

HIGHER-ORDER SPECTRAL (HOS) SCHEMES

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Short Course on “*Modeling of Nonlinear Ocean Waves*”



PLANS FOR TODAY LECTURE

1 LECTURE 1

- Introduction (D.D.)
- BIEM (C.V.)
- Spectral CG-method (D.D.)

2 LECTURE 2

- Higher-Order Spectral (HOS) methods (D.D.)
- Dirichlet-to-Neumann (D2N) operator technique (D.D.)
- Dynamic conformal mapping technique (C.V.)

And that's it!

Higher-Order Spectral (HOS) methods

HISTORY OF THE METHOD

HIGHER ORDER SPECTRAL (HOS): WE FOLLOW [DBLF12]

- ▶ Dommermuth & Yue's work [DY87]:
 - ▶ Dommermuth, D.G., & Yue, D.K.P. (1987). *A high-order spectral method for the study of nonlinear gravity waves*. JFM, 184, 267–288
- ▶ West *et al.* variant [WBJ⁺87]:
 - ▶ West, B. J., Brueckner, K. A., Janda, R. S., Milder, D. M., & Milton, R. L. (1987). *A new numerical method for surface hydrodynamics*. JGR, **92**(C11), 11803
- ▶ Bateman, Swan & Taylor (BST) [BST01]
- ▶ More recent implementations [DBLF07, DBLF12]:
 - ▶ Ducrozet, G., *et al.* (2012). *A modified High-Order Spectral method for wavemaker modeling in a numerical wave tank*. EJMFB, **34**, 19–34
 - ▶ Ducrozet, G., *et al.* (2007). *3-D HOS simulations of extreme waves in open seas*. NHESS, **7**(1), 109–122
- ▶ Comparative work [Sch08]: Schäffer, H. A. (2008). *Comparison of Dirichlet-Neumann operator expansions for nonlinear surface gravity waves*. Coast. Eng., **55**(4), 288–294

PROBLEM REFORMULATION

IN HOS WE USE THE VERTICAL VELOCITY AT THE F.S.

- ▶ Free surface boundary conditions in HOS methods:

$$\eta_t = (1 + |\nabla\eta|^2) \frac{\partial\phi}{\partial z} - \nabla\varphi \cdot \nabla\eta$$

$$\varphi_t = -g\eta - \frac{1}{2}|\nabla\varphi|^2 + \frac{1}{2}(1 + |\nabla\eta|^2) \left(\frac{\partial\phi}{\partial z}\right)^2$$

DYNAMIC VARIABLES:

$y = \eta(\underline{x}, t)$: free surface elevation

$\varphi(\underline{x}, t) := \phi(\underline{x}, y = \eta(\underline{x}, t), t)$: trace of the potential

$W(\underline{x}, t) := \frac{\partial\phi}{\partial z}(\underline{x}, y = \eta(\underline{x}, t), t)$: vertical velocity at the F.S.

Derivation hint:

$$\nabla\phi = \nabla\varphi - \nabla\eta \frac{\partial\phi}{\partial z}, \quad z = \eta(\underline{x}, t)$$

DEVELOPMENT IN HORIZONTAL VARIABLES

USE EIGENFUNCTIONS OF THE COMPUTATIONAL DOMAIN

- ▶ Decomposition in space and time:

$$\eta(\underline{x}, t) = \sum_i \sum_j A_{ij} \psi_{ij}(\underline{x})$$

$$\varphi(\underline{x}, t) = \sum_i \sum_j B_{ij} \psi_{ij}(\underline{x})$$

- ▶ Common choices of basis functions:

PERIODIC DOMAIN: $\psi_{ij}(\underline{x}) = e^{i\mathbf{k}_{ij} \cdot \underline{x}}$

WAVE TANK: $\psi_{ij}(\underline{x}) = \cos(k_1 x) \cos(k_2 y)$

Solution in the bulk:

$$\phi(\underline{x}, y, t) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} B_{ij}(t) \psi_{ij}(\underline{x}) \frac{\cosh(k_{ij}(y+h))}{\cosh(k_{ij}h)}, \quad k_{ij} := \sqrt{k_i^2 + k_j^2}$$

COMPUTATION OF THE VERTICAL VELOCITY W - I

DOUBLE EXPANSION OF THE VELOCITY POTENTIAL

1. Formal expansion in powers of η :

$$\phi(\underline{x}, \eta, t) = \sum_{m=1}^{\infty} \phi^{(m)}(\underline{x}, \eta, t)$$

2. Taylor expansion around $y = 0$:

$$\phi^{(m)}(\underline{x}, \eta, t) = \sum_{n=1}^{\infty} \frac{\eta^n}{n!} \frac{\partial^n \phi^{(n)}}{\partial y^n}(\underline{x}, 0, t)$$

We obtain:

$$\begin{aligned} \varphi(\underline{x}, t) = & \phi^{(1)}(\underline{x}, 0, t) + \eta \frac{\partial \phi^{(1)}}{\partial y}(\underline{x}, 0, t) + \dots \\ & \phi^{(2)}(\underline{x}, 0, t) + \eta \frac{\partial \phi^{(2)}}{\partial y}(\underline{x}, 0, t) + \dots \end{aligned}$$

COMPUTATION OF THE VERTICAL VELOCITY W - II

BY REGROUPING AT EVERY ORDER WE OBTAIN:

► Lower triangular system:

$$\phi^{(1)}(\underline{x}, 0, t) = \varphi(\underline{x}, t)$$

$$\phi^{(2)}(\underline{x}, 0, t) = -\eta(\underline{x}, t) \frac{\partial \phi^{(1)}}{\partial y}(\underline{x}, 0, t)$$

$$\phi^{(3)}(\underline{x}, 0, t) = -\eta(\underline{x}, t) \frac{\partial \phi^{(2)}}{\partial y}(\underline{x}, 0, t) - \frac{\eta^2(\underline{x}, t)}{2!} \frac{\partial^2 \phi^{(1)}}{\partial y^2}(\underline{x}, 0, t)$$

$$\phi^{(m)}(\underline{x}, 0, t) = - \sum_{k=1}^{m-1} \frac{\eta^k}{k!}(\underline{x}, t) \frac{\partial^k \phi^{(m-k)}}{\partial y^k}(\underline{x}, 0, t)$$

Derivatives ∂_y are computed using:

$$\phi(\underline{x}, y, t) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} B_{ij}(t) \psi_{ij}(\underline{x}) \frac{\cosh(k_{ij}(y+h))}{\cosh(k_{ij}h)}, \quad k_{ij} := \sqrt{k_i^2 + k_j^2}$$

COMPUTATION OF THE VERTICAL VELOCITY W - III

FINALLY, WE CONSTRUCT THE VERTICAL VELOCITY

- ▶ We expand in power series of η :

$$W(\underline{x}, t) = \sum_{m=1}^{\infty} W^{(m)}(\underline{x}, t)$$

- ▶ At every order we obtain:

$$W^{(1)}(\underline{x}, t) = \frac{\partial \phi^{(1)}}{\partial y}(\underline{x}, 0, t)$$

$$W^{(2)}(\underline{x}, t) = \frac{\partial \phi^{(2)}}{\partial y}(\underline{x}, 0, t) + \eta(\underline{x}, t) \frac{\partial^2 \phi^{(1)}}{\partial y^2}(\underline{x}, 0, t)$$

$$W^{(3)}(\underline{x}, t) = \frac{\partial \phi^{(3)}}{\partial y}(\underline{x}, 0, t) + \eta \frac{\partial^2 \phi^{(2)}}{\partial y^2} + \frac{\eta^2}{2!} \frac{\partial^3 \phi^{(1)}}{\partial z^3}$$

$$W^{(m)}(\underline{x}, t) = \sum_{k=0}^{m-1} \frac{\eta^k}{k!} \frac{\partial^{k+1} \phi^{(m-k)}}{\partial y^{k+1}}(\underline{x}, 0, t)$$

DIFFERENCE BETWEEN TWO FORMULATIONS

DOMMERMUTH & YUE VS WEST ET AL.

- ▶ Original treatment by DY:

$$(1 + |\nabla\eta|^2)W \approx W_M + |\nabla\eta|^2 W_M$$

- ▶ West *et al.* method:

$$(1 + |\nabla\eta|^2)W \approx W_M + |\nabla\eta|^2 W_{M-2}$$

REMARK [DBLF12]:

Formulation by DY is believed to be prone to numerical instabilities because of induced inhomogeneities in the method

COMPUTATIONAL COMPLEXITY ESTIMATION

FFT NUMBER COUNT

1. Solution of lower triangular system to obtain modal amplitudes B_{ij} : $M \frac{M+1}{2}$
2. Computation of the vertical velocity $W(\underline{x}, t)$: M FFTs since we can reuse some quantities from the previous step

TOTAL COUNT OF FFTS:

$$M \frac{M+1}{2} + M = M \frac{M+3}{2} = \mathcal{O}(M^2)$$

CONTRADICTION:

Yue in his Chapter about HOS method in [MSY05] gives the estimation $\mathcal{O}(M)$ behaviour for the number of FFTs. No further justification.

CPU TIME OF ONE TIME STEP:

$$\mathcal{O}\left(M \frac{M+3}{2} N \log(N)\right) = \mathcal{O}\left(M^2 N \log(N)\right), \text{ where } N = N_x \times N_y$$

CONVERGENCE STUDY

FROM PHD OF G. DUCROZET (2007)

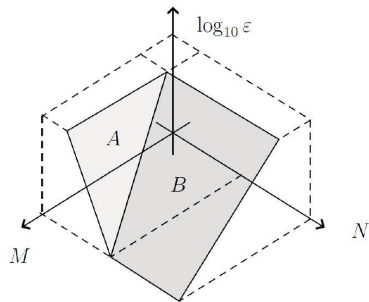
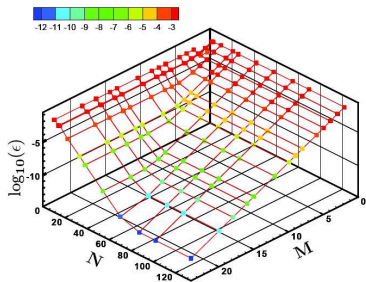


FIGURE: High amplitude Stokes wave solution ($ka = 0.3$)

STUDY OF 3D WAVE FOCUSSING

FROM PHD OF G. DUCROZET (2007)

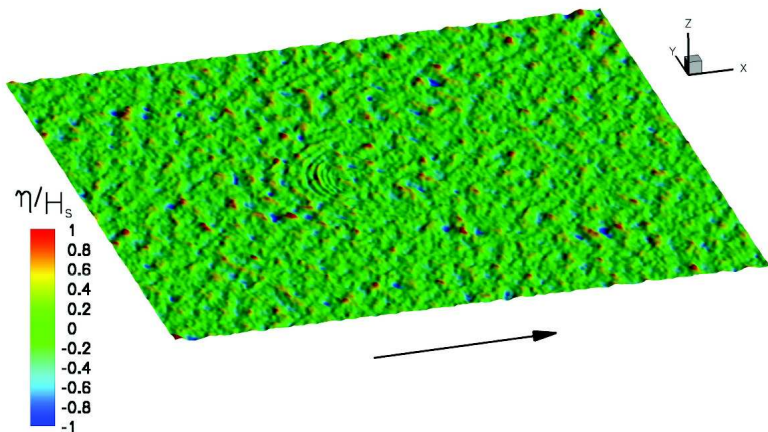


FIGURE: Initial condition

STUDY OF 3D WAVE FOCUSSING

FROM PHD OF G. DUCROZET (2007)

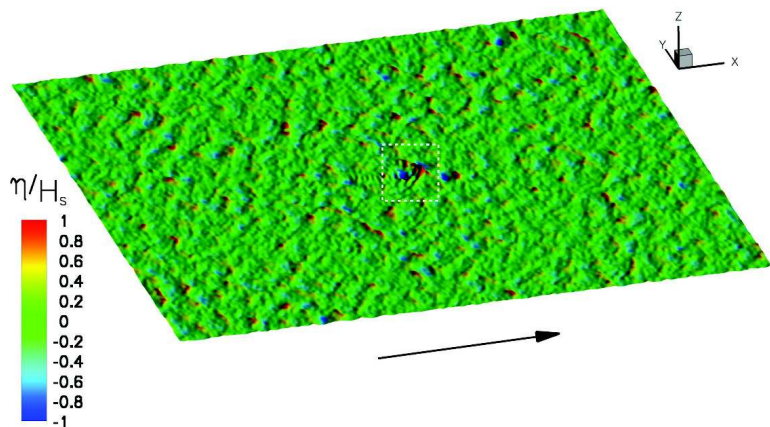


FIGURE: Free surface at $t = 20.3T_p$

STUDY OF 3D WAVE FOCUSSING

FROM PHD OF G. DUCROZET (2007)

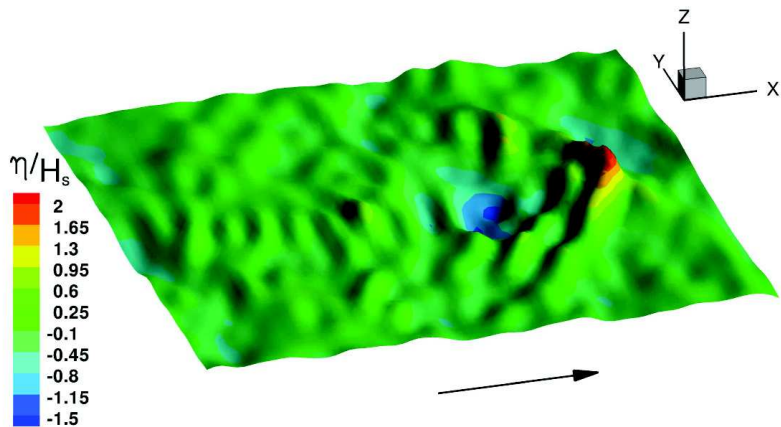


FIGURE: Zoom on free surface at $t = 20.3T_p$

Dirichlet-to-Neumann operator approach

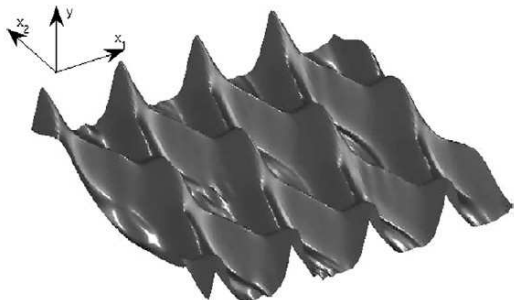
HISTORICAL PROSPECTIVE

BIBLIOGRAPHY REVIEW (THE LIST IS NOT EXHAUSTIVE!)

- ▶ Original work introducing the method [CS93]:
 - ▶ Craig, W., & Sulem, C. (1993). *Numerical simulation of gravity waves*. J. Comput. Phys., **108**, 73–83
- ▶ Travelling wave solutions [CN02]:
 - ▶ Craig, W., & Nicholls, D. P. (2002). *Traveling gravity water waves in two and three dimensions*. European Journal of Mechanics - B/Fluids, **21**(6), 615–641
- ▶ General/dynamic bottom extension [GN07]:
 - ▶ Guyenne, P., & Nicholls, D. P. (2007). *A high-order spectral method for nonlinear water waves over moving bottom topography*. SIAM J. Sci. Comput., **30**(1), 81–101
- ▶ Extension to rough bottoms [CGNS05]:
 - ▶ Craig, W., & Nicholls, D. P. (2002). *Traveling gravity water waves in two and three dimensions*. European Journal of Mechanics - B/Fluids, **21**(6), 615–641

HISTORICAL PROSPECTIVE (CONTINUED)

- ▶ Crescent waves in 3D [XG09]:
 - ▶ Xu, L., & Guyenne, P. (2009). *Numerical simulation of three-dimensional nonlinear water waves*. *J. Comput. Phys.*, **228**(22), 8446–8466
- ▶ Hydroelastic waves [GP12]:
 - ▶ Guyenne, P., & Parau, E. (2012). *Computations of fully nonlinear hydroelastic solitary waves on deep water*. *J. Fluid Mech.*, **713**, 307–329



DIRICHLET-TO-NEUMANN OPERATOR

REFORMULATION OF THE PROBLEM (WE FOLLOW CLOSELY [CS93])

DEFINITION OF D2N OPERATOR:

$$\mathbf{G}_\eta(\varphi) = \sqrt{1 + \eta_x^2} \left. \frac{\partial \phi}{\partial n} \right|_{y=\eta} \equiv (\phi_y - \eta_x \phi_x) \Big|_{y=\eta}$$

- ▶ Dynamic boundary conditions:

$$\begin{aligned} \eta_t - \mathbf{G}_\eta(\varphi) &= 0 \\ \varphi_t + g\eta + \frac{\varphi_x^2 - \mathbf{G}_\eta(\varphi)^2 - 2\eta_x \varphi_x \mathbf{G}_\eta(\varphi)}{2(1 + \eta_x^2)} &= 0 \end{aligned}$$

- ▶ Separate linear and nonlinear parts:

$$\begin{aligned} \eta_t - \mathbf{G}_0(\varphi) &= \mathbf{G}_\eta(\varphi) - \mathbf{G}_0(\varphi) \\ \varphi_t + g\eta &= -\frac{\varphi_x^2 - \mathbf{G}_\eta(\varphi)^2 - 2\eta_x \varphi_x \mathbf{G}_\eta(\varphi)}{2(1 + \eta_x^2)} \end{aligned}$$

- ▶ Linear terms can be integrated **exactly!**

GENERAL PROPERTIES OF D2N OPERATOR

ANALYTICAL PROPERTIES

1. D2N is a pseudo-differential analytical operator
2. D2N can be expanded in a Taylor series
3. This series is convergent for sufficiently small η in Lipschitz-norm
 - ▶ Proved by Coifman & Meyer (1985) [CM85]

LET US WRITE A FORMAL EXPANSION:

$$G_\eta = \sum_{j=0}^{\infty} G_\eta^{(j)} = G^{(0)} + G_\eta^{(1)} + G_\eta^{(2)} + \dots$$

- ▶ The first (linear) term can be trivially computed:

$$G^{(0)} = D \tanh(Dh), \quad D := -i\partial_x$$

- ▶ Can be efficiently computed in Fourier space:

$$\hat{G}^{(0)} e^{ikx} = k \tanh(kh) e^{ikx}$$

CONSTRUCTION OF RECURSIVE RELATIONS

- ▶ Choose particular solution: $\phi_k(x, y) \equiv e^{ikx} \cosh(k(h + y))$
- ▶ Substitute this solution into the definition of D2N operator:

$$\frac{\partial \phi_k}{\partial y} - \eta_x \frac{\partial \phi_k}{\partial x} = G_\eta \phi_k, \quad y = \eta$$

- ▶ Expand around $\eta = 0$:

$$\begin{aligned} & \sum_{j=2l} \frac{1}{j!} (k\eta)^j (k \sinh(kh) - ik\eta_x \cosh(kh)) e^{ikx} \\ & + \sum_{j=2l-1} \frac{1}{j!} (k\eta)^j (k \cosh(kh) - ik\eta_x \sinh(kh)) e^{ikx} \\ & = \left(\sum_{j=0}^{\infty} G_\eta^{(j)} \right) \left(\sum_{j=2l} \frac{1}{j!} (k\eta)^j \cosh(kh) e^{ikx} + \right. \\ & \qquad \qquad \qquad \left. \sum_{j=2l-1} \frac{1}{j!} (k\eta)^j \sinh(kh) e^{ikx} \right) \end{aligned}$$

RECURRENCE FORMULAS FOR D2N EXPANSION

RELATIVELY SIMPLE RECURSION FORMULA FOR $G_\eta^{(j)}$

► For j even:

$$\begin{aligned} G_\eta^{(j)} &= \frac{1}{j!} \left(\eta^j D^{j+1} \tanh(Dh) - i(\eta^j)_x \tanh(Dh) \right) \\ &\quad - \sum_{k < j, k=2l} G_\eta^{(k)} \frac{1}{(j-k)!} \eta^{j-k} D^{j-k} \\ &\quad - \sum_{k < j, k=2l-1} G_\eta^{(k)} \frac{1}{(j-k)!} \eta^{j-k} D^{j-k} \tanh(Dh) \end{aligned}$$

► For j odd:

$$\begin{aligned} G_\eta^{(j)} &= \frac{1}{j!} \left(\eta^j D^{j+1} - i(\eta^j)_x \right) - \sum_{k < j, k=2l-1} G_\eta^{(k)} \frac{1}{(j-k)!} \eta^{j-k} D^{j-k} \\ &\quad - \sum_{k < j, k=2l} G_\eta^{(k)} \frac{1}{(j-k)!} \eta^{j-k} D^{j-k} \tanh(Dh) \end{aligned}$$

A FEW TERMS IN D2N EXPANSION

TAYLOR SERIES FOR PSEUDO-DIFFERENTIAL OPERATORS

1. Linear solution:

$$G^{(0)} = D \tanh(Dh)$$

2. Quadratic terms:

$$G_{\eta}^{(1)} = D(\eta - \tanh(Dh)\eta \tanh(Dh))D$$

3. Third-order solution:

$$G_{\eta}^{(2)} = -\frac{1}{2}D(D\eta^2 \tanh(Dh) + \tanh(Dh)\eta^2 D - 2 \tanh(Dh)\eta D \tanh(Dh)\eta \tanh(Dh))D$$

Remarks:

- ▶ At any order contains only products of D , η and $\tanh(Dh)$
- ▶ Can be generalized to higher dimensions
- ▶ D2N expansion for the deep water case can be obtained by taking limit above as $h \rightarrow +\infty$

COMPARISON BETWEEN HOS & D2N METHODS

RECENT EXTENSIVE STUDY BY SCHAFFER WAVES

REFERENCE:

Schäffer, H. A. (2008). *Comparison of Dirichlet-Neumann operator expansions for nonlinear surface gravity waves*. *Coast. Eng.*, **55**(4), 288–294

MAIN CONCLUSIONS:

- ▶ One variation is due to the choice of definition for the Dirichlet–Neumann (DN) operator, where CS differs slightly
- ▶ With these variations, we find that given the surface potential, the vertical surface velocity is identically determined for WW, DY and BST with a slight variation for CS
- ▶ With regard to the **temporal derivative of the surface elevation**, the result from CS matches that of WW exactly, while BST matches that of DY

CONVERGENCE STUDY

WITH HIS FORMER PHD STUDENT LIWEI XU



FIGURE: *Philippe Guyenne, Univ. of Delaware*

REFERENCE:

Xu, L., & Guyenne, P. (2009). *Numerical simulation of three-dimensional nonlinear water waves*. *J. Comput. Phys.*, **228**(22), 8446–8466

CONVERGENCE STUDY

D2N OPERATOR SPECTRAL APPROXIMATION

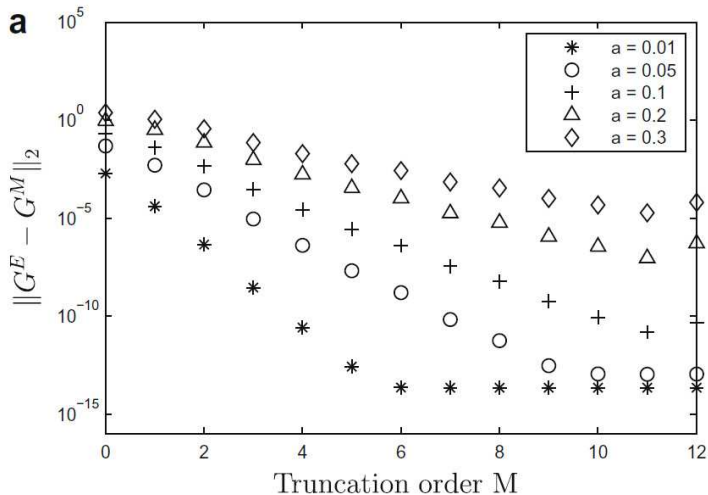


FIGURE: Influence of the nonlinearity

CONVERGENCE STUDY

D2N OPERATOR SPECTRAL APPROXIMATION

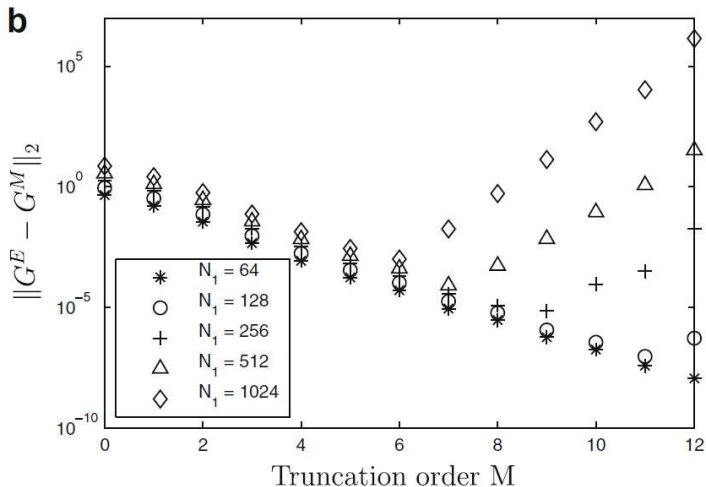


FIGURE: *Influence of the number of modes*

CONVERGENCE STUDY

D2N OPERATOR SPECTRAL APPROXIMATION

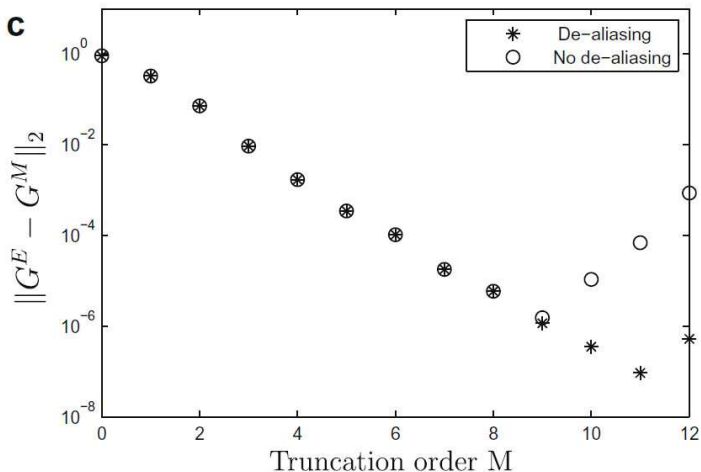


FIGURE: *Effect of the de-aliasing*

NUMERICAL RESULTS: SOLITARY WAVES COLLISION - I

(PH. GUYENNE, W. CRAIG, C. SULEM)

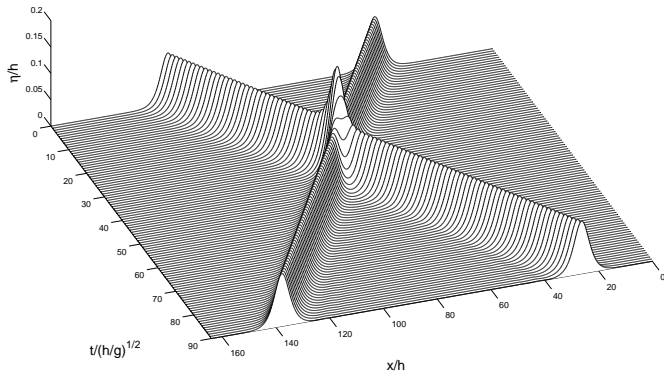


FIGURE: *Small amplitude ($a/h = 0.1$) solitary waves head-on collision.*

NUMERICAL RESULTS: SOLITARY WAVES COLLISION - II

(PH. GUYENNE, W. CRAIG, C. SULEM)

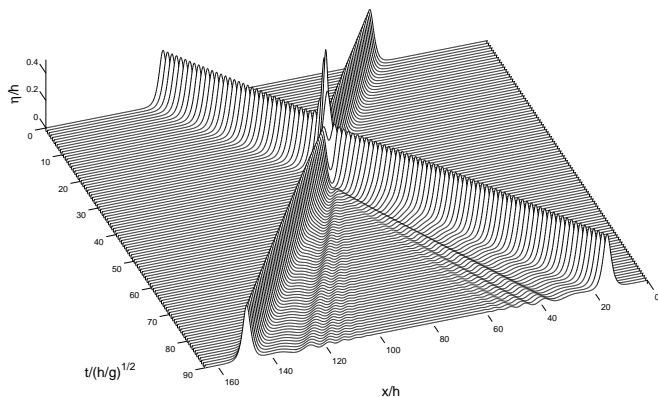


FIGURE: Moderate amplitude ($a/h = 0.4$) solitary waves head-on collision.

SOLITARY WAVES COLLISION – III

CARTOON OF THE COLLISION FOR $a/h = 0.4$: [HHGY04]

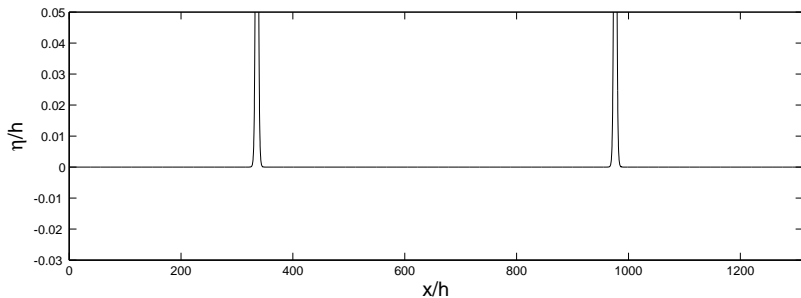


FIGURE: Initial condition: $t/\sqrt{g/h} = 0$

SOLITARY WAVES COLLISION – III

CARTOON OF THE COLLISION FOR $a/h = 0.4$: [HHGY04]

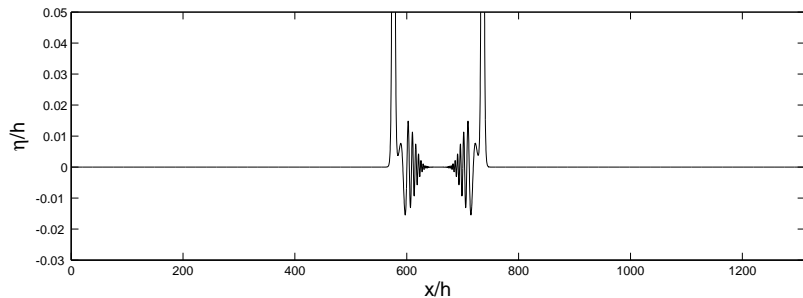


FIGURE: Just after interaction: $t/\sqrt{g/h} = 340$

SOLITARY WAVES COLLISION – III

CARTOON OF THE COLLISION FOR $a/h = 0.4$: [HHGY04]

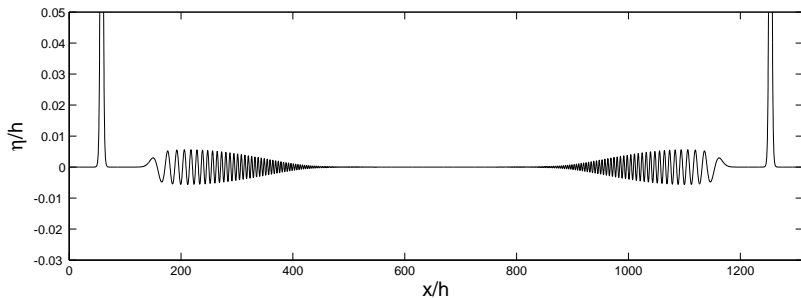


FIGURE: Long time evolution: $t/\sqrt{g/h} = 780$

CONCLUSIONS

PARTIAL CONCLUSIONS ABOUT HOS AND D2N METHODS

PROS:

- ▶ Relative simplicity
- ▶ Order can be made arbitrary high (in principle) thanks to recurrent relations

CONS:

- ▶ Pseudo-spectral \rightarrow periodic boundary conditions
- ▶ Cannot handle overturning surfaces (unlike BIEM)
- ▶ Increasing order of nonlinearity may lead numerical instabilities
- ▶ One cannot say the method is fully nonlinear






Thank you for your attention!







After the break let us listen to Claudio!
p.s.: A fully nonlinear approach to come...

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