Exam

A number of different exercises is proposed. You do not need to try to solve them all.

Exercise 1 Let R be a relation on a set X, that is, a subset of X^2. Let R^0 be the identity relation, that is, x is in relation with y, written xR^0y, if $x = y$. And let R^{n+1} be the composition of R and R^n. The composition of two relations R and S, written $R \circ S$, is the relation such that $x(R \circ S)y$ if there is z such that xRz and zRy. Hence, xR^ny ($n > 0$) if there are $n-1$ intermediate elements $x_1, \ldots, x_{n-1}$ such that $xRx_1, \ldots, x_{n-1}Ry$. A relation S is reflexive if, for all x, xSx. A relation S is transitive if, for all x, y, z, xSz whenever xSy and ySz. Prove that:

1. $R^+ = \bigcup_{i>0} R^i$ is transitive.
2. R^+ is the smallest transitive relation containing R.
3. $R^* = \bigcup_{i\geq 0} R^i$ is transitive and reflexive.
4. R^* is the smallest transitive and reflexive relation containing R.
5. The function $f(X) = X \circ X$ is monotone, that is, $f(X) \subseteq f(Y)$ if $X \subseteq Y$.
6. R^* is a fixpoint of f, that is, $f(R^*) = R^*$.

Exercise 2 We consider the set of λ-terms $t = x \mid \lambda xt \mid tt$, where x is a variable taken in an infinite set $\mathcal{X}$. Let $\to_\beta$ be the smallest relation such that:

- $(\lambda xt)u \to_\beta t\{x \mapsto u\}$, where $t\{x \mapsto u\}$ is the term obtained by replacing every free occurrence of x in t by u (and renaming bound variables if necessary)
- $\lambda xt \to_\beta \lambda xt'$ if $t \to_\beta t'$
- $tu \to_\beta t'u$ if $t \to_\beta t'$
- $tu \to_\beta tu'$ if $u \to_\beta u'$

Prove that:

1. β-reduction introduces no new variables: if $t \to_\beta t'$, then $\mathrm{FV}(t') \subseteq \mathrm{FV}(t)$.
2. If $v \to_\beta^* v'$, then $t\{y \mapsto v\} \to_\beta^* t\{y \mapsto v'\}$.

3. $\to_\beta$ is stable by substitution: if $t \to_\beta t'$, then $t\{y \mapsto v\} \to_\beta t'\{y \mapsto v\}$.

4. $\to_\beta$ is locally confluent, that is, if $t \to_\beta u$ and $t \to_\beta v$, then there is w such that $u \to_\beta^* w$ and $v \to_\beta^* w$.

This last property means that, locally, the order in which we make β-reductions is not important, since we can always find a common reduct. An important property of β-reduction is that it also holds globally: if $t \to_\beta^* u$ and $t \to_\beta^* v$, then there is w such that $u \to_\beta^* w$ and $v \to_\beta^* w$. Thus, β-reduction is in some sense deterministic: it can lead to only one result.

Exercise 3 Define a 2-tapes Turing machine $(\Sigma, Q, q_i, q_f, \delta)$ computing the function $n \mapsto 2n$, where Σ is the finite alphabet used by the machine with at least the symbols $\{\square, 0, 1\}$, Q is the finite set of states of the machine, q_i the initial state, q_f the final state, and δ is a partial function from $Q \times \Sigma^2$ to $Q \times \Sigma^2 \times \{-1, 0, +1\}$. To do this, you are free to introduce in Σ symbols different from $\{\square, 0, 1\}$, and in Q states different from $\{q_i, q_f\}$. The machine starts in position 0 in state q_i with, on tape 1, $\square$ followed by n 1's and then 0's, and on tape 2, $\square$ followed by 0's. It must end in position 0 in state q_f with, on tape 2, $\square$ followed by $2n$ 1's and then 0's.

Exercise 4 We consider a set $\mathcal{X}$ of variables, a set $\mathcal{F}$ of function symbols and a set $\mathcal{P}$ of predicate symbols. We assume that every (function or predicate) symbol f is equipped with an arity $\alpha_{\mathsf{f}} \in \mathbb{N}$, that is, in a term of the form $\mathsf{f}(t_1, \ldots, t_n)$, we always have $n = \alpha_{\mathsf{f}}$. A model is given by a set A and, for each function symbol f of arity n, a function $\mathsf{f}_A : A^n \to A$ and, for each predicate symbol P of arity n, a function $\mathsf{P}_A : A^n \to \mathsf{Bool}$ where $\mathsf{Bool} = \{\mathsf{true}, \mathsf{false}\}$. Then, given a finite valuation $\mu : \mathcal{X} \to A$, the interpretation of a formula ϕ, written $[\![\phi]\!]_\mu$, is defined as follows:

- $[\![x]\!]_\mu = \mu(x)$
- $[\![\mathsf{f}(t_1, \ldots, t_n)]\!]_\mu = \mathsf{f}_A([\![t_1]\!]_\mu, \ldots, [\![t_n]\!]_\mu)$
- $[\![\mathsf{P}(t_1, \ldots, t_n)]\!]_\mu = \mathsf{P}_A([\![t_1]\!]_\mu, \ldots, [\![t_n]\!]_\mu)$
- $[\![\bot]\!]_\mu = \mathsf{false}$
- $[\![\neg\phi]\!]_\mu = \mathsf{not}([\![\phi]\!]_\mu)$
- $[\![\phi \vee \psi]\!]_\mu = \mathsf{or}([\![\phi]\!]_\mu, [\![\psi]\!]_\mu)$
- $[\![\phi \wedge \psi]\!]_\mu = \mathsf{and}([\![\phi]\!]_\mu, [\![\psi]\!]_\mu)$
- $[\![\phi \Rightarrow \psi]\!]_\mu = \mathsf{impl}([\![\phi]\!]_\mu, [\![\psi]\!]_\mu)$
- $[\![\forall x \phi]\!]_\mu = \mathsf{forall}(\{[\![\phi]\!]_{\mu \cup \{x,a\}} \mid a \in A\})$ if $x \notin \mathrm{dom}(\mu)$
- $[\![\exists x \phi]\!]_\mu = \mathsf{exists}(\{[\![\phi]\!]_{\mu \cup \{x,a\}} \mid a \in A\})$ if $x \notin \mathrm{dom}(\mu)$

where $\mathrm{dom}(\mu)$ is the set of variables on which μ is defined, the boolean functions not, or, ... are defined as usual, and $\mathsf{forall}, \mathsf{exists} : \mathcal{P}(\mathsf{Bool}) \to \mathsf{Bool}$ are defined as follows:

- $\mathsf{forall}(S) = \mathsf{true}$ iff $\mathsf{false} \notin S$
- $\mathsf{exists}(S) = \mathsf{true}$ iff $\mathsf{true} \in S$

The universal closure of a formula ϕ with free variables $x_1, \ldots, x_n$ is $\overline{\forall}\phi = \forall x_1 \ldots \forall x_n \phi$. A formula ϕ is *valid* if, in every model A, $[\![\overline{\forall}\phi]\!] = \mathsf{true}$.

Let the provability relation $\vdash$ be the smallest relation on pairs (Γ, ϕ) where Γ is a finite set of formulas (the assumptions) and ϕ a formula, such that:

$$\text{(axiom)}\ \frac{\phi \in \Gamma}{\Gamma \vdash \phi}$$

$$(\bot\text{-elim})\ \frac{\Gamma \vdash \bot}{\Gamma \vdash \phi}$$

$$(\Rightarrow\text{-intro})\ \frac{\Gamma \cup \{\phi\} \vdash \psi}{\Gamma \vdash \phi \Rightarrow \psi}$$

$$(\Rightarrow\text{-elim})\ \frac{\Gamma \vdash \phi \Rightarrow \psi \quad \Gamma \vdash \phi}{\Gamma \vdash \psi}$$

$$(\wedge\text{-intro})\ \frac{\Gamma \vdash \phi \quad \Gamma \vdash \psi}{\Gamma \vdash \phi \wedge \psi}$$

$$(\wedge\text{-elim-left})\ \frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \phi}$$

$$(\wedge\text{-elim-right})\ \frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \psi}$$

$$(\vee\text{-intro-left})\ \frac{\Gamma \vdash \phi}{\Gamma \vdash \phi \vee \psi}$$

$$(\vee\text{-intro-right})\ \frac{\Gamma \vdash \psi}{\Gamma \vdash \phi \vee \psi}$$

$$(\vee\text{-elim})\ \frac{\Gamma \vdash \phi \vee \psi \quad \Gamma \cup \{\phi\} \vdash \chi \quad \Gamma \cup \{\psi\} \vdash \chi}{\Gamma \vdash \chi}$$

$$(\forall\text{-intro})\ \frac{\Gamma \vdash \phi \quad x \notin \mathrm{FV}(\Gamma)}{\Gamma \vdash \forall x \phi}$$

$$(\forall\text{-elim})\ \frac{\Gamma \vdash \forall x \phi}{\Gamma \vdash \phi\{x \mapsto t\}}$$

$$(\exists\text{-intro})\quad \frac{\Gamma \vdash \phi\{x \mapsto t\}}{\Gamma \vdash \exists x \phi}$$

$$(\exists\text{-elim})\quad \frac{\Gamma \vdash \exists x \phi \quad \Gamma \cup \{\phi\} \vdash \chi \quad x \notin \mathrm{FV}(\Gamma) \cup \mathrm{FV}(\chi)}{\Gamma \vdash \chi}$$

$$(\mathrm{EM})\quad \Gamma \vdash \phi \vee \neg\phi$$

Prove that the provability relation is correct, that is, in every model A, if $\Gamma \vdash \phi$ and $\Gamma = \{\phi_1, \ldots, \phi_n\}$, then $\phi_1 \wedge \ldots \wedge \phi_n \Rightarrow \phi$ is valid.