NUMERICAL METHODS FOR FULLY NONLINEAR FREE SURFACE WAVES

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Short Course on "*Modeling of Nonlinear Ocean Waves*"

Outline of the short course

1. Lecture 1
   - Introduction (D.D.)
   - BIEM (C.V.)
   - Spectral CG-method (D.D.)
2. Lecture 2
   - Higher-Order Spectral (HOS) methods (D.D.)
   - Dirichlet-to-Neumann (D2N) operator technique (D.D.)
   - Conformal mappings (C.V.)

Remark:

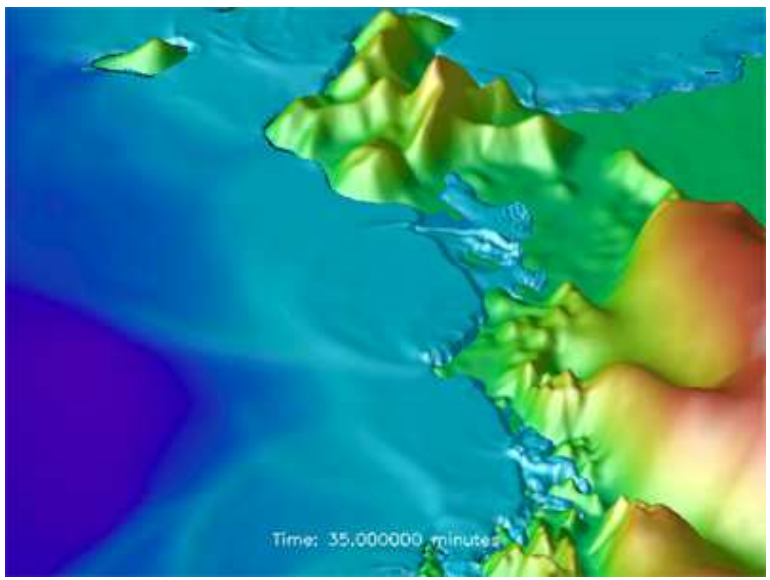
Focus only on numerical methods for the full Euler equations

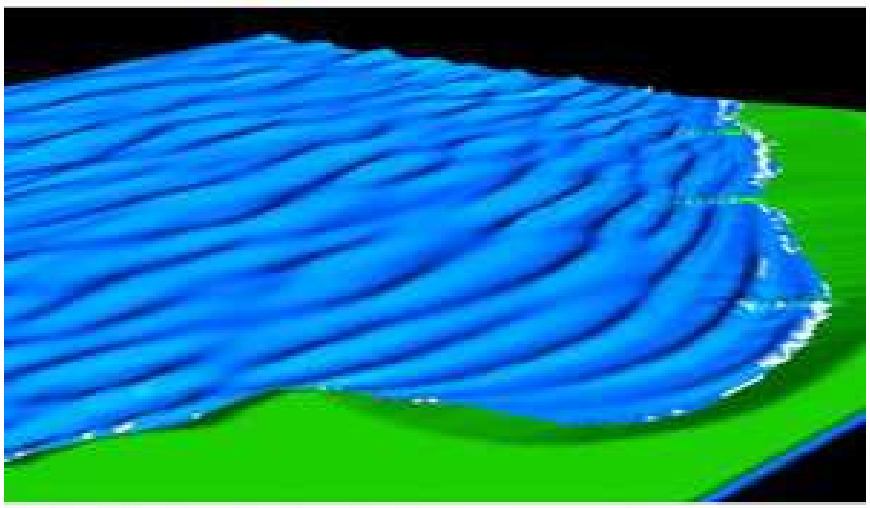
- No asymptotic models

What we will not cover in this course:

Self-learning is your friend!

- The first modern spectral method for water waves by Rienecker & Fenton (1981-1982) [RF81, FR82]
- Methods based on eigenfunctions expansions
  - Coupled-mode approach by Belibassakis & Athanassoulis [AB99, BA06]
- Finite difference-based methods
- Any kind of meshless methods





Physical assumptions

Mathematical modeling

Physical assumptions:

- Fluid is ideal (inviscid) ($Re = \infty$)
- Fluid is homogeneous ($\rho_w = const$)
- Flow is incompressible ($\nabla \cdot \underline{u} = 0$)
- Flow is potential ($\underline{u} = \nabla\phi$)

Interface conditions:

- Interface is a graph: $y = \eta(\underline{x}, t)$
- Air effect is neglected ($\rho_a \ll \rho_w$)
- Free surface is isobaric

Basic model:

- Incompressible Euler equations with free surface

EULER EQUATIONS

WITH FREE SURFACE

- Incompressibility:

$$\nabla \cdot \underline{u} = \nabla \cdot (\nabla \phi) = \nabla^2 \phi = 0$$

- Momentum conservation:

$$\underline{u}_t + (\underline{u} \cdot \nabla)\underline{u} + \frac{\nabla p}{\rho} = \underline{g}$$

$$\phi_t + \frac{1}{2}|\nabla \phi|^2 + gz + \frac{p}{\rho} = B(t)$$

BOUNDARY CONDITIONS:

$$\eta_t + \underline{u} \cdot \nabla \eta = v, \quad z = \eta(\underline{x}, t)$$

$$p = 0, \quad z = \eta(\underline{x}, t)$$

$$v_n = \underline{u} \cdot \underline{n}_b = 0, \quad z = -d(\underline{x})$$

- Continuity equation

$$\nabla^2\phi = 0, \quad (\underline{x}, y) \in \Omega \times [-d(\underline{x}), \eta(\underline{x}, t)],$$

- Kinematic bottom condition

$$\frac{\partial \phi}{\partial y} + \nabla\phi \cdot \nabla d = 0, \quad y = -d,$$

- Kinematic free surface condition

$$\frac{\partial \eta}{\partial t} + \nabla\phi \cdot \nabla\eta = \frac{\partial \phi}{\partial y}, \quad y = \eta(\underline{x}, t),$$

- Dynamic free surface condition

$$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + g\eta = 0, \quad y = \eta(\underline{x}, t).$$

FIGURE: Laplace

Hamiltonian structure

Petrov (1964) [Pet64]; Zakharov (1968) [Zak68]; Craig & Sulem (1993) [CS93]

Canonical variables:

$\eta(\underline{x}, t)$: free surface elevation

$\varphi(\underline{x}, t)$: velocity potential at the free surface

$\varphi(\underline{x}, t) := \phi(\underline{x}, y = \eta(\underline{x}, t), t)$

- Evolution equations:

$$\rho \frac{\partial \eta}{\partial t} = \frac{\delta \mathscr{H}}{\delta \varphi}, \qquad \rho \frac{\partial \varphi}{\partial t} = -\frac{\delta \mathscr{H}}{\delta \eta},$$

- Hamiltonian:

$$\mathscr{H} = \int_{-d}^{\eta} \frac{1}{2} |\nabla \phi|^2 \, \mathrm{d}y + \frac{1}{2} g \eta^2$$

Application to numerics:

Used in D2N operator methods: [CS93, GN07, XG09]

Luke's variational principles

J.C. Luke, JFM (1967) [Luk67]

- First improvement of the classical Lagrangian $\mathscr{L} := K - \Pi$:

$$\mathcal{L} = \int_{t_1}^{t_2} \int_{\Omega} \rho \mathscr{L} \, \mathrm{d}\underline{\mathrm{x}} \, \mathrm{d}t, \qquad \mathscr{L} := \int_{-d}^{\eta} \left(\phi_t + \frac{1}{2} |\boldsymbol{\nabla} \phi|^2 + g y \right) \mathrm{d}y$$

$$\begin{aligned}
\delta\phi: &\quad \Delta\phi = 0, \quad (\underline{\mathrm{x}}, y) \in \Omega \times [-d, \eta], \\
\delta\phi|_{y=-d}: &\quad \frac{\partial \phi}{\partial y} + \boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla} d = 0, \quad y = -d, \\
\delta\phi|_{y=\eta}: &\quad \frac{\partial \eta}{\partial t} + \boldsymbol{\nabla}\phi \cdot \boldsymbol{\nabla}\eta - \frac{\partial \phi}{\partial y} = 0, \quad y = \eta(\underline{\mathrm{x}}, t), \\
\delta\eta: &\quad \frac{\partial \phi}{\partial t} + \tfrac{1}{2} |\boldsymbol{\nabla}\phi|^2 + g\eta = 0, \quad y = \eta(\underline{\mathrm{x}}, t).
\end{aligned}$$

- Water wave problem formulation is recovered from $\mathcal{L}$

Application to numerics:

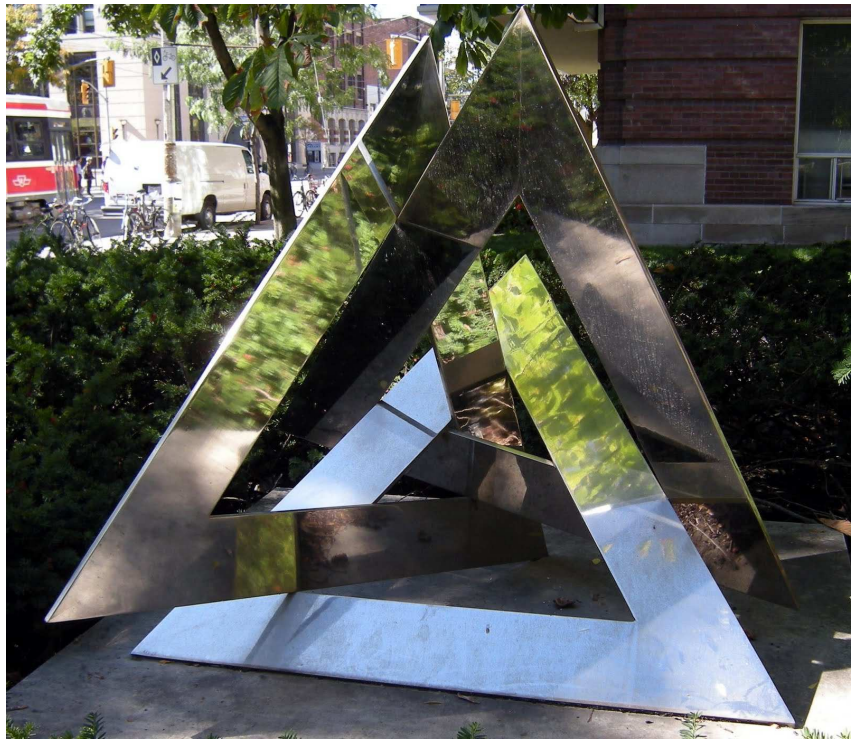
- Not fully explored...
- Coupled-mode technique by B. & A. (2006) [BA06]

WHY THIS PROBLEM IS DIFFICULT?

OUTLINE OF SOME NUMERICAL DIFFICULTIES

- Problem is highly nonlinear
- Computational domain is unknown ($y = \eta(\underline{x}, t)$ to be determined)
- Formulation is stiff (Hamiltonian structure)
- Taylor expansions involve very high derivatives
- Physical and numerical instabilities
- No dissipation to stabilize computation
- Overturning surface...

To be continued by Claudio...



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References I


G. A. Athanassoulis and K. A. Belibassakis.
A consistent coupled-mode theory for the propagation of small-amplitude water waves over variable bathymetry regions.
J. Fluid Mech, 389:275–301, 1999.


K. A. Belibassakis and G. A. Athanassoulis.
A coupled-mode technique for weakly nonlinear wave interaction with large floating structures lying over variable bathymetry regions.
Applied Ocean Research, 28(1):59–76, February 2006.


W. Craig and C. Sulem.
Numerical simulation of gravity waves.
J. Comput. Phys., 108:73–83, 1993.

References II


J. D. Fenton and M. M. Rienecker.
A Fourier method for solving nonlinear water-wave problems: application to solitary-wave interactions.
J. Fluid Mech., 118:411–443, April 1982.


P. Guyenne and D. P. Nicholls.
A high-order spectral method for nonlinear water waves over moving bottom topography.
SIAM J. Sci. Comput., 30(1):81–101, 2007.


J. C. Luke.
A variational principle for a fluid with a free surface.
J. Fluid Mech., 27:375–397, 1967.


A. A. Petrov.
Variational statement of the problem of liquid motion in a container of finite dimensions.
Prikl. Math. Mekh., 28(4):917–922, 1964.

References III


M. M. Rienecker and J. D. Fenton.
A Fourier approximation method for steady water waves.
J. Fluid Mech., 104:119–137, April 1981.


L. Xu and P. Guyenne.
Numerical simulation of three-dimensional nonlinear water waves.
J. Comput. Phys., 228(22):8446–8466, 2009.


V. E. Zakharov.
Stability of periodic waves of finite amplitude on the surface of a deep fluid.
J. Appl. Mech. Tech. Phys., 9:190–194, 1968.