

# Conformal Mapping Method for Free Surface Waves

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# General Features

## PROS

- Fully nonlinear (i.e., solves the Euler eqs.)
- Fast and accurate, particularly when combined with spectral methods
- Machine-precision conservation of flow invariants (energy, momentum, mass)
- Easy implementation
- Overturning waves (?)

## CONS

- 2D only, periodic only (but periodic does not always mean periodic...)
- It can be hard to introduce additional physical effects (forcing, damping, dissipation)

## RECENT DEVELOPMENTS

- Waves over rotational and irrotational currents
- Arbitrary bathymetry... in progress

# History

The general idea about conformal mapping as an approach to water waves dates back to

- Byatt-Smith 1970
- Schwartz 1974 [1]
- Cokelet 1977 [2]
- ...

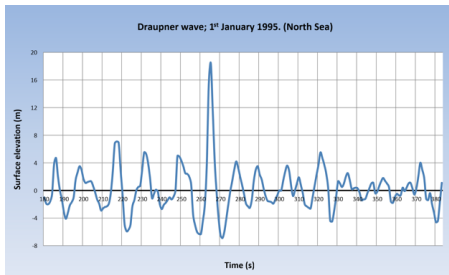
The modern formulation of the method stems from the seminal work by

- Dyachenko *et al.* 1996 [3]

Then used, consolidated and extended by many others

- Choi & Camassa 1999 [4]
- Li, Hyman & Choi 2004 [5]
- Choi 2009 [6]
- Milewski, Vanden-Broek & Wang 2010 [7]

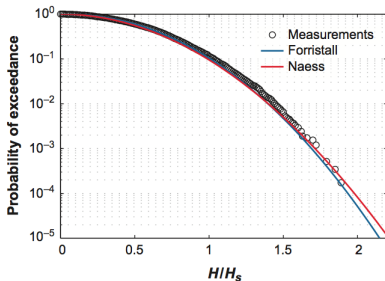
# Rogue waves: an example of what WE use it for



Hurricane Ivan at Marlin 03:00–20:00  
September 15, 2004

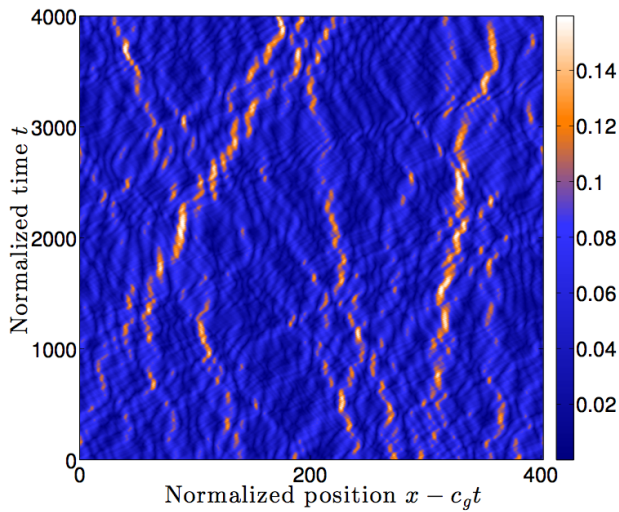
The high efficiency and accuracy of this method make it suitable for computing waves statistics over large ensembles.

If you want to know more about RWs, take a look at Dysthe *et al* 2008 [8].





## Another example: large-scale wave dynamics



(From Viotti, Dutykh, Dudley & Dias 2013, *submitted*)

LET'S START WITH THE METHOD NOW...

# Mathematical formulation

Kinematic and dynamic boundary condition on the free surface:

$$\left. \begin{aligned} \zeta_t &= -\phi_x \zeta_x + \phi_y, \\ \phi_t &= -\frac{1}{2} (\phi_x^2 + \phi_y^2) - g\zeta + \gamma \frac{\zeta_{xx}}{(1 + y_x^2)^{3/2}}, \end{aligned} \right\} \text{ at } y = \zeta(x, t)$$

Irrotational incompressible flow in the fluid volume:

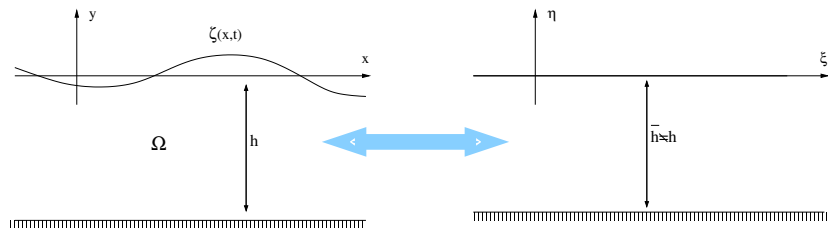
$$\nabla^2 \phi = 0, \quad \text{inside } \Omega$$

Bottom boundary condition:

$$\lim_{y \rightarrow -\infty} |\nabla \phi| = 0 \quad \text{Deep Water}$$

$$\psi = C \text{ at } y = -h(x) \quad \text{Finite Bottom}$$

# Conformal mapping



$$x = x(\xi, \eta, t), \quad y = y(\xi, \eta, t)$$

Conformal mapping:

$$x + iy = X(\xi + i\eta) + iY(\xi + i\eta) = Z(\xi + i\eta), \quad Z \text{ is an analytic function.}$$

Complex potential:  $\phi$  and  $\psi$  form a conjugate pair in the  $(x, y)$  space (harmonic functions + CR), so they do in the conformal space

$$\phi + i\psi = \Phi(Z(\xi + i\eta))$$

# Determine $Z(\xi, \eta, t) = X + iY$

The harmonic-conjugate pair  $(X, Y)$  needs to satisfy

$$X_{\xi\xi} + X_{\eta\eta} = 0, \quad Y_{\xi\xi} + Y_{\eta\eta} = 0$$

plus Cauchy–Riemann relations

$$X_{\xi} = Y_{\eta}, \quad X_{\eta} = -Y_{\xi}.$$

The vertical displacement of the free surface and the bathymetry provide **known boundary conditions for  $Y$**

$$Y(\xi, 0, t) = y(\xi, t), \quad Y(\xi, -\bar{h}, t) = h(\xi, t),$$

whereas the boundary conditions

$$X(\xi, 0, t) = x(\xi, t), \quad X(\xi, -\bar{h}, t) = \chi(\xi, t),$$

are **unknown for  $X$** .

# Determine $Z(\xi, \eta, t) = X + iY$

$Y$  is determined by solving the Laplace equation inside a uniform strip with Dirichlet data on both sides  $\Rightarrow$  easily done by Fourier Transform:

$$Y(\xi, 0, t) = y(\xi, t) = \sum_{n=0}^{+\infty} \hat{Y}_n e^{ink_0 \xi},$$

$$Y(\xi, -\bar{h}, t) = -h,$$

It is easy to verify that the most general harmonic function satisfying the above conditions has the form

$$Y(\xi, \eta, t) = -\eta \frac{h}{h} + \hat{Y}_0 \left( 1 + \frac{\eta}{h} \right) + \sum_{n=1}^{+\infty} \hat{Y}_n \frac{\sinh nk_0(\eta + \bar{h})}{\sinh nk_0 \bar{h}} e^{ink_0 \xi}.$$

# Determine $Z(\xi, \eta, t) = X + iY$

$X$  is determined by enforcing  $C - R$  relations:

$$X(\xi, \eta, t) = \frac{\hat{Y}_0 - h}{\bar{h}} \xi + x_0(t) + \sum_{n=1}^{+\infty} \hat{Y}_n n k_0 \frac{\cosh n k_0 (\eta + \bar{h})}{\sinh n k_0 \bar{h}} e^{i n k_0 \xi}$$

$\bar{h}$  determines the period in physical space. By choosing  $\bar{h} = \hat{Y}_0 - h$  the period  $2\pi/k_0$  is the same in both spaces.

On the free surface we can write in compact notation:

- DEEP WATER:  $x(\xi, t) = \xi - \mathcal{H}[y(\xi, t)]$ ,

$$\mathcal{H}[y(\xi, t)] \equiv \int_{-\infty}^{\infty} \frac{y(\xi', t)}{\xi - \xi'} d\xi'.$$

- FINITE DEPTH:  $x(\xi, t) = \xi - \mathcal{T}_c[y(\xi, t)] + x_0(t)$ ,

$$\mathcal{T}_c[y(\xi, t)] \equiv \int_{-\infty}^{\infty} y(\xi', t) \coth[\pi(\xi - \xi')/2\bar{h}] d\xi'.$$

## Determine $\Phi + i\Psi$

The harmonic-conjugate pair  $(\Phi, \Psi)$  is subject to the different set of known boundary conditions

$$\Phi_{\xi\xi} + \Phi_{\eta\eta} = 0, \quad \Psi_{\xi\xi} + \Psi_{\eta\eta} = 0$$

$$\Phi(\xi, 0, t) = \phi(\xi, t), \quad \Psi(\xi, -\bar{h}, t) = Q(t),$$

The same procedure now yields

$$\phi_\xi = -\mathcal{T}_s[\psi_\xi] + \mathcal{U}, \quad \psi_\xi = -\mathcal{T}_s[\phi_\xi - \mathcal{U}],$$

where  $\mathcal{U} \equiv m(\psi)/\bar{h}$ .



# Inversion of the mapping: physical space $\rightarrow$ conformal space

Typically one needs to assign initial conditions in physical space. Then it is necessary to invert the conformal mapping by numerical iterations.

Want to find:

$$y(\xi) \quad \text{such that} \quad y(x(\xi; y(\xi))) = \zeta_0(x)$$

## Fixed-point iterations

$$\begin{aligned} y^0 &= \zeta_0(\xi), \\ y^{n+1} &= \zeta_0(x(\xi; y^n)) \end{aligned}$$

Linear (i.e., slow) convergence:  $err^{n+1} = C err^n$ .  
(But it typically needs to be done only once.)

# Evolution equations in conformal variables

An (almost) straightforward application of chain-rule differentiation leads to the transformed Euler system (for details see, e.g., Choi & Camassa 1999 [4]):

$$\begin{aligned}y_t &= -x_\xi \left( \frac{\psi_\xi}{J} \right) + y_\xi \mathcal{T}_c \left[ \frac{\psi_\xi}{J} \right] + y_\xi q(t), \\ \phi_t &= -\frac{1}{2} \frac{\phi_\xi^2 - \psi_\xi^2}{J} + \phi_\xi \mathcal{T}_c \left[ \frac{\psi_\xi}{J} \right] - gy + \frac{x_\xi y_{\xi\xi} - y_\xi y_{\xi\xi}}{J^{3/2}} + C(t), \\ x_\xi &= 1 - \mathcal{T}_c[y_\xi], \\ \psi_\xi &= -\mathcal{T}_s[\phi_\xi - \mathcal{U}].\end{aligned}\quad (J \equiv x_\xi^2 + y_\xi^2)$$

Discrete Fourier Transform:

$$\eta(x, t) = \sum_{n=-N}^N \hat{\eta}_n(t) e^{ink_0 x}, \quad \phi(x, \eta(x), t) = \sum_{n=-N}^N \hat{\phi}_n(t) e^{ink_0 x}$$

Nonlinear terms computed pseudospectrally (cost  $N \log N$  each)

**Warning:** strong nonlinearities  $\Rightarrow$  strong dealiasing

## SOME EXAMPLES

# Random sea simulations (Viotti *et al.* 2013)

Discrete Fourier expansion:

$$\eta(x, t) = \sum_{n=-N}^N \hat{\eta}_n(t) e^{ink_0 x}, \quad \phi(x, \eta(x), t) = \sum_{n=-N}^N \hat{\phi}_n(t) e^{ink_0 x}$$

Initial condition:

$$\hat{\eta}_n = [2P_0(nk_0)]^{1/2} e^{ink_0 \varphi_n}, \quad \hat{\phi}_n = ic(nk_0) \hat{\eta}_n, \quad n = 0, 1, \dots, N$$

where

$$\left\{ \begin{array}{l} \varphi_n = \text{r.v. in } [0, 2\pi], \quad c(k) = \sqrt{g/k} \\ P_0(k) = \frac{\mathcal{P}_0}{\sqrt{2\pi}\sigma_0} \exp \left[ -\frac{1}{2} \left( \frac{k-k_0}{\sigma_0} \right)^2 \right] \end{array} \right.$$

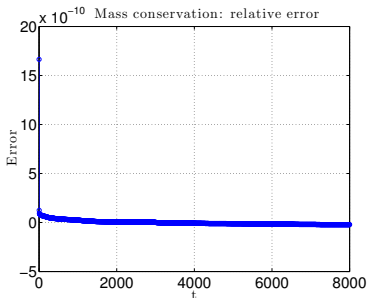
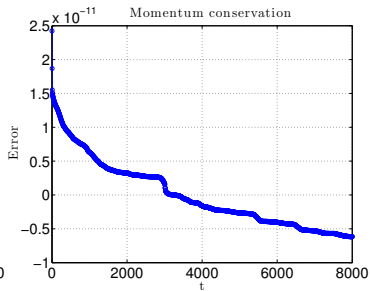
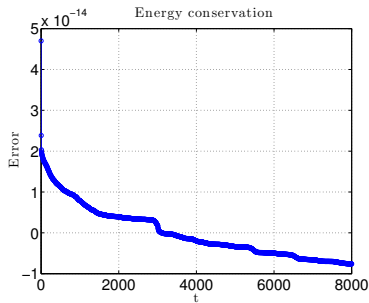
Benjamin-Feir Index (Janssen 2003)

$$\text{BFI}_0 = \sqrt{2s} \frac{k_0}{\sigma_0}, \quad s = k_0 \mathcal{P}_0^{1/2} = k_0 \langle \eta^2 \rangle^{1/2}$$

# 2D random sea - 1

$N = 65536$ ,  $L = 256\pi$ , steepness = 0.04

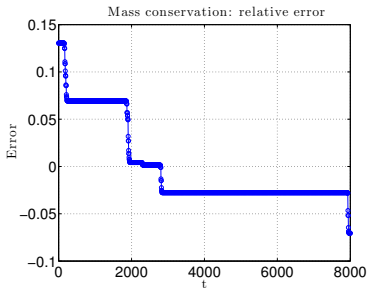
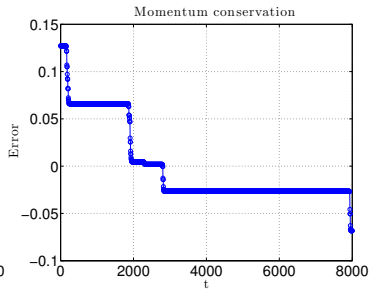
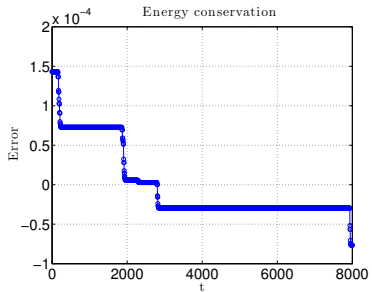
# Checking conservation laws



# 2D random sea - 2

$N = 65536$ ,  $L = 256\pi$ , steepness = 0.04

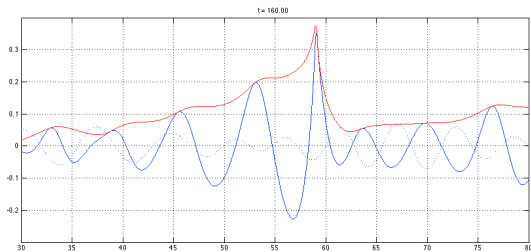
# Checking conservation laws



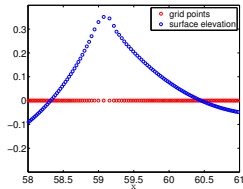


# Extreme waves

Resolution is not uniform in physical space, this is particularly notable for very steep waves.

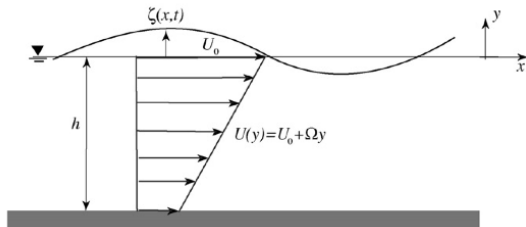


Loss of numerical resolution is more critical around wave crests.



- Waves over homogeneous shear flow
- Arbitrary bathymetry... work in progress

# Waves over a uniform shear current (Choi 2009) [6]



(From Choi 2009)

Governing eqs.

$$\hat{\phi}_{xx} + \hat{\phi}_{yy} = 0 \quad \text{for } -h \leq y \leq \zeta,$$

$$\zeta_t + (U_0 + \Omega\zeta + \hat{\phi}_x)\zeta_x = \hat{\phi}_y \quad \text{at } y = \zeta(x, t),$$

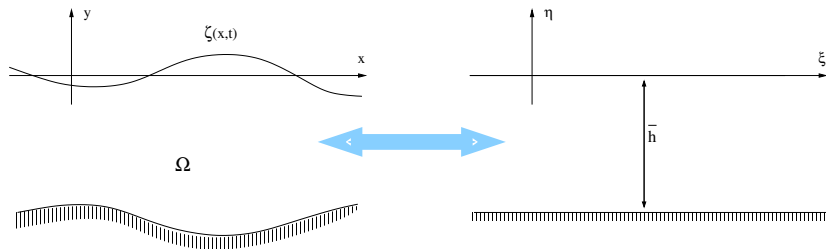
$$\hat{\phi}_t + g\zeta + \frac{1}{2}(\hat{\phi}_x^2 + \hat{\phi}_y^2) + (U_0 + \Omega\zeta)\hat{\phi}_x - \Omega\hat{\psi} = \frac{\sigma}{\rho} \frac{\zeta_{xx}}{(1 + \zeta_x^2)^{3/2}} \quad \text{at } y = \zeta(x, t),$$

$$\hat{\phi}_y = 0 \quad \text{at } y = -h,$$

Mapped governing eqs.

$$\phi_t = -gy - \frac{1}{2J}(\phi_\xi^2 - \psi_\xi^2) + \phi_\xi T \left[ \frac{\Psi_\xi}{J} \right] - \frac{1}{J}(U_0 + \Omega y)x_\xi \phi_\xi + \Omega\psi + \frac{\sigma}{\rho} \frac{x_\xi y_{\xi\xi} - y_\xi x_{\xi\xi}}{(x_\xi^2 + y_\xi^2)^{3/2}}.$$

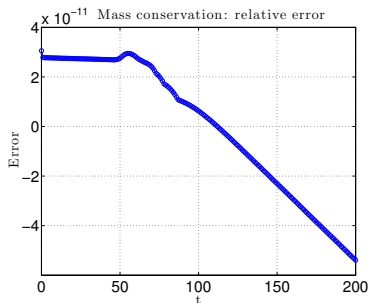
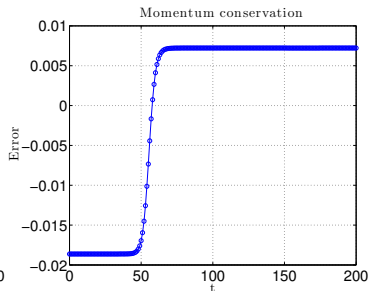
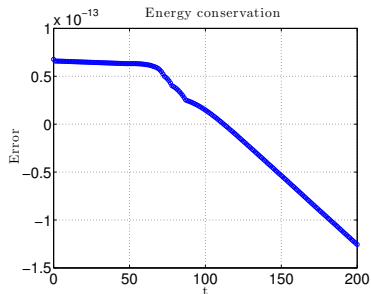
# Arbitrary bathymetry... in progress!



# Soliton over submerged step

$N = 16384$ ,  $L = 128\pi$ ,  $Fr = 1.15$

# Checking conservation laws



THANK YOU!



L. W. Schwartz.

Computer extension and analytic continuation of Stokes' expansion for gravity waves.

*J. Fluid Mech.*, 62:553–578, 1974.



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W. Choi and R. Camassa.

Fully nonlinear internal waves in a two-fluid system.

*Journal of Fluid Mech.*, 396(-1):1–36, 1999.



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