Fine properties of functions: an introduction

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April 30, 2005

Introduction

These lecture notes, intended as support to an intensive course at Şcoala Normală Superioară din Bucureşti, cover classical properties of function spaces such as: a.e. differentiability of Lipschitz functions (Rademacher's theorem), maximal functions (the Hardy-Littlewood-Wiener theorem), functions of bounded variation, area and coarea formulae, Hausdorff measure and capacity, isoperimetric inequalities, Hardy and bounded mean oscillations spaces (and their duality), trace theory, precise representatives.

Several textbooks cover part of these topics:

Herbert Federer, Geometric measure theory, Springer, 1969

Vladimir Maz'ja, Sobolev spaces, Springer, 1980

Leon Simon, Lectures on Geometric Measure Theory, Proceedings of the Centre for Mathematical Analysis, Australian National University, 1983

William P. Ziemer, Weakly differentiable functions, Springer, 1989

Lawrence C. Evans and Ronald F. Gariepy, Measure Theory and Fine Properties of Functions, Studies in Advanced Mathematics, CRC Press, 1992

Elias M. Stein, Harmonic Analysis: real variable methods, orthogonality, and oscillatory integrals, Princeton University press, 1993

However, there is no single source covering the basic facts the working analyst needs. This is the main purpose of the notes. These notes are also an invitation to reading the above wonderful books.

The background required is a good knowledge of standard measure theory: Radon-Nikodym and Hahn decomposition theorems, Riesz representation theorem. Also, the standard theory of distributions and basics about Sobolev spaces are presumes known. All other standard tools (Vitali and Whitney covering theorems, for example) are proved in the text.

Proofs are to be considered *cum grano salis*: there are a number of typos left. The reader is acknowledged in advance for his patience in this respect.

Lyon
April 2005

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Part I

Basic Tools

Chapter 0

The distribution function

Throughout this course, we consider on $\mathbb{R}^N$ the usual Lebesgue measure dx. The measure of a set $A \subset \mathbb{R}^N$ will be simply denoted by $|A|$.
Let $f : \mathbb{R}^N \to \mathbb{C}$ be a measurable function. We consider the **distribution function** of f,

$$F(t) = |\{ x \in \mathbb{R}^N ; |f(x)| > t \}|. \tag{1}$$

Clearly, $F : [0, \infty) \to [0, \infty]$ is decreasing and thus measurable. F is related to various norms of f via

Proposition 1. *For $1 \leq p < \infty$ we have*

a) $\|f\|_{L^p}^p = p \int_0^\infty t^{p-1} F(t) dt;$

*b) (***Chebyshev's inequality***)* $F(t) \leq \dfrac{\|f\|_{L^p}^p}{t^p}.$

Proof. a) We have

$$\|f\|_{L^p}^p = \int |f(x)|^p dx = \int \int_0^{|f(x)|} pt^{p-1} dt dx = p \int_0^\infty t^{p-1} \int_{\{x; |f(x)| > t\}} dx dt = p \int_0^\infty t^{p-1} F(t) dt. \tag{2}$$

b) Chebyshev's inequality follows from

$$\|f\|_{L^p}^p \geq \int_{\{x; |f(x)| > t\}} |f(x)|^p dx \geq \int_{\{x; |f(x)| > t\}} t^p dx = t^p F(t). \tag{3}$$

□

By copying the proof of a) above, we obtain the following

Proposition 2. *Let* $\Phi : [0,\infty] \to [0,\infty]$, $\Phi \in C^1$, *be a non decreasing function s. t.* $\Phi(0) = 0$. *Then*

$$\int_{\mathbb{R}^N} \Phi(|f(x)|)dx = \int_0^\infty \Phi'(t)F(t)dt. \tag{4}$$

0.1 Lorentz spaces

One may read the property a) in Proposition 1 as $\|f\|_{L^p}^p = p\|tF^{1/p}(t)\|^p_{L^p((0,\infty);dt/t)}$. This suggests a more general definition: a measurable function f belongs to the **Lorentz space** $L^{p,q}$ ($1 \le p < \infty$, $1 \le q \le \infty$) if

$$\|f\|_{L^{p,q}} = \|tF^{1/p}(t)\|_{L^q((0,\infty);dt/t)} < \infty. \tag{5}$$

Despite this notation, $\|\cdot\|_{L^{p,q}}$ is not a norm (but almost: it is a quasi-norm). When $q = p$, the corresponding Lorentz space coincides with L^p. When $q = \infty$, the corresponding space $L^{p,\infty}$ is called the **weak** L^p, also denoted by L^p_w (the **Marcinkiewicz space**). Clearly, a function f belongs to L^p_w if and only if its distribution function F satisfies a Chebyshev type inequality: $F(t) \le \dfrac{C}{t^p}$ for each $t > 0$.

It is well known that there is no inclusion relation for the L^p spaces. However, for fixed p, the Lorentz spaces are monotonic in q:

Proposition 3. *Let* $1 \le q < r \le \infty$. *Then* $L^{p,q} \subset L^{p,r}$.

Proof: Assume first that $r = \infty$. If $f \in L^{p,q}$, then

$$\|f\|^q_{L^{p,q}} = \int_0^\infty t^{q-1}F^{q/p}(t)dt \ge \int_0^s t^{q-1}F^{q/p}(t)dt \ge \int_0^s t^{q-1}F^{q/p}(s)dt = \frac{1}{q}s^qF^{q/p}(s). \tag{6}$$

Thus $F^{1/p}(s) \le \dfrac{C}{s}$, i.e. $f \in L^{p,\infty}$.

Let now $r < \infty$ and let $f \in L^{p,q}$. Then, using Hölder's inequality and the case $r = \infty$, we find

$$\|tF^{1/p}(t)\|_{L^r((0,\infty);dt/t)} \le \|tF^{1/p}(t)\|^{q/r}_{L^q((0,\infty);dt/t)}\|tF^{1/p}(t)\|^{1-q/r}_{L^\infty((0,\infty);dt/t)} < \infty. \tag{7}$$

Incidentally, we proved the stronger statement

$$\|f\|_{L^{p,r}} \le C\|f\|_{L^{p,q}}, \quad 1 \le p < \infty, 1 \le q \le r \le \infty. \tag{8}$$

We complete the scale of Lorentz spaces by setting $L^{\infty,q} = L^\infty$ for all q. The above inequality, combined with the fact that $\|\cdot\|_{p,q}$ is a quasi-norm yields immediately the following

Corollary 1. *The inclusion* $L^{p,q} \subset L^{p,r}$ *is continuous,* $1 \leq q < r \leq \infty$.

Remark 1. *One should understand the Lorentz spaces as "microscopic" versions of the* L^p *spaces. We mean that the properties of* $L^{p,q}$ *are very close to those of* L^p. *Here is an example: if* Ω *is a bounded set in* $\mathbb{R}^N$, *one may define in an obvious way the spaces* $L^{p,q}(\Omega)$. *It is easy to prove that, if* $p_1 < p_2$, *then* $L^{p_2,q_2}(\Omega) \subset L^{p_1,q_1}(\Omega)$ *for all the possible values of* q_1, q_2. *This is exactly the inclusion relation we have for the standard* L^p *spaces.*

Chapter 1

Elementary interpolation

Theorem 1. *(**Marcinkiewicz' interpolation theorem; simplified version**) Let $1 < q < \infty$ and let $T : L^1 \cap L^q(\mathbb{R}^N) \to \mathcal{D}'$ be linear and s. t.*

$$\|Tf\|_{L^1_w} \le C_1 \|f\|_{L^1}, \quad \forall\ f \tag{1.1}$$

and

$$\|Tf\|_{L^q_w} \le C_q \|f\|_{L^q}, \quad \forall\ f. \tag{1.2}$$

(In other words, T extends by density as a continuous operator from L^1 into L^1_w and from L^q into L^q_w.) Then T is a continuous operator from L^p into L^p, for each $1 < p < q$, i. e.

$$\|Tf\|_{L^p} \le C \|f\|_{L^p}, \quad \forall\ f \in L^1 \cap L^q. \tag{1.3}$$

Before proceeding to the proof of the theorem, let us note that L^q embeds into L^q_w, and thus we have the following consequence, which is the form we usually make use of the above theorem

Corollary 2. *Let $1 < q < \infty$ and let $T : L^1 \cap L^q(\mathbb{R}^N) \to \mathcal{D}'$ be linear and s. t.*

$$\|Tf\|_{L^1_w} \le C_1 \|f\|_{L^1}, \quad \forall\ f \tag{1.4}$$

and

$$\|Tf\|_{L^q} \le C_q \|f\|_{L^q}, \quad \forall\ f. \tag{1.5}$$

Then T extends as a continuous operator from L^p into L^p, $1 < p < q$.

Proof. Let $t > 0$ and let $f \in L^1 \cap L^q$. We are going to estimate the distribution function of Tf. For this purpose, we cut f at height t, i. e. we write $f = f_1 + f_2$, where $f_1(x) = \begin{cases} f(x), & \text{if } |f(x)| > t \\ 0, & \text{otherwise} \end{cases}$ and $f_2 = f - f_1$. Since $Tf = Tf_1 + Tf_2$, we have $|Tf| > t \Longrightarrow |Tf_1| > t/2$ or $|Tf_2| > t/2$, and thus

$$|\{|Tf| > t\}| \le |\{|Tf_1| > t/2\}| + |\{|Tf_2| > t/2\}| \le \frac{2C_1}{t} \|f_1\|_{L^1} + \frac{2^q C_q^q}{t^q} \|f_2\|^q_{L^q}. \tag{1.6}$$

Therefore,

$$\|Tf\|_{L^p}^p = p\int t^{p-1}|\{|Tf| > t\}| \leq 2pC_1 \int t^{p-2}\|f_1\|_{L^1} + 2^q p C_q^q \int t^{p-q-1}\|f_2\|_{L^q}^q. \tag{1.7}$$

Next, if F is the distribution function of f, then the distribution function of f_1 is $\begin{cases} F(\alpha), & \text{if } \alpha \geq t \\ F(t), & \text{if } \alpha < t, \end{cases}$ and the one of f_2 is $\begin{cases} 0, & \text{if } \alpha \geq t \\ F(\alpha) - F(t), & \text{if } \alpha < t \end{cases}$. Thus

$$\|f_1\|_{L^1} = tF(t) + \int_t^\infty F(\alpha)d\alpha \quad \text{and } \|f_2\|_{L^q}^q = q\int_0^t \alpha^{q-1}F(\alpha)d\alpha - t^q F(t). \tag{1.8}$$

By combining (1.6) and (1.8) and applying Fubini's theorem (to interchange the order of integration over α and t), we find that

$$\|Tf\|_{L^p}^p \leq p\left(\frac{2C_1}{p-1} + \frac{2^q C_q^q}{q-p}\right)\|f\|_{L^q}^q. \tag{1.9}$$

□

Remark 2. *We see that the estimate we obtain for the norm of T from L^p into L^p blows up as $p \to 1$ or $p \to q$. This is not a weakness of the proof. If this norm remains bounded as, say, $p \to 1$, then T must continuous from L^1 into L^1, which may not be the case.*

There is a way to improve the estimate (1.9): instead of cutting f at height t, we cut it at height at, where $a > 0$ is fixed. The above computations yield this time :

$$\|Tf\|_{L^p}^p \leq p\|f\|_{L^p}^p\left(\frac{2C_1}{p-1}a^{1-p} + \frac{2^q C_q^q}{q-p}a^{q-p}\right). \tag{1.10}$$

Optimizing the above r. h. s. over $a > 0$ (it is minimal when $a = 1/2(C_1/C_q)^{1/(q-1)}$), we find the following

Theorem 2. *With the notations and under the hypotheses of the preceding theorem, let $\theta \in (0,1)$ be the (unique) number s. t. $\dfrac{1}{p} = \dfrac{\theta}{1} + \dfrac{1-\theta}{q}$. Then the norm of T from L^p into L^p satisfies*

$$\|T\|_{L^p \to L^p} \leq c_{p,q}\|T\|_{L^1 \to L_w^1}^{\theta}\|T\|_{L^p \to L_w^p}^{1-\theta}. \tag{1.11}$$

This conclusion is reminiscent from the one of the Riesz-Thorin convexity theorem.

Chapter 2

Hardy's inequality

We present two (equivalent) forms of Hardy's inequality. The first one generalizes the usual (and historically first) Hardy's inequality $\int_0^\infty \frac{F^2(x)}{x^2}dx \le 4\int_0^\infty (F'(x))^2dx,\ F \in C_0^\infty((0,\infty))$. The second one will be needed later in the study of the Lorentz spaces.

Theorem 3. (**Hardy**) *Let* $1 \le p < \infty$ *and* $r > 0$ *and let* $f : (0,\infty) \to \mathbb{R}$.
If $\int_0^\infty |f(x)|^p x^{p-r-1}dx < \infty$, *then* $f \in L^1_{loc}([0,\infty))$.
With $F(x) = \int_0^x f(t)dt$, *we have*

$$\int_0^\infty |F(x)|^p x^{-r-1}dx \le \left(\frac{p}{r}\right)^p \int_0^\infty |f(x)|^p x^{p-r-1}dx. \tag{2.1}$$

Proof. In view of the conclusions, we may assume $f \ge 0$. In this case, it suffices to prove (2.1). We want to apply Jensen's inequality in order to estimate the integral $\left(\int_0^x f(t)dt\right)^p$. We consider, on $(0,x)$, the normalized measure $\mu = \frac{r}{p}x^{-r/p}t^{r/p-1}dt$. Then (with $u \mapsto u^p$ playing the role of the convex function)

$$\left(\int_0^x f(t)dt\right)^p = \left(\frac{p}{r}\right)^p x^r\left(\int_0^x f(t)t^{1-r/p}d\mu\right)^p \le \left(\frac{p}{r}\right)^p x^r \int_0^x [f(t)t^{1-r/p}]^p d\mu, \tag{2.2}$$

which yields

$$\left(\int_0^x f(t)dt\right)^p \le \left(\frac{p}{r}\right)^{p-1} x^{r(1-1/p)} \int_0^x f^p(t) t^{p-r-1+r/p} dt. \tag{2.3}$$

Thus

$$\int_0^\infty F^p(x) x^{-r-1} dx \le \left(\frac{p}{r}\right)^{p-1} \int_0^\infty x^{-1-r/p} \int_0^x f^p(t) t^{p-r-1+r/p} dt dx = \left(\frac{p}{r}\right)^p \int_0^\infty f^p(t) t^{p-r-1} dt, \tag{2.4}$$

by Fubini's theorem. □

Corollary 3. *(***Hardy***) With* $1 \le p < \infty$ *and* $r > 0$*, we have*

$$\int_0^\infty \left|\int_x^\infty f(t)dt\right|^p x^{r-1} dx \le \left(\frac{p}{r}\right)^p \int_0^\infty |f(x)|^p x^{p+r-1} dx. \tag{2.5}$$

Proof. We may assume $f \ge 0$. We apply the preceding theorem to the map g given by $g(t) = t^{-2} f(t^{-1})$ and find that

$$\int_0^\infty \left(\int_0^x t^{-2} f(t) dt\right)^p x^{-r-1} dx \le \left(\frac{p}{r}\right)^p \int_0^\infty f^p(x^{-1}) x^{-p-r-1} dx = \left(\frac{p}{r}\right)^p \int_0^\infty f^p(y) y^{p+r-1} dx. \tag{2.6}$$

We next perform, in the integral $\int_0^x t^{-2} f(t) dt$, the substitution $t = s^{-1}$, next we substitute, in the first integral in (2.6), $y = x^{-1}$, and obtain the desired result. □

Chapter 3

Coverings

3.1 The Vitali covering lemma (simplified version)

Let $\mathcal{F}$ be a finite family of balls in $\mathbb{R}^N$.

Lemma 1. *(***Vitali's lemma***). $\mathcal{F}$ contains a subfamily $\mathcal{F}'$ of disjoint balls such that*

$$\sum_{B\in\mathcal{F}'} |B| \geq C|\bigcup_{B\in\mathcal{F}} B|.$$

Here, C depends only on the space dimension N, not on the family $\mathcal{F}$.

Proof. Let B_1 be the largest ball in $\mathcal{F}$. Let B_2 be the largest ball in $\mathcal{F}$ that does not intersect B_1, B_3 the largest ball in $\mathcal{F}$ that does not intersect neither B_1 nor B_2, and so on. Let $\mathcal{F}' = \{B_1, B_2, ...\}$. Note that, for each $B \in \mathcal{F}$, there is some j s. t. $B \cap B_j \neq \emptyset$. For each ball B in $\mathcal{F}'$, let $\tilde{B}$ be the ball having the same center as B and the radius thrice the one of B. We claim that $\bigcup_{B\in\mathcal{F}} B \subset \bigcup_{B\in\mathcal{F}'} \tilde{B}$. Indeed, let $B \in \mathcal{F}$ and let j be the smallest integer such that $B \cap B_j \neq \emptyset$. Since $B \cap B_{j-1} = \emptyset$, the radius of B is at most the one of B_j, for otherwise we would have picked B instead of B_j at step j in the construction of $\mathcal{F}'$. Since $B \cap B_j \neq \emptyset$, we find that $B \subset \tilde{B}_j$.
It follows that

$$|\bigcup_{B\in\mathcal{F}} B| \leq |\bigcup_{B\in\mathcal{F}'} \tilde{B}| \leq 3^N \sum_{B\in\mathcal{F}'} |B|, \tag{3.1}$$

which is the desired result with $C = 3^{-N}$. □

3.2 Whitney's covering

Throughout this section, the norm we consider on $\mathbb{R}^N$ is the $\|\cdot\|_\infty$ one.

Let $F \subset \mathbb{R}^N$ be a not empty closed set and let $\mathcal{O} = \mathbb{R}^N \setminus F$. If C is a (closed) cube, let $l(C)$ be its size, i. e. the length of its edges.

Lemma 2. *(**Whitney's covering lemma**) There is a family $\mathcal{F}$ of closed cubes s. t.*
a) $\mathcal{O} = \bigcup_{C \in \mathcal{F}} C$;
b) distinct cubes in $\mathcal{F}$ have disjoint interiors;
c) $c^{-1} l(C) \le \operatorname{dist}(C, F) \le c\, l(C)$ for each $C \in \mathcal{F}$.

Here, c depends only on N.

Proof. We may assume that $0 \in F$. For $j \in \mathbb{Z}$, let $\mathcal{F}_j$ be the grid of cubes of size 2^j, with sides parallel to the coordinate axes, s. t. 0 be one of the vertices. Note that each cube $C \in \mathcal{F}_j$ is contained in exactly one predecessor $C' \in \mathcal{F}_{j+1}$. In addition, each cube has an ancestor containing 0. Thus, the non increasing sequence $\operatorname{dist}(C, F)$, $\operatorname{dist}(C', F)$, $\operatorname{dist}(C'', F), \ldots$, becomes 0 starting with a certain range. We throw away all the cubes contained in $\bigcup_j \mathcal{F}_j$ s. t. $\operatorname{dist}(C, F) \le l(C)$ and call $\mathcal{F}$ the family of all kept cubes C s. t. their predecessors C' were thrown away.
Note that, by definition, if $C \in \mathcal{F}$, then $\operatorname{dist}(C, F) > l(C)$, while there are some $y \in C'$ and $z \in F$ s. t. $\|y - z\|_\infty \le 2l(C)$. Let $x \in C$; then $\operatorname{dist}(C, F) \le \|x - z\|_\infty \le 3l(C)$, so that c) holds with $c = 3$.
Let $x \in \mathcal{O}$. If j is sufficiently close to $-\infty$, we have $\operatorname{dist}(C, F) > l(C)$ whenever $C \in \mathcal{F}_j$ and $x \in C$. Pick any such j and C and set $k = \sup\{l \in \mathbb{N}\ ;\ \operatorname{dist}(C^{(l)}, F) > l(C^{(l)})\}$. Then k is finite and it is clear from the definition that $x \in C^{(k)} \in \mathcal{F}$. Thus a) holds.
Finally, if $C, D \in \bigcup_j \mathcal{F}_j$ are distinct cubes s. t. $\mathring{C} \cap \mathring{D} \neq \emptyset$, then one of these cubes is contained in the other one. Assume, e. g., that $C \subset D$. Then $C' \subset D$. Therefore, we can not have at the same time $C \in \mathcal{F}$ and $D \in \mathcal{F}$, for otherwise $l(C') \ge \operatorname{dist}(C', F) \ge \operatorname{dist}(D, F) > l(D) \ge l(C')$. □

For a cube C, let C_* be the cube concentric with C and of size $3/2 l(C)$.

Proposition 4. *Let F, $\mathcal{O}$ and $\mathcal{F}$ be as in the proof of the above lemma. Then:*
a) $\mathcal{O} = \bigcup_{C \in \mathcal{F}} \mathring{C}_*$;
b) we have $d^{-1} l(C_*) \le \operatorname{dist}(C_*, F) \le d\, l(C_*)$ for each $C \in \mathcal{F}$;
c) if $x \in C_*$, then $e^{-1} \operatorname{dist}(x, F) \le \operatorname{dist}(C_*, F) \le e \operatorname{dist}(x, F)$;
d) each point $x \in \mathcal{O}$ belongs to at most M cubes C_*.

Here, d, e and M depend only on N.

Proof. On the one hand, we have $\operatorname{dist}(C_*, F) \le \operatorname{dist}(C, F) \le 3l(C) = 2l(C_*)$. On the other hand, if $x \in C_*$, then there is some $y \in C$ s. t. $\|x - y\|_\infty \le 1/2 l(C)$. In addition, $\operatorname{dist}(y, F) > l(C)$.

Thus $\text{dist}(x, F) \geq 1/2l(C) = 1/3l(C_*)$. Thus b) holds with $d = 3$. Property c) is a straightforward consequence of b).
Clearly, $C_* \subset \mathcal{O}$, by b), which implies a) with the help of a) of Whitney's lemma.
If $x \in C_*$, then (by b) and c)) $2/3(de)^{-1}\text{dist}(x, F) \leq l(C)$, and therefore $C_* \subset B(x, r)$, with $r = de \ \text{dist}(x, F)$. Thus, if k is the number of cubes C_* containing x, we have

$$(r/2)^N = |B(x,r)| \geq |\bigcup_{C_* \cap F \neq \emptyset} C_*| \geq |\bigcup_{C_* \cap F \neq \emptyset} C| = \sum_{C_* \cap F \neq \emptyset} |C| \geq k(1/3(de)^{-1}\text{dist}(x, F))^N,$$

whence conclusion d). □

Proposition 5. *With the above notations, there is, in $\mathcal{O}$, a partition of the unit* $1 = \sum_{C \in \mathcal{F}} \varphi_C$ *s. t.:*

a) for each C, $\text{supp } \varphi_C \subset C_*$*;*
b) $|\partial^\alpha \varphi_C(x)| \leq C_\alpha |C|^{-|\alpha|/N} \leq C'_\alpha \text{dist}(x, F)^{-|\alpha|}$ *when* $x \in \text{supp } \varphi_C$.

Here, the constants C_α do not depend on $\mathcal{O}$, x and C.

Proof. Fix a function $\zeta \in C^\infty(\mathbb{R}^N; [0, 1])$ s. t. $\zeta = 1$ in $B(0, 1/2)$ and $\text{supp } \zeta \subset B(0, 3/4)$. If $C \in \mathcal{F}$ is of size $2l$ and center x, set $\zeta_C = \zeta((\cdot - x)/l)$. Note that $\text{supp } \zeta_C \subset C_*$ and that $\zeta_C = 1$ in C. Moreover, $|\partial^\alpha \varphi_C| \leq C_\alpha l^{-|\alpha|}$. Set $\varphi = \sum \zeta_C$, which satisfies $1 \leq \varphi \leq M$, by a) in Whitney's lemma and d) in the above proposition. Finally, set $\varphi_C = \zeta_C/\varphi$. Properties a) and b) follow immediately by combining the conclusions of Whitney's lemma and of the above proposition. □

Chapter 4

The maximal function

If f is locally integrable, we define the (**uncentered**) **maximal function** of f,

$$\mathcal{M}f(x) = \sup\{ \frac{1}{|B|} \int_B |f(y)|dy \,; B \text{ ball containing } x \}. \tag{4.1}$$

In this definition, one may consider cubes instead of balls. This will affect the value of $\mathcal{M}f$, but not its size. E.g., if we consider instead

$$\mathcal{M}'f(x) = \sup\{ \frac{1}{|Q|} \int_Q |f(y)|dy \,; Q \text{ cube containing } x \}, \tag{4.2}$$

then we have $C^{-1}\mathcal{M}'f \leq \mathcal{M}f \leq C\mathcal{M}'f$, where C is the ratio of the volumes of the unit cube and of the unit ball. Thus the integrability properties of $\mathcal{M}f$ remain unchanged if we change the definition. Similarly, one may consider balls centered at x; this definition yields the **centered maximal function**.
A basic property of $\mathcal{M}f$ is that it is lower semi continuous, i.e. the level sets $\{ x \,; \mathcal{M}f(x) > t \}$ are open.

4.1 Maximal inequalities

When $f \in L^\infty$, we clearly have $\mathcal{M}f \in L^\infty$. However, it is not obvious whether, for $1 \leq p < \infty$ and $f \in L^p$, the maximal function has some integrability properties or even whether it is finite a.e.

Theorem 4. *(***Hardy-Littlewood-Wiener***) Let $1 \leq p \leq \infty$ and $f \in L^p$. Then:*
a) $\mathcal{M}f$ is finite a.e.;
b) if $1 < p \leq \infty$, then $\mathcal{M}f \in L^p$ and $\|\mathcal{M}f\|_{L^p} \leq C\|f\|_{L^p}$;

c) if $p = 1$, then $\mathcal{M}f \in L^1_w$ and $\|\mathcal{M}f\|_{L^1_w} \leq C\|f\|_{L^1}$, i.e. $|\{\, x\,;\, \mathcal{M}f(x) > t\,\}| \leq \dfrac{C\|f\|_{L^1}}{t}$ for each $t > 0$.

Here, C denotes a constant independent of f.

Proof. When $p = \infty$, the statement is clear and we may take $C = 1$. Let next $p = 1$. We fix some $t > 0$. Let $\mathcal{O} = \{\, x\,;\, \mathcal{M}f(x) > t\,\}$, which is an open set. Thus $|\mathcal{O}| = \sup\{\, |K|\,;\, K \text{ compact } \subset \mathcal{O}\,\}$. Let K be any compact in $\mathcal{O}$. From the definition of $\mathcal{M}f$, for each $x \in K$ there is some ball B containing x such that $\dfrac{1}{|B|}\displaystyle\int_B |f(y)|dy > t$. These balls cover K, so that we may extract a finite covering. Using Vitali's lemma, we may find a finite family $\mathcal{F}' = \{B_j\}$ such that

$$B_j \cap B_k = \emptyset \ \text{ for } j \neq k,\ \frac{1}{|B_j|}\int_{B_j} |f(x)|dx > t,\ \sum_j |B_j| \geq C|K|. \tag{4.3}$$

Thus

$$\|f\|_{L^1} \geq \int_{\bigcup_j B_j} |f(x)|dx = \sum_j \int_{B_j} |f(x)|dx \geq t\sum_j |B_j| \geq Ct|K|. \tag{4.4}$$

Taking the supremum over K in the above inequality, we find that $|\mathcal{O}| \leq \dfrac{C\|f\|_{L^1}}{t}$, i.e. the property c). Letting $t \to \infty$, we find a) for $p = 1$.

We now prove a) for $1 < p < \infty$. Let $f \in L^p$. We split f as $f = f_1 + f_2$, where $f_1(x) = \begin{cases} f(x), & \text{if } |f(x)| \geq 1 \\ 0, & \text{if } |f(x)| < 1 \end{cases}$ and $f_2 = f - f_1$. Then $f_1 \in L^1$ and $f_2 \in L^\infty$. Since $\mathcal{M}f \leq \mathcal{M}f_1 + \mathcal{M}f_2$, we obtain a).

Finally, we prove b) for $1 < p < \infty$. Let $t > 0$. We use a splitting of f similar to the above one: $f = f_1 + f_2$, with $f_1(x) = \begin{cases} f(x), & \text{if } |f(x)| \geq t/2 \\ 0, & \text{if } |f(x)| < t/2 \end{cases}$ and $f_2 = f - f_1$. Then $\mathcal{M}f_2 \leq \|f_2\|_{L^\infty} \leq \dfrac{t}{2}$. It follows that $\mathcal{M}f(x) > t \Rightarrow \mathcal{M}f_1(x) > \dfrac{t}{2}$. Therefore

$$|\{\, x\,;\, \mathcal{M}f(x) > t\,\}| \leq |\{\, x\,;\, \mathcal{M}f_1(x) > \frac{t}{2}\,\}| \leq \frac{2C\|f_1\|_{L^1}}{t}, \tag{4.5}$$

using c). We find that

$$\|\mathcal{M}f\|^p_{L^p} = p\int_0^\infty t^{p-1}|\{\, x\,;\, \mathcal{M}f(x) > t\,\}|dt \leq C\int_0^\infty t^{p-2}\|f_1\|_{L^1}dt. \tag{4.6}$$

If F is the distribution function of f, then we have

$$\|f_1\|_{L^1} = \int_0^\infty |\{\, x\,;\, |f_1(x)| > s\,\}| ds = \frac{t}{2} F(\frac{t}{2}) + \int_{t/2}^\infty F(s) ds. \tag{4.7}$$

Thus

$$\int_0^\infty t^{p-2} \|f_1\|_{L^1} dt = \frac{1}{2} \int_0^\infty t^{p-1} F(\frac{t}{2}) dt + \int_0^\infty t^{p-2} \int_{t/2}^\infty F(s) ds dt = Cp \int_0^\infty t^{p-1} F(t) dt = C\|f\|_{L^p}^p, \tag{4.8}$$

by Fubini's theorem. □

The constant in the last line of computations equals $\frac{2^{p-1}}{p} + \frac{1}{p(p-1)}$. We obtain the following

Corollary 4. *For $1 < p \le 2$ we have $\|\mathcal{M}f\|_{L^p} \le \frac{C}{p-1}\|f\|_{L^p}$ for some constant C depending only on N.*

Remark 3. *The maximal function is* **never** *in L^1 (except when $f = 0$). Indeed, if $f \ne 0$, there is some $R > 0$ s. t. $\int_{B(0,R)} |f| > 0$. Then, for $|x| \ge R$, we have $\mathcal{M}f(x) \ge \frac{1}{|B(x, 2|x|)|} \int_{B(0,R)} |f| \ge \frac{C}{|x|^N}$ and thus $Mf \notin L^1$.*

Remark 4. *By the above remark, if $f \in L^1$, $f \ne 0$, the best we can hope is that $\mathcal{M}f \in L^1_{loc}$. However, this may not be true. Indeed, let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{1}{x \ln^2 x} \chi_{[0,1/2]}$. Then $f \in L^1$. However, for $x \in (0, 1/2)$, $\mathcal{M}f(x) \ge \frac{1}{x} \int_0^x f(t) dt = \frac{1}{x|\ln x|}$, so that $\mathcal{M}f \notin L^1_{loc}$.*

4.2 Lebesgue's differentiation theorem

Theorem 5. *(**Lebesgue-Besicovitch**) If $f \in L^1_{loc}$, then for a.e. $x \in \mathbb{R}^N$ we have*

$$\lim_{r\to 0} \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} f(y)dy = f(x).$$

Proof. We start by recalling the following simple measure theoretic

Lemma 3. *(**Borel-Cantelli**) Let (A_n) be a sequence of measurable sets such that $\sum_n |A_n| < \infty$. Then $|\overline{\lim} A_n| = 0$, where $\overline{\lim} A_n = \bigcap_n \bigcup_{m\geq n} A_m$.*

Let $f(x,r) = \frac{1}{|B(x,r)|} \int\limits_{B(x,r)} f(y)dy$. The conclusion of the theorem being local, it suffices to prove it with f replaced by $f\varphi$ for any compactly supported smooth function φ. We may thus assume that $f \in L^1$. Let $n \geq 1$ and let f_n be a smooth compactly supported function such that $\|f - f_n\|_{L^1} \leq \frac{1}{2^n}$. Let also $g_n = f - f_n$. Since f_n is uniformly continuous, there is some δ_n such that $|f_n(x,r) - f_n(x)| \leq \frac{1}{n}$ for $r \leq \delta_n$ and $x \in \mathbb{R}^N$. Thus, if for some $r \leq \delta_n$ we have $|f(x,r) - f(x)| > \frac{2}{n}$, then we must have $|g_n(x,r) - g_n(x)| > \frac{1}{n}$, so that either $|g_n(x)| > \frac{1}{2n}$ or $|g_n(x,r)| > \frac{1}{2n}$. In the latter case, we have $\mathcal{M}g_n(x) > \frac{1}{2n}$. Therefore

$$\{\, x\,;\, |f(x,r) - f(x)| > \frac{2}{n} \text{ for some } r \leq \delta_n \,\} \subset A_n = \{\, x\,;\, |g_n(x)| > \frac{1}{2n} \text{ or } \mathcal{M}g_n(x) > \frac{1}{2n} \}. \quad (4.9)$$

By the maximal and Chebysev's inequalities, we find that $|A_n| \leq \frac{Cn}{2^n}$. If $x \notin \overline{\lim} A_n$, then clearly $\lim_{r\to 0} f(x,r) = f(x)$. The theorem follows from the above lemma, since $\sum_n \frac{n}{2^n} < \infty$. □

The same argument yields the following variants of the differentiation theorem:

Theorem 6. *If $f \in L^1_{loc}$, then for a.e. $x \in \mathbb{R}^N$ we have*

$$\lim_{x\in Q,\, |Q|\to 0} \frac{1}{|Q|} \int\limits_Q f(y)dy = f(x). \quad (4.10)$$

Here, we may choose the Q's to be balls or cubes (or, more generally, balls for some norm).

Theorem 7. *If $f \in L^1_{loc}$, then for a.e. $x \in \mathbb{R}^N$ we have*

$$\lim_{x \in Q, |Q| \to 0} \frac{1}{|Q|} \int_Q |f(y) - f(x)| dy = 0. \tag{4.11}$$

We end this section with two simple consequences of the maximal inequalities and of the above theorem:

Corollary 5. *Let $f \in L^1_{loc}$. Then $\mathcal{M}f \geq |f|$ a.e.*

Corollary 6. *Let $1 < p \leq \infty$. Then $\|f\|_{L^p} \leq \|\mathcal{M}f\|_{L^p} \leq C\|f\|_{L^p}$.*

4.3 Pointwise inequalities for convolutions

Proposition 6. *Let φ be such that $|\varphi| \leq g$ for some $g \in L^1$, g non increasing and radially symmetric. Then*

$$|f * \varphi(x)| \leq \|g\|_{L^1} \mathcal{M}f(x). \tag{4.12}$$

Proof. Since $|f * \varphi| \leq |f| * g$, it suffices to prove the proposition for $|f|$ and g. We start with a special case: we assume g to be piecewise constant; the general case will follow by approximation, using e.g. the Beppo Levi theorem. We assume thus that there is a sequence of radii $r_1 < r_2 < ...$ and a sequence of non negative numbers $a_1, a_2, ...$ such that $g = \sum_{k \geq j} a_k$ on $B(0, r_j)$. Then

$$\int_{\mathbb{R}^N} |f(x-y)||g(y)| dy = \sum_j a_j \int_{B(0,r_j)} |f(x-y)| dy \leq \sum_j a_j |B(0, r_j)| \mathcal{M}f(x) = \|g\|_{L^1} \mathcal{M}f(x), \tag{4.13}$$

which is the desired estimate. □

Let $\varphi \in \mathcal{S}(\mathbb{R}^N)$. For $t > 0$, let $\varphi_t(x) = t^{-N}\varphi(\frac{x}{t})$. As a consequence of the above proposition, we derive the following

Corollary 7. *We have*

$$|f * \varphi_t(x)| \leq C\mathcal{M}f(x). \tag{4.14}$$

Here, C depends only on φ, not on t or f.

Proof. Since $\varphi \in \mathcal{S}$, we have $|\varphi(x)| \leq g(x) = \dfrac{C}{1 + |x|^{N+1}}$. Then clearly $\varphi_t \leq g_t$. Since g is in L^1 and decreasing, so is g_t. Moreover, we have $\|g_t\|_{L^1} = \|g\|_{L^1}$. The corollary follows now from the above proposition. □

Chapter 5

The Calderón-Zygmund decomposition

If $f \in L^1$, then the set where f is large is relatively small, i.e. $|\{ x\,;\, |f(x)| > t \}| \leq \dfrac{\|f\|_{L^1}}{t}$. The following result provides a nice covering of this set.

Theorem 8. *(The* **Calderón-Zygmund decomposition***) Let $f \in L^1(\mathbb{R}^N)$ and $t > 0$. Then there is a sequence of disjoint cubes (C_n) such that:*
a) $|f(x)| \leq t$ a. e. in $\mathbb{R}^N \setminus (\bigcup\limits_n C_n)$;
b) for each n we have $C^{-1}t \leq \dfrac{1}{|C_n|}\displaystyle\int\limits_{C_n} |f(x)|dx \leq Ct$;
c) $\displaystyle\sum_n |C_n| \leq \frac{C\|f\|_{L^1}}{t}$.

Here, C depends only on the space dimension N, not on f or t.

Proof. The construction looks like the Whitney decomposition. Fix some $l > 0$ such that $l^N > \dfrac{\|f\|_{L^1}}{t}$. We cover $\mathbb{R}^N$ with disjoint cubes of size l. We call $\mathcal{F}_1$ the family of all these cubes. We bisect the cubes in $\mathcal{F}_1$ and call $\mathcal{F}_2$ the family of cubes obtained in this way. We keep bisecting and obtain in the same way the families $\mathcal{F}_j$, $j \geq 2$. We start by throwing all the cubes in $\mathcal{F}_1$. For $j \geq 2$, we keep a cube C in $\mathcal{F}_j$ if all its ancestors were thrown and $\dfrac{1}{|C|}\displaystyle\int\limits_C |f(x)|dx > t$. Let $\mathcal{F} = (C_n)$ be the family of all kept cubes and $A = \bigcup\limits_n C_n$. If $x \notin A$, then all the cubes containing x were thrown. Thus $|f(x)| \leq t$ a.e. in $\mathbb{R}^N \setminus A$, by the Lebesgue differentiation theorem. Let now

$C \in \mathcal{F}$. If $j \geq 3$, then the (unique) cube Q in $\mathcal{F}_{j-1}$ containing C was thrown, so that

$$\frac{1}{|C|}\int\limits_C |f(x)|dx \leq \frac{1}{|C|}\int\limits_Q |f(x)|dx = \frac{2^N}{|Q|}\int\limits_Q |f(x)|dx \leq 2^N t. \tag{5.1}$$

This inequality holds also for $j = 2$, by our choice of l. Thus b) holds with $C = 2^N$. Finally, c) follows from

$$\|f\|_{L^1} \geq \sum_n \int\limits_{C_n} |f(x)|dx \geq C^{-1} \sum_n |C_n| t. \tag{5.2}$$

□

A variant of the above theorem is the following

Theorem 9. *Let $f \in L^1(\mathbb{R}^N)$ and $t > 0$. Let (C_n) as above. Then $f = g + \sum_n h_n$, where:*

a) $g \in L^1$, $|g| \leq Ct$ a. e. and $g = f$ in $\mathbb{R}^N \setminus (\bigcup_n C_n)$;

b) supp $h_n \subset C_n$*;*

c) for each n we have $\int_{C_n} h_n(x)dx = 0$;

d) for each n we have $\frac{1}{|C_n|}\int\limits_{C_n} |h_n(x)|dx \leq Ct$;

e) $\|g\|_{L^1} + \sum_n \|h_n\|_{L^1} \leq C\|f\|_{L^1}$.

Proof. Let $g(x) = \begin{cases} f(x), & \text{if } x \notin A \\ \frac{1}{|C_n|}\int\limits_{C_n} f(y)dy, & \text{if } x \in C_n \end{cases}$ and $h_n(x) = f(x) - \frac{1}{|C_n|}\int\limits_{C_n} f(y)dy$ for $x \in C_n$. It is easy to check that this decomposition has all the desired properties. □

Part II

Hardy and bounded mean oscillations spaces

Chapter 6

Substitutes of L^1

6.1 The space $L \log L$

As we have already seen, if $f \in L^1$, we can expect at best $\mathcal{M}f \in L^1_{loc}$, but even this could be wrong if we only assume $f \in L^1$. We present below a necessary and sufficient condition for having $\mathcal{M}f \in L^1_{loc}$.

A measurable function f belongs to $L \log L$ iff $\int |f| \ln(1 + |f|) < \infty$. The space $L \log L_{loc}$ is defined as the set of measurable functions s. t. $f_{|K} \in L \log L$ for each compact K.

Theorem 10. *Let $f \in L^1$. Then $\mathcal{M}f \in L^1_{loc} \Longleftrightarrow f \in L \log L_{loc}$.*

Remark 5. *Set $\Phi(t) = t \ln(1 + t)$, $t \geq 0$. If F is the distribution function of f, then $\int |f| \ln(1 + |f|) = \int \Phi'(t) F(t) dt$. It is easy to see that $\Phi'(\lambda s) \leq \max\{1, \lambda\} \Phi'(s)$ when $\lambda, s > 0$. Thus*

$$\int |\lambda f| \ln(1 + |\lambda f|) = \int \Phi'(t) F(t/|\lambda|) dt = |\lambda| \int \Phi'(\lambda s) F(s) ds \leq C_\lambda \int |f| \ln(1 + |f|) \tag{6.1}$$

for each $\lambda \in \mathbb{R}$. On the other hand, we have $|\{|f + g| > t\}| \leq |\{|f| > t/2\}| + |\{|f| > t/2\}|$. Therefore, if F is the distribution function of f and G the one of g, we have, with $h = f + g$,

$$\int |h| \ln(1 + |h|) \leq \int \Phi'(t)(F(t/2) + G(t/2)) dt \leq 4 \int |f| \ln(1 + |f|) + 4 \int |g| \ln(1 + |g|). \tag{6.2}$$

Thus $L \log L$ is a vector space. Similarly, $L \log L_{loc}$ is a vector space.

Proof. Assume that $f \in L \log L_{loc}$. Fix a compact $K \subset \mathbb{R}^N$. Let F be the distribution function of f and let G be the distribution function of $\mathcal{M}f_{|K}$. Note that $G \in L^\infty$. Since $\|\mathcal{M}f\|_{L^1} = \int G(t) dt$, it suffices to check that $\int_2^\infty G(t) dt < \infty$. By combining (4.5) and (4.7), we find that, with some universal constant C, we have $G(t) \le C(F(t/2) + 1/t \int\limits_{t/2}^{\infty} F(s)ds)$. Integrating this inequality, we obtain

$$\int\limits_2^{\infty} G(t)dt \le C \int\limits_1^{\infty} (\ln s + 2)F(s)ds \le c \int\limits_0^{\infty} \Phi'(s)F(s)ds = c \int |f| \ln(1 + |f|).$$

Conversely, assume that $\mathcal{M}f \in L^1_{loc}$. Since $f \in L^1$, that is $\int F(s)ds < \infty$, it suffices to prove that $\int\limits_1^{\infty} (\Phi'(s) - \Phi'(1))F(s)ds < \infty$. Let, for a fixed $t > 0$, $\mathcal{O} = \{\mathcal{M}f > t\}$. Note that c) of the Hardy-Littlewood-Wiener theorem implies that $\mathcal{O} \neq \mathbb{R}^N$. Let $\mathcal{O} = \bigcup\limits_{C \in \mathcal{F}} C$ be a Whitney covering of $\mathcal{O}$. Recall that, if $C \in \mathcal{F}$, then there is an $x \in \mathbb{R}^N \setminus \mathcal{O}$ s. t. $\mathrm{dist}(x, C) \le 3l(C)$, and thus $C \subset B(x, (3 + \sqrt{N})l(C))$. Since $\mathcal{M}f(x) \le t$, we find that $\int\limits_C |f| \le \int\limits_{B(x,(3+\sqrt{N})l(C))} |f| \le c\, l(C)^N t$, that is $\int\limits_{\{\mathcal{M}f > t\}} |f| \le ctG(t)$. Note that c does not depend on t. Since $\{|f| > t\} \subset \{\mathcal{M}f > t\}$, we find that $\int\limits_{\{|f|>t\}} |f| \le ctG(t)$. Invoking again (4.7), we obtain

$$\int\limits_t^{\infty} F(s)ds \le tF(t) + \int\limits_t^{\infty} F(s)ds = \int\limits_{\{|f|>t\}} |f| \le ctG(t), \tag{6.3}$$

so that

$$\|\mathcal{M}f\|_{L^1} \ge \int\limits_1^{\infty} G(t)dt \ge c^{-1} \int\limits_1^{\infty} \int\limits_t^{\infty} F(s)/t ds dt = c^{-1} \int\limits_1^{\infty} \ln s\ F(s)ds \ge d \int\limits_1^{\infty} (\Phi'(s) - \Phi'(1))F(s)ds, \tag{6.4}$$

for some constant d. □

6.2 The Hardy space $\mathcal{H}^1$

There is a different way to come around the difficulty that $\mathcal{M}f$ is never in L^1. Maximal functions are especially interesting because they provide pointwise estimates for convolutions. Instead of asking $\mathcal{M}f$ to be in L^1, one could ask upper bounds for convolutions convolutions to be in L^1. Here it is how it works. Fix a smooth map $\Phi \in \mathcal{S}(\mathbb{R}^N)$ s. t. $\int \Phi \neq 0$. Set, for $t > 0$, $\Phi_t = t^{-N}\Phi(\cdot/t)$. For $u \in \mathcal{S}'$, let $\mathcal{M}_\Phi u = \sup_{t>0} |u * \Phi_t|$. We define, for $1 \leq p \leq \infty$,

$$\mathcal{H}^p_\Phi = \{u \in \mathcal{S}' \ ; \ \mathcal{M}_\Phi u \in L^p\}.$$

Note that we may assume, without loss of generality, that (H1) $\int \Phi = 1$. On the other hand, $\mathcal{M}_\Phi u = \mathcal{M}_{\Phi_s} u$, since $f * (u_s)_t = f * u_{st}$. The condition $\int \Phi = 1$ reads also $\hat{\Phi}(0) = 1$; replacing, if necessary, Φ by Φ_s for appropriate s, we may assume that (H2) $1/2 \leq |\hat{\Phi}(\xi)| \leq 3/2$ for $|\xi| \leq 2$. We will always implicitly assume that the different test functions Φ, Ψ we will consider below are admissible, in the sense that they satisfy (H1) and (H2).

This definition brings nothing new when $1 < p \leq \infty$.

Proposition 7. *For $1 < p \leq \infty$, we have $\mathcal{H}^p_\Phi = L^p$ and $\|u\|_{L^p} \sim \|\mathcal{M}_\Phi u\|_{L^p}$.*

Proof. Recall that, if $u \in L^p$, then $|u * \Phi_t| \leq C\mathcal{M}u$, and thus $\mathcal{M}_\Phi u \in L^p$. Conversely, assume that $\mathcal{M}_\Phi u \in L^p$. Then the family $(u * \Phi_t)_t$ is bounded in L^p and thus contains a sequence $(u * \Phi_{t_n})$ with $t_n \to 0$, weakly-* convergent in L^p. Since, on the other hand, $u * \Phi_t \to u$ in $\mathcal{S}'$ (here, we use the assumption $\int \Phi = 1$), we find that $u \in L^p$. Now, if $u \in L^p$, then (possibly along a subsequence (t_n)), $u * \Phi_t \to u$ a. e. and thus $|u| \leq \mathcal{M}_\Phi u \leq C\mathcal{M}u$, which together with the maximal theorem implies the equivalence of norms. □

We next note some simple properties of $\mathcal{H}^1_\Phi$.

Proposition 8. *a) $u \mapsto \|\mathcal{M}_\Phi u\|_{L^1}$ is a norm on $\mathcal{H}^1_\Phi$;*
b) $\mathcal{H}^1_\Phi \subset L^1$, with continuous inclusion;
c) $\mathcal{H}^1_\Phi$ is a Banach space.

Proof. The only property to be checked for a) is that $\|\mathcal{M}_\Phi u\|_{L^1} = 0 \Longrightarrow u = 0$. If $\|\mathcal{M}_\Phi u\|_{L^1} = 0$, then $u * \Phi_t = 0$ for each t; by taking the limit in $\mathcal{S}'$ as $t \to 0$, we find that $u = 0$.
If $u \in \mathcal{H}^1_\Phi$, then the family $(u * \Phi_t)_t$ is bounded in L^1 and thus contains a sequence $(u * \Phi_{t_n})$ with $t_n \to 0$, weakly-* convergent to some Radon measure μ. As above, this implies that $u = \mu$, and thus u is a Radon measure. We will prove that $|\mu|$ is absolutely continuous with respect to the Lebesgue measure. Let $\varepsilon > 0$. Then there is a $\delta > 0$ s. t. $\int_A |\mathcal{M}_\Phi u| < \varepsilon$ whenever A is a Borel set s. t. $|A| < \delta$. If B is a Borel set s. t. $|B| < \delta$, then there is an open set $\mathcal{O}$ containing B s. t.

$|\mathcal{O}| < \delta$. Then

$$|\mu|(B) \le |\mu|(\mathcal{O}) = \sup\{|\int \varphi \, d\mu| \; ; \; \varphi \in C_0(\mathcal{O}), \; |\varphi| \le 1\} \le \int_{\mathcal{O}} \mathcal{M}_\Phi u < \varepsilon, \tag{6.5}$$

since

$$|\int \varphi \, d\mu| = \lim |\int u * \Phi_{t_n} \varphi| \le \int \mathcal{M}_\Phi u |\varphi| \le \int_{\mathcal{O}} \mathcal{M}_\Phi u. \tag{6.6}$$

Thus $u \in L^1$. Moreover, $|u| \le \mathcal{M}_\Phi u$, since the Lebesgue-Besicovitch differentiability theorem implies, for a. e. $x \in \mathbb{R}^N$, that

$$|u(x)| = \lim_{x\in B, \; |B|\to 0} \frac{1}{|B|} \int_B |u| = \lim_{x\in B, \; |B|\to 0} \frac{1}{|B|} |\mu|(B) \le \lim_{x\in B, \; |B|\to 0} \frac{1}{|B|} \int_B \mathcal{M}_\Phi u = \mathcal{M}_\Phi u(x); \tag{6.7}$$

here, the limit is taken over all the balls. In conclusion, $\|u\|_{L^1} \le \|\mathcal{M}_\Phi u\|_{L^1}$, which implies b).

In order to prove that $\mathcal{H}^1_\Phi$ is a Banach space, it suffices to check that an absolutely convergent series has a sum in $\mathcal{H}^1_\Phi$. Assume that $\sum \|\mathcal{M}_\Phi f_n\|_{L^1} < \infty$. Then $\sum \|f_n\|_{L^1} < \infty$ and thus $\sum f_n$ converges in L^1 to some f. Clearly, $\mathcal{M}_\Phi(f - \sum_0^k f_n) \le \sum_{k+1}^\infty \mathcal{M}_\Phi f_n \to 0$ as $k \to \infty$ and thus $\sum f_n = f$ in $\mathcal{H}^1_\Phi$. □

There are two problems with the definition of $\mathcal{H}^1_\Phi$. The first one is that this space depends, in principle, on Φ. The second one is that it is not clear at all how to check that a given function belongs to $\mathcal{H}^1_\Phi$. A partial answer to the second question will be given in the next chapter. The answer to the first question is given by the following

Theorem 11. (**Fefferman-Stein**) *Let Φ, Ψ be two admissible functions. Then $\mathcal{H}^1_\Phi = \mathcal{H}^1_\Psi$ and $\|\mathcal{M}_\Phi f\|_{L^1} \sim \|\mathcal{M}_\Psi f\|_{L^1}$ for each $f \in L^1$.*

In view of this result, we may define $\mathcal{H}^1 = \mathcal{H}^1_\Phi$ for some admissible Φ, and endow it with the norm $\|\mathcal{M}_\Phi f\|_{L^1}$.

The proof of the theorem is long and difficult; the remaining part of this chapter is devoted to it.

6.3 More maximal functions

Let $f \in L^1$ (we will always consider such f's, in view of the preceding proposition) and let $\Phi \in \mathcal{S}$. We set

$F(x,t) = |f * \Phi_t(x)|$
$F^*(x) = \mathcal{M}_\Phi f(x) = \sup\{F(x,t)\ ;\ t > 0\}$
$F_a^*(x) = \sup\{F(y,t)\ ;\ t > 0, |x-y| < at\}$ (here, $a > 0$ is fixed)
$F_{a,\varepsilon,M}^*(x) = \sup\{F(y,t)\dfrac{t^M}{(\varepsilon + t + \varepsilon|y|)^M}\ ;\ a^{-1}|x-y| < t < \varepsilon^{-1}\}$ (here, ε, M are fixed positive constants).

Finally, let, for $\alpha, \beta \in \mathbb{N}^N$, $p_{\alpha,\beta}$ be the semi norm $p_{\alpha,\beta}(\varphi) = \sup|x^\alpha \partial^\beta \varphi|$, which is finite for $\varphi \in \mathcal{S}$. We consider any **finite** family $\mathcal{F}$ of such semi norms and set
$F^{\mathcal{F}}(x) = \sup\{\mathcal{M}_\varphi f(x)\ ;\ \varphi \in \mathcal{S}, p_{\alpha,\beta}(\varphi) \le 1,\ \forall\ p_{\alpha,\beta} \in \mathcal{F}\}$
$F_{a,\varepsilon,M}^{\mathcal{F}}(x) = \sup\{|f * \varphi(y,t)|\dfrac{t^M}{(\varepsilon + t + \varepsilon|y|)^M}\ ;\ a^{-1}|x-y| < t < \varepsilon^{-1}, p_{\alpha,\beta}(\varphi) \le 1,\ \forall\ p_{\alpha,\beta} \in \mathcal{F}\}$

We note the following elementary properties
$F^* \le F_a^* \le F_b^*$ if $0 < a < b$
(*) $F^* \le c_{\Phi,\mathcal{F}} F^{\mathcal{F}}$
$\lim_{\varepsilon\to 0} F_{a,\varepsilon,M}^* = F_a^*$
$\lim_{\varepsilon\to 0} F_{a,\varepsilon,M}^* \ge F^{\mathcal{F}}$
if $\mathcal{F} \supset \mathcal{F}_0$, then $F_{\mathcal{F}} \le F_{\mathcal{F}_0}$.

The main theorem is an immediate consequence of (*) and of the following

Theorem 12. *There is a finite family $\mathcal{F}_0$ s. t.*

$$c^{-1}\int F^* \le \int F^{\mathcal{F}_0} \le c\int F^*.$$

Here, c depends on Φ and $\mathcal{F}_0$, but not on f. An immediate consequence of the theorem is the following

Corollary 8. *If $\mathcal{F} \supset \mathcal{F}_0$, then, with a constant c independent of f, we have*

$$c^{-1}\int F^* \le \int F^{\mathcal{F}} \le c\int F^*.$$

We end this section with a simple statement we will need in the proof of Theorem 5.

Lemma 4. *With a constant c depending only on N, we have*

$$\int F_{b,\varepsilon,M}^* \le c(b/a)^N \int F_{a,\varepsilon,M}^*, \quad 0 < a < b.$$

Proof. Set, for $\alpha > 0$, $\mathcal{O}_a = \{F_{a,\varepsilon,M}^* > \alpha\}$ and define similarly $\mathcal{O}_b$. Then $x \in \mathcal{O}_b$ iff there are y, t s. t. $|y-x| < bt$, $t < \varepsilon^{-1}$ and $F(y,t)\dfrac{t^M}{(\varepsilon + t + \varepsilon|y|)^M} > \alpha$. It follows immediately that $x \in B(y,bt) \subset \mathcal{O}_b$ and that $B(y,at) \subset \mathcal{O}_a$. Let now $K \subset \mathcal{O}_b$ be a fixed compact. Since the balls $B(y,bt)$ cover $\mathcal{O}_b$, we may find a finite collection of such balls that cover K. In addition, Vitali's lemma implies that we may find a finite collection of such balls, say $(B(y_i, bt_i))$, mutually disjoint and s. t. $\sum |B(y_i, bt_i)| \geq c|K|$, where K depends only on N. Since $a < b$, the corresponding balls $(B(y_i, at_i))$ are mutually disjoint and contained in $\mathcal{O}_a$. Thus $b^N \sum t_i^N \geq c|K|$, while $a^N \sum t_i^N \leq c|\mathcal{O}_a|$. We find that $|K| \leq c(b/a)^N |\mathcal{O}_a|$; by taking the sup over K, we find that

$$|\{F^*_{b,\varepsilon,M} > \alpha\}| \leq c(b/a)^N |\{F^*_{a,\varepsilon,M} > \alpha\}|. \tag{6.8}$$

The conclusion of the lemma follows by integrating the above inequality over $\alpha > 0$. □

6.4 Transition from one admissible function to another

Lemma 5. *Let $\mathcal{F}_0$ be a finite collection of semi norms. Then there is another finite collection $\mathcal{F}$ of semi norms s. t., for each $\varphi \in \mathcal{S}$, $\sup\{p_{\alpha,\beta}(\varphi)\ ;\ p_{\alpha,\beta} \in \mathcal{F}_0\} \leq c \sup\{p_{\alpha,\beta}(\hat{\varphi})\ ;\ p_{\alpha,\beta} \in \mathcal{F}\}$.*

Proof. We have, for each α, β,

$$|x^\alpha \partial^\beta \varphi(x)| = (2\pi)^{-N} \Big| \int e^{\imath x\cdot\xi} (\imath\partial)^\alpha [(\imath\xi)^\beta \hat{\varphi}](\xi) d\xi \Big| \leq c \sup (1 + |\xi|)^{N+1} |(\imath\partial)^\alpha [(\imath\xi)^\beta \hat{\varphi}]| \leq c \sup_{\mathcal{F}_{\alpha,\beta}} p_{\gamma,\delta}(\hat{\varphi}),$$

for some appropriate family $\mathcal{F}_{\alpha,\beta}$. We may thus take $\mathcal{F} = \bigcup_{p_{\alpha,\beta} \in \mathcal{F}_0} \mathcal{F}_{\alpha,\beta}$. □

Remark 6. *We may, of course, reverse the roles of φ and $\hat{\varphi}$ in the above lemma.*

Lemma 6. *Let Φ be an admissible function, let $M > 0$ and let $\mathcal{F}_0$ be a finite collection of semi norms. Then there is another finite collection $\mathcal{F}$ s. t. we may write each $\varphi \in \mathcal{S}$ as $\varphi = \sum_{k=0}^{\infty} \Phi_{2^{-k}} * \eta^k$. Here, the series is convergent in $\mathcal{S}$, the functions η^k (which depend on φ) belong to $\mathcal{S}$ and satisfy*

$$p_{\alpha,\beta}(\eta^k) \leq c 2^{-kM} \sup\{p_{\gamma,\delta}(\varphi)\ ;\ p_{\gamma,\delta} \in \mathcal{F}\}, \quad \forall\ p_{\alpha,\beta} \in \mathcal{F}_0. \tag{6.9}$$

Here, c and L do not depend on φ.

Proof. Assume (6.9) proved, for the moment. Since $p_{\alpha,\beta}(\Phi_{2^{-k}}) = 2^{k(N - |\alpha| + |\beta|)} p_{\alpha,\beta}(\Phi)$, we find, by taking M sufficiently large, that the series giving φ is convergent in $\mathcal{S}$.

In view of the preceding lemma and of the remark that follows it, it suffices to prove, for each $p_{\alpha,\beta}$ and for an appropriate $\mathcal{F}$, the estimate

$$p_{\alpha,\beta}(\widehat{\eta^k}) \leq c 2^{-kM} \sup\{p_{\gamma,\delta}(\hat{\varphi})\ ;\ p_{\gamma,\delta} \in \mathcal{F}\}. \tag{6.10}$$

Fix a function $\zeta \in C_0^\infty$ s. t. $\zeta(\xi) = 1$ when $|\xi| \leq 1$ and $\zeta(\xi) = 0$ when $|\xi| \geq 2$. Define $\zeta_0 = \zeta$ and, for $k \geq 1$, $\zeta_k(\xi) = \zeta(2^{-k}\xi) - \zeta(2^{-k+1}\xi)$. Clearly, for $k \geq 1$, $\zeta_k(\xi) = 0$ unless if $2^{k-1} \leq |\xi| \leq 2^{k+1}$, and, for each ξ, $\sum_{k=0}^{\infty}\zeta_k(\xi)=1$ (despite the series that appears here, for each ξ the sum contains at most three non vanishing terms). Thus, noting that $\widehat{\Phi_{2^{-k}}}(\xi)=\hat\Phi(2^{-k}\xi)$, we find that

$$\hat\varphi(\xi)=\sum_{k=0}^{\infty}\hat\varphi(\xi)\zeta_k(\xi)=\sum_{k=0}^{\infty}\widehat{\Phi_{2^{-k}}}(\xi)\frac{\hat\varphi(\xi)\zeta_k(\xi)}{\widehat{\Phi_{2^{-k}}}(\xi)}\equiv\sum_{k=0}^{\infty}\widehat{\Phi_{2^{-k}}}(\xi)\Psi^k(\xi). \tag{6.11}$$

We first note that Ψ^k is well-defined. Indeed, Φ being admissible, we have $1/2\le|\widehat{\Phi_{2^{-k}}}(\xi)|\le 3/2$ if $\xi\in$supp ζ_k. Moreover, ζ_k being compactly supported, so is Ψ^k. Finally, $\Psi^k\in C_0^\infty$, and thus $\Psi^k=\widehat{\eta^k}$ for some $\eta^k\in\mathcal{S}$. It remains to establish (6.10), i. e.

$$p_{\alpha,\beta}(\Psi^k)\le c2^{-kM}\sup\{p_{\gamma,\delta}(\hat\varphi)\ ;\ p_{\gamma,\delta}\in\mathcal{F}\} \tag{6.12}$$

for some appropriate $\mathcal{F}$.

Set $\Psi=1/\hat\Phi\in C^\infty(\overline{B}(0,2))$. Since for $\xi\in$supp ζ_k we have $2^{-k}\xi\in\overline{B}(0,2)$, we find, for such ξ,

$$|\partial^\beta(1/\widehat{\Phi_{2^{-k}}})(\xi)|=2^{-|\beta|k}|\partial^\beta\Psi(2^{-k}\xi)|\le c_\beta.$$

Similarly, $|\partial^\beta\zeta_k|\le c_\beta$. Therefore, for $k\ge1$ and $\xi\in$supp ζ_k

$$|\xi^\alpha\partial^\beta\Psi^k(\xi)|\le c\sum_{\gamma\le\beta}|\partial^\gamma\hat\varphi(\xi)|\le c(1+|\xi|)^{-M}\sup_{p_{\alpha,\beta}\in\mathcal{F}}p_{\alpha,\beta}(\hat\varphi)\le c2^{-kM}\sup_{p_{\alpha,\beta}\in\mathcal{F}}p_{\alpha,\beta}(\hat\varphi),$$

provided we choose $\mathcal{F}$ properly. A similar conclusion holds for $k=0$, completing the proof of the lemma. □

We set $\mathcal{C}_0=\{x\ ;\ |x|\le2\}$ and, for $j\in\mathbb{N}^*$, $\mathcal{C}_j=\{x\ ;\ 2^{j-1}\le|x|<2^j\}$.

Corollary 9. *For each $M>0$, we may choose, in the above lemma, $\mathcal{F}$ is sufficiently rich in order to have*

$$\int_{\mathcal{C}_j}|\eta^k|\le c2^{-M(k+j)} \tag{6.13}$$

provided $p_{\alpha,\beta}(\varphi)\le1$ when $p_{\alpha,\beta}\in\mathcal{F}$.

Proof. If $\mathcal{F}_0$ is sufficiently rich and φ satisfies $p_{\alpha,\beta}(\varphi)\le1$ when $p_{\alpha,\beta}$ belongs to the corresponding $\mathcal{F}$, then $\sup(1+|x|)^{N+M}|\eta^k(x)|\le c2^{-kM}$. In this case,

$$\int_{\mathcal{C}_j}|\eta^k|\le\int_{\mathcal{C}_j}(1+|x|)^{-N-M}\sup_{\mathcal{C}_j}(1+|x|)^{N+M}|\eta^k(x)|\le c2^{-M(k+j)}. \tag{6.14}$$

□

6.5 Proof of Theorem 12

Proof. **Step 1.** $F^*_{1,\varepsilon,M}$ controls $F^{\mathcal{F}}_{2,\varepsilon,M}$

Lemma 7. *Assume that $\mathcal{F}$ is sufficiently rich and that $0 < \varepsilon < 1$. Then there is a constant c, which may depend on $\mathcal{F}$ and M, but not on f or ε, s. t.*

$$\int F^{\mathcal{F}}_{2,\varepsilon,M} \leq c \int F^*_{1,\varepsilon,M}. \tag{6.15}$$

Proof. Fix an $\varepsilon > 0$. By the transition lemma, we have

$$|f * \varphi_t(z)| \leq \sum_k \int |f * \Phi_{2^{-k}t}(z-y)||\eta^k_t(y)|dy = t^{-N} \sum_k \int |f * \Phi_{2^{-k}t}(z-y)||\eta^k(y/t)|dy. \tag{6.16}$$

If $2^{-1}|z-x| < t < \varepsilon^{-1}$ and $y \in t\mathcal{C}_j$ (the sets $\mathcal{C}_j$ were defined in the preceding section), we find that $|y-x| < 2^{k+j+2}2^{-k}t$, while $2^{-k}t < \varepsilon^{-1}$. Thus

$$|f * \Phi_{2^{-k}t}(z-y)| = F(z-y, 2^{-k}t) \leq \frac{(\varepsilon + 2^{-k}t + \varepsilon|z-y|)^M}{(2^{-k}t)^M} F^*_{2^{k+j+2},\varepsilon,M}(x), \tag{6.17}$$

so that

$$|f * \Phi_{2^{-k}t}(z-y)| \leq \frac{(\varepsilon + 2^{-k}t + \varepsilon|z| + 2^j t)^M}{(2^{-k}t)^M} F^*_{2^{k+j+2},\varepsilon,M}(x). \tag{6.18}$$

We find that

$$|f * \varphi_t(z)| \leq t^{-N} \sum_{k,j} \frac{(\varepsilon + 2^{-k}t + \varepsilon|z| + \varepsilon 2^j t)^M}{(2^{-k}t)^M} F^*_{2^{k+j+2},\varepsilon,M}(x) \int_{t\mathcal{C}_j} |\eta^k(y/t)|. \tag{6.19}$$

Therefore, for each fixed $L > 0$ we have, if $\mathcal{F}$ is sufficiently rich,

$$|f * \varphi_t(z)| \leq c \sum_{k,j} \frac{(\varepsilon + 2^{-k}t + \varepsilon|z| + \varepsilon 2^j t)^M}{(2^{-k}t)^M} F^*_{2^{k+j+2},\varepsilon,M}(x) 2^{-L(k+j)}. \tag{6.20}$$

The definition of $F^{\mathcal{F}}_{2,\varepsilon,M}$ implies that, with $A = \{\varphi \ ; \ p_{\alpha,\beta}(\varphi) \leq 1, \forall \ \varphi \in \mathcal{F}\}$,

$$F^{\mathcal{F}}_{2,\varepsilon,M}(x) = \sup_{\varphi \in A;\ 2^{-1}|z-x|<t<\varepsilon^{-1}} |f * \varphi_t(z)| \frac{t^M}{(\varepsilon + t + \varepsilon|z|)^M} \leq c \sum_{k,j} 2^{M(j+2k)} F^*_{2^{k+j+2},\varepsilon,M}(x) 2^{-L(k+j)} \tag{6.21}$$

(here, we use the assumption $\varepsilon < 1$). Recalling that L is arbitrary, we take $L = 2M + N + 1$ and find that

$$\int F^{\mathcal{F}}_{2,\varepsilon,M}(x) \leq c \sum_{k,j} 2^{-Mj-(N+1)(k+j)} \int F^*_{2^{k+j+2},\varepsilon,M} \leq c \sum_{k,j} 2^{-Mj-(k+j)} \int F^*_{1,\varepsilon,M} \leq c \int F^*_{1,\varepsilon,M}. \tag{6.22}$$

□

Step 2. $\mathcal{M}_\Phi f$ controls $F^*_{1,\varepsilon,M}$

Lemma 8. *If $M > N$, then $F^*_{1,\varepsilon,M} \in L^1$.*

Proof. We note that $|f * \Phi_t| \le \|f\|_{L^1}\|\Phi_t\|_{L^\infty} \le ct^{-N}$. Thus

$$F^*_{1,\varepsilon,M}(x) \le c \sup_{|y-x|<t<\varepsilon^{-1}} \frac{t^{M-N}}{(\varepsilon + t + \varepsilon|y|)^M} \le c_\varepsilon \sup_{|y-x|<\varepsilon^{-1}} \frac{1}{(1+|y|)^M}, \tag{6.23}$$

and the latter function belongs to L^1. □

Lemma 9. *Assume that $M > N$ and that $\varepsilon < 1$. Then, with some constant c that may depend on M, but not on ε or f, we have*

$$\int F^*_{1,\varepsilon,M} \le c \int \mathcal{M}_\Phi f. \tag{6.24}$$

Proof. For each x, there are t and y s. t. $|x-y| < t < \varepsilon^{-1}$ and

$$F^*_{1,\varepsilon,M}(x) \ge F(y,t)\frac{t^M}{(\varepsilon + t + \varepsilon|y|)^M} \ge \frac{3}{4}F^*_{1,\varepsilon,M}(x).$$

Let δ be a small constant to be fixed later. We claim that, if δ is sufficiently small and $\mathcal{F}$ is sufficiently rich (i. e., as in the preceding step), then

$$|z-y| < \delta t \Longrightarrow |F(y,t)\frac{t^M}{(\varepsilon + t + \varepsilon|y|)^M} - F(z,t)\frac{t^M}{(\varepsilon + t + \varepsilon|z|)^M}| \le c\delta F^{\mathcal{F}}_{2,\varepsilon,M}(x) + \frac{1}{4}F^*_{1,\varepsilon,M}(x). \tag{6.25}$$

The above implication is an immediate consequence of the following inequalities

$$|\frac{t^M}{(\varepsilon + t + \varepsilon|y|)^M} - \frac{t^M}{(\varepsilon + t + \varepsilon|z|)^M}| \le \frac{1}{4}\frac{t^M}{(\varepsilon + t + \varepsilon|y|)^M}, \tag{6.26}$$

respectively

$$|F(y,t) - F(z,t)|\frac{t^M}{(\varepsilon + t + \varepsilon|z|)^M} \le c\delta F^{\mathcal{F}}_{2,\varepsilon,M}(x). \tag{6.27}$$

Inequality (6.26) is elementary and left to the reader (it works when $\delta < (4/3)^{1/M} - 1$). As for (6.27), we start by noting that

$$\frac{\partial f * \Phi_t}{\partial x_j} = \frac{1}{t} f * \left(\frac{\partial \Phi}{\partial x_j}\right)_t, \tag{6.28}$$

and thus

$$|f * \Phi_t(y,t) - f * \Phi_t(z,t)| \le c\frac{|y-z|}{t} \sup_{1\le j\le N} \sup_{|w-z|<\delta t} |f * \left(\frac{\partial \Phi}{\partial x_j}\right)_t|(w). \tag{6.29}$$

Assuming, without loss of generality, that $\delta < 1$, we find that

$$|F(y,t) - F(z,t)| \frac{t^M}{(\varepsilon + t + \varepsilon|z|)^M} \le c\delta F^{\mathcal{F}}_{2,\varepsilon,M}(x) \sup_{|w-z|<\delta t} \frac{(\varepsilon + t + \varepsilon|w|)^M}{(\varepsilon + t + \varepsilon|z|)^M} \le 2^M c\delta F^{\mathcal{F}}_{2,\varepsilon,M}(x), \tag{6.30}$$

whence (6.27).
For each x, one of the two happens:
either (i) $c\delta F^{\mathcal{F}}_{2,\varepsilon,M}(x) \le \frac{1}{4} F^*_{1,\varepsilon,M}(x)$,
or (ii) $c\delta F^{\mathcal{F}}_{2,\varepsilon,M}(x) > \frac{1}{4} F^*_{1,\varepsilon,M}(x)$.
Let A, respectively B, be the set of points s. t. (i), respectively (ii) holds. If $x \in A$, then we have $F(z,t) \ge \frac{1}{4} F^*_{1,\varepsilon,M}(x)$ whenever $|z-y| < \delta t$, and thus

$$\sqrt{\mathcal{M}_\Phi f(z)} \ge \sqrt{F(z,t)} \ge \frac{1}{2}\sqrt{F^*_{1,\varepsilon,M}(x)}$$

for each such z. Thus

$$\sqrt{F^*_{1,\varepsilon,M}(x)} \le \frac{c}{|\{|z-y| < \delta t\}|} \int_{\{|z-y|<\delta t\}} \sqrt{\mathcal{M}_\Phi f(z)}. \tag{6.31}$$

Noting that $\{|z-y| < \delta t\} \subset \{|z-x| < 2t\}$, we find that, in case (i),

$$\sqrt{F^*_{1,\varepsilon,M}(x)} \le \frac{c_\delta}{|\{|z-x| < 2t\}|} \int_{\{|z-x|<2t\}} \sqrt{\mathcal{M}_\Phi f(z)} \le \mathcal{M}(\sqrt{\mathcal{M}_\Phi f})(x). \tag{6.32}$$

Therefore,

$$\int_A F^*_{1,\varepsilon,M} \le c \int_A (\mathcal{M}(\sqrt{\mathcal{M}_\Phi f}))^2 \le c \int \mathcal{M}_\Phi f, \tag{6.33}$$

by the maximal inequalities. (Here, the different constants may depend on δ.)
Concerning the set B, we have

$$\int_B F^*_{1,\varepsilon,M} \le 4c\delta \int_B F^{\mathcal{F}}_{2,\varepsilon,M} \le c'\delta \int F^*_{1,\varepsilon,M}. \tag{6.34}$$

We finally fix δ sufficiently small in order to have (6.26), (6.27), $\delta < 1$ and $c'\delta < 1/2$. Then

$$\int_B F^*_{1,\varepsilon,M} \leq \frac{1}{2}\int_B F^*_{1,\varepsilon,M} + \frac{1}{2}\int_A F^*_{1,\varepsilon,M}, \tag{6.35}$$

so that

$$\int F^*_{1,\varepsilon,M} \leq c\int \mathcal{M}_\Phi f, \tag{6.36}$$

by combining (6.33) and (6.35). □

Step 3. Conclusion

By letting $\varepsilon \to 0$ in Lemma 9, we find that $\displaystyle\int F^*_1 \leq c\int \mathcal{M}_\Phi f$. Next, letting $\varepsilon \to 0$ in Lemma 9 yields $\displaystyle\int F^{\mathcal{F}} \leq c\int F^*_1$. Finally, it suffices to note that $\mathcal{M}_\Phi f \leq cF^{\mathcal{F}}$. □

Chapter 7

Atomic decomposition

7.1 Atoms

For the moment, we do not even know if $\mathcal{H}^1$ contains a non zero function! In this section, we will give examples of functions in $\mathcal{H}^1$: the atoms. In the next section, we will show that this example is "generic". To motivate the definition of atoms, we start with the following simple

Proposition 9. *If $f \in \mathcal{H}^1$, then $\displaystyle\int f = 0$.*

Proof. Argue by contradiction and assume, e. g., that $\displaystyle\int f = 1$. Pick some $R > 0$ s. t. $\displaystyle\int\limits_{B(0,R)} f > 2/3$ and $\displaystyle\int\limits_{\mathbb{R}^N \setminus B(0,R)} |f| < 1/3$. Let $\Phi \in C_0^\infty$ be s. t. $\Phi = 1$ in $B(0,1)$ and $0 \leq \Phi \leq 1$. For $x \in \mathbb{R}^N$ s. t. $|x| > R$, let $t = |x| + R$, so that $t \sim |x|$. Then

$$\mathcal{M}_\Phi f(x) \geq f * \Phi_t(x) \geq t^{-N} \int\limits_{B(0,R)} f - t^{-N} \int\limits_{\mathbb{R}^N \setminus B(0,R)} |f| \geq \frac{1}{3} t^{-N} \geq c|x|^{-N}, \tag{7.1}$$

and thus $\mathcal{M}_\Phi f \notin L^1$. □

Remark 7. *$\mathcal{H}^1$ is a strict subspace of $\{f \in L^1 \; ; \; \displaystyle\int f = 0\}$. To see this, it suffices to modify the example in Remark 4 as follows: set $f_1 : \mathbb{R} \to \mathbb{R}$, $f_1(x) = \dfrac{1}{x \ln^2 x} \chi_{[0,1/2]}$ and let $f(x) = f_1(x) - f_1(3 - x)$. Then $f \in L^1$ and $\displaystyle\int f = 0$. However, if we pick $\Phi \in C_0^\infty$ s. t. $0 \leq \Phi \leq 1$,* supp

$\Phi \subset [0,2]$ *and* $\Phi = 1$ *in* $[0,1]$*, then, for* $x \in [0,1/2]$ *we have*

$$\mathcal{M}_\Phi f(x) \geq x^{-1} \int_0^x f_1(x)dx = \frac{1}{x|\ln x|}, \tag{7.2}$$

and thus $\mathcal{M}_\Phi f \notin L^1$.

Definition 1. *An* **atom** *is a function* $a : \mathbb{R}^N \to \mathbb{R}$ *s. t.:*
(i) supp $a \subset B$*, where* B *is a ball;*
(ii) $|a| \leq |B|^{-1}$*;*
(iii) $\int a = 0$.

We may replace balls by cubes, in this definition, since if a is an atom with respect to a ball B, then $c_N a$ is an atom with respect to any minimal cube containing B and conversely; here, c_N depends only on N.

Proposition 10. *If a is an atom, then* $\mathcal{M}_\Phi f \in L^1$ *and* $\|\mathcal{M}_\Phi f\|_{L^1} \leq c$ *for some constant depending only on* Φ.

Proof. Since $\mathcal{M}_\Phi f \leq c\mathcal{M} f \leq \|f\|_{L^\infty}$, we have $\mathcal{M}_\Phi f \leq c|B|^{-1}$. Therefore,

$$\int_{B^*} \mathcal{M}_\Phi f \leq c|B^*||B|^{-1} = c2^N; \tag{7.3}$$

here, B^* is the ball concentric to B and twice larger.
If $x \notin B^*$, we use the information (iii) and find that, with R the radius of B, we have

$$|f * \Phi_t(x)| = |\int_B a(y)[\Phi_t(x-y) - \Phi_t(x)]dy| \leq |B|^{-1} \sup_{z\in B} |\nabla(\Phi_t)(x-z)| \int_B |y|dy. \tag{7.4}$$

Taking into account the fact that $|x - z| \sim |x|$ and the inequality $|\nabla\Phi(x)| \leq c|x|^{-N-1}$, we obtain $|f * \Phi_t(x)| \leq \frac{cR}{|x|^{N+1}}$, and thus $\mathcal{M}_\Phi f(x) \leq \frac{cR}{|x|^{N+1}}$. Integrating the latter inequality, we find that

$$\int_{\mathbb{R}^N \setminus B^*} \mathcal{M}_\Phi f \leq c \tag{7.5}$$

and the desired inequality follows from (7.3) and (7.5). □

Corollary 10. *Let* $f = \sum \lambda_k a_k$*, where each* a_k *is an atom and* $\sum |\lambda_k| < \infty$*. Then* $f \in \mathcal{H}^1$ *and* $\|f\|_{\mathcal{H}^1} \leq c \sum |\lambda_k|$.

More generally, we could weaken condition (ii) in the definition of an atom as follows

Definition 2. *Let $1 < q \le \infty$. A $_q$***atom** *is a function satisfying (i), (iii) and (ii') $\|a\|_{L^q} \le |B|^{1/q-1}$.*

Thus, the usual atoms are $_\infty$atoms.

Proposition 11. *If a is a $_q$atom, then $\|a\|_{\mathcal{H}^1} \le c_q$. If, in addition, $q \le 2$, then $c_q \le \dfrac{c}{q-1}$.*

Proof. We may assume that $q < \infty$. We repeat the reasoning in the preceding proposition. On the one hand, we have

$$\int_{B^*} |\mathcal{M}_\Phi f| \le c \int_{B^*} |\mathcal{M} f| \le c|B^*|^{1-1/q} \Big(\int |\mathcal{M} f|^q \Big)^{1/q} \le c_q; \tag{7.6}$$

here, we use Hölder's inequality and the maximal theorem. In addition, we see that $c_q \le \dfrac{c}{q-1}$ if $q \le 2$.
When $x \notin B^*$, we find that

$$\mathcal{M}_\Phi f(x) \le \frac{c}{|x|^{N+1}} \int_B |f(y)||y| dy \le \frac{c}{|x|^{N+1}} \|f\|_{L^q} \Big(\int_B |y|^{q'} \Big)^{1/q'} \le \frac{c_q R}{|x|^{N+1}}; \tag{7.7}$$

here, c_q remains bounded when $q \le 2$. Thus

$$\int_{\mathbb{R}^N \setminus B^*} \mathcal{M}_\Phi f \le c \tag{7.8}$$

with c independent of $q \le 2$. We conclude by combining (7.6) and (7.8). □

7.2 Atomic decomposition

The following result tells that the atoms represent "generic" $\mathcal{H}^1$ functions.

Theorem 13. (**Coifman-Latter**) *Let $f \in \mathcal{H}^1$. Then we may write $f = \sum \lambda_k a_k$, where each a_k is an atom and $\sum |\lambda_k| \sim \|f\|_{\mathcal{H}^1}$.*

Proof. It suffices to write, in the sense of distributions, $f = \sum \lambda_k a_k$, with $\sum |\lambda_k| \le c\|f\|_{\mathcal{H}^1}$. Indeed, if we are able to do this, then on the one hand the series $\sum \lambda_k a_k$ is convergent in $\mathcal{H}^1$, thus in $\mathcal{D}'$, and therefore its sum has to be f, by uniqueness of the limit. On the other hand, we always have $\| \sum \lambda_k a_k \|_{\mathcal{H}^1} \leq c \sum |\lambda_k|$.

We fix a large family $\mathcal{F}$ of semi norms as in the preceding section. Let $F^{\mathcal{F}}$ be the corresponding maximal function, i. e.,

$$F^{\mathcal{F}}(x) = F^{\mathcal{F}} f(x) = \sup\{\mathcal{M}_\Phi f(x) \ ; \ p_{\alpha,\beta}(\Phi) \leq 1, \forall\ p_{\alpha,\beta} \in \mathcal{F}\}.$$

Let, for $j \in \mathbb{Z}$, $\mathcal{O}_j = \{F^{\mathcal{F}} > 2^j\}$; clearly, $\mathcal{O}_j$ is an open set and $\mathcal{O}_{j+1} \subset \mathcal{O}_j$. In addition, $\mathcal{O}_j \neq \mathbb{R}^N$, since $F^{\mathcal{F}} \in L^1$. Set $f_j = f\chi_{\mathcal{O}_j}$.

Lemma 10. *As $j \to \infty$, $f_j \to 0$ in L^1. As $j \to -\infty$, $f_j - f \to 0$ in L^∞.*

Proof. We have $\|f_j\|_{L^1} = \int\limits_{\mathcal{O}_j} |f| \to 0$ as $j \to \infty$, since $|\mathcal{O}_j| \to 0$ as $j \to \infty$. On the other hand,

$$\|f_j - f\|_{L^\infty} = \sup_{\mathbb{R}^N \setminus \mathcal{O}_j} |f| \leq \sup_{\mathbb{R}^N \setminus \mathcal{O}_j} \mathcal{M} f \leq c \sup_{\mathbb{R}^N \setminus \mathcal{O}_j} F^{\mathcal{F}} \leq c 2^j \to 0 \tag{7.9}$$

as $j \to \infty$. □

Corollary 11. *Set $g_j = f_j - f_{j+1}$. Then $\sum\limits_{-\infty}^{\infty} g_j = f$ in the distribution sense.*

Let (C_k^j) be a Whitney covering of $\mathcal{O}_j$ and let φ_k^j be the corresponding partition of the unit in $\mathcal{O}_j$. Recall that, with $1 < a < b$ depending only on N, we have
(i) $C_k^{j*} \subset \mathcal{O}_j$ (where C_k^{j*} is the cube concentric with C_k^j and having a times its size);
(ii) $C_k^{j**} \not\subset \mathcal{O}_j$ (where C_k^{j**} is the cube concentric with C_k^j and having b times its size);
(iii) at most M cubes C_k^{j*} meet at some point, where M depends only on N;
(iv) supp $\varphi_k^j \subset C_k^{j*}$;
(v) $|\partial^\alpha \varphi_k^j| \leq c_\alpha$ size$(C_k^j)^{-\alpha}$;
(vi) $\varphi_k^j \geq 1/M$ in C_k^j.
The last property implies
(vii) $\int \varphi_k^j \sim$ size(C_k^j).

We have $f_j = \sum f\varphi_k^j$ (the series that appears is well-defined, at least in the sense of distributions, since on each compact there are finitely many non vanishing terms). We define the coefficient c_k^j by the condition $\int (f - c_k^j)\varphi_k^j = 0$. We have $f_j = \sum (f - c_k^j)\varphi_k^j + R_j$, where $R_j = \sum c_k^j \varphi_k^j$.

Lemma 11. *We have $\sum\limits_{-\infty}^{\infty}(R_j - R_{j+1}) = 0$ in the sense of distributions.*

Proof. We have

$$\int |c_k^j \varphi_k^j| = |c_k^j \int \varphi_k^j| = |\int f\varphi_k^j| \le \int_{C_k^{j*}} |f|, \tag{7.10}$$

and thus

$$\|R_j\|_{L^1} \le \sum \int_{C_k^{j*}} |f| \le M \int_{\mathcal{O}_j} |f|; \tag{7.11}$$

here, M is the constant in (iii). The series $\sum c_k^j \varphi_k^j$ and $\sum f\varphi_k^j$ being convergent in L^1, we have

$$\sum \int c_k^j \varphi_k^j = \sum \int f\varphi_k^j = \int f \sum \varphi_k^j = \int_{\mathcal{O}_j} f. \tag{7.12}$$

Thus,

$$\lim_{k\to\infty} \sum_{j=-k}^{j=k} (R_j - R_{j+1}) = \lim_{k\to\infty} \int_{\mathcal{O}_{-k}\setminus\mathcal{O}_{k+1}} f = \int_{\{F^{\mathcal{F}}>0\}} f = \int f = 0, \tag{7.13}$$

since f is vanishing in the set $\{F^{\mathcal{F}} = 0\}$. □

Corollary 12. *We have* $f = \sum_{-\infty}^{\infty} \left[\sum_k (f - c_k^j)\varphi_k^j - \sum_l (f - c_l^{j+1})\varphi_l^{j+1} \right]$ *in the sense of distributions.*

Using the fact that $\varphi_l^{j+1} = \sum_k \varphi_l^{j+1}\varphi_k^j$ (since $\mathcal{O}_{j+1} \subset \mathcal{O}_j$), we may further decompose the general term of the above series as follows

$$\sum_k (f - c_k^j)\varphi_k^j - \sum_l (f - c_l^{j+1})\varphi_l^{j+1} = \sum_k (f - c_k^j)\varphi_k^j - \sum_{k,l} [(f - c_l^{j+1})\varphi_k^j - c_{k,l}^j]\varphi_l^{j+1} - \sum_{k,l} c_{k,l}^j \varphi_l^{j+1}; \tag{7.14}$$

here, the coefficients $c_{k,l}^j$ are chosen s. t. $\int [(f - c_l^{j+1})\varphi_k^j - c_{k,l}^j]\varphi_l^{j+1} = 0$.

Actually, the last sum in (7.14) vanishes. The reason is that, for fixed l, we have, with $c = \int \varphi_l^{j+1} \ne 0$,

$$c \sum_k c_{k,l}^j = \int \sum_k c_{k,l}^j \varphi_l^{j+1} = \int \sum_k (f - c_l^{j+1})\varphi_k^j \varphi_l^{j+1} = \int (f - c_l^{j+1})\varphi_l^{j+1} = 0; \tag{7.15}$$

commuting the series with the integral in the above computations is justified by the fact that, when l is fixed, we have only finitely many non vanishing terms.

Thus $f = \sum_{j=-\infty}^{\infty} \sum_k b_k^j$, where

$$b_k^j = (f - c_k^j)\varphi_k^j - \sum_l [(f - c_l^{j+1})\varphi_k^j - c_{k,l}^j]\varphi_l^{j+1} = f\varphi_k^j \chi_{\mathbb{R}^N \setminus \mathcal{O}_{j+1}} - c_k^j \varphi_k^j + \sum_l [c_l^{j+1}\varphi_k^j \varphi_l^{j+1} + c_{k,l}^j \varphi_l^{j+1}]. \tag{7.16}$$

Let $C > 0$ be a large constant to be specified later. We set, with l_k^j the size of C_k^j, $\lambda_k^j = C(l_k^j)^N 2^j$ and $a_k^j = (\lambda_k^j)^{-1} b_k^j$, so that

$$f = \sum_{j=-\infty}^{\infty} \sum_k \lambda_k^j a_k^j; \tag{7.17}$$

this is going to be the atomic decomposition of f. Clearly, the functions a_k^j satisfy, by construction, the cancellation property (iii) required in the definition of an atom. It remains to establish three facts: a) that the support of a_k^j is contained in some ball B; b) that $|a_k^j| \le |B|^{-1}$ (here, the choice of the constant C will count); c) that $\sum_{j,k} |\lambda_k^j| \le C\|f\|_{\mathcal{H}^1}$. These information are easily obtained by combining the conclusions of the following lemmata.

Lemma 12. *There is a constant $b > 0$ depending only on N s. t.* supp $b_k^j \subset B_k^j$*, where B_k^j is the ball concentric with C_k^j and of radius $b\, l_k^j$.*

Proof. If $b_k^j(x) \ne 0$, then either $x \in$ supp $\varphi_k^j \subset C_k^{j*}$, or there is some l s. t. supp φ_k^j intersects supp φ_l^{j+1} and s. t. $\varphi_l^{j+1}(x) \ne 0$. In the latter case, we have, on the one hand, $x \in C_l^{j+1*}$. On the other hand, if $y \in$ supp $\varphi_k^j \cap$ supp $\varphi_l^{j+1} \subset C_k^{j*} \cap C_l^{j+1*}$, then

$$l_l^{j+1} \le c_1 \text{ dist } (y, \mathbb{R}^N \setminus \mathcal{O}_{j+1}) \le c_1 \text{ dist } (y, \mathbb{R}^N \setminus \mathcal{O}_j) \le c_2\, l_k^j. \tag{7.18}$$

In both cases, we may find b s. t. the conclusion of the lemma holds. □

Lemma 13. *We have $|c_k^j| \le c2^j$, and $|c_{k,l}^j| \le c2^j$. Here, c depends only on N and $\mathcal{F}$.*

Proof. By definition, we have $c_k^j = \int f\varphi_k^j / \int \varphi_k^j$. As already noted, we have $\int \varphi_k^j \sim (l_k^j)^N$. Let $x \in C_k^{j**} \setminus \mathcal{O}_j$; thus $F^{\mathcal{F}} f(x) \le 2^j$, by the definition of $\mathcal{O}_j$. Set $t = l_k^j$ and $\varphi(z) = \varphi_k^j(x - zt)$. Clearly,

$$c_k^j = \int f\varphi_k^j / \int \varphi_k^j = t^N \int f(y)\varphi_t(x-y)dy / \int \varphi_k^j \sim \int f(y)\varphi_t(x-y)dy, \tag{7.19}$$

so that

$$|c_k^j| \le cF^{\mathcal{F}} f(x) \sup\{p_{\alpha,\beta}(\varphi)\ ;\ p_{\alpha,\beta} \in \mathcal{F}\} \le c2^j \sup\{p_{\alpha,\beta}(\varphi)\ ;\ p_{\alpha,\beta} \in \mathcal{F}\}. \tag{7.20}$$

Therefore, it suffices to prove that $p_{\alpha,\beta}(\varphi) \leq c_{\alpha,\beta}$, with $c_{\alpha,\beta}$ depending only on N. We first note that, since $x \in C_k^{j**}$ and supp $\varphi_k^j \subset C_k^{j*}$, there is some constant $R > 0$ depending only on N s. t. supp $\varphi \subset B(0, R)$. Thus we may consider only the case $\alpha = 0$. Now

$$\sup |\partial^\beta \varphi| = t^{|\beta|} \sup |\partial^\beta \varphi_k^j| \leq t^{|\beta|} c_\beta (l_k^j)^{-|\beta|} = c_\beta, \tag{7.21}$$

by the properties of Whitney's partition of the unit.
The argument for $c_{k,l}^j$ is similar. Since

$$c_{k,l}^j = \int f \varphi_k^j \varphi_l^{j+1} / \int \varphi_l^{j+1} - c_l^{j+1} \int \varphi_k^j \varphi_l^{j+1} / \int \varphi_l^{j+1}, \tag{7.22}$$

we find that

$$|c_{k,l}^j| \leq |c_l^{j+1}| + |\int f \varphi_k^j \varphi_l^{j+1} / \int \varphi_l^{j+1}|; \tag{7.23}$$

the latter term appears only if $\varphi_k^j \varphi_l^{j+1} \not\equiv 0$.

It suffices to prove that, when $\varphi_k^j \varphi_l^{j+1} \not\equiv 0$, we have

$$|\int f \varphi_k^j \varphi_l^{j+1} / \int \varphi_l^{j+1}| \leq c 2^j. \tag{7.24}$$

To this purpose, we pick an $x \in C_l^{j+1**} \setminus \mathcal{O}_{j+1}$ and define, with $t = l_l^{j+1}$, $\Phi(z) = \varphi_k^j(x - zt)\varphi_l^{j+1}(x - zt)$. Since supp $\varphi_k^j \varphi_l^{j+1} \subset$ supp φ_l^{j+1}, we have (with the same R as above) supp $\Phi \subset B(0, R)$. In addition,

$$|\int f \varphi_k^j \varphi_l^{j+1} / \int \varphi_l^{j+1}| = \frac{t^N}{\int \varphi_l^{j+1}} |f * \Phi_t(x)| \leq c 2^j \sup\{p_{\alpha,\beta}(\Phi) \ ; \ p_{\alpha,\beta} \in \mathcal{F}\}. \tag{7.25}$$

Therefore, it suffices to prove that $p_{0,\beta}(\Phi) \leq c_\beta$, with c_β depending only on N. Using the properties of the Whitney decomposition, we have

$$|\partial^\beta \Phi(z)| \leq c t^{|\beta|} (l_k^j + l_l^{j+1})^{-|\beta|} \leq c_\beta, \tag{7.26}$$

since we have already noted that $l_l^{j+1} \leq c l_k^j$ if the supports of φ_k^j and φ_l^{j+1} do intersect. □

Lemma 14. *With some constant c depending only on N and of the family $\mathcal{F}$ of semi norms, we have $|f| \leq c 2^j$ in the support of b_k^j.*

Proof. In view of the second equality in (7.16), we have

$$|b_k^j| \leq |f| \chi_{\mathbb{R}^N \setminus \mathcal{O}_{j+1}} + |c_k^j| + \sum_l (|c_l^{j+1}| + |c_{k,l}^j|) \leq c\ 2^j + |f| \chi_{\mathbb{R}^N \setminus \mathcal{O}_{j+1}}. \tag{7.27}$$

The desired conclusion is obtained by noting that $|f| \leq c F^{\mathcal{F}} f$ a. e., and thus $|f| \leq c\ 2^j$ in $\mathbb{R}^N \setminus \mathcal{O}_{j+1}$. □

By combining the above results, we find immediately that a_k^j are atoms, provided we chose C sufficiently large (depending only on N and $\mathcal{F}$).
We may now conclude as follows: we have

$$\sum |\lambda_k^j| \leq c \sum_{j,k} 2^j (l_k^j)^N = c \sum_{j,k} 2^j |C_k^j| = c \sum_j 2^j |\mathcal{O}_j|. \tag{7.28}$$

On the other hand,

$$\|F^{\mathcal{F}} f\|_{L^1} = \int |\{F^{\mathcal{F}} f > \alpha\}| d\alpha \geq \sum_{-\infty}^{\infty} \int_{2^{j-1}}^{2^j} |\{F^{\mathcal{F}} f > \alpha\}| d\alpha \geq \sum_{-\infty}^{\infty} 2^{j-1} |\mathcal{O}_j|. \tag{7.29}$$

If we take $\mathcal{F}$ sufficiently rich, we find, by combining (7.28) with (7.29), that

$$\|f\|_{\mathcal{H}^1} \sim \|F^{\mathcal{F}} f\|_{L^1} \geq c \sum |\lambda_k^j|. \tag{7.30}$$

The proof of the theorem is complete. □

Corollary 13. *On* $\mathcal{H}^1$

$$\|f\| = \inf\{\sum |\lambda_k| \ ; \ f = \sum \lambda_k a_k, \text{the } a_k's \text{ are atoms}\}$$

is a norm equivalent to the usual ones.

Chapter 8

The substitute of L^∞: BMO

8.1 Definition of BMO

Definition 3. *A function $f \in L^1_{loc}$ belongs to BMO (=***bounded mean oscillation***) if*

$$\|f\|_{BMO} = \sup\{\frac{1}{|C|}\int_C |f - \frac{1}{|C|}\int_C f| \ ; \ C \text{ cube with sides parallel to the axes}\} < \infty. \tag{8.1}$$

Despite the notation, $\|\cdot\|_{BMO}$ is not a norm, since $\|f\|_{BMO} = 0$ when f is a constant. However, it is easy to see that, if we identify two functions in BMO when their difference is constant (a. e.), then $\|\cdot\|_{BMO}$ is a norm on the quotient space (still denoted BMO).
It will be convenient to denote f_C the average of f on C, i. e., $f_C = \frac{1}{|C|}\int_C f$.

Proposition 12. *a) BMO is a Banach space.*
b) For each cube C and each constant m, we have

$$\int_C |f - m| \geq \frac{1}{2}\int_C |f - f_C|. \tag{8.2}$$

c) We may replace cubes by balls; the space remains the same and the norm is replaced by an equivalent one. Similarly, we may consider cubes in general position.
d) If $C \subset Q$ are parallel cubes of sizes $l \leq L$, then $|f_C - f_Q| \leq c(1 + \ln(L/l))\|f\|_{BMO}$.
e) If Ψ is a Lipschitz function of Lipschitz constant k, then $\|\Psi \circ f\|_{BMO} \leq 2k\|f\|_{BMO}$.

Warning: In e), we do not identify two functions if there difference is constant.

Proof. a) Let $\sum f_n$ be an absolutely convergent series in BMO. Let C be a cube. The series $\sum(f_{n|C} - (f_n)_C)$ is absolutely convergent (thus convergent) in L^1. Set $f^C = \sum(f_{n|C} - (f_n)_C)$.

Then $\int\limits_C f^C = 0$ and

$$\frac{1}{|C|}\int\limits_C |f^C| \le \sum \frac{1}{|C|}\int\limits_C |f_n - (f_n)_C| \le \sum \|f_n\|_{BMO}. \tag{8.3}$$

We now cover $\mathbb{R}^N$ with an increasing sequence of cubes (C_k). We set $f(x) = f^{C_k}(x) - (f^{C_k})_{C_0}$ if $x \in C_k$. We claim that the definition is correct (in the sense that it does not depend on the choice of C_k). This follows immediately from the equality

$$(f^C)_{C_0} = \sum [f_{n|C} - (f_n)_{C_0}], \tag{8.4}$$

valid whenever $C_0 \subset C$.
b) We have

$$|C||m - f_C| = |\int\limits_C (m - f)| \le \int\limits_C |m - f|, \tag{8.5}$$

and thus

$$\int\limits_C |f - f_C| \le \int\limits_C |f - m| + \int\limits_C |m - f_C| \le 2\int\limits_C |m - f|. \tag{8.6}$$

c) We prove the assertion concerning balls. The proof of the other statement is analog. Let B be a ball and let C, Q be cubes s. t. C is inscribed in B and B is inscribed in Q. Then

$$\frac{1}{2|B|}\int\limits_B |f - f_B| \le \frac{1}{|B|}\int\limits_B |f - f_Q| \le \frac{1}{|B|}\int\limits_Q |f - f_Q| \le \frac{c}{|Q|}\int\limits_Q |f - f_Q| \tag{8.7}$$

and similarly

$$\frac{1}{2|C|}\int\limits_C |f - f_C| \le \frac{1}{|C|}\int\limits_C |f - f_B| \le \frac{1}{|C|}\int\limits_B |f - f_B| \le \frac{c}{|B|}\int\limits_B |f - f_B|, \tag{8.8}$$

so that the supremum over the balls and the supremum over the cubes are equivalent quantities.
d) We have

$$|f_C - f_Q| = \frac{1}{|C|}|\int\limits_C (f - f_Q)| \le \frac{1}{|C|}\int\limits_C |f - f_Q| \le \frac{1}{|C|}\int\limits_Q |f - f_Q| \le (L/l)^N \|f\|_{BMO}, \tag{8.9}$$

which implies the desired estimate when $L/l \le 2$. If $L/l > 2$, let $j \in \mathbb{N}^*$ be s. t. $L \in [2^j l, 2^{j+1} l)$ and consider a sequence $C_0, \ldots, C_{j+1}$ of cubes s. t. $C_0 = C$, $C_{j+1} = Q$ and the size of each cube is at most the double of the size of its predecessor. Then

$$|f_C - f_Q| \le \sum_{l=0}^{l=j} |f_l - f_{l+1}| \le c\, j \|f\|_{BMO} \le c' \ln(L/l) \|f\|_{BMO}. \tag{8.10}$$

e) We have

$$\frac{1}{|C|}\int_C |\Psi\circ f - (\Psi\circ f)_C| \le \frac{2}{|C|}\int_C |\Psi\circ f - \Psi(f_C)| \le \frac{2k}{|C|}\int_C |f - f_C| \le 2k\|f\|_{BMO}. \tag{8.11}$$

□

Remark 8. *The space BMO is not trivial: L^∞ functions are in BMO and, if $f = g + const$, then $\|f\|_{BMO} \le 2\|g\|_{L^\infty}$. However, BMO is not reduced to L^∞ functions. Here is an example: let $f : \mathbb{R}^N \to \mathbb{R}$, $f(x) = \ln|x|$. Then $f \in BMO$. Indeed, let B be a ball of radius R and center x. If $|x| \le 2R$, then there is a ball B^* of radius $\rho \sim R$, containing B and centered at the origin. Then*

$$\frac{1}{|B|}\int_B |f - f_B| \le \frac{2}{|B|}\int_B |f - \ln\rho| \le \frac{2}{|B|}\int_{B^*} |f - \ln\rho| \le \frac{c}{|B^*|}\int_{B^*} |f - \ln\rho|. \tag{8.12}$$

Now it is easy to see that the last integral is finite and independent of ρ.
Assume now that $|x| > 2R$. Then $|\ln|y| - \ln|z|| \le c\dfrac{R}{|x|}$ whenever $y, z \in B$, and therefore

$$\frac{1}{|B|}\int_B |f - f_B| = \frac{1}{|B|^2}\int_B |\int_B (f(y) - f(z))dy|dz \le \frac{cR}{|x|} \le c. \tag{8.13}$$

We emphasize the following consequence of our above computation

$$\lim_{|x|\to\infty} \frac{1}{|B(x,R)|}\int_{B(x,R)} |\ln|y| - \ln|y|_{B(x,R)}|dy = 0, \quad \forall\ R > 0. \tag{8.14}$$

8.2 $\mathcal{H}^1$ and BMO

Theorem 14. *(**Fefferman**) BMO is the dual of $\mathcal{H}^1$ in the following sense:*
a) if $f \in BMO$, then the functional $T(g) = \displaystyle\int fg$, initially defined on the set of finite combinations of atoms, satisfies $|T(g)| \le c\|f\|_{BMO}$ and thus gives raise (by density) to a unique element of $(\mathcal{H}^1)^*$ of norm $\le c\|f\|_{BMO}$.
b) Conversely, let $T \in (\mathcal{H}^1)^*$. Then there is some $f \in BMO$ s. t. $T(g) = \displaystyle\int fg$ whenever g is a finite combination of atoms. In addition, $\|f\|_{BMO} \le c\|T\|_{(\mathcal{H}^1)^*}$.

Remark 9. *Since atoms are bounded and compactly supported, $\int fg$ makes sense when $f \in L^1_{loc}$ and g is an atom. Moreover, the definition is correct when $f \in BMO$, in the sense that if we replace f by $f + const$, then the value of the integral does not change, since atoms have zero integral.*

Proof. a) Assume first that f is bounded. Then $T(g) = \int fg$ is well-defined and continuous in $\mathcal{H}^1$, since the inclusion $\mathcal{H}^1 \subset L^1$ is continuous. If $g = \sum \lambda_k a_k$ is an atomic decomposition of g s. t. $\sum |\lambda_k| \le c\|g\|_{\mathcal{H}^1}$ and each a_k is supported in some B_k, then

$$|T(g)| \le \sum |\lambda_k| |\int f a_k| = \sum |\lambda_k| |\int (f - f_{B_k}) a_k| \le \sum |\lambda_k| \frac{1}{|B_k|} \int_{B_k} |f - f_{B_k}| \le c\|f\|_{BMO}\|g\|_{\mathcal{H}^1} \tag{8.15}$$

and a) follows.

When f is arbitrary, we apply (8.15) to the truncated function $f_n(x) = \begin{cases} n, & \text{if } f(x) \ge n \\ f(x), & \text{if } |f(x) < n \\ -n, & \text{if } f(x) \le -n \end{cases}$.
Noting that $f_n = \Psi_n \circ f$, where Ψ_n is Lipschitz of Lipschitz constant 1, we find that

$$|\int f_n g| \le c\|f\|_{BMO}\|g\|_{\mathcal{H}^1}. \tag{8.16}$$

When g is a finite combination of atoms, we have $|f_n g| \le |fg| \in L^1$ and $f_n g \to fg$ a. e. Thus

$$|\int fg| = \lim |\int f_n g| \le c\|f\|_{BMO}\|g\|_{\mathcal{H}^1}, \tag{8.17}$$

by dominated convergence. This implies a) in full generality.
b) Conversely, let $T \in (\mathcal{H}^1)^*$. Let B be a ball and let X_B be the space of L^2 functions supported in B having zero integral. If $g \in X_B$, then $\frac{1}{\|g\|_{L^2}|B|^{1/2}} g$ is a ${}_2$atom. Thus $\|g\|_{\mathcal{H}^1} \le c\|g\|_{L^2}|B|^{1/2}$. It follows that T restricted to X_B defines a linear continuous functional of norm $\le c\|T\||B|^{1/2}$. Thus, there is some $f^B \in X_B$ s. t. $\|f^B\|_{L^2} \le c\|T\||B|^{1/2}$ and $T(g) = \int f^B g$ when $g \in X_B$. We now cover $\mathbb{R}^N$ with an increasing sequence of balls B_n and set $f(x) = f^{B_n}(x) - (f^{B_n})_{B_0}$ if $x \in B_n$. This definition is correct. Indeed, if $j > k$, then f^{B_k} and $f^{B_j}_{|B_k} - (f^{B_j})_{B_k}$ yield the same functional $T_{|X_B}$ and thus must coincide. Therefore, f^{B_j} and f^{B_k} differ only by a constant in B_j (and thus in B_0), which implies that the definition of f is correct. Another obvious consequence of our argument is that, on each ball B, $f_{|B}$ and f^B differ with a constant. In other words, $f^B = f - f_B$.

We claim that, when g is a finite combination of atoms, we have $T(g) = \int fg$. Indeed, there is some n s. t. supp $g \subset B_n$. Since $g \in X_{B_n}$ for such n, we find that $T(g) = \int f^{B_n} g = \int fg$.
It remains to prove that $f \in BMO$ and that $\|f\|_{BMO} \le c\|T\|_{(\mathcal{H}^1)^*}$. This follows from the fact that, if B is any ball, then we have

$$\frac{1}{|B|}|f - f_B| = \frac{1}{|B|}|f^B| \le \frac{1}{\sqrt{|B|}}\|f^B\|_{L^2} \le c\|T\|. \tag{8.18}$$

□

8.3 BMO functions are almost bounded

Strictly speaking, the assertion in the title is not even nearly true, as shows the example $x \mapsto \ln|x|$. However, we will see that, on compacts, BMO functions are in each L^p, $p < \infty$, and even better.

Proposition 13. *Let $f \in BMO$. Then, for each ball B, $f \in L^p(B)$ and $\|f - f_B\|_{L^p(B)} \le c_p|B|^{1/p}\|f\|_{BMO}$. In addition, we have $c_p \le c\,p$ when $p \ge 2$.*

Proof. We copy the proof of b) in the preceding theorem. When $p = 1$, the conclusion is trivial, so that we may assume that $1 < p < \infty$. We may also assume that $f_B = 0$. Let q be the conjugate exponent of p. It is straightforward that, if $g \in L^q(B)$, then $\|g - g_B\|_{L^q(B)} \le 2\|g\|_{L^q(B)}$. On the other hand, if $g \in L^q(B)$ and $\int_B g = 0$, then $\frac{1}{|B|^{1-1/q}\|g\|_{L^q}} g$ is a ${}_q$atom and thus $\|g\|_{\mathcal{H}^1} \le C_q|B|^{1-1/q}\|g\|_{L^q}$. Here, C_q satisfies $C_q \le \frac{c}{q-1} \le c\,p$ when $q \le 2$ (and thus $p \ge 2$). Thus

$$\|f\|_{L^p(B)} = \sup\{\int f(g - g_B)\ ;\ \|g\|_{L^q(B)} \le 1\} \le 2c\,C_q\|f\|_{BMO}|B|^{1-1/q} = c_p\|f\|_{BMO}|B|^{1/p}. \tag{8.19}$$

□

Theorem 15. *(**John-Nirenberg**) There are constants $c_1, c_2 > 0$ s. t.*

$$|\{x \in B\ |f - f_B| > \alpha\}| \le c_1|B|\exp(-c_2\alpha/\|f\|_{BMO}). \tag{8.20}$$

Proof. It is immediate that, if the conclusion holds for f, it also holds for a multiple of f. We may therefore assume that $\|f\|_{BMO} = 1$. It is also clear that, if the conclusion holds for $\alpha \ge 2c\,e$, where c is the constant in the preceding proposition, then we may adjust the constants s. t. (8.20) holds for each α. We may therefore assume that $\alpha \geq 2c\,e$. We assume also that $f_B = 0$. Let $p = \dfrac{\alpha}{c\,e} \geq 2$. Then

$$|\{x \in B\ ;\ |f| > \alpha\}| \leq \frac{\|f\|^p_{L^p(B)}}{\alpha^p} \leq \frac{(cp)^p|B|}{\alpha^p} = |B| \exp(-c\,\alpha/e). \tag{8.21}$$

□

Theorem 16. *(**John-Nirenberg**) There are constants $C, k > 0$ s. t. if $f \in BMO$ and $\|f\|_{BMO} \leq 1$, then*

$$\frac{1}{|B|}\int\limits_B \exp(C|f - f_B|) \leq k. \tag{8.22}$$

Remark 10. *The normalization condition $\|f\|_{BMO} \leq 1$ is necessary. Indeed, if $\exp f \in L^1$, there is no reason to have $\exp(2f) \in L^1$. On the other hand, the constant C cannot be arbitrary large, as shown by the example $x \mapsto \ln|x|$.*

Proof. We may assume that $f_B = 0$. Then

$$\frac{1}{|B|}\int\limits_B (\exp(C|f|) - 1) = 1 + \frac{1}{|B|}\sum_{p=1}^{\infty} C^p \|f\|^p_{L^p(B)} \leq 1 + \sum_{p=1}^{\infty} \frac{(cCp)^p}{p!} \leq k < \infty \tag{8.23}$$

provided $C < (ce)^{-1}$, as it is easily seen using Stirling's formula. □

Chapter 9

L^p regularity for the Laplace operator

9.1 Preliminaries

Let E be the fundamental solution of the Laplace operator in $\mathbb{R}^N$, $N \geq 2$,
$E(x) = \begin{cases} \dfrac{1}{2\pi} \ln |x|, & \text{if } N = 2 \\ -\dfrac{1}{(N-2)|S^{N-1}||x|^{N-2}}, & \text{if } N \geq 3 \end{cases}$. If $f \in C_0^\infty$, then $u = E * f$ is a (classical) solution of the equation (*) $\Delta u = f$. If $f \in L_0^p$ for some $1 \leq p \leq \infty$, we may still define $u = E * f$ and we then have $u \in L_{loc}^p$. Indeed, if K is a compact and $L = \text{supp } f$, let $\Phi \in C_0^\infty$ be s. t. $\Phi = 1$ in the compact $K - L$. Then, in K, we have $E * f = (\Phi E) * f$, and thus

$$\|E * f\|_{L^p(K)} \leq \|(\Phi E) * f\|_{L^p} \leq \|\Phi E\|_{L^1} \|f\|_{L^p} \leq C_{K,L} \|f\|_{L^p}, \tag{9.1}$$

using Young's inequality and the fact that $E \in L_{loc}^1$.
In addition, u still satisfies (*), this time in the distribution sense. The reason is that we may approximate f with a sequence $(f_n) \subset C_0^\infty$ s. t. $f_n \to f$ in L^1 and supp $f_n \subset L'$, with L' a compact independent of n. Then (9.1) with $p = 1$ and L replaced by L' implies that $E * f_n \to E * f$ in L_{loc}^1, and thus in $\mathcal{D}'$. Since we also have $\Delta(E * f_n) = f_n \to f$ in $\mathcal{D}'$, we find that $\Delta(E * f) = f$.
Let now $1 \leq j, k \leq N$ and consider the operator $T = T_p : L_0^p \to \mathcal{D}'$, $Tf = \partial_j \partial_k (E * f)$. Note that the definition does not depend on p, in the sense that, if $f \in L_0^p \cap L_0^q$, then $T_p f = T_q f$.
We start by noting some simple properties of T that will be needed in the next section.

Lemma 15. *If $f \in L_0^2$, then*

$$Tf = \mathcal{F}^{-1}\left(\frac{\xi_j \xi_k}{|\xi|^2} \hat{f}\right). \tag{9.2}$$

Consequently, T has a continuous extension to L^2, given by the r. h. s. of (9.2).

In addition, T is self-adjoint in L^2, i. e.

$$\int Tf\ \overline{g} = \int f\ \overline{Tg}, \quad \forall\ f, g \in L^2. \tag{9.3}$$

Proof. The r. h. s. of (9.2) is continuous from L^2 into L^2 (and thus into $\mathcal{D}'$), by Plancherel's theorem. On the other hand, if L is a fixed compact, the l. h. s. is continuous from L^2_L (the space of L^2 functions supported in L) into $\mathcal{D}'$. Therefore, it suffices to prove the equality when $f \in C_0^\infty$. Since $(1+|x|^2)^{-N}E \in L^1$, we have $E \in \mathcal{S}'$, and thus $\partial_j\partial_k E \in \mathcal{S}'$. Therefore, $\widehat{Tf} = \widehat{\partial_j\partial_k E}\hat{f}$, and it suffices to prove that $\widehat{\partial_j\partial_k E} = \dfrac{\xi_j\xi_k}{|\xi|^2}$. We write $E = E_1 + E_2$, where $E_1 = \Phi E$, $E_2 = (1-\Phi)E$, $\Phi \in C_0^\infty$ and $\Phi = 1$ near the origin. Then $\widehat{\partial_j\partial_k E} = \widehat{\partial_j\partial_k E_1} + \widehat{\partial_j\partial_k E_2} \in C^\infty + L^2$, since $\partial_j\partial_k E \in \mathcal{E}'$, while $\partial_j\partial_k E_2 \in L^2$. On the other hand, $\Delta\partial_j\partial_k E = \partial_j\partial_k\delta$, and thus $|\xi|^2\widehat{\partial_j\partial_k E} = \xi_j\xi_k$. Thus $\widehat{\partial_j\partial_k E} = \dfrac{\xi_j\xi_k}{|\xi|^2} + \sum\limits_{|\alpha|\le 2} c_\alpha\partial^\alpha\delta$. The coefficients c_α must be zero, since $\widehat{\partial_j\partial_k E} \in C^\infty + L^2$, whence the first conclusion of the lemma.
As for (9.3), it follows from Plancherel's theorem:

$$\int Tf\ \overline{g} = (2\pi)^{-N}\int \widehat{Tf}\ \overline{\hat{g}} = (2\pi)^{-N}\int \frac{\xi_j\xi_k}{|\xi|^2}\hat{f}\ \overline{\hat{g}} = (2\pi)^{-N}\int \hat{f}\overline{\frac{\xi_j\xi_k}{|\xi|^2}\hat{g}} = (2\pi)^{-N}\int \hat{f}\ \overline{\widehat{Tg}} = \int f\ \overline{Tg}. \tag{9.4}$$

□

Lemma 16. *Assume that $f \in L^p_0$ and let $x \notin$ supp f. Then, with $K(x) = \dfrac{1}{|S^{N-1}|}\left(\dfrac{\delta_{j,k}}{|x|^N} - \dfrac{Nx_jx_k}{|x|^{N+2}}\right)$, we have*

$$Tf(x) = \int K(x-y)f(y)dy. \tag{9.5}$$

In addition, K satisfies

$$|K(x-y) - K(x)| \le \frac{C|y|}{|x|^{N+1}}, \qquad \text{if } |y| < 1/2|x|. \tag{9.6}$$

Proof. If $L =$ supp f and $\mathcal{O}$ is a relatively compact open set s. t. $\overline{\mathcal{O}} \cap L = \emptyset$, then the (pointwise) derivatives of $E(x-y)f(y)$ with respect to x satisfy

$$|\partial_x^\alpha(E(x-\cdot)f(\cdot)| \le c_\alpha|f(\cdot)| \in L^1, \quad x \in \mathcal{O}, \tag{9.7}$$

and thus $E * f \in C^\infty(\mathcal{O})$. Moreover, we may differentiate twice under the integral sign in the formula of $E * f$ to obtain, in $\mathcal{O}$, both the pointwise and the distributional derivative $\partial_j\partial_k(E*f)$ through the formula $\partial_j\partial_k(E*f) = \displaystyle\int \partial_j\partial_k E(x-y)f(y)dy$. Here, $\partial_j\partial_k E$ stands for the pointwise

derivative. Finally, we have $\partial_j\partial_k E = K$, whence the first conclusion.
To prove the inequality (9.6), we note that $|DK(z)| \le C|z|^{-N-1}$ and thus, for x, y s. t. $|y| < 1/2|x|$, we have

$$|K(x-y) - K(x)| \le |y| \sup_{z\in[x-y,x]} |DK(z)| \le C|y| \sup_{z\in[x-y,x]} |z|^{-N-1} \le \frac{C|y|}{|x|^{N+1}}. \tag{9.8}$$

□

Lemma 17. *Let $\Phi \in C_0^\infty$. Then*
a) $(T\Phi)_t = T\Phi_t$, for each $t > 0$.
b) $T\Phi \in L^\infty$.
c) $D(T\Phi) \in L^\infty$.
d) If $|y| < 1/2|x|$, then $|T\Phi(x-y) - T\Phi(x)| \le \dfrac{C|y|}{|x|^{N+1}}$.

Proof. a) Actually, this holds under the sole assumption that $\Phi \in L^2$. It suffices to check that $\widehat{(T\Phi)_t} = \widehat{T\Phi_t}$. This equality follows from

$$\widehat{(T\Phi)_t}(\xi) = \widehat{(T\Phi)}(t\xi) = \frac{\xi_j\xi_k}{|\xi|^2}\hat{\Phi}(t\xi) = \frac{\xi_j\xi_k}{|\xi|^2}\widehat{\Phi_t}(\xi) = \widehat{T\Phi_t}(\xi). \tag{9.9}$$

b) Since $|\widehat{T\Phi}| \le |\hat{\Phi}|$, we find that $\widehat{T\Phi} \in L^1$ and thus $T\Phi \in L^\infty$.
c) Similarly, $\widehat{D(T\Phi)} = \imath\xi\widehat{T\Phi}$, and thus $\widehat{D(T\Phi)} \in L^1$, which implies that $D(T\Phi) \in L^\infty$.
d) Let $R > 0$ be s. t. $\Phi = 0$ outside $B(0, R)$. If $|x| \le 3R$, then the conclusion follows from b). Assume $|x| > 3R$. Then both x and $x - y$ are outside the support of Φ and thus

$$|T\Phi(x-y) - T\Phi(x)| = |\int_{(B(0,R)} (K(x-y-z) - K(x-z))\Phi(z)dz| \le C \sup_{|z|\le R} |K(x-y-z) - K(x-z)| \le \frac{C|y|}{|x|^{N+1}}; \tag{9.10}$$

here, we rely on the inequality (9.6) and we take into account the fact that $|x-y-z| \sim |x-z| \sim |x|$. □

In the next section, we will prove the following

Theorem 17. *a) (**Calderón-Zygmund**) For $p = 1$, the operator T, initially defined on L^1_{loc}, has a continuous extension from L^1 into L^1_w.*
*b) (**Fefferman-Stein**) When restricted to $\mathcal{H}^1$, the extension of T to L^1 maps continuously $\mathcal{H}^1$ into $\mathcal{H}^1$.*
*c) (**Calderón-Zygmund**) For $1 < p < \infty$, the operator T, initially defined on L^p_{loc}, has a continuous extension from L^p into L^p.*
*d) (**Spanne-Peetre-Stein**) T maps BMO_0 continuously into BMO and thus L^∞_0 continuously into BMO.*

The most widely used form of the above result sais that a solution u of $\Delta u = f$ "gains two derivatives with respect to f":

Corollary 14. *Assume that $\Delta u = f$ in the distribution sense.*
a) If $f \in L^p_{loc}$ for some $1 < p < \infty$, then $u \in W^{2,p}_{loc}$.
b) If $f \in \mathcal{H}^1$, then $u \in W^{2,1}_{loc}$.

Proof. Let K be a compact in $\mathbb{R}^N$ and let $\Phi \in C_0^\infty$ be s. t. $\Phi = 1$ in an open neighborhood $\mathcal{O}$ of K. Set $g = \Phi f \in L^p_0$ and let $v = E * g$, which satisfies $\Delta v = f$ in $\mathcal{O}$. Then $\Delta(u - v) = 0$ in $\mathcal{O}$, and thus $u - v \in C^\infty(\mathcal{O})$, by Weyl's lemma. Now $v \in L^p_{loc}$, since $g \in L^p_0$, and the second order derivatives of v are in L^p if $f \in L^p_{loc}$ and $1 < p < \infty$, respectively in L^1 if $f \in \mathcal{H}^1$. In addition, it is easy to see that the distributional first order derivatives of $E * f$ are computed according to the formula $\partial_j(E * g)(x) = \displaystyle\int (\partial_{x_j} E)(x - y) f(y) dy$, where $\partial_{x_j} E$ stands for the pointwise derivative (this is obtained using an integration by parts when $f \in C_0^\infty$; the general case is obtained by approximation, with the help of Young's inequality). Since $\partial_{x_j} E \in L^1_{loc}$ and (in all the cases) $g \in L^p_0$, we find that $\partial_j v \in L^p_{loc}$. Therefore, $u \in W^{2,p}_{loc}$. □

The above results are optimal, in the following sense:

Proposition 14. *T does not map L^1_0 into L^1 and does not map L^∞_0 into L^∞.*

Proof. We fix a compact L in $\mathbb{R}^N$. We already noted that T maps continuously L^p_L into $\mathcal{D}'$. We claim that, if $T : L^p_L \to L^p$, then T has to be continuous. Indeed, let $f_n \to f$ in L^p_L be s. t. $T f_n \to g$ in L^p. Since $T f_n \to T f$ in $\mathcal{D}'$, we find that $T f = g$, and thus T has closed graph. Thus T is continuous.
Let now $p = 1$. We argue by contradiction. Let L be a ball containing the origin. We consider a sequence $(f_n) \subset C_0^\infty$ s. t. $\|f_n\|_{L^1} \le C$, supp $f_n \subset L$ and $f_n \to \delta$ in $\mathcal{D}'$ and set $u_n = E * f_n$. Then $u_n \to E$ in $\mathcal{D}'$ and $\|D^2 u_n\|_{L^1} \le C$. On the other hand, $D u_n = (DE) * f_n$ (where $DE \in L^1_{loc}$ is the pointwise derivative of E), and thus $\|D u_n\|_{L^1(L)} \le C$. Consequently, the sequence $(D u_n)$ is bounded in $W^{2,1}(L)$. The Sobolev embeddings imply that $(D u_n)$ is bounded also in $L^{N/(N-1)}(L)$. Since $D u_n \to DE$ in $\mathcal{D}'$, we find that $DE \in L^{N/(N-1)}(L)$; thus $|DE|^{N/(N-1)}$ is integrable near the origin. However, if we compute the (pointwise or distributional) gradient DE, we see that $|DE(x)| \sim |x|^{-(N-1)}$, a contradiction.
We next consider the case $p = \infty$. Argue again by contradiction. Recall that there is a function $u : \mathbb{R}^N \to \mathbb{R}$, $u \notin C^2$, s. t. $f = \Delta u$ (computed in the distributional sense) be continuous (example due to Weierstrass). We may assume, e. g., that $u \notin C^2(B(0,1))$. Let $g = \Phi f$, where $\Phi \in C_0^\infty$, $\Phi = 1$ in $B(0,1)$, supp $\Phi \subset B(0,2)$. Then $g \in L^\infty_0$, and thus $T f \in L^\infty$ (for all j, k). Let $(g_n) \subset C_0^\infty$ be s. t. $g_n \to g$ uniformly, supp $g_n \subset B(0,2)$. Then $T g_n \to T g$ uniformly. Since $T g_n \in C^\infty$, this implies that $T g$ is continuous. Thus $E * g \in C^2$. Since $\Delta(E * g) = \Delta u$ in $B(0,1)$, Weyl's lemma implies that $u \in C^2(B(0,1))$, a contradiction. □

9.2 Proof of Theorem 17

Proof. The plan of the proof is the following: a) we prove that T maps L^1 into L^1_w; this will rely on the Calderón-Zygmund decomposition. b) Marcinkiewicz' interpolation theorem, combined with the continuity of T from L^2 into L^2, will imply the result when $1 < p < 2$. c) For $\mathcal{H}^1$, the result is obtained via the atomic decomposition. d) The remaining cases, i. e. $2 < p \leq \infty$ or BMO_0, will be obtained by duality; we will exploit the fact that T is a symmetric operator.

Step 1. Continuity from L^1 into L^1_w

It suffices to prove the following estimate

$$|\{|Tf| > t\}| \leq \frac{C}{t}\|f\|_{L^1}, \quad \forall\, t > 0, \forall\, f \in L^1 \cap L^2. \tag{9.11}$$

Indeed, assume (9.11) proved, for the moment. Let $f \in L^1$ and consider a sequence $(f_n) \subset L^1 \cap L^2$ s. t. $f_n \to f$ in L^1. Then (9.11) applied to $f_n - f_m$ implies that $|\{|Tf_n - Tf_m| > t\}| \to 0$ when t is fixed and $m, n \to \infty$.Thus (Tf_n) is a Cauchy sequence in measure, and thus it converges in measure to some g. In particular, this means that g does not depend on the sequence (f_n), that $g = Tf$ if f happens to be in $L^1 \cap L^2$ and that $f \mapsto g$ is linear. Possibly after passing to a subsequence (f_{n_k}), we have $Tf_{n_k} \to g$ a. e., and thus $|\{|g| > t\}| \leq \liminf_k |\{|Tf_{n_k}| > t\}| \leq \dfrac{C}{t}\|f\|_{L^1}$. This implies that $f \mapsto g$ is the desired extension of T. (We needed this argument since L^1_w is not a normed space.)

We return to the proof of (9.11). Let $t > 0$. We write, as in Theorem 9 (with α replaced by t), a function $f \in L^1 \cap L^2$ as $f = g + \sum h_n$. We first note that $g \in L^2$, since $g \in L^1$ and $|g| \leq Ct$. We also claim that the series $\sum h_n$ is convergent in L^2. Indeed, the functions h_n are mutually orthogonal in L^2, and thus it suffices to prove that $\sum \|h_n\|^2_{L^2} < \infty$. Since $\|h_n\|_{L^2} = \|f - f_{C_n}\|_{L^2(C_n)} \leq \|f\|_{L^2(C_n)}$, we find that $\sum \|h_n\|^2_{L^2} \leq \sum \|f\|^2_{L^2(C_n)} \leq \|f\|^2_{L^2}$, whence the claim.

This allows us to write $Tf = T(g + \sum h_n) = Tg + \sum Th_n$. Thus

$$|\{|Tf| > t\}| \leq |\{|Tg| > t/2\}| + |\{|T\sum h_n| > t/2\}|. \tag{9.12}$$

On the one hand, we have $\displaystyle\int |g|^2 \leq Ct \int |g| \leq Ct\|f\|_{L^1}$, by the properties of the Calderón-Zygmund decomposition. Thus

$$|\{|Tg| > t/2\}| \leq \frac{C}{(t/2)^2}\|g\|^2_{L^2} \leq \frac{C}{t}\|f\|_{L^1}. \tag{9.13}$$

On the other hand, let, for each n, C_n^* be the cube concentric with C_n and twice bigger than it.

Then, with $A = \mathbb{R}^N \setminus \bigcup C_n^*$, we have

$$|\{|T\sum h_n| > t/2\}| \le |\bigcup C_n^*| + |\{x \in A\ ;\ \sum |Th_n| > t/2\}| \le C\sum |C_n| + \frac{C}{t}\sum \|Th_n\|_{L^1(A)}. \tag{9.14}$$

We denote by $\overline{x}_n$ the center of C_n and by l_n its size. For $x \in A$, we have

$$Th_n(x) = \int K(x-y)h_n(y)dy = \int [K(x-y) - K(x-\overline{x}_n)]h_n(y)dy, \tag{9.15}$$

and thus

$$|Th_n(x)| \le \frac{C}{|x-\overline{x}_n|^{N+1}}\int |y-\overline{x}_n||h_n|dy \le \frac{Cl_n}{|x-\overline{x}_n|^{N+1}}\|h_n\|_{L^1}. \tag{9.16}$$

Integrating the above inequality and summing over n, we find that

$$\sum \|Th_n\|_{L^1(A)} \le C\sum \|h_n\|_{L^1} \le C\|f\|_{L^1}, \tag{9.17}$$

by the properties of the Calderón-Zygmund decomposition.
We conclude the first step by combining (9.12), (9.13), (9.14), (9.17) and the fact that $\sum |C_n| \le \frac{C}{t}\|f\|_{L^1}$.

Step 2. Continuity in L^p, $1 < p < \infty$
We know that T, when defined in $L^1 \cap L^2$, is continuous from L^1 into L^1_w and from L^2 into L^2. Marcinkiewicz' interpolation theorem implies that T has a unique extension continuous from L^p into L^p when $1 < p < 2$. Let now $2 < p < \infty$. Part c) of the theorem follows if we prove that $\|Tf\|_{L^p} \le C\|f\|_{L^p}$ whenever $f \in L^p \cap L^2$. For such an f, we have, with $q < 2$ the conjugate exponent of p,

$$\|Tf\|_{L^p} = \sup_{g\in L^q\ ;\ \|g\|_{L^q}\le 1}\int Tf\ \overline{g} = \sup_{g\in L^q\cap L^2\ ;\ \|g\|_{L^q}\le 1}\int Tf\ \overline{g} = \sup_{g\in L^q\cap L^2\ ;\ \|g\|_{L^q}\le 1}\int f\overline{Tg} \le C\|f\|_{L^p}; \tag{9.18}$$

here, we use the continuity of T in L^q.

Step 3. Continuity in $\mathcal{H}^1$
In view of the properties of the atomic decomposition, it suffices to prove, with a constant C independent of a, the estimate

$$\|Ta\|_{\mathcal{H}^1} \le C, \quad \forall \text{ atom } a. \tag{9.19}$$

Let a be an atom supported in $B = B(\overline{x}, R)$. Let $\Phi \in C_0^\infty$ be s. t. $\int \Phi = 1$ and $\operatorname{supp}\Phi \subset B(0,1)$.
For each x, we have $\mathcal{M}_\Phi a(x) \le C\mathcal{M}a(x)$, and thus

$$\int_{B(\overline{x},2R)} \mathcal{M}_\Phi a \le C\int_{B(\overline{x},2R)} \mathcal{M}a \le \|\mathcal{M}a\|_{L^2}|B(\overline{x},2R)|^{1/2} \le C\|a\|_{L^2}|B|^{1/2} \le C. \tag{9.20}$$

We consider now an x outside $B(\overline{x}, 2R)$ and estimate $\Phi_t * (Ta)(x)$. We have (we take a, Φ real, here)

$$\Phi_t*(Ta)(x) = \int \Phi_t(x-y)Ta(y)dy = \int (T\Phi_t)(x-y)a(y)dy = \int_B [(T\Phi)_t(x-y)-(T\Phi)_t(x-\overline{x})]a(y)dy. \tag{9.21}$$

We next note that, when $y \in B$, we have $|x - y| < 1/2|x - \overline{x}|$. We intend to make use of the decay properties of $T\Phi$. To this purpose, we distinguish two possibilities concerning the size of t: (i) $t > |x - \overline{x}|$ and (ii) $t \leq |x - \overline{x}|$. In case (i), we use the fact that $T\Phi$ is Lipschitz, and find that

$$|(T\Phi)_t(x - y) - (T\Phi)_t(x - \overline{x})| \leq Ct^{-N-1}|y - \overline{x}|, \tag{9.22}$$

and thus

$$|\Phi_t * (Ta)(x)| \leq \frac{C}{t^{N+1}} \int_B |y - \overline{x}||a(y)|dy \leq \frac{Cl}{t^{N+1}} \leq \frac{Cl}{|x - \overline{x}|^{N+1}}. \tag{9.23}$$

In case (ii), we make use of Lemma 17 d), and obtain

$$|(T\Phi)_t(x - y) - (T\Phi)_t(x - \overline{x})| \leq \frac{C}{|x - \overline{x}|^{N+1}}|y - \overline{x}|, \tag{9.24}$$

which gives

$$|\Phi_t * (Ta)(x)| \leq \frac{C}{|x - \overline{x}|^{N+1}} \int_B |y - \overline{x}||a(y)|dy \leq \frac{Cl}{|x - \overline{x}|^{N+1}}. \tag{9.25}$$

(9.23) combined with (9.25) yields

$$\mathcal{M}_\Phi a(x) \leq \frac{Cl}{|x - \overline{x}|^{N+1}} \quad \text{when } x \notin B(\overline{x}, 2R). \tag{9.26}$$

Integration of (9.26) over $\mathbb{R}^N \setminus B(\overline{x}, 2R)$ combined with (9.20) gives the needed conclusion $\|\mathcal{M}_\Phi a\|_{L^1} \leq C$.

Step 4. Continuity of T in BMO_0

We note that $BMO_0 \subset L^2_0$, by the John-Nirenberg inequalities. We also note that the vector space $\mathcal{V}$ spanned by the atoms is contained in L^2. Thus, for $f \in BMO_0$, we have

$$\|Tf\|_{BMO} \sim \sup_{g\in\mathcal{V},\|g\|_{\mathcal{H}^1}\leq 1} \int Tf\ \overline{g} = \sup_{g\in\mathcal{V},\|g\|_{\mathcal{H}^1}\leq 1} \int f\ \overline{Tg} \leq C\|f\|_{BMO}; \tag{9.27}$$

here, we used the duality between $\mathcal{H}^1$ and BMO, the density of $\mathcal{V}$ in $\mathcal{H}^1$ and the continuity of T from $\mathcal{H}^1$ into $\mathcal{H}^1$. □

9.3 An equation involving the jacobian

We consider, in $\mathbb{R}^2$, the following equation that appears in Geometry

$$\Delta u = \det(Df, Dg), \quad f, g \in H^1(\mathbb{R}^2). \tag{9.28}$$

Theorem 18. *a) (***Wente***) Equation (9.28) has one and only one distribution solution $u \in C(\mathbb{R}^2)$ vanishing at infinity, i. e., s. t. $\lim_{|x|\to\infty} u(x) = 0$. In addition, $Du \in L^2$ and*

$$\|Du\|_{L^2} \le C\|Df\|_{L^2}^{1/2}\|Dg\|_{L^2}^{1/2}. \tag{9.29}$$

*b) (***Coifman-Lions-Meyer-Semmes***) In addition, we have $D^2u \in \mathcal{H}^1$.*

Proof. The main argument in the proof is that

$$\det(Df, Dg) \in \mathcal{H}^1 \quad \text{and} \quad \|\det(Df, Dg)\|_{\mathcal{H}^1} \le C\|Df\|_{L^2}\|Dg\|_{L^2}. \tag{9.30}$$

Assuming (9.30) proved for the moment, we reason as follows: let $h = \det(Df, Dg)$. Consider sequences $(f_n), (g_n) \subset C_0^\infty$ s. t. $f_n \to f$, $g_n \to g$ in H^1. Then $h_n = \det(Df_n, Dg_n) \to h$ in $\mathcal{H}^1$, by (9.30). Let $u_n = E * h_n$, which is a solution of $\Delta u_n = h_n$. We claim that $u_n \in C_\infty$ (the space of continuous functions vanishing at infinity) and that (u_n) is a Cauchy sequence for the sup norm. Indeed, let, for fixed n, $R = R_n > 0$ be s. t. $h_n(y) = 0$ if $|y| > R$. Then

$$|u_n(x)| = \frac{1}{2\pi}\left|\int \ln|x-y|h_n(y)dy\right| = \frac{1}{2\pi}\left|\int [\ln|x-y| - (\ln|\cdot|)_{B(x,R)}]h_n(y)dy\right|, \tag{9.31}$$

and thus

$$|u_n(x)| \le C_n \int_{B(0,R)} |\ln|x-y| - (\ln|\cdot|)_{B(x,R)}|dy = C_n \int_{B(x,R)} |\ln y - (\ln|\cdot|)_{B(x,R)}|dy \to 0 \text{ as } |x| \to \infty. \tag{9.32}$$

On the other hand, we have $\ln \in BMO$ and thus, using the $\mathcal{H}^1$-BMO duality,

$$|u_n(x) - u_m(x)| = \frac{1}{2\pi}\left|\int \ln|y|(h_n(x-y) - h_m(x-y))dy\right| \le C\|(h_n - h_m)(x-\cdot)\|_{\mathcal{H}^1} = C\|h_n - h_m\|_{\mathcal{H}^1}, \tag{9.33}$$

since the $\mathcal{H}^1$ norm is translation invariant. (Similarly, we have $|u_n| \le C\|h_n\|_{\mathcal{H}^1}$.)
To summarize, the sequence (u_n) is Cauchy in C_∞, and thus converges to some $u \in C_\infty$. This u is a distribution solution of (9.28). It is also the only solution of (9.28) in C_∞, for if v is another solution, their difference w is, by Weyl's lemma, a harmonic function vanishing at infinity, thus constant, by the maximum principle.
We now turn to the proof of b) and c) (assuming, again, (9.30) already proved). Note that c),

at least when $f, g \in C_0^\infty$, follows by combining (9.30) and the Fefferman-Stein regularity result concerning the equation $\Delta u = f$ with $f \in \mathcal{H}^1$. The general case is obtained by approximation, as above. Similarly, it suffices to establish b) when $f, g \in C_0^\infty$. Formally, estimate b) is clear, as shown by the following (wrong, in principle) computation:

$$\int |Du|^2 = -\int u\Delta u = -\int uh \leq \|u\|_{L^\infty}\|h\|_{L^1} \leq C\|u\|_{L^\infty}\|h\|_{\mathcal{H}^1} \leq C\|Df\|_{L^2}^2\|Dg\|_{L^2}^2. \quad (9.34)$$

The point is that this computation can be transformed into a rigorous one as follows: set $F(r) = \frac{1}{2\pi r}\int\limits_{|x|=r} |u|^2 dl$. Then $\lim_{r\to\infty} F(r) = 0$ and $F'(r) = \frac{1}{\pi r}\int\limits_{|x|=r} u \cdot u_r dl$. Thus, along a subsequence $r_n \to \infty$, we must have $r_n F'(r_n) \to 0$ (argue by contradiction; otherwise, we have $F(r) \geq C \ln r$ for large r). Then, for large n, we have $\Delta u = 0$ outside $B(0, r_n)$ and thus

$$\int |Du|^2 = \lim_n \int\limits_{B(0,r_n)} |Du|^2 = \lim_n \{\int\limits_{|x|=r} u \cdot u_r - \int u\Delta u\} = -\int u\Delta u \leq C\|Df\|_{L^2}^2\|Dg\|_{L^2}^2. \quad (9.35)$$

The only part of the proof left open is
Proof of (9.30)
In view of the conclusion we want to obtain, we may assume that $f, g \in C_0^\infty$. Let $\Phi \in C_0^\infty$ be supported in $B(0,1)$ and s. t. $\int \Phi = 1$. Then, with $h = \det(Df, Dg)$, we have

$$\Phi_t * h(x) = \int \Phi_t(y)h(x-y)dy = t^{-1}\int f(x-y)\det((D\Phi)_t)(y), Dg(x-y))dy, \quad (9.36)$$

as shown by an integration by parts. Next, if $k, l \in C_0^\infty$, then $\int \det(Dk, Dl) =$ (again, this follows by an integration by parts), and thus

$$\Phi_t * h(x) = t^{-1}\int [f(x-y) - f_{B(x,t)}]\det((D\Phi)_t)(y), Dg(x-y))dy. \quad (9.37)$$

Using the Hölder inequality and the inequality $|(D\Phi)_t| \leq Ct^{-2}$ together with the fact that Φ_t vanishes outside $B(0,t)$, we find that

$$|\Phi_t * h(x)| \leq t^{-3}\Big(\int\limits_{B(x,t)} |f - f_{B(x,t)}|^4\Big)^{1/4}\Big(\int\limits_{B(x,t)} |Dg|^{4/3}\Big)^{3/4}. \quad (9.38)$$

Applying Lemma 18 below to the function given by $v(y) = f(x - ty)$, we find that

$$\|f - f_{B(x,t)}\|_{L^4(B(x,t))} \leq C\|Df\|_{L^{4/3}(B(x,t))}, \quad (9.39)$$

and thus (9.38) yields

$$|\Phi_t * h(x)| \leq C\bigg(\frac{1}{|B(x,t)|}\int\limits_{B(x,t)} |Df|^{4/3}\bigg)^{3/4}\bigg(\frac{1}{|B(x,t)|}\int\limits_{B(x,t)} |Dg|^{4/3}\bigg)^{3/4}. \tag{9.40}$$

Recalling the definition of the maximal function, we obtain

$$\mathcal{M}_\Phi h(x) \leq C(\mathcal{M}|Df|^{4/3}(x))^{3/4}(\mathcal{M}|Dg|^{4/3}(x))^{3/4}, \tag{9.41}$$

and the Cauchy-Schwarz inequality implies that

$$\|\mathcal{M}_\Phi h\|_{L^1} \leq C\bigg(\int(\mathcal{M}|Df|^{4/3})^{3/2}\bigg)^{1/2}\bigg(\int(\mathcal{M}|Dg|^{4/3})^{3/2}\bigg)^{1/2}, \tag{9.42}$$

which may be rewritten as

$$\|\mathcal{M}_\Phi h\|_{L^1} \leq C\|\mathcal{M}|Df|^{4/3}\|_{L^{3/2}}^{3/4}\|\mathcal{M}|Dg|^{4/3}\|_{L^{3/2}}^{3/4} \leq C\||Df|^{4/3}\|_{L^{3/2}}^{3/4}\||Df|^{4/3}\|_{L^{3/2}}^{3/4} = C\|Df\|_{L^2}\|Dg\|_{L^2}; \tag{9.43}$$

that is, $\|h\|_{\mathcal{H}^1} \leq C\|Df\|_{L^2}\|Dg\|_{L^2}$, as claimed at the beginning of the proof. □

We next recall the following Sobolev embedding and the corresponding Poincaré inequality

Lemma 18. *$W^{1,4/3}(\mathbb{R}^2)$ is embedded into L^4 and, with $B = B(0,1)$ and $v \in W^{1,4/3}(B)$, we have*

$$\|v - v_B\|_{L^4(B)} \leq C\|Dv\|_{L^{4/3}(B)}. \tag{9.44}$$

Proof. The above Sobolev embedding will be proved, in a slightly better form, in the next chapter. We present a proof of (9.44), which is less standard. The starting point is the usual Poincaré inequality

$$\|v - v_B\|_{L^{4/3}(B)} \leq C\|Dv\|_{L^{4/3}(B)}. \tag{9.45}$$

We may assume, with no loss of generality, that $v_B = 0$. We extend v by reflections in a neighborhood of B by setting $\tilde{v}(x) = \begin{cases} v(x), & \text{if } x \in B \\ v(x/|x|^2), & \text{if } 1 < |x| < 3/2 \end{cases}$. The new function $\tilde{v}$ is in $W^{1,4/3}$ and satisfies $\|\tilde{v}\|_{L^{4/3}} \leq C\|v\|_{L^{4/3}}$ and $\|D\tilde{v}\|_{L^{4/3}} \leq C\|Dv\|_{L^{4/3}}$. Next let $\Phi \in C_0^\infty$ be s. t. $\Phi = 1$ in B and supp $\Phi \subset B(0,3/2)$. Set $w = \Phi\tilde{v} \in W^{1,4/3}(\mathbb{R}^2)$. Then

$$\|v\|_{L^4(B)} \leq \|w\|_{L^4} \leq C\|Dw\|_{L^{4/3}} \leq C(\|\tilde{v}\|_{L^{4/3}} + \|D\tilde{v}\|_{L^{4/3}}) \leq C(\|v\|_{L^{4/3}} + \|Dv\|_{L^{4/3}}) \leq C\|Dv\|_{L^{4/3}}. \tag{9.46}$$

□

Part III

Functions in Sobolev spaces

Chapter 10

Improved Sobolev embeddings

The usual form of the Sobolev embeddings states that $W^{1,p}(\mathbb{R}^N) \subset L^{Np/(N-p)}$, provided $1 \le p < N$. In this chapter, we will improve the conclusion to $W^{1,p}(\mathbb{R}^N) \subset L^{Np/(N-p),p}$ (when $1 < p < N$); this is slightly better, since $\dfrac{NP}{N-p} > p$, and thus $L^{Np/(N-p),p} \subset L^{Np/(N-p)}$.

10.1 An equivalent norm in Lorentz spaces

Let $f : \mathbb{R}^N \to \mathbb{C}$ be a measurable function and let $F : (0,\infty) \to [0,\infty]$ be its distribution function. Intuitively, we may think of F as a bijection of $(0,\infty)$ into itself. Then, if $p, q < \infty$ and if $f^* = F^{-1}$ (which is decreasing), we may (formally) compute the $L^{p,q}$ quasi-norm as follows:

$$\|f\|_{L^{p,q}}^q = \int t^{q-1} F^{q/p}(t)dt = -\int s^{q/p}(f^*)^{q-1}(s) f^{*\prime}(s)ds = p^{-1}\int s^{q/p-1}(f^*)^q(s)ds; \qquad (10.1)$$

here, the - sign at the beginning of the computation comes from the fact that F is decreasing. The second equality is obtained through the change of variable $F(t) = s$, the third one arises after an integration by parts.

The above equality maybe rewritten as

$$\|f\|_{L^{p,q}} = \|t^{1/p}F\|_{L^q((0,\infty);dt/t)} \sim \|t^{1/p}f^*\|_{L^q((0,\infty);dt/t)}. \qquad (10.2)$$

In this section, we will see that this formula is right!...provided we interpret it accurately.

Definition 4. *The* **non increasing rearrangement** $f^* : [0,\infty) \to [0,\infty]$ *of* f *is defined through the formula*

$$f^*(t) = \sup\{s > 0\ ;\ F(s) \le t\}. \qquad (10.3)$$

We note that, when F is a bijection, we have $f^* = F^{-1}$.

The elementary results we gather below explain, in particular, why f^* is called the non increasing rearrangement of f.

Proposition 15. *a) F is continuous from the right.*
b) $F(f^*(t)) \le t$ everywhere (in other words, $\inf = \min$ *in the definition of f^*).*
c) f^* is non increasing and continuous from the right.
d) f and f^* are equally distributed, i. e., $|\{x \in \mathbb{R}^N \; ; \; |f(x)| > t\}| = |\{s \in (0, \infty) \; ; \; f^*(s) > t\}|$ for each $t > 0$.
e) f^* depends continuously on f in the following sense: if (f_n) is a sequence of functions s. t. $|f_n(x)| \nearrow |f(x)|$ for a. e. $x \in \mathbb{R}^N$, then $f_n^*(t) \nearrow f^*(t)$ for each $t > 0$.
f) We have $(f+g)^*(2t) \le f^*(t) + g^*(t)$. More generally, $\left(\sum f_j\right)^* (\sum t_j) \le \sum (f_j)^*(t_j)$.

Proof. a) follows from the equality $\{|f| > t\} = \bigcup \{|f| > t + 1/n\}$, which implies that $F(t) = \lim F(t + 1/n)$.
b) Let $s = f^*(t)$. Then $F(s + \varepsilon) \le t$ for $\varepsilon > 0$, and thus $F(s) \le t$.
c) The fact that f^* is non increasing is clear from the definition. Concerning the second assertion, it suffices to prove that $f^*(t+0) \ge f^*(t)$. Let $t_n \searrow t$. Then $F(f^*(t_n)) \le t_n$, which implies that $F(f^*(t+0)) \le t_n$ and thus $F(f^*(t+0)) \le t$, that is $f^*(t+0) \ge f^*(t)$.
d) Since f^* is non increasing, we have $|\{s \in (0,\infty) \; ; \; f^*(s) > t\}| = \tau$, where τ is uniquely defined by $f^*(s) > t$ if $s < \tau$ and $f^*(s) \le t$ if $s > \tau$. In view of the conclusion we want, it suffices to check that $f^*(s) > t$ if $s < F(t)$ and that $f^*(s) \le t$ if $s > F(t)$. If $s < F(t)$, then $F(f^*(s)) \le s < F(t)$ and thus $f^*(s) > t$. On the other hand, if $s > F(t)$, then $t \ge f^*(s)$, by definition of $f^*(s)$.
e) We note that $|f| \le |g| \Longrightarrow f^* \le g^*$; therefore, the sequence (f_n^*) is non decreasing and $h(t) := \lim f_n^*(t) \le f^*(t)$ for each t. Hence, it suffices to prove that $h(t) \ge f^*(t)$, i. e., that $F(h(t)) \le t$. We note that, for each s, we have $F_n(s) \to F(s)$, since the set $\{|f| > s\}$ is the union of the non decreasing sequence $(\{|f_n| > s\})$. Thus $F_n(f_n^*(t)) \le t \Longrightarrow F_n(h(t)) \le t \Longrightarrow F(h(t)) \le t$, as needed.
f) Let $s = f^*(t)$ and $\tau = g^*(t)$. Then $|\{|f| > s\}| \le t$ and $|\{|g| > \tau\}| \le t$. Since $\{|f+g| > s + \tau\} \subset \{|f| > s\} \cup \{|g| > \tau\}$, we find that $|\{|f+g| > s+\tau\}| \le 2t$, i. e., $(f+g)^*(2t) \le s + \tau = f^*(t) + g^*(t)$. □

We next justify the equality (10.1).

Proposition 16. *For $1 \le p < \infty$ and $1 \le q \le \infty$, we have $\|f\|_{L^{p,q}} \sim \|t^{1/p} f^*\|_{L^q((0,\infty);dt/t)}$. For $p = \infty$, we have $\|f\|_{L^\infty} = \|f^*\|_{L^\infty}$.*

Proof. We start with the case $p = \infty$; we will prove the equality of the quasi-norms. Indeed,

$$\|f\|_{L^{\infty,q}} = \|f\|_{L^\infty} = \inf\{s \; ; \; F(s) = 0\} = \inf\{s \; ; \; F(s) \le 0\} = f^*(0) = \|f^*\|_{L^\infty}, \qquad (10.4)$$

since f^* is non increasing and continuous from the right.
Let now $p < \infty$ and $q = \infty$; once again, we will prove the equality of the quasi-norms.
" $\le$ " Let $C = \|f\|_{L^{p,\infty}} = \sup t F^{1/p}(t)$. Let $t > 0$. With $s = f^*(t)$, we want to prove that

$t^{1/p}s \leq C$. If $s = 0$, there is nothing to prove. If $s = \infty$, then $F(\tau) > t$ for each τ, and then $C = \infty$. If $s \in (0, \infty$, then $F(s - \varepsilon) > t$ for small $\varepsilon > 0$, and thus

$$t^{1/p}s < f^{1/p}(s - \varepsilon)s \leq \frac{Cs}{s - \varepsilon}, \tag{10.5}$$

and the desired conclusion follows by letting $\varepsilon \to 0$.
" $\geq$ " With $C = \sup t^{1/p}f^*(t)$, we will prove that $tF^{1/p}(t) \leq C$ for each $t > 0$. If $F(t) = 0$, there is nothing to prove. If $F(t) = \infty$, then $f^*(s) \geq t$ for each s, and thus $C = \infty$. Finally, if $u = F(t) \in (0, \infty)$, let, for small $\varepsilon > 0$, $u_\varepsilon = u - \varepsilon > 0$. Then $F(t) > u_\varepsilon$ and thus $f^*(u_\varepsilon) > t$. We find that

$$tF^{1/p}(t) \leq f^*(u - \varepsilon)u^{1/p}, \tag{10.6}$$

and we conclude by letting $\varepsilon \to 0$.
Finally, we consider the case $1 \leq p, q < \infty$. In view of the preceding proposition, it suffices to prove the equality $p\|f\|^q_{L^{p,q}} = \|t^{1/p}f^*\|^q_{L^q((0,\infty);dt/t)}$ when f is a step function; the general case will follow by monotone convergence, by approximating an arbitrary function f with a sequence (f_n) s. t. each f_n is a step function and $|f_n| \nearrow |f|$. In addition, since the quantities we consider do not distinguish between f and $|f|$, we may assume that $f \geq 0$. Let then $f = \sum a_n \chi_{A_n}$, where $a_1 > a_2 > \ldots > a_k > 0$ and the sets A_n are measurable and mutually disjoint. Set $b_n = |A_n|$, $c_l = b_1 + \ldots + b_l$, $c_0 = 0$ and $c_{k+1} = \infty$. Then, with $a_0 = \infty$ and $a_{k+1} = 0$, we have $F(t) = c_l$ if $t \in [a_{l+1}, a_l)$. On the other hand, $f^*(t) = a_{l+1}$ if $t \in [c_l, c_{l+1})$. Then

$$p\|f\|^q_{L^{p,q}} = p\sum_{l=0}^{k} \int\limits_{[a_{l+1},a_l)} t^{q-1}(c_l)^{q/p}dt = \frac{p}{q}\sum_{l=1}^{k}(c_l)^{q/p}[(a_l)^q - (a_{l+1})^q] \tag{10.7}$$

and

$$\|t^{1/p}f^*\|^q_{L^q((0,\infty);dt/t)} = \sum_{l=0}^{k} \int\limits_{[c_l,c_{l+1})} t^{q/p-1}(a_{l+1})^q = \frac{p}{q}\sum_{l=0}^{k-1}(a_{l+1})^q[(c_{l+1})^{q/p} - (c_l)^{q/p}], \tag{10.8}$$

so that the two quantities are equal (since $c_0 = 0$ and $a_{k+1} = 0$). □

10.2 Properties of f^*

Lemma 19. *For each $t > 0$ we have (with F the distribution function of f)*

$$\int\limits_0^{F(f^*(t))} f^*(s)ds = F(f^*(t))f^*(t) + \int\limits_{f^*(t)}^{\infty} F(s)ds. \tag{10.9}$$

In particular, $\displaystyle\int_{f^*(t)}^{\infty} F(s)ds \leq \int_0^t f^*(s)ds.$

Proof. Let $g(x) = \begin{cases} |f(x)|, & \text{if } |f(x)| > f^*(t) \\ 0, & \text{if } |f(x) \leq f^*(t) \end{cases}$, whose distribution function G is given by $\begin{cases} F(s), & \text{if } s \geq f^*(t) \\ F(f^*(t)), & \text{if } s < f^*(t) \end{cases}$. Let $\tau \geq F(f^*(t))$. Then $G(s) \leq \tau$ for each s and thus $g^*(\tau) = 0$. On the other hand, if $\tau < F(f^*(t))$, then clearly $g^*(\tau) = f^*(\tau)$. The equality $\|g\|_{L^1} = \|g^*\|_{L^1}$ reads then $\displaystyle\int G(s)ds = \int g^*(s)ds$, which is precisely the desired equality. □

Although it is actually part of the preceding proof, we emphasize for later use the following

Corollary 15. *Let, for* $\alpha > 0$, $f_\alpha(x) = \begin{cases} f(x), & \textit{if } |f(x)| > \alpha \\ 0, & \textit{otherwise} \end{cases}$. *Then*

$$\|f_{f^*(t)}\|_{L^1} \leq \int_0^t f^*(s)ds. \tag{10.10}$$

Lemma 20. *Let F be the distribution function of f. Then*

$$\int_0^{f^*(t)} \int_0^{F(s)} g^*(u)du\, ds \leq f^*(t) \int_0^t g^*(s)ds + \int_t^{\infty} f^*(u)g^*(u)du. \tag{10.11}$$

Proof. Let I be the l. h. s. of (10.11). Fubini's theorem implies that

$$I = \int_0^{\infty} g^*(u)|\{s\ ;\ s < f^*(t) \text{ and } u < F(s)\}|du. \tag{10.12}$$

Note that $u < F(s) \implies f^*(u) > s$, and therefore

$$\{s\ ;\ s < f^*(t) \text{ and } u < F(s)\} \subset (0, \min(f^*(u), f^*(t))) = \begin{cases} (0, f^*(t)), & \text{if } u \leq t \\ (0, f^*(u)), & \text{if } u > t \end{cases}. \tag{10.13}$$

Thus

$$I \leq \int_0^t g^*(u)f^*(t)du + \int_t^{\infty} f^*(u)g^*(u)du, \tag{10.14}$$

whence the result. □

Lemma 21. *Let $A \subset \mathbb{R}^N$ be a measurable set of measure t. Then*

$$\int_A |f| \le \int_0^t f^*(s)ds. \tag{10.15}$$

Proof. We may replace f by $f\chi_A$ (assuming thus f supported in A), since in this way the l. h. s. of (10.15) remains unchanged, while the r. h. s. is not increased. In this case, we have $F(s) \le t$ for each s, and thus $g^*(s) = 0$ if $s \ge t$. Therefore,

$$\int_A |f| = \|f\|_{L^1} = \|f^*\|_{L^1} = \int_0^\infty f^*(s)ds = \int_0^t f^*(s)ds. \tag{10.16}$$

□

10.3 Rearrangement and convolutions

The reason we considered f^* is that it is related convolution products. We start with some elementary, though tricky, results linking these objects.

Lemma 22. *Let f be s. t. $|f| \le \alpha$ and $f = 0$ outside a set E s. t. $|E| = t$. Let $h = f * g$. Then*

$$|f * g| \le \alpha \int_0^t g^*(s)ds. \tag{10.17}$$

Proof. We have

$$|f * g(x)| \le \int_E |f(y)||g(x-y)|dy \le \alpha \int_{x-E} |g| \le \alpha \int_0^t g^*(s)ds, \tag{10.18}$$

since $|x - E| = t$. □

Lemma 23. *Let $f \in L^\infty$ and set $\alpha = \|f\|_{L^\infty}$. Then*

$$|f * g| \le \int_0^\alpha \int_0^{F(t)} g^*(u)du\, dt. \tag{10.19}$$

Proof. Using a monotone convergence argument, we may assume that f is a step function. Each step function may be written as $f = \sum_{j=1}^{k} a_j \chi_{A_j}$, where $a_j > 0$ and $0 < |A_k| < \ldots < |A_1| < \infty$. With $b_j = a_1 + ... + a_j$, we have $f = b_j$ in $A_j \setminus A_{j+1}$. We set $b_0 = 0$ and $A_0 = \mathbb{R}^N$. Note that $\alpha = b_k$.
Since $|f * g| \leq \sum_j a_j \chi_{A_j} * |g|$, the preceding lemma implies that

$$|f * g| \leq \sum a_j \int_0^{|A_j|} g^*(t) dt. \tag{10.20}$$

On the other hand, we have $F(t) = |A_j|$ if $t \in [b_{j-1}, b_j)$ and thus

$$\int_0^{\alpha} \int_0^{F(t)} g^*(u) du\, dt = \sum \int_{b_{j-1}}^{b_j} \int_0^{|A_j|} g^*(u) du\, dt = \sum (b_j - b_{j-1}) \int_0^{|A_j|} g^*(u) du = \sum a_j \int_0^{|A_j|} g^*(t) dt. \tag{10.21}$$

□

Lemma 24. *(***O'Neil***) Let $h = f * g$. Then*

$$h^*(3t) \leq \frac{3}{t} \int_0^t f^*(s) ds \int_0^t g^*(s) ds + \int_t^{\infty} f^*(s) g^*(s) ds. \tag{10.22}$$

Proof. We may assume that $f, g \geq 0$. We split $f = f_1 + f_2$, $g = g_1 + g_2$ and $h = h_1 + h_2 + h_3$. Here,
(i) f is cut at height $f^*(t)$, i. e., we set $f_1(x) = \begin{cases} f(x), & \text{if } f(x) > f^*(t) \\ 0, & \text{if } f(x) \leq f^*(t) \end{cases}$ and $f_2 = f - f_1$;
(ii) similarly, g is cut at height $g^*(t)$;
(iii) $h_1 = f_2 * g$, $h_2 = f_1 * g_2$ and $h_3 = f_1 * g_1$.
We start by noting that $f_2 \leq f$, and thus the distribution function of f_2 is dominated by the one of f. Lemma 23 implies that

$$h_1 \leq \int_0^{f^*(t)} \int_0^{F(s)} g^*(u) du\, ds \leq f^*(t) \int_0^t g^*(s) ds + \int_t^{\infty} f^*(u) g^*(u) du, \tag{10.23}$$

by Lemma 10.11.
Concerning h_2, the inequality $\|h_2\|_{L^\infty} \leq \|f_1\|_{L^1}\|g_2\|_{L^\infty}$ combined with Corollary 15 yields

$$h_2 \leq g^*(t) \int_0^t f^*(s)ds. \tag{10.24}$$

We next note that h_3 satisfies

$$\|h_3\|_{L^1} \leq \|f_1\|_{L^1}\|g_1\|_{L^1} \leq \int_0^t f^*(s)ds \int_0^t g^*(s)ds. \tag{10.25}$$

To conclude, we start with the inequality $h^*(3t) \leq (h_1)^*(t) + (h_2)^*(t) + (h_3)^*(t)$. For h_1 and h_2, we use the fact that $k^*(t) \leq \|k^*\|_{L^\infty} = \|k\|_{L^\infty}$. For h_3, we rely on the inequality

$$k^*(t) \leq \frac{1}{t}\int_0^t k^*(s)ds \leq \frac{1}{t}\int_0^\infty k^*(s)ds = \frac{1}{t}\|k^*\|_{L^1} = \frac{1}{t}\|k\|_{L^1}. \tag{10.26}$$

We find that

$$h^*(3t) \leq \frac{1}{t}\int_0^t f^*(s)ds\int_0^t g^*(s)ds + f^*(t)\int_0^t g^*(s)ds + g^*(t)\int_0^t f^*(s)ds + \int_t^\infty f^*(s)g^*(s)ds; \tag{10.27}$$

we complete the proof noting that $f^*(t) \leq \frac{1}{t}\int_0^t f^*(s)ds$ and a similar inequality holds for g. $\square$

Theorem 19. *(**O'Neil**; simplified version) Let $1 < p, q, r < \infty$ be s. t. $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$. If $f \in L^p$ and $g \in L^{q,w}$, then $f * g \in L^{r,p}$.*

Remark 11. *This statement is to be compared with the usual Young inequality, which asserts that $f * g \in L^r$ if $f \in L^p$ and $g \in L^q$. Our hypothesis is weaker, since $L^q \subset L^{q,w}$, while the conclusion is stronger, since $L^{r,p} \subset L^r$ (because $p < r$).*

Proof. Let $h = f * g$. We have to prove that $\|t^{1/r}h^*(t)\|_{L^p((0,\infty);dt/t)} < \infty$. Clearly, this is equivalent to proving that $\|t^{1/r}h^*(3t)\|_{L^p((0,\infty);dt/t)} < \infty$. In view of the preceding lemma, this amounts to proving the following:

(i) $\|t^{1/r-1}\int_0^t f^*(s)ds\int_0^t g^*(s)ds\|_{L^p((0,\infty);dt/t)} < \infty$;

(ii) $\|t^{1/r}\int\limits_t^\infty f^*(s)g^*(s)ds\|_{L^p((0,\infty);dt/t)} < \infty$.

The fact that $g \in L^{q,w}$ is equivalent to the boundedness of the map $t \mapsto t^{1/q}g^*(t)$, and thus $g^*(t) \leq Ct^{-1/q}$. It follows that $\int\limits_0^t g^*(s)ds \leq C't^{1-1/q}$, and therefore (i) and (ii) reduce to

(i') $\|t^{1/r-1/q}\int\limits_0^t f^*(s)ds\|_{L^p((0,\infty);dt/t)} < \infty$;

(ii) $\|t^{1/r}\int\limits_t^\infty s^{-1/q}f^*(s)ds\|_{L^p((0,\infty);dt/t)} < \infty$.

To deal with (i'), we apply to f^* the first Hardy's inequality (Theorem 3) with r replaced by $p-1>0$ and find that

$$\|t^{1/r-1/q}\int\limits_0^t f^*(s)ds\|^p_{L^p((0,\infty);dt/t)} = \int\limits_0^\infty t^{-p}\Big(\int\limits_0^t f^*(s)ds\Big)^p dt \leq C\int\limits_0^\infty (f^*(s))^p ds = C\|f^*\|^p_{L^p} < \infty, \tag{10.28}$$

since $\|f^*\|_{L^p} = \|f\|_{L^p}$.
Concerning (ii'), the second Hardy's inequality (Corollary 3) with r replaced by p/r and f replaced by $s \mapsto s^{-1/q}f^*(s)$ in order to obtain

$$\|t^{1/r}\int\limits_t^\infty s^{-1/q}f^*(s)ds\|^p_{L^p((0,\infty);dt/t)} = \int\limits_0^\infty t^{p/r-1}\Big(\int\limits_t^\infty s^{-1/q}f^*(s)ds\Big)^p dt \leq C\int\limits_0^\infty (f^*(s))^p ds < \infty. \tag{10.29}$$

□

Corollary 16. *Set, with p, q, r as in O'Neil's theorem, $a = N/q$. If $f \in L^p$, then $f * |x|^{-a} \in L^{r,p}$.*

Proof. It suffices to prove that $|x|^{-a} \in L^{q,w}$. This is obvious, since $|\{|x|^{-a} > t\}| = Ct^{-N/a} = Ct^{-q}$. □

10.4 Improved Sobolev embeddings

In the remaining part of this chapter, we assume that $N \geq 2$.
We start with a simple

Lemma 25. *Let $u \in C_0^\infty(\mathbb{R}^N)$. Then*

$$|u(x)| \leq \frac{1}{|S^{N-1}|} \int \frac{|Du(y)|}{|x-y|^{N-1}} dy. \tag{10.30}$$

Proof. Let $v \in S^{N-1}$. Then

$$u(x) = -[u(x+tv)]_{t=0}^{t=\infty} = -\int_0^\infty \frac{d}{dt}(u(x+tv))dt = -\int_0^\infty (Du)(x+tv) \cdot v dt, \tag{10.31}$$

and therefore

$$|u(x)| \leq \int_0^\infty |Du(x+tv)| dt. \tag{10.32}$$

Integrating this inequality over $v \in S^{N-1}$ we find that

$$|S^{N-1}||u(x)| \leq \int_{S^{N-1}} \int_0^\infty |Du(x+tv)| dt ds_v. \tag{10.33}$$

We conclude by noting that the change of variables $y = x + tv$, $t > 0$, $v \in S^{N-1}$ yields

$$\int \frac{|Du(y)|}{|x-y|^{N-1}} dy = \int_{S^{N-1}} \int_0^\infty |Du(x+tv)| dt ds_v. \tag{10.34}$$

□

We next recall the following well-known result

Theorem 20. *(***converse to the dominated convergence***) Let $1 \leq p \leq \infty$. If $f_n \to f$ in L^p, then there is a subsequence (f_{n_k}) and a function $g \in L^p$ s. t. $f_{n_k} \to f$ a. e. and $|f_{n_k}| \leq g$.*

Proof. After passing, if necessary, to a subsequence, we may assume that $f_n \to f$ a. e. Consider a subsequence (f_{n_k}) s. t. $\|f_{n_k} - f_{n_{k+1}}\|_{L^p} \leq 2^{-k}$ and set $g = |f_{n_0}| + \sum_{k \geq 0} |f_{n_k} - f_{n_{k+1}}|$. Then

$$\|g\|_{L^p} \leq \|f_{n_0}\|_{L^p} + \sum_{k \geq 0} \|f_{n_k} - f_{n_{k+1}}\|_{L^p} < \infty \tag{10.35}$$

and, clearly, $|f_{n_k}| \leq g$ for each k. □

Theorem 21. *(***O'Neil***) Let $1 < p < N$ and set $p^* = \dfrac{Np}{N-p}$. If $u \in W^{1,p}(\mathbb{R}^N)$, then $u \in L^{p^*,p}(\mathbb{R}^N)$.*

Proof. The strategy consists in proving the following generalization of (10.30)

$$|u(x)| \leq \frac{1}{|S^{N-1}|} \int \frac{|Du(y)|}{|x-y|^{N-1}} dy, \quad \forall\, u \in W^{1,p}(\mathbb{R}^N). \tag{10.36}$$

Assume (10.36) proved, for the moment. Corollary 16 avec $a = N-1$ implies that $|Du| * |x|^{-(N-1)} \in L^{p^*,p}$. Since $|u| \leq C|Du| * |x|^{-(N-1)}$ a. e., we obtain that $u \in L^{p^*,p}$.

It remains to prove (10.36). This is done by approximation. Consider a sequence $(u_n) \subset C_0^\infty$ s. t. $u_n \to u$ in $W^{1,p}$. Possibly after passing to a subsequence, we may assume that $u_n \to u$ and $Du_n \to Du$ outside a null set A and that $|Du_n| \leq g \in L^p$. Since $g * |x|^{-(N-1)} \in L^{p^*,p} \subset L^p$, we find that $\displaystyle\int \frac{g(y)}{|x-y|^{N-1}} dy < \infty$ for x outside a null set B. When $x \notin A \cup B$, we find, by dominated convergence, that

$$|u(x)| = \lim |u_n(x)| \leq \liminf \frac{1}{|S^{N-1}|} \int \frac{|Du_n(y)|}{|x-y|^{N-1}} dy = \frac{1}{|S^{N-1}|} \int \frac{|Du(y)|}{|x-y|^{N-1}} dy. \tag{10.37}$$

This completes the proof of the theorem. □

10.5 The limiting case $p = 1$

When $p = 1$, Theorem 19 is no longer true. To see this, we choose $f = \chi_B \in L^1$ (here, B is the unit ball) and $g(x) = |x|^{-\alpha}$, which belongs to $L^{q,w}$ if $\alpha q = N$. We have $f * g(x) = \displaystyle\int_B \frac{1}{|x-y|^\alpha} dy$. If $|x| \geq 2$, we have $|x - y| \sim |x|$ when $|y| \leq 1$ and thus $f * g(x) \sim |x|^{-\alpha}$ when $|x| \geq 2$. Therefore,

$$\|f * g\|_{L^{q,1}} \geq \|f * g\|_{L^q} \geq C \int_{\{|x|\geq 2\}} \frac{1}{|x|^N} = \infty. \tag{10.38}$$

A remarkable fact is that the conclusion of Theorem 21 still holds; the proof requires an argument that does not involves convolution products. We start with one essential ingredient which is the isoperimetric inequality. We will not need the sharp (i. e., with the best constant) version, so that we will simply prove the following

Theorem 22. *(weak form of the isoperimetric inequality) Let $\mathcal{O}$ be a smooth bounded domain in $\mathbb{R}^N$ and let Σ be its boundary. Then*

$$|\mathcal{O}| \leq C|\Sigma|^{N/(N-1)}. \tag{10.39}$$

Proof. Let $\rho \in C_0^\infty$ be s. t. $\rho \geq 0$, $\displaystyle\int \rho = 1$ and supp $\rho \subset B(0,1)$. We apply the Sobolev inequality $\|u\|_{L^{N/(N-1)}} \leq C\|Du\|_{L^1}$ to the function $u = \chi_{\mathcal{O}} * \rho_\varepsilon$ and find that

$$\|\chi_{\mathcal{O}} * \rho_\varepsilon\|_{L^{N/(N-1)}} \leq C \sum_j \int_{\mathbb{R}^N} \Big|\int_{\mathcal{O}} \partial_j \rho_\varepsilon(x-y)dy\Big|dx = C \sum_j \int_{\mathbb{R}^N} \Big|\int_{\Sigma} n_j \rho_\varepsilon(x-y)ds_y\Big|dx; \tag{10.40}$$

here, $n_j = n_j(y)$ is the j$^{\text{th}}$ component of the outer normal $\mathbf{n}$ to $\mathcal{O}$ at y. Thus

$$\|\chi_{\mathcal{O}} * \rho_\varepsilon\|_{L^{N/(N-1)}} \leq C \int_{\Sigma} \int_{\mathbb{R}^N} \rho_\varepsilon(x-y)dxds_y = C|\Sigma|. \tag{10.41}$$

On the other hand, we have $\chi_{\mathcal{O}} \in L^{N/(N-1)}$ and thus $\|\chi_{\mathcal{O}} * \rho_\varepsilon\|_{L^{N/(N-1)}} \to \|\chi_{\mathcal{O}}\|_{L^{N/(N-1)}}$ as $\varepsilon \to 0$. This leads us to

$$|\mathcal{O}|^{(N-1)/N} = \|\chi_{\mathcal{O}}\|_{L^{N/(N-1)}} \leq C|\Sigma|, \tag{10.42}$$

which ends the proof. □

Theorem 23. *We have*

$$\|u\|_{L^{N/(N-1),1}} \leq C\|Du\|_{L^1}, \quad \forall\ u \in W^{1,1}(\mathbb{R}^N). \tag{10.43}$$

Proof. The strategy of the proof is the following: we first prove the inequality (10.43) when $u \in C_0^\infty$; the general case will be obtained from this one by passing to the limits.
Let $u \in C_0^\infty$; Sard's theorem insures that fact that, for a. e. $t > 0$, all the points x s. t. $|u(x)| = t$ satisfy $\nabla u(x) \neq 0$; in other words, the set $\Sigma_t = \{|u| = t\}$ is a smooth hyper surface. For any such t, set $\mathcal{O}_t = \{|u| > t\}$, which is a bounded open set. We claim that (*) $\mathcal{O}_t$ is a smooth domain with boundary Σ_t. Indeed, it is obvious that $\partial\mathcal{O}_t \subset \Sigma_t$. On the other hand, if $x \in \Sigma_t$ and we set $v = \nabla u(x)$, then Taylor's formula implies that $u(x + sv) - t$ has the sign of s when s is close to 0. Thus on the one hand $x \in \partial\mathcal{O}_t$, on the other hand $\mathcal{O}_t$ is locally on one side of Σ_t, which is the same as (*).
With F the distribution function of u, we have

$$F(t) = |\mathcal{O}_t| \leq C|\Sigma_t|^{N/(N-1)}, \tag{10.44}$$

by the weak isoperimetric inequality. Thus, with $H_t = \{u = t\}$, we have

$$\|u\|_{L^{N/(N-1),w}} = \int F^{(N-1)/N}(t)dt \leq C \int_0^\infty |\Sigma_t|dt = C \int_{\mathbb{R}} |H_t|dt = C \int |Du|; \tag{10.45}$$

the last equality follows from the coarea formula we will prove later.
We next turn to a general $u \in W^{1,1}$. Consider a sequence $(u_n) \subset C_0^\infty$ s. t. $u_n \to u$ in $W^{1,1}$ and

pointwise outside an exceptional zero measure set B. We claim that the corresponding distribution functions, F and F_n, satisfy $F(t) \leq \liminf F_n(t)$ for each t. Indeed, let $A = \{|u| > t\}$, $A_n = \{|u_n| > t\}$. If $x \in A \setminus B$, then $x \in A_n$ for sufficiently large n. Put it otherwise, $A \setminus B \subset \liminf(A_n \setminus B)$, and thus

$$F(t) = |A| = |A \setminus B| \leq \liminf |A_n \setminus B| = \liminf |A_n| = \liminf F_n(t). \tag{10.46}$$

Fatou's lemma implies than that

$$\|u\|_{L^{N/(N-1),1}} = \int F^{(N-1)/N}(t)dt \leq \liminf \int F_n^{(N-1)/N}(t)dt \leq C \int |Du|. \tag{10.47}$$

We have thus obtained (10.43) in full generality. □

10.6 The limiting case $p = N$

The conclusion of Theorem 19 is wrong when $1/p + 1/q = 1$ (and $p \neq 1$). Indeed, let $f(x) = \chi_{B(0,1/2)}|x|^{-N/p}|\ln|x||^{-\beta}$, where $\beta > 1/p$. Then $f \in L^p$. Let also $g(x) = |x|^{-N/q} \in L^{q,w}$. We claim that $f * g \notin L^\infty$ if β is well-chosen. We start by noting that Fatou's lemma implies that $f * g(0) \leq \liminf_{x \to 0} f * g(x)$. Therefore, $f * g \notin L^\infty$ if $f * g(0) = \infty$. Since $f * g(0) = \int_{\{|x| \leq 1/2\}} |x|^{-N}|\ln|x||^{-\beta}dx$, we find that $f * g(0) = \infty$ if $\beta \leq 1$.

Consequently, we may not use Theorem 19 in the proof of Theorem 21 when $p = N$. Actually, when $p = N$, the expected conclusion of Theorem 21, namely $W^{1,N} \subset L^\infty$, is wrong: it is easy to see that the function given by $f(x) = \chi_{B(0,1/2)}|\ln|x||^\alpha$, where $0 < \alpha < 1 - 1/N$, belongs to $W^{1,N}$, but not to L^∞. However, we will see that each function in $W^{1,N}$ is "almost" bounded. We start with a simple (and non optimal) result.

Proposition 17. *There are constants $c, C > 0$ s. t. $\frac{1}{|B|}\int_B \exp(c|u - u_B|) \leq C$ for each $u \in W^{1,N}(\mathbb{R}^N)$ s. t. $\|Du\|_{L^N} \leq 1$.*

Proof. The above estimate follows immediately from Theorem 16 and the following result. □

Proposition 18. *We have, for some C depending only on N,*

$$\frac{1}{|B|}\int_B |u - u_B| \leq C\|Du\|_{L^N}, \quad \forall\ u \in W^{1,N}. \tag{10.48}$$

Proof. It suffices to prove (10.48) when $u \in C_0^\infty$; the general case is obtained by passing to the limits in (10.48) when B is kept fixed. If $B = B(x, R)$, then

$$\frac{1}{|B|}\int\limits_B |u - u_B| = \frac{1}{|B|^2}\int\limits_B |\int\limits_{B(0,R)} (u(y+z) - u(y))dz|dy \le \frac{1}{|B|^2}\int\limits_B\int\limits_{B(0,R)} |u(y+z) - u(y)|dy\,dz. \tag{10.49}$$

Applying Taylor's formula in integral form, we find that

$$\frac{1}{|B|}\int\limits_B |u-u_B| \le \frac{1}{|B|^2}\int\limits_B\int\limits_{B(0,R)}\int\limits_0^1 |(Du)(y+tz)||z|dt\,dz\,dy \le \frac{C}{R^{2N-1}}\int\limits_B\int\limits_{B(0,R)}\int\limits_0^1 |(Du)(y+tz)|dt\,dz\,dy. \tag{10.50}$$

For each t and z, Hölder's inequality implies that

$$\int\limits_B |(Du)(y+tz)|dy \le \|Du\|_{L^N}|B(x,2R)|^{(N-1)/N} \le \frac{C}{R^{N-1}}\|Du\|_{L^N}, \tag{10.51}$$

so that

$$\frac{1}{|B|}\int\limits_B |u - u_B| \le \frac{C\|Du\|_{L^N}}{R^N}\int\limits_{B(0,R)}\int\limits_0^1 |(Du)(y+tz)|dt\,dz \le C\|Du\|_{L^N}. \tag{10.52}$$

□

We may actually replace $|u|$ by $|u|^{N/(N-1)}$ in the preceding exponential integrability result. The statement we give below includes the assumption that supp $u \subset B$. This is not a crucial assumption; if we want to remove it, it suffices to apply the theorem when B is replaced by B^* (the ball concentric with B and twice larger) and u is replaced by φu, where φ is a cutoff function supported in B^* and that equals 1 in B. However, the resulting inequality is less elegant.

Theorem 24. (**Trudinger**) *Let u be a $W^{1,N}$ function supported in B. If $\|Du\|_{L^N} \le 1$, then*

$$\frac{1}{|B|}\int\limits_B \exp\left(c|u|^{N/(N-1)}\right) \le C, \tag{10.53}$$

where $c, C > 0$ depend only on N.

Proof. We may assume that $B = B(0, R)$. We start by noting that (10.30) is valid for u as above. Indeed, u being compactly supported, it belongs to $W^{1,2N/3}$; we may therefore rely on (10.36). Let now $f = |Du|$, which belongs to L^N and is supported in B. Set $g = f * |x|^{-(N-1)}$. In view of

(10.30), it suffices to prove that $\frac{1}{|B|}\int\limits_B \exp\left(c\, g^{N/(N-1)}\right) \leq C$ provided that $\|f\|_{L^N} \leq 1$. The key result in proving this estimate is the following inequality

$$g(x) \leq C(\delta \mathcal{M}f(x) + (\ln(2R/\delta))^{(N-1)/N}), \quad \forall\, x \in B, \forall\, \delta \in (0, R]. \tag{10.54}$$

Assume (10.54) proved, for the moment. We consider, for $x \in B$, the two following possibilities:
(i) if x is s. t. $\mathcal{M}f(x) \leq R^{-1}$, we choose $\delta = R$ and find that $g(x) \leq C$;
(ii) if $\mathcal{M}f(x) > R^{-1}$, we choose $\delta = 1/\mathcal{M}f(x)$ and find that $g(x) \leq C(1 + \ln(R\mathcal{M}f(x))^{(N-1)/N})$.
Thus, in any event, we have $g(x)^{N/(N-1)} \leq C(1 + (\ln(R\mathcal{M}f(x))_+)$, so that $\exp\left(c\, g^{N/(N-1)(x)}\right) \leq C_1(1 + (R\mathcal{M}f(x))^{cC_2})$. Choosing c s. t. $cC_2 = N$, we find that

$$\frac{1}{|B|}\int\limits_B \exp\left(c\, g^{N/(N-1)}\right) \leq C\int\limits_B (1 + R^N \mathcal{M}f^N) \leq C(1 + \int \mathcal{M}f^N) \leq C(1 + \|f\|^N_{L^N}) \leq C, \tag{10.55}$$

by the maximal inequalities.
It thus remains to establish (10.54). We have

$$g(x) = \int\limits_{B(0,\delta)} \frac{f(x-y)}{|y|^{N-1}}dy + \int\limits_{B(x,R)\setminus B(0,\delta)} \frac{f(x-y)}{|y|^{N-1}}dy = I_1 + I_2. \tag{10.56}$$

To estimate I_1, we note that $I_1 = f * h(x)$, where $h(y) = \chi_{B(0,\delta)}|y|^{-(N-1)}$. Since h is integrable, radial and non increasing, we have $I_1 \leq \mathcal{M}f(x)\|h\|_{L^1} = C\delta\mathcal{M}f(x)$. We complete the proof of (10.54) by noting that Hölder's inequality combined with the fact that $\|f\|_{L^N} = 1$ yields

$$I_2 \leq \left(\int\limits_{B(x,R)\setminus B(0,\delta)} |y|^{-N}\right)^{(N-1)/N} \leq \left(\int\limits_{B(0,2R)\setminus B(0,\delta)} |y|^{-N}\right)^{(N-1)/N} = C(\ln(2R/\delta))^{(N-1)/N}. \tag{10.57}$$

□

Chapter 11

Traces

11.1 Definition of the trace

We discuss here the properties of the "restrictions" of Sobolev maps to hyper surfaces, e. g., to the boundary of a smooth domain. There is a standard reduction procedure which allows to replace a "smooth" (at least Lipschitz) hyper surface with a hyperplane; this is done by flattening locally the coordinates. Since this part works without any problem and we want to insist on the analytic part, we will simply consider in this chapter maps defined in the whole $\mathbb{R}^N$ and consider properties of their trace on the hyperplane $H = \{x = (x', x_N) \in \mathbb{R}^N \ ; \ x_N = 0\}$, which we identify with $\mathbb{R}^{N-1}$. We start by recalling the following

Proposition 19. *The map $u \mapsto u_{|H}$, initially defined from $C_0^\infty(\mathbb{R}^N)$ into $C_0^\infty(\mathbb{R}^{N-1})$, extends uniquely by density to a linear map (called* **trace map***) $u \mapsto \text{tr } u$ from $W^{1,p}(\mathbb{R}^N)$ into $L^p(\mathbb{R}^N)$, for $1 \le p < \infty$.*

There also is a statement concerning $W^{1,\infty}$ (we will see it in the last part), but in that case the trace is not defined in the same way, since C_0^∞ is not dense in $W^{1,\infty}$.

Proof. Fix a function $\varphi \in C_0^\infty(\mathbb{R})$ s. t. $\varphi(0) = 1$ and supp $\varphi \subset (-1, 1)$. If $u \in C_0^\infty$, then $v = u\varphi(x_N) \in C_0^\infty(\mathbb{R}^{N-1} \times (-1, 1))$ and $u_{|H} = v_{|H}$. In addition, it is clear that $\|v\|_{W^{1,p}} \le C\|u\|_{W^{1,p}}$. It therefore suffices to prove that $\|v_{|H}\|_{L^p} \le C\|v\|_{W^{1,p}}$. This follows from

$$\int\limits_H |v(x', 0)|^p dx' = \int\limits_H \left| \int\limits_0^1 \partial_N v(x', t) dt \right|^p dx' \le \int\limits_{H \times (0,1)} |Dv|^p \le \|Dv\|_{L^p}^p. \tag{11.1}$$

□

When $1 < p < \infty$, the above result is not sharp, in the following sense: if f is an arbitrary map in $L^p(\mathbb{R}^{N-1})$, we can not always find a map $u \in W^{1,p}$ s. t. tr $u = f$. In other words, the trace

map is not surjective between the spaces we consider.
In this chapter, we will determine the image of the trace map.

Definition 5. *For* $0 < s < 1$ *and* $1 \leq p < \infty$*, we define*

$$W^{s,p} = W^{s,p}(\mathbb{R}^N) = \{f \in L^p(\mathbb{R}^N) \ ; \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|f(x)-f(y)|^p}{|x-y|^{N+sp}} dx\, dy < \infty\}, \tag{11.2}$$

equipped with the norm

$$\|f\|_{W^{s,p}} = \|f\|_{L^p} + \left(\int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|f(x)-f(y)|^p}{|x-y|^{N+sp}} dx\, dy\right)^{1/p}. \tag{11.3}$$

We let the reader check that $W^{s,p}$ is a Banach space.
The main result of this chapter states that tr $W^{1,p}(\mathbb{R}^N) = W^{1-1/p,p}(\mathbb{R}^{N-1})$. We start with some preliminary results.

Lemma 26. $C^\infty(\mathbb{R}^N) \cap W^{s,p}(\mathbb{R}^N)$ *is dense into* $W^{s,p}(\mathbb{R}^N)$ *for* $0 < s < 1$ *and* $1 \leq p < \infty$*.*

Proof. Let ρ be a standard mollifier (i. e., $\rho \in C_0^\infty$, $\rho \geq 0$, $\displaystyle\int \rho = 1$, supp $\rho \subset B(0,1)$). We will prove that, if $f \in W^{s,p}$, then $f_\varepsilon = f * \rho_\varepsilon \to f$ in $W^{s,p}$. Clearly, $f_\varepsilon \to f$ in L^p. It remains to prove that, with $g_\varepsilon = f_\varepsilon - f$, we have

$$I_\varepsilon = \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|g_\varepsilon(x)-g_\varepsilon(y)|^p}{|x-y|^{N+sp}} dx\, dy = \int\limits_{\mathbb{R}^N}\int\limits_{\mathbb{R}^N} \frac{|g_\varepsilon(x+h)-g_\varepsilon(x)|^p}{|h|^{N+sp}} dx\, dh \to 0 \quad \text{as } \varepsilon \to 0. \tag{11.4}$$

In order to estimate I_ε, we start by noting that

$$g_\varepsilon(x+h) - g_\varepsilon(x) = \int\limits_{B(0,\varepsilon)} (f(x+h-y) - f(x+h) - f(x-y) + f(x))\rho_\varepsilon(y) dy. \tag{11.5}$$

Using in addition the fact that $\rho_\varepsilon \leq C\varepsilon^{-N}$, we find that

$$|g_\varepsilon(x+h) - g_\varepsilon(x)| \leq \frac{C}{\varepsilon^N} \int\limits_{B(0,\varepsilon)} |f(x+h-y) - f(x+h) - f(x-y) + f(x)| dy. \tag{11.6}$$

We next consider the two following cases:
(i) if $|h| < \varepsilon$, we have

$$|g_\varepsilon(x+h) - g_\varepsilon(x)| \leq \frac{C}{\varepsilon^N} \int\limits_{B(0,\varepsilon)} (|f(x+h-y) - f(x-y)| + |f(x) - f(x+h)| dy; \tag{11.7}$$

(ii) if $|h| \geq \varepsilon$, we use the inequality

$$|g_\varepsilon(x+h) - g_\varepsilon(x)| \leq \frac{C}{\varepsilon^N} \int\limits_{B(0,\varepsilon)} (|f(x+h-y) - f(x+h)| + |f(x) - f(x-y)|dy. \tag{11.8}$$

Thus $I_\varepsilon \leq \dfrac{C}{\varepsilon^{Np}}(J_1 + J_2 + J_3 + J_4)$, where

$$J_1 = \int\limits_{\mathbb{R}^N} \int\limits_{\{|h|<\varepsilon\}} \Big(\int\limits_{B(0,\varepsilon)} |f(x+h-y) - f(x-y)|dy \Big)^p |h|^{-(N+sp)} dh\, dx; \tag{11.9}$$

$$J_2 = \int\limits_{\mathbb{R}^N} \int\limits_{\{|h|<\varepsilon\}} \Big(\int\limits_{B(0,\varepsilon)} |f(x+h) - f(x)|dy \Big)^p |h|^{-(N+sp)} dh\, dx; \tag{11.10}$$

$$J_3 = \int\limits_{\mathbb{R}^N} \int\limits_{\{|h|\geq\varepsilon\}} \Big(\int\limits_{B(0,\varepsilon)} |f(x+h-y) - f(x+h)|dy \Big)^p |h|^{-(N+sp)} dh\, dx; \tag{11.11}$$

$$J_4 = \int\limits_{\mathbb{R}^N} \int\limits_{\{|h|\geq\varepsilon\}} \Big(\int\limits_{B(0,\varepsilon)} |f(x-y) - f(x)|dy \Big)^p |h|^{-(N+sp)} dh\, dx. \tag{11.12}$$

We will prove that $\varepsilon^{Np} J_j \to 0$, $j = 1, \ldots, 4$. The only ingredient we use in the proof is

$$\lim_{\varepsilon \to 0} \int\limits_{\mathbb{R}^N} \int\limits_{B(0,\varepsilon)} \frac{|f(x+y) - f(x)|^p}{|y|^{N+sp}} dy = 0; \tag{11.13}$$

this follows easily by dominated convergence.

We start with J_2. Noting that $\Big(\int\limits_{B(0,\varepsilon)} |f(x+h) - f(x)|dy \Big)^p = C\varepsilon^{Np} |f(x+h) - f(x)|^p$, we find that

$$\varepsilon^{-Np} J_2 = C \int\limits_{\mathbb{R}^N} \int\limits_{B(0,\varepsilon)} \frac{|f(x+h) - f(x)|^p}{|h|^{N+sp}} dh \to 0. \tag{11.14}$$

For J_1, Hölder's inequality with exponents p and p' implies that

$$\Big(\int\limits_{B(0,\varepsilon)} |f(x+h-y) - f(x-y)|dy \Big)^p \leq |B(0,\varepsilon)|^{p-1} \int\limits_{B(0,\varepsilon)} |f(x+h-y) - f(x-y)|^p dy, \tag{11.15}$$

and thus

$$\varepsilon^{-Np}J_1 \le C\varepsilon^{-N}\int\limits_{\mathbb{R}^N}\int\limits_{\{|h|<\varepsilon\}}\int\limits_{B(0,\varepsilon)} |f(x+h-y)-f(x-y)|^p dy\,|h|^{-(N+sp)}dh\,dx. \tag{11.16}$$

For fixed y and h, the change of variables $x-y=z$ leads to

$$\varepsilon^{-Np}J_1 \le C\int\limits_{\mathbb{R}^N}\int\limits_{\{|h|<\varepsilon\}} |f(z+h)-f(z)|^p|h|^{-(N+sp)}dh\,dz \to 0. \tag{11.17}$$

We next estimate J_3; the computation for J_4 is similar and will be omitted. Inequality (11.15) implies that

$$\varepsilon^{-Np}J_3 \le C\varepsilon^{-N}\int\limits_{\mathbb{R}^N}\int\limits_{\{|h|\ge\varepsilon\}}\int\limits_{B(0,\varepsilon)} |f(x+h-y)-f(x+h)|^p|h|^{-(N+sp)}dy\,dh\,dx. \tag{11.18}$$

In this integral, we fix y and h and make the change of variables $x+h=z$. Next we integrate in h and find that

$$\varepsilon^{-Np}J_3 \le C\int\limits_{\mathbb{R}^N}\int\limits_{B(0,\varepsilon)} \frac{|f(z-y)-f(z)|^p}{\varepsilon^{N+sp}}dy\,dz \le C\int\limits_{\mathbb{R}^N}\int\limits_{B(0,\varepsilon)} \frac{|f(z-y)-f(z)|^p}{|y|^{N+sp}}dy\,dz \to 0. \tag{11.19}$$

□

Lemma 27. *If* $u \in C(\mathbb{R}^N)\cap W^{1,p}$, *then* $\operatorname{tr} u = u_{|H}$.

Proof. Let ρ be a standard mollifier and set $u_\varepsilon = \rho(\varepsilon\cdot)(u*\rho_\varepsilon)$. Clearly, $u_\varepsilon \in C_0^\infty$ and $u_\varepsilon \to u$ in $W^{1,p}$. Thus $u_{\varepsilon|H} = \operatorname{tr} u_\varepsilon \to \operatorname{tr} u$ in L^p (and thus in $\mathcal{D}'$). On the other hand, $u_{\varepsilon|H}$ converges to $u_{|H}$ uniformly on compacts (and thus in $\mathcal{D}'$), whence the conclusion. □

The same argument leads to the following variant

Lemma 28. *Assume that* $u \in W^{1,p}$ *is continuous in a neighborhood of* H. *Then* $\operatorname{tr} u = u_{|H}$.

Lemma 29. *Let* $u \in C(\mathbb{R}^N)\cap C^1(\mathbb{R}^N\setminus H)$. *Assume that the pointwise differential* Du *of* u *satisfies* $Du \in L^p(\mathbb{R}^N)$. *Then* Du *is also the distributional differential of* u.

Proof. We have to prove that $\displaystyle\int D_j u\varphi = -\int u\partial_j\varphi$, $\varphi \in C_0^\infty$, $j=1,\ldots,N$. When $j \le N-1$, this follows simply by Fubini's theorem. Assume $j=N$. We integrate by parts $\displaystyle\int D_N u\varphi$ in the set $\{x\in\mathbb{R}^N\ ;\ |x_N|>\varepsilon\}$ and next let $\varepsilon\to 0$. We find that

$$\int D_N u\varphi = \lim_{\varepsilon\to 0}\left(\int\limits_{\{x_N=-\varepsilon\}} u\varphi ds_{x'} - \int\limits_{\{x_N=\varepsilon\}} u\varphi ds_{x'} - \int\limits_{\{|x_N|>\varepsilon\}} u\partial_N\varphi dx\right) = -\int u\partial_N\varphi dx. \tag{11.20}$$

□

11.2 Trace of $W^{1,p}$, $1 < p < \infty$

Theorem 25. *(**Gagliardo**) Let $p \in (1,\infty)$.*
a) If $u \in W^{1,p}(\mathbb{R}^N)$, then $\operatorname{tr} u \in W^{1-1/p,p}(\mathbb{R}^{N-1})$ and $\|\operatorname{tr} u\|_{W^{1-1/p,p}} \le C\|u\|_{W^{1,p}}$.
b) Conversely, let $f \in W^{1-1/p,p}(\mathbb{R}^{N-1})$. Then there is some $u \in W^{1,p}(\mathbb{R}^N)$ s. t. $\operatorname{tr} u = f$. In addition, we may pick u s. t. $\|u\|_{W^{1,p}} \le C\|\operatorname{tr} u\|_{W^{1-1/p,p}}$.

Remark 12. *Let $T : W^{1,p}(\mathbb{R}^N) \to W^{1-1/p,p}(\mathbb{R}^{N-1})$, $Tu = \operatorname{tr} u$. T is linear, and the above theorem implies that T is continuous and surjective. Then the last conclusion in b) follows from the open map theorem (each surjective linear continuous map between two Banach spaces has a bounded right inverse). However, we will see during the proof a stronger conclusion: we will construct in b) a* **linear** *right inverse, i. e. , the map $f \mapsto u$ in b) will be linear.*

Proof. a) By density, it suffices to prove that

$$\|u_{|H}\|_{W^{1-1/p,p}} \le C\|u\|_{W^{1,p}} \quad \forall\, u \in C_0^\infty. \tag{11.21}$$

We start by noting that we already know that $\|u_{|H}\|_{L^p} \le C\|u\|_{W^{1,p}}$; thus it suffices to establish, with $f(x') = u(x',0)$, the inequality

$$I = \int\limits_{\mathbb{R}^{N-1}} \int\limits_{\mathbb{R}^{N-1}} \frac{|f(x'+h') - f(x')|^p}{|h'|^{N+p-2}} dh'\, dx' \le C \int\limits_{\mathbb{R}^N} |Du(x)|^p dx. \tag{11.22}$$

The starting point is the inequality

$$|f(x'+h') - f(x')| \le |f(x'+h') - u(x'+h'/2, |h'|/2)| + |f(x') - u(x'+h', |h'|/2)|, \tag{11.23}$$

which implies that $I \le C(I_1 + I_2)$, where

$$I_1 = \int\limits_{\mathbb{R}^{N-1}} \int\limits_{\mathbb{R}^{N-1}} \frac{|f(x'+h') - u(x'+h'/2, |h'|/2)|^p}{|h'|^{N+p-2}}, I_2 = \int\limits_{\mathbb{R}^{N-1}} \int\limits_{\mathbb{R}^{N-1}} \frac{|f(x') - u(x'+h'/2, |h'|/2)|^p}{|h'|^{N+p-2}}. \tag{11.24}$$

If we perform, in I_1, the change of variables $x' + h' = y'$, next we change h' into $-h'$, we see that $I_1 = I_2$, and thus

$$I \le C \int\limits_{\mathbb{R}^{N-1}} \int\limits_{\mathbb{R}^{N-1}} \frac{|f(x') - u(x'+h'/2, |h'|/2)|^p}{|h'|^{N+p-2}} dh'\, dx'. \tag{11.25}$$

Changing h' into $2k'$ and applying the Leibniz-Newton formula, we find that

$$I \le C \int\limits_{\mathbb{R}^{N-1}} \int\limits_{\mathbb{R}^{N-1}} \left(\int\limits_0^{|k'|} |Du(x' + t(k'/|k'|), t)| \right)^p |k'|^{-(N+p-2)} dk'\, dx'. \tag{11.26}$$

Expressing k' in polar coordinates, we find that

$$I \le C \int_{\mathbb{R}^{N-1}} \int_{S^{N-2}} \int_0^\infty \Big(\int_0^s |Du(x'+t\omega,t)|dt \Big)^p s^{-p} ds\, ds_\omega\, dx'. \tag{11.27}$$

Applying, for fixed x' and ω, Hardy's inequality in to the double integral in s and t, we find that

$$I \le C \int_{\mathbb{R}^{N-1}} \int_{S^{N-2}} \int_0^\infty |Du(x'+t\omega,t)|^p dt\, ds_\omega\, dx'. \tag{11.28}$$

Integrating, in the above inequality, first in x', next in ω, we find that

$$I \le C \int_{\mathbb{R}^{N-1}} \int_0^\infty |Du(x',t)|^p dt\, dx' = \int_{\mathbb{R}^{N+}} |Du(x)|^p dx \le C\|Du\|_{L^p}^p. \tag{11.29}$$

b) It suffices to construct a linear map $f \mapsto u$, $f \in C^\infty(\mathbb{R}^{N-1}) \cap W^{1-1/p,p}$, $u \in W^{1,p}(\mathbb{R}^N)$, s. t. tr $u = f$ and $\|u\|_{W^{1,p}} \le C\|f\|_{W^{1-1/p,p}}$. We fix a standard mollifier ρ in $\mathbb{R}^{N-1}$ and an even function $\varphi \in C^\infty(\mathbb{R})$ s. t. $\varphi(0) = 1$, $0 \le \varphi \le 1$ and supp $\varphi \subset (-1/2, 1/2)$. We define, for $t \ne 0$, $v(x',t) = f * \rho_{|t|}(x')$ and $u(x',t) = v(x',t)\varphi(t)$. We extend u to $\mathbb{R}^N$ by setting $u(x',0) = f(x')$. Clearly, the map $f \mapsto u$ is linear and $u \in C^\infty(\mathbb{R}^N \setminus H)$. In addition, $u \in C(\mathbb{R}^N)$ when f is continuous. We also note that, for a fixed $t \ne 0$, Young's inequality implies that $\|f * \rho_{|t|}\|_{L^p} \le \|f\|_{L^p}$, and thus $\|u\|_{L^p} \le \|f\|_{L^p}$. Since u is even with respect to x_N, it suffices to prove, in view of Lemmata 27 and 29, that the usual differential Du of u satisfies

$$\int_{\mathbb{R}^{N-1}} \int_0^\infty |Du(x',t)|^p dt\, dx' \le C \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}^{N-1}} \frac{|f(x'+y') - f(x')|^p}{|y|^{N+p-2}} dy'\, dx' + C\|f\|_{L^p}^p. \tag{11.30}$$

For $1 \le j \le N-1$, we have $|\partial_j u| \le |\partial_j v|$. On the other hand, $|\partial_N u| \le C|v|\chi_{\mathbb{R}^{N-1}\times(-1/2,1/2)} + |\partial_N v|$. Since $\||v|\chi_{\mathbb{R}^{N-1}\times(-1/2,1/2)}\|_{L^p} \le \|u\|_{L^p}$, it suffices to prove the estimate

$$\int_{\mathbb{R}^{N-1}} \int_0^\infty |Dv(x',t)|^p dt\, dx' \le C \int_{\mathbb{R}^{N-1}} \int_{\mathbb{R}^{N-1}} \frac{|f(x'+y') - f(x')|^p}{|y|^{N+p-2}} dy'\, dx'. \tag{11.31}$$

Let $1 \le j \le N-1$. Since $\displaystyle\int \partial_j \rho = 0$, we have

$$\partial_j v(x',t) = t^{-N} \int f(y')(\partial_j \rho)((x'-y')/t) dy' = t^{-N} \int [f(y') - f(x')](\partial_j \rho)((x'-y')/t) dy', \tag{11.32}$$

so that

$$|\partial_j v(x',t)| \le \frac{C}{t^N} \int\limits_{B(0,t)} |f(x'+y') - f(x')|dy'. \tag{11.33}$$

We next claim that $\displaystyle\int \frac{d}{dt}[\rho_t(x')]dx' = 0$. This follows from the fact that $\displaystyle\int \rho_t \equiv 1$. Thus

$$|\partial_N v(x',t)| = |\int [f(y') - f(x')]\frac{d}{dt}[\rho_t(x'-y')]dy'| \le \frac{C}{t^N} \int\limits_{B(0,t)} |f(x'+y') - f(x')|dy', \tag{11.34}$$

since $\left|\dfrac{d}{dt}\rho_t\right| \le Ct^{-N}$. We find that $|Dv(x',t)| \le \dfrac{C}{t^N} \displaystyle\int\limits_{B(0,t)} |f(x'+y') - f(x')|dy'$, and therefore it suffices to establish the estimate

$$I = \int\limits_{\mathbb{R}^{N-1}} \int\limits_0^\infty \Big(\int\limits_{B(0,t)} |f(x'+y') - f(x')|dy' \Big)^p t^{-Np} dt\, dx' \le C \int\limits_{\mathbb{R}^{N-1}} \int\limits_{\mathbb{R}^{N-1}} \frac{|f(x'+y') - f(x')|^p}{|y|^{N+p-2}} dy'\, dx'. \tag{11.35}$$

This is done as in the proof of lemma 26: Hölder's inequality applied to the integral over $B(0,t)$ implies that

$$I \le C \int\limits_{\mathbb{R}^{N-1}} \int\limits_0^\infty \int\limits_{B(0,t)} |f(x'+y') - f(x')|^p dy'\, t^{-N-p+1} dt\, dx'. \tag{11.36}$$

Fubini's theorem yields

$$I \le C \int\limits_{\mathbb{R}^{N-1}} \int\limits_{\mathbb{R}^{N-1}} \int\limits_{|y'|}^\infty t^{-N-p+1} dt\, dx'\, dy' = C \int\limits_{\mathbb{R}^{N-1}} \int\limits_{\mathbb{R}^{N-1}} \frac{|f(x'+y') - f(x')|^p}{|y|^{N+p-2}} dy'\, dx'. \tag{11.37}$$

□

Corollary 17. *Let $f \in W^{1-1/p,p}(\mathbb{R}^N)$ and set , for $t \neq 0$, $u(x',t) = f * \rho_{|t|}(x')\varphi(t)$. Then $u \in W^{1,p}$ and* $\operatorname{tr} u = f$.

11.3 Trace of $W^{1,1}$

We start with some auxiliary results needed in the proof of the fact that the trace of $W^{1,1} = L^1$.

Lemma 30. *Let $u \in W^{1,p} \cap W^{1,q}$. Then the two traces of u (one in $W^{1,p}$, the other one in $W^{1,q}$), coincide.*

Proof. If ρ is a standard mollifier, then $u_\varepsilon = \rho(\varepsilon\cdot)u * \rho_\varepsilon$ converges (as $\varepsilon \to 0$) to u both in $W^{1,p}$ and in $W^{1,q}$. Since, for $u_\varepsilon \in C_0^\infty$, both traces coincide, we obtain the result by passing to the limits. □

The same argument leads to the following result.

Lemma 31. *Let $u \in W^{1,p}$. For $\lambda \neq 0$ and $x' \in \mathbb{R}^{N-1}$, we have* tr $u(\lambda \cdot -x') = (\text{tr } u)(\lambda \cdot -x')$.

Lemma 32. *Let f be the characteristic function of a cube in $\mathbb{R}^{N-1}$. Then $f \in W^{1-1/p,p}$ for $1 < p < 2$.*

Proof. We may assume that $C = (-l, l)^N$. If we consider in $\mathbb{R}^{N-1}$ the $\|\cdot\|_\infty$ norm, then

$$\|f\|^p_{W^{1-1/p,p}} \sim \int\limits_{|x'|<l} \int\limits_{|y'|>l} \frac{dx'\,dy'}{|x'-y'|^{N+p-2}}. \tag{11.38}$$

If $|x'| < l$ and $|y'| > l$, then $y' \in \mathbb{R}^{N-1} \setminus B(x', l-|x'|)$, and therefore

$$\int\limits_{|y'|>l} \frac{dy'}{|x'-y'|^{N+p-2}} \leq \int\limits_{|z'|>l-|x'|} \frac{dz'}{|z'|^{N+p-2}} = C \int\limits_{l-|x'|}^{\infty} r^{-p} = C(l-|x'|)^{1-p}. \tag{11.39}$$

Since $p < 2$, we find that

$$\|f\|^p_{W^{1-1/p,p}} \leq C \int\limits_{|x'|<l} (l-|x'|)^{1-p} \leq C \int\limits_{\{|x_j|\leq x_1<l,\ j=1,\ldots,N-2\}} \frac{dx'}{(l-x_1)^{p-1}} < \infty. \tag{11.40}$$

□

Lemma 33. *Let C be a cube of size l in $\mathbb{R}^{N-1}$ and set $a = \dfrac{1}{|C|}\chi_C$. Then there is a map $u \in W^{1,1}$ s. t.* tr $u = a$ *and*

$$\|u\|_{L^1} \leq c\,l \quad \text{and } \|Du\|_{L^1} \leq c. \tag{11.41}$$

Proof. We start with the case where C is the unit cube (or any other cube of size 1). We fix a $p \in (1,2)$. Since $a \in W^{1-1/p,p}$, we have $a = \text{tr } u_0$ for some $u \in W^{1,p}$. In addition, Corollary 17 implies that we may assume u_0 compactly supported. Thus $u \in W^{1,1}$ and tr $u_0 = a$ (computed in $W^{1,1}$). Let now C be an arbitrary cube, which we may assume with sides parallel to the unit cube Q. Let $C = x' + (0,l)^{N-1}$. Set $u = l^{-(N-1)}u_0(l^{-1}(\cdot - x'))$. Then $u \in W^{1,1}$ and tr $u = a$. Inequality (11.41) follows from the identities $\|u\|_{L^1} = l\|u_0\|_{L^1}$ and $\|Du\|_{L^1} = \|Du_0\|_{L^1}$. □

Theorem 26. *(***Gagliardo***) Let $f \in L^1(\mathbb{R}^{N-1})$. Then there is some $u \in W^{1,1}(\mathbb{R}^N)$ s. t.* tr $u = f$ *and $\|u\|_{W^{1,1}} \leq C\|f\|_{L^1}$.*

Remark 13. *This time, the map $f \mapsto u$ we construct is* **not** *linear.*

Proof. The main ingredient is the following: if $f \in L^1$, then we may write, in L^1, $f = \sum \lambda_n a_n$, where:
(i) each a_n is of the form $a_n = \frac{1}{|C_n|}\chi_{C_n}$;
(ii) each C_n is of size at most 1;
(iii) $\sum |\lambda_n| \leq C\|f\|_{L^1}$.
Assuming that this can be achieved, here is the end of the proof: the preceding lemma implies that each a_n is the trace of some $u_n \in W^{1,1}$ s. t. $\|u_n\|_{W^{1,1}} \leq C$. The linearity of the trace and property (iii) imply that the map $u = \sum \lambda_n u_n \in W^{1,1}$ satisfies $\operatorname{tr} u = f$ and $\|u\|_{W^{1,1}} \leq C\|f\|_{L^1}$.
It remains to perform the decomposition $f = \sum \lambda_n a_n$. For each $j \in \mathbb{N}$, let $\mathcal{F}_j$ be the grid of cubes of size 2^{-j}, with sides parallel to the coordinate axes and having the origin among the edges. We define the linear map $T_j : L^1 \to L^1$, $T_j f(x) = f_C$ if $x \in C \in \mathcal{F}_j$. Clearly, T_j is of norm 1. We claim that, for each $f \in L^1$, we have $T_j f \to f$ in L^1 as $j \to \infty$. This is clear when $f \in C_0^\infty$; the case of a general f follows by approximation using the fact that $\|T_j\| = 1$. We may thus find an increasing sequence of indices, (j_k), s. t. $\|f_{j_0}\|_{L^1} + \sum \|f_{j_{k+1}} - f_{j_k}\|_{L^1} \leq C\|f\|_{L^1}$. We claim that $f_{j_{k+1}} - f_{j_k} = \sum \lambda_n^k a_n^k$, where each a_n^k is of the form $\frac{1}{|C|}\chi_C$ for some cube of size at most 1 and $\sum |\lambda_n^k| = \|f_{j_{k+1}} - f_{j_k}\|_{L^1}$. Indeed, $f_{j_{k+1}} - f_{j_k}$ is constant on each cube $C \in \mathcal{F}_{j_k}$, and thus $f_{j_{k+1}} - f_{j_k} = \sum_{C \in \mathcal{F}_{j_k}} (f_{j_{k+1}} - f_{j_k})_{|C}\chi_C$, so that $f_{j_{k+1}} - f_{j_k} = \sum_{C \in \mathcal{F}_{j_k}} \lambda_C \frac{1}{|C|}\chi_C$, with $\lambda_C = (f_{j_{k+1}} - f_{j_k})_{|C}|C|$.
We find that

$$\|f_{j_{k+1}} - f_{j_k}\|_{L^1} = \sum_{C \in \mathcal{F}_{j_k}} \int_C |f_{j_{k+1}} - f_{j_k}| = \sum_{C \in \mathcal{F}_{j_k}} |C||(f_{j_{k+1}} - f_{j_k})_{|C}| = \sum_{C \in \mathcal{F}_{j_k}} |\lambda_C|. \tag{11.42}$$

Similarly, we may write $f_{j_0} = \sum \lambda_n^0 a_n^0$, where each a_n^0 is of the form $\frac{1}{|C|}\chi_C$ for some cube of size at most 1 and $\sum |\lambda_n^0| = \|f_{j_0}\|_{L^1}$.
Finally, we write $f = \sum_k \sum_n \lambda_n^k a_n^k$, and this decomposition has the properties (i)-(iii). □