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NUMERICAL METHODS FOR FULLY NONLINEAR FREE SURFACE WAVES

DENYS DUTYKH¹ & CLAUDIO VIOTTI¹

¹University College Dublin
School of Mathematical Sciences
Belfield, Dublin 4, Ireland

Short Course on “*Modeling of Nonlinear Ocean Waves*”



OUTLINE OF THE SHORT COURSE

1 LECTURE 1

- Introduction (D.D.)
- BIEM (C.V.)
- Spectral CG-method (D.D.)

2 LECTURE 2

- Higher-Order Spectral (HOS) methods (D.D.)
- Dirichlet-to-Neumann (D2N) operator technique (D.D.)
- Conformal mappings (C.V.)

REMARK:

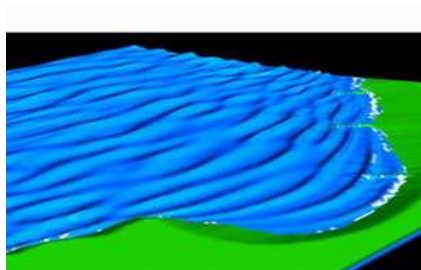
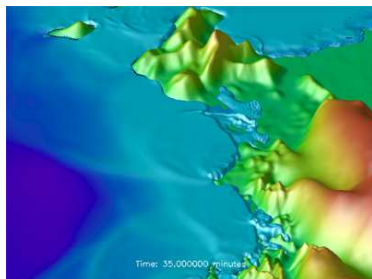
Focus only on numerical methods for the full Euler equations

- ▶ No asymptotic models

WHAT WE WILL NOT COVER IN THIS COURSE:

SELF-LEARNING IS YOUR FRIEND!

- ▶ The first modern spectral method for water waves by Rienecker & Fenton (1981-1982) [RF81, FR82]
- ▶ Methods based on eigenfunctions expansions
 - ▶ Coupled-mode approach by Belibassakis & Athanassoulis [AB99, BA06]
- ▶ Finite difference-based methods
- ▶ Any kind of meshless methods



PHYSICAL ASSUMPTIONS

MATHEMATICAL MODELING

PHYSICAL ASSUMPTIONS:

- ▶ Fluid is ideal (inviscid) ($Re = \infty$)
- ▶ Fluid is homogeneous ($\rho_w = \text{const}$)
- ▶ Flow is incompressible ($\nabla \cdot \underline{u} = 0$)
- ▶ Flow is potential ($\underline{u} = \nabla \phi$)

INTERFACE CONDITIONS:

- ▶ Interface is a graph: $y = \eta(\underline{x}, t)$
- ▶ Air effect is neglected ($\rho_a \ll \rho_w$)
- ▶ Free surface is isobaric

BASIC MODEL:

- ▶ Incompressible Euler equations with free surface



EULER EQUATIONS

WITH FREE SURFACE

- ▶ Incompressibility:

$$\nabla \cdot \underline{u} = \nabla \cdot (\nabla \phi) = \nabla^2 \phi = 0$$

- ▶ Momentum conservation:

$$\underline{u}_t + (\underline{u} \cdot \nabla) \underline{u} + \frac{\nabla \rho}{\rho} = \mathbf{g}$$

$$\phi_t + \frac{1}{2} |\nabla \phi|^2 + gz + \frac{\rho}{\rho} = B(t)$$



BOUNDARY CONDITIONS:

$$\eta_t + \underline{u} \cdot \nabla \eta = v, \quad z = \eta(\underline{x}, t)$$

$$\rho = 0, \quad z = \eta(\underline{x}, t)$$

$$v_n = \underline{u} \cdot \underline{n}_b = 0, \quad z = -d(\underline{x})$$

WATER WAVE PROBLEM

POTENTIAL FLOW FORMULATION

- ▶ Continuity equation

$$\nabla^2 \phi = 0, \quad (\underline{x}, y) \in \Omega \times [-d(\underline{x}), \eta(\underline{x}, t)],$$

- ▶ Kinematic bottom condition

$$\frac{\partial \phi}{\partial y} + \nabla \phi \cdot \nabla d = 0, \quad y = -d,$$

- ▶ Kinematic free surface condition

$$\frac{\partial \eta}{\partial t} + \nabla \phi \cdot \nabla \eta = \frac{\partial \phi}{\partial y}, \quad y = \eta(\underline{x}, t),$$

- ▶ Dynamic free surface condition

$$\frac{\partial \phi}{\partial t} + \frac{1}{2} |\nabla \phi|^2 + g\eta = 0, \quad y = \eta(\underline{x}, t).$$



FIGURE: Laplace

HAMILTONIAN STRUCTURE

PETROV (1964) [PET64]; ZAKHAROV (1968) [ZAK68]; CRAIG & SULEM (1993) [CS93]

CANONICAL VARIABLES:

$\eta(\underline{x}, t)$: free surface elevation

$\varphi(\underline{x}, t)$: velocity potential at the free surface

$$\varphi(\underline{x}, t) := \phi(\underline{x}, y = \eta(\underline{x}, t), t)$$

- ▶ Evolution equations:

$$\rho \frac{\partial \eta}{\partial t} = \frac{\delta \mathcal{H}}{\delta \varphi}, \quad \rho \frac{\partial \varphi}{\partial t} = -\frac{\delta \mathcal{H}}{\delta \eta},$$

- ▶ Hamiltonian:

$$\mathcal{H} = \int_{-d}^{\eta} \frac{1}{2} |\nabla \phi|^2 dy + \frac{1}{2} g \eta^2$$

APPLICATION TO NUMERICS:

Used in D2N operator methods: [CS93, GN07, XG09]

LUKE'S VARIATIONAL PRINCIPLES

J.C. LUKE, JFM (1967) [LUK67]

- ▶ First improvement of the classical Lagrangian $\mathcal{L} := K - \Pi$:

$$\mathcal{L} = \int_{t_1}^{t_2} \int_{\Omega} \rho \mathcal{L} \, d\underline{x} \, dt, \quad \mathcal{L} := \int_{-d}^{\eta} (\phi_t + \frac{1}{2} |\nabla \phi|^2 + gy) \, dy$$

$$\delta\phi: \Delta\phi = 0, \quad (\underline{x}, y) \in \Omega \times [-d, \eta],$$

$$\delta\phi|_{y=-d}: \frac{\partial\phi}{\partial y} + \nabla\phi \cdot \nabla d = 0, \quad y = -d,$$

$$\delta\phi|_{y=\eta}: \frac{\partial\eta}{\partial t} + \nabla\phi \cdot \nabla\eta - \frac{\partial\phi}{\partial y} = 0, \quad y = \eta(\underline{x}, t),$$

$$\delta\eta: \frac{\partial\phi}{\partial t} + \frac{1}{2} |\nabla\phi|^2 + g\eta = 0, \quad y = \eta(\underline{x}, t).$$

- ▶ Water wave problem formulation is recovered from \mathcal{L}

APPLICATION TO NUMERICS:

- ▶ Not fully explored...
- ▶ Coupled-mode technique by B. & A. (2006) [BA06]

WHY THIS PROBLEM IS DIFFICULT?

OUTLINE OF SOME NUMERICAL DIFFICULTIES

- ▶ Problem is highly nonlinear
- ▶ Computational domain is unknown ($y = \eta(\underline{x}, t)$ to be determined)
- ▶ Formulation is stiff (Hamiltonian structure)
- ▶ Taylor expansions involve very high derivatives
- ▶ Physical and numerical instabilities
- ▶ No dissipation to stabilize computation
- ▶ Overturning surface...



To be continued by Claudio...



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A consistent coupled-mode theory for the propagation of small-amplitude water waves over variable bathymetry regions.

J. Fluid Mech, 389:275–301, 1999.



K. A. Belibassakis and G. A. Athanassoulis.

A coupled-mode technique for weakly nonlinear wave interaction with large floating structures lying over variable bathymetry regions.

Applied Ocean Research, 28(1):59–76, February 2006.



W. Craig and C. Sulem.

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J. Comput. Phys., 108:73–83, 1993.

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P. Guyenne and D. P. Nicholls.

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J. C. Luke.

A variational principle for a fluid with a free surface.

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A. A. Petrov.

Variational statement of the problem of liquid motion in a container of finite dimensions.

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-  M. M. Rienecker and J. D. Fenton.
A Fourier approximation method for steady water waves.
J. Fluid Mech., 104:119–137, April 1981.
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Numerical simulation of three-dimensional nonlinear water waves.
J. Comput. Phys., 228(22):8446–8466, 2009.
-  V. E. Zakharov.
Stability of periodic waves of finite amplitude on the surface of a deep fluid.
J. Appl. Mech. Tech. Phys., 9:190–194, 1968.

Boundary Element Method for Three Dimensional Water Waves

C. Viotti, D. Dutykh

School of Mathematical Sciences, UCD, Dublin, Ireland

- Grilli, Skourup & Svendsen 1989 [1]
- Grilli, Guyenne & Dias 2001 [2]
- Fochesato & Dias 2006 [3]

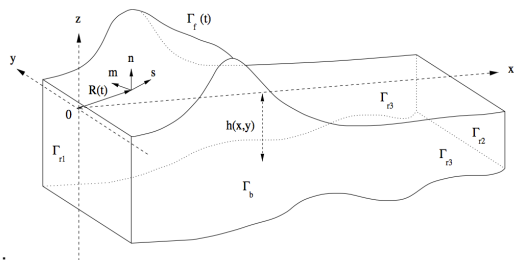
This numerical approach is probably the most general among those based on potential flow theory...

- 2D & 3D
- Overtopping waves (up to wave breaking)
- Arbitrary bathymetry
- Arbitrary boundary conditions (wave makers, absorbing beach,)

...of course, this comes at a cost

- Complicated
- Expensive

Governing equations and boundary conditions



On the free surface:

$$\frac{DR}{Dt} = \mathbf{u} = \nabla\phi$$

$$\frac{D\phi}{Dt} = \frac{1}{2}|\nabla\phi|^2 - g\mathbf{z} - \frac{p_a}{\rho}$$

where:

$$\frac{D\phi}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla$$

On the other boundaries:

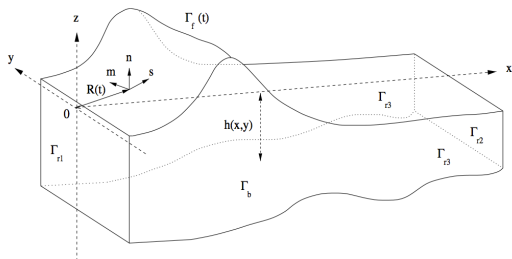
$$\nabla\phi \cdot \mathbf{n} = 0 \quad \text{or} \quad \nabla\phi \cdot \mathbf{n} = V_b$$

Inside the fluid volume Ω :

$$\nabla^2\phi = 0$$

For time marching need to find $\nabla\phi$ from the knowledge of ϕ .

Velocity potential



Laplace equation for the potential

$$\nabla^2 \phi(\mathbf{x}) = 0,$$

with boundary conditions

$$\left\{ \begin{array}{ll} \phi = \bar{\phi} & \text{on } \Gamma_f(t) \text{ (free surface)} \\ \frac{\partial \phi}{\partial n} = 0 & \text{on } \Gamma_b \text{ (bottom)} \\ \frac{\partial \phi}{\partial n} = V_i & \text{on } \Gamma_i, \quad i = 1, 2, 3 \text{ (lateral boundaries).} \end{array} \right.$$

Green's function

Free space Green's function

$$\nabla^2 G(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x}'), \quad \lim_{|\mathbf{x}| \rightarrow \infty} G = 0$$

3D Green's function

$$G(\mathbf{x}, \mathbf{x}') = \frac{1}{4\pi r}, \quad \frac{\partial G}{\partial n}(\mathbf{x}, \mathbf{x}') = -\frac{1}{4\pi} \frac{\mathbf{r} \cdot \mathbf{n}}{r^3}, \quad \mathbf{r} = \mathbf{x} - \mathbf{x}', \quad r = |\mathbf{r}|.$$

Combine with $\nabla^2 \phi = 0$ to obtain $\nabla^2 \phi G - \phi \nabla^2 G = -\delta \phi$.

Integrate over Ω :

$$\begin{aligned} \int_{\Omega} \nabla^2 \phi(\mathbf{x}') G(\mathbf{x}, \mathbf{x}') - \phi(\mathbf{x}') \nabla^2 G(\mathbf{x}, \mathbf{x}') dx' &= - \int_{\Omega} \delta(\mathbf{x}') \phi(\mathbf{x}') dx \\ \int_{\Omega} \nabla \cdot (\nabla \phi(\mathbf{x}') G(\mathbf{x}, \mathbf{x}') - \phi(\mathbf{x}') \nabla G(\mathbf{x}, \mathbf{x}')) dx' &= - \int_{\Omega} \delta(\mathbf{x}') \phi(\mathbf{x}') dx \end{aligned}$$

Then apply Divergence Theorem:

$$\int_{\Gamma} (\nabla \phi G - \phi \nabla G) \cdot \mathbf{n} d\Gamma = -\phi$$

Boundary Integral Equation

$$\begin{aligned}\alpha(\mathbf{x})\phi(\mathbf{x}) &= \int_{\Gamma} \frac{\partial\phi}{\partial n}(\mathbf{x}')G(\mathbf{x}, \mathbf{x}') - \phi(\mathbf{x})\frac{\partial G}{\partial n}(\mathbf{x}, \mathbf{x}')d\Gamma \\ &= \int_{\Gamma_s} \frac{\partial\phi}{\partial n}(\mathbf{x}')G(\mathbf{x}, \mathbf{x}') - \phi(\mathbf{x})\frac{\partial G}{\partial n}(\mathbf{x}, \mathbf{x}')d\Gamma_s \\ &+ \int_{\Gamma_b} \frac{\partial\phi}{\partial n}(\mathbf{x}')G(\mathbf{x}, \mathbf{x}') - \phi(\mathbf{x})\frac{\partial G}{\partial n}(\mathbf{x}, \mathbf{x}')d\Gamma_b \\ &+ \sum_i \int_{\Gamma_b} \frac{\partial\phi}{\partial n}(\mathbf{x}')G(\mathbf{x}, \mathbf{x}') - \phi(\mathbf{x})\frac{\partial G}{\partial n}(\mathbf{x}, \mathbf{x}')d\Gamma_i\end{aligned}$$

$$\alpha(\mathbf{x}) = \begin{cases} 1 & \text{for } \mathbf{x} \text{ inside the domain} \\ 1/2 & \text{for } \mathbf{x} \text{ on a smooth part of the boundary} \\ 1/4 & \text{for } \mathbf{x} \text{ on a edge} \\ 1/8 & \text{for } \mathbf{x} \text{ on a corner} \end{cases}$$

BIE is solved for the unknown variables on the boundary: $\frac{\partial\phi}{\partial n}$ on the free surface, and ϕ on the other boundaries.

Boundary Element Method (BEM) Discetization

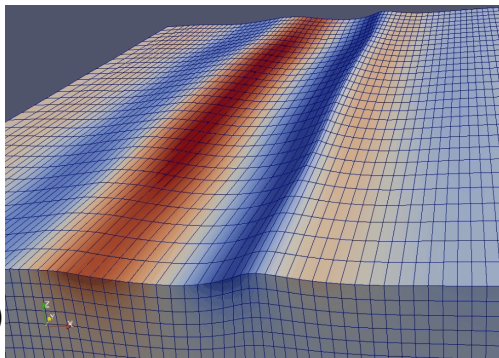
- N_{Γ} **collocation nodes** (i.e., points where all variables are sampled)
- M_{Γ} **finite elements** (discrete portions of the boundary, where all variables are reconstructed from neighboring nodal values)
- $k = 1, \dots, M_{\Gamma}$: **boundary element** index
- $N_j(\xi, \eta)$: **shape function** associated with the j -th node.
- (ξ, η) : local coordinates of the finite elements.

Represent geometry and field variable within each element:

$$\mathbf{x}(\xi, \eta) = N_j(\xi, \eta)\mathbf{x}_j^k,$$

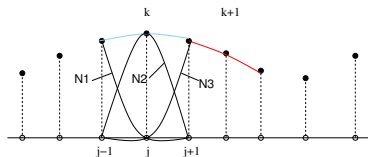
$$q(\xi, \eta) = N_j(\xi, \eta)q_j^k,$$

$j =$
 $1, \dots, m$ (nodes of the k -th element)



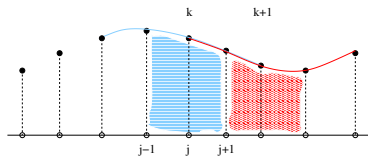
Construction of shape functions $N_j(\xi, \eta)$

The (too) simple way: polynomial interpolation using the nodal values which belong **only** to the current element



Inconvenient: only C^0 continuity is obtained, this was seen to enhance numerical instabilities (Grilli *et al.* 1989)

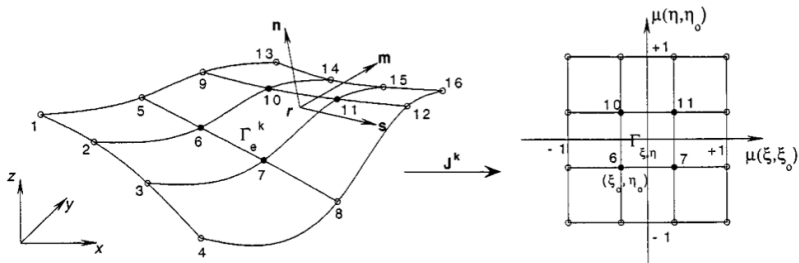
The better way: use also nodal values **outside** the current element: middle-interval-interpolation (MII)



Smoother reconstruction, enforce C^n regularity.

3D-MII Elements

Example (Grilli, Guyenne & Dias 2001): 16-node cubic 3D-MII element



$$N_j(\xi, \eta) = N'_{b(j)}(\mu(\xi, \xi_0))N'_{d(j)}(\mu(\eta, \eta_0)), \quad b, d = 1, \dots, 4; \quad j = 4(d-1) + b$$

$$N'_1(\mu) = \frac{1}{16}(1 - \mu)(9\mu^2 - 1), \quad N'_2(\mu) = \frac{9}{16}(1 - \mu^2)(1 - 3\mu),$$

$$N'_3(\mu) = \frac{9}{16}(1 - \mu^2)(1 + 3\mu), \quad N'_4(\mu) = \frac{1}{16}(1 + \mu)(9\mu^2 - 1),$$

Discretization of the Boundary Integral Equation

BIE is discretized by using shape-functions reconstructions of the geometry and field variables, and summing the contributions from all finite elements:

$$\begin{aligned}\int_{\Gamma(\mathbf{x})} \frac{\partial \phi}{\partial n} G_l \, d\Gamma &= \sum_{j=1}^{N_\Gamma} \left\{ \sum_{k=1}^{M_\Gamma} \int_{\Gamma_{\xi,\eta}} N_j(\xi, \eta) G(\mathbf{x}(\xi, \eta), \mathbf{x}_l) |\mathbf{J}^k(\xi, \eta)| \, d\xi \, d\eta \right\} \frac{\partial \phi}{\partial n}(\mathbf{x}_j) \\ &= \sum_{j=1}^{N_\Gamma} \left\{ \sum_{k=1}^{M_\Gamma} D_{lj}^k \right\} \frac{\partial \phi_j}{\partial n} = \sum_{j=1}^{N_\Gamma} K_{lj}^d \frac{\partial \phi_j}{\partial n} \\ \int_{\Gamma(\mathbf{x})} \phi \frac{\partial G_l}{\partial n} \, d\Gamma &= \sum_{j=1}^{N_\Gamma} \left\{ \sum_{k=1}^{M_\Gamma} \int_{\Gamma_{\xi,\eta}} N_j(\xi, \eta) \frac{\partial G(\mathbf{x}(\xi, \eta), \mathbf{x}_l)}{\partial n} \Big|_{\mathbf{J}^k(\xi, \eta)} \, d\xi \, d\eta \right\} \phi(\mathbf{x}_j) \\ &= \sum_{j=1}^{N_\Gamma} \left\{ \sum_{k=1}^{M_\Gamma} E_{lj}^k \right\} \phi_j = \sum_{j=1}^{N_\Gamma} K_{lj}^n \phi_j\end{aligned}$$

(Remember: the Jacobian is obtained from $\mathbf{x}(\xi, \eta) = N_j(\xi, \eta) \mathbf{x}_j^k$, $j = 1, \dots, m$)

This procedure yields the Dirichlet and Neumann $N_\Gamma \times N_\Gamma$ matrices:

$$\mathbf{K}^d, \quad \mathbf{K}^n.$$

- *Singular integrals.* Integral's kernel (Green's function) is singular for $l = j$ in the previous formulae. Particular treatment must be applied in this case: singularity extraction.
- *Edges and corners.* Double-node scheme: nodes on the edges are "counted twice". The value of ϕ is imposed to be the same, whereas $\frac{\partial \phi}{\partial n}$ has two values, one for each of the two local orientation of the surface.

Global linear system

The discretization procedure yields the global linear system:

$$\alpha_l u_l = \sum_{j=1}^{N_\Gamma} \{K_{lj}^d q_j - K_{lj}^n u_j\}$$

By moving nodal unknowns to the left-hand side and keeping known terms on the right-hand side we obtain:

$$\{C_{pl} + K_{pl}^n\} u_p - K_{gl}^d q_g = K_{pl}^d \bar{q}_p - \{C_{gl} + K_{gl}^n\} \bar{u}_g$$

where $l = 1, \dots, N_\Gamma$; $g = 1, \dots, N_g$ refers to nodes with a Dirichlet condition on the free surface, and $p = 1, \dots, N_p$ refers to nodes with a Neumann condition on the rest of the boundary. \mathbf{C} is a diagonal matrix containing the coefficients α_l .

- The linear system obtained is **dense**
- Iterative methods are typically the choice (GMRES, with a suitable preconditioner)

Time marching: 2nd Ord. Eulerian-Lagrangian formulation

$$\bar{\mathbf{R}}(t + \Delta t) = \mathbf{R}(t) + \Delta t \frac{D\mathbf{R}}{Dt}(t) + \frac{(\Delta t)^2}{2} \frac{D^2\mathbf{R}}{Dt^2}(t) + \mathcal{O}[(\Delta t)^3]$$

$$\bar{\phi}(\mathbf{R}(t + \Delta t)) = \phi(t) + \Delta t \frac{D\phi}{Dt}(t) + \frac{(\Delta t)^2}{2} \frac{D^2\phi}{Dt^2}(t) + \mathcal{O}[(\Delta t)^3]$$

- The first order coefficients are given by the b.c. on the free surface, and require the solution of the BIE.
- The second order coefficients are given by Lagrangian time differentiation of the free surface b.c.,

$$\begin{aligned} \frac{D^2\mathbf{R}}{Dt^2} &= \frac{D\mathbf{u}}{Dt} \\ \frac{D^2\phi}{Dt^2} &= \mathbf{u} \cdot \frac{D\mathbf{u}}{Dt} - g\omega - \frac{1}{\rho} \frac{Dp_a}{Dt} \end{aligned}$$

and require the calculation of $\frac{\partial\phi}{\partial t}$ of $\frac{\partial^2\phi}{\partial t\partial n}$. This is done by solving an analogous BIE for ϕ_t .

The two BIE's share the same geometry, hence they share the **same matrix** of the linear system, which then needs to be assembled **once!**

- Fast Multipole Method (Fochesato & Dias 2006 [3])
- Adaptive regridding

 S. T. Grilli, J. Skourup, and I. A. Svendsen.

An efficient boundary element method for nonlinear water waves.
Engineering Analysis with Boundary Elements, 6:97–107, 1989.

 S. T. Grilli, P. Guyenne, and F. Dias.

A fully non-linear model for three-dimensional overturning waves over an arbitrary bottom.

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 C. Fochesato and F. Dias.

A fast method for nonlinear three-dimensional free-surface waves.
Proc. Royal Soc. A, 462:2715–2735, 2006.

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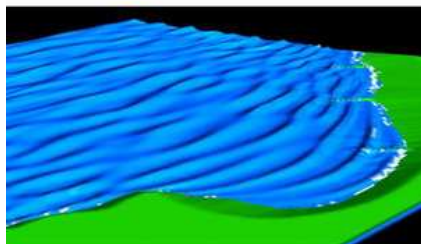
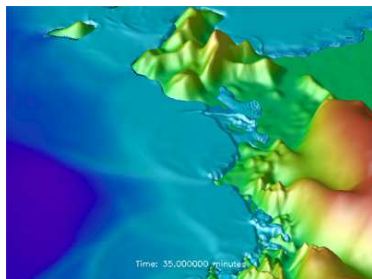
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J.C. LUKE, JFM (1967) [LUK67]

- ▶ First improvement of the classical Lagrangian $\mathcal{L} := K - \Pi$:

$$\mathcal{L} = \int_{t_1}^{t_2} \int_{\Omega} \rho \mathcal{L} \, d\underline{x} \, dt, \quad \mathcal{L} := \int_{-d}^{\eta} (\phi_t + \frac{1}{2} |\nabla \phi|^2 + gy) \, dy$$

$$\delta\phi: \Delta\phi = 0, \quad (\underline{x}, y) \in \Omega \times [-d, \eta],$$

$$\delta\phi|_{y=-d}: \frac{\partial\phi}{\partial y} + \nabla\phi \cdot \nabla d = 0, \quad y = -d,$$

$$\delta\phi|_{y=\eta}: \frac{\partial\eta}{\partial t} + \nabla\phi \cdot \nabla\eta - \frac{\partial\phi}{\partial y} = 0, \quad y = \eta(\underline{x}, t),$$

$$\delta\eta: \frac{\partial\phi}{\partial t} + \frac{1}{2} |\nabla\phi|^2 + g\eta = 0, \quad y = \eta(\underline{x}, t).$$

- ▶ Water wave problem formulation is recovered from \mathcal{L}

APPLICATION TO NUMERICS:

- ▶ Not fully explored...
- ▶ Coupled-mode technique by B. & A. (2006) [BA06]

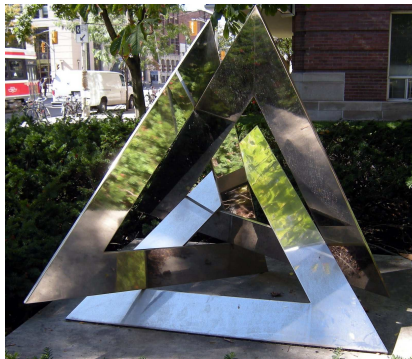
WHY THIS PROBLEM IS DIFFICULT?

OUTLINE OF SOME NUMERICAL DIFFICULTIES

- ▶ Problem is highly nonlinear
- ▶ Computational domain is unknown ($y = \eta(\underline{x}, t)$ to be determined)
- ▶ Formulation is stiff (Hamiltonian structure)
- ▶ Taylor expansions involve very high derivatives
- ▶ Physical and numerical instabilities
- ▶ No dissipation to stabilize computation
- ▶ Overturning surface...



To be continued by Claudio...



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