polyhedra  $K^* \subset \Delta(\mathfrak{p})$ . All such  $K^*$  define a polytopal convex subdivision  $\mathcal{C}_{\mathfrak{p}}^*$  of  $\Delta(\mathfrak{p})$ . When  $\mathfrak{p} = \mathfrak{p}_{R_F}$ , this polytopal convex subdivision coincides with the polytopal convex subdivision  $\mathcal{C}^*(F)$  (this presentation was originally developed in [PaRo], section 2). Consider for example a cell  $K^*$  of  $\mathcal{C}_{\mathfrak{p}_{R_F}}^*$  with maximal dimension, *i.e.*  $\dim K^* = \dim(\Delta(F))$  : such a cell  $K^*$  corresponds to a bounded facet  $\tilde{K}^*$  of  $\tilde{\Delta}(\mathfrak{p}_{R_F})$  with normal vector  $(x_{K^*}, 1)$  (for some  $x_{K^*}$  in  $\mathbb{R}^n$ , even in fact in  $\mathbb{Q}^n$  since  $\tilde{\Delta}(\mathfrak{p}_{R_F})$  is rational) ; suppose that the vertices of  $\tilde{K}^*$  are points  $(\alpha_j, -\check{p}(\alpha_j))$ ,  $j = 1, \dots, k$ , where all  $\alpha_j$  are taken among the  $\nu_{F,C}$ . One has then

$$\begin{aligned} K^* &= \left\{ \xi \in \Delta(F); \right. \\ &\quad \left. -\check{p}_{R_F}(\alpha_j) - \langle \xi - \alpha_j, x_{K^*} \rangle \geq -\check{p}(\nu_{F,C}) + \langle \nu_{F,C} - \alpha_j, x_{K^*} \rangle \quad \forall C, \forall j \right\} \\ &= \left\{ \xi \in \Delta(F); -\check{p}_{R_F}(\alpha_j) + \langle \xi, x_{K^*} \rangle \geq \sup_C (\langle \nu_{F,C}, x_{K^*} \rangle - \check{p}_{R_F}(\nu_{F,C})) \quad \forall j \right\} \\ &= \left\{ \xi \in \Delta(F); \check{p}_{R_F}(\alpha_j) + p_{R_F}(x_{K^*}) \leq \langle \xi, x_{K^*} \rangle \quad \forall j \right\} \\ &= \left\{ \xi \in \Delta(F); \check{p}_{R_F}(\xi) + p_{R_F}(x_{K^*}) \leq \langle \xi, x_{K^*} \rangle \right\} = \{ \xi \in \Delta(F); H_F(\xi, x_{K^*}) = 0 \}. \end{aligned}$$

Such an argument could be repeated for faces  $K^*$  of  $\mathcal{C}_{\mathfrak{p}_{R_F}}^*$  of lower dimension, in order to conclude that  $\mathcal{C}_{\mathfrak{p}_{R_F}}^* = \mathcal{C}^*(F)$ .

In fact, the two polytopal convex subdivisions  $\mathcal{C}(F)$  (of  $\mathbb{R}^n$ ) and  $\mathcal{C}^*(F)$  (of the Newton polyhedron  $\Delta(F) \subset (\mathbb{R}^n)^* \simeq \mathbb{R}^n$ ) are *dual* to each other, which justifies the notation.

**DEFINITION 3.9** (duality between convex subdivisions). Let  $\mathcal{C}$  be a polytopal convex subdivision of  $\mathbb{R}^n$  and  $\mathcal{C}^*$  be a polytopal convex subdivision of some convex polyhedron  $\Delta \subset (\mathbb{R}^n)^*$ . The subdivision  $\mathcal{C}^*$  is said to be *dual* of  $\mathcal{C}$  if and only if there is a bijective correspondence  $K \leftrightarrow K^*$  between cells in  $\mathcal{C}$  and cells in  $\mathcal{C}^*$  such that, for any cells  $K_1$  and  $K_2$  in  $\mathcal{C}$  such that  $K_1 \prec K_2$  :

- one has  $K_2^* \prec K_1^*$  in  $\mathcal{C}^*$  ;
- the two cones

$$\begin{aligned} &\{t(x - y); x \in K_2, y \in K_1, t \geq 0\} \subset \mathbb{R}^n \\ &\{t(\eta - \xi); \xi \in K_2^*, \eta \in K_1^*, t \geq 0\} \subset (\mathbb{R}^n)^* \end{aligned}$$

are polar<sup>9</sup> to each other.

<sup>9</sup>The *polar cone* of a cone  $\Gamma \subset \mathbb{R}^n$  is the cone  $\overset{\circ}{\Gamma} := \{\xi \in (\mathbb{R}^n)^*; \langle \xi, x \rangle \leq 0 \quad \forall x \in \Gamma\}$ . The cone  $\check{\Gamma} = -\overset{\circ}{\Gamma}$  is called the *dual cone* of  $\Gamma$ .

**THEOREM 3.10** (duality between  $\mathcal{C}(F)$  and  $\mathcal{C}^*(F)$ , relation with the *spine*, [PaR]). *Let  $F$  be a Laurent polynomial in  $\mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ . The two polytopal convex subdivisions  $\mathcal{C}(F)$  (of  $\mathbb{R}^n$ ) and  $\mathcal{C}^*(F)$  (of  $\Delta(F) \subset (\mathbb{R}^n)^*$ ), induced as described above by the tropical deformation  $\mathfrak{p}_{R_F}$  of the Ronkin function  $R_F$ , are dual to each other through the correspondence*

$$(3.32) \quad \begin{aligned} K^* &= \bigcap_{x \in K} K_x^* & \forall K \in \mathcal{C}(F) \\ (K^*)^* &= \bigcap_{\xi \in K^*} K^\xi & \forall K^* \in \mathcal{C}^*(F). \end{aligned}$$

Moreover

- the spine  $V_{\text{trop}}(\mathfrak{p}_{R_F}) \subset \mathcal{A}_{V_{\mathbb{T}}(F)}$  can be described as

$$(3.33) \quad V_{\text{trop}}(\mathfrak{p}_{R_F}) = \bigcup_{\substack{K \in \mathcal{C}(F) \\ \dim K < n}} K$$

- one has

$$(3.34) \quad \begin{aligned} &\{K^* \in \mathcal{C}^*(F); \dim K^* = 0\} = \\ &= \left\{ \nu_{F,C}; C \text{ connected component of } \mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(F)} \right\}; \end{aligned}$$

the correspondence  $K \leftrightarrow K^*$  induces a bijection between the family of connected components of  $\mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(F)}$  and the set of  $n$ -dimensional cells of  $\mathcal{C}(F)$ ; moreover, for any connected component  $C$  of  $\mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(F)}$ , the  $n$ -dimensional dual cell  $(\{\nu_{F,C}\})^*$  of  $\mathcal{C}(F)$  is, among  $n$ -dimensional cells of  $\mathcal{C}(F)$ , the only one which contains the component  $C$ .

**REMARK 3.11.** This proposition could have indeed interesting consequences from the combinatorics point of view. Remember that the convex subdivision  $\mathcal{C}^*(F)$  can be “visualized” in terms of the *roof* of the tropical Laurent polynomial  $\mathfrak{p}_{R_F}$  (which stands for the tropical deformation of the Ronkin function  $R_F$ ). Therefore, some combinatorial information about  $\mathcal{C}(F)$  can be deduced from the description of such *roof*  $\tilde{\Delta}(\mathfrak{p}_{R_F})$ : for example, for all  $k \in \{0, \dots, n\}$ ,

$$(3.35) \quad \begin{aligned} \#\{K \in \mathcal{C}(F); \dim K = k\} &= \#\{K^* \in \mathcal{C}^*(F); \dim K^* = n - k\} \\ &= \#\{\text{bounded faces of } \tilde{\Delta}(\mathfrak{p}_{R_F}) \text{ with dimension } n - k\}. \end{aligned}$$

In particular, the number of connected components of  $\mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(F)}$  equals the number of vertices of the extended Laurent polyhedron  $\tilde{\Delta}(\mathfrak{p}_{R_F})$ . Deciding whether the amoeba is *solid*<sup>10</sup> or not amounts to decide whether vertices of the extended Newton polyhedron  $\tilde{\Delta}(\mathfrak{p}_{R_F})$  are all above (or not) vertices of  $\Delta(F)$ . Of course, the main stumbling block (when one wants to really profit from this remark) is that the explicit computation of the coefficients  $\tau_{F,C}$ , on which the construction of the tropical deformation  $\mathfrak{p}_{R_F}$  (therefore, of course, of its extended Newton polyhedron  $\tilde{\Delta}(\mathfrak{p}_{R_F})$ ) depends, cannot be carried through easily<sup>11</sup>.

<sup>10</sup>That is the set of connected components of  $\mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(F)}$  reduces to the set  $\{C_\sigma; \sigma \text{ vertex of } \Delta(F)\}$ . This leads to a quite natural conjecture: does  $\text{Supp}(F) = \{\text{vertices of } \Delta(F)\}$  imply that the amoeba  $\mathcal{A}_{V_{\mathbb{T}}(F)}$  is *solid*? (see [Niss]). Note that the converse assertion is false, as we will see later on in the course.

<sup>11</sup>See the next subsection for approximations  $R_F$  that will lead to approximation of the spine of  $\mathcal{A}_{V_{\mathbb{T}}(F)}$ .

PROOF. If  $K_1 \prec K_2$  in  $\mathcal{C}(F)$ , then  $K_1^*$  and  $K_2^*$  defined by (3.32) fulfill the reverse inclusion  $K_2^* \prec K_1^*$ . Since any  $\xi \in \Delta(F)$  is such that  $\xi \in (K^\xi)^*$  (since  $\xi \in (K^\xi)_x$  for any  $x$  such that  $x \in K^\xi$ ), one has  $K \subset K^{**}$  for any  $K \in \mathcal{C}(F)$ . Repeating this argument with  $\mathcal{C}^*(F)$  instead of  $\mathcal{C}(F)$ , one gets that  $K \leftrightarrow K^*$  defined as (3.32) realizes a bijective correspondence reversing the order  $\prec$ .

If  $K_1 \prec K_2$ ,  $x \in K_2$ ,  $y \in K_1$ ,  $\xi \in K_2^*$ ,  $\eta \in K_1^*$ , then

$$\langle x - y, \eta - \xi \rangle = H_F(\xi, x) - H_F(\xi, y) - H_F(\eta, x) + H_F(\eta, y) = -H_F(\eta, x) \leq 0.$$

This implies that the polar cone of  $\{t(x - y); x \in K_2, y \in K_1, t \geq 0\}$  contains the cone

$$\{t(\eta - \xi); \xi \in K_2^*, \eta \in K_1^*, t \geq 0\}.$$

It remains to prove the other inclusion. Let  $K_2 = K^\xi$  and  $K_1 \prec K_2$ . We need to show that the polar cone of  $\{t(x - y); x \in K_2, y \in K_1, t \geq 0\}$  is contained in the cone

$$\{t(\eta - \xi); \xi \in K_2^*, \eta \in K_1^*, t \geq 0\}.$$

Take  $u$  in the polar cone of  $\{t(x - y); x \in K_2, y \in K_1, t \geq 0\}$ . Remember that there exists  $0 < t_0 \ll 1$  (independent of  $x$ , depending on  $\xi$  and  $\eta$ ), such that  $t \mapsto H_F(\xi + tu, x)$  remains affine on  $[0, t_0]$  (for any  $x$ ). As before<sup>12</sup>, one introduces  $\xi + t_0 u/2$  and claim that it belongs to  $K_1^*$ ; thus  $u = (2/t_0) \times ((\xi + t_0 u/2) - \xi)$  belongs to the cone  $\{t(\eta - \xi); \xi \in K_2^*, \eta \in K_1^*, t \geq 0\}$  and we are done. To prove the claim, proceed as follows : since  $t \mapsto H_F(\xi + tu, x)$  is linear, positive, then, for  $x \notin K_2 = K^\xi$ , one has  $H_F(\xi + t_0 u/2, x) > 0$ . Since  $x \mapsto H_F(\xi + t_0 u/2, x)$  certainly vanishes at some  $x = -\check{\rho}_{R_F}(\nu_{F,C})$ , such  $x$  necessarily belongs to  $K_2 = K^\xi$ ; if one takes  $y \in K_1$ , then, since  $u$  is in the polar cone of  $\{t(x - y); x \in K_2, y \in K_1, t \geq 0\}$ , one has

$$\frac{t_0}{2} \langle u, x - y \rangle = H_F(\xi + t_0 u/2, y) - H_F(\xi + t_0 u/2, x) = H_F(\xi + t_0 u/2, y) \leq 0,$$

that is  $H_F(\xi + t_0 u/2, y) = 0$  for any  $y \in K_1$ , thus  $\xi + t_0 u/2 \in K_1^*$ . This proves the first assertion of the proposition, that is that the two polytopal convex subdivisions  $\mathcal{C}(F)$  (of  $\mathbb{R}^n$  and  $\mathcal{C}^*(F)$  (of  $\Delta(F)$ ) are dual to each other.

The spine  $V_{\text{trop}}(\mathfrak{p}_{R_F})$  of  $\mathcal{A}_{V_{\mathbb{T}}(F)}$  is the union of the boundaries of the sets

$$(3.36) \quad \tilde{C} := \{x \in \mathbb{R}^n; p_{R_F}(x) = \tau_{F,C} + \langle \nu_{F,C}, x \rangle\} \supset C.$$

Since  $\bigcup_C \tilde{C} = \mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(F)}$ , one has  $V_{\text{trop}}(\mathfrak{p}_{R_F}) \subset \mathcal{A}_{V_{\mathbb{T}}(F)}$ . In fact, the collection of all  $\tilde{C}$  form exactly the collection of all  $n$ -dimensional cells in  $\mathcal{C}(F)$ ; one has  $(\tilde{C})^* = \{\nu_{F,C}\}$ . This implies that the union of all cells of  $\mathcal{C}(F)$  with dimension strictly less than  $n$ , that is the union of all faces of all cells  $\tilde{C}$ , coincides with the union of boundaries of all  $\tilde{C}$ , that is with the spine  $V_{\text{trop}}(\mathfrak{p}_{R_F})$  of the amoeba  $\mathcal{A}_{V_{\mathbb{T}}(F)}$ . This proves (3.33). Cells  $\tilde{C}$  with dimension  $n$  in  $\mathcal{C}(F)$  correspond *via*  $K \leftrightarrow K^*$  to 0-dimensional cells  $\{\nu_{F,C}\}$  in  $\mathcal{C}^*(F)$ . That is  $(\{\nu_{F,C}\})^* = \tilde{C} \supset C$  as claimed.  $\square$

In fact, one has the following more precise result :

**PROPOSITION 3.4 ([PaR]).** *For each connected component of  $\mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(F)}$ , let  $x_C$  some arbitrary point in  $C \subset \tilde{C}$ . The union, for all  $C$ , of all segments from  $x_C$  to the boundary of  $\tilde{C} \supset C$  (defined by (3.36)) contains the amoeba  $\mathcal{A}_{V_{\mathbb{T}}(F)}$ . This*

<sup>12</sup>When proving the third item ensuring  $\mathcal{C}(F)$  was a polytopal convex subdivision of  $\mathbb{R}^n$ .

means that the amoeba retracts continuously to its spine just following retractions of each  $C$  on the boundary of  $\tilde{C}$  along such segments from  $x_C$  to the boundary of the  $n$ -dimensional cell  $\tilde{C}$  of the polytopal convex subdivision  $\mathcal{C}(F)$  of  $\mathbb{R}^n$ .

PROOF. Let  $C$  be a connected component of  $\mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(F)}$  and  $x_C \in C$ . Let  $v \in \mathbb{R}^n \setminus \{0\}$ . Suppose that the half-line  $\{x_C + tv; t \geq 0\}$  never hits the boundary of the unique  $n$ -dimensional cell  $\tilde{C}$  of the convex polytopal subdivision  $\mathcal{C}(F)$  that contains  $C$ . This implies that, pour any  $t \geq 0$ ,  $x_C + tv$  remains in  $\tilde{C}$ , which means that :

$$\langle \nu_{F,C}, x_C + tv \rangle > \langle \nu_{F,C'}, x_C + tv \rangle \quad \forall C', \quad \forall t \geq 0.$$

Dividing by  $t$  when  $t > 0$  and letting  $t$  go to  $+\infty$ , it follows that

$$\forall C', \quad \langle \nu_{F,C}, v \rangle \geq \langle \nu_{F,C'}, v \rangle.$$

Since all vertices of  $\Delta(F)$  are among the  $\nu_{F,C'}$  (cf. Proposition 3.3), this implies that

$$\langle \nu_{F,C}, v \rangle \geq \max_{\xi \in \Delta(F)} \langle \xi, v \rangle.$$

We claim that this implies that the half-line  $\{x_C + tv; t \geq 0\}$ , which by hypothesis does not hit the boundary of  $\tilde{C}$ , does not hit the amoeba  $\mathcal{A}_{V_{\mathbb{T}}(F)}$  either. Let us now prove this claim.

Let  $a \in \mathbb{Z}^n \setminus \{0\}$ , precisely such that

$$(3.37) \quad \langle \nu_{F,C}, a \rangle \geq \max_{\xi \in \Delta(F)} \langle \xi, a \rangle.$$

Using the argument which was developed for the proof of Proposition 3.2, one can see that, for any  $z_{x_C} \in \text{Log}^{-1}(x_C)$ , the number of zero-poles of the Laurent polynomial

$$f_{z_{x_C}, -a}(Y) = F(z_{x_C,1}Y^{-a_1}, \dots, z_{x_C,n}Y^{-a_n})$$

in the unit disk (this Laurent polynomial is zero-pole free on the unit circle since  $x_C \in C$  lies in the complement of the amoeba  $\mathcal{A}_{V_{\mathbb{T}}(F)}$ ) equals  $-\langle \nu_{F,C}, a \rangle$ , which implies that the polynomial function  $\zeta \in \mathbb{C} \rightarrow \zeta^{\langle \nu_{F,C}, a \rangle} f_{z_{x_C}, -a}(\zeta)$  (this is indeed a polynomial function since condition (3.37) holds) restricts as an invertible holomorphic function to some open neighborhood of the closed unit disk  $\overline{D(0,1)}$  of the complex plane. The maximum principle (applied to  $\zeta \mapsto \zeta^{-\langle \nu_{F,C}, a \rangle} [f_{z_{x_C}, -a}(\zeta)]^{-1}$  in  $\overline{D(0,1)}$ ), implies that

$$\frac{1}{\min_{|\zeta|=1} (e^{\langle \nu_{F,C}, ta \rangle} |f_{z_{x_C}, -a}(\zeta e^{-t})|)} \leq \frac{1}{\min_{|\zeta|=1} |f_{z_{x_C}, -a}|} \quad \forall t \geq 0,$$

that is :

$$\min_{|\zeta|=1} |f_{z_{x_C}, -a}(\zeta e^{-t})| \geq e^{\langle \nu_{F,C}, ta \rangle} \min_{|\zeta|=1} |f_{z_{x_C}, -a}| \quad \forall t \geq 0.$$

Taking into account such an inequality for any  $z_{x_C} \in \text{Log}^{-1}(x_C)$ , it follows that

$$\min_{z \in \text{Log}^{-1}(x_C + ta)} |F(z)| \geq e^{\langle \nu_{F,C}, ta \rangle} \min_{z \in \text{Log}^{-1}(x_C)} |f(z)| \quad \forall t \geq 0.$$

Such an inequality remains true for all  $a \in \mathbb{Q}^n$  such that (3.37) is fulfilled (since it holds for any  $t \geq 0$ ). Since one may approach  $v$  by such  $a \in \mathbb{Q}^n$ , such an inequality

remains valid (by continuity) for  $v$ , thus, one has

$$\min_{z \in \text{Log}^{-1}(x_C + tv)} |F(z)| \geq e^{\langle \nu_{F,C}, tv \rangle} \min_{z \in \text{Log}^{-1}(x_C)} |f(z)| > 0 \quad \forall t \geq 0.$$

It follows that the half-line  $\{x_C + tv; t \geq 0\}$  does not hit the amoeba  $\mathcal{A}_{V_{\mathbb{T}}(F)}$  as claimed, which concludes the proof of Proposition 3.4.  $\square$

**3.1.4. A digression about O. Viro's direct approach.** As seen in section 1.1.4, one can consider alternatively  $(\mathbb{C}^n)^*$  when  $\mathbb{C} \simeq (\text{Trop}_{\mathbb{C}}, \widetilde{\boxplus}_{\mathbb{C}}, \boxtimes_{\mathbb{C}})$  is equipped with its structure of hyperfield of tropical numbers (introduced by O. Viro in [Vir] and presented precisely in section 1.1.4). Let

$$P = \sum_{\alpha \in \text{Supp}(P)} a_{\alpha} X^{\alpha} \in \mathbb{C}[X_1, \dots, X_n]$$

be a polynomial in  $n$  variables with complex coefficients. Instead of considering such a polynomial from the ‘‘classical’’ algebraic point of view, one can as well adopt the point of view inherent to such an hyperfield structure on  $\mathbb{C} \simeq \text{Trop}_{\mathbb{C}}$  and consider then the ‘‘tropical’’ polynomial (in  $n$  tropical complex variables) :

$$(3.38) \quad P_{\text{Trop}_{\mathbb{C}}} := \widetilde{\boxplus}_{\mathbb{C}, \alpha \in \text{supp}(P)} a_{\alpha} \boxtimes_{\mathbb{C}} X_1^{\boxtimes_{\mathbb{C}} \alpha_1} \cdots X_n^{\boxtimes_{\mathbb{C}} \alpha_n}.$$

One can associate to such a tropical polynomial (derived from  $P$ , keeping the same monomials, together with the same coefficients, but now interpreted tropically instead of classically) its *tropical complex zero set*, that is

$$(3.39) \quad V(P_{\text{Trop}_{\mathbb{C}}}) := \left\{ Z \in (\text{Trop}_{\mathbb{C}})^n; 0 \in P_{\text{Trop}_{\mathbb{C}}}(Z) \right\}.$$

Note that the particularity of such a polynomial  $P_{\text{Trop}_{\mathbb{C}}}$  is that, since there is no redundancy in the exhaustive definition of the support  $\text{Supp}(P)$  of  $P$ , all exponents of monomials involved in the definition of  $P_{\text{Trop}_{\mathbb{C}}}$  are supposed to be distinct<sup>13</sup>. The deformation result (Proposition 3.4) for complex archimedean amoebas of hypersurfaces should be compared to the following result by O. Viro ([Vir], Theorem 5.A).

**THEOREM 3.12** (amoebas of tropical complex polynomials *versus* amoebas of tropical real polynomials). *Let*

$$P = \sum_{\alpha \in \text{Supp}(P)} a_{\alpha} X^{\alpha} \in \mathbb{C}[X_1, \dots, X_n]$$

*be a polynomial in  $n$  variables,  $P_{\text{Trop}_{\mathbb{C}}}$  be the clean tropical complex polynomial defined as  $P$ 's tropical companion by (3.38) and  $V(P_{\text{Trop}_{\mathbb{C}}})$  its ‘‘zero set’’ in  $\text{Trop}_{\mathbb{C}}^n$  defined as (3.39). The image of  $V(P_{\text{Trop}_{\mathbb{C}}}) \cap \mathbb{T}^n$  by the logarithmic map  $\text{Log}$  (which is called the amoeba of the tropical complex polynomial  $P_{\text{Trop}_{\mathbb{C}}}$ ) coincides with the (real) tropical variety corresponding to the tropical polynomial*

$$\mathfrak{p} = \boxplus_{\alpha \in \text{Supp} P} (\log |a_{\alpha}| \boxtimes x_1^{\boxtimes \alpha_1} \cdots x_n^{\boxtimes \alpha_n}).$$

*The same result holds if one replaces  $P_{\text{Trop}_{\mathbb{C}}}$  by any cleanable tropical complex polynomial.*

<sup>13</sup>In the terminology adopted in [Vir], such a polynomial is said to be *clean*. A tropical (complex) polynomial of the form  $\widetilde{\boxplus}_{\mathbb{C}, \alpha \in A} a_{\alpha} \boxtimes_{\mathbb{C}} X_1^{\boxtimes_{\mathbb{C}} \alpha_1} \cdots X_n^{\boxtimes_{\mathbb{C}} \alpha_n}$  is said to be *cleanable* if its (multivalued) evaluation on  $(\text{Trop}_{\mathbb{C}})^n$  coincides with that of a *clean* tropical polynomial.



REMARK 3.13. In this direct approach from the complex tropical world to the real tropical one, the support of the Laurent polynomial (not only its Newton polyhedron  $\Delta(P)$ ) is preserved and the coefficients leading to the (real) tropical polynomial defining as tropical zero set the amoeba of the complex tropical polynomial correspond exactly to the logarithms of the absolute values of the corresponding coefficients of the original polynomial  $P$ . Nor the problem of the selection of multiplicities  $\nu_{P,C}$  among points in  $\Delta(P) \cap \mathbb{Z}^n$ , neither difficulties inherent to the determination of the constant coefficients  $\tau_{P,C}$  (such as in (3.8), see the next subsection) are present in such a direct process.

PROOF. At all points  $z$  in  $\mathbb{T}^n$  that are not mapped on the real tropical hypersurface defined as the corner set of the tropical polynomial function

$$(x_1, \dots, x_n) \mapsto \max_{\alpha \in \text{Supp}(P)} (\log |a_\alpha| + \langle \alpha, x \rangle),$$

the absolute value of one of the monomials in  $P$  (evaluated at  $z$ ) is strictly greater than the absolute value of all other monomials evaluated at the same point  $z$ . Therefore, one cannot have  $0 \in P_{\text{Trop}_\mathbb{C}}(Z)$  if  $Z$  is the point in  $(\text{Trop}_\mathbb{C})^n$  corresponding to  $z$ .

On the other hand, let  $x$  belong to the real tropical hypersurface defined by  $\mathfrak{p}$  and  $z \in \text{Log}^{-1}(x)$ ; then there are at least two distinct monomials  $m_1$  and  $m_2$  in  $P_{\text{Trop}_\mathbb{C}}$ , necessarily with distinct exponents (among those which absolute values are maximal, hence equal, when evaluated at the point  $Z$  corresponding to  $z$  in  $(\text{Trop}_\mathbb{C})^n$ ) such that their absolute values are equal when evaluated at  $Z$ . One can check immediately that  $0 \in m_1(Z) \widetilde{\boxplus}_\mathbb{C} m_2(Z)$  (taking into account the fact that  $a \widetilde{\boxplus}_\mathbb{C} (-a) = D(0, a)$  for any non zero tropical complex number  $a$ ). The tropical sum (in  $\text{Trop}_\mathbb{C}$ ) of all monomials  $m_j(Z)$  involved in  $P_{\text{Trop}_\mathbb{C}}$  and evaluated at  $Z$  thus contains 0, which shows that  $Z$  is indeed in the tropical complex zero set  $V(P_{\text{Trop}_\mathbb{C}})$ . This proves the reverse inclusion and concludes the proof of the theorem.  $\square$

EXAMPLE 3.14 (a tropical complex line  $(\text{Trop}_\mathbb{C})^2$  in the sense of Viro and its amoeba). Let  $a, b, c$ , three non zero complex numbers and  $(u, v)$  a point in  $(\text{Trop}_\mathbb{C})^2$  such that  $l_{a,b,c}(u, v) := au \widetilde{\boxplus}_\mathbb{C} bv \widetilde{\boxplus}_\mathbb{C} c \ni 0$ , i.e.  $au \widetilde{\boxplus}_\mathbb{C} bv \ni -c$ . One can distinguish the four following cases, in order that such situation occurs :

- if  $|au| < c$  and  $bv = -c$  (or  $|bv| < c$  and  $au = -c$ , then  $au \widetilde{\boxplus}_\mathbb{C} bv = \{-c\}$  ;
- if  $|au| > c$  and  $au + bv = 0$ , then  $au \widetilde{\boxplus}_\mathbb{C} bv = \overline{D(0, |au|)} \ni -c$  ;
- if  $au = c$  and  $bv = -c$  (or  $au = -c$  and  $bv = -c$ ), then  $au \widetilde{\boxplus}_\mathbb{C} bv = \overline{D(0, |c|)} \ni -c$  ;
- if  $au = ce^{i\theta}$  and  $bv = ce^{i\varphi}$ , with either  $0 \leq \theta \leq \pi \leq \varphi \leq \theta + \pi$  or  $\theta - \pi \leq \varphi \leq \pi \leq \theta < 2\pi$ , then the arc  $au \widetilde{\boxplus}_\mathbb{C} bv$  of the circle with center 0 and radius  $|c|$  contains  $-c$ .

One sees immediately that the image of such a tropical complex line by  $\text{Log}$  is the tropical line defined as the tropical real variety of the polynomial  $ax \boxplus by \boxplus c$  (see figure 1.1), in accordance with Theorem 3.12. As for the coamoeba of this tropical complex line, that is the image by the map  $\text{Arg}$  of such a tropical complex line, it is obtained in  $(\mathbb{R}/(2\pi\mathbb{Z}))^2$  as the translate by  $(-\arg a, -\arg b)$  of the coamoeba of the Laurent polynomial  $X + Y + 1$  (see figure 2.2, left, to visualize the representation of this coamoeba in  $] -\pi, \pi]^2$ ). Topologically speaking, such a tropical complex line  $L = L_{a,b,c} = V(l_{a,b,c, \text{Trop}_\mathbb{C}})$  is homeomorphic to a sphere with three points excluded.

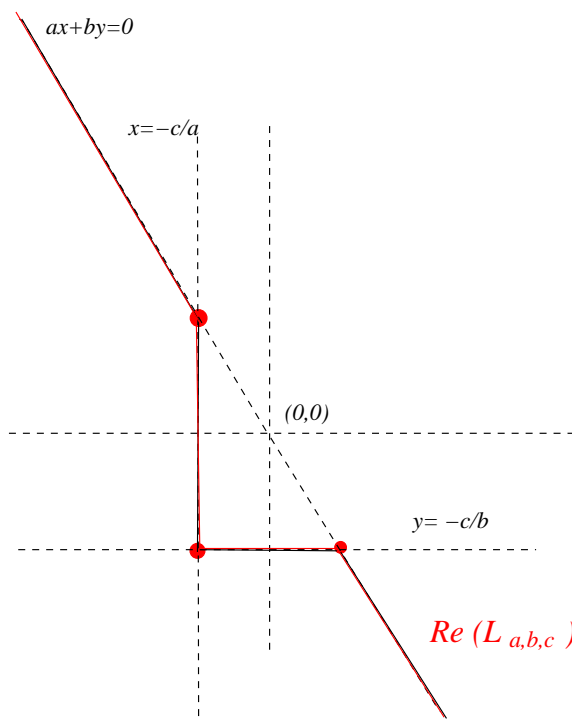


FIGURE 3.2. The tropical line  $ax + by + c = 0$ ,  $a > 0, b > 0, c > 0$ , in  $\text{Trop}_{\mathbb{R}}$

Preimages of the three open rays of the amoeba are homeomorphic to real tori, the union of these three real tori with the preimage of the node of the amoeba being homeomorphic to a sphere with three points excluded, that is to a *pair of pants*. The real part of such a tropical complex line in the tropical world  $\text{Trop}_{\mathbb{R}}^2$  looks like a piecewise broken line as represented on figure 3.2 (in the particular case where  $a, b, c$  are strictly positive real numbers).

**3.1.5. About the approximation of the tropical coefficients  $\tau_{F,C}$ .** The tropical coefficients  $\tau_{F,C}$  that appear in the tropical deformation (3.21) of the Ronkin function  $R_F$  need to be clarified if one wants to take profit from the possibility to deform the amoeba by retraction in a continuous way onto its spine. The knowledge of such  $\tau_{F,C}$  governs indeed the construction of the spine  $V_{\text{trop}}(\mathfrak{p}_{R_F})$ , since such coefficients presid (together to the corresponding multiplicities  $\nu_{F,C}$ ) to the construction of the tropical deformation  $\mathfrak{p}_{R_F}$  itself.

Let  $C$  be a connected component of  $\mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(F)}$ . For  $x \in C$ , the integral

$$\frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} \log |F(\exp(x + i\theta))| d\theta_1 \dots d\theta_n$$

(which equals  $\tau_{F,C} + \langle \nu_{F,C}, x \rangle$ ) can be computed as a limit of Riemann sums (since the integrant is a smooth  $(2\pi\mathbb{Z})^n$ -periodic function of  $\theta$ ). Namely, if

$$(3.40) \quad F_k(X) = \prod_{(l_1, \dots, l_n) \in \mathbb{F}_k^n} F(e^{2i\pi l_1/k} X_1, \dots, e^{2i\pi l_n/k} X_n) \quad (\mathbb{F}_k := \mathbb{Z}/k\mathbb{Z}, k \in \mathbb{N}^*),$$

is the sequence of Laurent polynomials attached to  $F$  introduced in Lemma 2.10 (and all with the same archimedean amoeba than  $F$ ), one has, approximating (multi)-integrals of smooth functions by Riemann sums :

$$(3.41) \quad \forall x \in C, \quad R_F(x) = \lim_{k \rightarrow +\infty} \frac{1}{k^n} R_{F_k}(x).$$

Let  $x \in \mathbb{C}$ . It follows from Lemma 2.10 that for  $k \geq k(x) \gg 1$ , one among the strictly positive numbers  $\{|c_{k,\alpha}| e^{\langle \alpha, x \rangle}; \alpha \in \text{Supp}(F_k)\}$  (the  $c_{k,\alpha}$  denote here the coefficients of  $F_k$ ), depending of course on  $k$  and also on  $x \in C$  (namely on the distance from  $x$  to the amoeba  $\mathcal{A}_{V_{\mathbb{T}}(F)}$ ), dominates strictly the sum of all others. In fact, this result can be made more precise as follows, so that to tell which monomial dominates among those of  $F_k$ , when absolute values of such monomials are evaluated at  $z_x \in \text{Log}^{-1}(x)$  for  $x \in C$  :

LEMMA 3.15 (Lemma 2.10 revisited). *Let  $F \in \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$  be a Laurent polynomial and  $C$  be a connected component of the archimedean amoeba  $\mathcal{A}_{V_{\mathbb{T}}(F)}$ , with associated multiplicity  $\nu_{F,C} \in \Delta(F)$ . Then, for any  $x \in C$ , for any  $k \gg k(x)$ , one has*

$$(3.42) \quad |c_{k,k^n \nu_{F,C}}| e^{\langle k^n \nu_{F,C}, x \rangle} > \sum_{\substack{\alpha \in \text{Supp}(F_k) \\ \alpha \neq k^n \nu_{F,C}}} |c_{k,\alpha}| e^{\langle \alpha, x \rangle},$$

where

$$F_k(X) = \sum_{\alpha \in \text{Supp}(F_k)} c_{k,\alpha} X^\alpha, \quad k \in \mathbb{N}^*.$$

PROOF. Take  $x \in C$  and  $z_x \in \text{Log}^{-1}(x)$ . As in the proof of Lemma 2.10, let us introduce, for  $k \in \mathbb{N}^*$ , the Laurent polynomial in one variable

$$(3.43) \quad \begin{aligned} f_{z_x, j, k} &:= F_k(z_{x,1}, \dots, z_{x,j-1}, Y, z_{x,j+1}, \dots, z_{x,n}) \\ &= \prod_{(l_1, \dots, l_n) \in \mathbb{F}_k^n} F\left(z_{x,1} e^{2i\pi l_1/k}, \dots, z_{x,j-1} e^{2i\pi l_{j-1}/k}, e^{2i\pi l_j/k} Y, z_{x,j+1} e^{2i\pi l_{j+1}/k}, \dots, z_{x,n} e^{2i\pi l_n/k}\right). \end{aligned}$$

The  $k^n$  factors that appear in its expression are all of the form  $f_{z'_x, j}(ue^{-x_j} Y)$ , where

$$f_{z'_x, e_j}^u = F_k(z'_{x,1}, \dots, z'_{x,j-1}, z'_{x,j} Y, z'_{x,j+1}, \dots, z'_{x,n})$$

for some  $z'_x \in \text{Log}^{-1}(x)$  and  $u \in \mathbb{S}^1$ . As seen in Section 3.1.1,  $\nu_{F,C,j}$  represents the number of zero-poles in the disc  $D(0,1)$  of such  $f_{z'_x, e_j}$ , for any  $z'_x \in \text{Log}^{-1}(x)$  and any  $u \in \mathbb{S}^1$ . So  $k^n \nu_{F,C,j}$  represents the number of zero-poles of  $f_{z_x, j, k}$  in the disc  $D(0, e^{-x_j})$ . This number should be also equal to the exponent of the dominant term  $k\nu_j(x)$  (we use again the notations introduced at the end of the proof of Lemma 2.10). Then, one has, repeating from this point the final argument in the proof of Lemma 2.10, that  $k\nu(x) = k^n \nu_{F,C}$ . The exponent of the dominant monomial among those of  $F_k$  for  $k \geq k(x) \gg 1$  is therefore  $k^n \nu_{F,C}$ , which concludes the proof of Lemma 3.15.  $\square$

REMARK 3.16. The final argument used in the proof of Lemma 3.15 gives in fact a bit more : it tells us also that whenever, for some  $x \in \mathbb{R}^n$  and some  $k \in \mathbb{N}^*$ , the finite list

$$\{|c_{k,\alpha}| e^{\langle \alpha, x \rangle}; \alpha \in \text{Supp}(F_k)\}$$

is lobsided (in the archimedean sense), then the exponent  $\alpha$  involved in its dominant term is necessarily of the form  $k^n \nu_{F,C}$  for some connected component  $C$  of the complement of the amoeba  $A_{V_{\mathbb{T}}(F)}$ .

Lemma 3.15 provides an approximation result concerning the spine  $V_{\text{trop}}(\mathfrak{p}_{R_F})$  of the archimedean amoeba  $\mathcal{A}_{V_{\mathbb{T}}(F)}$ , that is, indirectly, an approximation result, once of course multiplicities  $\nu_{F,C}$  are known, about the corresponding unknown constant coefficients  $\tau_{F,C}$ .

PROPOSITION 3.5 (approximation of the spine of the amoeba of a hypersurface [Purb]). *Let  $F \in \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$  be a Laurent polynomial,  $R_F$  be its Ronkin function, and  $[\mathcal{C}^*(F)]_0$  be the collection of 0-dimensional cells<sup>14</sup> in the polytopal convex subdivision  $\mathcal{C}^*(F) = \mathcal{C}_{\mathfrak{p}_{R_F}}^*$  of  $\Delta(F)$  defined in terms of the roof of the tropical polynomial  $\mathfrak{p}_{R_F}$  as in subsection 3.1.3. Let, for any  $k \in \mathbb{N}^*$ ,*

$$(3.44) \quad \begin{aligned} F_k(X) &= \prod_{(l_1, \dots, l_n) \in \mathbb{F}_k^n} F(e^{2i\pi l_1/k} X_1, \dots, e^{2i\pi l_n/k} X_n) = \\ &= \sum_{\{\nu\} \in [\mathcal{C}^*(F)]_0} \tau_{F_k, k^n \nu} X^{k^n \nu} + \sum_{\substack{\alpha \in \text{Supp}(F_k) \\ \{\alpha\} \notin k^n [\mathcal{C}^*(F)]_0}} c_{k,\alpha} X^\alpha. \end{aligned}$$

and

$$(3.45) \quad \mathfrak{p}_{R_F, k} := \bigsqcup_{\nu \in [\mathcal{C}^*(F)]_0} (\tau_{F_k, k^n \nu} \boxtimes x_1^{\boxtimes k^n \nu_1} \boxtimes \dots \boxtimes x_n^{\boxtimes k^n \nu_n}).$$

Then, one has

$$(3.46) \quad \lim_{k \rightarrow +\infty} \sup_{\substack{a \in V_{\text{trop}}(\mathfrak{p}_{R_F}) \\ a_k \in V_{\text{trop}}(\mathfrak{p}_{R_F, k})}} \|a - a_k\| = 0.$$

PROOF. Let  $\{\nu\}$  and  $\{\nu'\}$  two distinct 0-dimensional cells in  $\mathcal{C}^*(F)$ . Consider the hyperplane  $H_{\nu, \tilde{\nu}}$  in  $\mathbb{R}^n$  defined as :

$$H_{\nu, \tilde{\nu}} = \{x \in \mathbb{R}^n; \tau_{F, \nu} + \langle \nu, x \rangle = \tau_{F, \tilde{\nu}} + \langle \tilde{\nu}, x \rangle\}$$

and, for any  $k > 1$ , all parallel hyperplanes in  $\mathbb{R}^n$  :

$$H_{k, \nu, \tilde{\nu}} = \{x \in \mathbb{R}^n; \tau_{F_k, k^n \nu} + k^n \langle \nu, x \rangle = \tau_{F_k, k^n \tilde{\nu}} + k^n \langle \tilde{\nu}, x \rangle\}.$$

One has, by definition of tropical hypersurfaces as corner sets of the Laurent tropical polynomial functions they arise from,

$$V_{\text{trop}}(\mathfrak{p}_{R_F}) = \bigcup_{\substack{\nu, \tilde{\nu} \in [\mathcal{C}^*(F)]_0 \\ \nu \neq \tilde{\nu}}} H_{\nu, \tilde{\nu}} \quad \text{and} \quad V_{\text{trop}}(\mathfrak{p}_{R_F, k}) = \bigcup_{\substack{\nu, \tilde{\nu} \in [\mathcal{C}^*(F)]_0 \\ \nu \neq \tilde{\nu}}} H_{k, \nu, \tilde{\nu}} \quad \forall k > 1.$$

We know already that

$$V_{\text{trop}}(\mathfrak{p}_{R_F, k}) \subset \left\{ x \in \mathbb{R}^n; \{|c_{k,\alpha}| e^{\langle \alpha, x \rangle}; \alpha \in \text{Supp}(F_k)\} \text{ is not lobsided} \right\}$$

<sup>14</sup>Such cells correspond to the points  $\nu_{F,C}$ ,  $C$  being a connected component of the complement in  $\mathbb{R}^n$  of the amoeba  $\mathcal{A}_{V_{\mathbb{T}}(F)}$ , see Theorem 3.10.

since we know from the proof of Lemma 3.15 (see Remark 3.16) that, if  $x$  is such that, for some  $k \in \mathbb{N}^*$ , the finite list

$$\{|c_{k,\alpha}|e^{\langle \alpha, x \rangle}; \alpha \in \text{Supp}(F_k)\}$$

is lobsided (in the archimedean sense), than its dominant term needs necessarily to be taken from the shorter list  $\{|\tau_{F_k, k^n \nu}|e^{\langle k^n \nu, x \rangle}; \{\nu\} \in [\mathcal{C}^*(F)]_0\}$ .

This observation can be made indeed more precise, which will lead to the assertion (3.46). Let  $\eta \in ]0, 1[$ . Fix a pair  $(\{\nu\}, \{\tilde{\nu}\})$  of 0-dimensional cells in  $[\mathcal{C}^*(F)]_0$ , corresponding to two distinct connected components  $C_\nu$  and  $C_{\tilde{\nu}}$  of  $\mathbb{R}^n \setminus \mathcal{A}_{V_T(F)}$ . Fix a point  $x_\nu$  in  $C_\nu$  and a point  $x_{\tilde{\nu}}$  in  $C_{\tilde{\nu}}$ . It follows from Lemma 2.10 (see in fact also Remark 2.11), combined with the precision given by Lemma 3.15, that, for  $k \geq k(\eta)$ , one has :

$$\begin{aligned} \forall z_\nu \in \text{Log}^{-1}(x_\nu), \quad & \sum_{\substack{\alpha \in \text{Supp}(F_k) \\ \alpha \neq k^n \nu}} |c_{k,\alpha} z_\nu^\alpha| < \eta |\tau_{F_k, k^n \nu} z_\nu^{k^n \nu}|; \\ \forall z_{\tilde{\nu}} \in \text{Log}^{-1}(x_{\tilde{\nu}}), \quad & \sum_{\substack{\alpha \in \text{Supp}(F_k) \\ \alpha \neq k^n \tilde{\nu}}} |c_{k,\alpha} z_{\tilde{\nu}}^\alpha| < \eta |\tau_{F_k, k^n \tilde{\nu}} z_{\tilde{\nu}}^{k^n \tilde{\nu}}|. \end{aligned}$$

Taking logarithms and averaging respectively over  $\text{Log}^{-1}(x_\nu)$  or  $\text{Log}^{-1}(x_{\tilde{\nu}})$ , it follows that, for  $k \geq k(\eta)$ ,

$$\begin{aligned} \frac{\log |\tau_{F_k, k^n \nu}|}{k^n} + \langle \nu, x_\nu \rangle + \frac{\log(1-\eta)}{k^n} &\leq p_{R_F}(x_\nu) \leq \frac{\log |\tau_{F_k, k^n \nu}|}{k^n} + \langle \nu, x_\nu \rangle + \frac{\log(1+\eta)}{k^n} \\ \frac{\log |\tau_{F_k, k^n \tilde{\nu}}|}{k^n} + \langle \tilde{\nu}, x_{\tilde{\nu}} \rangle + \frac{\log(1-\eta)}{k^n} &\leq p_{R_F}(x_{\tilde{\nu}}) \leq \frac{\log |\tau_{F_k, k^n \tilde{\nu}}|}{k^n} + \langle \tilde{\nu}, x_{\tilde{\nu}} \rangle + \frac{\log(1+\eta)}{k^n}. \end{aligned}$$

Let  $\epsilon$  arbitrary small. For  $k$  large enough (depending on some  $\eta = \eta(\epsilon)$  sufficiently small), both functions

$$\begin{aligned} x \in C_\nu &\longmapsto \frac{p_{R_F, k}(x) - p_{R_F}(x)}{k^n} \\ x \in C_{\tilde{\nu}} &\longmapsto \frac{p_{R_F, k}(x) - p_{R_F}(x)}{k^n} \end{aligned}$$

are constant (respectively in  $C_\nu$  or  $C_{\tilde{\nu}}$ ) and can be thus continued to the whole  $\mathbb{R}^n$  as constant functions, the value of these constants being less than  $\epsilon/k^n \leq \epsilon$ . This implies that the distance between the two parallel hyperplanes  $H_{\nu, \tilde{\nu}}$  and  $H_{k, \nu, \tilde{\nu}}$  is bounded by  $K_{\nu, \tilde{\nu}} \epsilon$ , where the constant  $K_{\nu, \tilde{\nu}}$  depends only on the position of the distinct points  $\nu$  and  $\tilde{\nu}$  in  $\Delta(F)$ . The proof of the proposition follows.  $\square$

**REMARK 3.17** (Mahler measure). When  $P \in \mathbb{K}[X_1, \dots, X_n]$ , where  $\mathbb{K} \subset \mathbb{C}$  is a number field, an important constant attached to  $P$  (toward arithmetic considerations), besides its degree (which provides some geometric information), is its *Mahler measure* :

$$(3.47) \quad h(P) := \frac{1}{(2\pi)^n} \int_{[0, 2\pi]^n} \log |P(\exp(i\theta))| d\theta_1 \dots d\theta_n.$$

Almost nothing is known in general about properties of arithmetic nature carried by such a number (see *e.g.* [CM] for a presentation of some technics inspired by analytic continuation). When  $\mathbf{P}$  is in particular a homogeneous polynomial in  $\mathbb{Z}[X_0, \dots, X_n]$  (thus defining a codimension 1-arithmetic cycle in the scheme

$\text{Proj}(\mathbb{Z}[X_0, \dots, X_n])$ , the Mahler measure

$$h(\mathbf{P}) := \frac{1}{(2\pi)^{n+1}} \int_{[0, 2\pi]^{n+1}} \log |\mathbf{P}(e^{i\theta_0}, \dots, e^{i\theta_n})| d\theta_0 \dots d\theta_n$$

plays a major role (in Arakelov theory) in terms of the contribution at archimedean places to the *projective height* of the arithmetic cycle  $\{P = 0\}$  (see [PhSo], section 1.2, and also [Lel1], for the comparison with the averaging on the unit sphere  $\mathbb{S}^n$  instead of the real torus  $(\mathbb{S}^1)^{n+1}$ ). The Mahler measure of  $P \in \mathbb{K}[X_1, \dots, X_n]$  is defined as the value of the Ronkin function  $R_P$  at  $x = 0$ , independently of the fact that 0 lies or not in the amoeba  $\mathcal{A}_{V_{\mathbb{T}}(F)}$ . When  $F$  is a Laurent polynomial in  $\mathbb{Z}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$  (with Newton polyhedron  $\Delta(F)$  such that  $\dim(\Delta(F)) = n$ ), the volume of  $\Delta(F)$  (multiplied by  $n!$ ) becomes a new candidate for the *degree*, and it seems natural to conjecture that the list of all numbers  $\tau_{F,C}$ ,  $C$  being a component of  $\mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(F)}$ , will play (with respect to the *logarithmic height*) a similar role than that played by the Mahler measure in the projective frame : that is, with respect to the contribution at archimedean places to the *toric height* of the arithmetic cycle  $\text{div}(F)$ , considered this time in the arithmetic toric scheme corresponding to the complete toric variety attached to the fan in  $\mathbb{R}^n$  which is dual to the Newton polyhedron  $\Delta(F)$  (see [Fult]). Here is a reason for which being able to approximate the  $\tau_{F,C}$  would indeed be quite important.

### 3.2. Ronkin functions and Monge-Ampère operators

Presentation of the results quoted in this section amounts essentially to M. Passare and H. Rullgård [PaR]. Some fundamental material about real Monge-Ampère measures can be found in the seminal paper [RT]. I also took a lot of profit, when writing this course, from the preprint [Rull] (H. Rullgård's part of the thesis) and from the survey by G. Mikhalkin [Mik1].

**3.2.1. The real Monge-Ampère measure of a convex function  $f$  :**  $\mathbb{R}^n \rightarrow \mathbb{R}$ . Let  $f : \mathbb{R}^n = (\text{Trop} \setminus \{-\infty\})^n \rightarrow \mathbb{R}$  be a convex (hence continuous) function. When  $f$  is, in addition, smooth (*i.e.* at least  $C^2$ ), the measure

$$\text{Hess}[f](x_1, \dots, x_n) dx_1 \dots dx_n = \det \left[ \frac{\partial^2 f}{\partial x_j \partial x_k} \right]_{1 \leq j, k \leq n} dx_1 \dots dx_n$$

defines a positive measure in  $\mathbb{R}^n$  (which is absolutely continuous with the Lebesgue measure  $dx = dx_1 \dots dx_n$ ). Consider the map

$$\nabla f : x \in \mathbb{R}^n \mapsto \xi = \nabla f(x) \in (\mathbb{R}^n)^*.$$

If  $\xi \in \text{Im}(\nabla f)$ , the smooth convex function

$$x \mapsto f(x) - \langle \xi, x \rangle$$

necessarily achieves its absolute minimum (that is its minimum in  $\mathbb{R}^n$ ) at some point  $x_0 \in \mathbb{R}^n$ .

The above observation suggests a possible definition of  $\nabla f$  when  $f$  fails to be smooth (apart of course from being continuous, since convex). In such case, the punctual definition of  $\nabla f$  does not make sense anymore. Nevertheless, one can define  $\nabla f$  as

a *multivalued function*, in the same vein than one defines the multivalued tropical addition in various contexts (*cf.* Sections 1.1.3 and 1.1.4). Namely :

$$(3.48) \quad \nabla f(x) := \{ \xi \in (\mathbb{R}^n)^* ; x \mapsto f(x) - \langle \xi, x \rangle \text{ realizes its absolute minimum at } \xi \}.$$

We claim that the subset of points  $\xi$  in  $\mathbb{R}^n$  such that  $\xi \in \nabla f(x_1) \cap \nabla f(x_2)$  for at least two distinct values  $x_1, x_2$  in  $\mathbb{R}^n$  is a Lebesgue negligible set in  $(\mathbb{R}^n)^*$  : such point would be indeed a critical point of the Legendre-Fenchel transform

$$\check{f} : \xi \in (\mathbb{R}^n)^* \mapsto \sup_{\mathbb{R}^n} (\langle \xi, x \rangle - f(x))$$

since the two affine functions  $\xi \mapsto \langle \xi, x_j \rangle - f(x_j)$  (with distinct slopes) would then realize this sup at  $\xi$  ; since the Legendre-Fenchel transform (as a locally bounded convex function) is almost everywhere differentiable in  $\mathbb{R}^n$ , our claim follows<sup>15</sup>.

Let us come back to the case where  $f$  is smooth. The changing of variables formula in Lebesgue integration theory (combined with the above observation that almost any  $\xi \in (\mathbb{R}^n)^*$  cannot be achieved twice as some  $\nabla f(x)$ ) implies then that, for any  $A \subset \mathbb{R}^n$ ,

$$(3.49) \quad \int_{\nabla f(A)} d\xi = \int_A \text{Hess}[f](x) dx = \text{vol}_n \{ \xi \in (\mathbb{R}^n)^* ; x \mapsto f(x) - \langle \xi, x \rangle \text{ realizes its absolute minimum in } A \}.$$

When  $f : \mathbb{R}^n \mapsto \mathbb{R}$  is convex (hence continuous), but not smooth any more, (3.49) suggests the following definition :

**DEFINITION 3.18** (Monge-Ampère (real) measure attached to a convex function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ ). The Monge-Ampère measure  $\mu[f, \dots, f]$  attached to a convex function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is the positive measure on  $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$  defined as :

$$(3.50) \quad \mu[f, \dots, f](A) = \int_{\nabla f(A)} d\xi : \text{vol}_n \{ \xi \in (\mathbb{R}^n)^* ; x \mapsto f(x) - \langle \xi, x \rangle \text{ realizes its absolute minimum in } A \}.$$

**REMARK 3.19** (justification of the notation). What motivates the notation  $\mu[f, \dots, f]$  (instead of  $\mu[f]$ ) is the multilinearity of the determinant that appear when taking the Hessian in the right-hand side of (3.49). We will profit later on from such an observation to polarize the operation :  $(f, \dots, f) \mapsto \mu[f, \dots, f]$  (*cf.* Definition 3.23 below).

**EXAMPLE 3.20** (the example of  $p_{R_F}$ ). Let  $F \in \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$  be a Laurent polynomial such that  $\dim \Delta(F) = n$  and  $p_{R_F}$  be the Laurent tropical polynomial function defined as the evaluation of the tropical deformation  $\mathfrak{p}_{R_F}$  of the Ronkin function  $R_F$  (see (3.22)). It follows from Lemma 3.8 that

$$(3.51) \quad \mu[p_{R_F}, \dots, p_{R_F}](\mathbb{R}^n) = \text{vol}_n(\Delta(F))$$

since  $\nabla p_{R_F}(\mathbb{R}^n) = \Delta(F) = \Delta(\mathfrak{p}_{R_F})$  (see (3.50)). Consider now the dual polytopal convex subdivisions (respectively of  $\mathbb{R}^n$  and  $\Delta(F)$ )  $\mathcal{C}(F)$  and  $\mathcal{C}^*(F)$  introduced in

<sup>15</sup>See [RT], section 2, in particular Theorem 2.5, for a recap of such results due to A.D. Aleksandrov (1955) and Rademacher (1918).

Section 3.1.3 (*cf.* in particular Theorem 3.10). For any  $\xi \in \Delta(F)$ , the subset of  $\mathbb{R}^n$  where

$$x \mapsto p_{R_F}(x) - \langle \xi, x \rangle$$

achieves its absolute minimum (in  $\mathbb{R}^n$ ) coincides with the subset of  $\mathbb{R}^n$  where the positive function

$$x \mapsto p_{R_F}(x) + \check{p}_{R_F}(\xi) - \langle \xi, x \rangle$$

achieves its absolute minimum (namely 0), that is the convex cell  $K^\xi$  of the polytopal convex subdivision  $\mathcal{C}(F)$ . The positive measure  $\mu[p_{R_F}, \dots, p_{R_F}]$  is supported by the spine  $V_{\text{trop}}(\mathfrak{p}_{R_F})$  of the archimedean amoeba  $\mathcal{A}_{V_{\text{T}}(F)}$ . Moreover, for any borelian  $A \subset V_{\text{trop}}(\mathfrak{p}_{R_F})$ , one has

$$(3.52) \quad \mu[p_{R_F}, \dots, p_{R_F}](A) = \text{vol}_n \{ \xi \in \Delta(F); K^\xi \cap A \neq \emptyset \}.$$

Suppose now that  $A = \{a\} \in [\mathcal{C}(F)]_0$  is a 0-dimensional cell of the polytopal convex subdivision  $\mathcal{C}(F)$  (whose union of cells with dimension strictly less than  $n$  corresponds to the spine  $V_{\text{trop}}(\mathfrak{p}_{R_F})$  of the amoeba  $\mathcal{A}_{V_{\text{T}}(F)}$ , *cf.* Theorem 3.10). Let  $(\{a\})^*$  be its dual  $n$ -dimensional cell in  $\mathcal{C}(F^*)$ .

$$\{a\} = ((\{a\})^*)^* = \bigcap_{\xi \in (\{a\})^*} K^\xi$$

(*cf.* (3.32)). One has then :

$$\xi \in (\{a\})^* \implies a \in K^\xi.$$

Therefore

$$(\{a\})^* \subset \{ \xi \in \Delta(F); K^\xi \cap \{a\} \neq \emptyset \}.$$

It follows that, for any such 0-dimensional cell of  $\mathcal{C}(F)$ , one has :

$$\text{vol}_n ((\{a\})^*) \leq \mu[p_{R_F}, \dots, p_{R_F}](\{a\}).$$

Since

$$\sum_{\{a\} \in [\mathcal{C}(F)]_0} \text{vol}_n ((\{a\})^*) = \text{vol}_n (\Delta(F)) = \mu[p_{R_F}, \dots, p_{R_F}](\mathbb{R}^n),$$

one has then the formula :

$$(3.53) \quad \mu[p_{R_F}, \dots, p_{R_F}] = \sum_{\{a\} \in [\mathcal{C}(F)]_0} \text{vol}_n ((\{a\})^*) \delta_a,$$

where  $\delta_a$  denotes the Dirac mass at the node  $a$  of  $V_{\text{Trop}}(\mathfrak{p}_{R_F})$ , that is at the node  $a$  of the spine of the amoeba  $\mathcal{A}_{V_{\text{T}}(F)}$ . Note that the coefficients in (3.53) are such that

$$(3.54) \quad n! \text{vol}_n ((\{a\})^*) \in \mathbb{N}^* \quad \forall \{a\} \in [\mathcal{C}(F)]_0$$

since the volume of the standard simplex  $\Delta^{[n]}$  (with vertices the origin and the extremities of the vectors from the canonical basis of  $\mathbb{R}^n$ ) equals  $1/n!$ .

**EXAMPLE 3.21** (the example of  $R_F$ ). Let  $F \in \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$  be again a Laurent polynomial such that  $\dim \Delta(F) = n$  and let  $R_F : \mathbb{R}^n \rightarrow \mathbb{R}$  be its Ronkin function (considered here in  $\mathbb{R}^n$  and not in the tube  $\mathbb{R}^n + i\mathbb{R}^n$  as in Definition 3.4 or (3.19)). Since  $R_F \geq p_{R_F}$  everywhere in  $\mathbb{R}^n$ , the function

$$(3.55) \quad x \in \mathbb{R}^n \mapsto R_F(x) - \langle \xi, x \rangle \geq p_{R_F}(x) - \langle \xi, x \rangle$$

is certainly bounded from below for any  $\xi \in \Delta(F)$  (thanks to assertion 1, first point, in Lemma 3.8). Since  $R_F \equiv p_{R_F}$  in any connected component  $C$  (in particular in



$C = C_{\sigma_a}$  as in the proof of assertion 1, second point, in Lemma 3.8) of  $\mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(F)}$ , exactly the same argument than that carried in the proof of assertion 1, second point, in Lemma 3.8, shows indeed that the function (3.55) is bounded from below in  $\mathbb{R}^n$  if and only if  $\xi \in \Delta(F)$ . The positive Monge-Ampère measure  $\mu[R_F, \dots, R_F]$  has this time support equal<sup>16</sup> to the amoeba. As in the previous Example 3.20 :

$$(3.56) \quad \mu[R_F, \dots, R_F](\mathbb{R}^n) = \mu[p_{R_F}, \dots, p_{R_F}](\mathbb{R}^n) = \text{vol}_n(\Delta(F))$$

since  $\nabla R_F(\mathbb{R}^n) = \Delta(F) = \Delta(\mathfrak{p}_{R_F})$  (see again (3.50)). When  $R_F$  is deformed into the tropical Laurent polynomial function  $p_{R_F}$  (evaluation of the tropical Laurent polynomial  $\mathfrak{p}_{R_F}$ ), it appears that the Monge-Ampère measure (with total mass  $\text{vol}_n(\Delta(F))$ ), which is initially spread over the whole amoeba (as the current  $[\mathcal{A}_F] = dd^c R_F$  introduced in Section 3.1.2 is spread in the whole tube  $\mathcal{A}_{V_{\mathbb{T}}(F)} + i\mathbb{R}^n$  in  $\mathbb{C}^n$ ), concentrates during the deformation (from  $R_F$  to  $p_{R_F}$ ) toward a positive atomic measure which charges only the 0-dimensional cells of the polytopal subdivision  $\mathcal{C}(F)$ , that is the nodes of the spine  $V_{\text{trop}}(\mathfrak{p}_F)$  of the amoeba  $\mathcal{A}_{V_{\mathbb{T}}(F)}$  (see (3.53)). Such a remark will have indeed interesting geometric consequences. The deformation from  $R_F$  toward  $p_{R_F}$  (from complex to tropical) is indeed *mass preserving* with respect to the Monge-Ampère real measure, since the total mass remains equal to  $\text{vol}_n(\Delta(F))$  during the deformation (see (3.56)).

REMARK 3.22 (about the euclidean volume of the archimedean amoeba  $\mathcal{A}_{V_{\mathbb{T}}(F)}$ ). When  $n = 2$ , the euclidean area  $\text{vol}_2(\mathcal{A}_{V_{\mathbb{T}}(F)})$  of the archimedean amoeba of a Laurent polynomial  $F \in \mathbb{C}[X_1^{\pm 1}, X_2^{\pm 2}]$  is bounded, namely (see [PaR], Theorem 7 and Corollary 1) :

$$(3.57) \quad \text{vol}_2(\mathcal{A}_{V_{\mathbb{T}}(F)}) \leq \pi^2 \text{vol}_2(\Delta(F)).$$

Amoebas with maximal area (that is equality in (3.57)) correspond to Harnack curves in the real plane (see [MRu]). On the other hand, as  $n \geq 3$ , the euclidean volume  $\text{vol}_n(\mathcal{A}_{V_{\mathbb{T}}(F)})$  appears to be in the general case infinite.

**3.2.2. Mixed Monge-Ampère real measure attached to  $n$  convex functions on  $\mathbb{R}^n$ .** It will be important for us later to *polarize* the definition of the real Monge-Ampère measure, following the observation made in Remark 3.19.

DEFINITION 3.23 (mixed Monge-Ampère measure of  $n$  convex functions). The *mixed Monge-Ampère measure*  $\mu[f_1, \dots, f_n]$  of  $n$  convex functions  $f_j : \mathbb{R}^n \rightarrow \mathbb{R}$  is defined as the positive measure :

$$(3.58) \quad \begin{aligned} \mu[f_1, \dots, f_n] &:= \\ &= \frac{1}{n!} \sum_{k=1}^n (-1)^{n-k} \sum_{1 \leq j_1 < \dots < j_k \leq n} \mu \left[ \sum_{l=1}^k f_{j_l}, \dots, \sum_{l=1}^k f_{j_l} \right]. \end{aligned}$$

PROOF. We need just to justify here the definition of  $\mu[f_1, \dots, f_n]$  and the positivity of this measure. Due to the multilinearity of the determinant, the polarization formula (3.58) defines a positive measure (absolutely continuous with respect to the Lebesgue measure  $dx$  on  $\mathbb{R}^n$ ) when  $f_1, \dots, f_n$  are indeed  $C^2$  convex functions. Any convex function  $f$  can be approached (for the topology of uniform convergence on all

<sup>16</sup>One has clearly the inclusion  $\text{Supp}(\mu[R_F, \dots, R_F]) \subset \mathcal{A}_{V_{\mathbb{T}}(F)}$ . The reason for the equality lies here in the fact that the amoeba (on the opposite to its spine) has non empty interior. We will justify the equality  $\text{Supp}(\mu[R_F, \dots, R_F]) = \mathcal{A}_{V_{\mathbb{T}}(F)}$  in the next section, relating precisely the Monge-Ampère measure  $\mu[R_F, R_F, \dots, R_F]$  with the current  $[\mathcal{A}_F]$ .

compact subsets of  $\mathbb{R}^n$ ) by smooth convex functions : take a smooth positive function  $\varphi$  in  $\mathbb{R}^n$ , with integral over  $\mathbb{R}^n$  equal to 1 and such that  $\text{Supp}(\varphi) \subset B_n(0, 1)$ , and let, for any  $\epsilon > 0$ , the regularized convolved function :

$$f * \varphi_\epsilon : x \in \mathbb{R}^n \mapsto \frac{1}{\epsilon^n} \int_{\mathbb{R}^n} f(x-y)\varphi(y/\epsilon) dy ;$$

then

$$\lim_{\epsilon \rightarrow 0_+} f * \varphi_\epsilon = f$$

as functions from  $\mathbb{R}^n$  to  $\mathbb{R}$ , for the topology of uniform convergence on all compact subsets of  $\mathbb{R}^n$  ; moreover, since  $\varphi$  is both positive and smooth, all such regularized convolved functions  $f * \varphi_\epsilon$ ,  $\epsilon > 0$ , are convex and inherit the smooth character of  $\varphi$ . Continuity of the operator  $f \rightarrow \mu[f, \dots, f]$  from the space of convex real functions (equipped with the topology of uniform convergence on any compact subset of  $\mathbb{R}^n$ ), into the space of positive Borel measures in  $\mathbb{R}^n$  (equipped with the weak topology<sup>17</sup>) imply that the limit, when  $\epsilon$  tends to  $0_+$ , of  $\mu[f_1 * \varphi_\epsilon, \dots, f_n * \varphi_\epsilon]$  (considered here as a positive Borel measure in  $\mathbb{R}^n$ ), exists and defines a positive measure in  $\mathbb{R}^n$ , that is precisely the measure  $\mu[f_1, \dots, f_n]$  obtained in (3.58).  $\square$

### 3.2.3. Real Monge Ampère versus complex Monge-Ampère operators.

When  $f$  is a smooth convex function in  $\mathbb{R}^n = (\text{Trop} \setminus \{-\infty\})^n$ , one can compose it with  $z \in \mathbb{T}^n \mapsto \text{Log } z \in \mathbb{R}^n$  and consider the smooth function  $H_f = f \circ \text{Log}$  from  $\mathbb{T}^n$  to  $\mathbb{R}$ . An easy computation (see for example [Rash0], section 3) shows that

$$(3.59) \quad \frac{\partial^2 H_f}{\partial z_j \partial \bar{z}_k}(z) = \frac{1}{4} \frac{1}{z_j \bar{z}_k} \left[ \frac{\partial^2 f}{\partial x_j \partial x_k} \right]_{x=\text{Log}(z)},$$

which implies that, in terms of  $(n, n)$ -currents in  $\mathbb{T}^n$  :

$$(3.60) \quad \begin{aligned} (dd^c[H_f])^{\wedge n} &= \frac{n!}{4^n \pi^n} \frac{1}{|z_1 \dots z_n|^2} \left[ \det \left[ \frac{\partial^2 f}{\partial x_j \partial x_k} \right]_{j,k} \right]_{x=\text{Log}(z)} \bigwedge_{j=1}^n (i dz_j \wedge d\bar{z}_j) \\ &= n! \left[ \det \left[ \frac{\partial^2 f}{\partial x_j \partial x_k} \right]_{j,k} \right]_{x=\text{Log}(z)} \bigwedge_{j=1}^n \left( \frac{id[e^{x_j + i\theta_j}] \wedge d[e^{x_j - i\theta_j}]}{4\pi e^{2x_j}} \right) \\ &= n! \left[ \det \left[ \frac{\partial^2 f}{\partial x_j \partial x_k} \right]_{j,k} \right]_{x=\text{Log}(z)} \bigwedge_{j=1}^n \left( dx_j \wedge \frac{d\theta_j}{2\pi} \right). \end{aligned}$$

The action of the (non-linear!) operator  $u \mapsto (dd^c u)^{\wedge n}$  on plurisubharmonic functions in a open subset of  $\mathbb{C}^n$  is known as the *complex Monge-Ampère* operator. It follows from (3.60) that, for any borelian set  $A \in \mathcal{B}(\mathbb{R}^n)$ , one has

$$(3.61) \quad \mu[f, \dots, f](A) = \frac{1}{n!} \int_{\text{Log}^{-1}(A)} (dd^c[H_f])^{\wedge n} := \frac{1}{n!} \langle (dd^c[H_f])^{\wedge n}, \chi_{\text{Log}^{-1}(A)} \rangle,$$

where the right-hand side in (3.61) is understood as the positive  $(n, n)$ -current  $(dd^c[H_f])^{\wedge n}$  (with measure coefficients<sup>18</sup>) acting, which makes sense precisely since

<sup>17</sup>Real Borel measures on  $\mathbb{R}^n$  being considered as continuous  $\mathbb{R}$ -linear forms  $T$  of the vector space  $\mathcal{K}(\mathbb{R}^n, \mathbb{R})$  of real continuous functions in  $\mathbb{R}^n$  with compact support, equipped with its inductive limit topology. For a proof of the fact that  $f \rightarrow \mu[f, \dots, f]$  is continuous, see e.g [RT].

<sup>18</sup>See the introduction to positive currents and convexity in the complex sense in subsection 3.1.2 (refer also to appendix A).

the coefficients of this current are Borel measures, on the positive measurable function  $\chi_{\text{Log}^{-1}(A)}$ .

Such formula (3.61) extends immediately by polarization<sup>19</sup> : namely, if  $f_1, \dots, f_n$  denote  $n$  smooth convex functions in  $\mathbb{R}^n$ , then, for any borelian  $A \in \mathcal{B}(\mathbb{R}^n)$ , one has :

$$(3.62) \quad \mu[f_1, \dots, f_n](A) = \frac{1}{n!} \int_{\text{Log}^{-1}(A)} \bigwedge_{j=1}^n dd^c[H_{f_j}] = \frac{1}{n!} \left\langle \bigwedge_{j=1}^n dd^c H_{f_j}, \chi_{\text{Log}^{-1}(A)} \right\rangle.$$

EXAMPLE 3.24 (the Ronkin function  $R_F$  seen from a dual point of view [Rull]). Let  $F \in \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$  be a Laurent polynomial,  $f_1, \dots, f_{n-1}$  be  $(n-1)$  smooth convex functions in  $\mathbb{R}^n$ . Let (for  $\epsilon > 0$ )  $R_F^\epsilon$  be the smooth convex function defined as :

$$(3.63) \quad \begin{aligned} R_F^\epsilon : x \in \mathbb{R}^n &\longmapsto \frac{1}{(2\pi)^n} \int_{[0, 2\pi]^n} \log(\sqrt{|F(\exp(x + i\theta))|^2 + \epsilon^2}) d\theta_1 \dots d\theta_n \\ &= \int_{(\mathbb{S}^1)^n} \log(\sqrt{|F(\exp(x + i\theta))|^2 + \epsilon^2}) d\sigma_{(\mathbb{S}^1)^n}(\theta). \end{aligned}$$

It follows from (3.62) and Fubini's theorem that, for any borelian  $A \in \mathcal{B}(\mathbb{R}^n)$ ,

$$(3.64) \quad \begin{aligned} \mu[f_1, \dots, f_{n-1}, R_F^\epsilon](A) &= \\ &= \frac{1}{n!} \int_{\text{Log}^{-1}(A)} \left[ \bigwedge_{j=1}^{n-1} dd^c[H_{f_j}] \wedge \left( \int_{(\mathbb{S}^1)^n} dd^c \left[ \log(\sqrt{|F(uz)|^2 + \epsilon^2}) \right] d\sigma_{(\mathbb{S}^1)^n}(u) \right) \right] \\ &= \frac{1}{n!} \int_{(\mathbb{S}^1)^n} \left[ \int_{\text{Log}^{-1}(A)} \left[ \left( \bigwedge_{j=1}^{n-1} dd^c[H_{f_j}] \right) \wedge dd^c \left[ \log(\sqrt{|F(uz)|^2 + \epsilon^2}) \right] \right] d\sigma_{(\mathbb{S}^1)^n}(u) \right] \\ &= \frac{1}{n!} \int_{\text{Log}^{-1}(A)} \left( \bigwedge_{j=1}^{n-1} dd^c[H_{f_j}] \right) \wedge dd^c \left[ \log(\sqrt{|F(z)|^2 + \epsilon^2}) \right]. \end{aligned}$$

When  $\epsilon$  tends to 0, the left-hand side of (3.64) tends to  $\mu[f_1, \dots, f_{n-1}, R_F]$  because of the continuity of the real Monge-Ampère operator acting from the space of convex functions in  $\mathbb{R}^n$  into the space of positive Borel measures (see the argument justifying Definition 3.23). On the other hand, Lelong-Poincaré equation (see subsection 3.1.2 and Appendix A) implies that the right-hand side of (3.64) tends, when  $\epsilon$  tends to 0, toward :

$$\int_{\text{Log}^{-1}(A)} \left( \bigwedge_{j=1}^{n-1} dd^c[H_{f_j}] \right) \wedge [F^{-1}(0)],$$

the current  $[F^{-1}(0)]$  being understood here as the integration current with multiplicities have been taken into account, namely  $[F^{-1}(0)] := dd^c \log |F|$ . We may summarize the results obtained here by introducing the averaged current  $[\mathcal{A}_F]$  introduced in subsection 3.1.2 (see (3.16) or (3.17)). One has then, for any smooth convex functions  $f_j : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $j = 1, \dots, n-1$ , trivially extended to the tube

<sup>19</sup>The right-hand side is easy to polarize since the operator  $u \mapsto (dd^c)^{\wedge n} [u]$  is precisely expressed here in its factorized form  $u \mapsto dd^c u \wedge \dots \wedge dd^c u$ .

$\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n$  by setting  $f_j(x + iy) = f_j(x)$  for all  $x, y \in \mathbb{R}^n$ , for any borelian set  $A \in \mathcal{B}(\mathbb{R}^n)$ ,

$$(3.65) \quad \mu[f_1, \dots, f_{n-1}, R_F](A) = \frac{1}{n!} \left\langle [\mathcal{A}_F], \chi_{A+i\mathbb{R}^n} \bigwedge_{j=1}^{n-1} dd^c f_j \right\rangle.$$

When  $f_1, \dots, f_{n-1}$  are convex (hence continuous) functions in  $\mathbb{R}^n$ , considered as extended as continuous functions  $\mathbb{C}^n$  by  $f_j(x + iy) = f_j(x)$  for all  $x, y \in \mathbb{R}^n$ , the  $(n, n)$  positive current

$$[(F \circ \exp)^{-1}(0)] \wedge \bigwedge_{j=1}^{n-1} dd^c f_j = dd^c [\log |F \circ \exp|] \wedge \bigwedge_{j=1}^{n-1} dd^c f_j$$

in  $\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n$  can be naturally constructed thanks to successive integrations by parts performed on the closed analytic subset  $(F \circ \exp)^{-1}(0)$  (where multiplicities have been taken into account, as in (3.15)), as follows :

$$[(F \circ \exp)^{-1}(0)] \wedge \bigwedge_{j=1}^{k+1} dd^c f_j = dd^c \left[ f_{k+1} \left( [(F \circ \exp)^{-1}(0)] \wedge \bigwedge_{j=1}^k dd^c f_j \right) \right]$$

for  $k = 1, \dots, n-2$ . Then, for any convex functions  $f_j : \mathbb{R}^n \rightarrow \mathbb{R}$  extended trivially to  $\mathbb{C}^n$  setting  $f_j(x + iy) = f_j(x)$  for all  $(x, y) \in \mathbb{R}^n$ , one can define the positive  $(n, n)$ -current :

$$(3.66) \quad \begin{aligned} [\mathcal{A}_F] \wedge \bigwedge_{j=1}^{n-1} dd^c f_j &= \left( \int_{(\mathbb{S}^1)^n} [(F \circ u \exp)^{-1}(0)] d\sigma_{(\mathbb{S}^1)^n}(u) \right) \wedge \bigwedge_{j=1}^{n-1} dd^c f_j : \\ &= \int_{(\mathbb{S}^1)^n} \left( [(F \circ u \exp)^{-1}(0)] \wedge \bigwedge_{j=1}^{n-1} dd^c f_j \right) d\sigma_{(\mathbb{S}^1)^n}(u). \end{aligned}$$

Formula (3.65) holds in fact when  $f_1, \dots, f_{n-1}$  are arbitrary convex functions in  $\mathbb{R}^n$ , and can be rewritten shortly as :

$$(3.67) \quad \frac{1}{n!} \left( [\mathcal{A}_F] \wedge \bigwedge_{j=1}^{n-1} dd^c f_j \right) = d\mu[f_1, \dots, f_{n-1}, R_F](x) \bigwedge_{j=1}^n dx_j \wedge dy_j.$$

It is important to point out that one could choose as “test functions” the evaluations  $l_{a_j}$  of tropical Laurent polynomials with Newton polyedron the simplex  $\Delta_n$ , such as

$$l_{a_j} = a_{j0} \boxplus (a_{j1} \boxtimes x_1) \boxplus \dots \boxplus (a_{jn} \boxtimes x_n),$$

the  $a_{jk}$ ,  $j = 1, \dots, n-1$ ,  $k = 0, \dots, n$ , being generic in  $\text{Trop} \setminus \{-\infty\}$ . The “test” of the Ronkin function would be then the positive measure-valued function

$$(a_1, \dots, a_{n-1}) \mapsto \mu[l_{a_1}, \dots, l_{a_{n-1}}, R_f]$$

defined in a suitable Grassmanian.

**3.2.4. An application of formula (3.62) to the proof of D. Bernstein’s theorem.** When  $P_1 \in \mathbb{C}[X_1, \dots, X_n], \dots, P_n \in \mathbb{C}[X_1, \dots, X_n]$  are polynomials with generic coefficients such that their Newton polyedrons are respectively :

$$\Delta(P_j) = D_j \Delta^{[n]}, \quad j = 1, \dots, n,$$

where  $D_j = \deg P_j \in \mathbb{N}^*$  and  $\Delta^{[n]}$  denotes the standard simplex in  $\mathbb{R}^n$  (with vertices the origin and the extremities of the canonical basis), then the homogeneous polynomials  ${}^h P_j(X_0, \dots, X_n) = X_0^{D_j} P_j(X_1/X_0, \dots, X_n/X_0) \in \mathbb{C}[X_0, \dots, X_n]$ ,  $j = 1, \dots, n$ , define effective Weil divisors  $\overline{\text{div}}(P_j) = \overline{\text{div}}({}^h P_j)$  on the complete algebraic variety  $\mathbb{P}^n(\mathbb{C})$ . The supports of these Weil divisors intersect all together in  $\mathbb{P}^n(\mathbb{C})$  only at isolated points, which all lie in the torus  $\mathbb{T}^n \subset \mathbb{C}^n \subset \mathbb{P}^n(\mathbb{C})$  (in particular, they do not intersect all together in  $\mathbb{P}^n(\mathbb{C}) \setminus \mathbb{T}^n$ , that is “at infinity” in the toric compactification  $\mathbb{P}^n(\mathbb{C})$  of the complex torus  $\mathbb{T}^n$ ). Moreover, thanks to Bézout theorem in  $\mathbb{P}^n(\mathbb{C})$  (see *e.g.* [CLO] or [Eis]), one has :

$$(3.68) \quad \deg [\overline{\text{div}}(P_1) \bullet \dots \bullet \overline{\text{div}}(P_n)] = \prod_{j=1}^n \deg [\overline{\text{div}}(P_j)] = D_1 \cdots D_n$$

(the notation  $\bullet$  stands here for the intersection product, see [Eis] or also [Fult]). Note that  $D_1 \cdots D_n$  equals  $n! \times \text{vol}_n[D_1 \Delta^{[n]}, \dots, D_n \Delta^{[n]}]$ , where

$$(\Delta_1, \dots, \Delta_n) \mapsto \text{vol}_n(\Delta_1, \dots, \Delta_n)$$

denotes the polarization of the map :

$$(\Delta, \dots, \Delta) \mapsto \text{vol}_n[\Delta, \dots, \Delta] := \text{vol}_n(\Delta)$$

(defined on the additive semi-group of convex polyedra with vertices in  $\mathbb{Z}^n$  equipped with the Minkovski addition) ; namely :

$$(3.69) \quad \text{vol}_n(\Delta_1, \dots, \Delta_n) = \frac{1}{n!} \sum_{k=1}^n (-1)^{n-k} \sum_{1 \leq j_1 < \dots < j_k \leq n} \text{vol}_n(\Delta_{j_1} + \dots + \Delta_{j_k}).$$

Therefore, one has :

$$(3.70) \quad \#\{z \in \mathbb{T}^n ; P_1(z) = \dots = P_n(z) = 0\} = D_1 \cdots D_n = n! \times \text{vol}_n[D_1 \Delta^{[n]}, \dots, D_n \Delta^{[n]}]$$

(points being here counted with their multiplicities).

A similar result holds when  $F_1, \dots, F_n$  are Laurent polynomials in  $\mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ , with respective Newton polyedra<sup>20</sup>  $\Delta_1, \dots, \Delta_n$ , such that :

$$(3.71) \quad \dim(\Delta_1 + \dots + \Delta_n) = n.$$

Let then  $\Delta = \Delta_1 + \dots + \Delta_n$ ,  $\Sigma(\Delta)$  be the dual fan of this convex polyedron (see [Fult]) and  $\mathcal{X}(\Sigma(\Delta)) \supset \mathbb{T}^n$  be the complete  $n$ -dimensional toric variety (see [Fult]), which is associated to this rational fan (and realizes a compactification of  $\mathbb{T}^n$  equipped with a group action of  $\mathbb{T}^n$  on it). When the coefficients  $c_{1,\alpha}, \dots, c_{k,\alpha}$  of all the

$$F_j = \sum_{\alpha \in \Delta_j \cap \mathbb{Z}^n} c_{j,\alpha} X^\alpha$$

are generic, that is when they do not satisfy some algebraic equation

$$(3.72) \quad \Phi_{\Delta_1, \dots, \Delta_n}(\{c_{1,\alpha} ; \alpha \in \Delta_1 \cap \mathbb{Z}^n\}, \dots, \{c_{n,\alpha} ; \alpha \in \Delta_n \cap \mathbb{Z}^n\}) = 0,$$

where the polynomial  $\Phi$  depends only on  $\Delta_1, \dots, \Delta_n$ , then the supports of the Weil divisors  $\overline{\text{div}}(F_j)$  that are induced by the Laurent polynomials  $F_j$  on the toric variety  $\mathcal{X}(\Sigma(\Delta))$  intersect all together only at isolated points (hence in finite number), which all lie in the torus  $\mathbb{T}^n \subset \mathcal{X}(\Sigma(\Delta))$  (in particular, they do not intersect all

<sup>20</sup>In fact, for each  $j$ , the polyedron  $\Delta_j = \Delta(F_j)$  figures here the “toric degree” of  $F_j$ .

together in  $\mathcal{X}(\Sigma(\Delta)) \setminus \mathbb{T}^n$ , that is “at infinity” in the toric compactification  $\mathcal{X}(\Sigma(\Delta))$  of the complex torus  $\mathbb{T}^n$ ). This can be rephrased in terms of the Log map as follows :

$$(3.73) \quad \liminf_{\substack{\|x\| \rightarrow +\infty \\ x \in \mathbb{R}^n}} \left( \inf_{z_x \in \text{Log}^{-1}(x)} \sum_{j=1}^n \frac{|F_j(z_x)|}{\exp(\sup_{\xi \in \Delta_j} \langle \xi, x \rangle)} \right) > 0$$

(see for example [VY], Proposition 2.2). Moreover, the function

$$(3.74) \quad (\tilde{F}_1, \dots, \tilde{F}_n) \mapsto \#\{z \in \mathbb{T}^n ; \tilde{F}_1(z) = \dots = \tilde{F}_n(z) = 0\}$$

(the zeroes being here counted with multiplicities) remains locally constant, hence constant (since it is an integer-valued function) when the coefficients of the Laurent polynomials  $\tilde{F}_j$  (with respective Newton polyedra  $\Delta_j$ ) vary in the complement of the algebraic hypersurface defined by (3.72). Moreover, one has the following result (due to David Bernstein), in the same vein than the classical projective Bézout formula (3.68) in  $\mathbb{P}^n(\mathbb{C})$  quoted above. The approach we describe here was proposed in [PaR].

**THEOREM 3.25** (toric Bézout theorem [Bern]). *Let  $F_1, \dots, F_n$  be  $n$  Laurent polynomials with respective Newton polyedra  $\Delta_1, \dots, \Delta_n$  such that (3.71) holds. Suppose that the coefficients of  $F_1, \dots, F_n$  are generic, that is (more precisely), lie outside the algebraic hypersurface defined by (3.72). Then, one has :*

$$(3.75) \quad \#\{z \in \mathbb{T}^n ; F_1(z) = \dots = F_n(z) = 0\} = n! \times \text{vol}_n(\Delta_1, \dots, \Delta_n)$$

(common zeroes being counted here taking into account their multiplicities).

**PROOF.** Take  $f_j = R_{F_j}$ ,  $j = 1, \dots, n$ , in formula (3.62), where the  $F_j$  have generic coefficients, that is coefficients which lie outside the algebraic hypersurface (3.72) attached to  $\Delta_1, \dots, \Delta_n$ . One can observe, following the same computations than those leading to formula (3.67) in Example 3.24, that stating (3.62) for any borelian  $A \in \mathcal{B}(\mathbb{R}^n)$  in such case ( $f_j = R_{F_j}$ ,  $j = 1, \dots, n$ ) is equivalent to formulate the following equality between positive  $(n, n)$ -currents in the tube domain  $\mathbb{C}^n = \mathbb{R}_x^n + i\mathbb{R}_y^n$  :

$$(3.76) \quad \frac{1}{n!} \left( [\mathcal{A}_{F_1}] \wedge \dots \wedge [\mathcal{A}_{F_n}] \right) = d\mu[R_{F_1}, \dots, R_{F_n}](x) \bigwedge_{j=1}^n dx_j \wedge dy_j.$$

The integration of these two equal currents over the whole tube domain  $\mathbb{R}_x^n + i\mathbb{R}_y^n$ , using the right-hand side in (3.76), gives :

$$(3.77) \quad \mu[R_{F_1}, \dots, R_{F_n}](\mathbb{R}^n) = \mu[p_{R_{F_1}}, \dots, p_{R_{F_n}}](\mathbb{R}^n) = \text{vol}_n(\Delta_1, \dots, \Delta_n).$$

Using now the left-hand side of (3.76), the integration over the whole tube  $\mathbb{R}_x^n + i\mathbb{R}_y^n$  gives :

$$(3.78) \quad \frac{1}{n!} \int_{(\mathbb{S}^1)^n} \dots \int_{(\mathbb{S}^1)^n} \left( \int_{\mathbb{T}^n} [F_1^{-1}(u_1(\cdot))(0)] \wedge \dots \wedge [F_n^{-1}(u_n(\cdot))(0)] \right) \bigotimes_{j=1}^n d\sigma_{(\mathbb{S}^1)^n}(u_j),$$

where the integration currents involved here do take into account multiplicities, namely :

$$[F_j^{-1}(u_j(\cdot))(0)] := dd^c[\log |F_j(u_j z)|] \quad \forall u_j \in (\mathbb{S}^1)^n \quad \forall j = 1, \dots, n.$$

Since the  $F_j$  are taken with generic coefficients, then, for  $(u_1, \dots, u_n)$  generic in  $(\mathbb{S}^1)^n \times \dots \times (\mathbb{S}^1)^n$ , the integral :

$$\int_{\mathbb{T}^n} [F_1^{-1}(u_1(\cdot))(0)] \wedge \dots \wedge [F_n^{-1}(u_n(\cdot))(0)]$$

equals the generically constant value of the map (3.74). Formula (3.75) then follows from the comparison of the two different computations for the total mass of the positive current which was defined in (3.78) in the tube  $\mathbb{R}_x^n + i\mathbb{R}_y^n$ .  $\square$

**REMARK 3.26.** The interest of such proof of D. Bernstein's theorem is that it provides a proof of a result in complex projective geometry, based on tools which are relevant to convexity ideas in the real setting  $\mathbb{R}^n$ . The mixed volume  $\text{vol}_n(\Delta_1, \dots, \Delta_n)$  in the right-hand side of (3.75) is here understood as the mass of the mixed (real) Monge-Ampère measure  $\mu[R_{F_1}, \dots, R_{F_n}]$  attached to the Ronkin functions  $R_{F_j}$ . During the mass-preserving deformation that retracts the amoeba onto its spine (Proposition 3.4), this Monge-Ampère mass concentrates on the zero-dimensional cells of the subdivision  $\mathcal{C}(F)$ , that is the nodes of the spine  $V_{\text{trop}}(\mathfrak{p}_F)$  (see Examples 3.20 and 3.21). Such a proof could be understood as an indication that results holding within the frame of tropical (real) geometry could be transposed (thanks precisely to "inverse" retracting deformation) to complex algebraic geometry in the affine complex scheme  $\mathbb{T}^n$ . We will present in the next chapter other results in this direction (within the frame  $n = 2$ ).

**3.2.5. Miscellaneous remarks about the higher codimension case.** In this subsection, one considers, instead of a principal ideal  $(F)$  as before, an ideal  $(F_1, \dots, F_m)$  in  $\mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$  such that the sequence  $(F_1, \dots, F_m)$  is quasi-regular, that is

$$V_{\mathbb{T}}(F_1, \dots, F_m) = \{z \in \mathbb{T}^n ; F_1(z) = \dots = F_m(z) = 0\}$$

has pure codimension equal to  $m$ , or, equivalently, the  $F_j$ 's,  $j = 1, \dots, m$ , define a *complete intersection* in the affine algebraic variety  $\mathbb{T}^n$ .

Let, for any  $u_0 \in (\mathbb{S}^1)^n$ ,  $[(F \circ u_0 \exp)^{-1}(0)]$  be the integration current (multiplicities being taken into account) attached to the  $(n-m)$ -cycle in  $\mathbb{C}^n$  defined as the (proper) intersection of the principal divisors  $\text{div}(F_j \circ u_0 \exp)$ ,  $j = 1, \dots, m$ . One has (see e.g. [De1]), for any  $u_0 \in (\mathbb{S}^1)^n$ , the *Monge-Ampère equation* :

$$(3.79) \quad [(F \circ u_0 \exp)^{-1}(0)] = (dd^c[\log |F \circ u_0 \exp|])^{\wedge m},$$

that is, more precisely :

$$(3.80) \quad \begin{aligned} [(F \circ u_0 \exp)^{-1}(0)] &= \lim_{\epsilon \rightarrow 0^+} (dd^c[\log(|F \circ u_0 \exp| + \epsilon)])^{\wedge m} \\ &= \lim_{\substack{(\epsilon_1, \dots, \epsilon_m) \rightarrow 0 \\ \epsilon_j > 0, j=1, \dots, m}} \left( \bigwedge_{j=1}^m dd^c[\log |F \circ u_0 \exp|^2 + \epsilon_j] \right), \end{aligned}$$

the limit being in both cases understood with respect to the topology of the space  $\mathcal{D}^{m,m}(\mathbb{C}^n)$  of  $(m, m)$ -currents in  $\mathbb{C}^n$ .

Let  $R_F$  be the convex function in  $\mathbb{R}^n$  defined as

$$(3.81) \quad \begin{aligned} x \in \mathbb{R}^n &\mapsto R_F(x) := \frac{1}{(2\pi)^n} \int_{[0,2\pi]^n} \log |F(e^{x_1+i\theta_1}, \dots, e^{x_n+i\theta_n})| d\theta_1 \dots d\theta_n \\ &= \int_{(\mathbb{S}^1)^n} \log |(F \circ u \exp)(x)| d\sigma_{(\mathbb{S}^1)^n}. \end{aligned}$$

The argument used in the proof of Proposition 2.4 indeed motivates the definition of the *Ronkin function*  $R_F$  in such terms.

Thanks to the continuity of the real Monge-Ampère operator acting from the space of convex functions in  $\mathbb{R}^n$  into the space of positive Borel measures (see again the argument justifying Definition 3.23), one has, for any real convex (hence continuous) functions  $f_1, \dots, f_{n-m}$  in  $\mathbb{R}^n$  :

$$(3.82) \quad \begin{aligned} \mu[f_1, \dots, f_{n-m}, R_F, \dots, R_F](x) &\bigwedge_{j=1}^n dx_j \wedge dy_j = \\ &= \frac{1}{n!} \lim_{\substack{(\epsilon_1, \dots, \epsilon_m) \rightarrow 0 \\ \epsilon_j > 0, j=1, \dots, m}} \left( \bigwedge_{j=1}^m [\mathcal{A}_F^{\epsilon_j}] \wedge \bigwedge_{j=1}^{n-m} dd^c f_j \right), \end{aligned}$$

where  $[\mathcal{A}_F^{\epsilon_j}]$ ,  $j = 1, \dots, m$ , denotes the positive current

$$[\mathcal{A}_F^{\epsilon_j}] := \int_{(\mathbb{S}^1)^n} dd^c [\log(|F \circ u \exp| + \epsilon_j)] d\sigma_{(\mathbb{S}^1)^n}(u)$$

(the limit in the right-hand side of (3.82) being understood in the sense of a limit of currents in the weak sense). Formula (3.82) thus appears as a generalization of formula (3.67) (when  $m > 1$  instead of  $m = 1$ ).

Besides the Ronkin function  $R_F$ , one can introduce the *Ronkin current*  $\mathcal{R}_F$  defined as the  $(m-1, m-1)$ -current :

$$(3.83) \quad \mathcal{R}_F := \int_{(\mathbb{S}^1)^n} \log |F \circ u \exp| (dd^c[\log |F \circ u \exp|])^{\wedge m-1} d\sigma_{(\mathbb{S}^1)^n}(u)$$

(for a justification of the multiplicative operation between positive currents, see *e.g.* [De0], Théorème 4.5 (corollary 4.11)). Since the action of the  $dd^c$  operator commutes with the averaging process over  $(\mathbb{S}^1)^n$ , it follows from (3.79) that :

$$(3.84) \quad \begin{aligned} dd^c \mathcal{R}_F &= \int_{(\mathbb{S}^1)^n} (dd^c[\log |F \circ u \exp|])^{\wedge m} d\sigma_{(\mathbb{S}^1)^n}(u) \\ &= \int_{(\mathbb{S}^1)^n} [(F \circ u \exp)^{-1}(0)] d\sigma_{(\mathbb{S}^1)^n}(u). \end{aligned}$$

The  $(m, m)$  current  $[\mathcal{A}_F] = dd^c \mathcal{R}_F$  is a closed positive  $(m, m)$ -current such that :

$$(3.85) \quad \text{Supp}([\mathcal{A}_F]) = \mathcal{A}_{V_{\mathbb{T}}(F_1, \dots, F_m)} + i\mathbb{R}^n.$$

As a consequence of Theorem 3.7, the connected components (in  $\mathbb{C}^n$ ) of the complement of the tubular domain  $\mathcal{A}_{V_{\mathbb{T}}(F_1, \dots, F_m)} + i\mathbb{R}^n$  are  $(n-m)$ -pseudoconvex (see Definition 3.5), as noticed in [Rash] (see also [Henr] and [Mik1]). This geometric property reflects in some geometric property of the connected components of



$\mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(F_1, \dots, F_m)}$  : namely *any such connected component has no supporting  $m$ -cap*<sup>21</sup>. Note that, when  $m > 1$ , there is no hope to get convexity of the connected components of  $\mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(F_1, \dots, F_m)}$  : for example, when  $n = 3$  and  $m = 2$ , the amoeba looks essentially (at least a tentacle of it) as some ‘‘Eiffel tower’’. Such a property of  $(n - m)$ -pseudoconvexity seems to be the geometric property that is revealed by this currential approach.

The current  $\mathcal{R}_F$  constructed above provides an example of a  $(m - 1, m - 1)$ -current which  $dd^c$  (namely here the current  $[\mathcal{A}_F]$ ) is a  $d$ -closed positive current with support precisely the tubular domain  $\mathcal{A}_{V_{\mathbb{T}}(F_1, \dots, F_m)} + i\mathbb{R}^n$ . A most accurate candidate for a positive  $d$ -closed current with support exactly the tubular domain  $\mathcal{A}_{V_{\mathbb{T}}(F_1, \dots, F_m)} + i\mathbb{R}^n$  (in order to respect the correspondence between the real Monge-Ampère measure in  $\mathbb{R}^n$  and the complex Monge-Ampère measure in  $\mathbb{C}^n = \mathbb{R}^n + i\mathbb{R}^n$  that is materialized through the ‘‘multiplicative’’ formula (3.82)) would be the current  $[\mathcal{A}_{F_1}] \bullet \dots \bullet [\mathcal{A}_{F_m}]$  defined (formally) at follows :

$$(3.86) \quad \begin{aligned} & [\mathcal{A}_{F_1}] \bullet \dots \bullet [\mathcal{A}_{F_m}] := \\ &= \int \dots \int_{(\mathbb{S}^1)^n \times \dots \times (\mathbb{S}^1)^n} (\mathbf{1}_{(F \circ u_1 \exp)^{-1}(0)} \cdot dd^c [\log |F \circ u_1 \exp|]) \wedge \\ & \wedge \bigwedge_{j=2}^m (\mathbf{1}_{\mathbb{C}^n \setminus (F \circ u_j \exp)^{-1}(0)} \cdot dd^c [\log |F \circ u_j \exp|]) d\sigma_{(\mathbb{S}^1)^n}(u_1) \cdots d\sigma_{(\mathbb{S}^1)^n}(u_m). \end{aligned}$$

The notation  $T = \mathbf{1}_Z \cdot T + \mathbf{1}_{\mathbb{C}^n \setminus Z} \cdot T$  (when  $T$  is a  $d$ -closed positive  $(1, 1)$ -current in  $\mathbb{C}^n$  and  $Z \subset \mathbb{C}^n$  a closed analytic subset) stands here with the splitting of a  $d$ -closed positive current as a sum of two closed positive currents provided by El Mir’s theorem [EIM] (see also [De0], chapter III, section 2). Analytic continuation methods (see *e.g.* [ASWY]), when combined with the use of Bernstein-Sato type algebraic functional equations for polynomials in  $\exp X$  (see *e.g.* [BeY]), provide a justification for the formal definition of  $[\mathcal{A}_{F_1}] \bullet \dots \bullet [\mathcal{A}_{F_m}]$  such as proposed in (3.86), namely :

$$(3.87) \quad \begin{aligned} & [\mathcal{A}_{F_1}] \bullet \dots \bullet [\mathcal{A}_{F_m}] := \\ &= \left[ \int \dots \int_{(\mathbb{S}^1)^n \times \dots \times (\mathbb{S}^1)^n} \bar{\partial} [|F \circ u_1 \exp|^\mu] \wedge \frac{\partial [|F \circ u_1 \exp|^2]}{|F \circ u_1 \exp|^2} \wedge \right. \\ & \wedge \left. \left( \bigwedge_{j=2}^m |F \circ u_j \exp|^\lambda dd^c [\log |F \circ u_j \exp|] \right) d\sigma_{(\mathbb{S}^1)^n}(u_1) \cdots d\sigma_{(\mathbb{S}^1)^n}(u_m) \right]_{\lambda=0} \Bigg|_{\mu=0} \\ &= \left[ \int \dots \int_{(\mathbb{S}^1)^n \times \dots \times (\mathbb{S}^1)^n} \bar{\partial} [|F \circ u_1 \exp|^\lambda] \wedge \frac{\partial [|F \circ u_1 \exp|^2]}{|F \circ u_1 \exp|^2} \wedge \right. \\ & \wedge \left. \left( \bigwedge_{j=2}^m |F \circ u_j \exp|^{\lambda^2} dd^c [\log |F \circ u_j \exp|] \right) d\sigma_{(\mathbb{S}^1)^n}(u_1) \cdots d\sigma_{(\mathbb{S}^1)^n}(u_m) \right]_{\lambda=0}. \end{aligned}$$

<sup>21</sup>An open  $k$ -dimensional closed ball  $\bar{B}$  in a  $k$ -real plane in  $\mathbb{R}^n$  is called a *supporting  $k$ -cap* for an open subset  $C \subset \mathbb{R}^n$  if  $\bar{B} \cap C$  is non-empty and compact, and there exists  $v \in \mathbb{R}^n$ , such that, for  $\epsilon > 0$  small enough,  $(\bar{B} + \epsilon v) \cap C = \emptyset$ . This notion is connected with the notion of  *$k$ -convexity*, that we will introduce later on in this section.

Such a current is (as  $dd^c \mathcal{R}_F$ ) a  $d$ -closed positive current with support the tube  $\mathcal{A}_{V_{\mathbb{T}}(F_1, \dots, F_m)} + i\mathbb{R}^n$ . It fits better than  $dd^c \mathcal{R}_F = [\mathcal{A}_F]$  with the “multiplicative” formula (3.82). Such a remark could provide some hint toward the deformation the archimedean amœba  $\mathcal{A}_{V_{\mathbb{T}}(F_1, \dots, F_m)}$  on a codimension  $m$ -tropical variety in the tropical world ( $\text{Trop} \setminus \{-\infty\} \simeq \mathbb{R}^n$  (which is still an open question, see *e.g.* [Rull], section 8, for various comments and suggestions).

REMARK 3.27 (a promising alternative point of view). The approach developed within this course (separating real coordinates  $(x_1, \dots, x_n)$  corresponding to the world where archimedean amœbas, together with their tropical deformations, live, from argument coordinates  $\theta$  on which averaging is performed) can also be understood from a strictly “real” point of view, introducing *superforms* in  $\mathbb{R}^n$  (where one has at his disposal only  $n$  real variables<sup>22</sup>), in order precisely to recover the strength of multivariate complex analysis that the rich interplay between holomorphic and antiholomorphic coordinates provides (in particular with respect to the key notion of *positivity*). The differential operator  $dd^\sharp$  will play the role of the Lelong operator  $dd^c$ , and a theory of Monge-Ampère operators (Lelong-Poincaré equation, Monge-Ampère equation, *etc.*) can thus be carried through in such an ultrametric context. Such a point of view has been developed recently in [Lag1] (see in particular Remark 1.7 there) and carried through towards intersection theory in the ultrametric context, *i.e.* for example on Berkovich analytic spaces (see [CLD]).

Another stumbling block when dealing with archimedean amœbas in the higher codimensional case concerns the definition of entities that could play the role of multiplicities  $\nu_{F,C}$  such as defined in Definition 3.1 in the codimension one case. When  $F_1, \dots, F_m$  are  $m$  Laurent polynomials in  $\mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$  defining a complete intersection in  $\mathbb{T}^n$ , the pendant of the rational differential form  $dF/F$  (which is  $\bar{\partial}$ -closed in  $\mathbb{T}^n \setminus V_{\mathbb{T}}(F)$ ) is the so-called *Bochner-Martinelli form* :

$$\begin{aligned}
 \Omega(F_1, \dots, F_m) &:= \\
 (3.88) \quad &= \frac{(-1)^{(m(m-1)/2)} (m-1)!}{(2i\pi)^m} \frac{\sum_{j=1}^m (-1)^{j-1} \widehat{F}_j d\bar{F}_1 \wedge \dots \wedge \widehat{d\bar{F}_j} \wedge \dots \wedge d\bar{F}_m}{|F|^{2m}} \wedge \\
 &\wedge dF_1 \wedge \dots \wedge dF_m = \\
 &= d \left[ \log |F| \wedge (dd^c \log |F|)^{m-1} \right]
 \end{aligned}$$

(the hat over  $d\bar{F}_j$  means that this  $(0,1)$ -differential form is deleted in the wedge product). Such a  $(m, m-1)$ -form is  $\bar{\partial}$ -closed in  $\mathbb{T}^n \setminus V_{\mathbb{T}}(F)$ .

A basis for the homology group  $H_m((\mathbb{S}^1)^n, \mathbb{Z})$  is provided by the  $\binom{n}{m}$   $m$ -cycles :

$$\begin{aligned}
 \gamma_J &: (\theta_1, \dots, \theta_m) \in [0, 2\pi]^m \mapsto (1, \dots, 1, \overbrace{e^{i\theta_1}}^{j_1}, 1, \dots, 1, \overbrace{e^{i\theta_m}}^{j_m}, 1, \dots, 1), \\
 J &:= \{j_1, \dots, j_m\}, \quad 1 \leq j_1 < \dots < j_m \leq n.
 \end{aligned}$$

<sup>22</sup>Instead of the  $n$  complex variables  $z_1, \dots, z_n$ , together with their “ghosts”, namely the  $n$  “neutral” (with respect to the  $\bar{\partial}$ -operator) antiholomorphic variables  $\bar{z}_1, \dots, \bar{z}_n$ .

Let  $C$  be a connected component of the complement  $\mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(F_1, \dots, F_m)}$ . For each  $J = \{i_1, \dots, i_m\}$  with  $1 \leq i_1 < \dots < i_m \leq n$ , for each  $\gamma \in H_{m-1}(C, \mathbb{Z})$ , the cycle  $\log^*[\gamma(x) \times \gamma_J(\theta)]$  defines an element in  $H_{2m-1}(\text{Log}^{-1}(C), \mathbb{Z})$ . Moreover, the map :

$$(3.89) \quad \begin{aligned} & \nu_{(F_1, \dots, F_m), C} : \gamma \in H_{m-1}(C, \mathbb{Z}) \mapsto \\ & \mapsto \left( \int_{\log^*[\gamma(x) \times \gamma_J(\theta)]} \Omega(F_1, \dots, F_m) \right)_{1 \leq j_1 < \dots < j_m \leq m} \in \mathbb{Z}^{\binom{n}{m}} \end{aligned}$$

stands for an element of the dual cohomology group  $H^{m-1}(C, H_m((\mathbb{S}^1)^n, \mathbb{Z}))$ . In the particular case where  $m = 1$ , for a given connected component  $C$  of  $\mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(F)}$ , this map coincides with the map  $k \in H_0(C, \mathbb{Z}) \simeq \mathbb{Z} \mapsto k \nu_{F, C} \in \mathbb{Z}^n$ . For such reason, the morphism (3.89) (depending on the component  $C$  of  $\mathbb{R}^n \setminus \mathcal{A}_{V_{\mathbb{T}}(F_1, \dots, F_m)}$ ) plays a role which is similar to that played by  $k \in H_0(C, \mathbb{Z}) \mapsto k \nu_{F, C} \in \mathbb{Z}^n \simeq H_1((\mathbb{S}^1)^n, \mathbb{Z})$  in the case  $m = 1$ .

## Amœbas from the topological point of view

In this chapter, we will focus on various tools which revealed to be of significant importance towards applications of the concept of archimedean amœba in real algebraic geometry (see *e.g.* [Mik3]). The notion of *contour* ([PT1], [PST], [Mik1], [Mik3]), its analysis through the study of the *logarithmic Gauß map*, the compactification of amœbas of hypersurfaces thanks to the so-called *algebraic moment map* of the toric variety  $\mathcal{X}(\Sigma(\Delta(F)))$  (as define in [GKZ], chapter 6), more generally the (topological as well as geometric) description of the amœba  $\mathcal{A}_{V_{\mathbb{T}}(F)}$  with respect to the behavior “at infinity” of the hypersurface  $V_{\mathbb{T}}(F)$  in the “adjusted” toric compactification  $\mathcal{X}(\Sigma(\Delta(F)))$ , will be among the major points presented here. Basic notions about toric geometry (fans, construction of toric varieties, homogeneous coordinates and realisation as geometric quotient in the simplicial case, various notions of moment maps) used in this chapter are recalled allthrough this chapter when needed.

### 4.1. Contour of archimedean amœbas and logarithmic Gauß map

Let  $I \subset \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$  be an ideal and  $\mathcal{A}_{V_{\mathbb{T}}(I)} \subset \mathbb{R}^n$  be its archimedean amœba. Assume here that  $V_{\mathbb{T}}(I)$  is a purely dimensional algebraic subset of  $\mathbb{T}^n$ , with codimension  $k = 1, \dots, n - 1$  (such as for example in the case where  $I = (F)$  is a principal ideal in  $\mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ ).

Consider the restriction  $\text{Log}_{|V_{\mathbb{T}}(I)}$  of the Log map to  $V_{\mathbb{T}}(I)$ . Critical values of  $\text{Log}_{|V_{\mathbb{T}}(I)}$  are defined as the images of points  $z \in V_{\mathbb{T}}(I)$  where :

- either  $z$  belongs to  $(V_{\mathbb{T}}(I))^{\text{sing}}$  ;
- either  $z$  belongs to the complex manifold  $(V_{\mathbb{T}}(I))^{\text{reg}}$  and the smooth map  $\text{Log}_{|(V_{\mathbb{T}}(I))^{\text{reg}}}$  is such that

$$\text{rank}(d_z[\text{Log}_{|(V_{\mathbb{T}}(I))^{\text{reg}}}] < n_I := \dim(\text{Span}(\mathcal{A}_{V_{\mathbb{T}}(I)})),$$

where  $\text{Span}(\mathcal{A}_{V_{\mathbb{T}}(I)})$  denotes the  $\mathbb{R}$ -vector subspace generated by the amœba  $\mathcal{A}_{V_{\mathbb{T}}(I)}$  in  $\mathbb{R}^n$ .

Note that, if  $2(n - k) < n_I$ , *i.e.*  $2k > 2n - n_I$ , all points in  $\mathcal{A}_{V_{\mathbb{T}}(I)}$  are critical values of  $\text{Log}_{|V_{\mathbb{T}}(I)}$  (for example, when  $n = 3$  and  $V_{\mathbb{T}}(I)$  is an algebraic curve in  $\mathbb{T}^3$ ).

**DEFINITION 4.1** (contour of an archimedean amœba [PT1]). The *contour* (denoted as  $\text{cont}(\mathcal{A}_{V_{\mathbb{T}}(I)})$ ) of the archimedean amœba  $\mathcal{A}_{V_{\mathbb{T}}(I)}$  is the subset of  $\mathcal{A}_{V_{\mathbb{T}}(I)}$  defined as the set of critical values of  $\text{Log}_{|V_{\mathbb{T}}(I)}$ .

**REMARK 4.2.** One has of course always the inclusion :

$$\text{Log}((V_{\mathbb{T}}(I))^{\text{sing}}) \subset \text{cont}(\mathcal{A}_{V_{\mathbb{T}}(I)})$$

since images of singular points in  $V_{\mathbb{T}}(I)$  are automatically critical values of  $\text{Log}_{|V_{\mathbb{T}}(I)}$ . All points in the boundary  $\partial\mathcal{A}_{V_{\mathbb{T}}(I)}$  also belong to the contour, that is :

$$\partial(\mathcal{A}_{V_{\mathbb{T}}(I)}) \subset \text{cont}(\mathcal{A}_{V_{\mathbb{T}}(I)}).$$

When  $I = (F)$  is principal ( $F$  being reduced, *i.e.* without multiple factors), a convenient way to describe the contour of the amoeba  $\mathcal{A}_{V_{\mathbb{T}}(F)}$  is to introduce the *logarithmic Gauß map*. Let us consider first a point  $z_0 \in (V_{\mathbb{T}}(F))^{\text{reg}}$  ; in a neighborhood of such point, the algebraic hypersurface  $V_{\mathbb{T}}(F)$  can be parametrized thanks to  $n - 1$  complex parameters  $t_1, \dots, t_{n-1}$  close to 0, with  $z(0, \dots, 0) = z_0$ . Choose an arbitrary determination of  $z \mapsto \log z = (\log z_1, \dots, \log z_n)$  in a neighborhood  $W(z_0)$  of  $z_0$  and consider the restriction  $\log_{|V_{\mathbb{T}}(F)}$  of this map about  $z_0$  (in  $V_{\mathbb{T}}(F)$ ) :

$$(t_1, \dots, t_{n-1}) \mapsto (\log(z_1(t)), \dots, \log(z_n(t))).$$

The complex tangent space  $T_{\log(z_0)}(\log_{|V_{\mathbb{T}}(F)})$  is generated by the  $n - 1$  vectors :

$$\left( \frac{1}{z_{01}} \frac{\partial z_1}{\partial t_j}(0, \dots, 0), \dots, \frac{1}{z_{0n}} \frac{\partial z_n}{\partial t_j}(0, \dots, 0) \right), \quad j = 1, \dots, n - 1.$$

The complex normal to this complex tangent space corresponds therefore to the point

$$\left[ z_{01} \frac{\partial F}{\partial z_1}(z_0) : \dots : z_{0n} \frac{\partial F}{\partial z_n}(z_0) \right] \in \mathbb{P}^{n-1}(\mathbb{C})$$

(we use here the fact that  $F$  is taken as reduced) and does not depend on the choice of the determination for log about the point  $z_0$ .

**DEFINITION 4.3** (logarithmic Gauß map [**Krap0**]). Let  $F \in \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$  be a reduced Laurent polynomial (*i.e.* without multiple factors). The rational map from  $V_{\mathbb{T}}(F)$  to  $\mathbb{P}^{n-1}(\mathbb{C})$  defined by

$$(4.1) \quad \forall z \in (V_{\mathbb{T}}(F))^{\text{reg}}, \quad \gamma_F(z) := \left[ z_1 \frac{\partial F}{\partial z_1}(z) : \dots : z_n \frac{\partial F}{\partial z_n}(z) \right] \in \mathbb{P}^{n-1}(\mathbb{C})$$

is called the *logarithmic Gauß map*  $\gamma_F$  attached with the Laurent polynomial  $F$ .

The logarithmic Gauß map provides a description of the contour of the archimedean amoeba  $\mathcal{A}_{V_{\mathbb{T}}(F)}$  of a principal ideal  $(F)$  in  $\mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ . Namely one has the following :

**PROPOSITION 4.1** (logarithmic Gauß map and contour, for the amoeba of a principal ideal  $(F)$  [**Mik3**] (see also [**MaP**])). *Let  $F \in \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$  be a reduced Laurent polynomial in  $n$  variables. One has :*

$$(4.2) \quad \text{cont}(\mathcal{A}_{V_{\mathbb{T}}(F)}) = \overline{\text{Log} \left( (\gamma_F)_{|(V_{\mathbb{T}}(F))^{\text{reg}}}^{-1}(\mathbb{P}^{n-1}(\mathbb{R})) \right)} = \text{Log}(\gamma_F^{-1}(\mathbb{P}^{n-1}(\mathbb{R}))).$$

**PROOF.** The proof we give here is quoted from [**MaP**]. Since the map  $\text{Log} : V_{\mathbb{T}}(F) \rightarrow \mathbb{R}^n$  is continuous, the contour of the amoeba  $\mathcal{A}_{V_{\mathbb{T}}(F)}$  is the closure (in  $\mathbb{R}^n$ ) of the image by  $\text{Log}$  of the set of points  $z_0 \in (V_{\mathbb{T}}(F))^{\text{reg}}$  which are critical points for  $\text{Log}_{|(V_{\mathbb{T}}(F))^{\text{reg}}}$ . Let  $z_0 \in (V_{\mathbb{T}}(F))^{\text{reg}}$  be such a point. Take some arbitrary determination for log in a neighborhood  $W(z_0)$  of  $z_0$  in  $\mathbb{T}^n$ . Then the normal direction to the  $(n - 1)$ -dimensional complex tangent plane

$$T_{\log(z_0)} \left[ \log((V_{\mathbb{T}}(F))^{\text{reg}}) \right]$$

can be represented by some vector  $\tilde{\gamma}_F(z_0)$  in  $\mathbb{C}^n \setminus \{(0, \dots, 0)\}$ . Therefore the fiber  $T_{\log z_0}[\log(V_{\mathbb{T}}(F) \cap W(z_0))]$  of the complex tangent bundle  $T[\log(V_{\mathbb{T}}(F) \cap W(z_0))]$  is the complex hyperplane :

$$T_{\log z_0}[\log(V_{\mathbb{T}}(F) \cap W(z_0))] = \{x + i\theta \in \mathbb{C}^n = \mathbb{R}_x^n + i\mathbb{R}_\theta^n; \langle \tilde{\gamma}_F(z_0), x + i\theta \rangle = 0\}.$$

If  $\tilde{\gamma}_F(z_0)$  can be chosen in  $\mathbb{R}^n \setminus \{(0, \dots, 0)\}$ , the projection of this complex hyperplane on the space  $\mathbb{R}_x^n$  is indeed not surjective, which implies that  $z_0$  is a critical point for the map  $\text{Log}_{|(V_{\mathbb{T}}(F))^{\text{reg}}}$  (which is the composition of  $\log_{|(V_{\mathbb{T}}(F))^{\text{reg}} \cap W_{z_0}}$  with the projection  $x + i\theta \in \mathbb{R}_x^n + i\mathbb{R}_\theta^n \mapsto x \in \mathbb{R}_x^n$ ). If  $\tilde{\gamma}_F(z_0) = \tilde{a} + i\tilde{b}$ , with  $\tilde{a}$  and  $\tilde{b}$  linearly independent in  $\mathbb{R}^n$ , one can find, whenever  $x \in \mathbb{R}^n$  is fixed, some  $\theta \in \mathbb{R}^n$  such that

$$\langle \tilde{\gamma}_F(z_0), x + i\theta \rangle = 0 \iff \begin{cases} \langle a, \theta \rangle = -\langle b, x \rangle \\ \langle b, \theta \rangle = \langle a, x \rangle \end{cases}$$

(the rank of the linear system of two equations in  $n$  unknown  $\theta_j$  in the right-hand side being equal to 2). This concludes the proof of the first equality in (4.2). The second equality follows from the reasoning above, combined with the fact that  $(V_{\mathbb{T}}(F))^{\text{sing}}$  (*i.e.* the algebraic variety defined by  $F$  and its partial derivatives) equals exactly the polar set of  $\gamma_F$ .  $\square$

As a consequence, one can observe the following :

**COROLLARY 1** (the case of Laurent polynomials with real coefficients). Let  $F \in \mathbb{R}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$  be a reduced Laurent polynomial with real coefficients. Then one has

$$\text{Log}(V_{\mathbb{T}}(F) \cap (\mathbb{R}^*)^n) \subset \text{cont}(\mathcal{A}_{V_{\mathbb{T}}(F)}).$$

**PROOF.** Any point in  $(V_{\mathbb{T}}(F))^{\text{reg}} \cap (\mathbb{R}^*)^n$  is carried by the logarithmic Gauß map (4.1) to a point that belongs to  $\mathbb{P}^{n-1}(\mathbb{R})$ , which proves that it is a critical point for  $\text{Log}_{|(V_{\mathbb{T}}(F))^{\text{reg}}}$  (see Proposition 4.1). Its image by  $\text{Log}$  lies then in the contour of the amoeba  $\mathcal{A}_{V_{\mathbb{T}}(F)}$ . The same is true for any point in  $(V_{\mathbb{T}}(F))^{\text{sing}} \cap (\mathbb{R}^*)^n$ , since such a point is considered among critical points of  $\text{Log}_{V_{\mathbb{T}}(F)}$ .  $\square$

## 4.2. Generic degree of the logarithmic Gauß map

**4.2.1. Toric compactifications adjusted to a given Laurent polynomial.** Let  $F \in \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$  be a reduced Laurent polynomial. We assume here  $\dim(\Delta(F)) = n$ . In order to associate (if possible) to the Gauß logarithmic rational map :

$$\gamma_F : V_{\mathbb{T}}(F) \rightarrow \mathbb{P}^{n-1}(\mathbb{C})$$

a topological degree, one needs first to extend it as a rational map from a compact algebraic variety into  $\mathbb{P}^{n-1}(\mathbb{C})$ . In order to proceed to such an extension, one will introduce the most “suitable<sup>1</sup>” compactification of  $\mathbb{T}^n$ , in accordance with the Newton polyedron  $\Delta(F)$  of  $F$ . Our main objective is indeed to “stick” as much as possible to the geometric configuration of such  $\Delta(F)$  (which is, as we have seen already, intimately related to the geometric and topologic configuration of the amoeba  $\mathcal{A}_{V_{\mathbb{T}}(F)}$  and its spine).

<sup>1</sup>In a sense that will be precised later on.

In order to achieve our goal, we appeal here again to the concept of *complete toric  $n$ -dimensional algebraic variety*, which was already briefly introduced in order to state D. Bernstein's theorem (*cf.* subsection 3.2.4 above) : a complete  $n$ -dimensional toric algebraic variety is a  $n$ -dimensional algebraic variety (in the sense of algebraic geometry<sup>2</sup>), which is complete<sup>3</sup> and, moreover :

- is normal ;
- contains  $\mathbb{T}^n$  as a dense subset ;
- (most important, this justifies the terminology “toric”) is equipped with a group action of  $\mathbb{T}^n$ , such that this action coincides with pointwise multiplication on the dense open orbit  $\mathbb{T}^n$ .

Consider the *dual fan*  $\Sigma(\Delta(F))$  attached to  $\Delta(F)$ . This fan is the convex subdivision of  $\mathbb{R}^n$  which cells are the dual cones<sup>4</sup> to the faces of the Newton polyhedron  $\Delta(F)$ . Since  $\Delta(F)$  is assumed here to be  $n$ -dimensional, all such cones are *strict* (that is do not contain any linear subspace). Among the cones in this fan, the cones with dimension one (dual to the facets of  $\Delta(F)$ ) are of particular interest : denote them as  $\xi_1, \dots, \xi_{n+r}$  (note that  $r \in \mathbb{N}^*$  since  $\Delta(F)$  is  $n$ -dimensional).

In general the fan  $\Sigma(\Delta(F))$  is not *simplicial*, which would mean that the 1-dimensional faces of each cone  $\tau \neq \{0\}$  in the fan are generated by linear independent vectors<sup>5</sup>. But it is possible, without adding extra rays, to refine such a fan (*i.e.* introduce some extra subdivisions of its conic cells) in order to obtain a refined fan  $\tilde{\Sigma}(\Delta(F))$ , which is now a simplicial one. One can do even better : introducing this time extra rays, one can construct a new refinement  $\tilde{\tilde{\Sigma}}(\Delta(F))$  of  $\tilde{\Sigma}(\Delta(F))$ , that is of  $\Sigma(\Delta(F))$ , such that the new refined fan  $\tilde{\tilde{\Sigma}}(\Delta(F))$  is *primitive* or *simple*, which means that the 1-dimensional faces of each  $n$ -dimensional cone  $\tau$  of it are directed by respective vectors  $\vec{\eta}_{\tau,1}, \dots, \vec{\eta}_{\tau,n}$ , each with integer coprime coordinates, such that determinant  $\det(\vec{\eta}_{\tau,1}, \dots, \vec{\eta}_{\tau,n}) = \pm 1$ . All such refinement processes can be done algorithmically, but of course not in a canonical way ; we refer for example to [Fult] (sequence of exercices pp. 47-48) for a description of such an algorithmic process. Moreover, given any such refinement  $\tilde{\tilde{\Sigma}}(\Delta(F))$  of the fan  $\Sigma(\Delta(F))$ , there is a proper birational surjective morphism from the toric variety  $\mathcal{X}(\tilde{\tilde{\Sigma}}(\Delta(F)))$  to the toric variety  $\mathcal{X}(\Sigma(\Delta(F)))$  (see [Fult], section 2.6). Though the complete algebraic toric variety  $\mathcal{X}(\Sigma(\Delta(F)))$  is in general not projective, it is always birational to a projective variety.

Each rational cone  $\tau$  in the fan  $X(\Sigma(\Delta(F)))$  with dimension  $k$  (which is thus dual to some  $(n - k)$ -dimensional face  $\gamma$  of  $\Delta(F)$ ) has a dual cone :

$$\check{\tau} = \{u \in (\mathbb{R}^n)^* ; \langle u, x \rangle \geq 0 \quad \forall x \in \tau\},$$

(which is always  $n$ -dimensional since  $\tau$  is strict), to which it is possible to associate the affine normal  $n$ -dimensional algebraic variety :

$$X_{\check{\tau}} := \text{Spec}(\mathbb{C}[X^{\alpha} ; \alpha \in \check{\tau} \cap (\mathbb{Z}^n)^*]).$$

<sup>2</sup>If one wants to keep more close to the pointsetwise point of view, one could think of it as a complex analytic space.

<sup>3</sup>Topologically speaking, from the pointset point of view, this means compact.

<sup>4</sup>Remember that the opposite cones are called *polar cones* and that some among them are recession cones for the connected components of  $\mathbb{R}^n \setminus \mathcal{A}_{V_{\tau}(F)}$  (see Proposition 3.3).

<sup>5</sup>Think for example at the case where  $\Delta(F)$  is a pyramid in  $\mathbb{R}^3$ , in which one vertex is the intersection point of 4 ( $> \dim \mathbb{R}^3$ ) edges.

Each  $X_\tau$  is in fact a normal variety, in which one can embed all  $X_{\tau'}$  for  $\tau' \prec \tau$ , in particular  $X_{\{0\}} = \text{Spec}(\mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]) = \mathbb{T}^n$ . Moreover, there is an action of  $\mathbb{T}^n$  on  $X_\tau$ , which coincides with the pointwise multiplication on  $\mathbb{T}^n \subset X_\tau$  (which thus becomes a dense open orbit for this action, in fact the only open one). So  $X_\tau$  itself inherits a structure of algebraic  $n$ -dimensional toric variety. All such  $X_\tau$ ,  $\tau \in \Sigma(\Delta(F))$ , can be glued together in accordance with the action of  $\mathbb{T}^n$  (see [Fult]) in order to realize a complete algebraic toric variety, called  $\mathcal{X}(\Sigma(\Delta(F)))$ .

Geometrically speaking, this toric algebraic variety  $\mathcal{X}(\Sigma(\Delta(F)))$  can only be understood as a (singular) compact normal algebraic variety containing the torus  $\mathbb{T}^n$ , and equipped with an action of the torus which coincides with pointwise multiplication on  $\mathbb{T}^n$ . It stands then as a (non smooth) algebraic compactification of  $\mathbb{T}^n$ . Such an algebraic variety is in fact birational to any other one which could be constructed along the same lines, but starting from a refinement either simplicial or primitive ( $\tilde{\Sigma}(\Delta(F))$  or  $\bar{\Sigma}(\Delta(F))$ ) of the dual fan  $\Sigma(\Delta(F))$ . The rays  $\xi_1, \dots, \xi_{n+r}$  of the fan correspond to toric hypersurfaces  $Y_1, \dots, Y_{n+r}$ , which are  $\mathbb{T}^n$ -invariant, and which union is such that, geometrically speaking,

$$\mathcal{X}(\Sigma(\Delta(F))) = \mathbb{T}^n \sqcup \bigcup_{j=1}^{n+r} Y_j.$$

The union of the toric hypersurfaces  $Y_j$ ,  $j = 1, \dots, n+r$ , can thus be understood as the world “at infinity” in the (in general singular) toric compactification  $\mathcal{X}(\Sigma(\Delta(F)))$  of the complex torus  $\mathbb{T}^n$ . Each toric hypersurface  $Y_j$ ,  $j = 1, \dots, n+r$ , is obtained glueing together all the  $X_\tau$  such that  $\xi_j \prec \tau$ . The orbit corresponding to the ray  $\xi_j$  is the topological relative interior of  $Y_j$ . A convenient way to visualize the correspondence between faces  $\gamma$  of  $\Delta(F)$  and closures of orbits in  $\mathcal{X}(\Sigma(\Delta(F)))$  under the action of  $\mathbb{T}^n$  (corresponding to dual cones  $\tau = \gamma^*$  in the fan  $\Sigma(\Delta(F))$ ) is to use the following continuous *algebraic moment map*<sup>6</sup>:

$$(4.3) \quad z \in \mathbb{T}^n \mapsto \frac{\sum_{\alpha \in \Delta(F) \cap \mathbb{Z}^n} \alpha |z^\alpha|}{\sum_{\alpha \in \Delta(F) \cap \mathbb{Z}^n} |z^\alpha|} \in \Delta(F)$$

(see [Fult], chapter 4). Such a map can be continued as a continuous map  $\mu_F$  from the toric variety  $\mathcal{X}(\Sigma(\Delta(F)))$  into the Newton polyhedron  $\Delta(F)$  (we denote this moment map as  $\mu_F$ , but of course it depends only on the Newton polyhedron  $\Delta(F)$ ). The algebraic moment map thus obtained is a surjective map from the compact algebraic variety  $\mathcal{X}(\Sigma(\Delta(F)))$  into the Newton polyhedron  $\Delta(F)$ . More precisely : for any proper face  $\gamma$  of  $\Delta(F)$ ,  $\mu_F^{-1}(\gamma)$  equals the closure of the orbit associated with the dual cone  $\tau = \gamma^*$  in the fan  $\Sigma(\Delta(F))$ . Moreover, if one considers the action of the real torus  $(\mathbb{S}^1)^n$  on  $\mathcal{X}(\Sigma(\Delta(F)))$  (induced by that of the complex torus  $\mathbb{T}^n$  on this toric variety), then, for any point  $u$  belonging to some  $k$ -dimensional face  $\gamma$  of  $\Delta(F)$  ( $0 \leq k \leq n-1$ )  $\mu_F^{-1}(u)$  is an orbit (under the action of  $(\mathbb{S}^1)^n$  on  $\mathcal{X}(\Sigma(\Delta(F)))$ )

<sup>6</sup>The *moment map* is the continuation to  $\mathcal{X}(\Sigma(\Delta(F)))$  of the continuous map :

$$z \in \mathbb{T}^n \mapsto \frac{\sum_{\alpha \in \Delta(F) \cap \mathbb{Z}^n} \alpha |z^\alpha|^2}{\sum_{\alpha \in \Delta(F) \cap \mathbb{Z}^n} |z^\alpha|^2} \in \Delta(F)$$



which is isomorphic to  $(\mathbb{S}^1)^k$ . For any  $u$  in the interior of  $\Delta(F)$ ,  $\mu_F^{-1}(u)$  correspond to the orbit under the action of  $(\mathbb{S}^1)^n$  of a point  $z$  in the complex torus  $\mathbb{T}^n$ .

The situation is particular nice when one starts with a primitive refinement  $\tilde{\Sigma}(\Delta(F))$ . In such case, the toric algebraic variety  $\mathcal{X}(\tilde{\Sigma}(\Delta(F)))$  is realized (see *e.g.* [Dan] or [Eh10]) as a  $n$ -dimensional compact toric complex manifold by glueing together copies of  $\mathbb{C}^n$  thanks to monomial changes of coordinates from one copy into another one (exactly on the model leading to the classical construction of the projective space  $\mathbb{P}^n(\mathbb{C})$ ).

The situation remains also rather nice (while more complicated) when the refinement  $\tilde{\Sigma}(\Delta(F))$  is just simplicial (but not primitive). In this case, the algebraic toric variety  $X(\tilde{\Sigma}(\Delta(F)))$  (still birational to  $\mathcal{X}(\Sigma(\Delta(F)))$ ) is realized (see [Co1], [CLO1]) as the geometric quotient :

$$(4.4) \quad X(\tilde{\Sigma}(\Delta(F))) = \frac{\mathbb{C}^{n+r} - \left\{ Z \in \mathbb{C}^{n+r}; \prod_{\xi_j \neq \tau} Z_j = 0 \quad \forall \tau \in \tilde{\Sigma}(\Delta(F)), \dim \tau = n \right\}}{G},$$

where  $G$  stands for the group action of the multiplicative group

$$G := \left\{ (t_1, \dots, t_{n+r}) \in \mathbb{C}^{n+r}; \prod_{j=1}^{n+r} t_j^{\xi_j^k} = 1, k = 1, \dots, n \right\},$$

and  $\vec{\xi}_j := (\xi_{j1}, \dots, \xi_{jn})$  denotes, for  $j = 1, \dots, n+r$ , the vector with integer coprime coordinates which generate the ray  $\xi_j$ . The equations  $\{Z_j = 0\}$ ,  $j = 1, \dots, n+r$ , provide respectively (modulo the group action in (4.4)) equations for the hypersurfaces  $\tilde{Y}_j$ ,  $j = 1, \dots, n+r$ , corresponding to the rays  $\xi_1, \dots, \xi_{n+r}$  of the refined fan (the same than those of the original fan  $\Sigma(\Delta(F))$ ). The ‘‘coordinates’’  $Z_1, \dots, Z_{n+r}$  can be considered as *homogeneous coordinates* (see [Co1]). The *homogenization* of the Laurent polynomial  $F = \sum_{\alpha \in \text{Supp } F} c_\alpha X^\alpha$  is given as the polynomial in  $Z_1, \dots, Z_{n+r}$  that appear in the numerator of the expression :

$$(4.5) \quad \mathfrak{F}(Z_1, \dots, Z_d) := \sum_{\alpha \in \text{Supp}(F)} c_\alpha \prod_{j=1}^{n+r} Z_j^{\langle \alpha, \vec{\xi}_j \rangle},$$

after reducing it to a common denominator. The zero set of this numerator  ${}^h F(Z_1, \dots, Z_{n+r}) \in \mathbb{C}[Z_1, \dots, Z_{n+r}]$  defines then (modulo the group action in (4.4)) the equation of the algebraic hypersurface  $\overline{V_{\mathbb{T}}(F)} \sim$  (the Zariski closure of the algebraic hypersurface  $V_{\mathbb{T}}(F) \subset \mathbb{T}^n$  in  $\mathcal{X}(\tilde{\Sigma}(\Delta(F)))$ ), expressed in such homogeneous coordinates.

One needs to mention here two important facts about the position of the Zariski closure  $\overline{V_{\mathbb{T}}(F)}$  of  $V_{\mathbb{T}}(F)$  in the toric variety  $\mathcal{X}(\Sigma(\Delta(F)))$ , with respect to the toric hypersurfaces  $Y_j$ ,  $j = 1, \dots, n+r$ , corresponding to the rays  $\xi_1, \dots, \xi_{n+r}$  of the dual fan. In order to check these two facts, one can for example use (as described above) a simplicial refinement  $\tilde{\Sigma}(\Delta(F))$  and related homogeneous coordinates. These two key fact both emphasise the fact that the toric compactification  $\mathcal{X}(\Sigma(\Delta(F)))$  indeed respects the information carried by the Newton polyedron  $\Delta(F)$  (as the amoeba  $\mathcal{A}_{V_{\mathbb{T}}(F)}$  also does) :

- (1) on one hand, the algebraic hypersurface  $\overline{V_{\mathbb{T}}(F)} \subset \mathcal{X}(\tilde{\Sigma}(\Delta(F)))$  avoids all points  $\mu_F^{-1}(\sigma)$  corresponding to the vertices  $\sigma$  of  $\Delta(F)$  ;

- (2) on the other hand, the algebraic hypersurface  $\overline{V_{\mathbb{T}}(F)} \subset \mathcal{X}(\tilde{\Sigma}(\Delta(F)))$  intersects any of the algebraic toric curves  $L_{\gamma} = \mu_F^{-1}(\gamma)$ , where  $\gamma$  is an edge of  $\Delta$  (corresponding to some  $(n-1)$ -dimensional cone in the dual fan).

EXAMPLE 4.4 (the particular case  $n=2$ ). In that case  $\overline{V_{\mathbb{T}}(F)} \subset X(\Sigma(\Delta(F)))$  is an algebraic curve that intersect at discrete points each of the toric curves at infinity  $Y_j$ ,  $j=1, \dots, n+r$ , which correspond in that case to the orthogonal directions  $\xi_1, \dots, \xi_{n+r}$  to the edges  $\gamma_1, \dots, \gamma_{n+r}$  of the Newton polyedron  $\Delta(F)$ .

The moment map  $\mu_F$  provides also an important notion, that of *compactified amœba*.

DEFINITION 4.5 (compactified amœba of an algebraic hypersurface in  $\mathbb{T}^n$ ). Let  $F \in \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$  be a Laurent polynomial such that  $\dim(\Delta(F)) = n$ ,  $X_{\Delta}$  the complete  $n$ -dimensional algebraic toric variety attached to the dual fan of  $\Delta(F)$ ,  $\mu_F$  its algebraic moment map. The *compactified amœba* of the Laurent polynomial  $F$  is

$$(4.6) \quad \mathcal{A}_{V_{\mathbb{T}}(F)}^c := \mu_F(\overline{V_{\mathbb{T}}(F)}).$$

Using a simplicial refinement of the fan  $\Sigma(\Delta(F))$  and homogenisation of the Laurent polynomial  $F$  as before, one can point out the following result.

PROPOSITION 4.2. *Let  $F$  as in Définition (4.5) and  $\gamma$  be a  $k$ -dimensional face ( $0 < k \leq n-1$ ) of the Newton polyedron  $\Delta(F)$ . Let  $F_{\gamma}$  be the Laurent polynomial obtained as*

$$F_{\gamma}(X) = \sum_{\alpha \in \text{Supp } F \cap \gamma} c_{\alpha} X^{\alpha},$$

where  $F(X) = \sum_{\alpha \in \text{Supp } (F)} c_{\alpha} X^{\alpha}$ . Then the intersection  $\mathcal{A}_{V_{\mathbb{T}}(F)}^c \cap \gamma$  is the compactified amœba of the Laurent polynomial  $F_{\gamma}$  (considered as a Laurent polynomial in  $k$  variables with support in the lattice induced by  $\mathbb{Z}^n$  on the face  $\gamma$ ), the compactification being done with respect to the toric  $k$ -dimensional variety  $\mu_F^{-1}(\gamma)$ .

**4.2.2. The generic degree of the Gauß logarithmic map.** We are now able, once the configuration needed for compactification fixed, to derive some information about the topological degree of the logarithmic Gauß map, since it can be continued up to the compactification of  $V_{\mathbb{T}}(F)$  as a dominant rational map.

PROPOSITION 4.3 (the degree of the Gauß map in the smooth and transversal case, [Mik3]). *Let  $F \in \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$  be a reduced Laurent polynomial such that  $\dim \Delta(F) = n$  and the origin is not a vertex of  $\Delta(F)$ . Assume that for some primitive refinement of the fan  $\Sigma(\Delta(F))$  (with rays  $\xi_1, \dots, \xi_{n+\bar{r}}$ , including the rays  $\xi_1, \dots, \xi_{n+\bar{r}}$  of the fan  $\Sigma(\Delta(F))$ ), corresponding to homogeneous coordinates  $Z_1, \dots, Z_{n+\bar{r}}$  on the  $n$ -dimensional toric manifold  $\mathcal{X}(\tilde{\Sigma}(\Delta(F)))$ , the  $(n+1)$  polynomials*

$$(4.7) \quad {}^h F(Z_1, \dots, Z_{n+\bar{r}}), {}^h \left[ z_1 \frac{\partial F}{\partial z_1} \right] (Z_1, \dots, Z_{n+\bar{r}}), \dots, {}^h \left[ z_n \frac{\partial F}{\partial z_n} \right] (Z_1, \dots, Z_{n+\bar{r}})$$

(note that the  $z_j \partial F / \partial z_j$  share  $\Delta(F)$  as Newton polyedron since the origin is not a vertex) have no common zero in

$$\mathbb{C}^{n+\bar{r}} - \left\{ Z \in \mathbb{C}^{n+\bar{r}}; \prod_{\xi_j \notin \tau} Z_j = 0 \quad \forall \tau \in \tilde{\Sigma}(\Delta(F)), \dim \tau = n \right\}.$$

The same property then holds for any simplicial refinement of the fan  $\Sigma(\Delta(F))$  and is equivalent to say that the Zariski closure of  $V_{\mathbb{T}}(F)$  in any toric  $n$ -dimensional manifold  $\mathcal{X}(\tilde{\Sigma}(\Delta(F)))$  (for any primitive refinement of the fan  $\Sigma(\Delta(F))$ ) is a complex submanifold which intersects transversally in  $\mathcal{X}(\tilde{\Sigma}(\Delta(F)))$  all closed orbits at infinity (corresponding to all cones of  $\tilde{\Sigma}(\Delta(F))$ ) with dimension between 1 and  $n - 1$ ). That is, such a property depends only of  $F$  and its Newton polyedron, not on the refinement of the fan  $\Sigma(\Delta(F))$ . The logarithmic Gauß map then extends to a dominant rational map  $\bar{\gamma}_F$  between the Zariski closure of  $V_{\mathbb{T}}(F)$  in  $\mathcal{X}(\Sigma(\Delta(F)))$  and  $\mathbb{P}^{n-1}(\mathbb{C})$ . The degree of this dominant rational map equals

$$(4.8) \quad \deg \bar{\gamma}_F = n! \operatorname{vol}_n(\Delta).$$

PROOF. Let  $\tilde{\Sigma}(\Delta(F))$  be a primitive refinement of the fan  $\Sigma(\Delta(F))$  for which the conditions hold and  $\widetilde{V_{\mathbb{T}}(F)}$  the Zariski closure of  $V_{\mathbb{T}}(F)$  in the  $n$ -dimensional compact toric complex manifold  $\mathcal{X}(\tilde{\Sigma}(\Delta(F)))$ . It is immediate to check that the hypothesis imply that  $\widetilde{V_{\mathbb{T}}(F)}$  is a complex submanifold of  $\mathcal{X}(\tilde{\Sigma}(\Delta(F)))$ , which is even transverse to all closed orbits at infinity (corresponding to cones with dimension between 1 and  $n - 1$  in the primitive refined fan  $\tilde{\Sigma}(\Delta(F))$ ). Any Laurent polynomial  $z_j \partial F / \partial z_j$ ,  $j = 1, \dots, n$ , considered as a rational function in  $\mathbb{T}^n$ , extends as a rational function on the  $n$ -dimensional toric manifold  $\mathcal{X}(\tilde{\Sigma}(\Delta(F)))$ : just express it for example in homogeneous coordinates, then in affine coordinates in the different copies of  $\mathbb{C}^n$  which are glued together *via* monoidal maps in order to realize the toric manifold  $\mathcal{X}(\tilde{\Sigma}(\Delta(F)))$ . Then the logarithmic Gauß map extends as a rational map from the  $(n - 1)$ -dimensional compact algebraic manifold  $\widetilde{V_{\mathbb{T}}(F)}$  into the  $(n - 1)$ -compact algebraic manifold  $\mathbb{P}^{n-1}(\mathbb{C})$ . Since there is a birational map  $f_{\tilde{\iota}, \iota}$  between the toric variety  $\mathcal{X}(\Sigma(\Delta(F)))$  and its resolution of singularities  $\mathcal{X}(\tilde{\Sigma}(\Delta(F)))$ , one obtains then a rational map  $\bar{\gamma}_F$  between the Zariski closure of  $V_{\mathbb{T}}(F)$  in  $\mathcal{X}(\Sigma(\Delta(F)))$ , considered here as a  $(n - 1)$ -dimensional algebraic subvariety of this  $n$ -dimensional (in general singular) toric variety, and  $\mathbb{P}^{n-1}(\mathbb{C})$ . The hypothesis which is made on the polynomials (4.7) also implies that :

$$(4.9) \quad \liminf_{\substack{\|x\| \rightarrow +\infty \\ x \in \mathbb{R}^n}} \left( \inf_{z_x \in \operatorname{Log}^{-1}(x)} \frac{|F(z_x)| + \sum_{j=1}^n |z_{x,j}(\partial F / \partial z_j)(z_x)|}{\exp(\sup_{\xi \in \Delta} \langle \xi, x \rangle)} \right) > 0.$$

As noticed in (3.73) above (see [VY], Proposition 2.2), the analytic condition (4.9) (which appears here as a particular case) is equivalent to say that the Laurent polynomials  $(F, X_1 F'_{X_1}, \dots, X_n F'_{X_n})$  (all sharing the same Newton polyedron  $\Delta(F)$ ) fulfill the generic hypothesis needed for D. Bernstein's theorem (Theorem 3.25) to hold). Therefore, for generic  $\lambda = (\lambda_0, \dots, \lambda_{n-1}) \in \mathbb{C}^n \setminus \{(0, \dots, 0)\}$ , the  $n$  Laurent polynomials

$$(4.10) \quad F(X), \lambda_j X_1 \frac{\partial F}{\partial X_1} - \lambda_1 X_j \frac{\partial F}{\partial X_j}, \quad j = 2, \dots, n$$

also fulfill the hypothesis of D. Bernstein's theorem. The supports of the corresponding effective Weil divisors in  $\mathcal{X}(\Sigma(\Delta(F)))$  share exactly  $n! \operatorname{vol}_n(\Delta(F))$  isolated zeroes (counted with multiplicities), all such common zeroes lying in the dense orbit  $\mathbb{T}^n$ . One has then :

$$\#(\bar{\gamma}_F^{-1}([\lambda_0 : \dots : \lambda_{n-1}])) = n! \operatorname{vol}_n(\Delta(F))$$

for  $[\lambda_0 : \cdots : \cdots : \lambda_{n-1}]$  generic in  $\mathbb{P}^{n-1}(\mathbb{C})$ , which proves that  $\bar{\gamma}_F$  is dominant, with degree equal to  $n! \operatorname{vol}_n(\Delta(F))$ .  $\square$

REMARK 4.6. Note that, provided the hypothesis in Proposition 4.3 is fulfilled, the continuation of  $\gamma_F$  as a rational map from the Zariski closure  $\overline{V_{\mathbb{T}}(F)}$  of  $V_{\mathbb{T}}(F)$  (in any simplicial  $n$ -dimensional toric variety  $\mathcal{X}(\tilde{\Sigma}(\Delta(F)))$ ), into  $\mathbb{P}^{n-1}(\mathbb{C})$ , remains of course a dominant rational map with degree  $n! \operatorname{vol}_n(\Delta(F))$ : one need just to use that the morphisms  $f_{i,\iota}$  are birational. For the sake of simplicity, we will keep the notation  $\bar{\gamma}_F$  for such rational continuation to  $\overline{V_{\mathbb{T}}(F)}$  (independently of the simplicial refinement of  $\Sigma(\Delta(F))$  which is used).

REMARK 4.7 (What about the singular case?). Consider a reduced Laurent polynomial in  $\mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$  such that  $\dim(\Delta(F)) = n$  and the origin is not a vertex of this polyedron. If the Zariski closure  $\overline{V_{\mathbb{T}}(F)} \sim$  of  $V_{\mathbb{T}}(F)$  in some  $n$ -dimensional toric complex manifold  $\mathcal{X}(\tilde{\Sigma}(\Delta(F)))$  (corresponding to a primitive refinement of the dual fan  $\Sigma(\Delta(F))$ ) happens to be singular at some point at infinity in  $\mathcal{X}(\tilde{\Sigma}(\Delta(F)))$ , that is in  $\mathcal{X}(\tilde{\Sigma}(\Delta(F))) \setminus \mathbb{T}^n$ <sup>7</sup>, then, whatever how  $(\lambda_0, \dots, \lambda_{n-1})$  is taken generic in  $\mathbb{R}^n \setminus \{(0, \dots, 0)\}$ , the  $n$  Laurent polynomials (4.10) will never fulfill the conditions that are needed for D. Bernstein's theorem to hold. If by chance  $\gamma_F$  can still be continued as a dominant rational map  $\bar{\gamma}_F$  to  $\mathcal{X}(\Sigma(\Delta(F)))$ , it is clear in that case that the degree of this map will be strictly less than  $n! \operatorname{vol}_n(\Delta(F))$ . We will characterize in the next subsection the situation where such degree could be indeed equal to one.

### 4.3. Discriminants and amoebas

Given an algebraic hypersurface

$$V_{\mathbb{T}}(F) \subset \mathbb{T}^n$$

(defined as the zero set of a reduced Laurent polynomial  $F$  in  $n$  variables), a natural question one could ask is whether (and when?) the logarithmic Gauß map  $\gamma_F : V_{\mathbb{T}}(F) \rightarrow \mathbb{P}^{n-1}(\mathbb{C})$  defined in (4.3) can be extended as a dominant rational map  $\bar{\gamma}_F$  from some  $(n-1)$ -dimensional compact algebraic variety  $\overline{V_{\mathbb{T}}(F)}$  into  $\mathbb{P}^{n-1}(\mathbb{C})$ , with the smallest possible degree, that is  $\deg \bar{\gamma}_F = 1$  (which would mean that  $\bar{\gamma}_F$  realizes a birational map between  $\mathbb{P}^{n-1}(\mathbb{C})$  and the algebraic variety  $\overline{V_{\mathbb{T}}(F)}$ ). If it is the case<sup>8</sup>, then the inverse  $\bar{\gamma}_F^{-1} : \mathbb{P}^{n-1}(\mathbb{C}) \rightarrow \overline{V_{\mathbb{T}}(F)}$  of such rational map, once composed with Log, will indeed provide a parametrization of the contour of  $\mathcal{A}_{V_{\mathbb{T}}(F)}$  (thanks to Proposition 4.1). In fact, a complete answer to such a bunch of questions was almost already settled by J. Horn in 1889 [Horn], in his study of convergence of hypergeometric series [Horn], then revisited in [GKZ], [Krap0], [PST] and [PT1]. It lies deeply on the theory of *sparse elimination*, more particularly on the various notions of  *$\mathcal{A}$ -discriminantal varieties* and  *$\mathcal{A}$ -discriminants* or *determinants* (*ordinary*, *reduced*, or *principal*). In fact, the theory of sparse elimination (as developed in [GKZ]) was a motivation for the “resurgence” of the concept of archimedean amoeba (which probably goes back to Newton and, later on, Puiseux).

<sup>7</sup>This of course automatically happens as soon as  $\dim(\overline{V_{\mathbb{T}}(F)})^{\text{sing}} > 0$ , since Liouville's theorem prevents any algebraic subset of  $\mathcal{X}(\tilde{\Sigma}(\Delta(F)))$  with strictly positive dimension from remaining in  $\mathbb{T}^n$ .

<sup>8</sup>We will see moreover that  $F \in \mathbb{R}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$  in such case.

**4.3.1.  $\mathcal{A}$ -discriminants (ordinary and reduced) ; Horn-Krapanov theorem (statement).** In order to state the first result (formalized in [Krap0], see also [PT1]), we need to recall the concepts of  $\mathcal{A}$ -discriminant variety and  $\mathcal{A}$ -discriminant (*ordinary* or *reduced*). Let  $N \geq 1$ . We will consider here a finite subset  $\mathcal{A} \subset \mathbb{Z}^N$  with cardinal  $N + n + 1$  such  $\mathcal{A}$  affinely generates the lattice  $\mathbb{Z}^N$  over  $\mathbb{Z}$ . Note that this condition requires this time  $n \geq 1$  and implies of course also that  $\dim \Delta_{\mathcal{A}} = N$ .

DEFINITION 4.8 ( $\mathcal{A}$ -discriminantal variety,  $\mathcal{A}$ -discriminant [GKZ]). Let  $\mathcal{A}$  be a finite set in  $\mathbb{Z}^N$  satisfying the conditions required above. The (ordinary)  $\mathcal{A}$ -discriminantal variety  $\nabla_{\mathcal{A}}$  is the affine algebraic irreducible<sup>9</sup> variety which is defined as the Zariski closure (in the affine scheme  $\mathbb{C}^{\mathcal{A}}$ , which could as well be considered as the affine scheme of Laurent polynomials in  $N$  variables with support contained in  $\mathcal{A}$ ) of the set :

$$(4.11) \quad \left\{ (c_{\alpha})_{\alpha \in \mathcal{A}} \in \mathbb{C}^{\mathcal{A}} ; \exists z \in \mathbb{T}^N, \sum_{\alpha \in \mathcal{A}} c_{\alpha} z^{\alpha} = 0, d_z \left[ \sum_{\alpha \in \mathcal{A}} c_{\alpha} z^{\alpha} \right] = 0 \right\}.$$

In case such an algebraic variety is an hypersurface, a (irreducible) equation for it is called a (ordinary)  $\mathcal{A}$ -discriminant. If not, the  $\mathcal{A}$ -discriminant is taken to be equal to 1.

EXAMPLE 4.9 (classical discriminants ( $N = 1$ )). Consider, when  $N = 1$  and  $n \geq 1$ , the set  $\mathcal{A} = \{0, \alpha_1, \alpha_2, \dots, \alpha_n, d\}$ , where  $1 \leq \alpha_n < \dots < \alpha_1 < d$  are strictly positive coprime integers. The convex hull  $\Delta_{\mathcal{A}}$  equals  $[0, d]$  and the  $2 + n = 1 + n + 1$  elements of  $\mathcal{A}$  affinely generate  $\mathbb{Z}$  over  $\mathbb{Z}$ . In this case, the  $\mathcal{A}$ -discriminant (evaluated on  $c_d = 1$ ) coincides with the resultant of the two polynomials :

$$X^d + \sum_{j=1}^n z_j X^{\alpha_j} - 1, \quad dX^{d-1} + \sum_{j=1}^n \alpha_j z_j X^{\alpha_j - 1}.$$

For the polynomials  $X^2 + z_1 X - 1$  ( $d = 2$ ) and  $X^3 + z_1 X^2 + z_2 X - 1$  ( $d = 3$ ), one gets respectively :

$$\Delta_2(z_1) = z_1^2 + 4, \quad \Delta_3(z_1, z_2) = 27 + 4z_1^3 - 4z_2^3 + 18z_1 z_2 - z_1^2 z_2^2.$$

Note also that any equation (in  $w$ ) of the form

$$w^d + \sum_{j=1}^m z_j w^{m_j} + z_0 = 0$$

reduces (after replacing  $w$  by  $\alpha w$ ) to an equation of the form :

$$w^d + \sum_{j=1}^n \tilde{z}_j w^{m_j} - 1 = 0,$$

so that the example presented here covers the case of ordinary discriminants.

<sup>9</sup>The reason for irreducibility lies in the fact that the variety (4.11) inherits a structure of  $\mathbb{C}^{\mathcal{A}}$ -vectorial bundle over the irreducible affine variety  $\mathbb{T}^N$ . Note also that the  $\mathcal{A}$ -discriminantal variety is defined over  $\mathbb{Q}$  : it remains indeed invariant after conjugaison on the variables, when they are taken in a number field  $\mathbb{K}$ .

The above Definition 4.8 of the  $\mathcal{A}$ -discriminant variety does not take into account affine dependance that indeed exists between elements in  $\mathcal{A}$ . In order to introduce a refined version of such a  $\mathcal{A}$ -discriminantal variety (taking into account such affine dependance), one needs to proceed as follows. Such affine dependance is materialized by the  $(N + n + 1) - N - 1 = n$ -lattice of affine relations between elements in  $\mathcal{A}$ , that is :

$$(4.12) \quad M_{\mathcal{A}} = \left\{ \nu \in \mathbb{Z}^{\mathcal{A}} ; \sum_{\substack{\alpha \in \mathcal{A} \\ \alpha \neq \alpha_1}} \nu_{\alpha} (\alpha - \alpha_1) = 0 \right\},$$

for which one can pick up a  $\mathbb{Z}$ -basis  $\{b\} = \{b_0, \dots, b_{n-1}\}$ , where :

$$(4.13) \quad b_k = (b_{k,\alpha})_{\alpha \in \mathcal{A}} = (b_{k,\alpha_1}, \dots, b_{k,\alpha_{N+n+1}}) = (b_{k,1}, \dots, b_{k,N+n+1}), \quad k = 0, \dots, n-1.$$

Note that

$$(4.14) \quad \sum_{j=1}^{N+n+1} b_{k,j} = \sum_{\alpha \in \mathcal{A}} b_{k,\alpha} = 0,$$

since each  $b_k$  lies in the lattice of affine relations (4.12) between elements of  $\mathcal{A}$ . Observe that the  $\mathcal{A}$ -discriminantal variety  $\nabla_{\mathcal{A}}$  remains invariant if the Laurent polynomial  $F(X) = \sum_{\alpha \in \mathcal{A}} c_{\alpha} X^{\alpha}$  is replaced by  $t \bullet F := t_0 F(t_1 X, \dots, t_N X)$ , where  $(t_0, \dots, t_N) \in \mathbb{T}^{N+1}$ . If  $\mathbb{C}^{\mathcal{A}}$  is now considered as the affine scheme of Laurent polynomials in  $N + n + 1$  variables with support included in  $\mathcal{A}$ , we define thus an action of  $\mathbb{T}^N$  on  $\mathbb{C}^{\mathcal{A}}$  under which  $\nabla_{\mathcal{A}}$  remains invariant, which arises from the following homomorphism of tori :

$$(4.15) \quad \begin{aligned} \varphi : (t_0, t') \in \mathbb{T}^{N+1} = \text{Spec}(\mathbb{C}[X_0^{\pm 1}, \dots, X_n^{\pm 1}]) \\ \mapsto (t_0 (t')^{\alpha})_{\alpha \in \mathcal{A}} \in \mathbb{T}^{\mathcal{A}} = \text{Spec}(\mathbb{C}[T_{\alpha}^{\pm 1} ; \alpha \in \mathcal{A}]) \end{aligned}$$

The kernel of  $\varphi^* : \mathbb{Z}^{\mathcal{A}} \rightarrow \mathbb{Z}^N$  is precisely the  $n$ -lattice  $M_{\mathcal{A}}$ . Consider the affine  $n$ -dimensional affine toric manifold :

$$X_{\mathcal{A}} = \text{Spec}(\mathbb{C}[T^{\nu} ; \nu \in M_{\mathcal{A}}]) = \text{Spec} \mathbb{C}[T^{\pm b_0}, \dots, T^{\pm b_{n-1}}] = \frac{\text{Spec}(\mathbb{C}[T_{\alpha}^{\pm 1} ; \alpha \in \mathcal{A}])}{\text{Im } \varphi}.$$

Note that the choice of the basis  $\{b\}$  makes precise an identification between  $\mathbb{T}^n = \text{Spec}(\mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}])$  and  $X_{\mathcal{A}} = \text{Spec} \mathbb{C}[T^{\pm b_0}, \dots, T^{\pm b_{n-1}}]$ . Let  $\pi$  be the projection :

$$\pi : \mathbb{T}^{\mathcal{A}} = \text{Spec}(\mathbb{C}[T_{\alpha}^{\pm 1} ; \alpha \in \mathcal{A}]) \rightarrow X_{\mathcal{A}}.$$

Because of the invariance of  $\nabla_{\mathcal{A}}$  (hence of  $\nabla_{\mathcal{A}} \cap \mathbb{T}^{\mathcal{A}}$ ) under the action of the homomorphism of tori  $\varphi$  defined as (4.15), one can write

$$\nabla_{\mathcal{A}} \cap \mathbb{T}^{\mathcal{A}} = \pi^{-1}[\nabla_{\mathcal{A}}^{\text{red}}],$$

where  $\nabla_{\mathcal{A}}^{\text{red}}$  is a well defined algebraic subvariety of the toric  $n$ -dimensional manifold  $X_{\mathcal{A}}$ , with the same codimension than that of  $\nabla_{\mathcal{A}}$  in  $\mathbb{C}^{\mathcal{A}}$ .

DEFINITION 4.10 ( $\mathcal{A}$ -reduced discriminantal variety, discriminant [Krap0]). The algebraic subvariety  $\nabla_{\mathcal{A}}^{\text{red}}$  of  $X_{\mathcal{A}}$  which is uniquely defined by

$$(4.16) \quad \nabla_{\mathcal{A}} \cap \mathbb{T}^{\mathcal{A}} = \pi^{-1}[\nabla_{\mathcal{A}}^{\text{red}}]$$

is called the *reduced  $\mathcal{A}$ -discriminantal subvariety*. When it is an hypersurface (that is when  $\nabla_{\mathcal{A}}$  is an hypersurface in  $\mathbb{C}^{\mathcal{A}}$ ), a (reduced) equation for it in  $\mathbb{T}^n$  (using the

identification  $\mathbb{T}^n \simeq X_{\mathcal{A}}$  which is conditioned to the choice of a basis  $\{b_0, \dots, b_{n-1}\}$  for the lattice  $M_{\mathcal{A}}$  is called a  $\mathcal{A}$ -*reduced discriminant*.

REMARK 4.11 ( $\mathcal{A}$ -discriminant *versus* reduced  $\mathcal{A}$ -discriminant). When  $\nabla_{\mathcal{A}}$  is an hypersurface, it is easy to deduce an expression for the reduced  $\mathcal{A}$ -discriminant from that of the  $\mathcal{A}$  determinant (modulo the identification of  $\mathbb{T}^n$  with the affine toric variety  $X_{\mathcal{A}}$  which is conditioned to the choice of a basis  $\{b^0, \dots, b^{n-1}\}$  for the  $\mathbb{Z}$ -lattice  $M_{\mathcal{A}}$ ). Choose a non zero  $(n, n)$  minor of the matrix :

$$\begin{pmatrix} b_{0,\alpha_1}, & \dots & \dots & b_{0,\alpha_{N+n+1}} \\ \vdots & \vdots & \vdots & \vdots \\ b_{n-1,\alpha_1}, & \dots & \dots & b_{n-1,\alpha_{N+n+1}} \end{pmatrix},$$

which corresponds to  $n$  columns labelled with  $\alpha_\iota$ ,  $\iota \in I \subset \{1, \dots, N+n+1\}$ . In the expression of the  $\mathcal{A}$ -discriminant (in  $(c_\alpha)_{\alpha \in \mathcal{A}}$ ), just substitute 1 instead of the  $N+1 = N+1+n-n = N+1$  undeterminate coefficients  $c_{\alpha_\iota}$ ,  $\iota \notin I$ , thus killing the  $N+1$  quasi homogeneities. The new expression one obtains is a polynomial in  $n$  variables which stands for an expression of the reduced  $\mathcal{A}$ -discriminant (up to the identification between  $\mathbb{T}^n$  and the affine toric variety  $X_{\mathcal{A}}$ ).

EXAMPLE 4.12 (Classical determinants ( $N = 1$ ) revisited). Consider  $n+1$  strictly positive integers  $d > m_1 > m_2 > \dots > m_n = 1$ . Using Remark 4.11 above, one can specialize  $c_d = 1$  and  $c_0 = -1$  in the expression of the  $\mathcal{A}$ -discriminant  $\nabla_{\mathcal{A}}$  corresponding to  $\mathcal{A} = \{d, m_1, m_2, \dots, m_n, 0\}$  in order to recover the reduced  $\mathcal{A}$ -discriminant  $\nabla_{\mathcal{A}}^{\text{red}}$ . The polynomial  $\nabla_{\mathcal{A}}^{\text{red}}$  corresponds to the resultant of the two polynomials :

$$X^d + \sum_{j=1}^{n-1} z_j X^{m_j} + X - 1, \quad dX^{d-1} + \sum_{j=1}^{n-1} m_j z_j X^{m_j-1} + 1,$$

that is to the usual discriminant (see Example 4.9 above).

Given a fixed choice of a basis  $b = \{b^0, \dots, b^{n-1}\}$  of vectors in  $\mathbb{Z}^{\mathcal{A}}$  as (4.13) for the lattice  $M_{\mathcal{A}}$  ( $b^j = (b_1^j, \dots, b_{N+n+1}^j)$  for  $j = 0, \dots, n-1$ ), let us pair with it a rational map from  $\mathbb{P}^{n-1}(\mathbb{C})$  into  $\mathbb{T}^n$  defined as follows:

(4.17)

$$\Psi^{\{b\}}([\lambda_0 : \dots : \lambda_{n-1}]) = \left( \prod_{k=1}^{N+n+1} \left( \sum_{j=0}^{n-1} b_k^j \lambda_k \right)^{b_k^0}, \dots, \prod_{k=1}^{N+n-1} \left( \sum_{j=0}^{n-1} b_k^j \lambda_k \right)^{b_k^{n-1}} \right).$$

Note that, since  $\sum_{k=1}^{N+n+1} b_k^j = 0$  for any  $j = 0, \dots, n-1$  (see (4.14)), each component of  $\Psi^{\{b\}}$  is homogeneous with degree 0.

DEFINITION 4.13 (Horn uniformisation map, or Gale transform of  $\mathcal{A}$ ). Let  $\mathcal{A}$  be a collection of  $N+n+1 > N+1$  points in  $\mathbb{Z}^N$  such that  $\mathcal{A}$  affinely generates  $\mathbb{Z}^N$  over  $\mathbb{Z}$  (note that this implies necessarily  $\dim \Delta_{\mathcal{A}} = N$ ). Let  $\{b\} = \{b^0, \dots, b^{n-1}\}$  be a basis of the lattice  $M_{\mathcal{A}}$  (see (4.12)) of affine relations between elements of  $\mathcal{A}$ . The rational map  $\Psi^{\{b\}}$  from  $\mathbb{P}^{n-1}(\mathbb{C})$  into  $\mathbb{P}^n(\mathbb{C})$  defined as (4.17) is known as the *Horn uniformisation map* (or the *Gale transform*) of the set  $\mathcal{A}$ , with respect to the choice of the  $\mathbb{Z}$ -basis  $\{b\}$  for the  $n$ -lattice  $M_{\mathcal{A}}$ .

One can now state what could be seen as a modern presentation of Horn's theory in our context of archimedean amoebas :

**THEOREM 4.14** (toric hypersurfaces with birational logarithmic Gauß map [Krap0]). *Let  $n \in \mathbb{N}^*$  and  $V_{\mathbb{T}}(F)$  be an algebraic hypersurface in  $\mathbb{T}^n$  defined by a reduced Laurent polynomial  $F \in \mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$ . The logarithmic Gauß map  $\gamma_F$  extends to a birational map  $\bar{\gamma}_F$  between some  $(n-1)$ -complete algebraic variety  $\overline{V_{\mathbb{T}}(F)}$  and  $\mathbb{P}^{n-1}(\mathbb{C})$  if and only if there exists  $N \geq 1$  and a finite set  $\mathcal{A} \subset \mathbb{Z}^N$  with cardinal  $N+n+1$ , such that :*

- the finite set  $\mathcal{A}$  affinely generates  $\mathbb{Z}^N$  over  $\mathbb{Z}$  ;
- the algebraic hypersurface  $V_{\mathbb{T}}(F)$  coincides (up to the identification between  $\mathbb{T}^n$  and  $X_{\mathcal{A}}$  which the choice of  $\{b\}$  induces) with the reduced  $\mathcal{A}$ -discriminantal variety  $\nabla_{\mathcal{A}}^{\text{red}}$ , which in that case is an algebraic hypersurface ; thus  $F$  coincides (up to multiplication by a monomial) with the reduced  $\mathcal{A}$ -discriminant (which belongs to  $\mathbb{Z}[X_1, \dots, X_n]$ ).

Moreover, for a given basis  $\{b\}$  of the  $n$ -lattice  $M_{\mathcal{A}}$ , the inverse map  $\bar{\gamma}_F^{-1}$  coincides with the Gale transform  $\Psi^{\{b\}}$  defined in (4.17) and the map :

$$[\lambda_0 : \dots : \lambda_{n-1}] \in \mathbb{P}^{n-1}(\mathbb{R}) \longmapsto \text{Log}(\Psi^{\{b\}}([\lambda_0 : \dots : \lambda_{n-1}]))$$

provides (thanks to Proposition 4.1) a parametrisation of the contour of the archimedean amoeba  $\mathcal{A}_{V_{\mathbb{T}}(\nabla_{\mathcal{A}}^{\text{red}})}$ .

**4.3.2. Mixed sparse resultants, principal  $\mathcal{A}$ -determinants.** Due to lack of time, we will not give in this course the proof of Theorem 4.14. We refer instead the reader to the presentation which is given in [Krap0] as well as in [GKZ], Chapter 9, 3-C (page 288 and followings). Nevertheless, in order to explore later on the relation between Horn's theory of hypergeometric functions and the theory of complex amoebas (which is the guideline of this course, in between complex and tropical geometry through the concept of "deformation"), let us pursue here the catalog of various definitions in relation with *sparse elimination* that will be needed later on in the discriminant context. The most important one is that of  $(\mathcal{A}_0, \dots, \mathcal{A}_N)$ -mixed resultant (see [GKZ], chapter 8).

**DEFINITION 4.15** (mixed sparse resultant). Let  $\mathcal{A}_0, \dots, \mathcal{A}_N$ , be  $N+1$  finite subsets in  $\mathbb{Z}^N$  such that :

- each set  $\mathcal{A}_k$ ,  $k = 0, \dots, N$  affinely generates  $\mathbb{R}^N$  over  $\mathbb{R}$  ;
- the union of all the  $\mathcal{A}_k$ ,  $k = 0, \dots, N$ , affinely generates the lattice  $\mathbb{Z}^N$  over  $\mathbb{Z}$ .

The algebraic variety in  $\mathbb{C}^{\mathcal{A}_0} \times \dots \times \mathbb{C}^{\mathcal{A}_N}$  obtained as the Zariski closure (in this product of affine varieties) of the (non closed) subvariety defined as

$$(4.18) \quad \left\{ (c^{(0)}, \dots, c^{(N)}) \in \prod_{j=0}^N \mathbb{C}^{\mathcal{A}_j} ; \exists z \in \mathbb{T}^N, \sum_{\alpha \in \mathcal{A}_j} c_{\alpha_j}^{(j)} z^{\alpha} = 0, j = 0, \dots, N \right\}.$$

is an irreducible hypersurface defined over  $\mathbb{Q}$ . It admits an irreducible defining equation with integer coprime coefficients, denoted (up to sign) as :

$$\mathcal{R}_{\mathcal{A}_0, \dots, \mathcal{A}_N},$$

and called the  $(\mathcal{A}_0, \dots, \mathcal{A}_N)$ -mixed resultant.



REMARK 4.16. The reason for the irreducibility of the hypersurface thus defined lies, as in Definition 4.8, in the fact that the subvariety (4.18) inherits a structure of vectorial bundle over the irreducible affine variety  $\mathbb{T}^N$ . The fact that the  $(\mathcal{A}_0, \dots, \mathcal{A}_N)$ -mixed resultant has integral coefficients follows also from Galois theory. In fact, one knows more : the GCD of the coefficients can be chosen equal to 1. The new point here (compare to Definition 4.8) is that the geometric conditions imposed on the  $\mathcal{A}_k$ ,  $k = 0, \dots, N$ , force the closure of the subvariety defined as (4.18) to be an hypersurface in  $\prod_{k=0}^N \mathbb{C}^{\mathcal{A}_k}$ . We refer to [GKZ], Chapter 3, Proposition 3.1, for more details. The  $(\mathcal{A}_0, \dots, \mathcal{A}_N)$ -mixed resultant is homogeneous in each block of coordinates and its degree in the block  $c_{\alpha_k}^{(k)}$  equals :

$$N! \times \text{vol}_N(\Delta_{\mathcal{A}_0}, \dots, \widehat{\Delta}_{\mathcal{A}_k}, \dots, \Delta_{\mathcal{A}_N}).$$

Given a finite subset  $\mathcal{A}$  of  $\mathbb{Z}^N$  that affinely generates the lattice  $\mathbb{Z}^N$  over  $\mathbb{Z}$ , one may naturally propose the following definition, this time intimately connected (much more than that of  $\mathcal{A}$ -discriminant) with the logarithmic Gauß map.

DEFINITION 4.17 (principal  $\mathcal{A}$ -determinant, [GKZ], chapter 10). Let  $\mathcal{A}$  be a finite subset  $\mathcal{A}$  of  $\mathbb{Z}^N$  that affinely generates the lattice  $\mathbb{Z}^N$  over  $\mathbb{Z}$ . The *principal  $\mathcal{A}$ -determinant*  $E_{\mathcal{A}}$  is defined as the following polynomial map on  $\mathbb{C}^{\mathcal{A}}$  by :

(4.19)

$$\forall c = (c_{\alpha})_{\alpha \in \mathcal{A}} \in \mathbb{C}^{\mathcal{A}},$$

$$E_{\mathcal{A}}(c) = \mathcal{R}_{\mathcal{A}, \dots, \mathcal{A}} \left( \sum_{\alpha \in \mathcal{A}} c_{\alpha} X^{\alpha}, X_1 \frac{\partial}{\partial X_1} \left( \sum_{\alpha \in \mathcal{A}} c_{\alpha} X^{\alpha} \right), \dots, X_n \frac{\partial}{\partial X_n} \left( \sum_{\alpha \in \mathcal{A}} c_{\alpha} X^{\alpha} \right) \right).$$

REMARK 4.18 (factorization of  $\mathcal{A}$ -principal determinants). Thanks to Definition 4.17, any principal  $\mathcal{A}$ -determinant ( $\mathcal{A} \subset \mathbb{Z}^N$  being a finite subset that affinely generates the lattice  $\mathbb{Z}^N$  over  $\mathbb{Z}$ ) factorizes as a product of powers of lower order  $\mathcal{A}^{\sigma}$ -discriminants (see Definition 4.8),  $\mathcal{A}^{\sigma}$  being the intersection of the lattice  $\mathbb{Z}^N$  with some face  $\sigma$  of the  $N$ -dimensional Newton polyhedron  $\Delta_{\mathcal{A}}$  (the lattice  $\mathbb{Z}^N$  being now replaced by the induced lattice on the affine subspace of  $\mathbb{R}^N$  generated by the face  $\sigma$ ).

**4.3.3. “Solidity” of amoebas and singularities of non-confluent hypergeometries : a statement.** Instead of proving Theorem 4.14, the related result we would like to state here and prove in the next subsections (as the achievement of this last chapter) is the following :

THEOREM 4.19 (“solidity” of amoebas of principal  $\mathcal{A}$ -determinant [PST]). *The archimedean amoeba of any principal  $\mathcal{A}$ -determinant  $E_{\mathcal{A}}$  (hence of any  $\mathcal{A}$ -discriminant  $\delta$  thanks to Remark 4.18 above) in  $n$  variables is solid (that is its complement in  $\mathbb{R}^n$  has exactly as many connected components than there are vertices for the Newton polyhedron  $\Delta[\delta]$ ).*

This result will be reformulated as Theorem 4.27 (and thus proved) in subsection 4.3.6. As we mentioned it earlier (the proof one could read in [Krap0] or [GKZ], chapter 9, indeed confirms it), its companion Theorem 4.14 appears as a byproduct of the theory of hypergeometric series (as developed in [Horn] and [GKZ]). Theorem 4.19 in fact also (essentially in the same vein, may be even more in the spirit of this course) : it appears as a consequence of the fact that singularities of

non-confluent hypergeometric functions (in the Horn sense) lie precisely (in good cases) on the so-called *resultant locus* of the related *Horn system* (see Definition 4.21 below). Examples of such resultant loci are precisely provided by zero sets of principal  $\mathcal{A}$ -determinants, as it follows from the intimate relation (that we will suggest here) between hypergeometric functions derived in the Horn sense and solutions of  $\mathcal{A}$ -hypergeometric systems such as introduced in [GGR], [GKZ0] (see also [GKZ]). The solidity of the amoeba of a principal  $\mathcal{A}$ -determinant  $E_{\mathcal{A}}$  (hence of a  $\mathcal{A}$ -discriminant, which can be viewed as a factor of some  $E_{\mathcal{A}}$ , see Remark 4.18) will then follow from the fact that the singular set of a non confluent hypergeometric Horn system has a solid amoeba. We therefore will need to study the singular locus of hypergeometric functions which are recovered by means of analytic continuatation from Horn hypergeometric Puiseux series solutions of the corresponding Horn system. As for Laurent series, the notion of *support* of a hypergeometric series, together with Abel's lemma<sup>10</sup> (in its two-sided version, cf. Lemma 4.28 below), will play here a fundamental role.

**4.3.4. Puiseux hypergeometric series : the Horn's approach.** We recall in this subsection basic material about Horn's approach [Horn] to the study of hypergeometric series (in the Horn's sense). See for example [Sad] for a detailed introduction (used to prepare the notes of this course). In the next subsection, we will present (and relate it to the Horn's point of view) the notion of *Gelfand-Krapanov-Zelevinsky* (in short "GKZ")  $\mathcal{A}$ -hypergeometric system ( $\mathcal{A}$  being a finite subset in  $\mathbb{Z}^N$  which affinely spans  $\mathbb{Z}^N$  over  $\mathbb{Z}$ , as in Definition 4.13), see [GKZ], [Krap] (or also [Cat]) for a presentation of that companion concept.

Let us start first with a presentation of the Horn's point of view [Horn].

DEFINITION 4.20 (Puiseux formal hypergeometric Horn series in  $n$  variables). A *Puiseux formal hypergeometric series (in the Horn's sense)* in  $n$  complex variables is a formal power series centered at the origin in  $\mathbb{C}^n$ , of the form :

$$(4.20) \quad F(z) = z^\gamma \sum_{k \in \mathbb{Z}^n} a_k z^k,$$

where  $\gamma = (\gamma_1, \dots, \gamma_n) \in \mathbb{C}^n$ ,  $\operatorname{Re} \gamma_j \in [0, 1[$  for any  $j = 1, \dots, n$ , and the coefficients  $a_k$  are such that, for any  $j = 1, \dots, n$ , there exists a rational function  $R_j \in \mathbb{C}(X_1, \dots, X_n)$  such that

$$(4.21) \quad a_{k+e_j} = a_k R_j(\gamma + k) \quad \forall k \in \mathbb{Z}^n$$

(here  $(e_1, \dots, e_n)$  denotes the canonical basis of  $\mathbb{Z}^n$ ).

Letting

$$(4.22) \quad R_j(X) = \frac{P_j(X)}{Q_j(X + e_j)}, \quad j = 1, \dots, n,$$

one can check that such a formal power series (4.20) (with the  $a_k$  governed by the inductive relations (4.21)) satisfies :

$$(4.23) \quad \left[ z_j P_j(z_1 \partial / \partial z_1, \dots, z_n \partial / \partial z_n) - Q_j(z_1 \partial / \partial z_1, \dots, z_n \partial / \partial z_n) \right] (F) \equiv 0.$$

<sup>10</sup>Note that Abel's lemma was precisely invoked in order to justify the convexity of the connected complements of the archimedean amoeba of an algebraic hypersurface (each such component being the image by  $\operatorname{Log}$  of a maximal domain of convergence of a Laurent series, see Proposition 2.1).

Moreover, if one looks for the formal solutions  $F$  of the differential system (4.23) that have a formal Puiseux expansion such as (4.20) (centered at the origin), one recovers Puiseux formal hypergeometric series for which the sequence of coefficients  $(a_k)_{k \in \mathbb{Z}^n}$  in (4.20) is precisely governed by the relations (4.21). This leads naturally to the following definition.

DEFINITION 4.21 (Horn system, non-confluency). Let

$$P_1(\theta), \dots, P_n(\theta), Q_1(\theta), \dots, Q_n(\theta),$$

be  $2n$  differential operators with constant coefficients in  $\theta = (\theta_1, \dots, \theta_n)$ , where  $\theta_j = X_j \partial / \partial X_j$ ,  $j = 1, \dots, n$ . Assume that  $P_j$  and  $Q_j$  are coprime for any  $j = 1, \dots, n$ . The left ideal in the Weyl algebra  $\mathbb{C}\langle X, \partial / \partial X \rangle$  generated by the  $n$  differential operators involved in the differential system (4.23) is called the *Horn system* attached to the set of polynomials  $P_1, \dots, P_n, Q_1, \dots, Q_n$ . *Non-confluency* of this system means that  $P_j$  and  $Q_j$  have the same degree for any  $j = 1, \dots, n$ .

The fact that the Horn system (4.23) admits at least one non trivial formal solution which can be developed as a formal hypergeometric Puiseux series in  $z$  of the form (4.20) (centered at the origin in  $\mathbb{C}^n$ ), where the coefficients  $a_k$  are governed by the relations (4.21), requires indeed compatibility conditions, namely that the rational expressions  $R_j$  defined in (4.22) fulfill the so called *solvability conditions* :

$$(4.24) \quad R_i(X + e_j)R_j(X) = R_j(X + e_i)R_i(X), \quad 1 \leq i, j \leq n.$$

Such solvability conditions will be from now on imposed.

Other constraints on the  $R_j$  do indeed exist. It follows from Ore-Sato's theorem (see [GGR], I, sections 2 and 3) that any formal Puiseux hypergeometric series of form (4.20), where the  $a_k$  are ruled by rational fractions such as in (4.21), has necessarily its coefficients  $a_k$  of the form

$$(4.25) \quad a_k = \left( \prod_{j=1}^n t_j^{k_j} \right) \times U(k) \times \prod_{l=1}^p \Gamma(\langle A_l, k + \gamma \rangle + c_l) \quad \forall k \in \mathbb{Z},$$

where  $t_1, \dots, t_n$  are arbitrary non zero complex numbers,  $U$  is an arbitrary element in  $\mathbb{C}(X_1, \dots, X_n)$ ,  $A_1, \dots, A_p$  are  $p \geq 1$  arbitrary elements in  $\mathbb{Z}^n$ , and  $c_1, \dots, c_p$  are  $p$  arbitrary complex numbers<sup>11</sup>. The expression of the  $R_j$ ,  $j = 1, \dots, n$ , which govern in that case the Puiseux development (4.20), is thus given by

$$R_j(X + \gamma) = t_j \frac{U(Y + e_j)}{U(Y)} \times \prod_{l=1}^p \frac{\Gamma(\langle A_l, X + e_j + \gamma \rangle + c_l)}{\Gamma(\langle A_l, X + \gamma \rangle + c_l)}, \quad j = 1, \dots, n.$$

In case  $U \equiv 1$ , the condition of non-confluency for the related Horn system therefore reflects in the condition

$$(4.26) \quad \sum_{l=1}^p A_l = 0.$$

Note that, in such case, the polynomials  $P_j$  and  $Q_j$  involved in the related Horn system are all necessarily representable as product of affine forms  $\langle A, X \rangle + c$ ,  $A \in \mathbb{Z}^n$ ,  $c \in \mathbb{C}$ .

<sup>11</sup>Note that, because of the well known reflection formula  $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$ , it is indifferent to take the  $\Gamma$  factors in the numerator (such as in (4.25)) or, as it is done more classically, speaking about hypergeometric series, in the denominator (as for example in [GGR]).

EXAMPLE 4.22 (Horn system attached to a lattice  $M_{\mathcal{A}}$  in  $\mathbb{Z}^{N+n+1}$ , Horn uniformization revisited, *cf.* Definition 4.13, see *e.g.* [GKZ], [Krap0], section 2, or also [DMS], section 2, for the presentation which is followed here). Consider a  $(N+1, N+n+1)$  matrix  $\mathbf{A} \in \mathcal{M}^{N+1, N+n+1}(\mathbb{Z})$  with integer coefficients ( $N+1$  rows labelled from 0 to  $N$ ,  $N+n+1$  columns labelled from 1 to  $N+n+1$ ), with as first row the line  $\mathbf{a}_0 := \mathbf{ones}(1, N+n+1)$ , with  $\text{rank}(\mathbf{A}) = N+1$ , together with a matrix  $\mathbf{B} \in \mathcal{M}^{N+n+1, n}(\mathbb{Z})$  such that  $\mathbf{A} \cdot \mathbf{B} = 0$  (lines of  $\mathbf{B}$  are labelled from 1 to  $N+n+1$ , while columns of  $\mathbf{B}$  are labelled from 0 to  $n-1$ ). For each  $k = 1, \dots, N+n+1$ , let  $\mathbf{b}_k \in \mathbb{Z}^n$  be the row of  $\mathbf{B}$  with label  $k$ . Consider the lattice  $M_{\mathcal{A}} = \mathbf{B} \cdot \mathbb{Z}^n \subset \mathbb{Z}^{N+n+1}$ , which is a lattice with rank  $n$  in  $\mathbb{R}^{N+n+1}$ . A  $\mathbb{Z}$ -basis for this lattice is provided by the  $n$  columns  $\mathbf{b}^0, \dots, \mathbf{b}^{n-1}$  of the matrix  $\mathbf{B}$ . Notations used here are in accordance with the notations used previously in Definition 4.13. The different column vectors  $\mathbf{a}^j = (a_k^j)_{k=1, \dots, N+n+1}$ ,  $j = 1, \dots, N+n+1$  (below the first row  $\mathbf{a}_0$ ) stand here for a collection  $\mathcal{A}$  of  $N+n+1$  points in  $\mathbb{Z}^N$  which affinely generate  $\mathbb{Z}^N$  over  $\mathbb{Z}$  (as in Definition 4.13). The lattice  $M_{\mathcal{A}}$  corresponds (see Definition 4.13) to the matrix of affine relations between elements in  $\mathcal{A}$ . The column vectors  $\mathbf{b}^0, \dots, \mathbf{b}^{n-1}$  of  $\mathbf{B}$  provide in fact a basis for this lattice  $M_{\mathcal{A}}$ . Fix a vector  $c = (c_1, \dots, c_{N+n+1}) \in \mathbb{C}^{N+n+1}$ . Consider the  $2n$  polynomials (all representable as products of affine forms with integer coefficients) in the  $n$  variables  $X_0, \dots, X_{n-1}$ :

$$(4.27) \quad \begin{aligned} P_j(X_0, \dots, X_{n-1}) &= \prod_{b_k^j < 0} \prod_{l=0}^{|b_k^j|-1} (\langle b_k, X \rangle + c_k - l), \quad j = 0, \dots, n-1 \\ Q_j(X_0, \dots, X_{n-1}) &= \prod_{b_k^j > 0} \prod_{l=0}^{b_k^j-1} (\langle b_k, X \rangle + c_k - l), \quad j = 0, \dots, n-1. \end{aligned}$$

The differential operators

$$X_j P_j(\theta_0, \dots, \theta_{n-1}) - Q_j(\theta_0, \dots, \theta_{n-1}), \quad j = 0, \dots, n-1,$$

( $\theta_j = X_j \partial / \partial X_j$ ,  $j = 0, \dots, n-1$ ) are called the *Horn operators* attached to the  $n$ -rank lattice  $M_{\mathcal{A}} \subset \mathbb{Z}^{N+n+1}$ . Note that

$$(4.28) \quad \sum_{b_k^j > 0} b_k^j + \sum_{b_k^j < 0} b_k^j = 0 \quad \forall j = 0, \dots, n-1$$

since  $\mathbf{A} \cdot \mathbf{B} = 0$  and  $a_0 = \mathbf{ones}(1, N+n+1)$ . The corresponding Horn system (denoted as  $\mathcal{H}(M_{\mathcal{A}}, c)$ ) is here non-confluent because of (4.28). Such a Horn system  $\mathcal{H}(M_{\mathcal{A}}, c)$  (considered here on  $\mathbb{T}_{z_0, \dots, z_{n-1}}^n$ ) can be pulled back into a differential system (that we will still consider as being of the Horn type) in the coordinates  $(Z_1, \dots, Z_{N+n+1})$  *via* the monomial map

$$(4.29) \quad Z^{\mathbf{B}} : Z = (Z_1, \dots, Z_{N+n+1}) \in \mathbb{T}^{N+n+1} \longrightarrow \left( \prod_{k=1}^{N+n+1} Z_k^{b_k^0}, \dots, \prod_{k=1}^{N+n+1} Z_k^{b_k^{n-1}} \right) \in \mathbb{T}^n.$$

Consider for that in the Weyl algebra  $\mathbb{C}\langle Y_1, \dots, Y_{N+n+1}, \partial / \partial Y_1, \dots, \partial / \partial Y_{N+n+1} \rangle$  the  $n$  differential operators

$$\partial_Y^{(\mathbf{b}^j)^+} - \partial_Y^{(\mathbf{b}^j)^-} := \prod_{k=1}^{N+n+1} \partial_{Y_k}^{\max(b_k^j, 0)} - \prod_{k=1}^{N+n+1} \partial_{Y_k}^{-\min(b_k^j, 0)}, \quad j = 0, \dots, n-1,$$

and the  $N + 1$  Euler operators :

$$(4.30) \quad \text{Eul}_k^{\mathbf{A}} := \sum_{j=1}^{N+n+1} a_k^j Y_j \partial / \partial Y_j, \quad k = 0, \dots, N,$$

corresponding to the  $N + 1$  rows of  $\mathbf{A}$ . Then, given a point  $Z_0 \in \mathbb{T}_{Z_1, \dots, Z_{N+n+1}}^{N+n+1}$ , it is equivalent to say that a function  $F$  is a formal solution of the Horn system  $\mathcal{H}(M_{\mathcal{A}}, c)$  about  $Z_0^{\mathbf{B}}$  and that  $Z^c(Z^{\mathbf{B}})^*[F]$  is a formal solution about  $Z_0$  of the differential system corresponding to the ideal

$$(4.31) \quad \mathcal{H}_{M_{\mathcal{A}}}(c) := \left\langle \partial_Y^{(\mathbf{b}^j)^+} - \partial_Y^{(\mathbf{b}^j)^-}, j = 0, \dots, n-1 \right\rangle + \left\langle \text{Eul}_k^{\mathbf{A}} - (\mathbf{A} \cdot c)_k, k = 0, \dots, N \right\rangle$$

(see detailed computations in [DMS], proof of Lemma 5.1).

The expression (4.23) of a Horn system thus involves  $n$  differential operators :

$$\mathcal{D}_j := X_j P_j(X_1 \partial / \partial X_1, \dots, X_n \partial / \partial X_n) - Q_j(X_1 \partial / \partial X_1, \dots, X_n \partial / \partial X_n), \quad j = 1, \dots, n$$

(or  $N + n + 1$  operators such as those obtained *via* a monomial change of variables from  $\mathbb{T}^{N+n+1}$  into  $\mathbb{T}^n$  as for example  $\mathcal{H}_{M_{\mathcal{A}}}(c)$  in (4.31)). Such differential operators with polynomial coefficients lie in the Weyl algebra  $\mathbb{C}\langle X_1, \dots, X_n, \partial / \partial X_1, \dots, \partial / \partial X_n \rangle$ . Let

$$\sigma(\mathcal{D}_j) : (z, \xi) \longmapsto \sum_{|\alpha|=m_j} c_\alpha(z) \xi^\alpha$$

be the *principal symbol* of the differential operator

$$\mathcal{D}_j = \sum_{|\alpha|=m_j} c_\alpha(X) \partial^\alpha + \sum_{|\beta|<m_j} c_\beta(X) \partial^\beta.$$

The theory of  $\mathcal{D}$ -modules (see [Bj1], chapter 5) shows that the singularities of hypergeometric functions  $F$  (defined by means of analytic continuation of Horn hypergeometric Puiseux series (4.20), the  $a_k$  being governed by the relations (4.21)), lie in the *strong support* of the  $\mathbb{C}\langle X_1, \dots, X_n, \partial / \partial X_1, \dots, \partial / \partial X_n \rangle$ -module

$$\mathcal{M}_{\mathcal{D}} = \frac{\mathbb{C}\langle X, \partial / \partial X \rangle}{\sum_{j=1}^n \mathcal{D}_j \mathbb{C}\langle X, \partial / \partial X \rangle},$$

namely the algebraic subset  $\mathcal{U}(\mathcal{M}_{\mathcal{D}}) \subset \mathbb{C}^n$  defined as the image through the projection  $\mathbb{C}^n \times \mathbb{P}^{n-1}(\mathbb{C}) \rightarrow \mathbb{C}^n$  of the so-called *characteristic variety*

$$\text{charact}(\mathcal{M}_{\mathcal{D}}) := \left\{ (z, \xi) \in \mathbb{C}^{2n}; \sigma(P)(z, \xi) = 0 \quad \forall P \in \sum_{j=1}^n \mathcal{D}_j \mathbb{C}\langle X, \partial / \partial X \rangle \right\}$$

of  $\mathcal{M}_{\mathcal{D}}$ , that is :

$$\mathcal{U}(\mathcal{M}_{\mathcal{D}}) := \left\{ z \in \mathbb{C}^n; \exists \xi \in \mathbb{C}^n \setminus \{0\}, \sigma(P)(z, \xi) = 0 \quad \forall P \in \sum_{j=1}^n \mathcal{D}_j \mathbb{C}\langle X, \partial / \partial X \rangle \right\}$$

(since the stalk at  $z \in \mathbb{C}^n \setminus \mathcal{U}(\mathcal{M}_{\mathcal{D}})$  of the  $\mathbb{C}\langle X, \partial / \partial X \rangle$  module  $\mathcal{M}_{\mathcal{D}}$  is isomorphic to  $\mathcal{O}_z^{\oplus r(z)}$  for some integer  $r(z) \in \mathbb{N}^*$ , see Theorem 7.1 in [Bj1]).

We assume here that  $\mathcal{U}(\mathcal{M}_{\mathcal{D}})$  is a proper algebraic subset of  $\mathbb{C}^n$ . This indeed implies that the  $\mathbb{C}\langle X, \partial / \partial X \rangle$ -module  $\mathcal{M}_{\mathcal{D}}$  is *holonomic*, *i.e.* the dimension of its characteristic variety  $\text{charact}(\mathcal{M}_{\mathcal{D}})$  equals exactly  $n$  (see [Bj1], chapter 3). Moreover, the space of analytic solutions of the Horn system (4.23) about each non

singular point  $z \in \mathbb{C}^n \setminus \mathcal{W}(\mathcal{M}_{\mathcal{D}})$  is finitely dimensional, with dimension  $r(z) = r$ , equal to the dimension over  $\mathbb{C}(z_1, \dots, z_n)$  of  $\mathcal{M}_{\mathcal{D}}$  (considered now as a quotient  $\mathbb{C}(z_1, \dots, z_n)$ -vectorial space).

By Hartogs theorem, singularities of eventual (holomorphic) solutions  $F$  of the Horn system (4.23) (that is functions defined by means of analytic continuation of Horn hypergeometric Puiseux series (4.20), the  $a_k$  being governed by the relations (4.21)) then lie necessarily on the union of codimension one irreducible components of the proper algebraic subset  $\mathcal{W}(\mathcal{M}_{\mathcal{D}})$ . Such union of codimension one irreducible components is called the *singular set of the Horn system*. Let  $\sigma_j$ ,  $j = 1, \dots, n$ , be the principal symbol of the operator  $\mathcal{D}_j$ . In case the polynomial

$$\text{Res}_{\xi}[\sigma_1(X, \xi), \dots, \sigma_n(X, \xi)]$$

(the resultant being here understood in the sense of the classical Macaulay resultant of  $n$  homogeneous polynomials in  $n$  variables with prescribed degrees  $m_1, \dots, m_n$ , see e.g. [Lang0], chapter IX), considered as a polynomial in  $z$ , is not identically zero, its zero set

$$\{z \in \mathbb{C}^n ; \text{Res}_{\xi}[\sigma_1(z, \xi), \dots, \sigma_n(z, \xi)] = 0\}$$

contains the singular set of the Horn system, thus the singular set of any hypergeometric function  $F$  which is solution of this system.

**4.3.5. Hypergeometric series : the GKZ point of view.** Let  $\mathcal{A} \subset \mathbb{Z}^N$  be a finite collection of  $N + n + 1$  points which affinely generate  $\mathbb{Z}^N$  as a lattice and is such  $\dim \Delta_{\mathcal{A}} = N$ . Form the matrix  $\mathbf{A} \in \mathcal{M}^{N+1, N+n+1}(\mathbb{Z})$  with first row  $\mathbf{a}_0$  equal to ones  $(1, N+n+1)$  and column vectors  $\mathbf{a}^j := (a_k^j)_{k=1, \dots, N+n+1}$ ,  $j = 1, \dots, N + n + 1$ , below the first row, corresponding to the list of coordinates of the points of  $\mathcal{A}$  in  $\mathbb{Z}^N$  (as in Definition 4.13, see also Example 4.22 above). Let  $\mathbf{B} \in \mathcal{M}^{N+n+1, n}(\mathbb{Z})$  such that the columns  $\mathbf{b}^0, \dots, \mathbf{b}^{n-1}$  of  $\mathbf{B}$  correspond to a basis over  $\mathbb{Z}$  of the  $n$ -rank lattice  $M_{\mathcal{A}}$  of affine relations between points in  $\mathcal{A}$ .

**DEFINITION 4.23** (toric ideal corresponding to a finite subset  $\mathcal{A} \subset \mathbb{Z}^N$ ). Let  $\mathcal{A} = \{a^j, j = 1, \dots, N + n + 1\}$  be a finite subset of  $N + n + 1$  points in  $\mathbb{Z}^N$  which affinely generate  $\mathbb{Z}^N$  over  $\mathbb{Z}$ . Let  $\mathbf{A} \in \mathcal{M}^{N, N+n+1}(\mathbb{Z})$  be the matrix which columns are the vectors  $(1, \mathbf{a}^j) = (1, a_1^j, \dots, a_N^j)$ ,  $j = 1, \dots, N + n + 1$ . For any  $u \in \text{Ker}(\mathbf{A}) \cap \mathbb{Z}^{N+n+1}$ , set  $u = u^+ - u^- = \max(u, 0) - \max(-u, 0)$  (coordinate by coordinate). The *toric ideal*  $\mathcal{I}(\mathcal{A})$  attached to  $\mathcal{A}$  is defined as the binomial ideal in  $\mathbb{C}[Y_1, \dots, Y_{N+n+1}]$  generated by the binomials  $Y^{u^+} - Y^{u^-}$ , for  $u \in \text{Ker} \mathbf{A} \cap \mathbb{Z}^{N+n+1}$  or<sup>12</sup> the ideal generated by the differential operators  $\partial_Y^{u^+} - \partial_Y^{u^-}$ , for any such  $u$ , in  $\mathbb{C}\langle \partial/\partial Y_1, \dots, \partial/\partial Y_{N+n+1} \rangle$ .

**REMARK 4.24** (toric ideal versus lattice ideal). Let  $\mathcal{A} \subset \mathbb{Z}^N$  be a finite subset with cardinal  $N + n + 1$  as in Definition 4.23. Let  $M_{\mathcal{A}}$  be the  $\mathbb{Z}$ -module (with rank  $n$ ) of affine relations between the elements in  $\mathcal{A}$ . When  $\mathbb{Z}^{N+n+1}/M_{\mathcal{A}}$  is a free abelian group, the toric ideal  $\mathcal{I}(\mathcal{A})$  is prime and coincides then with the ideal  $\mathcal{J}(M_{\mathcal{A}})$  generated by the  $Y^{v^+} - Y^{v^-}$ , where  $v \in M_{\mathcal{A}}$  (which is known as the *lattice ideal* of the lattice of affine relations  $M_{\mathcal{A}}$ ). Computation of toric or lattice ideals can be performed for example with `Singular`. If  $\mathbf{b}^0, \dots, \mathbf{b}^{n-1}$  denotes a basis of the lattice  $M_{\mathcal{A}}$  (as the column vectors of  $\mathbf{B}$  do within the picture presented above), the ideal

<sup>12</sup>Which amounts to be equivalent, but from the point of view of differential operators.

generated by the  $n$  binomials  $Y^{(\mathbf{b}^j)^+} - Y^{(\mathbf{b}^j)^-}$  (or the corresponding differential operators  $\partial^{(\mathbf{b}^j)^+} - \partial^{(\mathbf{b}^j)^-}$ ) is called a *lattice basis ideal* for the lattice  $M_{\mathcal{A}}$ .

A unifying foundation for the theory of multivariate hypergeometric series, inspired by the treatment of Gauß hypergeometric series, was proposed in [GGR] and [GKZ0] (see also [GKZ]). It is based on the notions of GKZ-differential systems (either  $\mathcal{A}$ -hypergeometric systems or hypergeometric systems attached to the lattice  $M_{\mathcal{A}}$  of affine relations of a finite subset  $\mathcal{A} \subset \mathbb{Z}^N$ ).

DEFINITION 4.25 ( $\mathcal{A}$ -hypergeometric systems, hypergeometric systems attached to a lattice of affine relations  $M_{\mathcal{A}}$ ). Let  $\mathcal{A} \subset \mathbb{Z}^N$  a collection of  $N + n + 1$  elements as in Definition 4.23. The *GKZ- $\mathcal{A}$ -hypergeometric system* (with parameter  $d = (d_0, \dots, d_N)$ ) attached to  $\mathcal{A}$  is the left ideal in the Weyl algebra  $\mathbb{C}\langle Y_1, \dots, Y_{N+n+1}, \partial/\partial Y_1, \dots, \partial/\partial Y_{N+n+1} \rangle$  defined as

$$(4.32) \quad H_{\mathcal{A}}(d) = \left\langle \partial_Y^{u^+} - \partial_Y^{v^-} ; u \in \text{Ker } \mathbf{A} \cap \mathbb{Z}^{N+n+1} \right\rangle + \left\langle \text{Eul}_k^{\mathbf{A}} - d_k ; k = 0, \dots, N \right\rangle,$$

where the Euler operators  $\text{Eul}_k^{\mathbf{A}}$ ,  $k = 0, \dots, N$ , have been previously defined in (4.30). The *GKZ hypergeometric system* (with parameter the same  $d$ ) attached to the lattice of relations  $M_{\mathcal{A}}$  is defined as

$$(4.33) \quad H_{M_{\mathcal{A}}}(d) = \left\langle \partial_Y^{v^+} - \partial_Y^{v^-} ; v \in M_{\mathcal{A}} \right\rangle + \left\langle \text{Eul}_k^{\mathbf{A}} - d_k ; k = 0, \dots, N \right\rangle.$$

REMARK 4.26. In order to emphasize the intimate relation with hypergeometric functions, let us make the two following observations :

- if  $c = (c_1, \dots, c_{N+n+1})$  is taken such that  $\mathbf{A} \cdot c = d$  ( $c$  being determined modulo  $M_{\mathcal{A}} \otimes_{\mathbb{Z}} \mathbb{R}$ ), then the hypergeometric series defined as the formal sum

$$Z = (Z_1, \dots, Z_{N+n+1}) \mapsto \sum_{v \in M_{\mathcal{A}}} \frac{Z^{v+\gamma}}{\Gamma(v_1 + c_1 + 1) \cdots \Gamma(v_{N+n+1} + c_{N+n+1} + 1)}$$

is a formal solution of the system  $H_{M_{\mathcal{A}}}(d)$  ; this explains the terminology “ $\Gamma$ -hypergeometric series” used in [GGR] to denote the solutions of such GKZ systems  $H_{M_{\mathcal{A}}}$  ;

- the definition of the Horn system  $\mathcal{H}_{M_{\mathcal{A}}}(c)$  (which is derived from the Horn system  $\mathcal{H}(M_{\mathcal{A}}, c)$ , see Example 4.22) is closely related with that of GKZ system  $H_{M_{\mathcal{A}}}(\mathbf{A} \cdot c)$  ; one has clearly the inclusions

$$(4.34) \quad \mathcal{H}_{M_{\mathcal{A}}}(c) \subset H_{M_{\mathcal{A}}}(\mathbf{A} \cdot c) \subset H_{\mathcal{A}}(\mathbf{A} \cdot c)$$

for any choice of  $c \in \mathbb{C}^{N+n+1}$ .

The key property shared by these GKZ systems  $\mathcal{H}_{M_{\mathcal{A}}}(c)$ ,  $H_{\mathcal{A}}(\mathbf{A} \cdot c)$ ,  $H_{M_{\mathcal{A}}}(\mathbf{A} \cdot c)$  (which share the same characteristic variety<sup>13</sup>) is that they are holonomic whatever the value of  $c$  is. Moreover, one has the following important result :

PROPOSITION 4.4 (singular locus of GKZ  $\mathcal{A}$ -systems and principal  $\mathcal{A}$ -determinant). *Let  $\mathcal{A} \subset \mathbb{Z}^N$  be a finite set with cardinal  $N + n + 1$  such that  $\mathcal{A}$*

<sup>13</sup>Torsion eventually present in the module  $\mathbb{Z}^{N+n+1}/M_{\mathcal{A}}$  will be responsible only for multiplicities or embedded components.

affinely generates  $\mathbb{Z}^N$  over  $\mathbb{Z}$  and  $\dim \Delta_{\mathcal{A}} = N$ . For any  $d \in \mathbb{C}^{N+1}$ , the Zariski closure (in  $\mathbb{C}^{N+n+1}$ ) of the set

$$\begin{aligned} \mathcal{U}(\mathcal{M}_{H_{\mathcal{A}}(d)}) &:= \\ &= \left\{ Z \in \mathbb{C}^{N+n+1}; \exists \xi \in \mathbb{C}^{N+n+1} \setminus \{0\}, \sigma(P)(z, \xi) = 0 \quad \forall P \in H_{\mathcal{A}}(d) \right\}, \end{aligned}$$

hence the singular locus<sup>14</sup> of the GKZ  $\mathcal{A}$ -system  $H_{\mathcal{A}}(d)$ , equals the zero set of the principal  $\mathcal{A}$ -determinant  $E_{\mathcal{A}}$  defined in Definition 4.17.

PROOF. It follows precisely from the definition of the  $\mathcal{A}$ -principal determinant. We refer to [GKZ0] for more details (see also [Cat]).  $\square$

In order to prove Theorem 4.19, it is therefore enough to prove that the singular locus of a Horn system has a solid amoeba (see Theorem 4.27 below) Because of the correspondence between solutions of the Horn systems  $\mathcal{H}(\mathcal{A}, c)$  and  $\mathcal{H}_{M_{\mathcal{A}}}(c)$  which has been explicited in Example 4.22, and since a monomial change of coordinates such as  $Z \mapsto Z^{\mathbf{B}}$  in (4.29) does not affect the solidity of amoebas, that is the fact its genus equals zero, and transforms  $\mathcal{A}$ -hypergeometric series into Horn series, the fact that the singular locus of the Horn system  $\mathcal{H}(\mathcal{A}, c)$  has a solid amoeba will imply that the same is indeed true for the amoeba of the hypersurface defined by the principal  $\mathcal{A}$ -determinant  $E_{\mathcal{A}}$ .

**4.3.6. Singular locus of Horn systems.** The subsection is devoted to the proof of the following result<sup>15</sup>:

**THEOREM 4.27 ([PST]).** *The singular locus  $S$  of any compatible non-confluent Horn system in  $\mathbb{C}^n$ , when it is a proper subset of  $\mathbb{C}^n$  (the  $P_j$  and  $Q_j$  being defined as product of affine factors with integer slopes, in accordance with Ore-Sato theorem), has a solid amoeba, that is the recession cone for each connected component  $C$  of  $\mathbb{R}^n \setminus \mathcal{A}_S$  has a non-empty interior.*

PROOF. We will restrict ourselves here to the case where the Horn system is given by polynomials  $P$  and  $Q$  that can be expressed (as in (4.27)) as products of affine forms  $\langle A_k, X \rangle + c_k$ , where the coefficients  $c_k$  are *generic* in the following sense:

- the rank of the matrix generated by the row vectors  $A_1, \dots, A_p$  (which add together as the null vector because of the non-confluency hypothesis) involved in  $P_1, \dots, P_n$  and  $Q_1, \dots, Q_n$  equals  $n$ <sup>16</sup>;
- for each  $I \subset \{1, \dots, p\}$  such that  $\text{rank } A_I = n$ , then, if  $\gamma = \gamma_I$  denotes the solution of the Cramer system  $A_I \cdot \gamma + c_I = 0$ , one has

$$\{z \in \mathbb{C}^n; \langle A_j, \gamma_I + z \rangle + c_j = 0\} \cap (\gamma_I + \mathbb{Z}^n) = \emptyset \quad \forall j \notin I.$$

<sup>14</sup>That is (see the arguments developed in the presentation of Horn systems in subsection 4.3.4), the algebraic hypersurface in  $\mathbb{C}^{N+n+1}$  along which are located the singularities of the solutions of the GKZ system  $H_{\mathcal{A}}(d)$ .

<sup>15</sup>Which implies, as we already observed in the previous section, Theorem 4.19.

<sup>16</sup>Otherwise, the corresponding hypergeometric series solutions of the Horn system could be considered as hypergeometric series in strictly less than  $n$  variables, and the problem would reduce to the case where  $\mathbb{C}^n$  is replaced by  $\mathbb{C}^{n-1}$ .



Since we intend to deal with Horn systems of the form  $\mathcal{H}(M_{\mathcal{A}}, c)$  and the resultant locus of Horn systems such as  $\mathcal{H}_{M_{\mathcal{A}}}(c)$  does not depend on the choice of  $c$  (see Proposition 4.4), such genericity conditions will indeed not be restrictive for our purpose.

Let  $C$  be an open connected component of  $\mathbb{R}^n \setminus \mathcal{A}_S$ , where  $S$  denotes the singular locus of the Horn system (which coincides in this case with the zero set of its resultant). Since  $C$  is convex, the fundamental group  $\pi_1(\text{Log}^{-1}(C))$  is the free group isomorphic to the direct product of the fundamental groups of  $n$  or less punctured discs  $D_{z_j}(0, \epsilon_j) \setminus \{0\}$ ,  $\epsilon_j \ll 1$ , each of them being generated by the homotopy class of the loop  $\eta_j$ ,  $j = 1, \dots, n$ , about the origin in the  $\mathbb{C}_{z_j}$  plane.

Let us prove first that any solution  $z \mapsto y(z)$  of the Horn system  $\mathcal{H}(M_{\mathcal{A}}, c)$  in the domain  $\text{Log}^{-1}(C)$  can be expressed *a priori* in such a domain as a polynomial function in Puiseux monomials  $z^\alpha$  and logarithmic functions  $\log z_1, \dots, \log z_n$ , with single-valued coefficient functions  $h_{\alpha, \beta}$  as coefficients, hence can be represented in the form of the finite sum

$$(4.35) \quad y(z) = \sum_{\alpha, \beta} h_{\alpha, \beta}(z) z^\alpha (\log z)^\beta,$$

where the single-valued coefficient functions  $h_{\alpha, \beta}$  can be expanded in  $\text{Log}^{-1}(C)$  as Laurent series in the coordinates  $z_1, \dots, z_n$ . This can be achieved thanks to an argument based on the use of *monodromy*. Consider for that purpose a basis  $\{y_1, \dots, y_r\}$  for the space of holomorphic solutions of the Horn system  $\mathcal{H}(M_{\mathcal{A}}, c)$  on a simply connected domain in  $\text{Log}^{-1}(C)$ . Inspired by the argument used for example in the proof of Theorem 2.4.12 in [SST], one introduces differential operators with rational coefficients that constitute a basis  $\{1, \partial^{[1]}, \dots, \partial^{[r-1]}\}$  for the  $\mathbb{C}(z_1, \dots, z_n)$ -vector space  $\mathcal{M}_{\mathcal{D}}$  realized as the  $r$ -dimensional quotient space of the Weyl algebra  $\mathbb{C}\langle X, \partial/\partial X \rangle$  by the left ideal generated by the differential operators  $\mathcal{D}_j$  attached to polynomials  $P_1, \dots, P_n$  and  $Q_1, \dots, Q_n$  of the form (4.27) involved in the Horn system  $\mathcal{H}(M_{\mathcal{A}}, c)$ . Then

$$z \in \text{Log}^{-1}(C) \mapsto \begin{pmatrix} y_1(z) & \dots & \dots & y_r(z) \\ \partial^{[1]}y_1(z) & \dots & \dots & \partial^{[1]}y_r(z) \\ \vdots & \vdots & \vdots & \vdots \\ \partial^{[r-1]}y_1(z) & \dots & \dots & \partial^{[r-1]}y_r(z) \end{pmatrix}$$

defines a (multi-valued) matricial holomorphic function in  $\text{Log}^{-1}(C)$ . Since the family  $\{y_1, \dots, y_r\}$  is a basis of solutions of the hypergeometric system in some simply connected subset in  $\text{Log}^{-1}(C)$ , there exists, for each  $j = 1, \dots, n$ , an invertible matrix  $V_j = e^{2i\pi W_j}$  with complex coefficients (all matrices  $V_j$  being commuting since the fundamental group of  $\text{Log}^{-1}(C)$  is commutative), such that:

$$\forall z \in \text{Log}^{-1}(C), \eta_j^*[\Phi](z) = \Phi(z) \cdot V_j$$

(as multi-valued matricial functions). Hence the matricial function

$$z \in \text{Log}^{-1}(C) \mapsto \Phi(z) \cdot z_1^{-W_1} \dots z_n^{-W_n}$$

becomes single-valued (the monodromy being killed precisely by the monomial factors in  $z_1, \dots, z_n$ ). This leads to the expression (4.35) for a multi-valued solution  $z \mapsto y(z)$  of the Horn system in  $\text{Log}^{-1}(C)$ . In fact, provided the parameters  $c$  in

the choice of the Horn system  $\mathcal{H}(M_{\mathcal{A}}, c)$  are generic, one may assert that no logarithmic factor appears in the expression (4.35); this follows indeed from the fact that no logarithmic factor appears in the expression of the solution of any ordinary hypergeometric differential equation (in one variable) whose vector of coefficients  $c$  is generic. Any (multi-valued) solution of the Horn system  $\mathcal{H}(M_{\mathcal{A}}, c)$  (with  $c$  generic) in  $\text{Log}^{-1}(C)$  can thus be assumed to be represented in  $\text{Log}^{-1}(C)$  as a finite linear combination of Puiseux series<sup>17</sup>

$$y_{\gamma} : z \in \text{Log}^{-1}(C) \longmapsto y(z) = z^{\gamma} \sum_{k \in \text{Supp}(y_{\gamma}) - \gamma \subset \mathbb{Z}^n} a_k z^k$$

(also solution of the same system, since any Horn system has polynomial coefficients), where the subset

$$\text{Supp}(y_{\gamma}) := \gamma + \{k \in \mathbb{Z}^n; a_k \neq 0\}, \quad \gamma \in \mathbb{C}^n,$$

is called the *support* of the Puiseux hypergeometric series  $y_{\gamma}$  (the union of the supports of all such  $y_{\gamma}$ , solutions of the same Horn system, involved in a linear combination with complex coefficients  $y$  being then the support of  $y$ ). Note that the convex envelope (in  $\mathbb{C}^n$ ) of the support of a Puiseux hypergeometric series  $y_{\gamma}$  solution of the Horn system  $\mathcal{H}(M_{\mathcal{A}}, c)$  (hence of a linear combination with constant coefficients of such  $y_{\gamma}$  all solutions of the same Horn system) is always a polyedral set (see *e.g.* T. Sadykov's dissertation thesis [Sad]). Moreover, when the coefficients  $c$  of the Horn system  $\mathcal{H}(M_{\mathcal{A}}, c)$  happen to be generic (and the row vectors  $A_1, \dots, A_p$  define a matrix with rank  $n$ ), the recession cone (in  $\mathbb{R}^n$ ) of the support of such an hypergeometric series  $\sum_{\gamma} c_{\gamma} y_{\gamma}$  (solution of  $\mathcal{H}(M_{\mathcal{A}}, c)$  as a multi-valued function in some  $\text{Log}^{-1}(C)$ ) is a strongly convex affine polyedral cone in  $\mathbb{R}^n$ , *i.e.* a cone in  $\mathbb{R}^n$  that does not contain any line through the origin. We will admit here this crucial fact from the geometry of convex polytopes (*cf.* for example Proposition 1.12 in [Zieg]) and profit from it to derive a contradiction with the fact that the recession cone of  $C$  has empty interior in  $\mathbb{R}^n$ . We will for that need to invoke the following (bilateral) Abel's lemma.

LEMMA 4.28 (two sided Abel's lemma [PST]). *Let  $z \mapsto L(z)$  be a (non necessarily pure) Puiseux series in  $n$  variables which satisfies the non-confluent Horn system  $\mathcal{H}(M_{\mathcal{A}}, c)$  (with  $c$  generic and the rank of the subspace generated by  $A_1, \dots, A_p$  equal to  $n$ ), with non empty domain of convergence  $\Omega$ . Let  $\sigma(L)$  be the recession cone (in  $\mathbb{R}^n$ ) of the convex hull of  $\text{Supp}(L)$ . Then, for any  $z \in \Omega$  and some conenient  $z' \in \mathbb{T}^n \setminus \Omega$ , one has*

$$(4.36) \quad \text{Log}(z) - (\sigma(L))^{\sim} \subset \text{Log}(\Omega) \subset \text{Log}(z') - (\sigma(L))^{\sim}.$$

*Therefore the recession cone of  $\text{Log}(\Omega)$  coincides with the polar cone of the recession cone  $\sigma(L)$ .*

PROOF OF LEMMA 4.28. The first inclusion in (4.36) follows from the classical Abel's lemma (already a crucial ingredient in this course, for example when dealing with Laurent series in the proof of Proposition 2.1). Most interesting here is the second inclusion in (4.36). As mentioned earlier, the recession cone  $\sigma(L)$  of the convex hull of  $\text{Supp}(L)$  is a strongly convex polyedral cone in  $\mathbb{R}^n$ , let us say generated by vectors  $u^1, \dots, u^N$  in  $\mathbb{R}^n$ . Pick up then  $\xi \in \mathbb{R}^n$  such that  $\langle u^j, \xi \rangle > 0$  for

<sup>17</sup>Such a Puiseux series solution to a given Horn system is called a *pure* solution of this system.

$j = 1, \dots, N$  (strong convexity of  $\sigma(L)$  allows it). For the sake of simplicity<sup>18</sup>, one may even suppose here that the support of  $L$  lies in  $\mathbb{R}^n$  and coincides with  $\sigma(L) \cap \mathbb{Z}^n$ . It follows from the theory of (non-confluent) generalized hypergeometric functions in one variable (see *e.g.* section 1.1 in [GGR]; note that we deal here with the case labelled there as  $p = q + 1$ , since  $\sum_{k=1}^p \langle A_k, u^j \rangle = 0$  thanks to the non-confluency hypothesis) that the domain of convergence of the restricted hypergeometric series

$$\sum_{l=0}^{\infty} a_{lu^j} z^{lu^j}$$

lies in  $\{z \in \mathbb{T}^n; |z^{u^j}| < r_j\}$  for some  $r_j > 0$ , which implies that

$$\text{Log } \Omega \subset \{v \in \mathbb{R}^n; \langle v, u^j \rangle < \log r_j, j = 1, \dots, N\}.$$

Take then  $z' \in \mathbb{T}^n$  such that

$$\log z' = \xi \max_{1 \leq j \leq N} \frac{\log r_j}{\langle u^j, \xi \rangle},$$

so that

$$\forall j = 1, \dots, N, \langle u^j, \text{Log } z' \rangle \geq \log r_j.$$

Then, one has  $z' \in \mathbb{T}^n \setminus \Omega$  and the second inclusion in (4.36) is fulfilled.  $\square$

Let us now come back to the proof of Theorem 4.27. Consider a sequence  $(x^{[k]})_{k \in \mathbb{N}}$  of points in  $\partial C$  such that the recession cone  $\tilde{\Gamma}_{(x^{[k]})_k, C}$  of the intersection

$$\bigcap_{k \in \mathbb{N}} H_{x^{[k]}}(C)$$

of all  $H_{x^{[k]}}(C)$  (where  $H_{x^{[k]}}(C)$  denotes the supporting halfspace of the convex subset  $C$  at the point  $x^{[k]} \in \partial C$ ) coincides with the recession cone  $\Gamma_C$  of  $C$  (which is assumed for the moment to be of dimension strictly less than  $n$ , that is with empty interior). It is clear that such a sequence exists thanks to the definition of the recession cone  $\Gamma_C$  (see Proposition 3.3).

Since all points  $x^{[k]}$ ,  $k \in \mathbb{N}$ , belong to the image by  $\text{Log}$  of the set of singularities of the Horn system, there exists, for each  $k \in \mathbb{N}$ , a germ of (multivalued) solution  $y_{[k]}$  of the hypergeometric system  $\mathcal{H}(M_{\mathcal{A}}, c)$  that certainly cannot be continued analytically through at least one point in  $\text{Log}^{-1}(x^{[k]})$ , the analytic continuation  $L_{[k]}$  of this germ having (as seen in the preliminary part of the proof devoted to the description of solutions in  $\text{Log}^{-1}(C)$  of the hypergeometric system  $\mathcal{H}(M_{\mathcal{A}}, c)$ ) a development as a Puiseux hypergeometric series (non necessarily pure) whose domain of convergence contains  $\text{Log}^{-1}(C)$ . Choose complex coefficients  $\lambda_k$ ,  $k \in \mathbb{N}$ , such that, for any  $K \in \mathbb{N}$ , the Puiseux hypergeometric series  $L^{[K]} := \sum_0^K \lambda_k L_{[k]}$  (which is also solution in  $\text{Log}^{-1}(C)$  of the hypergeometric system  $\mathcal{H}(M_{\mathcal{A}}, c)$  as well as all the  $L_{[k]}$ ,  $k \in \mathbb{N}$ , are) is not identically zero. Denote as  $\Omega_{[K]} \supset \text{Log}^{-1}(C)$  the domain of convergence of the Puiseux hypergeometric series  $L^{[K]}$  ( $K \in \mathbb{N}$ ). For any  $K \in \mathbb{N}$ , the recession cone  $\Gamma_{\text{Log}(\Omega_{[K]})}$  of  $\text{Log}(\Omega_{[K]})$  is, by construction, a subset

<sup>18</sup>Since the coefficients  $c$  in the Horn system  $\mathcal{H}(M_{\mathcal{A}}, c)$  are generic, this is not really restrictive.

of the recession cone of the intersection of halfspaces  $\bigcap_{k=1}^K H_{x^{[k]}}(C)$ . It follows then from Lemma 4.28 that

$$-(\sigma(L_{[K]}))^\circ = \text{Log}(\Omega_{[K]}) \subset \text{recession cone of } \bigcap_{k=0}^K H_{x^{[k]}}(C).$$

As a consequence, the cone

$$\bigcup_{k=0}^{\infty} \sigma(L_{[K]})$$

fails to be strongly convex, since the recession cones of the  $\bigcap_{k=0}^K H_{x^{[k]}}(C)$ ,  $K \in \mathbb{N}$ , approach (when  $K$  goes to  $+\infty$ ) the recession cone  $\Gamma_C$  of  $C$ , and  $\Gamma_C$  is assumed here to have empty interior. But each of the cones  $\sigma(L_{[K]})$ ,  $K \in \mathbb{N}$ , has its boundary lying in a subset of the union of the zero sets of the polynomials  $P_1, \dots, P_n, Q_1, \dots, Q_n$  involved in the differential operators  $\mathcal{D}_j$  defining the Horn system. The union of all these zero sets being an arrangement of hyperplanes, all cones  $\sigma(L_{[K]})$ ,  $K \in \mathbb{N}$ , need to be taken among a finite set of distinct strongly polyedral cones in  $\mathbb{R}^n$ . Therefore, one at least among these cones fails to be strongly convex if their (finite) union is not, which is apparently the case. This leads indeed to a contradiction with the strong convexity of all cones  $\sigma(L_{[K]})$  for any  $K \in \mathbb{N}$ .  $\square$



## A brief overview about the concept of distribution

### A.1. Interpreting functions in a dual way : the concept of *distribution*

In order to motivate the concept of *distribution* in an open subset  $U \subset \mathbb{R}^n$ , one needs to have in mind that the numerical precise “evaluation” of a given function  $f : U \mapsto \mathbb{C}$  at a specific point  $x_0 \in U$  is indeed quite irrealist from the practical point of view : think for example about the case  $U = \mathbb{R}$ ,  $x_0 = \pi$  ; the number  $\pi$  is numerically tractable only through its rational approximations (which are unknown beyond some fixed barrier), so that  $f(\pi)$  cannot precisely be numerically evaluated. When  $f$  is locally integrable<sup>1</sup> in  $U$ , it is therefore more relevant (from the practical point of view) to interpret  $f(x_0)$  as

$$f(x_0) \simeq \int_U f(x) \varphi_\epsilon(x_0 + x) dx = \int_U f(x - x_0) \varphi_\epsilon(x) dx, \quad 0 < \epsilon \ll 1,$$

where  $dx$  stands for the Lebesgue measure in  $U$  and  $\varphi_\epsilon$  is *test function*, that is a function  $\varphi_\epsilon : U \rightarrow [0, \infty[$ ,  $C^\infty$  in  $x$ , with compact support  $K_\epsilon \subset \{\|x\| < \epsilon\}$ , such that  $\int_U \varphi_\epsilon(x) dx = \int_{K_\epsilon} \varphi_\epsilon(x) dx = 1$ . Then,  $f(x_0)$  is interpreted as some “averaged value” of  $f$  about the point  $x_0$ , this averaging being more and more precise as  $\epsilon$  shrinks towards 0.

Following this point of view, a locally integrable function  $f : U \rightarrow \mathbb{C}$  is known through all its “tests”

$$\langle T, \varphi \rangle := \int_U f(x) \varphi(x) dx,$$

where  $\varphi$  belongs to the  $\mathbb{R}$ -vector space  $\mathcal{D}(U)$  defined as the  $\mathbb{R}$ -vector space of  $C^\infty$  functions  $\varphi : U \rightarrow \mathbb{C}$  with compact support lying in  $U$ . Note that the vector space  $\mathcal{D}(U, \mathbb{C})$  is indeed very rich : for any open subset  $V \subset U$ , for any  $K \subset\subset V \subset U$ , there is always a function  $\varphi_{K,V} : U \rightarrow [0, 1]$ ,  $C^\infty$ , with compact support included in  $V$ , such that  $\varphi_{K,V} \equiv 1$  in a some open neighborhood of  $K$  in  $V$  (this is a consequence of Urysohn’s lemma).

One can write  $\mathcal{D}(U, \mathbb{C})$  as  $\bigcup_{K \subset\subset U} \mathcal{D}_K(U, \mathbb{C})$ , where  $\mathcal{D}_K(U, \mathbb{C})$  denotes the  $\mathbb{R}$ -vector space of  $C^\infty$  functions  $\varphi : U \rightarrow \mathbb{C}$  with support lying in the compact  $K$ . On  $\mathcal{D}_K(U, \mathbb{C})$ , one can define a topology (even defined by a metric  $d_K$ ), namely that of *uniform convergence on  $K$  of functions, together with all their derivatives*. One may then equip  $\mathcal{D}(U, \mathbb{C}) = \bigcup_{K \subset\subset U} \mathcal{D}_K(U, \mathbb{C})$  with the so called *inductive limit* topology of the (metrizable) topologies on the  $\mathcal{D}_K(U, \mathbb{C})$ ,  $K \subset\subset U$ . Such an inductive limit topology fails unfortunately to be metrizable (because of Baire’s theorem).

<sup>1</sup>It may even happen that  $f$  is not punctually defined, but just locally integrable, as for example  $f : z \mapsto \log|h(z)|$ , where  $h$  denotes a meromorphic function in some open subset  $U \subset \mathbb{C}^n \simeq \mathbb{R}^{2n}$ .

The  $\mathbb{R}$ -vector space  $\mathcal{D}'(U, \mathbb{C})$ , which elements are called  $\mathbb{C}$ -valued distributions in  $U$ , is defined as the topological  $\mathbb{R}$ -dual space of  $\mathcal{D}(U, \mathbb{C})$  (equipped precisely with this non metrizable inductive limit topology). Though the topology of  $\mathcal{D}(U, \mathbb{C})$  is not metrizable, it remains enough, from the practical point of view, just to know what means the fact that a sequence  $(\varphi_k)_{k \geq 0}$  of elements in  $\mathcal{D}(U, \mathbb{C})$  converges towards some element  $\varphi \in \mathcal{D}(U, \mathbb{C})$  : it just means that, for  $k$  large enough, all  $\varphi_k$  lie in the same  $\mathcal{D}_{K_0}(U, \mathbb{C})$  for some  $K_0 \subset\subset U$ , and that the sequence  $(\varphi_k)_{k \geq 0}$  precisely converges towards  $\varphi$  in  $\mathcal{D}_{K_0}(U, \mathbb{C})$  (for the distance  $d_{K_0}$  defining the topology of uniform convergence on  $K_0$  for functions, together with all their derivatives). A distribution  $T \in \mathcal{D}'(U, \mathbb{C})$  is then a  $\mathbb{R}$ -linear form  $T : \mathcal{D}(U, \mathbb{C}) \rightarrow \mathbb{C}$ , such that, whenever a sequence  $(\varphi_k)_{k \geq 0}$  (of elements in  $\mathcal{D}(U, \mathbb{C})$ ) converges towards the null function in  $\mathcal{D}(U, \mathbb{C})$  (in the sense precised above), then  $\lim_{k \rightarrow \infty} \langle T, \varphi_k \rangle = 0$ .

As a dual space of a  $\mathbb{R}$ -vectorial topological space,  $\mathcal{D}'(U, \mathbb{C})$  (when  $U$  denotes an open subset of  $\mathbb{R}^n$ ) can be equipped with a so called *weak* topology. Instead of defining this topology, it is enough for practical applications just to retain the following *principle of convergence for sequences of distributions* in  $\mathcal{D}'(U, \mathbb{C})$  : A sequence  $(T_k)_{k \geq 0}$  of elements in  $\mathcal{D}'(U, \mathbb{C})$  converges (with respect to the weak topology on  $\mathcal{D}'(U, \mathbb{C})$ ) towards some element  $T \in \mathcal{D}'(U, \mathbb{C})$  if and only if, for all test function  $\varphi \in \mathcal{D}(U, \mathbb{C})$ , one has  $\lim_{k \rightarrow +\infty} \langle T_k, \varphi \rangle = \langle T, \varphi \rangle$ . An important point to have in mind is the following, which is again a consequence of Baire's theorem : whenever a sequence  $(T_k)_{k \geq 0}$  of distributions in  $\mathcal{D}'(U, \mathbb{C})$  is such that, for any  $\varphi \in \mathcal{D}(U, \mathbb{C})$ ,  $\lim_{k \rightarrow +\infty} \langle T_k, \varphi \rangle$  exists in  $\mathbb{C}$ , then  $T : \varphi \in \mathcal{D}(U, \mathbb{C}) \mapsto \lim_{k \rightarrow +\infty} \langle T_k, \varphi \rangle$  defines also a an element of  $\mathcal{D}'(U, \mathbb{C})$ , that is a distribution in  $U$ .

When  $T \in \mathcal{D}'(U, \mathbb{C})$ , its *support*  $\text{Supp}(T)$  is the closed subset of  $U$  defined as the complement (in  $U$ ) of the largest open subset  $V \subset U$  such that  $T|_V = 0$ , that is :

$$\forall \varphi \in \mathcal{D}(U, \mathbb{C}), \text{Supp } \varphi \subset\subset V \implies \langle T, \varphi \rangle = 0.$$

Its *singular support*  $\text{SS}(T)$  is the closed subset of  $U$  defined as the complement (in  $U$ ) of the largest open subset  $V \subset U$  such that  $T|_V$  can be defined by a  $C^\infty$  function  $f_V$  in  $V$ , that is :

$$\forall \varphi \in \mathcal{D}(U, \mathbb{C}), \text{Supp } \varphi \subset\subset V \implies \langle T, \varphi \rangle = \int_V f_V(x) \varphi(x) dx.$$

Of course, one has always  $\text{SS}(T) \subset T$ , the inclusion being in general strict : for example, if  $p$  is a Laurent tropical polynomial function in  $\mathbb{R}^n$  (such as (1.12)), and if one considers the distribution :

$$T_p : \varphi \in \mathcal{D}(\mathbb{R}^n, \mathbb{C}) \mapsto \int_{\mathbb{R}^n} p(x) \varphi(x) dx,$$

one has  $\text{Supp}(T_p) = \mathbb{R}^n$  and  $\text{SS}(T_p) = V_{\text{trop}}(p)$  (cf. Definition 1.2).

## A.2. Example of distributions

Let  $U \subset \mathbb{R}^n$  be an open subset of  $\mathbb{R}^n$ . A locally integrable function  $f : U \rightarrow \mathbb{C}$  defines a distribution in  $U$ , namely the distribution :

$$f : \varphi \in \mathcal{D}(U, \mathbb{C}) \mapsto \int_U f(x) \varphi(x) dx.$$

EXAMPLE A.1. When  $f_1, \dots, f_m$  are holomorphic functions in some open subset  $U \subset \mathbb{C}^n$ , then  $\log(|f_1|^2 + \dots + |f_m|^2)$  defines a distribution in  $U$ .

Recall that a *complex Radon measure*  $T$  in  $U \subset \mathbb{R}^n$  denotes a continuous linear form on the  $\mathbb{R}$ -vector space  $\mathcal{C}(U, \mathbb{C})$  of continuous functions with compact support in  $U$ , equipped with the (metrizable) topology of uniform convergence on any compact subspace. There are then four unique positive borelian measures  $\mu^+, \mu^-, \nu^+, \nu^-$  on  $U$  (with fine mass on any compact subset), such that

$$(A.1) \quad \begin{aligned} \forall \varphi \in \mathcal{C}(U, \mathbb{C}), \quad \langle T, \varphi \rangle &= \int_U \varphi(x) d\mu^+(x) - \int_U \varphi(x) d\mu^-(x) \\ &+ i \left( \int_U \varphi(x) d\nu^+(x) - \int_U \varphi(x) d\nu^-(x) \right). \end{aligned}$$

This is a consequence of F. Riesz's theorem, that connects integration theory from the pointset point of view with integration theory from the functional point of view. Any such Radon measure  $T = \mu^+ - \mu^- + i(\nu^+ - \nu^-)$  defines a distribution in  $U$  since  $\mathcal{D}(U, \mathbb{C})$  embeds continuously in  $\mathcal{C}(U, \mathbb{C})$ : Take the action of  $T$  on a test-function as (A.1). Any complex Radon measure  $T$  in  $U$  defines then a distribution.

EXAMPLE A.2 (the Dirac mass at a point). When  $x_0 \in U \subset \mathbb{R}^n$ , the Dirac mass

$$\delta_{x_0} : \varphi \in \mathcal{D}(U, \mathbb{C}) \mapsto \varphi(x_0)$$

defines a distribution in  $U$ .

Any distribution  $T \in \mathcal{D}'(U, \mathbb{C})$  which is such that

$$\forall \varphi \in \mathcal{D}(U, \mathbb{R}), \quad \left( \varphi \geq 0 \text{ in } U \right) \implies \langle T, \varphi \rangle \geq 0$$

( $T$  is then said to be a *positive distribution* in  $U$ ) is in fact a *positive measure* in  $U$ , that is of the form

$$T : \varphi \in \mathcal{D}(U, \mathbb{C}) \mapsto \int_U \varphi(x) d\mu^+(x),$$

where  $\mu^+$  is a positive Borel measure in  $U$  with finite mass on any compact subset.

There are of course distributions in  $U \subset \mathbb{R}^n$  which are neither locally integrable functions nor measures.

EXAMPLE A.3 (differential operators and distributions with punctual support). Any differential operator with constant coefficients

$$P = P(D) \in \mathbb{C} \left[ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right]$$

induces distributions in  $U$ : given  $x_0 \in U$ ,

$$P(D)[\delta_{x_0}] : \varphi \in \mathcal{D}(U, \mathbb{C}) \mapsto \left( P(-D)[\varphi] \right)_{x=x_0}$$

is a distribution<sup>2</sup>. Moreover, any distribution  $T$  in  $U$  such that  $\text{Supp}(T) = \{x_0\}$  is of this form. That is,  $\mathbb{C}$ -valued distributions with support a given point  $x_0$  in  $U$  form a  $\mathbb{C}$ -algebra which is isomorphic to  $\mathbb{C}[X_1, \dots, X_n]$ .

<sup>2</sup>One will justify in the next section why it is denoted as  $P(D)[\delta_{x_0}]$ .



EXAMPLE A.4 (Principal Value distributions in  $U \subset \mathbb{C}^n$ ). A quite important example of a distribution which is not a locally integrable function, nor a complex Radon measure, nor of the form  $P(D)[\delta_{x_0}]$ , occurs in the complex setting. Let  $U$  be an open subset of  $\mathbb{C}^n$  and  $h = f/g : U \mapsto \mathbb{P}^1(\mathbb{C})$  be a meromorphic function in  $U$ . Then, for any  $\varphi \in \mathcal{D}(U, \mathbb{C})$ , the limit, when  $\epsilon$  tends to  $0_+$ , of

$$\epsilon \mapsto \int_{|g| \geq \epsilon} \frac{f(z)}{g(z)} \varphi(z) dx_1 dy_1 \dots dx_n dy_n$$

exists, thus defines a distribution in  $U$ , called *Principal Value* of  $f/g$ , denoted as  $\text{VP}[h]$ . When  $h^{-1}(\infty) \neq \emptyset$ ,  $\text{VP}[h]$  is neither a locally integrable function nor even a complex Radon measure in  $U$ .

When  $U = \mathbb{R}^n$ , a distribution  $T \in \mathcal{D}'(\mathbb{R}^n, \mathbb{C})$  is said to be *temperate* if, after lifted on sphere  $\mathbb{S}^n \setminus \{(0, \dots, 0, 1)\}$  via the inverse stereographic projection, it can be continued as a distribution on the whole compact sphere  $\mathbb{S}^n$ . Such *temperate distributions* on  $\mathbb{R}^n$  form the  $\mathbb{R}$ -vector space  $\mathcal{S}'(\mathbb{R}^n, \mathbb{C})$  ( $\mathcal{S}$  stands here for “spherical”). This space  $\mathcal{S}'(\mathbb{R}^n, \mathbb{C})$  is the  $\mathbb{R}$ -dual space of the *Schwartz space*  $\mathcal{S}(\mathbb{R}^n, \mathbb{C})$  of  $C^\infty$  functions that decrease at infinity in  $\mathbb{R}^n$  (as well as all their derivatives) faster than any polynomial function (the Gauß function  $x \mapsto \exp(-\|x\|^2)$  being the prototype of an element in such a class). The interest of  $\mathcal{S}'(\mathbb{R}^n, \mathbb{C}) \subset \mathcal{D}'(\mathbb{R}^n, \mathbb{C})$  lies in the fact that the *Fourier transform* (which is a fundamental tool in mathematical analysis or in physics) realises a  $\mathbb{R}$ -isomorphism of  $\mathcal{S}(\mathbb{R}^n, \mathbb{C})$  onto itself, thus, thanks to duality, induces also a  $\mathbb{R}$ -isomorphism between  $\mathcal{S}'(\mathbb{R}^n, \mathbb{C})$  onto itself.

### A.3. Differentiating distributions

Let  $U$  be an open subset in  $\mathbb{R}^n$ . When  $\varphi \in \mathcal{D}(U, \mathbb{C})$  and  $f : U \rightarrow \mathbb{C}$  is a  $C^1$  function, the formula of integration by parts (*i.e.* the so-called fundamental theorem of analysis in dimension 1) ensures that :

$$(A.2) \quad \int_U \frac{\partial f}{\partial x_j} \varphi(x) dx = - \int_U f(x) \frac{\partial \varphi}{\partial x_j} dx, \quad j = 1, \dots, n.$$

This formula suggests the possibility of differentiating distributions  $T \in \mathcal{D}'(U, \mathbb{C})$ . One needs just to observe that, when  $T \in \mathcal{D}'(U, \mathbb{C})$  is a distribution in  $U$ , the  $\mathbb{R}$ -linear maps

$$(A.3) \quad \frac{\partial T}{\partial x_j} : \varphi \in \mathcal{D}(U, \mathbb{C}) \mapsto - \left\langle T, \frac{\partial \varphi}{\partial x_j} \right\rangle, \quad j = 1, \dots, n,$$

define distributions in  $U$ . Moreover the maps

$$T \in \mathcal{D}'(U, \mathbb{C}) \mapsto \frac{\partial T}{\partial x_j} \in \mathcal{D}'(U, \mathbb{C}), \quad j = 1, \dots, n,$$

are continuous ( $\mathcal{D}'(U, \mathbb{C})$  being equipped with its weak topology).

Let us give here four fundamental examples. The two last ones (especially the third one) will reveal to be capital in this “tropical” course (since it involves the log function).

EXAMPLE A.5 (derivative of the Heaviside function). Here is the “historical” example, at the origin of the invention of distribution theory by Paul Dirac around the beginning of the nineties. Let  $Y : \mathbb{R} \mapsto [0, 1]$  be the *Heaviside function*

defined by  $Y(t) = 0$  when  $t < 0$  and  $Y(t) = 1$  when  $t \geq 0$ . Denote also as  $Y$  the corresponding distribution :

$$\varphi \in \mathcal{D}(\mathbb{R}, \mathbb{C}) \longmapsto \int_{\mathbb{R}} Y(t) \varphi(t) dt = \int_{[0, \infty]} \varphi(t) dt.$$

One has

$$\forall \varphi \in \mathcal{D}'(\mathbb{R}, \mathbb{C}), \langle Y', \varphi \rangle = -\langle Y, \varphi' \rangle = -\int_{[0, \infty]} \varphi'(t) dt = -[\varphi(t)]_0^\infty = \varphi(0).$$

One has therefore the formule  $Y' = \delta_0$ , where  $\delta_0$  denotes the Dirac mass at the origin, that is the positive measure :

$$\varphi \in \mathcal{C}(U, \mathbb{C}) \longmapsto \varphi(0).$$

The formula  $Y' = \delta_0$  is known as the *jump formula*.

EXAMPLE A.6 (Cauchy-Riemann operator and Cauchy formula). Consider the function  $z \in \mathbb{C}^* \mapsto 1/z \in \mathbb{C}$ . It is holomorphic in  $\mathbb{C}^*$ , that is  $C^\infty$  and solution of the Cauchy-Riemann equation

$$\frac{\partial}{\partial \bar{z}} \left[ \frac{1}{z} \right] := \frac{1}{2} \left( \frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \left[ \frac{1}{z} \right] \equiv 0 \quad \forall z \in \mathbb{C}^*.$$

As a function which is locally integrable in  $\mathbb{C}$ , it defines also a distribution :

$$\frac{1}{z} : \varphi \in \mathcal{D}(\mathbb{R}^2, \mathbb{C}) \longmapsto \iint_{\mathbb{R}^2} \frac{\varphi(\xi, \eta)}{\xi + i\eta} d\xi d\eta.$$

We claim that, in the sense of distributions in  $\mathbb{C}$ , one has the *Cauchy formula*

$$(A.4) \quad \frac{\partial}{\partial \bar{z}} \left[ \frac{1}{z} \right] = \pi \delta_{(0,0)},$$

$\delta_{(0,0)}$  being the Dirac mass at the origin  $(0, 0)$ , that is

$$(A.5) \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^2, \mathbb{C}), \varphi(0) = -\frac{1}{\pi} \iint_{\mathbb{R}^2} \frac{\partial \varphi}{\partial \bar{\zeta}}(\xi, \eta) \frac{d\xi d\eta}{\zeta}.$$

Formula (A.5) results from the so-called *Cauchy-Pompeiu formula*, which is itself an avatar of *Green-Riemann's formula* : for any open disk  $D(z_0, R)$  in  $\mathbb{C}$ , for any complex function  $\varphi$  defined and  $C^1$  in a neighborhood of  $\bar{D}$ , one has :

$$(A.6) \quad \forall z \in D(z_0, R), \varphi(z) = \frac{1}{2i\pi} \int_{\gamma_{z_0, R}} \frac{\varphi(\zeta)}{\zeta - z} d\zeta - \frac{1}{\pi} \iint_{\bar{D}} \frac{\partial \varphi}{\partial \bar{\zeta}}(\xi, \eta) \frac{d\xi d\eta}{\zeta - z},$$

where  $\gamma_{z_0, R} : \theta \in [0, 2\pi] \mapsto z_0 + Re^{i\theta}$ . Note that, when  $\varphi$  is holomorphic in a neighborhood of  $\bar{D}$ , then the double integral in (A.6) disappears and one recovers *Cauchy representation* formula for holomorphic functions. When  $\varphi \in \mathcal{D}(\mathbb{R}^2, \mathbb{C})$ , one can take  $z_0 = 0$  and  $R$  large enough so that  $D(0, R)$  contains the support of  $\varphi$  ; the contour integral in (A.6) then disappears and one gets (A.5), taking  $z = z_0 = 0$ .

EXAMPLE A.7 (Lelong-Poincaré equation, Weil divisors). Let  $U$  be a connected open subset of  $\mathbb{C}$  and let  $h : U \rightarrow \mathbb{P}^1(\mathbb{C})$  be a meromorphic function in  $U$  which is not identically zero in  $U$ . One can associate to  $h$  a (principal) *Weil divisor* defined as

$$(A.7) \quad \text{div}(h) = \sum_{\alpha \in h^{-1}(0)} \mu_\alpha \{ \alpha \} - \sum_{\beta \in h^{-1}(\infty)} \nu_\beta \{ \beta \},$$

where  $\mu_\alpha$ , for  $\alpha \in h^{-1}(0)$ , denotes the multiplicity of  $\alpha$  as a zero of  $h$ , and  $\nu_\beta$ , for  $\beta \in h^{-1}(\infty)$ , denotes the order of  $\beta$  as a pole of  $h$ . On the other hand,  $\log |h| : U \rightarrow \mathbb{R}$  (defined almost everywhere in  $U$ ) defines a locally integrable function in  $U$ , that is a distribution :

$$\log |h| : \varphi \in \mathcal{D}(U, \mathbb{C}) \longmapsto \int \int_U \log |h(\xi + i\eta)| \varphi(\xi, \eta) d\xi d\eta.$$

One can check immediately, as a consequence of (A.4), that, in the sense of distributions,

$$(A.8) \quad \Delta[\log |h|] = 2\pi \left( \sum_{\alpha \in h^{-1}(0)} \mu_\alpha \delta_\alpha - \sum_{\beta \in h^{-1}(\infty)} \nu_\beta \delta_\beta \right).$$

Note that the right-hand side of (A.8) can be interpreted as the “dual” of the right-hand side of (A.7) : instead of considering points such as  $\{\alpha\}$  or  $\{\beta\}$  (that is 0-cycles in  $U$ ), one considers in the right-hand side of (A.8) evaluation of test functions at these points, that is action on test-functions of the Dirac masses  $\delta_\alpha$  or  $\delta_\beta$ , which of course amounts to be the same (from the dual point of view). Formula (A.8) is known as *Lelong-Poincaré* formula. What makes it interesting is that it connects some analytic object (the distribution  $\log |h|$  involved in the left-hand side) with some geometric object (the principal Weil divisor involved in the right-hand side).

EXAMPLE A.8 (Lelong Poincaré formule, Cartier divisors). Consider an open subset  $U$  of a Riemann surface and a *Cartier divisor* on  $U$ , that is a collection of pairs  $(U_\iota, h_\iota)$ , where the open subsets  $U_\iota$  are such that  $\bigcup_\iota U_\iota = U$ , and, for any index  $\iota$ ,  $h_\iota : U_\iota \rightarrow \mathbb{P}^1(\mathbb{C})$  is a meromorphic function on  $U_\iota$ , such that, for any pair  $(\iota, \iota')$  such that  $U_\iota \cap U_{\iota'} \neq \emptyset$ ,  $h_\iota/h_{\iota'}$  is an invertible holomorphic function in  $U_\iota \cap U_{\iota'}$ . Consider, for each  $\iota$ , a strictly positive  $C^\infty$  function  $\rho_\iota : U_\iota \rightarrow \mathbb{C}$ , such that

$$(A.9) \quad \forall \iota, \iota', \quad U_\iota \cap U_{\iota'} \neq \emptyset \implies \frac{|h_\iota|}{\rho_\iota} \equiv \frac{|h_{\iota'}|}{\rho_{\iota'}} \text{ in } U_\iota \cap U_{\iota'}.$$

Such data  $(\rho_\iota)_\iota$  can be interpreted as choosing an hermitian metric  $|\cdot|_\rho$  on the holomorphic line bundle<sup>3</sup> over  $U$  which corresponds to the Cartier divisor  $(U_\iota, h_\iota)_\iota$ . One defines globally (almost everywhere on  $U$ ) a function  $\log |h|_\rho : U \rightarrow \mathbb{R}$ , setting:

$$\forall \iota, \forall z \in U_\iota, \quad \log |h(z)|_\rho := \log \frac{|h_\iota(z)|}{\rho_\iota}.$$

Take the action of the operator  $dd^c := (i/\pi) \partial \bar{\partial}$ , where  $\partial$  transforms  $(p, q)$ -differential forms ( $0 \leq p + q \leq 2$ ) into  $(p + 1, q)$  forms,  $\bar{\partial}$  transforms  $(p, q)$ -differential forms into  $(p, q + 1)$ -differential forms ( $d^2 = \partial^2 = \bar{\partial}^2 = 0$ ,  $\partial \bar{\partial} = -\bar{\partial} \partial$ ). Then one has :

$$(A.10) \quad dd^c(\log |h|_\rho) + dd^c(\log \rho) = \sum_{\alpha \in h^{-1}(0)} \mu_\alpha \delta_\alpha - \sum_{\beta \in h^{-1}(\infty)} \nu_\beta \delta_\beta.$$

The smooth globally defined  $(1, 1)$ -form  $dd^c(\log \rho)$  (defined as  $dd^c(\log \rho_\iota)$  in each  $U_\iota$ ) is called the *first Chern form*  $c_1(|\cdot|_\rho)$  of the hermitian line bundle corresponding

<sup>3</sup>By a *holomorphic line bundle* over  $U$ , one means a collection of 1-dimension  $\mathbb{C}$ -vectorial spaces  $L_z$ ,  $z \in U$  (called fibers) that depend “holomorphically” on  $z$ . Choosing a smooth hermitian metric on this bundle (thus equipping it with a structure of *hermitian holomorphic line bundle* over  $U$ ) amounts to choose a metric  $|\cdot|_{\rho, z}$  on each fiber, so that it depends in a smooth way on  $z \in U$ .

to the choice of the metric  $(\rho_\iota)_L$  on the holomorphic line bundle corresponding to the Cartier divisor  $(U_\iota, h_\iota)_L$ . Note that

$$\sum_{\alpha \in h^{-1}(0)} \mu_\alpha \{\alpha\} - \sum_{\beta \in h^{-1}(\infty)} \nu_\beta \{\beta\}$$

represents the Weil divisor  $\text{div}(h)$  associated to the Cartier divisor  $(U_\iota, h_\iota)_L$ . Formula (A.10) can thus be rewritten in some abridged way as

$$dd^c(-\log|h|_\rho) + \text{div}(h) = c_1(|\cdot|_\rho),$$

and is known as the *geometric Lelong-Poincaré equation*. The function  $G_\rho := -\log|h|_\rho$ , which is such that  $dd^c G_\rho + \text{div}(h)$  is a smooth form, is called a *Green function* with respect to the Cartier divisor  $(U_\iota, h_\iota)_L$ . Such a notion reveals to be fundamental in Arakelov's theory (see *e.g.* [Lang] for an introduction).

#### A.4. Fundamental solutions and hypoellipticity

For any partial differential operator with constant coefficients in  $n$  variables

$$P = P(D) \in \mathbb{C} \left[ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right],$$

a fundamental theorem due to B. Malgrange and L. Ehrenpreis asserts that one can always find a temperate distribution  $T \in \mathcal{S}'(\mathbb{R}^n, \mathbb{C})$  such that  $P(D)[T] = \delta_0$ . Such a distribution is called a *fundamental solution* of  $P(D)$ .

EXAMPLE A.9. When  $n = 2$ , example A.6 shows that a fundamental solution for the Cauchy-Riemann operator  $\partial/\partial\bar{z} = 1/2(\partial/\partial x + i\partial/\partial y)$  is the locally integrable function  $z \mapsto 1/\pi \times 1/z$ . Example A.7 shows that a fundamental solution for the Laplace operator  $\Delta$  in  $\mathbb{R}^2$  is the locally integrable function  $z \mapsto (\log|z|^2)/(2\pi)$ . In  $\mathbb{R}^n$  ( $n \geq 3$ ), a fundamental solution for the Laplace operator is  $-c_n \|x\|^{2-n}$ , where  $c_n = \Gamma(n/2)\pi^{-n/2}/(2(n-2))$ .

A differential operator that admits a fundamental solution  $T$  such that  $\text{SS}(T) = \{0\}$  is called *hypoelliptic*. Hypoelliptic operators do not propagate singularities, that is, one has

$$\text{SS}(T) \subset \text{SS}(P(D)[T]) \quad \forall T \in \mathcal{D}'(U, \mathbb{C})$$

when  $T \in \mathcal{D}'(U, \mathbb{R}^n)$ ,  $U$  being an open subset of  $\mathbb{R}^n$ .

As a consequence, if a  $(k, 0)$ -differential form ( $0 \leq k \leq n$ ) with distribution coefficients (*i.e.* a  $(k, 0)$ -current)

$$T = \sum_{\#I=k} T_I dz_I$$

on a complex  $n$ -dimensional manifold  $\mathcal{X}$  satisfies  $\bar{\partial}T = 0$ , then all distributions  $T_I$ ,  $\#I = k$ , are holomorphic functions on  $\mathcal{X}$ ; the differential form  $T$  is then called an *abelian form* on  $\mathcal{X}$ . This emphasises the central role of the Cauchy-Riemann operator  $\bar{\partial}$  in complex geometry.



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