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David Dureisseix

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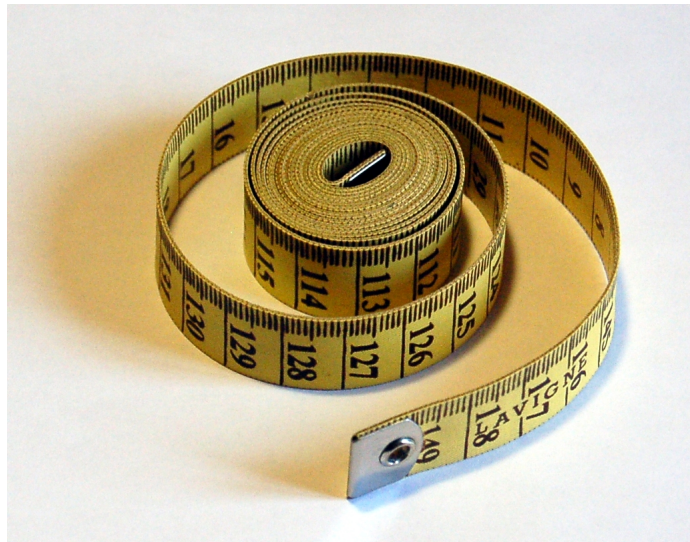
# An introduction to dimensional analysis

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May 8, 2023

This document is a short (and hopefully concise) introduction to dimensional analysis and is not expected to be printed. Indeed, it relies on URL links (in colored text) to refer to information sources and complementary studies, so it does not provide a large bibliography, nor many pictures.

It has been realized with the kind help of Ton Lubrecht and Marie-Pierre Noutary.



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# 1 Goals of dimensional analysis

**Dimensional analysis** (DA) is a wide-spread tool in fluid mechanics, but less used for structural engineering. Nevertheless, it has been implicitly present for a long time in certain applications. One example is stress concentration graphs [4], another the design of small scale models (for experimental quantitative studies) [similitude](#).

Since these notes were initially part of a course on ‘friction and lubrication’, the mechanical components that are studied, belong to the theme of fluid-structure interaction. As such, some applications of dimensional analysis for fluids mechanics might transpose to solids mechanics. Nevertheless, not so much applications concerning the ‘friction and lubrication’ theme are developed in these notes, since dimensional analysis application range is wider than this particular course.

Apart for its instructional purpose on physical quantities and the system of units [2, 10], DA enables one to check the homogeneity of a formula. DA is often used to make physical equations dimensionless and to focus on dimensionless characteristic numbers that are useful for detecting the physical regime of the studied phenomena (Reynolds number, Mach number...). Such an analysis may eventually lead to a simplification of the model, a reduction of the number of parameters, an extension of the domain of application and the use of small scale models.

# 2 Physical quantities and their units

A physical quantity (a length, a pressure, a temperature...) is usually a compound of two parts: (i) a value (i.e. a number), and (ii) a unit which is characteristic of the kind of physical quantity that is looked at. We usually indicate that a symbol chosen to represent this quantity is equal to the value times the unit, e.g. a length is typically

$$L = 2.5 \text{ km}$$

Such quantities are sometimes called *denominate numbers*.

Here we wish to point out that this is indeed a true multiplication between the value and the unit. Changing the unit does change the value but not the physical quantity:

$$L = 2.5 \text{ km} = 2500 \text{ m} = 2.5 \times 10^9 \mu\text{m}$$

Moreover, we can divide by a unit to extract a value:

$$L/\text{m} = 2500$$

(it is a true division of a physical quantity by a unit). Note that this is the recommended notation for labeling the axis in a graph: values as tics, or table entries, see appendix B.1). Additionally, we can imagine dividing a physical quantity by a number:

$$L/2500 = \text{m} = 1 \text{ m}$$

To convince yourself, you can also have a look on [this slot](#) of the Khan Academy.

name	abbrev.	usage	remarks
knot	kt	unit for speed, nautical	1 knot = 1 nautical mile per hour
horsepower	hp	unit of power	1 hp = 746 W
acre	ac	unit of area	1 acre = 4840 square yards
barrel		unit for volume, now much useful for oil, US	1 barrel = 42 US gallons
barrel		unit for volume, for beer	1 barrel = 36 gallons, or 4 firkins, or 288 pints
slug	SL	unit for mass, old British system	mass which provides a force of 1 pound when subjected to an acceleration of 1 ft/s <sup>2</sup>

Table 1: Some strange units (still in use)

For the purpose of good communication, we expect that all the people working together on a project understand the same physical quantities, i.e. they are aware of the units (and their abbreviations) used by the others. To avoid misunderstanding, the standardization imposes rules to denote the physical quantities and their units<sup>1</sup>. A milestone in this standardization procedure is the 11th general conference on weights and measures, held in 1960: This is the first advent of the International System of units (SI).

Clearly, there are still other units in use, for instance the [English units](#), the [Imperial units](#)... that are now supposed to be superseded by the SI<sup>2</sup>. Other, even less regulated units, are linked to various professions; try to guess which professions use these somewhat strange units in table 1, after [this link](#).

## 2.1 A bit of History: Definition of the meter

The first definition dates back to march 26, 1791:  $1\text{ m} = 1/10\,000\,000 \times$  half the length of the earth meridian. 16 standard meters (mètres-étalons) were made; only 4 remain today<sup>3</sup>. The idea was therefore: The value of the length of something measured in meters, is the number of times the standard length fits alongside. For instance, if  $L_0$  was the standard meter, then  $L/L_0 = L/\text{m} = 2\,500$  (one can align 2 500 times the standard length to cover the measured distance). Note that if the standard models where not ‘exactly’ equal, or if their length varies in time due to temperature, hygrometry... one is running into trouble. So, with the advent of more and more precise measurements, this ‘definition’ itself was not accurate enough. Nowadays, the meter is defined as the length of the path traveled by light in vacuum during a time interval of  $1/299\,792\,458$  of a second [3] (this definition may change again in the future... see appendix A).

## 2.2 A simple example of computing on values *and* on units

Consider the case where you have to evaluate the velocity of a peripheral point on a rotating solid. The angular velocity of the solid is  $\omega$  and the distance of

<sup>1</sup>try to browse [this one](#), [this other one](#), or even [this one](#)

<sup>2</sup>see [this link](#) to be aware that this is not yet the case

<sup>3</sup>have a look at [this](#), to locate [this one](#)

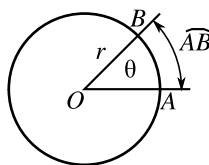


Figure 1: Angle measurement

the considered point to the axis of rotation is  $R$ ; you probably already know that the particular velocity is  $V = R\omega$ . Now we need a numerical application: Say that the radius is  $R = 10$  cm, and the angular velocity is  $\omega = 1500$  rpm; what is the velocity?

### 2.2.1 Angular measurement

First, have a look at the unit of angular velocity. rpm stands for ‘revolution per minute’; it is a classical unit, but not as standard as the french counterpart, for which tr/min stands for ‘tour par minute’<sup>4</sup>.

‘tour’ is actually a unit for measuring angles. This angle measurement is our first special case: The corresponding basic SI unit is the radian (abbreviated with rad<sup>5</sup>) and is defined as follows. Consider a circle of radius  $r$  and center  $O$  with two points  $A$  and  $B$  defining the angle  $\theta$ , figure 1. The measure of this angle in radians is defined as the ratio of the arc length  $\widehat{AB}$  over the radius:  $\theta = \widehat{AB}/r$ . Therefore, the radian is a unit which is... nothing (a length divided by a length)<sup>6</sup>. Hence the following is valid:

$$\theta = \pi \text{ rad} = \pi$$

nevertheless, the advice is: Keep the symbol rad to keep in memory that this is an angle! Other angle measures (not SI) are: the ‘tour’ (tr and the ‘grad’ (gon),

$$1 \text{ tr} = 2\pi \text{ rad} = 400 \text{ gon}$$

Note also that the second of arc '' and the minute of arc ' are measures of angle, not of time:

$$1' = (1/60) \text{ tr}, \quad 1'' = (1/60)'$$

The basic unit for the rotation rate is therefore rad/s. This is different from a frequency (the number of cycles per time unit) though they have formally the same unit: The [hertz](#) <sup>7</sup> (abbreviated as Hz) is better used for the frequency,

$$1 \text{ Hz} = 1 \text{ s}^{-1}$$

<sup>4</sup>note that the duration unit ‘minute’ is abbreviated as min and not mn!

<sup>5</sup>and not rd which is the ‘rad’: An old unit for radiation quantity, now superseded by the ‘gray’

<sup>6</sup>There is also a measure for the solid angle which is the [steradian](#), defined as a surface to surface ratio

<sup>7</sup>It seems that the units, even when derived from a name, are not expected to exhibit an uppercase letter... except when this is a name, such as ‘degree Celsius’... but the symbol does... a bit strange, however.

### 2.2.2 Using your favorite pocket calculator

Lets go back to the numerical application for the velocity

$$V = R\omega$$

To obtain  $V$ , simply use your pocket calculator. Note that, up to now, this kind of computer only computes using values, not yet on physical quantities (the units are not taken into account). So we can use it as: enter the value for the radius (10), multiply by the value for the rotation rate (1 500), ask for the result which is hopefully 15 000.

The problem is now: What is the associated unit?

The answer is: The unit of the result is the same computing sequence applied to the units of the input. Since we did a multiplication, the correct answer should be:

$$V = (10 \times 1\,500)(\text{cm} \times \text{tr}/\text{min}) = 15\,000 \text{ cm} \cdot \text{tr}/\text{min}$$

Though correct, this result is quite cumbersome. Better is to use a more standard unit such as m/s. To change the units, perform the following calculations:

$$\text{cm} = 10^{-2}\text{m}, \quad \text{tr} = 2\pi \text{ rad}, \quad \text{min} = 60 \text{ s}$$

so

$$V = 15\,000 \text{ cm} \cdot \text{tr}/\text{min} = 15\,000 \times (10^{-2}\text{m}) \times (2\pi \text{ rad})/(60 \text{ s}) = 5\pi \text{ m/s}$$

### 2.3 Additional remarks and (funny) special cases

Note that if additions have to be made, only quantities with the same physical meaning can be added; the following development is therefore correct:

$$1 \text{ m} + 10 \text{ cm} = (1 + 0.1) \text{ m} = 1.1 \text{ m}$$

If you consider analytical functions such as sin, log, exp... their argument should be without unit. For angles, the rad can be perfectly used in this way!

Some units are compounds of a base unit with fractional exponents. This is perfectly consistent with the previous use. For instance, the crack propagation in fracture mechanics uses the stress intensity factor (in crack mode I), defined as

$$K_I = 1.12\sigma\sqrt{\pi a}/\Phi$$

where  $\sigma$  is the tensile nominal stress,  $a$  is the crack length, and  $\Phi$  is a coefficient depending on the crack shape. You can therefore check that the unit of  $K_I$  is

$$[K_I] = \text{Nm}^{-3/2}$$

The crack growth itself is often modeled with a Paris law of the form

$$da/dN = C[\Delta K_I]^m$$

where  $N$  is the number of cycles, and  $\Delta K_I$  a variation of  $K_I$  during a cycle.  $m$  is a material parameter that should have no unit, and  $C$  is a second material parameter; you can check that the unit of  $C$  is

$$[C] = \text{m}(\text{Nm})^{3m/2}$$

(it depends itself on the value of  $m!$ ).

To avoid the use of many zeros before the comma (for huge quantities) or after it (for small quantities), some prefixes are standardized. They are merely multiplicative values without unit. Apart the classical deca (1 da = 10), hecto (1 h = 100), deci (1 d = 0.1), centi (1 c = 0.01), the other standard ones are defined for each change of amplitude of 3 orders of magnitude, sometimes with [funny names](#).

Concerning multiplicative factors, you can also think about the percentages: Percent merely means one over one hundred. Therefore one simply has  $5\% = 0.05$ ,  $4\text{‰} = 0.004$ ...

Finally, concerning the computation on values as well as on units, the special case where the value is zero can be mentioned. The [zero property](#) states that anything multiplied with zero is zero, so we can write:

$$V = 0 \text{ m/s} = 0$$

To be sure that this rule applies, care must be taken for the temperature measurements... see [appendix C](#).

### 3 Physical laws do not depend on a particular system of units

even if you don't now what exactly the laws are... (this is a consequence of the more fundamental statement that all physical laws can be represented in a form equally valid for all observers).

What is a physical law? It is a relationship between physical quantities (which are products of a value and a unit). For the previous example of the velocity, it can be expressed as

$$V = f(R, \omega)$$

where  $f$  is a function of two parameters, this indicates the relation between the physical quantities  $V$ ,  $R$ , and  $\omega$ . Indeed, the same function could be applied to the values alone, and to the units alone, as exemplified before. Therefore, the physical law  $f$  does not depend on the choice of the system of units, provided it is consistent: Whatever this choice is for  $R$  and  $\omega$ , the unit of  $V$  can be deduced.

#### 3.1 The SI base units

The SI is first based on a set of independent base units. What is the number of independent base units? The smallest possible. This means that each time a physical law is identified and modeled, it is a relationship between different physical units that are therefore not independent. For instance, the newton is a unit for a force, but one knows that gravity produces a force  $F$  from a mass  $M$  and an acceleration  $g$ :

$$F = Mg$$

is a physical law. The same function applies to the units, so

$$\text{N} = \text{kg} \cdot \text{m/s}^2$$



which can be viewed as the definition of N. N is therefore a derived unit (not a base one) built on kg, m and s. Today, the number of all the base units is 7<sup>8</sup>.

A useful notation, though not so standard, consists in using brackets to denote the extraction of the unit in the SI system from a physical quantity<sup>9</sup>. For instance,  $[V] = \text{m/s}$ ,  $[F] = \text{N} = \text{kg}\cdot\text{m}\cdot\text{s}^{-2}$ . Therefore the value of a physical quantity in the SI unit is for instance  $V/[V]$  or  $F/[F]$ .

Now consider the initial physical law  $V = f(R, \omega)$ . It is valid for the units in a consistent system of units, like SI. Therefore, one can write:  $[V] = f([R], [\omega])$ , and it is also valid for the values in a consistent system of units, i.e.  $V/[V] = f(R/[R], \omega/[\omega])$ . This last expression relates dimensionless quantities!

### 3.2 Special choice of base units

For deriving a dimensional analysis, the [Vashy-Buckingham IIs theorem](#) is famous and the general approach for rigorous derivations. A simpler approach which is used herein is the following: Since a physical law is independent of the system of units, one can use a dedicated system of units. In the same spirit as for the artifacts, the ‘mètres-étalons’ may be chosen in the arguments of the physical function law itself. Indeed, the question of the values for the various quantities is therefore how many ‘étalons’ are there in the quantity? Note that some fundamental requirements are:

- (i) the output is independent of the inputs (think of a test rig: To design and operate it, you need to specify a certain number of input parameters, such as lengths, velocities... and the measured outputs are the results of the test; for a spring you can prescribe the displacement and measure the force, or conversely, but not prescribe both),
- (ii) the inputs chosen as base units should be independent, i.e. they should provide all the units required for the physical problem that is studied. Their number may depend on the overall physics that is studied; for instance, if one studies geometry only, there is only one unit required: the one used for length. If one deals with kinematics, there should be two units to be able to measure length and time. If forces are also involved, (dynamics), there should be three units that allow to measure length, time and force (or mass, since  $\text{N} = \text{kg}\cdot\text{m}\cdot\text{s}^{-2}$ ).

Lets use the example of our physical law  $V = f(R, \omega)$ . Clearly, we need something for the lengths, as well as for the time. Two basic units are required, so we can try  $R$  and  $\omega$ . Now we wish to write the same physical law for the values of each involved quantity: How many  $R$ 's in  $R$ ?  $R/R = 1$ , so 1 is the value of  $R$  in this new system of units; how many  $\omega$ 's in  $\omega$ ?  $\omega/\omega = 1$ , so 1 is also the value of  $\omega$  in this new system of units. Finally, what is the value of  $V$ ? Here we can see that the choice of the two base units is enough to describe the physics: The value of  $V$  is  $V/(R\omega)$  (justified by  $[R\omega] = \text{m/s}$ ). The physical law for the values in this new system of units is therefore:

$$\frac{V}{R\omega} = f\left(\frac{R}{R}, \frac{\omega}{\omega}\right) = f(1, 1) = \text{constant}$$

<sup>8</sup>see [this link](#), or [this one](#)

<sup>9</sup>in particular, we get for an angle  $[\theta] = \text{rad} = 1$

Without information on the detailed model for the physical law, we can nevertheless conclude that it reads

$$V = \text{constant} \times R\omega$$

(which is indeed consistent with the detailed result that found constant = 1).

This case is a little bit special and we can consider a more generic case: If we consider that the solid with a rotational movement is very flexible, the centrifugal forces may change the form of the solid, and so, the velocity that is studied. Determining (or choosing) the input parameters is a difficult task, it relies on the expertise of the modeler. Here, we can consider that the stiffness of the material  $E$  (Young modulus) is an important parameter, as well as the Poisson coefficient  $\nu$  for the elastic behavior, and the density  $\rho$ . The physical law is therefore

$$V = f(R, \omega, E, \nu, \rho)$$

These parameters are usually belonging to 3 possible families: (i) the geometrical parameters, (ii) the load parameters, (iii) the material parameters. With this new physics, 3 base units are required. Lets try to choose  $R$ ,  $\omega$  and  $E$ . You can check that the same framework as before leads to

$$\frac{V}{R\omega} = f\left(1, 1, 1, \nu, \frac{\rho}{ER^{-2}\omega^{-2}}\right) = \bar{f}\left(\nu, \frac{\rho}{ER^{-2}\omega^{-2}}\right) \quad (1)$$

The number of arguments is reduced from 5 to only 2<sup>10</sup>! The dimensional analysis actually does not introduce additional information, but rather condenses the information. Nothing more can be said about the expression (1) up to this point. To particularize the unknown function  $\bar{f}$  some more modeling, or assumptions, or experiments have to be made.

The initial choice of important parameters in the description of the physical law may lead to different results. Note that for this example only one geometric parameter,  $R$ , was in the list of arguments. This means that the final result is only valid for a class of cases where only the size of the solid (given with the sole parameter  $R$ ) can change, and not its shape. On the other hand, if the complete shape was to be taken into account, all the (independent) geometric parameters defining the geometry (think of a fully parameterized geometry with a CAD tool) should have been arguments of function  $f$ , but still only 3 parameters could have been gotten rid of.

### 3.3 Does the result depend on your initial special choice?

Consider again the physical law  $V = f(R, \omega, E, \nu, \rho)$ . But try now as base units  $R$ ,  $E$  and  $\rho$ . You can check again that this is a valid choice and that it leads to:

$$\frac{V}{\sqrt{E/\rho}} = f\left(1, \frac{\omega}{R^{-1}\sqrt{E/\rho}}, 1, \nu, 1\right) = \bar{g}\left(\frac{\omega}{R^{-1}\sqrt{E/\rho}}, \nu\right) \quad (2)$$

Still 2 arguments, but not really the same expression as before (1)... One or the other expression may be more or less interesting for the user, but are they

<sup>10</sup>note also that  $\nu$  has not been changed. Indeed it was initially already a dimensionless coefficient, that has the same value on every system of units!

equivalent?

$$V = R\omega \times \bar{f}\left(\nu, \frac{\rho}{ER^{-2}\omega^{-2}}\right) = \sqrt{E/\rho} \times \bar{g}\left(\frac{\omega}{R^{-1}\sqrt{E/\rho}}, \nu\right) \quad ?$$

To simplify this expression, let's denote  $x = (R^2\omega^2)/(E/\rho)$  and  $y = 1/\sqrt{x}$ ; this can be re-casted as: Do we get

$$\bar{f}(\nu, x) = \frac{1}{\sqrt{x}} \times \bar{g}\left(\frac{1}{\sqrt{x}}, \nu\right) \quad \text{or} \quad \bar{g}(y, \nu) = \frac{1}{y} \times \bar{f}\left(\nu, \frac{1}{y^2}\right) \quad ?$$

Now do not forget that  $\bar{f}$  was an unknown function; so is  $\bar{g}$ . Therefore, we only see now what is the relation between these two unknown functions... If we know one, we now know the other one. So all is perfectly consistent.

### 3.4 Example of stress concentration factors (SCF)

Consider for instance the case of a cylindrical beam in traction ( $P$  is the traction force) with a change in diameter from  $D$  to  $d$ , and with a radius at the junction  $r$ , figure 2. You probably know [6] that the maximum stress in the material is reached near the diameter change, and that its value is larger than that obtained in the case of a smooth beam, i.e. a beam without diameter change, and with the smallest diameter  $d$ :

$$\sigma_{\max} > \sigma_{\text{nom}} = \frac{P}{\pi d^2/4}$$

The question is: How to obtain this maximum stress needed for designing the beam? One may rely on experiments, simulations... but in order to build a compendium of results available to rapidly estimate the maximum stress for various geometries, materials, forces... many tests or computations will be required. To decrease this effort, one can try dimensional analysis.

The maximum stress is obtained with a certain physical law, once we decide what are the input parameters. We may consider that geometry is given with  $d$ ,  $D$ ,  $r$  but not the length! (we consider only diameter changes and other 'defects' far away one to the other, so that they do not interfere). Note that we suspect herein that  $r$  is an influential parameter (if it is the case, and if we did not use it as an entry, we would make a modeling mistake...) We also did not add to the inputs the length  $t = (D - d)/2$  since it is not independent of the other ones.

Concerning the material parameters, for an linear isotropic elastic homogeneous material (an additional assumption), we could use Young modulus  $E$  and Poisson coefficient  $\nu$ . We may suspect that  $E$  has no influence (at least, it has none on  $\sigma_{\text{nom}}$ )... to be assumed with the user knowledge, to be checked by experiments or simulations... A usual assumption in this case is also that for classical range of the values of  $\nu$ , it has only a small influence on the result.

Finally, concerning the load, indeed  $P$  has an influence.

Choosing the important parameters is one of the harder tasks. For now on, we consider that

$$\sigma_{\max} = f(d, D, r, P)$$

The second stage is to choose the base units among the inputs. For instance:  $d$  for length,  $P$  for force. Are these independent?  $[d] = \text{m}$ ,  $[P] = \text{N} = \text{kg}\cdot\text{m}\cdot\text{s}^{-2}$ ,

so indeed they are. Are they sufficient to describe the physics of our problem? Let us try:  $[\sigma_{\max}] = \text{Pa} = \text{N/m}^2 = [P/d^2]$ , so the physical law can be rewritten as

$$\frac{\sigma_{\max}}{Pd^{-2}} = f\left(1, \frac{D}{d}, \frac{r}{d}, 1\right) = \bar{f}\left(\frac{D}{d}, \frac{r}{d}\right) \quad (3)$$

From 4 parameters, only 2 are useful! This can be depicted on a single abacus with  $\sigma_{\max}/(Pd^{-2})$  on the vertical axis, and  $D/d$  and  $r/d$  as abscissa and parameter.

Let us look at a classical SCF chart, figure 2 (or see an other version [here](#)). This is not exactly what was expected...  $K_t$  is plotted as a function of  $d/D$  and  $r/t$ . But wait: The SCF is defined as

$$K_t = \sigma_{\max}/\sigma_{\text{nom}} = \sigma_{\max}/(P \times 4d^{-2}/\pi)$$

this is the dimensionless stress in equation (3) with a multiplicative constant  $\pi/4$ ;  $d/D$  is the inverse of  $D/d$  and  $r/t = r/d \times 2/(D/d - 1)$ . Therefore, we indeed get

$$K_t = \bar{g}\left(\frac{d}{D}, \frac{r}{t}\right) = \frac{\pi}{4} \bar{f}\left(\frac{D}{d}, \frac{r}{d}\right)$$

Finally note that  $r$  has a tremendous influence on the maximum stress. Forgetting it at the beginning would have lead to unreliable results. We can also note that with the first dimensionless result, the maximum stress is proportional to the load  $P$  (linear elastic regime only! see genuine assumptions).

## 4 Some examples for practicing dimensional analysis: Train yourself

### 4.1 Bone is a porous media

A [porous media](#) is the mixture of a fluid and a solid: There is a fluid that can migrate *inside* the solid. Bone is a porous media<sup>11</sup> and some consider this phenomena as a way to address the question of [bone remodeling](#). The following example concerns a [mice ulna](#) [9]. A simplified (1D) case leads to the analytical pore pressure evolution [equation](#) [7]:

$$\frac{\partial^2 p}{\partial x^2} = \frac{\tau}{L^2} \dot{p} + \frac{\alpha}{E} \dot{\sigma}$$

where  $p$  is the pore pressure (the pressure of the fluid inside the solid),  $x \in [0, L]$  is the coordinate,  $\tau = L^2/H(1/Q + \alpha^2/E)$  is the characteristic time,  $L = 2\text{cm}$  is the characteristic length,  $\alpha = 0.78$  is a coupling coefficient (no dimension),  $E = 15\text{GPa}$  is Young modulus,  $\sigma$  is the external mechanical load (Pa),  $Q = 15\text{GPa}$  is the so-called compressibility modulus or [Biot](#) modulus, and  $H = 1.1 \times 10^{-13}\text{m}^4/\text{N/s}$  the intrinsic permeability.

The problem is not well conditioned if SI units are used (see the difference of values for  $E$  or  $Q$  and  $H$ ...) hence leading to bad numerical results. The question is therefore: Find a new system of units (a length, a mass, a time)

<sup>11</sup>Not convinced? See [this image](#).

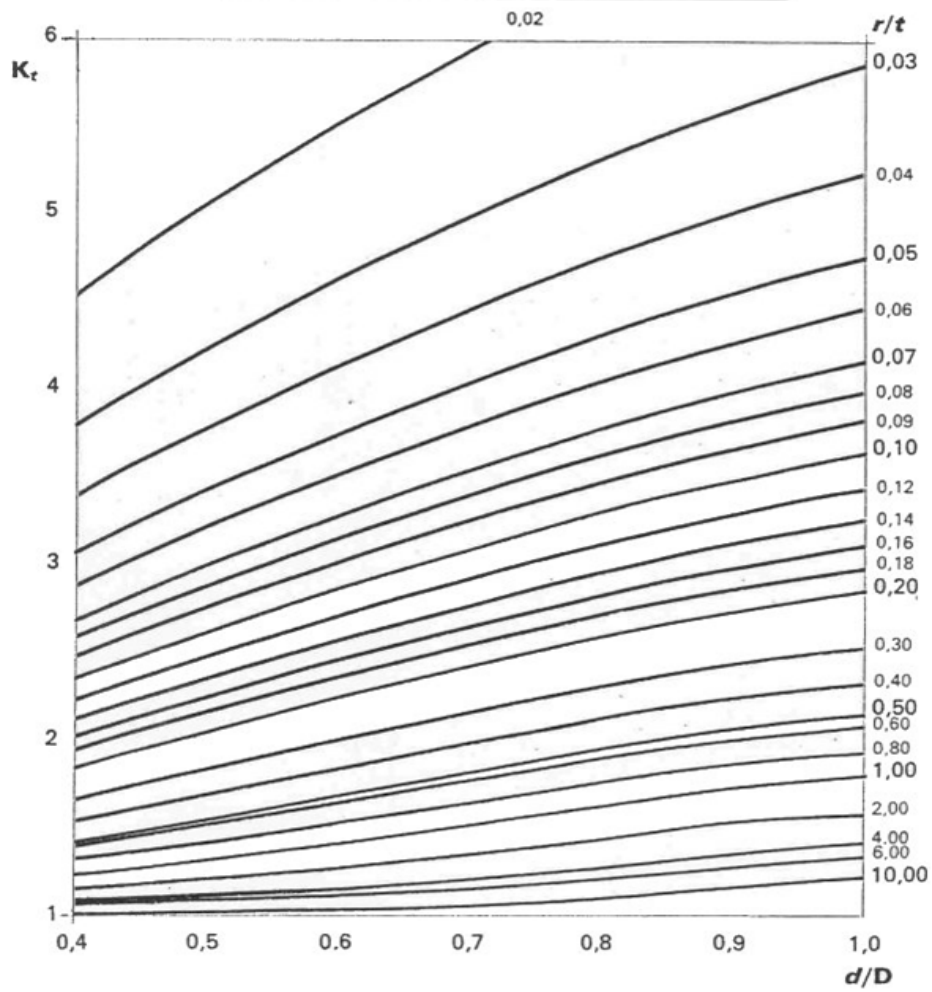
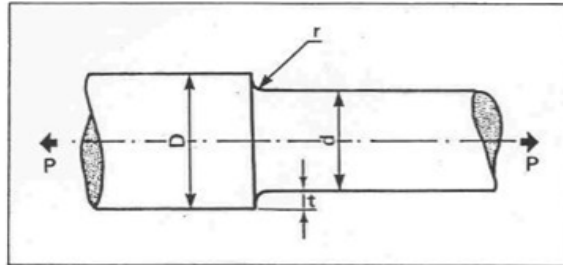


Figure 2: A classical SCF chart, after [4], with permission

such that the following quantities have a value 1: Young modulus  $E$ , intrinsic permeability  $H$ , characteristic time  $\tau$ .

For checking purpose, I found a characteristic time (for the pressure accommodation) of  $\tau \approx 0.39$  s, a length  $\approx 0.025$  m, a mass  $\approx 5.8 \times 10^7$  kg.

## 4.2 Porous ceramics is a porous media (!)

Porous ceramic materials are sometimes used as filter for hot fluids (adding the thermal behavior to the porous behavior). For a simplified (1D) case, the temperature evolution is governed by a (tricky) [advection-diffusion equation](#) [8]:

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{C_d}{k} \dot{\theta} - \frac{\rho_F c_F}{k} H \frac{P_d}{L_d} \frac{\partial \theta}{\partial x} - \frac{1 - 3\alpha_F T_0}{k} H \frac{P_d^2}{L_d^2}$$

where  $\theta$  is the temperature,  $x \in [0, L_d]$  the coordinate. Using  $T_0$  as a reference temperature (dimensionless temperature is  $\theta/T_0$ ),  $L_d$  as a reference length (dimensionless coordinate is  $x/L_d$ ),  $P_d$  as a reference pressure, the dimensionless version of the previous equation is

$$\frac{\partial^2(\theta/T_0)}{\partial(x/L_d)^2} + \mathcal{P}_e \left[ \frac{\partial(\theta/T_0)}{\partial(x/L_d)} - \frac{\partial(\theta/T_0)}{\partial(t/\tilde{\tau}_T)} \right] = -\mathcal{B}_r \quad (4)$$

These models involve many parameters (this is the drawback of multiphysics coupled problems!) that are:

Reference temperature	$T_0 = 293$ K
Characteristic pore pressure	$P_d = 36$ MPa
Characteristic length	$L_d = 1.4$ m
Permeability of the porous media	$H = 2 \times 10^{-10}$ m <sup>3</sup> s/kg
Thermal conductivity of the porous media	$k = 68$ W/m/K
Specific mass of the fluid	$\rho_F = 1\,000$ kg/m <sup>3</sup>
Specific heat of the fluid	$c_F = 4\,182$ J/kg/K
Heat capacity of the porous media	$C_d = 2.1 \times 10^6$ J/K/m <sup>3</sup>
Thermal expansion coefficient of the fluid	$\alpha_F = 2.6 \times 10^{-4}$ K <sup>-1</sup>

$\mathcal{P}_e$  is [Péclet number](#) (it quantifies the order of magnitude of the heat convectively transported by the fluid compared to the heat supplied by diffusion).  $\mathcal{B}_r$  is [Brinkman number](#) (it quantifies the order of magnitude of the heat source due to the viscous dissipation compared to the heat supplied by conduction). These are both dimensionless numbers.

The question is: Give the expression and value of the reference advection time  $\tilde{\tau}_T$  in the simpler dimensionless equation (4).

For checking purpose, I found:  $\mathcal{P}_e = 445$ ,  $\mathcal{B}_r = 37$  and  $\tilde{\tau}_T = 136$  s.

## 4.3 For biomechanics too

Maybe you have seen a TV series named ‘[on n’est pas que des cobayes](#)’ on channel ‘France 5’. During one episode, on march 2012, they wonder who is stronger, man or ant. Let us study the same question, with an additional precision: If they were of the same size, who would be stronger?

First, collect data:

- Up to now, the strongest man on earth may well be [Hossein Reza Zadeh](#) (Iran). He is a weightlifter and holds the world record for clean & jerk, super heavyweight class (over 105 kg, more precisely  $M = 152$  kg for  $H = 1.86$  m) with 263.5 kg lift at the 2004 Olympics Games in Athens (25/08/2004), so he can lift approx. 1.7 times his own weight.
- The strongest insect on earth may well be the [Formica Rufa](#) (red ant) that can lift approx. up to 60 times its own weight for a size of approx.  $h = 9$  mm.

So, who's the strongest?

If we define a performance index  $p$  as being the ratio of the force needed to lift, to the mass,  $p = F/M$ , we can use a crude dimensional analysis to get orders of magnitude. Consider that mass is a function of size  $H$  and mean density  $\rho$ :  $M = f(H, \rho)$ , and that the lift force is a function of the size  $H$  and of the strength limit  $R_p$  for the tendons, muscles or bones (depending which is the first to break),  $F = g(H, R_p)$ .

For the first physical law, use as base units  $H$  and  $\rho$ , for the second one, we may use  $H$  and  $R_p$ . Try these and give an expression for the performance index  $p$ . Now you can answer the question: What happens to the performance index of the human if its size is reduced to that of an ant?

For checking purpose, I found that it is multiplied by a factor of approx. 200. Therefore, if the real Hussein can lift 1.7 times his own weight, the resized one could lift 340 times his own weight.

So, who's the strongest?

As an addendum, I recently read a related [article](#) [5] with the same flavor; maybe you read it too?

## 4.4 Drag

Maybe you know about the  $C_x$  (drag) coefficient? It is related to the aerodynamic resistance to movement. Consider a car with a velocity  $V$ ; can we quantify and compare the resisting horizontal force  $F$  due to the action of the air (without wind) on the car?

The influential parameters are all that define the geometry of the car body (say lengths  $l_i$ ), the velocity  $V$ , the air density  $\rho$ :  $F = f(l_i, V, \rho)$ . Usually, the base units (need for 3) are: the projected surface on the direction of the velocity  $S$  ('maître-couple'), the velocity  $V$  and the density  $\rho$ .

Apply the dimensional analysis to find that

$$F = \frac{1}{2} \rho V^2 S C_x$$

where  $C_x$  is a dimensionless coefficient, called [drag](#) coefficient, function of the adimensional geometry of the vehicle. Different [cars](#) have different drag coefficients. The world record for efficiency seems to be the [PAC-Car II](#) prototype (Shell Eco-marathon in Nogaro, 2005) with  $S C_x = 0.019$  m<sup>2</sup>.

For fuel consumption issues, note also that the needed power at velocity  $V$  is

$$\mathcal{P} = FV = \frac{1}{2} \rho V^3 S C_x$$

so the velocity has a huge influence... Moreover, we may not be really interested in the value of the  $C_x$  coefficient, but rather in the value of  $SC_x$ , so beware of the advertising arguments! To convince yourself, you may [compare](#) the  $C_x$  of

- the [Empire state building](#), with a drag coefficient of 1.3 to 1.5,
- the [Eiffel tower](#), with a larger drag coefficient of 1.8 to 2!

Certainly not the same resistance to wind... consider the values for  $S$ ...

## 5 Using small scale models

When designing (sometimes unique) large-scale structures, one needs to perform experiments. Nevertheless, building a real-scale prototype is often too costly, so the question is to build a model at a reduced size, perform measurements on this model, and deduce what would have been the same measurements on the full-scale case. Changing the scale, and eventually other parameters, can be assessed with dimensional analysis.

### 5.1 Example

Consider for instance that one wishes to design a new passenger ship hull. One is interested in the resistant force  $F$  that needs to be overcome by the propulsion system. The full fluid-structure with a free surface problem is complicated enough to rely on experiments rather than on simulations. Since a passenger ship is often a unique product design with a large cost, as for the [Queen Mary II](#), the prototype is the final product and experiments should be done on cheaper small scale models. This problem is somehow more intricate than the drag force for the car; the influential parameters could now be: the various lengths  $l_i$  that parameterize the hull geometry (including the position of the free surface, the waterline), the gravity  $g$  (that influences the waves generated by the movement), the viscosity  $\eta$  and density  $\rho$  of the fluid, the velocity  $V$ . Therefore, the physical law reads

$$F = f(V, l_i, g, \eta, \rho)$$

The chosen base units could be a particular length  $L$ , velocity  $V$  and density  $\rho$ . Show that the dimensionless drag force is

$$\frac{F}{\rho V^2 L^2} = \bar{f}\left(\frac{l_i}{L}, \frac{g}{L^{-1} V^2}, \frac{\eta}{\rho V L}\right)$$

Since  $\bar{f}$  is an unknown function, the force can be estimated from a measurement on a small scale model if the arguments of the function  $\bar{f}$  are the same for the full scale problem and the model. The equality of the argument values are the similitude conditions. The first of these conditions is to have the same non-dimensionalized geometry lengths, meaning that the scale should be preserved without distortion of the geometry. The second one is the equality of the non-dimensionalized gravity (which is called the [Froude number](#)); since gravity will be the same for the real scale and small scale cases (!), the coefficient  $L^{-1} V^2$  should be the same. The last one is the equality of the non-dimensionalized viscosity (which is called the [Reynolds number](#)).



Show that measuring the force as a function of the velocity on the small scale model (in a towing tank, ‘bassin des carènes’, see for instance [this one](#) at ECN), with subscript  $s$  (i.e.  $F_s = h(V_s)$  is an experimentally plotted function), therefore provides the force as a function of the velocity on the real scale case, with subscript  $r$ :

$$F_r = \frac{\rho_r}{\rho_s} \left( \frac{L_r V_r}{L_s V_s} \right)^2 \times h \left( \sqrt{\frac{L_s}{L_r}} V_r \right)$$

One can notice that equalling all the arguments may not be a trivial task: for instance, using the same fluid would lead to have the coefficient  $LV$  equal, but Froude said that  $L^{-1}V^2$  should also be equal, which is not possible (unless scale 1 is used...) Considering other influential parameters, such as surface tension (negligible here!), would have lead to an other dimensionless condition with an non-dimensional number (the Weber [number](#)) and would have complicated the similitude conditions.

## 5.2 Large scale model as well...

The same tool can be used in the reverse direction: If a real-scale prototype is too small to be built at the early design stage (micro-mechanisms that would require specific manufacturing tools not available at that time of the design process), an enlarged model can be manufactured more easily, and experiments can be conducted on it. Going back to the real-scale case is the same problem as before.

## References

- [1] G. Berry. Manifeste pour la réhabilitation du Pavillon des Poids et Mesures. *Viridis Candela, Correspondancier du Collège de ‘Pataphysique*, 1:33–60, Novembre 2007. Accessed: 2015-04-28. URL: <http://www-sop.inria.fr/members/Gerard.Berry/Pataphysique/BerryPoidsEtMesures.pdf>.
- [2] M. Debacq. Unités, dimensions et nombres adimensionnels. Lecture, March 2008. URL: <https://hal.archives-ouvertes.fr/cel-02020110>.
- [3] M. Dursapt. *Appelez-moi mètre*. Book-e-book, 2022.
- [4] J.-P. Faurie, P. Monnier, and A. Niku-Lari. *Guide du dessinateur, les concentrations de contraintes*. CETIM, 1977.
- [5] R. Lehoucq and J.-S. Steyer. Superman est-il une super-fourmi ? *Pour la Science*, 443:82–83, Septembre 2014.
- [6] J. Lemaitre, P.-A. Boucard, and F. Hild. *Résistance mécanique des solides. Matériaux et structures*. Sciences Sup. Dunod, 2007. EAN13 : 9782100507122. URL: <http://www.dunod.com/sciences-techniques/sciences-fondamentales/physique-et-astrophysique/bts-iut/resistance-mecanique-des-solides>.
- [7] E. Malachanne, F. Jourdan, and D. Dureisseix. Numerical model of bone remodeling sensitive to loading frequency through a poroelastic behavior and

- internal fluid movements. *Journal of the Mechanical Behavior of Biomedical Materials*, 4:849–857, 2011. URL: <http://hal.archives-ouvertes.fr/hal-00595986/en/>, doi:10.1016/j.jmbbm.2011.03.004.
- [8] D. Néron and D. Dureisseix. A computational strategy for thermo-poroelastic structures with a time-space interface coupling. *International Journal for Numerical Methods in Engineering*, 75(9):1053–1084, 2008. URL: <http://hal.archives-ouvertes.fr/hal-00325946/fr/>, doi:10.1002/nme.2283.
- [9] A. G. Robling, F. M. Hinant, D. B. Burr, and C. H. Turner. Improved bone structure and strength after long-term mechanical loading is greatest if loading is separated into short bouts. *Journal of Bone and Mineral Research*, 17(8):1545–1554, 2002. doi:10.1359/jbmr.2002.17.8.1545.
- [10] A. A. Sonin. The physical basis of dimensional analysis. 2nd Edition, Department of Mechanical Engineering, MIT, Cambridge, 2001. URL: <http://goo.gl/2BaQM6>.



velocity ( $\times 10$ m/s)	error (%)
1.0	10.1
2.1	15.0
4.2	18.1
6.1	22.5

velocity / (10 m/s)	error / %
1.0	10.1
2.1	15.0
4.2	18.1
6.1	22.5

Table 2: Two table presentations (the one on the right is standardized and less ambiguous)

## B.2 Physical law valid for special units only

You may sometimes find some expressions that are only valid for a certain choice of units. This does not mean that they are not a physical law, but only that, for practical purpose, they are tuned for using special units. Though easy to use, they nevertheless require some practice to avoid mistakes. On the other hand; the true expression conforming to a general physical law may be more cumbersome and require some coefficients with their own units.

For instance, a lathe (a machine tool) requires to tune the rotation rate  $N$  to obtain a good cutting velocity  $V$  depending on the material of the tool and on the part to be cut, to create a good cylinder of diameter  $d$ . The practical expression is  $N = 1000 \times V/(\pi d)$ , valid only if  $V$  is given in  $\text{m}\cdot\text{s}^{-1}$ ,  $N$  in  $\text{tr}\cdot\text{s}^{-1}$  and  $d$  in mm. The correct corresponding relation with our development is  $N/(\text{tr/s}) = 1000 (V/(\text{m/s})) / (\pi(d/\text{mm}))$ , or  $N = 1000 (\text{mm} \times \text{tr/s}) / (\pi \text{m/s}) V/d = V/(d/2)$  that does not involve any units; therefore this is the true physical law.

## B.3 Not-so-clear abbreviations for units

Some compound units may be a combination of many base unit. When the units are multiplied and divided in various arrangements, using exponents may clarify the case. For instance, a specific heat capacity is measured with an energy per unit of mass and per unit of temperature: The notation  $\text{J}\cdot\text{kg}^{-1}\cdot\text{K}^{-1}$  is less ambiguous than, and therefore preferred to,  $\text{J}/\text{kg}\cdot\text{K}$ , or  $\text{J}/(\text{kg}\cdot\text{K})$ , or  $\text{J}/\text{kg}/\text{K}$ .

Concerning the mechanical torque, the case is delicate. Indeed a torque is a force (for which a SI unit is the **newton**, abbreviated as N) multiplied with a length. Therefore its unit is  $\text{N}\cdot\text{m}$ . A first remark is: Avoid permutation of these two units, since  $\text{m}\cdot\text{N}$  could be easily read as ‘milli newton’ (mN)! The second remark is that a force times a distance is also a mechanical work, for which the unit is the joule (abbreviated as J) <sup>12</sup>.

Formally, a torque and a work are homogeneous, but advice is: keep the symbol  $\text{N}\cdot\text{m}$  for the torque, and **J** for the work to keep in mind the nature of the physical quantities that are manipulated.

## C The temperature case

Care must be taken when dealing with temperatures. Indeed, the absolute temperature should be measured in **kelvin (K)**<sup>13</sup>. Other units, such as **degree**

<sup>12</sup>Thanks to Aurélien Pommel, student in mechanical engineering at INSA Lyon for debugging this...

<sup>13</sup>Note that no ° symbol is used with K.

[Celsius \(°C\)](#) and [degree Fahrenheit \(°F\)](#) are used only for relative temperatures (i.e. temperature differences). If one tries to use these for an absolute temperature, the multiplication rule between value and unit does not hold anymore: a ‘formula’ for the conversion would be  $^{\circ}\text{C} = (5/9) \times (^{\circ}\text{F} - 32) = \text{K} - 273.15$ . On the other hand, for a temperature difference, one gets:  $^{\circ}\text{C} = (5/9) ^{\circ}\text{F} = \text{K}$  which is consistent.

## D For $\text{\LaTeX}$ or pdf $\text{\LaTeX}$ fans

Consider using the recent [siunitx](#) package. It has been used to typeset this document.