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## Approximation by PDEs

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# Approximation by PDEs

Master MA 2019

J. Vovelle

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
1.1	Discrete conservation laws	2
1.2	The symmetric simple exclusion process	3
1.3	Interacting particle systems	4
<b>2</b>	<b>Martingales in continuous time</b>	<b>5</b>
2.1	Conditional expectation	5
2.2	Martingales	6
2.3	A digression on the Calderón-Zygmund decomposition	6
<b>3</b>	<b>Markov processes</b>	<b>7</b>
3.1	Definition	7
3.2	Semi-group property, Chapman-Kolmogorov property and Markov property	9
3.3	Invariant measure	10
3.4	Infinitesimal generator	11
3.5	Martingale property of Markov processes	12
<b>4</b>	<b>Evolution of a random interface</b>	<b>16</b>
4.1	Change of scale and limit behavior	16
4.2	Markov property	18
4.2.1	From discrete-time to continuous-time Markov process	18
4.2.2	Markov property for the symmetric simple exclusion process	20
4.3	Deterministic limit	22
4.4	Averaging	25
<b>5</b>	<b>Conservation laws and the Finite Volume method</b>	<b>28</b>
5.1	Discrete conservation laws, continuous limit	28
5.2	Discrete fluxes	29
5.3	Discrete fluxes for linear equations	30
5.4	General monotone fluxes	30
5.5	Constants as solutions	31
5.6	Comparison principle	32
5.6.1	Periodic discrete conservation law	32
5.6.2	Comparison principle and consequences	32
5.6.3	Asymptotic behavior	35
5.7	Energy estimate	35

5.8	Approximate weak solutions . . . . .	38
5.9	Convergence in the linear case . . . . .	42
5.10	Error estimate in the linear case . . . . .	43
	5.10.1 Functions of bounded variations . . . . .	43
	5.10.2 Comments on the error estimate . . . . .	47
5.11	Error estimate in the linear case: proof . . . . .	48
	5.11.1 Reduction of the problem . . . . .	48
	5.11.2 Error estimate . . . . .	52
<b>6</b>	<b>Interacting particle systems</b> . . . . .	<b>55</b>
6.1	Empirical measure . . . . .	55
	6.1.1 An alternative description of the system . . . . .	55
	6.1.2 Empirical measure . . . . .	56
6.2	Equilibria . . . . .	57
	6.2.1 Generator . . . . .	57
	6.2.2 Invariant measure . . . . .	58
	6.2.3 Simple exclusion process . . . . .	59
	6.2.4 Zero range processes . . . . .	60
	6.2.5 Weak local equilibrium . . . . .	62
6.3	Some hydrodynamic limits . . . . .	63
<b>7</b>	<b>Solution to the exercises</b> . . . . .	<b>66</b>

### Abstract

In this course, we will see how to understand and describe the large scale limit of various discrete evolution systems (random and deterministic) with the help of partial differential equations. This will be the occasion to use, and discover, some standard tools from the theory of PDEs, of numerical analysis, and of statistical physics.

## 1 Introduction

### 1.1 Discrete conservation laws

Suppose that we are given a family  $\mathcal{T}$  of open polyhedral sets forming a partition of the space  $\mathbb{R}^d$ : for all distinct  $K, L \in \mathcal{T}$ , we assume that  $K \cap L = \emptyset$  and that  $\overline{K} \cap \overline{L}$  is contained in an hyperplane of  $\mathbb{R}^d$ . The partition is understood up to a negligible set: the Lebesgue measure of  $\mathbb{R}^d \setminus \bigcup_{K \in \mathcal{T}} K$  is zero. The picture 1 below gives the example of a triangulation of the plane.

We consider the following evolution of an extensive quantity  $u$ : let  $0 = t_0 < t_1 < \dots < t_n < \dots$  be some discrete times, let  $U_K^n$  denote the amount of the quantity  $u$  in the cell  $K$  at time  $t_n$ . We assume that  $U_K^{n+1}$  is given by the formula

$$U_K^{n+1} = U_K^n + \Delta t_n \sum_{L \in \mathcal{N}(K)} |K|L| Q_{L \rightarrow K}^n. \quad (1.1)$$

The notations used in (1.1) are the following ones:  $\Delta t_n$  is the length  $t_{n+1} - t_n$  of the time interval,  $\mathcal{N}(K)$  is the set of neighbors of  $K$ :  $L \in \mathcal{T}$  is a neighbor of  $K$  if  $K|L := \overline{K} \cap \overline{L}$  is non-empty and of finite  $(N - 1)$ -dimensional Hausdorff measure  $|K|L|$  (in particular,  $K$  is not a neighbor of  $K$ ). The quantity  $\Delta t_n |K|L| Q_{L \rightarrow K}^n$  represents a certain flux of the quantity  $u$  that has passed through the interface  $K|L$  from the cell  $L$  to the cell  $K$  between the times  $t_n$  and  $t_{n+1}$ . We have put in factor the term  $\Delta t_n |K|L|$  because we prefer to work with densities, rather than with

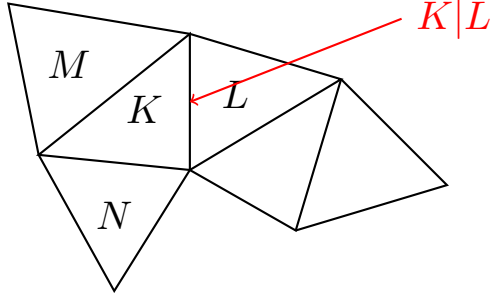


Figure 1: A mesh in  $\mathbb{R}^2$

scale-dependent quantities (the typical scales here depend on the size of the cells and of  $\Delta t_n$  and will tend to zero at some point later on). For the same reason, it is more appropriate to introduce  $|K|$ , the Lebesgue measure of the cell  $K$ , and to work with the scaled quantity  $u_K^n = U_K^n/|K|$ , which satisfies the equation

$$u_K^{n+1} = u_K^n + \frac{\Delta t_n}{|K|} \sum_{L \in \mathcal{N}(K)} |K|L|Q_{L \rightarrow K}^n. \quad (1.2)$$

Assume that the densities of flux  $Q_{L \rightarrow K}^n$  satisfy the following condition:

$$Q_{L \rightarrow K}^n = -Q_{K \rightarrow L}^n, \quad (1.3)$$

for all  $n \in \mathbb{N}$ , for all  $K, L \in \mathcal{T}$  being neighbors. The condition (1.3) ensures that the (algebraic) quantity of  $u$  that was given by the cell  $K$  to the cell  $L$  is the quantity of  $u$  received by the cell  $L$  from the cell  $K$ . Under (1.3), the evolution given by (1.2) is conservative: we will show in particular that, when it makes sense, the quantity

$$\sum_{K \in \mathcal{T}} |K|u_K^n$$

is constant with respect to  $n$ . Our objective will be to explain what is the limit of  $(u_K^n)$  when  $\Delta t_n$  and  $|K|$  tends to 0. We need to be more specific on our framework to achieve this goal. Let us simply say for the moment that what we will obtain in the end are some conservation laws

$$\partial_t u + \operatorname{div}_x(Q) = 0, \quad (1.4)$$

where  $Q(x)$  is a function of  $x$ ,  $u(x)$  and  $\nabla u(x)$ . The derivation of (1.4) is related to the analysis of the Finite Volume method, which is used to compute the solution of conservation laws such as (1.4) with the help of the discrete formulation (1.2).

## 1.2 The symmetric simple exclusion process

Let  $0 < N < L$  be some integers. Consider  $N$  particles located at one of the site  $1, \dots, L-1$  that evolve according to the following process: there is always one particle at site 0 and, for each site  $x \in \{1, \dots, L-1\}$ , we draw a random time  $T_x$  that follows an exponential law of parameter  $\lambda > 0$ , so that the family  $\{T_y\}$  is independent. Consider the point  $x_*$  at which  $x \mapsto T_x$  is minimal and let the particle at  $x_*$  jump from its original site  $x$  to a new site  $y$  with probability  $p(x_*, y)$ , the jump occurring under the restriction that the arrival site  $y$  is vacant. Then start over. This

process is called an exclusion process for the reason that jumps to occupied sites are excluded. It is termed *simple* to make the distinction with some more complicated situations, where the probability of a jump from  $\mathbf{x}$  to  $\mathbf{y}$  may depend not only on  $\mathbf{x}$  and  $\mathbf{y}$ , but on the whole interval  $[\mathbf{x}, \mathbf{y}]$  and on the disposition of particles in this interval. We also call the process *symmetric* when  $p(\mathbf{x}, \mathbf{x} + l) = p(\mathbf{x}, \mathbf{x} - l)$ , whenever the quantities are well defined. Here we will consider the case  $p(\mathbf{x}, \mathbf{y}) = 0$  if  $|\mathbf{x} - \mathbf{y}| \neq 1$ , so that only jumps to left or right immediate neighboring site are possible, and equi-probable. At the boundary, we assume  $p(0, 1) = 0$ ,  $p(L - 1, L - 2) = 1$ . We can put in correspondence this evolution of particles with the evolution of a random interface described as follows: we set  $H(0) = 0$  and, for  $\mathbf{x} \in \{1, \dots, L\}$ , define  $H$  as the discrete primitive function

$$H(\mathbf{x}) = \sum_{y=0}^{\mathbf{x}-1} (2\eta(y) - 1), \quad (1.5)$$

where  $\eta(y) \in \{0, 1\}$  is the number of particle at  $y$ . Then we interpolate linearly between those points. Conversely, we deduce  $\eta(\mathbf{x})$  from  $H$  by the “differentiation” formula  $\eta(\mathbf{x}) = [1 + H(\mathbf{x} + 1) - H(\mathbf{x})]/2$ . In the situation where the site  $\mathbf{x}$  is occupied and the site  $\mathbf{x} + 1$  is vacant, the shape (above  $\{\mathbf{x}, \mathbf{x} + 1, \mathbf{x} + 2\}$ ) of the function  $H$  is  $\wedge$ . If the particle at  $\mathbf{x}$  jumps at  $\mathbf{x} + 1$ , it becomes  $\vee$ , – and conversely. We consider then the following problem: assume that  $L$  and  $N$  are very large. For definiteness, we will take  $L = 2N$ , which ensures that  $H(L) = 0$ . Consider the change of scale

$$h_t^L(x) = L^{-1}H_t(Lx), \quad x \in (0, 1). \quad (1.6)$$

What can we say about the evolution of the profile  $t \mapsto h_t^L$ , for, possibly,  $t$  very large? We will see that, under adequate conditions on the initial data, and after the following parabolic change of time scale:

$$h_t^L(x) = h_{L^2t}^L = L^{-1}H_{L^2t}(Lx), \quad x \in (0, 1), t > 0, \quad (1.7)$$

we have a kind of law of large numbers: for all final time  $T > 0$ ,  $h^L$  is converging in probability in  $L^\infty(0, T; L^2(0, 1))$  to a deterministic profile  $h$  which is completely determined as a solution of the heat equation with homogeneous Dirichlet boundary conditions.

### 1.3 Interacting particle systems

We will now consider a problem similar to the previous one, with the difference that it is multi-dimensional and that jumps to occupied sites are not excluded. Let  $\Lambda_N$  be a finite subset of  $\mathbb{Z}^d$ . We consider a system of particles scattered on  $\Lambda_N$ , which interact as follows: let  $\mathbf{x}$  denote a typical site of  $\Lambda_N$  and let  $\eta_t(\mathbf{x})$  denote the number of particles located at site  $\mathbf{x}$  at time  $t$ . We will be interested in the evolution in time of the functions  $\mathbf{x} \mapsto \eta_t(\mathbf{x})$ . The state space is therefore  $\mathcal{E}_N := \mathbb{N}^{\Lambda_N}$ , the set of functions  $\Lambda_N \rightarrow \mathbb{N}$ . The evolution is described by the following algorithm: each site  $\mathbf{x}$  has its own clock that is independent from the clocks at other sites, and that rings after a time  $T_{\mathbf{x}}$  which is a random variable of exponential law of parameter  $\lambda(\eta(\mathbf{x}))$ . Assume that it is at the site  $\mathbf{x}_*$  that a clock is ringing first. If  $\eta(\mathbf{x}_*) > 0$ , then one particle of the site  $\mathbf{x}_*$  jumps to an other site  $\mathbf{y}$  chosen at random in  $\Lambda_N$ , according to a transition probability  $p(\mathbf{x}_*, \mathbf{y})$  (possibly, at that stage, some exclusion rules may be added, see Section 6.2.1). Then we start over. Let us consider the case where  $\Lambda_N$  is the discrete torus  $\mathbb{T}_N^d = \mathbb{Z}^d/N\mathbb{Z}^d$  and  $p$  is compatible and translation invariant: for all  $l \in N\mathbb{Z}^d$ ,  $m \in \mathbb{Z}^d$ ,

$$p(\mathbf{x} + l, \mathbf{y}) = p(\mathbf{x}, \mathbf{y}), \quad p(\mathbf{x} + m, \mathbf{y} + m) = p(\mathbf{x}, \mathbf{y}). \quad (1.8)$$

Let us zoom out (cf. (1.6)) by considering the function

$$[0, 1]^d \ni x \mapsto N^{-1}\eta_t([Nx]) \quad (1.9)$$

extended by periodicity. In (1.9),  $[Nx]$  is the element  $\mathbf{x}$  of  $\mathbb{T}_N^d \simeq \{0, \dots, N-1\}^d$  such that  $\mathbf{x}_i \leq Nx_i < \mathbf{x}_i + 1$  for all  $i = 1, \dots, d$ . May it be the case that, possibly after a change of time scale (cf. (1.7)), some averaging phenomena would lead to a given deterministic behavior? We will see that the question has to be refined, before being answered positively (at least in certain cases).

## 2 Martingales in continuous time

### 2.1 Conditional expectation

**Proposition 2.1** (Conditional expectancy). *Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space and let  $\mathcal{G} \subset \mathcal{F}$  be a sub- $\sigma$ -algebra of  $\mathcal{F}$ . Let  $X$  be real-valued random variable which is integrable:  $X \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ . Then there exists a unique  $\mathcal{G}$ -measurable and integrable random variable  $Z$  such that*

$$\mathbb{E}(\mathbf{1}_A X) = \mathbb{E}(\mathbf{1}_A Z), \quad \forall A \in \mathcal{G}. \quad (2.1)$$

We call  $Z$  the conditional expectancy of  $X$  knowing  $\mathcal{G}$ , denoted  $\mathbb{E}(X|\mathcal{G})$ .

Roughly speaking,  $\mathbb{E}(X|\mathcal{G})$  is the average of  $X$  with respect to all the events not relative to  $\mathcal{G}$ . The following facts or examples illustrate this fact.

**Fact 1.** If  $\mathcal{G} = \mathcal{F}$ , then  $\mathbb{E}(X|\mathcal{F}) = X$  a.s. If  $\mathcal{G}$  is the trivial  $\sigma$ -algebra  $\{\emptyset, \Omega\}$ , then  $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X)$ .

**Example 1.** When  $\mathcal{G}$  is the  $\sigma$ -algebra generated by an event  $A \in \mathcal{F}$ ,  $\mathcal{G} = \{\emptyset, A, A^c, \Omega\}$ , then

$$\mathbb{E}(X|\mathcal{G}) = \frac{\mathbb{E}(\mathbf{1}_A X)}{\mathbb{P}(A)} \mathbf{1}_A + \frac{\mathbb{E}(\mathbf{1}_{A^c} X)}{\mathbb{P}(A^c)} \mathbf{1}_{A^c}.$$

If  $X = \mathbf{1}_B$  where  $B \in \mathcal{F}$ , this gives  $\mathbb{E}(\mathbf{1}_B|\mathcal{G}) = \mathbb{P}(B|A)\mathbf{1}_A + \mathbb{P}(B|A^c)\mathbf{1}_{A^c}$ .

**Fact 2.** One has the following tower property: if  $\mathcal{H}$  is a sub- $\sigma$ -algebra of  $\mathcal{G}$ , then

$$\mathbb{E}(\mathbb{E}(X|\mathcal{G})|\mathcal{H}) = \mathbb{E}(X|\mathcal{H}) \text{ a.s.} \quad (2.2)$$

As a particular case, when  $\mathcal{H} = \{\emptyset, \Omega\}$ , we obtain  $\mathbb{E}[\mathbb{E}(X|\mathcal{G})] = \mathbb{E}[X]$ .

**Example 2.** Let  $X, Y$  be two independent random variable and let  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$  be a bounded Borel function. Then  $Z = \mathbb{E}(f(X, Y)|\sigma(Y))$  is  $\sigma(Y)$ -measurable, and it is known that such a function can be written  $h(Y)$ , where  $h$  is Borel. In general, when saying that a  $\sigma(Y)$ -measurable function has the form  $h(Y)$ , we have no particular information on  $h$ . Here, however, we know very well what is  $h$ : it is the function obtained by averaging with respect to “all that is not  $Y$ ”, *i.e.*

$$\mathbb{E}(f(X, Y)|\sigma(Y)) = h(Y), \quad h(y) := \mathbb{E}(f(X, y)). \quad (2.3)$$

**Example 4.** Let  $D$  denote the set of dyadic cubes in  $[0, 1]^d$ , and for  $n \in \mathbb{N}$ , let  $D_n$  denote the subset of dyadic cubes of length  $2^{-n}$ : all cubes in  $D_n$  are translation by an element of  $2^{-n}\mathbb{Z}^d$  of the basic cube  $[0, 2^{-n})^d$ . Let  $f: [0, 1]^d \rightarrow \mathbb{R}$  be integrable. The piecewise-constant function  $f_n$  equal to the averaged value of  $f$  over each cube  $Q$  in  $D_n$  can be seen as the conditional expectancy  $\mathbb{E}(f|\mathcal{F}_n)$  by taking  $\Omega = [0, 1]^d$ ,  $\mathbb{P}$  being the Lebesgue measure,  $\mathcal{F}$  the Borel  $\sigma$ -algebra, and  $\mathcal{F}_n$

being the  $\sigma$ -algebra generated by all the cubes in  $D_n$  (verification left as an exercise). There is a consistency property in this approximation process, which is the following one: for all  $m < n$ , averaging the finer approximation  $f_n$  over the coarser grid corresponding to  $D_m$  gives  $f_m$ :

$$\mathbb{E}(f_n | \mathcal{F}_m) = f_m \text{ a.s.} \quad (2.4)$$

The property (2.4) follows from the tower property (2.2) for example. It is an instance of a martingale property.

## 2.2 Martingales

**Definition 2.1** (Filtration). Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space. A family  $(\mathcal{F}_t)_{t \geq 0}$  of sub- $\sigma$ -algebras of  $\mathcal{F}$  is said to be a *filtration* if the family is increasing with respect to  $t$ :  $\mathcal{F}_s \subset \mathcal{F}_t$  for all  $0 \leq s \leq t$ . The space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  is called a filtered space.

**Definition 2.2** (Adapted process). Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  be a filtered space. A real-valued process  $(X_t)_{t \geq 0}$  is said to be *adapted* if, for all  $t \geq 0$ ,  $X_t$  is  $\mathcal{F}_t$ -measurable.

**Definition 2.3** (Martingale). Let  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$  be a filtered space. Let  $(X_t)_{t \geq 0}$  be an adapted real-valued process such that, for all  $t \geq 0$ ,  $X_t \in L^1(\Omega)$ . The process  $(X_t)_{t \geq 0}$  is said to be a *martingale* if, for all  $0 \leq s \leq t$ ,  $X_s = \mathbb{E}(X_t | \mathcal{F}_s)$ .

*Remark 2.1.* A martingale with continuous (*resp.*, càdlàg) trajectories is said to be a continuous (*resp.*, càdlàg) martingale.

*Remark 2.2.* With respect to a fixed time  $t > 0$ , conditioning on  $\mathcal{F}_s$  with  $s \leq t$  is a way to average  $X_t$  over all events which occurred between times  $s$  and  $t$ . For a martingale  $X$ , this will let the position  $X_s$  unchanged. We expect a martingale not to wander too much therefore. We will see and use several instance of this general principle. See Section 2.3 for a first example.

**Theorem 2.2** (Doob's martingale inequality). *Let  $p > 1$ . Let  $(M_t)_{t \in [0, T]}$  be a càdlàg, real-valued martingale, such that  $\mathbb{E}|M_T|^p < +\infty$ . Then the inequality*

$$\mathbb{E}[(M_T^*)^p] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}|M_T|^p, \quad M_T^* = \sup_{t \in [0, T]} |M_t|, \quad (2.5)$$

*is satisfied.*

## 2.3 A digression on the Calderón-Zygmund decomposition

Let  $f: [0, 1]^d \rightarrow \mathbb{R}$  be a non-negative, integrable function. Let  $\lambda > 0$  be a fixed threshold such that the integral of  $f$  over  $[0, 1]^d$  is smaller than  $\lambda/2$ . In terms of Example 3. in Section 2.1, this means  $\mathbb{E}[f] \leq \lambda/2$ . Consider (see Remark 2.2) that being below  $\lambda$  is “not wandering too much”, while being above  $\lambda$  is “wandering too much”. What is the behavior of the martingale  $(f_n)$  defined in the Example 3. in Section 2.1? Let  $T$  be the stopping time  $T = \inf\{n \geq 0; f_n > \lambda\}$ . We know that  $T > 0$  almost surely. If  $T = +\infty$ , then  $f_n \leq \lambda$  for all  $n$ , and thus  $f = \lim f_n \leq \lambda$ . Here we use the intuitive fact that  $f = \lim f_n$ . We have to specify the mode of convergence however and to justify the convergence. The convergence is almost sure. One can use the martingale convergence theorem for example (probabilistic approach) or the dyadic version of the Lebesgue differentiation theorem (analyst's approach). The set  $\{T < +\infty\}$  can be written as an at most countable collection  $(Q_i)_{i \in I}$  of dyadic cubes. Indeed, it is the union over  $n \geq 1$  of the sets  $\{T = n\}$ , and  $\{T = n\}$  is a union of dyadic cubes in  $D_n$ . If  $Q$  is one of the cubes that enter in

the decomposition of  $\{T = n\}$ , and if  $Q' \in D_{n-1}$  is the twice bigger cube containing  $Q$ , then the averaged value of  $f$  on  $Q'$  is smaller than  $\lambda$  (otherwise  $T < n$ ). It follows that

$$\lambda \leq \frac{1}{|Q|} \int_Q f(x) dx \leq \frac{1}{|Q|} \int_{Q'} f(x) dx = \frac{2^d}{|Q'|} \int_{Q'} f(x) dx \leq 2^d \lambda. \quad (2.6)$$

From these considerations on martingales, we can deduce the following statement.

**Lemma 2.3** (Calderón-Zygmund). *Let  $f: \mathbb{R}^d \rightarrow \mathbb{R}$  be a non-negative, integrable function. Let  $\lambda > 0$ . There exists an at most countable family  $(Q_i)_{i \in I}$  of dyadic cubes such that*

$$\forall i \in I, \quad \lambda \leq \frac{1}{|Q_i|} \int_{Q_i} f(x) dx \leq 2^d \lambda, \quad (2.7)$$

and  $f \leq \lambda$  a.e. on the complementary set  $\mathbb{R}^d \setminus \cup_{i \in I} Q_i$ .

*Proof of Lemma 2.3.* fix  $N$  large enough such that  $2^{-Nd} \|f\|_{L^1(\mathbb{R}^d)} \leq \lambda/2$ . Consider the countable decomposition of  $\mathbb{R}^d$  by all the dyadic cubes of size  $2^N$ . On each such cube  $R$ , we apply the analysis performed before the statement of the lemma. This analysis was done with the starting cube  $R = [0, 1)^d$ , but can be readily adapted to the general case. The final family of cube  $(Q_i)_{i \in I}$  is then the union of the families obtained on each such cube  $R$ .  $\square$

The Calderón-Zygmund lemma is applied to obtain a decomposition of  $f = g + b$ , where

$$g = \sum_{i \in I} \left[ \frac{1}{|Q_i|} \int_{Q_i} f(x) dx \right] \mathbf{1}_{Q_i} + f \mathbf{1}_{\mathbb{R}^d \setminus \cup_{i \in I} Q_i}$$

and

$$b = f - g = \sum_{i \in I} b_i, \quad b_i = \left[ f - \frac{1}{|Q_i|} \int_{Q_i} f(x) dx \right] \mathbf{1}_{Q_i}.$$

The function  $g$  is considered as the good part, since it is controlled in size by  $\lambda$ ; the function  $b$  is considered as the “bad” part. It is not controlled in size but has the properties that  $b_i$  is supported in  $Q_i$  and has zero integral. The Calderón-Zygmund is fundamental in harmonic analysis. Note that, if we come back again to the probabilistic approach (and restrict things to  $[0, 1)^d$ ), then  $g$  is simply  $f_T$ , while  $b = f - f_T$ .

### 3 Markov processes

We consider Markov processes taking values in a Polish space  $E$ . This is a framework that is quite general, all the more since we shall apply it in the case of a finite state space  $E$ . The generality of the approach used here may be useful elsewhere however.

#### 3.1 Definition

If  $E$  is a Polish space, we denote by  $\text{BM}(E)$  the Banach space of bounded Borel-measurable functions on  $E$  with the sup-norm

$$\|\varphi\|_{\text{BM}(E)} = \sup_{x \in E} |\varphi(x)|.$$

The set  $\text{BC}(E)$  is the subspace of continuous bounded functions. By Markov process, we mean the triplet constituted of a Markov semi-group, some probability kernels, and the associated



Markov processes. More precisely, we suppose first that we are given a Markov semi-group  $\mathbf{P} = (P_t)_{t \geq 0}$ , which is defined a priori as a family of endomorphisms of the space  $\text{BM}(E)$  that satisfy the initial condition  $P_0 = \text{Id}$ , the semi-group property  $P_t \circ P_s = P_{t+s}$  for  $t, s \geq 0$ , the preservation of positivity  $P_t \varphi \geq 0$  when  $\varphi \geq 0$ , while fixing the constant function  $\mathbf{1}$  equal to 1 everywhere:  $P_t \mathbf{1} = \mathbf{1}$  for all  $t \geq 0$ . The second element of our set of data is a probability kernel  $Q(t, x, B)$ : for all  $\varphi \in \text{BM}(E)$ , for all  $x \in E$ ,

$$P_t \varphi(x) = \int_E \varphi(y) Q(t, x, dy), \quad (3.1)$$

where, for every  $t \geq 0$ , for every  $x \in E$ ,  $Q(t, x, \cdot)$  is a probability measure and the dependence in  $x$  is measurable, in the sense that the right-hand side of (3.1) is a measurable function of  $x$ . The third and last element of our set of data is the set  $\mathbf{X} = \{(X_t^x)_{t \geq 0}; x \in E\}$  of Markov processes indexed by their starting points  $x$ :  $X_0^x = x$  almost surely. The finite-dimensional distributions of  $(X_t^x)_{t \geq 0}$  are given by

$$\begin{aligned} \mathbb{P}(X_0^x \in B_0, X_{t_1}^x \in B_1, \dots, X_{t_k}^x \in B_k) &= \int_{B_0} \cdots \int_{B_{k-1}} Q(t_k - t_{k-1}, y_{k-1}, B_k) \\ &\times Q(t_{k-1} - t_{k-2}, y_{k-2}, dy_{k-1}) \cdots Q(t_1, y_0, dy_1) \delta_x(dy_0), \end{aligned} \quad (3.2)$$

where  $0 \leq t_1 \leq \dots \leq t_k$  and  $B_0, \dots, B_k \in \mathcal{G}$  and the Markov property

$$\mathbb{E}[\varphi(X_{t+s}^x) | \mathcal{F}_t^X] = P_s \varphi(X_t^x) \quad (3.3)$$

is satisfied for all  $s, t \geq 0$ ,  $\varphi \in \text{BM}(E)$ , where  $(\mathcal{F}_t^X) = (\sigma(X^x(s)_{0 \leq s \leq t}))$  is the filtration generated by  $X^x$  (see the proof of (3.3) in Section 3.2 below). There is a certain redundancy in the description above since the existence of probability kernels satisfying (3.1) can be deduced from the properties of  $(P_t)_{t \geq 0}$ , [1, Proposition 1.2.3], while, starting from a semi-group and a family of probability kernels related by (3.1), the construction of a Markov process with the required finite dimensional distribution is established in [6, Theorem 1.1 p.157] for example. It is not limiting, however, to assume that all these elements are given altogether, all the more since the processes  $(X_t^x)_{t \geq 0}$  will generally have additional pathwise properties, being typically continuous or càdlàg. They may also satisfy the Markov property (3.3) with respect to a given filtration  $(\mathcal{F}_t)$  larger than  $(\mathcal{F}_t^X)$ . See Section 3.2 below.

*Remark 3.1* (From the probability kernels to the Markov process). Actually, it is established in [6, Theorem 1.1 p.157] that, for every Borel probability measure  $\mu$  on  $E$ , there exists a Markov process  $(X_t)_{t \geq 0}$  with finite-dimensional distributions given by

$$\begin{aligned} \mathbb{P}(X_0^x \in B_0, X_{t_1}^x \in B_1, \dots, X_{t_k}^x \in B_k) &= \int_{B_0} \cdots \int_{B_{k-1}} Q(t_k - t_{k-1}, y_{k-1}, B_k) \\ &\times Q(t_{k-1} - t_{k-2}, y_{k-2}, dy_{k-1}) \cdots Q(t_1, y_0, dy_1) \mu(dy_0). \end{aligned} \quad (3.4)$$

The difference between (3.2) and (3.4) is the initial distribution of  $(X_t)$ :  $X_0 = x$  almost surely in the first case,  $X_0 \sim \mu$  in the second case. The law at time  $t$  of  $X_t$  is then given by

$$\mathbb{P}(X_t \in B) = \int_E \int_B Q(t, y_0, dy_1) \mu(dy_0). \quad (3.5)$$

We will use the notations

$$\mathbb{P}_\mu(X_0^x \in B_0, X_{t_1}^x \in B_1, \dots, X_{t_k}^x \in B_k), \quad \mathbb{E}_\mu \varphi(X_t) \quad (3.6)$$

to denote (3.4) and the quantity

$$\int_E \int_E \varphi(y_1) Q(t, y_0, dy_1) \mu(dy_0)$$

respectively. When  $\mu = \delta_x$ , we simply use the notations  $\mathbb{P}_x$  and  $\mathbb{E}_x$ . For example,  $P_t \varphi(x)$  is  $\mathbb{E}_x \varphi(X_t)$ .

## 3.2 Semi-group property, Chapman-Kolmogorov property and Markov property

**From the semi-group property to the Markov property.** Taking for  $\varphi$  the characteristic function  $\varphi = \mathbf{1}_A$  in (3.1), we obtain  $Q(t, x, A) = P_t \mathbf{1}_A(x)$ . From the semi-group property  $P_{t+s} = P_t \circ P_s$  and the relation (3.1), we can then draw the following consequences

$$Q(t+s, x, A) = \int_E P_s \mathbf{1}_A(y) Q(t, x, dy) = \int_E Q(s, y, A) Q(t, x, dy). \quad (3.7)$$

The equality between the extreme sides of (3.7) is called the Chapman-Kolmogorov relation. In the approach to Markov processes described in the previous section 3.1, the Markov property (3.3) is deduced from the knowledge of the finite dimensional distributions (3.2) and (3.7). Although (3.2) is expressed in terms of sets  $B_0, \dots, B_k$ , the Markov property (3.3) is expressed in terms of functions  $\varphi \in \text{BM}(E)$ . To verify that (3.3) is a consequence of (3.2) and (3.7), we need to switch from measurable sets to bounded measurable functions. The procedure is standard: take  $\varphi = \mathbf{1}_A$  for once, and, conversely, given  $\varphi \in \text{BM}(E)$ , consider a sequence of simple functions  $(\varphi_n)$  that converges pointwise, while being bounded, to  $\varphi$  (the existence of such a sequence is true in a context more general than the one of Borel measure space, [17, Theorem 1.17]). The former mode of convergence is denoted  $\varphi_n \xrightarrow{\text{b.p.c.}} \varphi$ . So we say that there is bounded pointwise convergence of a sequence  $(\varphi_n)$  in  $\text{BM}(E)$  to  $\varphi \in \text{BM}(E)$  if  $\sup_n \|\varphi_n\|_{\text{BM}(E)} < +\infty$  and  $\varphi_n(x) \rightarrow \varphi(x)$  for all  $x \in E$ . Using b.p.c. sequences, we can give the following version of (3.2):

$$\mathbb{E}[\varphi_1(X_{t_1}) \cdots \varphi_n(X_{t_n})] = \int_E \cdots \int_E \varphi_n(y_n) Q(t_n - t_{n-1}, y_{n-1}, dy_n) \cdots \varphi_1(y_1) Q(t_1, x, dy_1), \quad (3.8)$$

for all  $\varphi_1, \dots, \varphi_n \in \text{BM}(E)$ . At the same time, the  $\sigma$ -algebra  $\mathcal{F}_t^X$  is generated by all the random variables  $X_{t_1}, \dots, X_{t_n}$ , where  $n \in \mathbb{N}^*$  and  $0 \leq t_1 < \dots < t_n \leq t$ . Therefore (3.3) is equivalent to the fact that

$$\mathbb{E}[\mathbf{1}_A \mathbf{1}_B(X_{t+s})] = \mathbb{E}[\mathbf{1}_A P_s \mathbf{1}_B(X_t)], \quad (3.9)$$

for all  $A$  of form

$$A = \{X_{t_1} \in B_1, \dots, X_{t_n} \in B_n\}.$$

Note that, for simplicity, we have dropped the explicit dependence on  $x$  in the notations. By (3.2), the left-hand side of (3.9) is

$$\int_{B_1} \cdots \int_{B_n} Q(t+s-t_n, y_n, B) Q(t_n - t_{n-1}, y_{n-1}, dy_n) \cdots Q(t_1, x, dy_1). \quad (3.10)$$

This term (3.10) is also

$$\int_{B_1} \cdots \int_{B_n} \int_E P_s \mathbf{1}_B(z) Q(t-t_n, y_n, dz) Q(t_n - t_{n-1}, y_{n-1}, dy_n) \cdots Q(t_1, x, dy_1). \quad (3.11)$$

by (3.7). We apply (3.8) with  $k = n + 1$  and

$$\varphi_1 = \mathbf{1}_{B_1}, \dots, \varphi_n = \mathbf{1}_{B_n}, \varphi_{n+1} = P_s \mathbf{1}_B,$$

to conclude that (3.11) is equal to  $\mathbb{E}[\mathbf{1}_A P_s \mathbf{1}_B(X_t)]$ .

**From the Markov property to the semi-group property.** It is often the case that a Markov process is given as a dynamical system. We start from an initial  $X_0$  with distribution  $\mu$ , a given probability measure on  $E$ , and then compute a trajectory  $(X_t)_{t \geq 0}$  which satisfies the Markov property

$$\mathbb{E}[\varphi(X_{t+s}) | \mathcal{F}_t] = P_s \varphi(X_t), \quad (3.12)$$

where  $P_t \varphi$  is defined by the identity  $P_t \varphi(x) = \mathbb{E}_x \varphi(X_t)$  (cf. Remark 3.1 for the notation  $\mathbb{E}_x$ ), and  $(\mathcal{F}_t)$  is a certain filtration. See Section 4.2 for instance. We emphasize the fact that, in this approach, what is really given is  $P_t^* \mu$ , defined as the law of  $X_t$  when  $X_0 \sim \mu$ : what is naturally given here is the evolution in time of the distribution of  $X_t$ . Then  $P_t \varphi(x)$  is simply obtained by the duality formula

$$\langle P_t^* \mu, \varphi \rangle = \langle \mu, P_t \varphi \rangle, \quad (3.13)$$

where we take  $\mu = \delta_x$ . To cast this framework into the one described in Section 3.1, we have essentially to show that  $(P_t)$  has the semi-group property, the probability kernel  $Q(t, x, A)$  being simply defined as  $\mathbb{P}_x(X_t \in A)$ . The semi-group property is obtained as follows: by the tower property (2.2), we have

$$P_{t+s} \varphi(x) = \mathbb{E}_x[\varphi(X_{t+s})] = \mathbb{E}_x[\mathbb{E}_x[\varphi(X_{t+s}) | \mathcal{F}_t]].$$

We use the Markov property (3.12) to obtain

$$P_{t+s} \varphi(x) = \mathbb{E}_x[P_s \varphi(X_t)] = (P_t \circ P_s) \varphi(x),$$

which is the desired identity.

### 3.3 Invariant measure

Let  $(\mathbf{P}, Q, \mathbf{X})$  be a Markov process as in Section 3.1. If  $\mu$  is a probability measure on  $E$ , we denote by  $P_t^* \mu$  the law at time  $t$  of  $X_t$ , when  $X_0 \sim \mu$ . The notation is explained by the fact that, for  $B$  a Borel subset of  $E$  and with  $\varphi = \mathbf{1}_B$ , we have by (3.1) and (3.5), the identity

$$\langle P_t^* \mu, \varphi \rangle = \langle \mu, P_t \varphi \rangle. \quad (3.14)$$

Since  $B \mapsto \langle \mu, P_t \mathbf{1}_B \rangle$  is a measure, the identity (3.14) is then true for all  $\varphi \in \text{BM}(E)$ .

**Definition 3.1** (Invariant measure). A probability measure  $\mu$  on  $E$  is said to be an invariant measure if  $P_t^* \mu = \mu$  for all  $t \geq 0$ .

If  $\mu$  is an invariant measure and one chooses  $X_0$  according to the law  $\mu$ , then  $X_t$  follows the same law  $\mu$  for all  $t \geq 0$ : an invariant measure is a fixed-point for the evolution in distribution of the Markov process.

**Exercise 3.2** (Invariant measure). Let  $X_0, X_1, \dots$  be the sequence of random variables on  $\mathbb{R}$  defined as follows:  $X_0$  is chosen at random, according to a law  $\mu_0$ , then,  $X_N$  being known, a random variable  $Z_{N+1}$  taking the values  $+1$  or  $-1$  with equi-probability is drawn independently on  $X_0, \dots, X_N$  and  $X_{N+1}$  given by

$$X_{N+1} = \frac{1}{2} X_N + Z_{N+1}.$$

1. What means  $\mu_0 = \delta_0$ ? What are then the law  $\mu_1, \mu_2$  of  $X_1$  and  $X_2$  respectively?
2. Consider the case  $\mu_0 = \frac{1}{2}\delta_{-2} + \frac{1}{2}\delta_{+2}$ . Compute  $\mu_1, \mu_2, \mu_3$ . Can you guess a general formula for  $\mu_N$ ?
3. Find an invariant measure.

The solution to Exercise 3.2 is [here](#).

### 3.4 Infinitesimal generator

Given a Markov process as in Section 3.1, we would like to define the associated infinitesimal generator. There are various possible approaches. In [1] for example, it is assumed that the process admits an invariant measure  $\mu$ . The semi-group can then be extended as a contraction semi-group on  $L^2(\mu)$ . By assuming additionally that this extension gives rise to a strongly continuous semi-group, [1, Property (vi), p.11], one can use the standard theory of strongly continuous semi-group, [16], to define the infinitesimal generator. One may wonder why not simply working in  $\text{BM}(E)$ , or  $\text{BC}(E)$ , which are Banach spaces, to apply the standard theory of strongly continuous semi-group. The difficulty is that the continuity property  $P_t\varphi \rightarrow \varphi$  when  $t \rightarrow 0$  is too stringent in that context, at least when  $E$  is infinite-dimensional. Consider for example the simple deterministic case where  $P_t\varphi$  is given as the composition  $\varphi \circ \Phi_t$  with a flow  $(\Phi_t)$ . Let  $E$  be the Hilbert space  $E = \ell^2(\mathbb{N})$ , with orthonormal basis  $(e_n)_{n \in \mathbb{N}}$ , and let  $\Phi_t$  be given by

$$\Phi_t(x) = \sum_{n=0}^{\infty} e^{-\lambda_n t} \langle x, e_n \rangle e_n,$$

where  $(\lambda_n)$  is an increasing sequence converging to  $+\infty$ . In general, one cannot control the distance  $\|\Phi_t(x) - x\|_{\ell^2(\mathbb{N})}$  uniformly in  $x$  (this is possible when  $x$  is restricted to a compact set), so even if  $\varphi$  is uniformly continuous, one does not expect the convergence

$$\limsup_{t \rightarrow 0} \sup_{x \in E} |P_t\varphi(x) - \varphi(x)| = 0.$$

We consider a different mode of convergence therefore, the bounded pointwise convergence defined in Section 3.2 (see also [6, p.111]). A function  $\varphi \in \text{BM}(E)$  is in the domain  $D(\mathcal{L})$  of the infinitesimal generator  $\mathcal{L}$  of  $(P_t)$  if there exists  $\psi \in \text{BM}(E)$  such that

$$\frac{P_t\varphi - \varphi}{t} \xrightarrow{\text{b.p.c.}} \psi, \tag{3.15}$$

when  $t \rightarrow 0$ . We then set  $\mathcal{L}\varphi = \psi$ . Note that, on the elements  $\varphi \in D(\mathcal{L})$ , the property of continuity

$$P_t\varphi \xrightarrow{\text{b.p.c.}} \varphi \tag{3.16}$$

when  $t \rightarrow 0$ , is satisfied. By the semi-group property, (3.16) implies more generally the property of continuity from the right  $P_t\varphi \xrightarrow{\text{b.p.c.}} P_{t_*}\varphi$  when  $t \downarrow t_*$ , for every  $t_* \geq 0$ . The semi-group property, and the continuity of  $P_t$  with respect to b.p. convergence, that can be deduced from (3.1), have also the following consequence: if  $\varphi \in D(\mathcal{L})$ , then  $P_t\varphi \in D(\mathcal{L})$  for all  $t \geq 0$  and

$$\frac{P_{t+h}\varphi - P_t\varphi}{h} \xrightarrow{\text{b.p.c.}} P_t\mathcal{L}\varphi = \mathcal{L}P_t\varphi, \tag{3.17}$$

when  $h \rightarrow 0^+$ .

### 3.5 Martingale property of Markov processes

Consider a Markov process as in Section 3.1, which is Markov with respect to a filtration  $(\mathcal{F}_t)$ , and has a generator  $\mathcal{L}$ , as defined as in Section 3.4. We make the following hypotheses:

1. *stochastic continuity*: we have  $P_t\varphi \xrightarrow{\text{b.p.c.}} P_{t_*}\varphi$  when  $t \rightarrow t_*$  for every  $\varphi \in \text{BC}(E)$  and every  $t_* \geq 0$ ,
2. *measurability*: for all  $x \in E$ , the application  $(\omega, t) \mapsto X_t^x(\omega)$  is measurable  $\Omega \times \mathbb{R}_+ \rightarrow E$ .

We have then the following result.

**Theorem 3.1.** *Let  $\varphi \in D(\mathcal{L}) \cap \text{BC}(E)$  and  $x \in E$ . Then*

$$M_t^x := \varphi(X_t^x) - \varphi(x) - \int_0^t \mathcal{L}\varphi(X_s^x)ds \quad (3.18)$$

is a  $(\mathcal{F}_t)$ -martingale. If furthermore  $|\varphi|^2$  is in the domain of  $\mathcal{L}$ , then the process  $(Z_t^x)$  defined by

$$Z_t^x := |M_t^x|^2 - \int_0^t (\mathcal{L}|\varphi|^2 - 2\varphi\mathcal{L}\varphi)(X_s^x)ds, \quad (3.19)$$

is a  $(\mathcal{F}_t)$ -martingale.

*Remark 3.2.* Since  $\|P_t\varphi\|_{\text{BM}(E)} \leq \|\varphi\|_{\text{BM}(E)}$ , the convergence  $P_t\varphi \xrightarrow{\text{b.p.c.}} P_{t_*}\varphi$  is equivalent to the convergence  $P_t\varphi(x) \rightarrow P_{t_*}\varphi(x)$  for all  $x \in E$ . The hypothesis of stochastic continuity can therefore be rephrased as the continuity, for the topology of the weak convergence of probability measures, of  $t \mapsto P_t^*\delta_x$ , where  $P_t^*\delta_x$  is the law of  $X_t^x$ . This continuity property is satisfied if  $(X_t^x)$  is stochastically continuous in particular: for all  $\delta > 0$ ,

$$\lim_{t \rightarrow t_*} \mathbb{P}(d_E(X_t^x, X_{t_*}^x) > \delta) = 0, \quad (3.20)$$

where  $d_E$  is the distance on  $E$ . Under (3.20), we can also assume, up to a modification of the process, that the measurability property of Item 2 is satisfied, [3, Proposition 3.2]. If the stochastic continuity property  $P_t\varphi \xrightarrow{\text{b.p.c.}} P_{t_*}\varphi$  is satisfied, then

$$P_t\varphi(x) - \varphi(x) = \int_0^t P_s\mathcal{L}\varphi(x)ds, \quad (3.21)$$

for all  $\varphi \in D(\mathcal{L})$ ,  $x \in E$ ,  $t \geq 0$ . Indeed both sides of (3.21) are continuous functions that coincide at  $t = 0$  and have same right-differential at all points.

*Remark 3.3* (Quadratic variation). If  $(X_t^x)$  is continuous, then

$$A_t^x := \int_0^t (\mathcal{L}|\varphi|^2 - 2\varphi\mathcal{L}\varphi)(X_s^x)ds \quad (3.22)$$

is the quadratic variation  $\langle M^x, M^x \rangle_t$ , [11, p.38], of  $(M_t^x)$ . In the general case where  $(X_t^x)$  is càdlàg,  $(A_t^x)$  is the compensator, [11, p.32], of the quadratic variation  $[M^x, M^x]_t$ , [11, p.51], of  $(M_t^x)$ . For instance, if  $(X_t = N_t)$  is a Poisson Process of rate  $\lambda$ , then  $\mathcal{L}\varphi(n) = \lambda(\varphi(n+1) - \varphi(n))$  and

$$A_t := \lambda \int_0^t (\varphi(N_s + 1) - \varphi(N_s))^2 ds.$$

Taking  $\varphi = \text{Id}$ , gives the standard fact that  $(N_t - \lambda t)$  is a martingale.

*Proof of Theorem 3.1.* Since  $x$  is fixed, we will suppress the superscript  $x$  in the proof. Let  $0 \leq s \leq t$ . By the Markov property, we have

$$\begin{aligned} \mathbb{E}[M_t|\mathcal{F}_s] - M_s &= \mathbb{E}[M_t - M_s|\mathcal{F}_s] \\ &= P_{t-s}\varphi(X_s) - \varphi(X_s) - \int_s^t [P_{\sigma-s}\mathcal{L}\varphi](X_s)d\sigma. \end{aligned}$$

We use (3.21) to conclude that  $\mathbb{E}[M_t|\mathcal{F}_s] - M_s = 0$ . The proof of the martingale property for (3.19) is divided in several steps. First, we fix two times  $0 \leq \tau < \tau' \leq T$ . We fix a subdivision  $\sigma = (t_i)_{0,n}$  of  $[0, \tau']$ , chosen in such a way that  $\tau$  is always one of the  $t_i$ , say  $\tau = t_l$  (the index  $l$  may hence vary with  $\sigma$ ). By  $C(\varphi)$ , we will denote any constant that depend on  $\varphi$  and is independent on  $\sigma$  and may vary from lines to lines. We also denote by  $A = \mathcal{O}(B)$  any estimate of the form  $|A| \leq C(\varphi)|B|$ . At last, we introduce the following notations: we denote by  $\delta_{t_i}K$  the increment  $K_{t_{i+1}} - K_{t_i}$  of a function  $t \mapsto K_t$ . We also denote by  $\mathbb{E}_{t_i}$  the conditional expectation with respect to  $\mathcal{F}_{t_i}$ . Our aim is to show that

$$A'_\tau = \lim_{|\sigma| \rightarrow 0} \sum_{i=0}^{n-1} \mathbb{E}_{t_i} [|\delta_{t_i}M|^2], \quad (3.23)$$

where the limit is taken in  $L^2(\Omega)$ . Indeed, taking (3.23) for granted,  $\mathbb{E}[Z_{\tau'} - Z_\tau|\mathcal{F}_\tau]$  is the limit when  $|\sigma| \rightarrow 0$  of the quantity

$$\mathbb{E}\left[|M_{t_n}|^2 - |M_{t_l}|^2 - \sum_{i=l}^{n-1} \mathbb{E}_{t_i} [|\delta_{t_i}M|^2] \middle| \mathcal{F}_\tau\right]. \quad (3.24)$$

Let us show that (3.24) = 0. To simplify the presentation<sup>1</sup>, we will treat the case  $t_l = \tau = 0$ ,  $\mathcal{F}_0 = \{\emptyset, \Omega\}$  (it makes sense to consider that  $\mathcal{F}_0$  is the trivial sigma algebra since  $M_0 = 0$ ). We have

$$\mathbb{E}[|M_{t_n}|^2] = \mathbb{E}\left[\left|\sum_{i=0}^{n-1} \delta_{t_i}M\right|^2\right]. \quad (3.25)$$

In (3.25), we can expand the square. The contribution of the double products is zero since, if  $j > i$ , then, using the fact that  $\delta_{t_i}M$  is  $\mathcal{F}_{t_j}$ -measurable, we have

$$\mathbb{E}[\delta_{t_i}M\delta_{t_j}M] = \mathbb{E}[\mathbb{E}_{t_j}[\delta_{t_j}M]\delta_{t_i}M] = 0.$$

The last identity follows from the martingale property  $\mathbb{E}_{t_j}[\delta_{t_j}M] = 0$ . This implies (3.24) = 0, and thus  $\mathbb{E}[Z_{\tau'} - Z_\tau|\mathcal{F}_\tau] = 0$ . The proof of (3.23) is divided into three steps.

**Step 1.** We show that  $A_{\tau'} = \lim_{|\sigma| \rightarrow 0} \sum_{i=0}^{n-1} \mathbb{E}_{t_i}[\delta_{t_i}A]$ , with a convergence in  $L^2(\Omega)$ . Since  $A_{\tau'} = \sum_{i=0}^{n-1} \delta_{t_i}A$ , we have to show that  $\lim_{|\sigma| \rightarrow 0} \sum_{i=0}^{n-1} \zeta_i = 0$  in  $L^2(\Omega)$ , where  $\zeta_i := \delta_{t_i}A - \mathbb{E}_{t_i}[\delta_{t_i}A]$ . The method is similar to the analysis of (3.24) above: we decompose

$$\mathbb{E}\left|\sum_{i=0}^{n-1} \zeta_i\right|^2 = \sum_{i=0}^{n-1} \mathbb{E}\left[|\zeta_i|^2\right] + 2 \sum_{0 \leq i < j < n} \mathbb{E}[\zeta_i\zeta_j]. \quad (3.26)$$

By conditioning with respect to  $\mathcal{F}_{t_j}$ , we get that each term in the last sum in (3.26) is trivial. Since  $\zeta_i = \mathcal{O}(\delta t_i)$ , the first sum in the right-hand side of (3.26) is  $\mathcal{O}(|\sigma|)$ . This gives the result.

<sup>1</sup>the general case is left as an exercise, use the tower property (2.2)

**Step 2.** We show that  $\mathbb{E}_{t_i} [|\delta_{t_i} \varphi(X)|^2] = \mathcal{O}(\delta t_i)$ . First, by the Markov Property, we have  $\mathbb{E}_{t_i} [\delta_{t_i} M] = 0$ . Using (3.18), which implies

$$\delta_{t_i} M = \delta_{t_i} \varphi(X) - \int_{t_i}^{t_{i+1}} \mathcal{L} \varphi(X_s) ds, \quad (3.27)$$

we deduce  $\mathbb{E}_{t_i} [\delta_{t_i} \varphi(X)] = \mathcal{O}(\delta t_i)$ . We can apply the previous estimate to  $|\varphi|^2$ , since  $|\varphi|^2$  is in the domain of  $\mathcal{L}$  by hypothesis (cf. (3.29) below), to get  $\mathbb{E}_{t_i} [\delta_{t_i} |\varphi|^2(X)] = \mathcal{O}(\delta t_i)$ . On the other hand, we have also the identity

$$|\delta_{t_i} \varphi(X)|^2 = \delta_{t_i} |\varphi|^2(X) - 2\varphi(X_{t_i}) \delta_{t_i} \varphi(X). \quad (3.28)$$

Taking expectation with respect to  $\mathcal{F}_{t_i}$  in (3.28) and using the fact that

$$\mathbb{E}_{t_i} [\varphi(X_{t_i}) \delta_{t_i} \varphi(X)] = \varphi(X_{t_i}) \mathbb{E}_{t_i} [\delta_{t_i} \varphi(X)]$$

gives the desired estimate  $\mathbb{E}_{t_i} [|\delta_{t_i} \varphi(X)|^2] = \mathcal{O}(\delta t_i)$ . We can insert this result in (3.27) to obtain also  $\mathbb{E}_{t_i} [|\delta_{t_i} M|^2] = \mathcal{O}(\delta t_i)$ .

**Step 3.** We conclude the proof. First, we note that (3.27) applied to  $\varphi^2$  gives

$$\delta_{t_i} M^{(2)} = \delta_{t_i} |\varphi|^2(X) - \int_{t_i}^{t_{i+1}} \mathcal{L} |\varphi|^2(X_s) ds, \quad (3.29)$$

where  $M_t^{(2)} := |\varphi|^2(X_t) - |\varphi|^2(x) - \int_0^t \mathcal{L} |\varphi|^2(X_s) ds$ . We combine (3.27), (3.28) and (3.29) to obtain the identity

$$\begin{aligned} |\delta_{t_i} M|^2 + 2\delta_{t_i} M \int_{t_i}^{t_{i+1}} \mathcal{L} \varphi(X_s) ds + \left| \int_{t_i}^{t_{i+1}} \mathcal{L} \varphi(X_s) ds \right|^2 \\ = \delta_{t_i} M^{(2)} + \int_{t_i}^{t_{i+1}} \mathcal{L} |\varphi|^2(X_s) ds - 2\varphi(X_{t_i}) \left( \delta_{t_i} M + \int_{t_i}^{t_{i+1}} \mathcal{L} \varphi(X_s) ds \right). \end{aligned} \quad (3.30)$$

Taking the conditional expectation  $\mathbb{E}_{t_i}$  in (3.30) and using the Markov property gives us

$$\mathbb{E}_{t_i} [|\delta_{t_i} M|^2] = \mathbb{E}_{t_i} [\delta_{t_i} A] - 2\mathbb{E}_{t_i} \left[ \int_{t_i}^{t_{i+1}} (\varphi(X_{t_i}) - \varphi(X_s)) \mathcal{L} \varphi(X_s) ds \right] + \mathcal{O}(|\delta t_i|^{3/2}). \quad (3.31)$$

Indeed, we have discarded the terms

$$\mathbb{E}_{t_i} \left[ 2\delta_{t_i} M \int_{t_i}^{t_{i+1}} \mathcal{L} \varphi(X_s) ds \right] \text{ and } \mathbb{E}_{t_i} \left[ \left| \int_{t_i}^{t_{i+1}} \mathcal{L} \varphi(X_s) ds \right|^2 \right],$$

which are respectively  $\mathcal{O}(|\delta t_i|^{3/2})$  and  $\mathcal{O}(|\delta t_i|^2)$ . To obtain the  $\mathcal{O}(|\delta t_i|^{3/2})$ -estimate, we use the bound

$$\left| 2\delta_{t_i} M \int_{t_i}^{t_{i+1}} \mathcal{L} \varphi(X_s) ds \right| \leq \eta |\delta_{t_i} M| + \eta^{-1} \left| \int_{t_i}^{t_{i+1}} \mathcal{L} \varphi(X_s) ds \right|^2,$$

then Step 2, and then we choose  $\eta = (\delta t_i)^{1/2}$ . We can repeat Step 2, where we consider the time interval  $[t_i, s]$  instead of  $[t_i, t_{i+1}]$ , to obtain the estimate  $\mathbb{E}_{t_i} [|\varphi(X_{t_i}) - \varphi(X_s)|^2] = \mathcal{O}(\delta t_i)$ , when  $t_i \leq s \leq t_{i+1}$ . Consequently, the last term in (3.31) is also  $\mathcal{O}(|\delta t_i|^{3/2})$ . By summing with respect to  $i$  in (3.31), we deduce finally that

$$\sum_{i=0}^{n-1} \mathbb{E}_{t_i} [\delta_{t_i} A] = \sum_{i=0}^{n-1} \mathbb{E}_{t_i} [|\delta_{t_i} M|^2] + \mathcal{O}(|\sigma|^{1/2}).$$

This equality, combined with Step 1, yields (3.23). This achieves the proof.  $\square$

The results of Theorem 3.1 can be extended to the case where the test function  $\varphi$  also depends on  $t$ . We will need this result only in the simple case where the test function has the form  $\theta(t)\varphi(x)$ .

**Corollary 3.2.** *Let  $\varphi \in D(\mathcal{L})$  satisfies  $|\varphi|^2 \in D(\mathcal{L})$ . Let  $\theta \in C^1(\mathbb{R}_+)$  and let  $\psi(t, x) = \theta(t)\varphi(x)$ . Then, for all  $x \in E$ , the process*

$$M_t^x := \psi(t, X_t^x) - \psi(0, x) - \int_0^t (\partial_t + \mathcal{L})\psi(s, X_s^x) ds \quad (3.32)$$

is a  $(\mathcal{F}_t)$ -martingale and the process  $(Z_t^x)$  defined by

$$Z_t^x := |M_t^x|^2 - \int_0^t ((\partial_t + \mathcal{L})|\psi|^2 - 2\psi(\partial_t + \mathcal{L})\psi)(s, X_s^x) ds, \quad (3.33)$$

is a  $(\mathcal{F}_t)$ -martingale.

*Proof of Corollary 3.2.* By the Markov property, we have

$$\begin{aligned} & \mathbb{E}[M_t^x - M_s^x | \mathcal{F}_s] \\ &= \theta(t)(P_{t-s}\varphi)(X_s^x) - \theta(s)\varphi(X_s^x) - \int_s^t (\theta'(\sigma)(P_{\sigma-s}\varphi)(X_s^x) + \theta(\sigma)\frac{d}{d\sigma}(P_{\sigma-s}\varphi)(X_s^x))d\sigma. \end{aligned}$$

By explicit integration, we see that  $(M_t^x)$  is a  $(\mathcal{F}_t)$ -martingale. We compute then

$$(\partial_t + \mathcal{L})|\psi|^2 - 2\psi(\partial_t + \mathcal{L})\psi = \theta^2(\mathcal{L}|\varphi|^2 - 2\varphi\mathcal{L}\varphi).$$

Let us examine the proof of the second part of Theorem 3.1. Since  $\theta$  is locally Lipschitz continuous, we have  $\theta(t) \simeq \theta(t_i) + \mathcal{O}(\delta t_i)$ , for  $t \in [t_i, t_{i+1}]$ . Using this approximation, it is easy to show, by adapting the proof of Theorem 3.1, that, in our context,  $(Z_t^x)$  is a martingale.  $\square$

**Exercise 3.3** (Markov process with finite state space). Let  $(\mathbf{P}, \mathbf{Q}, \mathbf{X})$  be a Markov process. Assume that the state space  $E$  is finite,  $E = \{x_1, \dots, x_L\}$ . We introduce the family of matrices  $A(t) = a_{ij}(t)$ , with  $a_{ij}(t) = Q(t, x_i, \{x_j\})$ , i.e.  $a_{ij}(t) = \mathbb{P}_{x_i}(X(t) = x_j)$ .

1. If  $\varphi: E \rightarrow \mathbb{R}$ , we still denote by  $\varphi$  the vector  $(\varphi(x_i))_{1 \leq i \leq L}$ . Give the expression of  $P_t\varphi$  as a product matrix-vector.
2. If  $\mu$  is a probability measure on  $E$ , we still denote by  $\mu$  the vector  $(\mu(\{x_i\}))_{1 \leq i \leq L}$ . Give the expression of  $P_t^*\mu$  as a product matrix-vector.
3. We assume that  $t \mapsto A(t)$ , from  $\mathbb{R}_+$  into  $\mathcal{M}_L(\mathbb{R})$  is of class  $C^1$ . Show that  $A(t) = e^{t\mathcal{L}}$ , where  $\mathcal{L} = A'(0)$  is the generator.
4. Give the equation satisfied by an invariant measure.

*The solution to Exercise 3.3 is here.*

**Exercise 3.4** (Markov process in discrete time). We consider now a Markov process  $(X_n)_{n \geq 0}$  in discrete time.

1. Assume that the state space  $E$  is finite. How can you rephrase the questions and answers of the previous exercise 3.3?



2. Give and prove the equivalent statement to Theorem 3.1. More precisely, let  $\mathcal{L} = P_1 - \text{Id}$ , let  $\varphi \in \text{BM}(E)$ . Show that

$$M_n = \varphi(X_n) - \varphi(X_0) - \sum_{k=0}^{n-1} \mathcal{L}\varphi(X_k) \quad (3.34)$$

and

$$Z_n := |M_n|^2 - \sum_{k=0}^{n-1} \Gamma[\varphi](X_k) \quad (3.35)$$

are martingales. In (3.35),  $\Gamma(\varphi)$  is a certain non-negative expression that you will have to identify.

The solution to Exercise 3.4 is [here](#).

## 4 Evolution of a random interface

In this part we establish the limit behavior of the symmetric simple exclusion process. More precisely, we show in Theorem 4.1 that, after an adequate change of scales, the random interface associated to the symmetric simple exclusion process converges in probability to the solution of a heat equation.

### 4.1 Change of scale and limit behavior

Let  $X_L$  denote the discrete interval  $X_L = \{0, \dots, L-1\}$ . Let  $E_L$  be the set of functions  $H: X_L \rightarrow \mathbb{R}$ . Let  $E_L^{(1)}$  be the convex subset of  $E_L$  constituted of the functions  $H$  such that  $H(0) = 0$  and  $|H(\mathbf{x}+1) - H(\mathbf{x})| = 1$  for all  $\mathbf{x} \in X_L$  (in the case  $\mathbf{x} = L-1$ , we use the convention  $H(L) = 0$ ). The space  $E_L^{(1)}$  is the state space for the process described in Section 1.2. To  $H \in E_L$ , we associate a function  $\hat{H}: [0, 1] \rightarrow \mathbb{R}$  defined by

$$\hat{H}(x) = L^{-1}H(\lfloor Lx \rfloor), \quad (4.1)$$

where  $p = \lfloor y \rfloor$ , defined for  $y \geq 0$ , is the integer such that  $p \leq y < p+1$ . The map  $H \mapsto \hat{H}$  is an isometry  $E_L \rightarrow L^2(0, 1)$  when  $E_L$  and  $L^2(0, 1)$  are endowed with the respective scalar products

$$\langle H, G \rangle_{E_L} = \frac{1}{L^3} \sum_{\mathbf{x}=0}^{L-1} H(\mathbf{x})G(\mathbf{x}), \quad \langle f, g \rangle_{L^2} = \int_0^1 f(x)g(x)dx. \quad (4.2)$$

Indeed, given  $H, G \in E_L$ , we compute

$$\langle \hat{H}, \hat{G} \rangle_{L^2(0,1)} = \sum_{\mathbf{x}=0}^{L-1} \int_{\frac{\mathbf{x}}{L}}^{\frac{\mathbf{x}+1}{L}} \hat{H}(y)\hat{G}(y)dy = \sum_{\mathbf{x}=0}^{L-1} \frac{1}{L^3} H(\mathbf{x})G(\mathbf{x}) = \langle H, G \rangle_{E_L}. \quad (4.3)$$

We will also work with the  $C^0$  norm defined by  $\|f\|_{C([0,1])} = \sup_{x \in [0,1]} |f(x)|$  for  $f$  continuous on  $[0, 1]$ . Since  $H \mapsto \hat{H}$  is an isometry, a natural left-inverse is given by calculating the adjoint operator. This is easily done, and we obtain a map  $L^2(0, 1) \rightarrow E_L$ , which, to  $h \in L^2(0, 1)$  associates the function in  $E_L$  given by

$$J_h: \mathbf{x} \mapsto L^2 \int_{\frac{\mathbf{x}}{L}}^{\frac{\mathbf{x}+1}{L}} h(x)dx. \quad (4.4)$$

However, we will work preferably with the related application  $h \mapsto \check{h}$ , defined on  $C([0, 1])$  by  $\check{h}(\mathbf{x}) = Lh(L^{-1}\mathbf{x})$ . The reason of this modification is apparent in Proposition 4.4. If  $h$  is Lipschitz continuous on  $[0, 1]$ , then  $|J_h(\mathbf{x}) - \check{h}(\mathbf{x})|$  is bounded by  $\text{Lip}(h)$ . This implies that

$$|\langle \hat{H}, h \rangle_{L^2(0,1)} - \langle H, \check{h} \rangle_{E_L}| \leq \text{Lip}(h)L^{-2} \sup_{\mathbf{x} \in X_L} |H(\mathbf{x})|, \quad (4.5)$$

for all  $H \in E_L$ . Let  $h_{\text{in}}$  be a continuous function on  $[0, 1]$ , which is 1-Lipschitz continuous and satisfies the boundary conditions  $h_{\text{in}}(0) = h_{\text{in}}(1) = 0$ . Given such a function  $h_{\text{in}}$ , we build an initial datum  $H_{\text{in}} \in E_L^{(1)}$  for the evolution of the random interface. We want  $h_{\text{in}}$  and  $\hat{H}_{\text{in}}$  to be close in a certain norm. It is simpler to consider things in the space  $E_L$ , in which case we require  $H_{\text{in}}$  and the function  $\check{h}_{\text{in}}$  to be at distance  $\mathcal{O}(1)$  for a certain norm. Note that the graph of a profile  $H \in E_L^{(1)}$  is a subset of the lattice

$$\mathcal{R} = \{(\mathbf{x}, H); \mathbf{x} \in \{0, \dots, L\}, H \in \mathbb{Z}_{\mathbf{x}}\},$$

where we have set  $\mathbb{Z}_{\mathbf{x}} = 2\mathbb{Z}$  if  $\mathbf{x}$  is even,  $\mathbb{Z}_{\mathbf{x}} = 2\mathbb{Z} + 1$  if  $\mathbf{x}$  is odd. To build  $H_{\text{in}}$ , we draw the graph  $Gr_L$  of  $\check{h}_{\text{in}}$ . Then we choose the closest points of  $Gr_L$  in  $\mathcal{R}$  to obtain the graph of  $H_{\text{in}}$ . We have then

$$\sup_{\mathbf{x} \in \{0, \dots, L\}} \left| \check{h}_{\text{in}}(\mathbf{x}) - H_{\text{in}}(\mathbf{x}) \right| \leq 1. \quad (4.6)$$

Since  $h_{\text{in}}$  is 1-Lipschitz continuous, this implies

$$\|\hat{H}_{\text{in}} - h_{\text{in}}\|_{L^2(0,1)} \leq \|\hat{H}_{\text{in}} - h_{\text{in}}\|_{C([0,1])} \leq 2L^{-1}. \quad (4.7)$$

Let  $(H_t)$  be the Markov process described in Section 1.2 (we will show below in Section 4.2 that it is a Markov process indeed) that starts from  $H_{\text{in}}$ . We fix a time  $T > 0$  and consider the solution to the heat equation on  $[0, 1]$  with Dirichlet homogeneous boundary conditions and initial datum  $h_{\text{in}}$ : this is the function  $h \in L^2(0, T; H_0^1(0, 1))$  such that  $\partial_t h \in L^2(0, T; H^{-1}(0, 1))$  and

$$\langle \partial_t h(t), g \rangle_{L^2(0,1)} + \langle \partial_x h(t), \partial_x g \rangle_{L^2(0,1)} = 0, \quad (4.8)$$

for all  $g \in H_0^1(0, 1)$  and a.e.  $t \in (0, T)$ , and  $h(0) = h_{\text{in}}$ , see [7, p.374]. We call such a function  $h$  a weak solution to the following problem:

$$\partial_t h - \partial_x^2 h = 0 \text{ in } (0, +\infty) \times (0, 1), \quad (4.9)$$

$$h(t, x) = 0 \text{ for } (t, x) \in (0, +\infty) \times (\{0\} \cup \{1\}), \quad (4.10)$$

$$h(0, x) = h_{\text{in}}(x) \text{ for } x \in (0, 1). \quad (4.11)$$

We will establish the following result.

**Theorem 4.1.** *Let  $h_{\text{in}}$  be a 1-Lipschitz continuous function on  $[0, 1]$  vanishing at 0 and 1 and satisfying<sup>2</sup>  $h_{\text{in}} \in H^2(0, 1)$ . Let  $h$  be the unique solution to (4.9)-(4.10)-(4.11) (see (4.8)). Let  $H_{\text{in}} \in E_L^{(1)}$  satisfy (4.7), and let  $(H_t)$  be the Markov process described in Section 1.2 that starts from  $H_{\text{in}}$ . Then the rescaled process  $(\hat{H}_{L^2 t})$  converges to  $h$  in probability when  $L$  tends to  $+\infty$ , in the sense that, for all  $T > 0$ , for all  $\delta > 0$ , one has*

$$\lim_{L \rightarrow +\infty} \mathbb{P} \left( \sup_{t \in [0, T]} \|\hat{H}_{L^2 t} - h(t, \cdot)\|_{L^2(0,1)} > \delta \right) = 0. \quad (4.12)$$

---

<sup>2</sup>check

## 4.2 Markov property

We will show in this part that the process  $(H_t)$  described in Section 1.2 is a Markov process and give the expression of its generator. First, the general procedure using “clocks”, that transform discrete-time Markov processes into continuous-time Markov processes is analyzed. In a second step, we study the discrete-time Markov process that gives rise to  $(H_t)$ .

### 4.2.1 From discrete-time to continuous-time Markov process

*Notation:* if  $Z = (Z_t)_{t \in I}$  is a process ( $I$  may be continuous or discrete), we denote by  $(\mathcal{F}_t^Z)_{t \in I}$  the filtration generated by  $Z$ :  $\mathcal{F}_t^Z$  is the  $\sigma$ -algebra generated by all random variables  $Z_{t_1}, \dots, Z_{t_m}$ , where  $m \in \mathbb{N}^*$ ,  $t_1, \dots, t_m$  in  $I$ ,  $t_k \leq t$  for all  $k$ .

**Proposition 4.2.** *Let  $E$  be a Polish space. Let  $(X_n)_{n \geq 0}$  be a discrete time-homogeneous Markov chain on  $E$  with transition operator  $P_n$ ,  $n \in \mathbb{N}$ . Let  $N(t)$  be a Poisson process of exponent  $\lambda > 0$  independent on  $(X_n)_{n \geq 0}$  and let  $\xi_t = X_{N(t)}$ . Let also  $\mathcal{F}_t = \mathcal{F}_t^\xi \vee \mathcal{F}_t^N$  be the minimal  $\sigma$ -algebra containing  $\mathcal{F}_t^\xi$  and  $\mathcal{F}_t^N$ . Then  $(\xi_t)_{t \geq 0}$  is a time-homogeneous Markov process with respect to  $(\mathcal{F}_t)_{t \geq 0}$ , with transition operator and infinitesimal generator given by*

$$\Pi_t = \exp(-\lambda t(\text{Id} - P_1)), \quad \mathcal{L} = -\lambda(\text{Id} - P_1), \quad (4.13)$$

respectively.

*Proof of Proposition 4.2.* Note first that  $P_n = P_1^n$  for all  $n \geq 0$ . This is the semi-group property in discrete time. Then, we want to establish the following kind of Markov property: for all  $A \in \mathcal{F}_t$ , for all  $\varphi \in \text{BM}(E)$ ,

$$\mathbb{E}[\mathbf{1}_A \varphi(X_{n+N(t)})] = \mathbb{E}[\mathbf{1}_A P_n \varphi(X_{N(t)})]. \quad (4.14)$$

Indeed, (4.14) means that  $\mathbb{E}[\varphi(X_{n+N(t)}) | \mathcal{F}_t] = P_1^n \varphi(X_{N(t)})$ . Assuming that (4.14) is satisfied for the moment, we use the decomposition

$$\mathbb{E}[\varphi(X_{N(t+s)}) | \mathcal{F}_t] = \sum_{n=0}^{\infty} \mathbb{E}[\varphi(X_{N(t+s)}) \mathbf{1}_{N(t+s)-N(t)=n} | \mathcal{F}_t].$$

By independence, this gives

$$\mathbb{E}[\varphi(X_{N(t+s)}) | \mathcal{F}_t] = \sum_{n=0}^{\infty} \mathbb{P}(N(t+s) - N(t) = n) \mathbb{E}[\varphi(X_{N(t)+n}) | \mathcal{F}_t].$$

In the last summand, we replace

$$\mathbb{P}(N(t+s) - N(t) = n) = e^{-\lambda s} \frac{(\lambda s)^n}{n!}, \quad \mathbb{E}[\varphi(X_{N(t)+n}) | \mathcal{F}_t] = P_1^n \varphi(X_{N(t)}).$$

The summation over  $n$  gives  $\mathbb{E}[\varphi(\xi_{t+s}) | \mathcal{F}_t] = (\Pi_s \varphi)(\xi_t)$ , where  $\Pi_t$  is defined by (4.13). Then we are in the situation described in the last paragraph of Section 3.2, cf. (3.12)-(3.13). It follows that  $(\xi_t)_{t \geq 0}$  is a time-homogeneous Markov process with respect to  $(\mathcal{F}_t)_{t \geq 0}$ . It is also clear that  $\mathcal{L} = -(\text{Id} - P_1)$ . To establish (4.14), we observe that each side of the equality defines a set-function, by dependance on  $A$ , which is a finite measure. By [2, Theorem 3.3], it is sufficient to prove (4.14) for all sets  $A$  in a class  $\mathcal{M}$  which is a  $\pi$ -system generating  $\mathcal{F}_t$ , in the sense that  $\sigma(\mathcal{M}) = \mathcal{F}_t$ . To that effect, we consider the class  $\mathcal{M}$  of sets of the form  $B \cap D \cap \{N(t) = m\}$ ,

where  $m \in \mathbb{N}$ ,  $B \in \mathcal{F}_m^X$ ,  $D \in \mathcal{F}_t^N$ . It is clear that  $\mathcal{M}$  is a  $\pi$ -system. The  $\sigma$ -algebra  $\mathcal{F}_t$  is generated by all the random variables  $X_{N(t_1)}, \dots, X_{N(t_j)}$  and  $N(s_1), \dots, N(s_k)$  for  $j, k \in \mathbb{N}^*$  and times  $t_i, s_i \leq t$ . By considering all the possible values taken by  $N(t_1), \dots, N(t_j)$  and  $N(t)$ , the event

$$\{X_{N(t_1)} \in \Gamma_1, \dots, X_{N(t_j)} \in \Gamma_j, N(s_1) \in E_1, \dots, N(s_k) \in E_k\},$$

where  $\Gamma_1, \dots, \Gamma_j \in \mathcal{B}(E)$ ,  $E_1, \dots, E_k \subset \mathbb{N}$ , can be written as a union over  $m_1 \in \mathbb{N}, \dots, m_j, m \in \mathbb{N}$  of the intersection  $A := A_1 \cap A_2 \cap \{N(t) = m\}$  of the events  $A_1 = \{X_{m_1} \in \Gamma_1, \dots, X_{m_j} \in \Gamma_j\}$  with the events

$$A_2 = \{N(t_1) = m_1, \dots, N(t_j) = m_j\} \cap \{N(s_1) \in E_1, \dots, N(s_k) \in E_k\}.$$

Since  $N$  is non-decreasing and  $t_i \leq t$ , the set  $A$  is possibly non-empty only if the integers  $m_i$  are all smaller than  $m$ . In the latter case, we have  $A \in \mathcal{M}$ . We conclude that  $\sigma(\mathcal{M}) = \mathcal{F}_t$ . For  $A = B \cap D \cap \{N(t) = m\} \in \mathcal{M}$ , we have then

$$\mathbb{E}[\mathbf{1}_A \varphi(X_{n+N(t)})] = \mathbb{P}(D \cap \{N(t) = m\}) \mathbb{E}[\mathbf{1}_B \varphi(X_{n+m})]$$

by independence. By the Markov property,  $\mathbb{E}[\mathbf{1}_B \varphi(X_{n+m})]$  is equal to  $\mathbb{E}[\mathbf{1}_B P_n \varphi(X_m)]$ . We use independence again to conclude.  $\square$

**Exercise 4.1** (Poisson process). Let  $(T_n)$  be a sequence of i.i.d. random variables with exponential law of parameter  $\lambda > 0$ :  $\mathbb{P}(T_n > t) = e^{-\lambda t}$ . We define a sequence of times  $S_0, S_1, \dots$  as follows: let  $S_0 = 0$  and, for  $n \geq 1$ ,  $S_n = T_1 + \dots + T_n$ . Given an interval  $I$  of  $\mathbb{R}_+$ , we denote by  $\Gamma(I)$  the number of times  $S_n$  in  $I$ :

$$\Gamma(I) = \#(\mathcal{S} \cap I), \quad \mathcal{S} = \{S_n; n \in \mathbb{N}\}.$$

It is often understood that some particular events take place at the times  $S_n$  and that we are counting the number of such occurrences in the time interval  $I$ , see also Remark 4.1 below.

1. Compute the (density of the) law of  $S_n$ .
2. Let  $N(t) = \Gamma([0, t])$ . Show that  $N(t)$  is càdlàg and non-decreasing and is Poisson of parameter  $\lambda t$  (*hint*: compute first  $\mathbb{P}(N(t) < n) = \mathbb{P}(S_n > t)$ ).
3. Establish the “memoryless property” of exponential random variables: for all  $t, s \geq 0$ ,

$$\mathbb{P}(T > t + s | T > s) = \mathbb{P}(T > t), \tag{4.15}$$

for all exponential random variable  $T$ .

4. Generalize (4.15) to the case where  $s$  is replaced by  $S$ , a random variable with values in  $\mathbb{R}_+$  independent on  $T$ . More precisely, establish the formula

$$\mathbb{P}(T > t + S) = \mathbb{P}(T > t) \mathbb{P}(T > S), \tag{4.16}$$

where  $T$  is exponential random variable,  $t \geq 0$ ,  $S$  a random variable with values in  $\mathbb{R}_+$  independent on  $T$ .

5. Our aim now is to prove the following result: *let  $I$  and  $J$  be two disjoint interval of  $\mathbb{R}_+$ . Then  $\Gamma(I)$  and  $\Gamma(J)$  are independent and  $\Gamma(J)$  is Poisson of parameter  $\lambda|J|$  (where  $|J|$  is the length of  $J$ ).*

- (a) If we reach the desired conclusion, then the law of  $\Gamma(J)$  should be unchanged if we modify the extremities of  $J$  (replacing for instance  $J = (s, t]$  by  $[s, t]$ , or all possible variations). Justify this fact.
- (b) Prove the result when  $I = [0, t]$ ,  $J = (t, \sigma]$  (*hint*: adapt the proof of (4.16)).
- (c) Establish the general case.

The solution to Exercise 4.1 is [here](#).

*Remark 4.1* (Terminology, point Poisson Process). In the framework of Exercise 4.1, the process  $(N(t))$  defined by  $N(t) = \Gamma([0, t])$  is often called a Poisson process. We insisted to introduce the random variables  $\Gamma(I)$ , where  $I$  is an interval of  $\mathbb{R}_+$  (actually,  $I$  may be a more complicated set) to relate the Poisson process on  $\mathbb{R}_+$  to general point Poisson processes. Let us define the latter on the space  $\mathbb{R}^d$ . Let  $\mu$  be a  $\sigma$ -finite Borel measure on  $\mathbb{R}^d$ . A Poisson point process  $\Gamma$  on  $\mathbb{R}^d$ , with mean measure  $\mu$ , is a random map  $\mathcal{B}(\mathbb{R}^d) \rightarrow \mathbb{N}$ , where  $\mathcal{B}(\mathbb{R}^d)$  are the Borel sets of  $\mathbb{R}^d$ , such that

1. for any Borel sets  $A_1, \dots, A_n$ , the random variables  $\Gamma(A_1), \dots, \Gamma(A_n)$  are independent,
2. the random variable  $\Gamma(A)$  is Poisson with parameter  $\mu(A)$ :

$$\mathbb{P}(\Gamma(A) = n) = e^{-\mu(A)} \frac{\mu(A)^n}{n!}.$$

See [12, p.11-12]. In Exercise 4.1, we draw an explicit set of points  $\mathcal{S} = \{S_n; n \in \mathbb{N}\}$  and define  $\Gamma(I)$  as the number of points of  $\mathcal{S}$  in  $I$ . We obtain then a Poisson point process with mean measure  $\mu$ , the one-dimensional Lebesgue measure restricted to  $\mathbb{R}_+$ . One may wonder how to construct point Poisson processes, and even if such random objects exist. We refer again to [12] again, p. 23.

#### 4.2.2 Markov property for the symmetric simple exclusion process

If  $T$  and  $T'$  are two exponential independent random variables of parameters  $\lambda$  and  $\lambda'$ , then  $T \wedge T'$  is also an exponential random variable of parameter  $\lambda + \lambda'$ . Indeed,

$$\mathbb{P}(T \wedge T' > t) = \mathbb{P}(\{T > t\} \cap \{T' > t\}) = \mathbb{P}(T > t)\mathbb{P}(T' > t) = e^{-\lambda t} e^{-\lambda' t} = e^{-(\lambda + \lambda')t}.$$

Instead of considering  $L$  independent clocks given by exponential random variables of parameter 1 at each site  $0, \dots, L - 1$  and then looking at the site where the first clock is ringing, we can select a particular location by just having one single clock of parameter  $L$ . When it rings, we choose a location  $\mathbf{x}$  uniformly in  $X_L$ . With that approach, we see that  $H_t = \mathbb{H}_{N(t)}$ , where  $(N(t))$  is a Poisson process of parameter  $L$  and  $(\mathbb{H}_n)$  is an independent process that evolves in discrete time. To describe this evolution, we introduce the following notation<sup>3</sup> let  $\Delta_D$  be the discrete Laplace operator defined on functions  $H \in E_L$  by

$$\Delta_D H(\mathbf{x}) = H(\mathbf{x} + 1) + H(\mathbf{x} - 1) - 2H(\mathbf{x}), \quad \forall \mathbf{x} \in \{2, \dots, L - 1\}, \quad (4.17)$$

$$\Delta_D H(1) = H(2) - 2H(1), \quad \Delta_D H(L - 1) = H(L - 2) - 2H(L - 1). \quad (4.18)$$

The index  $D$  in  $\Delta_D$  is for “Dirichlet”, since  $\Delta_D$  is actually the discrete Laplace operator with homogeneous Dirichlet boundary conditions. This can be seen in (4.18), which is consistent with

<sup>3</sup>Note that the notation  $\Delta_D$  is reserved for the discrete Laplace operator (4.17). The continuous Laplace operator on  $(0, 1)$  is denoted  $\partial_x^2$ .

(4.17) if  $H(0) = 0$  and if we set  $H(L) := 0$ . In (4.17)-(4.18), the value of  $\Delta_D H$  at  $\mathbf{x} = 0$  has not been defined. We set this value to be 0:  $\Delta_D H(0) = 0$ . In this way, denoting by  $E_L^{(0)}$  the subset of  $E_L$  constituted of the functions  $H$  such that  $H(0) = 0$ , we may consider  $\Delta_D$  as an operator  $E_L^{(0)} \rightarrow E_L^{(0)}$  (see Proposition 4.4 for instance). Let us come back to the description of the evolution in discrete time of  $(\mathbb{H}_n)$ . If  $H$  has no corner at  $\mathbf{x}$ , then  $\Delta_D H(\mathbf{x}) = 0$ . If  $H$  has a corner pointing upwards at  $\mathbf{x}$ , then  $\Delta_D H(\mathbf{x}) = -2$ . When the corner is pointing downwards, then  $\Delta_D H(\mathbf{x}) = +2$ . Given  $\mathbb{H}_n$ , the profile  $\mathbb{H}_{n+1}$  is obtained as follows: choose  $\mathbf{x}$  uniformly in  $X_L$ . Set  $\mathbb{H}_{n+1} = \mathbb{H}_n + \delta_{\mathbf{x}} \Delta_D \mathbb{H}_n$ , where

$$\delta_{\mathbf{x}}(\mathbf{y}) = \mathbf{1}_{\mathbf{x}=\mathbf{y}} \text{ if } \mathbf{x} \in X_L. \quad (4.19)$$

One can check that  $\mathbb{H}_n \mapsto \mathbb{H}_{n+1}$  is the desired transformation. Indeed, nothing happens if  $\mathbf{x} = 0$ , or, when  $\mathbf{x} \in \{1, \dots, L-1\}$ , if  $\mathbb{H}_n$  has no corner at  $\mathbf{x}$ . If  $\mathbb{H}_n$  has a corner at  $\mathbf{x}$ , then  $\mathbb{H}_{n+1}$  is the graph deduced from  $\mathbb{H}_n$  by flipping this corner. The process  $(\mathbb{H}_n)$  is Markov and time-homogeneous with transition operator  $P_1$  given by

$$P_1 \varphi(H) = \mathbb{E}_H \varphi(\mathbb{H}_1) = \frac{1}{L} \sum_{\mathbf{x}=0}^{L-1} \varphi(H + \delta_{\mathbf{x}} \Delta_D H). \quad (4.20)$$

From Proposition 4.2 and (4.20), we deduce the following result.

**Theorem 4.3.** *Let  $E_L^{(1)}$  be the set of functions  $H: X_L \rightarrow \mathbb{R}$  such that  $H(0) = 0$  and  $|H_{\mathbf{x}+1} - H_{\mathbf{x}}| = 1$  for all  $\mathbf{x} \in X_L$  (in the case  $\mathbf{x} = L-1$ , we use the convention  $H(L) = 0$ ). Let  $\Delta_D$  and  $\delta$  be defined by (4.17) and (4.19) respectively. The symmetric simple exclusion process  $(H_t)$  described in Section 1.2 is a Markov process with generator  $\mathcal{L}$  given by*

$$\mathcal{L} \varphi(H) = \sum_{\mathbf{x}=0}^{L-1} (\varphi(H + \delta_{\mathbf{x}} \Delta_D H) - \varphi(H)), \quad (4.21)$$

with domain the set of functions  $\varphi: E_L^{(1)} \rightarrow \mathbb{R}$ .

**Exercise 4.2** (Independent clocks). Justify the assertion used at the beginning of the present section 4.2.2. The situation is the following one: let  $X_L = \{0, \dots, L-1\}$ . We consider two different processes that, at a given time  $t$ , will have chosen  $N(t)$  points  $\mathbf{x}_1, \dots, \mathbf{x}_{N(t)}$  in  $X_L$ . To define the state space of this process, we introduce an arbitrary point  $\dagger$ . Then, we set  $X_L^\dagger = X_L \cup \{\dagger\}$ , and complete the finite sequence  $\mathbf{x}_1, \dots, \mathbf{x}_{N(t)}$  by the infinite sequence  $\dagger, \dagger, \dots$ , to get an element of  $(X_L^\dagger)^\mathbb{N}$ , the state space being then  $\mathbb{N} \times (X_L^\dagger)^\mathbb{N}$ . We want to compare the law of these processes, *i.e.* to compute

$$\mathbb{P}(N(t) \in E, \mathbf{x}_{k_1} \in A_1, \dots, \mathbf{x}_{k_n} \in A_n),$$

where  $E \subset \mathbb{N}$  and  $A_1, \dots, A_n \subset X_L^\dagger$ . Clearly, it is sufficient to compute

$$\mathbb{P}(N(t) \geq n, \mathbf{x}_1 \in A_1, \dots, \mathbf{x}_n \in A_n), \quad (4.22)$$

where  $A_1, \dots, A_n \subset X_L$ . Now that we have reached this point, the question is the following one: show that the value of (4.22) is the same in the two following cases:

- Case 1. Given some random variables  $T_{\mathbf{x}}^n$  indexed by  $\mathbf{x} \in X_L$  and  $n \in \mathbb{N}^*$ , the whole family of random variables that it constitutes being independent, set  $\hat{T}^n = \min(T_{\mathbf{x}}^n; \mathbf{x} \in X_L)$ ,  $N(t) = \max\{n \geq 1; \hat{T}^1 + \dots + \hat{T}^n \leq t\}$  (with the convention that  $\max(\emptyset) = 0$ ), and, if  $N(t) = n$ ,  $\mathbf{x}_k = \operatorname{argmin}(\hat{T}^k)$ ,  $1 \leq k \leq n$ .

- Case 2. The process  $(N(t))$  is a Poisson process of parameter  $L$ , and, given that  $N(t) \geq n$ ,  $\mathbf{x}_1, \dots, \mathbf{x}_n$ , are uniform on  $X_L$  and independent. Precisely, this means that

$$(4.22) = \sum_{k \geq n} e^{-Lt} \frac{(Lt)^k}{k!} \frac{|A_1|}{L} \dots \frac{|A_n|}{L}, \quad (4.23)$$

where  $|A|$  is the number of elements in  $A$ .

The solution to Exercise 4.2 is [here](#).

### 4.3 Deterministic limit

The result of Theorem (4.1) is a kind of law of large numbers (a “functional law of large numbers”). Indeed, let us introduce the average  $\langle H_t \rangle = \mathbb{E}[H_t]$ . The convergence (4.12) is a consequence of these two following facts:

1. after change of scale, the symmetric simple exclusion process is close to its average value with high probability: for all  $\delta > 0$ ,

$$\lim_{L \rightarrow +\infty} \mathbb{P} \left( \sup_{t \in [0, T]} \|\hat{H}_{L^2 t} - \langle \hat{H}_{L^2 t} \rangle\|_{L^2(0,1)} > \delta \right) = 0, \quad (4.24)$$

2. we have the deterministic convergence

$$\lim_{L \rightarrow +\infty} \sup_{t \in [0, T]} \|h(t, \cdot) - \langle \hat{H}_{L^2 t} \rangle\|_{L^2(0,1)} = 0, \quad (4.25)$$

where  $h$  is the solution to (4.9)-(4.10)-(4.11).

In this section, we will establish the convergence (4.25). Before we proceed, let us study  $\langle H_t \rangle$  more closely. Given  $\mathbf{x} \in \{1, \dots, L-1\}$ , we consider the evaluation map  $\pi_{\mathbf{x}}: H \mapsto H(\mathbf{x})$ . We have  $\langle H_t(\mathbf{x}) \rangle = \mathbb{E}\pi_{\mathbf{x}}(H_t) = P_t \pi_{\mathbf{x}}(H)$ . By definition of the generator  $\mathcal{L}$  the derivative in time is  $\partial_t \langle H_t(\mathbf{x}) \rangle = P_t \mathcal{L} \pi_{\mathbf{x}}(H)$ . The explicit formula (4.21) gives  $\mathcal{L} \pi_{\mathbf{x}}(H) = \Delta_D H(\mathbf{x})$ . By linearity of the operator  $\Delta_D$ , we deduce that  $\partial_t \langle H_t(\mathbf{x}) \rangle = \Delta_D \langle H_t(\mathbf{x}) \rangle$ . After examination of the various boundary conditions, we conclude that  $\langle H_t \rangle$  is solution to the following problem:

$$\partial_t \langle H_t \rangle - \Delta_D \langle H_t \rangle = 0 \text{ in } (0, +\infty) \times \{1, \dots, L-1\}, \quad (4.26)$$

$$\langle H_t(0) \rangle = 0 \text{ for all } t \in (0, +\infty), \quad (4.27)$$

$$\langle H_0(\mathbf{x}) \rangle = H_{\text{in}}(\mathbf{x}) \text{ for all } \mathbf{x} \in \{1, \dots, L-1\}. \quad (4.28)$$

Different approaches to the convergence result (4.25) are possible. Our proof will be based on a spectral decomposition that will be exploited also to establish the averaging property (4.24) in Section 4.4.

**Proposition 4.4** (Spectral basis). *The Laplace operator with homogeneous Dirchlet boundary conditions in dimension 1, which is the operator  $-\partial_x^2$ , with domain*

$$\mathcal{D}(-\partial_x^2) = \{h \in H^2(0, 1); h(0) = h(1) = 0\}$$

*admits a spectral basis  $(a_k)_{k \in \mathbb{N}^*}$ , where  $a_k(x) = \sqrt{2} \sin(\pi k x)$ . This constitutes an orthonormal basis of  $L^2(0, 1)$ . The eigenvalue associated to  $a_k$  is  $\mu_k = \pi^2 k^2$ .*

*Let  $E_L^{(0)}$  be the subset of  $E_L$  constituted of the functions  $H$  such that  $H(0) = 0$ . The discrete Laplace operator  $-\Delta_D: E_L^{(0)} \rightarrow E_L^{(0)}$  is self-adjoint and admits the spectral basis  $(\check{a}_k)_{1 \leq k \leq L-1}$ . The eigenvalue associated to  $\check{a}_k$  is  $\nu_k = 4 \sin^2\left(\frac{\pi k}{2L}\right)$ .*

*Proof of Proposition 4.4.* We simply give the proof of some assertions about the discrete case. If we extend any  $H \in E_L^{(0)}$  to the value  $L$  by setting  $H(L) = 0$ , then for  $H, G$  in  $E_L^{(0)}$ , we easily check that  $-\Delta_D H = D_- \circ D_+ H$ , where

$$D_+ H(\mathbf{x}) = H(\mathbf{x} + 1) - H(\mathbf{x}), \quad D_- H(\mathbf{x}) = H(\mathbf{x}) - H(\mathbf{x} - 1).$$

Then we use the formula  $\langle D_- H, G \rangle_{E_L} = -\langle H, D_+ G \rangle_{E_L}$  to get

$$\langle -\Delta_D H, G \rangle_{E_L} = \langle D_+ H, D_+ G \rangle_{E_L} = \langle H, -\Delta_D G \rangle_{E_L}. \quad (4.29)$$

This shows that  $-\Delta_D$  is self-adjoint. The elementary trigonometry formula

$$1 - \cos(2a) = 2 \sin^2(a) \quad (4.30)$$

gives  $-\Delta_D \check{a}_k = \nu_k \check{a}_k$ , with

$$\nu_k = (e^{i\pi k/L} + e^{-i\pi k/L} - 2) = 4 \sin^2\left(\frac{\pi k}{2L}\right).$$

Let  $1 \leq k, l \leq L-1$ . Using (4.29) and the fact that  $\nu_k \neq \nu_l$  if  $k \neq l$ , we obtain the orthogonality relation  $\langle \check{a}_k, \check{a}_l \rangle_{E_L} = 0$  when  $k \neq l$ . If  $k = l$ , the trigonometric identity (4.30) gives

$$\langle \check{a}_k, \check{a}_k \rangle_{E_L} = L^{-1} \sum_{\mathbf{x} \in X_L} (1 - \cos(2\pi k \mathbf{x}/L)) = 1 - L^{-1} \operatorname{Re} \left( \sum_{\mathbf{x}=0}^{L-1} e^{2i\pi k \mathbf{x}/L} \right) = 1.$$

The family  $(\check{a}_k)_{1 \leq k \leq L-1}$  is free since  $\langle \check{a}_k, \check{a}_l \rangle_{E_L} = \delta_{kl}$ . It constitutes a basis of  $E_L^{(0)}$  hence, since, clearly,  $\dim(E_L^{(0)}) = L-1$ . This concludes the proof.  $\square$

Regularity of functions can be expressed in terms of decay of the ‘‘Fourier’’ coefficients. This is what accounts for the following result.

**Lemma 4.5.** *Let  $h_{\text{in}}$  be a 1-Lipschitz continuous function on  $[0, 1]$  vanishing at 0 and 1. Let  $h$  be the unique solution to (4.9)-(4.10)-(4.11) (see (4.8)). Then*

$$\sup_{t \in (0, T)} |\langle h(t, \cdot), a_k \rangle_{L^2(0,1)}| \leq \frac{\sqrt{2}}{\pi k}, \quad (4.31)$$

for all  $k \geq 1$ , where  $(a_k)_{k \in \mathbb{N}^*}$  is the orthonormal basis defined in Proposition 4.4.

*Proof of Lemma 4.5.* At time  $t$ , we have  $\langle h(t, \cdot), a^k \rangle_{L^2(0,1)} = e^{-\mu_k t} \langle h_{\text{in}}, a^k \rangle_{L^2(0,1)}$ . It is sufficient to consider the case  $t = 0$ . The estimate (4.31) then follows from the fact that the function  $h_{\text{in}}$  is 1-Lipschitz continuous. Indeed, integration by parts gives

$$\langle h_{\text{in}}, a^k \rangle_{L^2(0,1)} = -\langle h'_{\text{in}}, A^k \rangle_{L^2(0,1)}, \quad A_k(x) = \int_0^x a_k(y) dy = \frac{\sqrt{2}}{\pi k} (1 - \cos(\pi k x)),$$

and the bound  $|\langle h_{\text{in}}, a^k \rangle_{L^2(0,1)}| \leq \sqrt{2}/\pi k$ .  $\square$

We need a result similar to Lemma 4.5 on functions of the discrete variable  $\mathbf{x} \in X_L$ .

**Lemma 4.6.** *There exists a constant  $C \geq 0$  such that*

$$|\langle H, \check{a}_k \rangle_{E_L}| \leq \frac{C}{k}, \quad |\langle \hat{H}, a_k \rangle_{L^2(0,1)}| \leq \frac{C}{k}, \quad (4.32)$$

for all  $H \in E_L^{(1)}$ , for all  $k \in \{1, \dots, L-1\}$ , where  $(a_k)_{k \in \mathbb{N}^*}$  is the orthonormal basis defined in Proposition 4.4. One can take  $C = \sqrt{2}$ .



*Proof of Lemma 4.6.* This time we use a discrete integration by parts:

$$\langle H, \check{a}_k \rangle_{E_L} = \frac{1}{L^3} \sum_{\mathbf{x}=1}^{L-1} (H(\mathbf{x}) - H(\mathbf{x}-1)) B_k(\mathbf{x}), \quad (4.33)$$

where  $B_k(\mathbf{x}) := \sqrt{2}L \sum_{y=\mathbf{x}}^{L-1} \sin(k\pi y/L)$  satisfies  $|B_k(\mathbf{x})| \leq \sqrt{2}L^2 k^{-1}$  for all  $\mathbf{x} \in X_L$ . Indeed, we compute

$$|B_k(\mathbf{x})| = \sqrt{2}L \left| \operatorname{Im} \sum_{y=\mathbf{x}}^{L-1} e^{i\pi k y/L} \right| \leq \sqrt{2}L \left| \frac{1 - e^{i\pi k(L-\mathbf{x})/L}}{1 - e^{i\pi k/L}} \right| \leq \frac{2L\sqrt{2}}{|1 - e^{i\pi k/L}|}.$$

We have

$$|1 - e^{i\pi k/L}| = 2 \sin(\pi k/(2L)) \geq 2k/L,$$

since  $\sin(x) \geq (2/\pi)x$  if  $x \in [0, \pi/2]$ , which gives  $|B_k(\mathbf{x})| \leq \sqrt{2}L^2 k^{-1}$  as desired. The product  $\langle \hat{H}, a_k \rangle_{L^2(0,1)}$  satisfies an identity similar to (4.33), with

$$B_k(\mathbf{x}) := L^2 \sqrt{2} \sum_{y=\mathbf{x}}^{L-1} \int_{y/L}^{(y+1)/L} \sin(\pi k z) dz.$$

Using the bound

$$|B_k(\mathbf{x})| = \frac{\sqrt{2}L^2}{\pi k} |\cos(\pi k) - \cos(\pi k \mathbf{x}/L)| \leq \frac{2\sqrt{2}L^2}{\pi k},$$

we obtain the second estimate in (4.32).  $\square$

*Proof of the convergence (4.25).* Let  $K(L)$  satisfy

$$\lim_{L \rightarrow +\infty} K(L) = +\infty, \quad K(L) = o(L^{1/3}).$$

By Lemma 4.5, Lemma 4.6 and the Parseval identity, we have

$$\sup_{t \in [0, T]} \|h(t, \cdot) - \langle \hat{H}_{L^2 t} \rangle_{L^2(0,1)}\|_{L^2(0,1)}^2 = \sup_{t \in [0, T]} \sum_{k=1}^{K(L)} \left| \langle h(t, \cdot) - \langle \hat{H}_{L^2 t} \rangle, a_k \rangle_{L^2(0,1)} \right|^2 + o(1) \quad (4.34)$$

when  $L \rightarrow +\infty$ . We will show that (4.34) can be approached, still with an  $o(1)$  error, by

$$\sup_{t \in [0, T]} \sum_{k=1}^{K(L)} \left| \langle h(t, \cdot), a_k \rangle_{L^2(0,1)} - \langle \langle H_{L^2 t} \rangle, \check{a}_k \rangle_{E_L} \right|^2. \quad (4.35)$$

We use (4.5) and the fact that  $H(\mathbf{x}) \leq L$  for all  $\mathbf{x} \in X_L$  when  $H \in E_L^{(1)}$ . Since  $\operatorname{Lip}(a_k) = \mathcal{O}(k)$ , we obtain

$$\sum_{k=1}^{K(L)} \left| \langle \langle \hat{H}_{L^2 t} \rangle, a_k \rangle_{L^2(0,1)} - \langle \langle H_{L^2 t} \rangle, \check{a}_k \rangle_{E_L} \right|^2 = \sum_{k=1}^{K(L)} |\mathcal{O}(L^{-1}k)|^2 = \mathcal{O}(L^{-2}K(L)^3) = o(1) \quad (4.36)$$

where the  $\mathcal{O}$  and  $o$  are uniform in  $t \in [0, T]$ . By Proposition 4.4, (4.35) is equal to

$$\sup_{t \in [0, T]} \sum_{k=1}^{K(L)} \left| e^{-\mu_k t} \langle h_{\text{in}}, a_k \rangle_{L^2(0,1)} - e^{-L^2 \nu_k t} \langle H_{\text{in}}, \check{a}_k \rangle_{E_L} \right|^2. \quad (4.37)$$

Let us now compare  $\langle h_{\text{in}}, a_k \rangle_{L^2(0,1)}$  to  $\langle H_{\text{in}}, \check{a}_k \rangle_{E_L}$ . By (4.6), we have

$$\sup_{x \in [0,1]} |\hat{h}_{\text{in}}(x) - \hat{H}_{\text{in}}(x)| \leq L^{-1} \sup_{\mathbf{x} \in X_L} |\check{h}_{\text{in}}(\mathbf{x}) - H_{\text{in}}(\mathbf{x})| \leq L^{-1}.$$

We have also

$$\sup_{x \in [0,1]} |\hat{h}_{\text{in}}(x) - h_{\text{in}}(x)| = \sup_{x \in [0,1]} |h_{\text{in}}([xL]/L) - h_{\text{in}}(x)| \leq L^{-1},$$

since  $h_{\text{in}}$  is 1-Lipschitz continuous. Finally, we can estimate the  $L^2$ -norm by the  $L^\infty$ -norm to obtain  $\|h_{\text{in}} - \hat{H}_{\text{in}}\|_{L^2(0,1)} \leq 2L^{-1}$  and

$$|\langle h_{\text{in}}, a_k \rangle_{L^2(0,1)} - \langle \hat{H}_{\text{in}}, a_k \rangle_{L^2(0,1)}| \leq 2L^{-1}. \quad (4.38)$$

By (4.5), we also have  $\langle H_{\text{in}}, \check{a}_k \rangle_{E_L} = \langle \hat{H}_{\text{in}}, a_k \rangle_{L^2(0,1)} + \mathcal{O}(L^{-1}k)$ . An estimate similar to (4.36) gives then shows that (4.37) is equal (up to  $o(1)$ ) to

$$\sup_{t \in [0,T]} \sum_{k=1}^{K(L)} \left| e^{-\mu_k t} - e^{-L^2 \nu_k t} \right|^2 \langle h_{\text{in}}, a_k \rangle_{L^2(0,1)}^2. \quad (4.39)$$

At that point, we need to compare the eigenvalues  $\mu_k$  to the rescaled eigenvalues  $L^2 \nu_k$ . The two standard inequalities  $\frac{2}{\pi}x \leq \sin(x) \leq x$ ,  $|\sin(x) - x| \leq x^3$ , for  $0 \leq x \leq \frac{\pi}{2}$ , have the consequence that there exists a constant  $C \geq 0$  such that

$$\frac{4}{\pi^2} \mu_k \leq L^2 \nu_k \leq \mu_k, \quad \mu_k - L^2 \nu_k \leq C \frac{k^3}{L}, \quad (4.40)$$

for all  $k \in \{1, \dots, L-1\}$ . We deduce that (4.39) is bounded from above by

$$C^2 T L^{-2} K(L)^6 \|h_{\text{in}}\|_{L^2(0,1)}^2,$$

which is  $o(1)$  since  $K(L) = o(L^{1/3})$  by hypothesis. This concludes the proof.  $\square$

#### 4.4 Averaging

In this section, we will establish the convergence (4.24). We use (4.3) and Proposition 4.4, which give

$$\|\hat{H}_{L^2 t} - \hat{H}_{L^2 t}\|_{L^2(0,1)}^2 = \sum_{k=1}^{L-1} |\langle H_{L^2 t} - \langle H_{L^2 t} \rangle, \check{a}_k \rangle_{E_L}|^2. \quad (4.41)$$

We need to analyze the behavior on  $[0, L^2 T]$  of the process  $\langle H_t, \check{a}_k \rangle_{E_L}$ , which is of the form  $\varphi_k(H_t)$ , with  $\varphi_k(H) = \langle H, \check{a}_k \rangle_{E_L}$ . The formula (4.21) for the generator  $\mathcal{L}$  of  $(H_t)$  gives  $\mathcal{L} \varphi_k(H) = \langle \Delta_D H, \check{a}_k \rangle_{E_L}$ . By Proposition 4.4 and the fact that  $\Delta_D$  is self-adjoint on  $E_L^{(0)}$ , we obtain  $\mathcal{L} \varphi_k(H) = -\nu_k \varphi_k(H)$ , when  $H \in E_L^{(0)}$ . Let us then apply the corollary 3.2 with  $\psi(t, H) = e^{\nu_k t} \varphi_k(H)$ . The quantity  $(\partial_t + \mathcal{L})\psi$  vanishes and we obtain that

$$M_t^{(k)} := e^{\nu_k t} \langle H_t, \check{a}_k \rangle_{E_L} - \langle H_{\text{in}}, \check{a}_k \rangle_{E_L} \quad (4.42)$$

and

$$Z_t^{(k)} := \left| M_t^{(k)} \right|^2 - \int_0^t e^{2\nu_k s} (\mathcal{L} |\varphi_k|^2 - 2\varphi_k \mathcal{L} \varphi_k)(H_s) ds \quad (4.43)$$

are both martingales. Since  $t \mapsto \mathbb{E} \left[ M_t^{(k)} \right]$  is constant, by the martingale property, and vanishes at  $t = 0$ , we have

$$0 = \mathbb{E} \left[ M_t^{(k)} \right] = e^{\nu_k t} \langle \langle H_t \rangle, \check{a}_k \rangle_{E_L} - \langle H_{\text{in}}, \check{a}_k \rangle_{E_L}.$$

Consequently our quantity of interest is  $\langle H_t - \langle H_t \rangle, \check{a}_k \rangle_{E_L} = e^{-\nu_k t} M_t^{(k)}$ . We can use the Doob's martingale inequality (Theorem 2.2 with  $p = 2$ ), and the trivial bound  $e^{-\nu_k t} \leq 1$ , to obtain the estimate

$$\mathbb{P} \left( \sup_{t \in [0, L^2 T]} |\langle H_t - \langle H_t \rangle, \check{a}_k \rangle_{E_L}| > a \right) \leq \frac{4}{a^2} \mathbb{E} |M_{L^2 T}^{(k)}|^2. \quad (4.44)$$

Since  $\mathbb{E} \left[ Z_t^{(k)} \right] = 0$ , (4.43) gives us the bound from above

$$\mathbb{P} \left( \sup_{t \in [0, L^2 T]} |\langle H_t - \langle H_t \rangle, \check{a}_k \rangle_{E_L}| > a \right) \leq \frac{4}{a^2} \int_0^{L^2 T} e^{2\nu_k s} (\mathcal{L} |\varphi_k|^2 - 2\varphi_k \mathcal{L} \varphi_k)(H_s) ds. \quad (4.45)$$

We will compute the ‘‘carré du champ’’  $\mathcal{L} |\varphi_k|^2 - 2\varphi_k \mathcal{L} \varphi_k$  to understand better what gives (4.45). Before we start, let us pause a moment to consider the inequalities that we have used. We come back to (4.44) in particular, where we have discarded the term  $e^{-\nu_k t}$ . We may have lost something here. If  $k$  is  $\mathcal{O}(1)$ , then  $\nu_k$  is of order  $L^{-2}$  for  $L$  large, and  $t \mapsto e^{-\nu_k t}$  is not smaller than a given positive constant on the time interval  $[0, L^2 T]$ . If  $k$  takes greater values, then things are different. However, as soon as  $k \geq K(L)$ , where  $K(L)$  is a quantity that grows to  $+\infty$  with  $L$ , but possibly very slowly, we can use the bound of Lemma 4.6 to get the estimate

$$\sum_{k=K(L)}^{L-1} |\langle H_{L^2 t} - \langle H_{L^2 t} \rangle, \check{a}_k \rangle_{E_L}|^2 \leq C^2 \sum_{k \geq K(L)} \frac{1}{k^2} \leq C^2 K(L)^{-1}. \quad (4.46)$$

We have only to consider the indexes  $k \leq K(L)$  hence. If this is not exactly a bounded range of indexes, we will see that the loss of the  $e^{-\nu_k t}$  factor in (4.44) is not a problem.

We go back to the computation of the carré du champ now. We can write  $\varphi_k$  as the sum over  $\mathbf{x} \in \{1, \dots, L-1\}$  of  $L^{-3} \check{a}_k(\mathbf{x}) \pi_{\mathbf{x}}$ , where  $\pi_{\mathbf{x}}$  is the evaluation at  $\mathbf{x}$ . We need to compute  $\mathcal{L}(\pi_{\mathbf{x}} \otimes \pi_{\mathbf{y}})$  therefore, where  $\pi_{\mathbf{x}} \otimes \pi_{\mathbf{y}}(H) := H(\mathbf{x})H(\mathbf{y})$ . By (4.21), this is

$$\mathcal{L}(\pi_{\mathbf{x}} \otimes \pi_{\mathbf{y}})(H) = \sum_{z=0}^{L-1} [(H(\mathbf{x}) + \delta_z(\mathbf{x}) \Delta_D H(\mathbf{x}))(H(\mathbf{y}) + \delta_z(\mathbf{y}) \Delta_D H(\mathbf{y})) - H(\mathbf{x})H(\mathbf{y})],$$

which is equal to  $H(\mathbf{y}) \Delta_D H(\mathbf{x}) + H(\mathbf{x}) \Delta_D H(\mathbf{y})$  if  $\mathbf{x} \neq \mathbf{y}$ , and to  $2H(\mathbf{x}) \Delta_D H(\mathbf{x}) + |\Delta_D H(\mathbf{x})|^2$  if  $\mathbf{x} = \mathbf{y}$ . We obtain

$$(\mathcal{L} |\varphi_k|^2 - 2\varphi_k \mathcal{L} \varphi_k)(H) = \frac{1}{L^6} \sum_{\mathbf{x}=1}^{L-1} |\check{a}_k(\mathbf{x})|^2 |\Delta_D H(\mathbf{x})|^2. \quad (4.47)$$

If  $H \in E_L^{(1)}$ , then  $|\Delta_D H(\mathbf{x})| \leq 2$  for all  $\mathbf{x}$ . This shows that the right-hand side of (4.47) is bounded by  $4L^{-3} \|\check{a}_k\|_{E_L}^2$ . Since  $\check{a}_k$  is normalized, we conclude finally that

$$0 \leq (\mathcal{L} |\varphi_k|^2 - 2\varphi_k \mathcal{L} \varphi_k)(H) \leq 4L^{-3}, \quad (4.48)$$

for all  $H \in E_L^{(1)}$ . Let  $\theta \in (0, 1/2)$  be fixed and let  $A_L$  denote the event

$$A_L = \bigcap_{1 \leq k < K(L)} \left\{ \sup_{t \in [0, L^2 T]} |\langle H_t - \langle H_t \rangle, \check{a}_k \rangle_{E_L}| \leq L^{-\theta} \right\}.$$

Let us choose  $K(L) = (\log(L))^{1/3}$ . We will show that we have then  $\lim_{L \rightarrow +\infty} \mathbb{P}(A_L) = 1$ . By (4.41) and (4.46) and, we see that

$$\sup_{t \in [0, T]} \|H_{L^2 t} - \langle H_{L^2 t} \rangle\|_{L^2(0,1)}^2 \leq C^2 (\log(L))^{-1/3} + (\log(L))^{1/3} L^{-2\theta},$$

when  $A_L$  is realized, so it is clearly sufficient to prove  $\lim_{L \rightarrow +\infty} \mathbb{P}(A_L) = 1$  to get the desired result. The union bound gives

$$\mathbb{P}(A_L^c) \leq \sum_{1 \leq k < K(L)} \mathbb{P} \left( \sup_{t \in [0, L^2 T]} |\langle H_t - \langle H_t \rangle, \check{a}_k \rangle_{E_L}| > L^{-\theta} \right).$$

Using (4.45) and (4.48), we obtain

$$\mathbb{P}(A_L^c) \leq 16L^{2\theta} \sum_{1 \leq k < K(L)} \int_0^{L^2 T} e^{\nu_k s} L^{-3} ds \leq 16L^{2\theta-1} \sum_{1 \leq k < K(L)} \frac{e^{\nu_k L^2 T}}{\nu_k L^2}. \quad (4.49)$$

From the inequality  $\frac{2}{\pi}x \leq \sin(x) \leq x$  for  $0 \leq x \leq \frac{\pi}{2}$ , we infer that  $\nu_k L^2$  is bounded between  $16k^2$  and  $4\pi^2 k^2$ . We deduce then from (4.49) that

$$\mathbb{P}(A_L^c) \leq L^{2\theta-1} \sum_{1 \leq k < K(L)} \frac{e^{4\pi^2 k^2 T}}{k^2} \leq S L^{2\theta-1} e^{4\pi^2 T (\log(L))^{2/3}},$$

where  $S = \sum_{k \geq 1} k^{-2} = \pi^2/6$  is finite. This shows that  $\lim_{L \rightarrow +\infty} \mathbb{P}(A_L) = 1$ , as required.

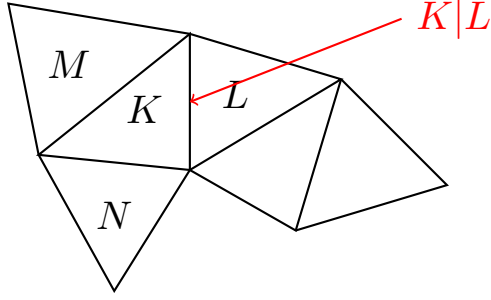


Figure 2: A mesh in  $\mathbb{R}^2$

## 5 Conservation laws and the Finite Volume method

### 5.1 Discrete conservation laws, continuous limit

We go back to Section 1.1 of the introductory part. We considered a discrete evolution equation

$$u_K^{n+1} = u_K^n + \frac{\Delta t_n}{|K|} \sum_{L \in \mathcal{N}(K)} |K|L| Q_{L \rightarrow K}^n. \quad (5.1)$$

The quantity  $u_K^n$  represents the density of a certain extensive quantity  $u$  in the space-time cell  $K \times (t_n, t_{n+1})$ . The time grid is constituted from the discrete times  $t_0 < t_1 < \dots < t_n < \dots$ , where  $t_n = n\Delta t$ ,  $n \in \mathbb{N}$  for a fixed time-step  $\Delta t$ . The space  $\mathbb{R}^d$  is partitioned as follows: we are given a family  $\mathcal{T}$  of disjoint open sets such that:

- for all distinct  $K, L \in \mathcal{T}$ , the interface  $\overline{K} \cap \overline{L}$  is contained in an hyperplane of  $\mathbb{R}^d$ ,
- up to a negligible set for the  $d$ -dimensional Lebesgue measure, the union of the sets  $K$  in  $\mathcal{T}$  is equal to  $\mathbb{R}^d$ .

We also use the following notations:

- $K|L$  is the intersection  $\overline{K} \cap \overline{L}$ ,
- $|K|$  is the  $d$ -dimensional Lebesgue measure of  $K$  and  $|K|L|$  is the  $d-1$ -dimensional Lebesgue measure of  $K|L$ ,
- $\mathcal{N}(K) = \{L \in \mathcal{T}; 0 < |K|L| < +\infty\}$  is the set of neighbors of  $K$ ,
- when  $K \in \mathcal{T}$  and  $L \in \mathcal{N}(K)$ ,  $n_{K \rightarrow L}$  is the outward unit normal to  $K$  along  $K|L$  and  $Q_{K \rightarrow L}^n$  is some numerical flux,  $\Delta t Q_{K \rightarrow L}^n$  representing the amount of  $u$  that has passed from  $K$  to  $L$  through the interface  $K|L$  on the time interval  $(t_n, t_{n+1})$ .

In the introductory section 1.1, we also assumed that the condition

$$Q_{L \rightarrow K}^n = -Q_{K \rightarrow L}^n, \quad (5.2)$$

for all  $n \in \mathbb{N}$ , for all  $K, L \in \mathcal{T}$  being neighbors, is satisfied. The condition (1.3) ensures that, in the time interval  $(t_n, t_{n+1})$ , the (algebraic) quantity of  $u$  transferred from the cell  $K$  to the cell

$L$  is the exact opposite of the quantity of  $u$  transferred from  $L$  to  $K$ : no loss or creation of  $u$  occurs at the interface  $K|L$ . Define

$$h = \sup_{K \in \mathcal{T}} \text{diam}(K), \quad u_{h,\Delta t} = \sum_{n \in \mathbb{N}} \sum_{K \in \mathcal{T}} u_K^n \mathbf{1}_{K \times (t_n, t_{n+1})}. \quad (5.3)$$

Under some additional conditions on the discrete fluxes  $Q_{K \rightarrow L}^n$ , we will study the limit when  $h, \Delta t \rightarrow 0$  of  $u_{h,k}$ . We will show that we obtain in the limit a conservation law

$$\partial_t u + \text{div}_x(Q) = 0, \quad (5.4)$$

where  $Q = Q(x, t)$ . There are various instances of such conservation laws. For example the heat equation  $\partial_t u - \text{div}(K \nabla u) = 0$  or the diffusion equation  $\partial_t u - \text{div}(D \nabla u) = 0$ , the flux being then given by the Fourier law,  $Q = -K \nabla_x u$ , or the Fick law,  $Q = -D \nabla u$  respectively. An other example is the continuity equation

$$\partial_t u + \text{div}_x(au) = 0, \quad (5.5)$$

where  $a$  is a vector-field over  $\mathbb{R}^d$ . The continuity equation can be rewritten

$$\partial_t u + a \cdot \nabla_x u + \text{div}_x(a)u = 0, \quad (5.6)$$

and coincides with the transport equation  $\partial_t u + a \cdot \nabla_x u = 0$  when  $a$  is divergence-free. We can also mention the Fokker-Planck equation of the kinetic theory of gases,

$$\partial_t f + v \cdot \nabla_x f + F(x) \cdot \nabla_v f = \text{div}_v(\nabla_v f + vf), \quad (5.7)$$

which is of the form (1.4), or more precisely  $\partial_t f + \text{div}_{x,v}(Q) = 0$ , with a flux

$$Q = \begin{pmatrix} vf \\ F(x)f - (\nabla_v f + vf) \end{pmatrix}.$$

In all these examples, the equations are linear. We can also consider the non-linear equations

$$\partial_t u - \Delta \phi(u) = 0, \quad (5.8)$$

or

$$\partial_t u + \text{div}_x(A(u)) = 0, \quad (5.9)$$

where  $A: \mathbb{R} \rightarrow \mathbb{R}^d$ . The hydrodynamic limits of particles in stochastic interaction that we will consider later can be of very different types, including in particular (5.8) and (5.9). Although both (5.8) and (5.9) may be considered in our framework, we will restrict our attention to models with (5.9) as continuous limit. We refer to [10] for the derivation of (5.8).

## 5.2 Discrete fluxes

We will begin this section with a discussion on expected discrete fluxes in some specific situations, before giving the description of our general framework.

### 5.3 Discrete fluxes for linear equations

Consider the continuity equation (5.5). Assume for simplicity that the vector field  $a$  is constant. Then (5.5) is equivalent to the transport equation (5.6). What one would observe by looking at the behavior of the solution to (5.6) on the interface  $K|L$  between the times  $t_n$  and  $t_{n+1}$  is a flow of  $u$  across  $K|L$  in the direction  $u$ . Let  $n_{K \rightarrow L}$  denote the unit normal to  $K$  along  $K|L$  in the direction of  $L$ . The value of  $|a \cdot n_{K \rightarrow L}|$  ponders the amplitude of the flux across  $K|L$ , while the sign of  $a \cdot n_{K \rightarrow L}$  determines the direction of the flow of  $u$  across  $K|L$ . It is quite natural then to set  $Q_{K \rightarrow L}^n = a \cdot n_{K \rightarrow L} u_K^n$  if  $a \cdot n_{K \rightarrow L} \geq 0$ . The condition of conservation (5.2) will be satisfied then if we also set  $Q_{K \rightarrow L}^n = -a \cdot n_{K \rightarrow L} u_L^n$  when  $a \cdot n_{K \rightarrow L} \leq 0$ . This can be summed up in the formula

$$Q_{K \rightarrow L}^n = (a \cdot n_{K \rightarrow L})^+ u_K^n - (a \cdot n_{K \rightarrow L})^- u_L^n. \quad (5.10)$$

A generalization of (5.11) in the case where  $a$  is a non-constant vector field is

$$Q_{K \rightarrow L}^n = a_{K \rightarrow L}^+ u_K^n - a_{K \rightarrow L}^- u_L^n, \quad (5.11)$$

where

$$a_{K \rightarrow L} = \frac{1}{|K|L|} \int_{K|L} a(x) \cdot n_{K \rightarrow L} d\sigma(x). \quad (5.12)$$

A further generalization of (5.10) can be given in the case where the flux  $A(u)$  in (5.9) actually depends on  $x$  also and is of the form  $A(x, u) = f(u)a(x)$ , where  $f$  is a *non-decreasing* locally Lipschitz function  $\mathbb{R} \rightarrow \mathbb{R}$  and  $a: \mathbb{R}^d \rightarrow \mathbb{R}^d$  is a divergence-free smooth vector field. Indeed, (5.9) can be rewritten as the non-linear transport equation  $\partial_t u + f'(u)a \cdot \nabla_x u = 0$  and, using the definition (5.12), the sign of  $f'(u)a_{K \rightarrow L}$  is the sign of  $a_{K \rightarrow L}$  since  $f'(u) \geq 0$ . In that situation, one can consider the flux

$$Q_{K \rightarrow L}^n = a_{K \rightarrow L}^+ f(u_K^n) - a_{K \rightarrow L}^- f(u_L^n). \quad (5.13)$$

### 5.4 General monotone fluxes

Consider the case of a general flux  $A$  in (5.9). By general flux  $A$ , we mean any function  $A: \mathbb{R} \rightarrow \mathbb{R}^d$  that is locally Lipschitz continuous. Sometimes, we also consider the extension to some fluxes  $A(x, u)$  depending also on the space variable. What kind of numerical flux may be compatible with such an expected limit as (5.9)? Inspired by the examples in Section 5.3, we look for some numerical fluxes  $Q_{K \rightarrow L}^n$  given by a relation

$$Q_{K \rightarrow L}^n = A_{K \rightarrow L}(u_K^n, u_L^n), \quad (5.14)$$

where  $A_{K \rightarrow L}$  is a function with the following properties:

1. compatibility with the flux  $A$ :

$$A_{K \rightarrow L}(v, v) = A(v) \cdot n_{K \rightarrow L}, \quad (5.15)$$

for all  $v \in \mathbb{R}$ ,

2. regularity: the function  $A_{K \rightarrow L}$  is locally Lipschitz continuous: for every  $R > 0$ , there exists a constant  $L_A(R) \geq 0$  such that

$$|A_{K \rightarrow L}(v, w) - A_{K \rightarrow L}(v', w')| \leq L_A(R)(|v - v'| + |w - w'|), \quad (5.16)$$

for all  $v, v', w, w' \in [-R, R]$  and for all neighboring cells  $K, L \in \mathcal{T}$ ,

3. monotony: for all  $v, w \in \mathbb{R}$ , the function  $A_{K \rightarrow L}(v, \cdot)$  is non-increasing, while the function  $A_{K \rightarrow L}(\cdot, w)$  is non-decreasing,

4. conservation property:

$$A_{K \rightarrow L}(v, w) = -A_{L \rightarrow K}(w, v), \quad (5.17)$$

for all  $v, w \in [-R, R]$  and for all neighboring cells  $K, L \in \mathcal{T}$ .

If we choose the definition (5.14) of the flux, then (5.17) yields the conservation property (5.2). There is some redundancy in the properties required above: (5.17) and the single fact that  $A_{K \rightarrow L}(v, \cdot)$  is non-increasing implies that  $A_{K \rightarrow L}(\cdot, w)$  is non-decreasing for instance. In the next two paragraphs we infer some consequences on the discrete evolution equation (5.1) of (5.14), (5.15), (5.16), (5.17) and the monotony properties of  $A_{K \rightarrow L}$ .

**Exercise 5.1** (Godunov flux, Engquist-Osher flux). Define  $A_{K \rightarrow L}^G(v, w)$  as follows: if  $v \leq w$ , then  $A_{K \rightarrow L}^G(v, w)$  is the maximum value of  $u \mapsto A(u) \cdot n_{K \rightarrow L}$  on the interval  $[v, w]$ . If  $w \leq v$ , then  $A_{K \rightarrow L}^G(v, w)$  is the minimum value of  $u \mapsto A(u) \cdot n_{K \rightarrow L}$  on the interval  $[w, v]$ . Define also  $A_{K \rightarrow L}^{EO}(v, w)$  by the formula

$$A_{K \rightarrow L}^{EO}(v, w) = \int_0^v (a(\xi) \cdot n_{K \rightarrow L})^+ d\xi - \int_0^w (a(\xi) \cdot n_{K \rightarrow L})^- d\xi,$$

where  $a(u) = A'(u)$ . Show that  $A_{K \rightarrow L}^G$  and  $A_{K \rightarrow L}^{EO}$  have the required properties and show that they coincide with the upwind flux (5.10) in the linear case  $A(u) = au$ .

*The solution to Exercise 5.1 is here.*

## 5.5 Constants as solutions

Any constant function  $u_K^n \equiv v$  is solution to (5.1). By (5.15), we have indeed

$$\sum_{L \in \mathcal{N}(K)} |K|L|Q_{K \rightarrow L}^n = \sum_{L \in \mathcal{N}(K)} |K|L|A(v) \cdot n_{K \rightarrow L} = \sum_{L \in \mathcal{N}(K)} \int_{K|L} A(v) \cdot n_{K \rightarrow L} d\sigma(x).$$

We use the Stokes formula

$$\sum_{L \in \mathcal{N}(K)} \int_{K|L} \Psi(x) \cdot n_{K \rightarrow L} d\sigma(x) = \int_K \operatorname{div} \Psi(x) dx, \quad (5.18)$$

to obtain

$$\sum_{L \in \mathcal{N}(K)} |K|L|Q_{K \rightarrow L}^n = 0,$$

as desired.

**Exercise 5.2** (Spatially dependent flux). Assume that  $A(x, u)$  satisfies the divergence-free condition  $(\operatorname{div}_x A)(x, u) = 0$  for all  $u \in \mathbb{R}$ . Assume also that  $Q_{K \rightarrow L}^n$  is given by (5.14), where  $A_{K \rightarrow L}$  satisfies the following generalized version of (5.15):

$$A_{K \rightarrow L}(v, v) = \frac{1}{|K|L|} \int_{K|L} A(x, v) \cdot n_{K \rightarrow L} d\sigma(x). \quad (5.19)$$

Show that constant are solutions.

*The solution to Exercise 5.2 is here.*



For later use, we record the identity

$$\sum_{L \in \mathcal{N}(K)} |K|L|A_{K \rightarrow L}(u_K^n, u_K^n) = 0, \quad (5.20)$$

valid for all neighboring cells  $K, L \in \mathcal{T}$ . We can use it to transform (5.1) into the identity

$$u_K^{n+1} = u_K^n + \frac{\Delta t_n}{|K|} \sum_{L \in \mathcal{N}(K)} |K|L|[A_{K \rightarrow L}(u_K^n, u_K^n) - A_{K \rightarrow L}(u_K^n, u_L^n)]. \quad (5.21)$$

On the formula (5.21), we can see the stabilizing effect of the monotony of the numerical flux. Imagine that  $u_K^n > u_L^n$  for all neighboring cells  $L$  of  $K$ . Then

$$A_{K \rightarrow L}(u_K^n, u_K^n) - A_{K \rightarrow L}(u_K^n, u_L^n) \leq 0$$

for all  $L$  since  $A_{K \rightarrow L}$  is non-increasing in its second argument, which implies that  $u_K^{n+1} \leq u_K^n$ . The estimates in the following two sections essentially use this.

## 5.6 Comparison principle

### 5.6.1 Periodic discrete conservation law

In all that follows we will consider for simplicity a periodic setting. We assume that the mesh  $\mathcal{T}$  is periodic, in the sense that there exists a mesh  $\mathcal{T}^\sharp$  of the hypercube  $(0, 1)^d$  such every  $K \in \mathcal{T}$  is the translation of an element  $K^\sharp$  of  $\mathcal{T}^\sharp$  by a vector of  $\mathbb{Z}^d$ . We also assume that  $K \mapsto u_K^0$  is periodic, in the sense that  $K \sim L$  (where the relation of equivalence  $K \sim L$  is defined by  $K = \ell + L$ ,  $\ell \in \mathbb{Z}^d$ ) implies  $u_K^0 = u_L^0$ . This will be the case if we assume, as will be done later, that

$$\forall K \in \mathcal{T}, u_K^0 = \frac{1}{|K|} \int_K u_0(x) dx, \quad (5.22)$$

where  $u_0: \mathbb{R}^d \rightarrow \mathbb{R}$  is  $\mathbb{Z}^d$ -periodic. We denote by  $\mathbb{T}^d$  the  $d$ -dimensional torus  $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$ .

### 5.6.2 Comparison principle and consequences

The value of  $u_{h, \Delta t}(t_n, \cdot)$  in (5.3) is 0. We slightly modify the second identity of (5.3) as

$$u_{h, \Delta t} = \sum_{n \in \mathbb{N}} \sum_{K \in \mathcal{T}} u_K^n \mathbf{1}_{K \times [t_n, t_{n+1})}. \quad (5.23)$$

With this new definition, we have  $u_{h, \Delta t}(t_n, x) = u_K^n$  if  $x \in K$ . We will use the following notation also: if  $F: \mathbb{R} \rightarrow \mathbb{R}$  is continuous,

$$\int_{\mathbb{T}^d} F(u_{h, \Delta t}(t_n, x)) dx = \sum_{K \in \mathcal{T}^\sharp} |K| F(u_K^n). \quad (5.24)$$

**Proposition 5.1** ( $L^1$ -contraction). *Let  $u_{h, \Delta t}$  and  $v_{h, \Delta t}$  be two sequences defined by (5.1), with a flux given as in Section 5.4. Define*

$$R_K^n(u_{h, \Delta t}) = \max \{|u_L^n|; L \in \mathcal{N}(K) \cup \{K\}\}, \quad |\partial K| := \sum_{L \in \mathcal{N}(K)} |K|L|.$$

Let  $n \in \mathbb{N}$  be fixed, and assume that the conditions

$$2 \frac{\Delta t |\partial K|}{|K|} L_A(R_K^n(u_{h,\Delta t})) \leq 1, \quad 2 \frac{\Delta t |\partial K|}{|K|} L_A(R_K^n(v_{h,\Delta t})) \leq 1 \quad (5.25)$$

are satisfied for all  $K \in \mathcal{T}$ , where  $L_A$  is defined in (5.16). We have then

$$\int_{\mathbb{T}^d} (u_{h,\Delta t}(t_{n+1}) - v_{h,\Delta t}(t_{n+1}))^+ dx \leq \int_{\mathbb{T}^d} (u_{h,\Delta t}(t_n) - v_{h,\Delta t}(t_n))^+ dx. \quad (5.26)$$

*Remark 5.1* (CFL condition). Recall that  $h$  is defined in (5.3) by  $h = \sup_{K \in \mathcal{T}} \text{diam}(K)$ . Suppose that there exists  $\alpha > 0$  such that

$$\alpha h^d \leq |K|, \quad |\partial K| \leq \frac{1}{\alpha} h^{d-1}, \quad (5.27)$$

for all  $K \in \mathcal{T}$ . Then (5.25) is satisfied if

$$\Delta t \leq Ch, \quad (5.28)$$

where  $C = 2\alpha^{-2} L_A(R)$ , and  $R$  is a bound for all the quantities  $R_K^n(w)$ ,  $w = u_{h,\Delta t}$  or  $v_{h,\Delta t}$  (we will see soon how to ensure that  $R$  is finite). The condition (5.28) puts a constraint of the size of the time step, depending on the size of the space-step  $h$ . It is called a Courant-Friedrichs-Lewy (CFL) condition.

**Exercise 5.3** (Spatially dependent flux). Give some examples of meshes in dimension  $d = 2$  which do not satisfy one of the two bounds in (5.27).

The solution to Exercise 5.3 is [here](#).

*Proof of Proposition 5.1.* Here, and later in the analysis of (5.1), we will use the notation

$$a \wedge b = \min(a, b), \quad a \vee b = \max(a, b). \quad (5.29)$$

We have then the formula

$$(u - v)^+ = u \vee v - v, \quad (5.30)$$

for all  $u, v \in \mathbb{R}$ . Our first goal is to estimate  $u_K^{n+1} \vee v_K^{n+1}$ . Let us consider the right-hand side of (5.21). It is a non-decreasing function of the variables  $u_L^n$ ,  $L \in \mathcal{N}(K)$ . With respect to the variable  $u_K^n$ , it can be written as a sum  $\text{Id} + f$ , where  $f$  is a locally Lipschitz continuous function. On the domain where  $\text{Lip}(f) \leq 1$ , it will be also a non-decreasing function of  $u_K^n$ . Actually, our function  $f$  here is has the form  $F(u, u, u)$ , where

$$F(u_1, u_2, u_3) = \frac{\Delta t_n}{|K|} \sum_{L \in \mathcal{N}(K)} |K| |L| [A_{K \rightarrow L}(u_1, u_2) - A_{K \rightarrow L}(u_3, u_L^n)]$$

is a non-decreasing function of  $u_1$ . We are only interested in the Lipschitz dependency of this function with respect to  $u_2$  and  $u_3$ , which, using (5.16), is bounded by the first term in (5.25). To sum up, as long as the first condition in (5.25) is satisfied, we have

$$u_K^{n+1} = H_K^n(u_K^n, u_L^n; L \in \mathcal{N}(K)), \quad (5.31)$$

where  $H_K^n$  is a non-decreasing function of its arguments. We deduce, under (5.25), that

$$u_K^{n+1} \vee v_K^{n+1} \leq H_K^n(u_K^n \vee v_K^n, u_L^n \vee v_L^n; L \in \mathcal{N}(K)), \quad (5.32)$$

for all  $K \in \mathcal{T}$ . Then we use (5.30) and (5.31) to obtain the inequality

$$(u_K^{n+1} - v_K^{n+1})^+ \leq H_K^n(u_K^n \vee v_K^n, u_L^n \vee v_L^n; L \in \mathcal{N}(K)) - H_K^n(v_K^n, v_L^n; L \in \mathcal{N}(K)). \quad (5.33)$$

We write (5.33) under the form

$$\begin{aligned} & (u_K^{n+1} - v_K^{n+1})^+ \\ & \leq (u_K^n - v_K^n)^+ + \frac{\Delta t_n}{|K|} \sum_{L \in \mathcal{N}(K)} |K|L|[\Phi_{K \rightarrow L}(u_K^n, u_K^n; v_K^n, v_K^n) - \Phi_{K \rightarrow L}(u_K^n, u_L^n; v_K^n, v_L^n)], \end{aligned} \quad (5.34)$$

where

$$\Phi_{K \rightarrow L}(v, w; v', w') := A_{K \rightarrow L}(v \vee v', w \vee w') - A_{K \rightarrow L}(v', w'). \quad (5.35)$$

We multiply (5.34) by  $|K|$  and sum over  $K \in \mathcal{T}^\#$ . It gives us the desired estimate (5.26), provided we can show that

$$\sum_{K \in \mathcal{T}^\#} \sum_{L \in \mathcal{N}(K)} |K|L|[\Phi_{K \rightarrow L}(u_K^n, u_K^n; v_K^n, v_K^n) - \Phi_{K \rightarrow L}(u_K^n, u_L^n; v_K^n, v_L^n)] = 0. \quad (5.36)$$

The cancellation property (5.36) follows from the two identities

$$\sum_{L \in \mathcal{N}(K)} |K|L|[\Phi_{K \rightarrow L}(u_K^n, u_K^n; v_K^n, v_K^n) = 0, \quad \sum_{K \in \mathcal{T}^\#} \sum_{L \in \mathcal{N}(K)} |K|L|\Phi_{K \rightarrow L}(u_K^n, u_L^n; v_K^n, v_L^n) = 0. \quad (5.37)$$

The left identity in (5.37) follows from (5.20). The second identity in (5.37) is a consequence of (5.17) and of the formula

$$\sum_{K \in \mathcal{T}^\#} \sum_{L \in \mathcal{N}(K)} a(K, L) = \frac{1}{2} \sum_{K \in \mathcal{T}^\#} \sum_{L \in \mathcal{N}(K)} (a(K, L) + a(L, K)), \quad (5.38)$$

satisfied by any periodic function  $a: \mathcal{T} \times \mathcal{T} \rightarrow \mathbb{R}$ . Indeed, if  $K_* \in \mathcal{T}^\#$  and  $L_* \in \mathcal{N}(K)$ , then the term  $a(L_*, K_*)$  in the right-hand side of (5.38) will appear in the sum on the left when  $K = L_*$  and  $L = K_*$  (in the case where the interface  $K_*|L_*$  is on the boundary of  $(0, 1)^d$ , we need to use the periodic character of  $a$  to complete this argument).  $\square$

From Proposition 5.1, we deduce first a comparison principle and an  $L^\infty$  estimate.

**Proposition 5.2** (Comparison principle,  $L^1$  estimate). *Under the hypotheses of Proposition 5.1, we have*

$$\int_{\mathbb{T}^d} |u_{h, \Delta t}(t_{n+1}) - v_{h, \Delta t}(t_{n+1})| dx \leq \int_{\mathbb{T}^d} |u_{h, \Delta t}(t_n) - v_{h, \Delta t}(t_n)| dx. \quad (5.39)$$

Besides, if  $v_{h, \Delta t}(t_n) \geq u_{h, \Delta t}(t_n)$  a.e. in  $\mathbb{T}^d$ , then  $v_{h, \Delta t}(t_{n+1}) \geq u_{h, \Delta t}(t_{n+1})$  a.e. in  $\mathbb{T}^d$ .

**Proposition 5.3** (Comparison principle,  $L^\infty$  estimate). *Assume  $|u_{h, \Delta t}(0)| \leq R$  a.e. in  $\mathbb{T}^d$ . Then, under the CFL condition*

$$\forall K \in \mathcal{T}, \quad 2 \frac{\Delta t |\partial K|}{|K|} L_A(R) \leq 1, \quad (5.40)$$

*we have the  $L^\infty$  bound  $|u_{h, \Delta t}(t)| \leq R$  a.e. in  $\mathbb{T}^d$ , for a.e.  $t \geq 0$ .*

*Proof of Proposition 5.2 and Proposition 5.3.* We exchange the roles of  $u_{h, \Delta t}$  and  $v_{h, \Delta t}$  to deduce the  $L^1$ -contraction (5.39) from (5.26) and we use the fact that  $v_{h, \Delta t}(t_n) \geq u_{h, \Delta t}(t_n)$  a.e. in  $\mathbb{T}^d$  if, and only if the integral over  $\mathbb{T}^d$  of  $(u_{h, \Delta t}(t_n) - v_{h, \Delta t}(t_n))^+$  vanishes to prove the comparison principle. The  $L^\infty$  bound  $|u_{h, \Delta t}(t_n)| \leq R$  is proved by recursion on  $n$ , using the comparison principle and the fact that the constant functions  $R$  and  $-R$  are solutions of (5.1).  $\square$

### 5.6.3 Asymptotic behavior

We consider the behavior of the numerical solution to (5.1) when the characteristic scales  $h$  and  $\Delta t$  get smaller and smaller. Let  $(\Delta t_k)$  be a sequence of positive reals that converge to 0, let  $(\mathcal{T}_k)$  be a sequence of meshes that are  $\mathbb{Z}^d$ -periodic and such that  $h_k := \sup_{K \in \mathcal{T}_k} \text{diam}(K)$  tends to 0 when  $k \rightarrow +\infty$ . We assume that (5.27) is satisfied for all  $k$ , for all  $K \in \mathcal{T}_k$ , where  $\alpha$  is independent on  $k$ . We also assume that (5.16) is satisfied with a Lipschitz constant  $L_A(R)$  independent on  $k$ . Let  $(u_K^0)_{K \in \mathcal{T}_k}$  be given by (5.22), where  $u_0 \in L^\infty(\mathbb{T}^d)$ . Consider the CFL condition

$$\Delta t_k L_A(R) \leq \alpha^2 h_k, \quad (5.41)$$

where  $R \geq \|u_0\|_{L^\infty(\mathbb{T}^d)}$ . Let  $T > 0$  be fixed. By Proposition 5.3, the solution  $u_{(k)} := u_{h_k, \Delta t_k}$  of (5.1) satisfies the bound  $\|u_{(k)}\|_{L^\infty(\mathbb{T}^d \times (0, T))} \leq R$  for all  $k$ . Consequently, there is a subsequence still denoted  $(u_{(k)})$  which converges to a certain function  $u$  in  $L^\infty(\mathbb{T}^d \times (0, T))$  for the weak-\* topology. We would like to show that  $u$  is solution to the conservation law (5.9). In the case where  $A$  is not a linear function, there two difficulties to this approach:

1. we use a weak mode of convergence (convergence in  $L^\infty(\mathbb{T}^d \times (0, T))$  for the weak-\* topology), which is not sufficient in all generality to deal with the convergence of non-linear terms,
2. the theory of the Cauchy Problem for (5.9) in  $L^\infty$  requires a specific treatment, via the use of entropy solutions.

We will establish the convergence of  $(u_{(k)})$  towards a solution of (5.9) in the linear case only, see Section 5.9. Some additional estimates on  $u_{(k)}$  are necessary for this, and we will give them in the following section 5.7, for a general numerical fluxes, associated, via (5.15), to a not-necessarily linear flux  $A$ . We refer to [9, Chapter 6] for the proof of convergence of (5.1) in the general case.

## 5.7 Energy estimate

Consider the parabolic equation

$$u_t + \text{div}(A(u)) - \eta \Delta u = 0 \text{ in } \mathbb{T}^d \times (0, +\infty). \quad (5.42)$$

Here  $\eta > 0$  is supposed to be small. The flux in (5.42) is  $A(u) - \eta \nabla u$ . This is a perturbation of the flux  $A(u)$ . The addition of the term  $-\eta \nabla u$  has a stabilizing effect, of the same nature as the stabilizing effect discussed at the end of Section 5.5, in relation with the monotony properties of the numerical fluxes. In (5.42), the additional term  $-\eta \Delta u$  has a positive contribution in the energy estimate: if we multiply (5.42) by  $u$  (say, a smooth solution) and integrate over  $\mathbb{T}^d \times (0, t)$ , we obtain

$$\int_0^t \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^d} u^2 dx ds + \int_0^t \int_{\mathbb{T}^d} u \text{div}(A(u)) dx ds - \eta \int_0^t \int_{\mathbb{T}^d} u \Delta u dx ds = 0. \quad (5.43)$$

We develop the term

$$u \text{div}(A(u)) = u A'(u) \cdot \nabla u = B'(u) \cdot \nabla u = \text{div}(B(u)), \quad B'(u) := u A'(u),$$

and, using periodicity, we obtain

$$\frac{1}{2} \int_{\mathbb{T}^d} |u(x, t)|^2 dx + \eta \int_0^t \int_{\mathbb{T}^d} |\nabla u|^2 dx ds \leq \frac{1}{2} \int_{\mathbb{T}^d} |u_0|^2 dx.$$

We will establish a similar result in the discrete case.

**Proposition 5.4** (Energy estimate). *Let  $u_0 \in L^\infty(\mathbb{T}^d)$  satisfy  $\|u_0\|_{L^\infty(\mathbb{T}^d)} \leq R$ . Define*

$$\mathcal{D}(t_N) = \frac{1}{2} \sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}} \sum_{L \in \mathcal{N}(K)} |K|L \int_{u_L^n}^{u_K^n} \{A_{K \rightarrow L}(u_K^n, u_L^n) - A_{K \rightarrow L}(z, z)\} dz.$$

*Assume that the following CFL condition is satisfied: there exists  $\xi \in ]0, 1]$  such that*

$$2\Delta t \frac{|\partial K|}{|K|} L_A(R) \leq 1 - \xi, \quad \forall K \in \mathcal{T}, \quad (5.44)$$

*for all  $K \in \mathcal{T}$ . Then we have the energy estimate*

$$\frac{1}{2} \|u_{h, \Delta t}(t_N)\|_{L^2(\mathbb{T}^d)}^2 + \xi \mathcal{D}(t_N) \leq \frac{1}{2} \|u_0\|_{L^2(\mathbb{T}^d)}^2, \quad (5.45)$$

*for all  $N \geq 1$ .*

*Remark 5.2.* the term  $\mathcal{D}(t_N)$  is non-negative. Indeed, using the monotony properties of  $A_{K \rightarrow L}$ , we have  $A_{K \rightarrow L}(u_K^n, u_L^n) - A_{K \rightarrow L}(z, z) \geq 0$  if  $u_L^n \leq z \leq u_K^n$ . Similarly,  $A_{K \rightarrow L}(u_K^n, u_L^n) - A_{K \rightarrow L}(z, z) \leq 0$  if  $u_K^n \leq z \leq u_L^n$ .

*Proof of Proposition 5.4.* We multiply the identity (5.21) by  $|K|u_K^n$  and we sum the result over  $K \in \mathcal{T}^\sharp$  and  $n \in \{0, \dots, N-1\}$ . We obtain an identity  $J_{\Delta t} + J_{\Delta x} = 0$ , where

$$J_{\Delta t} = \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}^\sharp} |K| u_K^n (u_K^{n+1} - u_K^n), \quad (5.46)$$

and

$$J_{\Delta x} = \sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}^\sharp} \sum_{L \in \mathcal{N}(K)} u_K^n \{A_{K \rightarrow L}(u_K^n, u_L^n) - A_{K \rightarrow L}(u_K^n, u_K^n)\}. \quad (5.47)$$

We use the formula  $a(b-a) = \frac{b^2-a^2}{2} - \frac{(a-b)^2}{2}$ , which is the “finite difference” version of the continuous identity  $u \partial_t u = \frac{1}{2} \partial_t u^2$ . It gives

$$J_{\Delta t} = \frac{1}{2} \|u_{h, \Delta t}(t_N)\|_{L^2(\mathbb{T}^d)}^2 - \frac{1}{2} \|u_{h, \Delta t}(0)\|_{L^2(\mathbb{T}^d)}^2 - \frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}^\sharp} |K| |u_K^{n+1} - u_K^n|^2. \quad (5.48)$$

We leave as an exercise the proof that (5.22) implies  $\|u_{h, \Delta t}(0)\|_{L^2(\mathbb{T}^d)} \leq \|u_0\|_{L^2(\mathbb{T}^d)}$ . From (5.48), we deduce that

$$\frac{1}{2} \|u_{h, \Delta t}(t_N)\|_{L^2(\mathbb{T}^d)}^2 + J_{\Delta x} \leq \frac{1}{2} \|u_0\|_{L^2(\mathbb{T}^d)}^2 + \frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}^\sharp} |K| |u_K^{n+1} - u_K^n|^2 \quad (5.49)$$

The remaining term in the right-hand side of (5.49) will be absorbed in  $J_{\Delta x}$ , by means of the CFL condition. The summation formula (5.38) and the conservation property (5.17) give the following expression of  $J_{\Delta x}$ :

$$\begin{aligned} J_{\Delta x} = & \frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}^\sharp} k \sum_{L \in \mathcal{N}(K)} u_K^n \{A_{K \rightarrow L}(u_K^n, u_L^n) - A_{K \rightarrow L}(u_K^n, u_K^n)\} \\ & - u_L^n \{A_{K \rightarrow L}(u_K^n, u_L^n) - A_{K \rightarrow L}(u_L^n, u_L^n)\}. \end{aligned} \quad (5.50)$$

Denote by  $\psi_{K \rightarrow L}$  an anti-derivative of  $z \mapsto z \frac{d}{dz} A_{K \rightarrow L}(z, z)$ . Integration by parts shows that

$$\begin{aligned} \psi_{K \rightarrow L}(v) - \psi_{K \rightarrow L}(w) &= v \{A_{K \rightarrow L}(v, v) - A_{K \rightarrow L}(v, w)\} \\ &\quad - w \{A_{K \rightarrow L}(w, w) - A_{K \rightarrow L}(v, w)\} + \int_w^v \{A_{K \rightarrow L}(v, w) - A_{K \rightarrow L}(z, z)\} dz. \end{aligned} \quad (5.51)$$

Taking  $w = u_L^n$ ,  $v = u_K^n$  in (5.51) shows that

$$J_{\Delta x} = \frac{1}{2} \sum_{n=0}^{N-1} k \left\{ \sum_{K \in \mathcal{T}^\sharp} \sum_{L \in \mathcal{N}(K)} \int_{u_L^n}^{u_K^n} \{A_{K \rightarrow L}(u_K^n, u_L^n) - A_{K \rightarrow L}(z, z)\} dz + R_{\Delta x}^n \right\}, \quad (5.52)$$

where the remainder term is

$$R_{\Delta x}^n = -\frac{1}{2} \sum_{K \in \mathcal{T}^\sharp} \sum_{L \in \mathcal{N}(K)} \psi_{K \rightarrow L}(u_K^n) - \psi_{K \rightarrow L}(u_L^n).$$

The cancellation property (5.20) and (5.38) give  $R_{\Delta x}^n = 0$ . We conclude that  $J_{\Delta x} = \mathcal{D}(t_N)$ . The estimate (5.45) will be established (as a consequence of (5.49)) if we can prove that

$$\frac{1}{2} \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}^\sharp} |K| |u_K^{n+1} - u_K^n|^2 \leq (1 - \xi) J_{\Delta x}. \quad (5.53)$$

We use Equation (5.21) and the Cauchy-Schwarz inequality to get

$$|u_K^{n+1} - u_K^n|^2 \leq \frac{(\Delta t)^2 |\partial K|}{|K|^2} \sum_{L \in \mathcal{N}(K)} |K| |L| |A_{K \rightarrow L}(u_K^n, u_L^n) - A_{K \rightarrow L}(u_K^n, u_K^n)|^2.$$

The CFL condition (5.44) gives then (5.53) with a term  $J_{\Delta x}^*$  instead of  $J_{\Delta x}$ , where

$$J_{\Delta x}^* = \frac{1}{4L_A(R)} \sum_{n=0}^N \Delta t \sum_{K \in \mathcal{T}^\sharp} \sum_{L \in \mathcal{N}(K)} |K| |L| |A_{K \rightarrow L}(u_K^n, u_L^n) - A_{K \rightarrow L}(u_K^n, u_K^n)|^2.$$

To conclude, we show that  $J_{\Delta x}^* \leq \mathcal{D}(t_N)$ . To that purpose, we use the following inequality:

$$\int_0^r B(z) dz \geq \frac{1}{2 \text{Lip}(B)} B(r)^2, \quad r \in [0, R] \quad (5.54)$$

valid for any non-decreasing Lipschitz continuous function  $B$  on  $[0, R]$ . To obtain (5.54), we simply use the formula

$$B(r)^2 = 2 \int_0^r B(s) B'(s) ds,$$

and bound  $B'(s)$  by  $\text{Lip}(B)$ . Suppose  $u_K^n \geq u_L^n$  for instance. Then (5.54) applied to  $B(z) := A_{K \rightarrow L}(u_K^n, u_L^n) - A_{K \rightarrow L}(u_K^n, z + u_L^n)$  and  $r = u_K^n - u_L^n$  will give

$$|A_{K \rightarrow L}(u_K^n, u_L^n) - A_{K \rightarrow L}(u_K^n, u_K^n)|^2 \leq 2L_A(R) \int_{u_L^n}^{u_K^n} \{A_{K \rightarrow L}(u_K^n, u_L^n) - A_{K \rightarrow L}(u_K^n, z)\} dz. \quad (5.55)$$

We use the fact that  $A_{K \rightarrow L}(u_K^n, z) \geq A_{K \rightarrow L}(z, z)$  since  $u_K^n \geq z$  to get the desired identity. The reasoning in the case  $u_K^n \leq u_L^n$  is similar.  $\square$

*Remark 5.3* (Discrete  $H^1$ -estimate in the time variable). Note that (5.53) and (5.45) give the estimate

$$\sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}^\sharp} |K| |u_K^{n+1} - u_K^n|^2 \leq (1 - \xi) \mathcal{D}(t_N) \leq \frac{1 - \xi}{\xi} \|u_0\|_{L^2(\mathbb{T}^d)}^2, \quad (5.56)$$

for all  $N \geq 1$ . Note also that the inequality  $J_{\Delta x}^* \leq \mathcal{D}(t_N)$  in the proof above and (5.45) give the estimate

$$\sum_{n=0}^N \Delta t \sum_{K \in \mathcal{T}^\sharp} \sum_{L \in \mathcal{N}(K)} |K| |L| |A_{K \rightarrow L}(u_K^n, u_L^n) - A_{K \rightarrow L}(u_K^n, u_K^n)|^2 \leq \frac{4L_A(R)}{\xi} \|u_0\|_{L^2(\mathbb{T}^d)}^2, \quad (5.57)$$

for all  $N \geq 1$ .

## 5.8 Approximate weak solutions

In this section, we will prove that  $u_{(k)}$  obtained in Section 5.6.3 is an approximate weak solution of (5.9).

**Definition 5.4** (Weak solution). Let  $u_0 \in L^\infty(\mathbb{T}^d)$ , assume that  $A: \mathbb{R} \rightarrow \mathbb{R}^d$  is a locally Lipschitz continuous function. Let  $T > 0$ . A function  $u \in L^\infty(\mathbb{T}^d \times (0, T))$  is said to be a weak solution to (5.9) on  $(0, T)$  with initial datum  $u_0$  if

$$\int_0^T \int_{\mathbb{T}^d} (u \varphi_t + A(u) \cdot \nabla_x \varphi) dx dt + \int_{\mathbb{T}^d} u_0(x) \varphi(x, 0) dx = 0, \quad (5.58)$$

for all test-function  $\varphi \in C_c^\infty(\mathbb{T}^d \times [0, T])$ .

**Notation:** if  $u: \mathbb{T}^d \rightarrow \mathbb{R}$  and  $1 \leq p < +\infty$ , we denote by  $\omega_{L^p}(u; h)$  the modulus of continuity in  $L^p(\mathbb{T}^d)$ :

$$\omega_{L^p}(u; h) = \sup_{|z| \leq h} \|u - u(\cdot + z)\|_{L^p(\mathbb{T}^d)}. \quad (5.59)$$

**Theorem 5.5** (Approximate weak solutions). Let  $u_0 \in L^\infty(\mathbb{T}^d)$  and let  $R \geq \|u_0\|_{L^\infty(\mathbb{T}^d)}$ . Assume that the CFL condition (5.44) is satisfied for all  $K \in \mathcal{T}^\sharp$ . Then  $u_{h, \Delta t}$  is an approximate weak solution to (5.9) on  $(0, T)$  with initial datum  $u_0$  in the sense that

$$\left| \int_0^T \int_{\mathbb{T}^d} (u_{h, \Delta t} \varphi_t + A(u_{h, \Delta t}) \cdot \nabla_x \varphi) dx dt + \int_{\mathbb{T}^d} u_0(x) \varphi(x, 0) dx \right| \leq \langle \mu_{h, \Delta t}^0, |\varphi| \rangle + \langle \mu_{h, \Delta t}^1, |\partial_t \varphi| \rangle + \langle \mu_{h, \Delta t}^2, |\nabla_x \varphi| \rangle, \quad (5.60)$$

for all test-function  $\varphi \in C_c^\infty(\mathbb{T}^d \times [0, T])$ , where  $\mu_{h, \Delta t}^i$ ,  $i \in \{0, 1, 2\}$  are some non-negative measures on  $\mathbb{T}^d \times [0, T]$  which satisfy the estimate

$$\mu_{h, \Delta t}^i(\mathbb{T}^d \times [0, T]) \leq C(\Delta t^{1/2} + h^{1/2} + \omega_{L^1}(u_0; h)), \quad (5.61)$$

where  $C$  is a constant depending only on the dimension  $d$ , on  $T$ , on the constant  $\alpha$  in (5.27), on  $R$ , on  $L_A(R)$  (cf. (5.16)), and on the constant  $\xi$  in (5.44).

*Remark 5.4* (Entropy solutions). When  $A$  is non-linear, weak solutions to (5.58) are non unique. The Cauchy Problem for (5.9) is solved in the class of weak entropy solutions. A function

$u \in L^\infty(\mathbb{T}^d \times (0, T))$  is said to be a weak entropy solution to (5.9) on  $(0, T)$  with initial datum  $u_0$  if

$$\int_0^T \int_{\mathbb{T}^d} (\eta(u)\varphi_t + \Phi(u) \cdot \nabla_x \varphi) dx dt + \int_{\mathbb{T}^d} \eta(u_0(x))\varphi(x, 0) dx \geq 0, \quad (5.62)$$

for all non-negative test-function  $\varphi \in C_c^\infty(\mathbb{T}^d \times [0, T])$  and all entropy, entropy-flux pair  $(\eta, \Phi)$ . This means that  $\eta$  is of class  $C^2$ , convex,  $\Phi$  is locally lipschitz continuous,  $\Phi'(u) = \eta'(u)A'(u)$  for a.e.  $u \in \mathbb{R}$ . Actually, it is sufficient to establish (5.58) for a family of generating entropy, entropy-flux pairs. One generally considers the Kruzhkov entropies  $\eta(u) = |u - r|$ , where the parameter  $r$  runs in  $\mathbb{R}$ . Such a  $\eta$  is not of class  $C^2$ , but the associated flux is well defined. We can also work with the semi Kruzhkov entropies  $\eta^\pm(u) = (u - r)^\pm$ . The associated fluxes are

$$\Phi^+(u; r) = A(u \vee r) - A(r), \quad \Phi^-(u; r) = A(u) - A(u \wedge r). \quad (5.63)$$

We can see on the expressions (5.34) and (5.63) (we take  $v_K^n \equiv r$  in (5.34)) that we have already established a discrete version of (5.62):

$$(u_K^{n+1} - r)^+ \leq (u_K^n - r)^+ + \frac{\Delta t_n}{|K|} \sum_{L \in \mathcal{N}(K)} |K|L|[\Phi_{K \rightarrow L}(u_K^n, u_K^n; r) - \Phi_{K \rightarrow L}(u_K^n, u_L^n; r)], \quad (5.64)$$

where  $\Phi_{K \rightarrow L}(v, w; r) = A_{K \rightarrow L}(v \vee r, w \vee r) - A_{K \rightarrow L}(r, r)$ . If we start from (5.64) and adapt in a suitable way the proof of Theorem 5.5, we can establish that  $u_{h, \Delta t}$  is an approximate weak entropy solution to (5.9) on  $(0, T)$  with initial datum  $u_0$  in the sense that

$$\begin{aligned} \int_0^T \int_{\mathbb{T}^d} (\eta^\pm(u_{h, \Delta t}; r)\varphi_t + \Phi^\pm(u_{h, \Delta t}; r) \cdot \nabla_x \varphi) dx dt + \int_{\mathbb{T}^d} \eta^\pm(u_0(x); r)\varphi(x, 0) dx \\ \geq -\langle \mu_{h, \Delta t}^0, |\varphi| \rangle - \langle \mu_{h, \Delta t}^1, |\partial_t \varphi| \rangle - \langle \mu_{h, \Delta t}^2, |\nabla_x \varphi| \rangle, \end{aligned} \quad (5.65)$$

for all non-negative test-function  $\varphi \in C_c^\infty(\mathbb{T}^d \times [0, T])$  and for all  $r \in \mathbb{R}$ , where  $\mu_{h, \Delta t}$  satisfies an estimate similar to (5.61). See [9].

*Proof of Theorem 5.5.* Let  $\varphi \in C_c^\infty(\mathbb{T}^d \times [0, T])$ . We first look at the error done at initial time. Define the error  $\varepsilon_0(\varphi)$  by the formula

$$\varepsilon_0(\varphi) = \int_{\mathbb{T}^d} (u_0(x) - u_{h, \Delta t}(x, 0))\varphi(x, 0) dx. \quad (5.66)$$

By decomposition of the integral in (5.66), we have

$$\varepsilon_0(\varphi) = \sum_{K \in \mathcal{T}^{\sharp}} \int_K (u_0(x) - u_K^0)\varphi(x, 0) dx.$$

For  $x \in K$ ,  $u_0(x) - u_{h, \Delta t}(x, 0)$  is the average over  $K$  of  $y \mapsto u_0(x) - u_0(y)$ . Using Fubini's theorem, this gives the inequality  $|\varepsilon_0(\varphi)| \leq \mu_{h, \Delta t}^0(|\varphi|)$ , where

$$\mu_{h, \Delta t}^0(\psi) = \sum_{K \in \mathcal{T}^{\sharp}} \int_K \frac{1}{|K|} \int_K |u_0(x) - u_0(y)|\psi(x, 0) dx dy$$

In particular, we have

$$|\mu_{h, \Delta t}^0(\mathbb{T}^d \times [0, T])| \leq \sum_{K \in \mathcal{T}^{\sharp}} \int_K \frac{1}{|K|} \int_K |u_0(x) - u_0(y)| dx dy.$$



This can be written

$$|\mu_{h,\Delta t}^0(\mathbb{T}^d \times [0, T])| \leq \sum_{K \in \mathcal{T}^\#} \frac{1}{|K|} \int_{x \in \mathbb{T}^d} \int_{y \in K} \mathbf{1}_K(x) |u_0(x) - u_0(y)| dx dy.$$

We do the change of variable  $y = x + z$  to obtain

$$|\mu_{h,\Delta t}^0(\mathbb{T}^d \times [0, T])| \leq \sum_{K \in \mathcal{T}^\#} \frac{1}{|K|} \int_{x \in \mathbb{T}^d} \int_{z \in K-x} \mathbf{1}_K(x) |u_0(x) - u_0(x+z)| dx dz,$$

and thus

$$|\mu_{h,\Delta t}^0(\mathbb{T}^d \times [0, T])| \leq \sum_{K \in \mathcal{T}^\#} \frac{1}{|K|} \int_{x \in \mathbb{T}^d} \int_{z \in B(0, h)} \mathbf{1}_K(x) |u_0(x) - u_0(x+z)| dx dz,$$

since  $K - x \subset B(0, h)$  if  $x \in K$ . We use the first bound of (5.27) and the fact that the sum over  $K$  of  $\mathbf{1}_K(x)$  is 1 for a.e.  $x$  to get

$$|\mu_{h,\Delta t}^0(\mathbb{T}^d \times [0, T])| \leq \frac{1}{\alpha h^d} \int_{x \in \mathbb{T}^d} \int_{z \in B(0, h)} |u_0(x) - u_0(x+z)| dx dz,$$

We can exchange the integrals in  $x$  and  $z$  then to obtain

$$|\mu_{h,\Delta t}^0(\mathbb{T}^d \times [0, T])| \leq \frac{|B(0, h)|}{\alpha h^d} \sup_{|z| \leq h} \int_{x \in \mathbb{T}^d} |u_0(x) - u_0(x+z)| dx dz.$$

This gives the first estimate

$$|\mu_{h,\Delta t}^0(\mathbb{T}^d \times [0, T])| \leq \alpha^{-1} |B(0, 1)| \omega(u_0; h). \quad (5.67)$$

Let us now study the term

$$\mathbf{I}_t = \int_0^T \int_{\mathbb{T}^d} u_{h,\Delta t} \varphi_t dx dt + \int_{\mathbb{T}^d} u_{h,\Delta t}(x, 0) \varphi(x, 0) dx.$$

Let  $N \in \mathbb{N}$  be such that  $t_{N-1} < T \leq t_N$ . Since  $\varphi$  is compactly supported in  $\mathbb{T}^d \times [0, T]$ , we can assume that  $T = t_N$ . We expand  $\mathbf{I}_t$  as

$$\mathbf{I}_t = \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}^\#} |K| u_K^n (\varphi_K(t_{n+1}) - \varphi_K(t_n)) + \sum_{K \in \mathcal{T}^\#} |K| u_K^0 \varphi_K(0),$$

where  $\varphi_K(t)$  is the average value of  $\varphi(\cdot, t)$  on the cell  $K$ . A discrete integration by parts gives

$$\mathbf{I}_t = - \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}^\#} |K| (u_K^{n+1} - u_K^n) \varphi_K(t_{n+1}). \quad (5.68)$$

We proceed similarly with the term

$$\mathbf{I}_x = \int_0^T \int_{\mathbb{T}^d} A(u_{h,\Delta t}) \cdot \nabla_x \varphi dx dt.$$

We expand  $I_x$  as

$$I_x = \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}^\#} \int_{t_n}^{t_{n+1}} \int_K A(u_K^n) \cdot \nabla_x \varphi dx dt.$$

By the Stokes formula, this gives

$$I_x = \sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}^\#} \sum_{L \in \mathcal{N}(K)} |K|L| A(u_K^n) \cdot n_{K \rightarrow L} \varphi_{K|L}^n, \quad (5.69)$$

where  $\varphi_{K|L}^n$  is the average of the function  $\varphi$  on  $K|L \times (t_n, t_{n+1})$ . We use the consistency property (5.15) to write  $A(u_K^n) \cdot n_{K \rightarrow L} = A_{K \rightarrow L}(u_K^n, u_L^n)$ . We also add a corrective term to the sum in (5.69) to obtain

$$I_x = \sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}^\#} \sum_{L \in \mathcal{N}(K)} |K|L| (A_{K \rightarrow L}(u_K^n, u_L^n) - A_{K \rightarrow L}(u_K^n, u_L^n)) \varphi_{K|L}^n. \quad (5.70)$$

By the anti-symmetry property of the term  $A_{K \rightarrow L}(u_K^n, u_L^n) \varphi_{K|L}^n$  (cf. (5.17)) and the summation formula (5.38), (5.69) and (5.70) coincide indeed. Let us now denote by  $\varphi_K^n$  the average value of the function  $\varphi$  over  $K \times (t_n, t_{n+1})$ . If we replace the quantities  $\varphi_K(t_{n+1})$  in (5.68) and  $\varphi_{K|L}^n$  in (5.70) by  $\varphi_K^n$ , then we obtain  $I_t + I_x = 0$ . This follows from (5.21). Consequently, we have  $I_t + I_x = \varepsilon^1(\varphi) + \varepsilon^2(\varphi)$ , where

$$\varepsilon^1(\varphi) = \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}^\#} |K| (u_K^{n+1} - u_K^n) (\varphi_K^n - \varphi_K(t_{n+1})),$$

and

$$\varepsilon^2(\varphi) = \sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}^\#} \sum_{L \in \mathcal{N}(K)} |K|L| (A_{K \rightarrow L}(u_K^n, u_L^n) - A_{K \rightarrow L}(u_K^n, u_L^n)) (\varphi_{K|L}^n - \varphi_K^n).$$

To conclude to (5.60), we need to examine the error terms  $\varepsilon^1(\varphi)$  and  $\varepsilon^2(\varphi)$ . Since

$$\int_{t_n}^{t_{n+1}} \{\varphi(t_{n+1}) - \varphi(t)\} dt = \int_{t_n}^{t_{n+1}} \int_t^{t_{n+1}} \varphi_t(s) ds dt = \int_{t_n}^{t_{n+1}} (t_{n+1} - s) \varphi_t(s) ds,$$

we have

$$|\varphi_K(t_{n+1}) - \varphi_K^n| \leq \frac{1}{|K|} \int_{t_n}^{t_{n+1}} \int_K |\varphi_t(x, t)| dx dt.$$

This gives  $|\varepsilon^1(\varphi)| \leq \langle \mu_{h, \Delta t}^1, |\partial_t \varphi| \rangle$ , where

$$\langle \psi, \mu_{h, \Delta t}^1 \rangle = \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}^\#} |u_K^n - u_K^{n+1}| \int_{t_n}^{t_{n+1}} \int_K \psi(x, t) dx dt. \quad (5.71)$$

In particular, the total mass of  $\mu_{h, \Delta t}^1$  is

$$\mu_{h, \Delta t}^1(\mathbb{T}^d \times [0, T]) = \sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}^\#} |K| |u_K^n - u_K^{n+1}|. \quad (5.72)$$

By the Cauchy-Schwarz inequality and (5.56), we have

$$[\mu_{h,\Delta t}^1(\mathbb{T}^d \times [0, T])]^2 \leq T \frac{1-\xi}{\xi} \|u_0\|_{L^2(\mathbb{T}^d)}^2 \Delta t. \quad (5.73)$$

Similarly, we develop

$$\varphi_{K|L} - \varphi_K = \frac{1}{|K|L||K|} \int_{K|L} \int_K |\varphi(x) - \varphi(y)| dx d\sigma(y)$$

and use the development  $\varphi(x) - \varphi(y) = \int_0^1 \nabla \varphi(ry + (1-r)x) \cdot (x-y) dr$  to obtain  $|\varepsilon^2(\varphi)| \leq \langle \mu_{h,\Delta t}^2, |\nabla \varphi| \rangle$ , where

$$\begin{aligned} \langle \psi, \mu_{h,\Delta t}^2 \rangle &:= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}^\#} \sum_{L \in \mathcal{N}(K)} |A_{K \rightarrow L}(u_K^n, u_K^n) - A_{K \rightarrow L}(u_K^n, u_L^n)| \\ &\quad \times \frac{1}{|K|} \int_{t_n}^{t_{n+1}} \int_{K|L} \int_K \int_0^1 \psi(ry + (1-r)x, t) |x-y| dr dx d\sigma(y) dt. \end{aligned} \quad (5.74)$$

We have  $|x-y| \leq h$  when  $x \in K$ ,  $y \in K|L$ , so

$$\mu_{h,\Delta t}^2(\mathbb{T}^d \times [0, T]) \leq h \sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}^\#} \sum_{L \in \mathcal{N}(K)} |K|L| |A_{K \rightarrow L}(u_K^n, u_K^n) - A_{K \rightarrow L}(u_K^n, u_L^n)|. \quad (5.75)$$

We use the Cauchy-Schwarz inequality and the estimate (5.57) to get the bound

$$[\mu_{h,\Delta t}^2(\mathbb{T}^d \times [0, T])]^2 \leq h^2 \Gamma L_A(R) \|u_0\|_{L^2(\mathbb{T}^d)}^2.$$

The factor  $\Gamma$  is

$$\Gamma = \sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}^\#} \sum_{L \in \mathcal{N}(K)} |K|L| = T \sum_{K \in \mathcal{T}^\#} |\partial K|.$$

By (5.27), we have the bound  $\Gamma \leq T \alpha^{-2} h^{-1}$ , which shows that

$$[\mu_{h,\Delta t}^2(\mathbb{T}^d \times [0, T])]^2 \leq T \alpha^{-2} L_A(R) \|u_0\|_{L^2(\mathbb{T}^d)}^2 h. \quad (5.76)$$

We can bound the  $L^2$ -norm of  $u_0$  by  $R$  in (5.73) and (5.76). This gives the desired estimate (5.61).  $\square$

## 5.9 Convergence in the linear case

We restrict now our analysis to the case of a linear flux  $A$ :  $A(u) = au$ . In this context, we consider a possibly non-constant vector field  $a$ . More precisely, we will assume that  $a \in C^1(\mathbb{T}^d; \mathbb{R}^d)$  and that  $a$  is divergence free:  $\operatorname{div}(a(x)) = 0$  for all  $x \in \mathbb{T}^d$ . We consider then the scheme (5.1) with a numerical flux given by (5.11)-(5.12), which is called the upwind, or upstream, flux. We have then (5.14) with some numerical flux functions

$$A_{K \rightarrow L}(v, w) = a_{K \rightarrow L}^+ v - a_{K \rightarrow L}^- w \quad (5.77)$$

which satisfies all the properties listed in Section 5.4, with  $L_A(R) = \|a\|_{L^\infty(\mathbb{T}^d)}$ . We will admit that Theorem 5.5 remains valid, in the sense that we have

$$\left| \int_0^T \int_{\mathbb{T}^d} (u_{h,\Delta t} \varphi_t + u_{h,\Delta t} a(x) \cdot \nabla_x \varphi) dx dt + \int_{\mathbb{T}^d} u_0(x) \varphi(x, 0) dx \right| \leq \langle \mu_{h,\Delta t}^0, |\varphi| \rangle + \langle \mu_{h,\Delta t}^1, |\partial_t \varphi| \rangle + \langle \mu_{h,\Delta t}^2, |\nabla_x \varphi| \rangle, \quad (5.78)$$

and (5.61). In the asymptotic situation  $\Delta t \rightarrow 0$ ,  $h \rightarrow 0$  described in Section 5.6.3, we can pass to the limit in (5.78). This shows that  $u$  is a weak solution to (5.5) on  $(0, T)$  with initial datum  $u_0$ . We use then the following theorem.

**Theorem 5.6.** *Let  $u_0 \in L^\infty(\mathbb{T}^d)$  and  $T > 0$ . The continuity equation (5.5) admits a unique weak solution on  $(0, T)$  with initial datum  $u_0$ . It is given explicitly by  $u(x, t) = u_0 \circ \Phi^t(x)$ , where  $(\Phi_t)$  is the flow associated to the ODE  $\dot{x} = a(x)$  and  $\Phi^t$  is the inverse<sup>4</sup> of  $x \mapsto \Phi_t(x)$ .*

**Exercise 5.5** (Uniqueness in transport equations). Prove Theorem 5.6. *The solution to Exercise 5.5 is here.*

## 5.10 Error estimate in the linear case

Our aim in this section and the following ones is to establish the following result.

**Theorem 5.7.** *Let  $u_0 \in L^\infty \cap \text{BV}(\mathbb{T}^d)$  and  $T > 0$ . Let  $A(x, u) = a(x)u$ , where  $a \in C^1(\mathbb{T}^d; \mathbb{R}^d)$  is divergence-free. Let  $u_{h,\Delta t}$  be the solution of the upwind Finite Volume method (5.1) with fluxes given by (5.14)-(5.77)-(5.12). Let  $u \in L^\infty(\mathbb{T}^d \times (0, T))$  be the weak solution to (5.5) on  $(0, T)$  with initial datum  $u_0$ . Assume that (5.27) and (5.44) are satisfied. Assume also that  $\delta t \leq C_0 h$  for a certain constant  $C_0$ . Then, there is a constant  $c(d) > 0$  depending on  $d$  only such that, for  $h, \Delta t \leq c(d)$ , we have the error estimate*

$$\|u_{h,\Delta t}(t) - u(t)\|_{L^1(\mathbb{T}^d)} \leq C |Du_0|(\mathbb{T}^d) h^{1/2}, \quad (5.79)$$

for all  $t \in [0, T]$ , where  $C$  is a constant depending only on the dimension  $d$ , on  $T$ , on  $C_0$ , on  $\|a\|_{C^1(\mathbb{T}^d)}$ , on the constant  $\alpha$  in (5.27) and on the constant  $\xi$  in (5.44).

We will make some comments on Theorem 5.7, but first we need a brief remainder on the space BV.

### 5.10.1 Functions of bounded variations

Let  $U$  be an open subset of  $\mathbb{R}^d$ . If  $\varphi \in C(U; \mathbb{R}^d)$ , we denote by  $\|\varphi\|_{C(U)}$  the sup over  $x \in U$  of the euclidean norm  $|\varphi(x)|$  of  $\varphi(x)$ .

**Definition 5.6** (Functions of bounded variation). Let  $U$  be an open subset of  $\mathbb{R}^d$ . A function  $u \in L^1(U)$  is said to have bounded variation in  $U$  if

$$\sup \left\{ \int_U u \operatorname{div} \varphi dx \right\} < +\infty \quad (5.80)$$

where the supremum is taken over all  $\varphi \in C_c^1(U; \mathbb{R}^d)$  such that  $\|\varphi\|_{C(U)} \leq 1$ . We denote by  $\text{BV}(U)$  the space of functions of bounded variations.

<sup>4</sup>an expression of  $\Phi^t$  is  $\Phi^t(x) = \Phi_{-t}(x)$ , this is a consequence of the group property  $\Phi_t \circ \Phi_s = \Phi_{t+s}$ ; when the sense of the time evolution does matter, for instance in the study of stochastic differential equations, it is important to define  $\Phi^t$  as the inverse of  $x \mapsto \Phi_t(x)$ , not as  $\Phi_{-t}(x)$

We denote by  $BV_{\text{loc}}(U)$  the space of functions having locally bounded variations, defined as the set of functions  $u \in L^1_{\text{loc}}(U)$  such that  $u \in BV(V)$  for all open subset  $V \subset\subset U$  (this last notations means that there exists a compact  $K$  of  $\mathbb{R}^d$  such that  $V \subset K \subset U$ ).

**Exercise 5.7** (Some functions of bounded variation). 1. Let  $U = (-1, 1)$ . Let  $u: U \rightarrow \mathbb{R}$  be defined as the integral over  $[0, x]$  of a function  $f \in L^1_{\text{loc}}(U)$ . Show that  $u \in BV_{\text{loc}}(U)$  and that  $u \in BV(U)$  if, and only if,  $f \in L^1(U)$ .

2. Let  $U = (-1, 1)$ . Let  $u: U \rightarrow \mathbb{R}$  be the Heavyside function:  $u(x) = 0$  if  $x < 0$ ,  $u(x) = 1$  if  $x > 0$ . Show that  $u \in BV(U)$ .

3. Let  $U = B(0, 1)$  in  $\mathbb{R}^2$ . Let  $u$  be the characteristic function of the disk  $B(0, 1/2)$ . Show that  $u \in BV(U)$ .

The solution to Exercise 5.7 is [here](#).

To enunciate the following structure theorem for functions of bounded variations, let us recall the following facts about measures.

1. (See [17, Chapter 6]). Let  $(X, \mathcal{A})$  be a measure space. A complex measure over  $(X, \mathcal{A})$  is a set function  $\mu: \mathcal{A} \rightarrow \mathbb{C}$  such that, for all  $A \in \mathcal{A}$ , one has

$$\mu(A) = \sum_{i=1}^{\infty} \mu(A_i), \quad (5.81)$$

for all countable partition  $(A_i)_{i \geq 1}$  of  $A$ , the sum in (5.81) being absolutely convergent. If  $\mu$  is a complex measure, the formula

$$|\mu|(A) = \sup \left\{ \sum_{i=1}^{\infty} |\mu(A_i)| \right\}, \quad (5.82)$$

where the supremum is taken over all countable partitions  $(A_i)_{i \geq 1}$  of  $A$ , defines a non-negative finite measure  $|\mu|$  on  $\mathcal{A}$  called the *total variation* of  $\mu$ .

2. (See [17, p. 130]). A complex measure  $\mu$  defined on the Borel subsets of a topological Hausdorff space  $X$  is said to be regular if for all Borel set  $A$ ,

$$|\mu|(A) = \sup \{ |\mu|(K); K \text{ compact } \subset A \} = \inf \{ |\mu|(V); V \text{ open } \supset A \}. \quad (5.83)$$

**Theorem 5.8** (Structure theorem for functions of bounded variations). *Let  $U$  be an open set in  $\mathbb{R}^d$ . Let  $u \in L^1(U)$ . Then  $u \in BV(U)$  if, and only if, there exists a non-negative regular finite measure  $\kappa$  on  $U$  and a Borel map  $n: U \rightarrow \mathbb{R}^d$  such that  $|n(x)| = 1$  for  $\kappa$ -a.e.  $x \in U$  and*

$$\int_U u \operatorname{div} \varphi dx = - \int_U \varphi \cdot n d\kappa, \quad (5.84)$$

for all  $\varphi \in C_c^1(U)$ . The sup in (5.80) is then equal to  $\kappa(U)$ .

*Proof of Theorem 5.8.* In essential, the proof is an application of the theorem of representation of Riesz. We take as a reference Theorem 6.19 in [17]. In [17], the result is given for a functional of complex-valued functions. Since we need to consider a functional of vector valued functions, we will come back on the main steps of the proof of Theorem 6.19 in [17]. For simplicity, we will use the same notations as Rudin. Consider the functional

$$\Phi(f) = - \int_U u \operatorname{div}(f) dx$$

It is defined for  $f \in C_c^1(U; \mathbb{R}^d)$ . By (5.80), it can be extended to a linear continuous functional (still denoted  $\Phi$ ) on  $C_0(U; \mathbb{R}^d)$ . We consider then the further extension to  $C_0(U; \mathbb{C}^d)$  defined by  $\Phi(f) := \Phi(f_1) + i\Phi(f_2)$ , where  $f_1$  is the real part of  $f$  and  $f_2$  the imaginary part of  $f$ . Our aim is to prove that there exists a non-negative regular finite measure  $\lambda$  on  $U$  and a Borel map  $g: U \rightarrow \mathbb{R}^d$  such that  $|g(x)| = 1$  for  $\lambda$ -a.e.  $x \in U$  and

$$\Phi(f) = \int_U f \cdot g d\lambda, \quad (5.85)$$

for all  $f \in C_0(U; \mathbb{R}^d)$ , where  $(f \cdot g)(x) = \sum_{i=1}^d f_i(x)g_i(x)$ . For  $f \in C_c(U; \mathbb{R}_+)$ , we set

$$\Lambda(f) = \sup \{ |\Phi(h)|; h \in C_c(U; \mathbb{C}^d), |h(x)| \leq f(x) \text{ for all } x \in U \}. \quad (5.86)$$

For a general  $f \in C_c(U; \mathbb{R})$ , we set then  $\Lambda(f) = \Lambda(f^+) - \Lambda(f^-)$ . This defines a linear continuous functional  $\Lambda$  on  $C_c(U; \mathbb{R})$  which is positive. By the representation theorem of Riesz, [17, Theorem 2.14], there exists a non-negative regular finite measure  $\lambda$  on  $U$  such that  $\Lambda(f) = \int_U f d\lambda$  for all  $f \in C_c(U; \mathbb{R})$ . In the previous assertion, the fact that  $\Lambda$  is linear is of course not obvious. This is the delicate point here. That we consider  $\Phi(h)$  with  $h \in C_c(U; \mathbb{C})$  or  $h \in C_c(U; \mathbb{C}^d)$  changes absolutely nothing to the proof that  $\Lambda(f+g) = \Lambda(f) + \Lambda(g)$ . We give the sketch of the proof for completeness. Let  $f, g \in C_c(U; \mathbb{R}_+)$ ,  $\varepsilon > 0$  and  $h_1, h_2 \in C_c(U; \mathbb{C}^d)$  such that

$$\Lambda(f) \leq |\Phi(h_1)| + \varepsilon, \quad \Lambda(g) \leq |\Phi(h_2)| + \varepsilon.$$

There are some complex numbers  $\alpha_1, \alpha_2 \in \mathbb{C}$  of modulus 1 such that  $|\Phi(h_i)| = \alpha_i \Phi(h_i)$ . Then the sum  $\Lambda(f) + \Lambda(g)$  is bounded by

$$\alpha_1 \Phi(h_1) + \alpha_2 \Phi(h_2) + 2\varepsilon = \Phi(\alpha_1 h_1 \alpha_2 h_2) + 2\varepsilon \leq \Lambda(f+g) + 2\varepsilon,$$

which shows that  $\Lambda(f) + \Lambda(g) \leq \Lambda(f+g)$ . To prove the converse inequality, consider  $h \in C_c(U; \mathbb{C}^d)$  satisfying the constraint  $|h| \leq f+g$  and set  $V = \{f+g > 0\}$  and

$$h_1 = \frac{f}{f+g} \mathbf{1}_V h, \quad h_2 = \frac{g}{f+g} \mathbf{1}_V h.$$

Then  $h_1, h_2 \in C_c(U; \mathbb{C}^d)$ ,  $h_1 + h_2 = h$ ,  $|h_1| \leq f$ ,  $|h_2| \leq g$ , which shows that  $|\Phi(h)| \leq \Lambda(f) + \Lambda(g)$ . Taking the sup over  $h$  gives the desired result. Once the representation of the functional  $\Lambda$  by the measure  $\lambda$  has been established, we obtain by definition of  $\Lambda$  the estimate

$$|\Phi(f)| \leq \int_U |f| d\lambda, \quad (5.87)$$

for all  $f \in C_c(U; \mathbb{C}^d)$ . Then we use the fact that there is a correspondence between the maps of the form

$$\Psi: f \mapsto \int_U f \cdot g d\lambda, \quad g \in L^\infty(U; \mathbb{C}^d) \quad (5.88)$$

and the continuous linear forms of  $E := L^1(U, \lambda; \mathbb{C}^d)$ . This is an extension of [17, Theorem 6.16] to the vector valued case. We admit this result, which can be proved by a systematic examination of the proof of [17, Theorem 6.16]. We also obtain the fact that  $\|\Psi\|_{E'} = \|g\|_{L^\infty(U)}$  in this correspondence, where  $\|g\|_{L^\infty(U)}$  is the essential supremum over  $x \in U$  of the euclidean norm  $|g(x)|$  of  $g(x)$  and  $\|\Psi\|_{E'}$  is the norm of the linear form  $\Psi$ . Since  $C_c(U; \mathbb{C}^d)$  is dense in  $L^1(U, \lambda; \mathbb{C}^d)$ ,  $\|\Psi\|_{E'}$  is equal to

$$\|\Psi\|_{E'} = \sup \{ |\Psi(f)|; f \in C_c(U; \mathbb{C}^d), \|f\|_{L^1(\lambda)} \leq 1 \}. \quad (5.89)$$

This gives us the desired representation (5.85). There remains to prove that  $|g(x)| = 1$  for  $\lambda$ -a.e.  $x \in U$ . By (5.87), we have  $\|\Phi\|_{E'} \leq 1$ , hence  $\|g\|_{L^\infty(U)} \leq 1$ . On the other hand, it follows from (5.85) and the Cauchy-Schwarz inequality  $|f \cdot g| \leq |f||g|$  that

$$|\Phi(f)| \leq \int_U |g| d\lambda, \quad f \in C_c(U; \mathbb{C}^d), \quad |f(x)| \leq 1.$$

Taking the sup over  $f$  in the previous inequality, and considering the definition (5.86), we see that  $\Lambda(1) \leq \int_U |g| d\lambda$ . Since  $\Lambda(1) = \lambda(U)$  at the same time,  $|g|$  is equal to 1  $\lambda$ -a.e.  $\square$

**Notation:** if  $u \in \text{BV}(U)$ , we denote by  $Du$  the (vector-valued) complex measure  $n\kappa$  in (5.84) and by  $|Du|$  the measure  $\kappa$ . The norm  $\|u\|_{\text{BV}(U)}$  of  $u$  is defined as

$$\|u\|_{\text{BV}(U)} = \|u\|_{L^1(U)} + |Du|(U). \quad (5.90)$$

**Exercise 5.8** (Some functions of bounded variation). Compute  $Du$ ,  $|Du|$  and  $\|u\|_{\text{BV}(U)}$  for the functions  $u$  considered in the exercise 5.7.

*The solution to Exercise 5.8 is here.*

**Definition 5.9** (Set of finite perimeter). 1. A Lebesgue measurable set  $E$  of  $\mathbb{R}^d$  is said to have finite perimeter in  $U$  if  $\mathbf{1}_E \in \text{BV}(\mathbb{R}^d)$ . In that case, we set  $P(E) = |D\mathbf{1}_E|(\mathbb{R}^d)$ .

2. Let  $U$  be an open subset of  $\mathbb{R}^d$ . A Lebesgue measurable set  $E$  of  $\mathbb{R}^d$  is said to have finite perimeter in  $U$  if  $\mathbf{1}_E \in \text{BV}(U)$ . In that case, we set  $P(E; U) = |D\mathbf{1}_E|(U)$ .

We now state without proof the following results.

**Theorem 5.9** (Local approximation by smooth functions). *Let  $U$  be an open set in  $\mathbb{R}^d$ . Let  $u \in \text{BV}(U)$ . There exists a sequence of functions  $u_k$  in  $\text{BV}(U) \cap C^\infty(U)$  such that*

1.  $u_k \rightarrow u$  in  $L^1(U)$ , and
2.  $|Du_k|(U) \rightarrow |Du|(U)$ .

*Remark 5.5.* Note that if  $u \in \text{BV}(U) \cap C^\infty(U)$  then  $u \in W^{1,1}(U)$  and

$$|Du|(U) = \int_U |\nabla u(x)| dx. \quad (5.91)$$

**Theorem 5.10** (Trace of functions of bounded variations). *Let  $U$  be an open bounded set in  $\mathbb{R}^d$ , with  $\partial U$  Lipschitz continuous. Let  $\sigma$  denote the surface measure on  $\partial U$  and  $n$  the outward unit normal to  $U$  on  $\partial U$ . There exists a bounded linear application*

$$\gamma: \text{BV}(U) \rightarrow L^1(\partial U, \sigma),$$

such that

$$\int_U u \operatorname{div} \varphi dx = - \int_U \varphi \cdot dDu + \int_{\partial U} (\gamma u) \varphi \cdot n d\sigma, \quad (5.92)$$

for all  $\varphi \in C^1(\mathbb{R}^d; \mathbb{R}^d)$ .

**Theorem 5.11** (Patch of functions of bounded variations). *Let  $U$  be an open bounded set in  $\mathbb{R}^d$ , with  $\partial U$  Lipschitz continuous. Let  $\sigma$  denote the surface measure on  $\partial U$  and  $n$  the outward unit normal to  $U$  on  $\partial U$ . Let  $v \in \text{BV}(U)$ ,  $w \in \text{BV}(\mathbb{R}^d \setminus \bar{U})$  and let  $u \in L^1(\mathbb{R}^d)$  be the function defined as  $u = v\mathbf{1}_U + w\mathbf{1}_{\mathbb{R}^d \setminus \bar{U}}$ . Then  $u \in \text{BV}(\mathbb{R}^d)$  and*

$$\int_{\mathbb{R}^d} \varphi \cdot dDu = \int_U \varphi \cdot dDv + \int_{\mathbb{R}^d \setminus \bar{U}} \varphi \cdot dDw + \int_{\partial U} (\gamma v - \gamma w) \varphi \cdot n d\sigma, \quad (5.93)$$

for all  $\varphi \in C_c(\mathbb{R}^d; \mathbb{R}^d)$ . The BV norm of  $u$  is

$$\|u\|_{\text{BV}(\mathbb{R}^d)} = \|v\|_{\text{BV}(U)} + \|w\|_{\text{BV}(\mathbb{R}^d \setminus \bar{U})} + \int_{\partial U} |\gamma v - \gamma w| d\sigma. \quad (5.94)$$

**Theorem 5.12** (Co-area formula for functions of bounded variations). *Let  $U$  be an open set in  $\mathbb{R}^d$ . Let  $u \in L^1(U)$  be a non-negative function. For  $t \in \mathbb{R}$ , we denote by  $E_t$  the super-level set  $\{u > t\}$ . Then, for a.e.  $t \in \mathbb{R}$ ,  $E_t$  has finite perimeter in  $U$ , and we have*

$$\|u\|_{L^1(U)} = \int_0^\infty \|\mathbf{1}_{E_t}\|_{L^1(U)} dt, \quad |Du|(U) = \int_0^\infty |D\mathbf{1}_{E_t}|(U) dt. \quad (5.95)$$

See [8, Theorem 5.3] for the proof of Theorem 5.9, [8, Theorem 5.6] for the proof of Theorem 5.10, [8, Theorem 5.8] for the proof of Theorem 5.11 and [8, Theorem 5.9] for the proof of Theorem 5.12. To complete this section, let us give the definition of the norm  $\|u\|_{\text{BV}(\mathbb{T}^d)}$  of a  $\mathbb{Z}^d$ -periodic function  $u: \mathbb{R}^d \rightarrow \mathbb{R}$ . First, by  $\text{BV}(\mathbb{T}^d)$  we denote the set of  $\mathbb{Z}^d$ -periodic functions  $u \in \text{BV}_{\text{loc}}(\mathbb{R}^d)$ . Let  $(e_i)_{1,d}$  denote the canonical basis of  $\mathbb{R}^d$ . Let  $Q$  denote the unit cube  $(0,1)^d$ . For  $\sigma$  almost all  $x \in \partial Q$ , there is a unique  $\check{x} \equiv x \pmod{\mathbb{Z}^d}$  in  $\partial Q$  (located on the opposite side...). If  $v: \partial Q \rightarrow \mathbb{R}$ , we denote by  $\check{v}$  the function that assign to  $x \mapsto v(\check{x})$ . This is simply the function  $x \mapsto v(x \pm e_i)$ , depending on the side of  $Q$  to which  $x$  belongs. We define then

$$|Du|(\mathbb{T}^d) = |Du|(Q) + \frac{1}{2} \int_{\partial Q} |\gamma u - \check{\gamma} u| d\sigma, \quad \|u\|_{\text{BV}(\mathbb{T}^d)} = \|u\|_{L^1(Q)} + |Du|(\mathbb{T}^d). \quad (5.96)$$

The integral on  $\partial Q$  in (5.96) accounts for the possible jumps of  $u$  across  $\partial Q$  and is coherent with (5.94). Observe also that, with the definition (5.96), we have also the following identity: for  $u \in L^1(\mathbb{T}^d)$ , denote by  $u_h$  the piecewise constant function defined by

$$u_h(x) = u_K := \frac{1}{|K|} \int_K u(x) dx, \quad x \in K. \quad (5.97)$$

Then we have

$$|Du_h|(\mathbb{T}^d) = \sum_{K \in \mathcal{T}^\sharp} \sum_{L \in \mathcal{N}(K)} |u_K - u_L|. \quad (5.98)$$

### 5.10.2 Comments on the error estimate

If  $1 < p < +\infty$ , one can establish the error estimate

$$\|u_{h,\Delta t}(t) - u(t)\|_{L^p(\mathbb{T}^d)} \leq C \|u_0\|_{W^{1,p}(\mathbb{T}^d)} h^{1/2}. \quad (5.99)$$

See [14]. The estimate (5.99) cannot be generalized when the flux in the conservation law (5.9) is non-linear, for the reason that  $W^{1,p}(\mathbb{T}^d)$  is not stable in the evolution: if  $u_0 \in W^{1,p}(\mathbb{T}^d)$ , there



may be some time  $t > 0$  such that the (entropy) solution  $u$  of (5.9) starting from  $u_0$  loses the  $W^{1,p}(\mathbb{T}^d)$  regularity at time  $t$ . This is a consequence of the apparition of discontinuities and is already clear in dimension  $d = 1$ . On the contrary, the space  $\text{BV}(\mathbb{T}^d)$  is stable in the evolution by (5.9). For general fluxes  $A$ , the error estimate (5.79) is observed in numerical practice, but has not been established yet, except when the mesh is a cartesian mesh, *i.e.* each cell is a product of one-dimensional cells of a one-dimensional mesh.

## 5.11 Error estimate in the linear case: proof

The following proof of the error estimate (5.79) is taken from [15]. A different proof, using probabilistic tools, has been given in [4].

### 5.11.1 Reduction of the problem

**Projection on piecewise constant functions and BV-norm.** We will use several times the following result.

**Proposition 5.13.** *Consider the map  $u \mapsto u_h$  defined by (5.97). There exists a constant  $C \geq 0$  only depending on  $d$  and on the constant  $\alpha$  in (5.27) such that, if  $u \in \text{BV}(\mathbb{T}^d)$ , then*

$$|Du_h|(\mathbb{T}^d) \leq C|Du|(\mathbb{T}^d) \text{ and } \|u_h - u\|_{L^1(\mathbb{T}^d)} \leq C|Du|(\mathbb{T}^d)h. \quad (5.100)$$

*Proof of Proposition 5.13.* Let  $K \in \mathcal{T}^\sharp$  and let  $L \in \mathcal{N}(K)$ . We will establish first the estimate

$$|u_K - u_L| \leq \frac{2^{d+1} \max(|K|, |L|)h}{|K||L|} |Du|(B(x_K, 2h)), \quad (5.101)$$

where  $x_K := \frac{1}{|K|} \int_K x dx$  is the center of gravity of  $K$ . Note that  $K, L \subset B(x_K, 2h)$ . By Theorem 5.9 applied with  $U = B(x_K, 2h)$ , we may suppose that  $u \in \text{BV} \cap C^1(B(x_K, 2h))$ , in which case (*cf.* (5.91))

$$|Du|(B(x_K, 2h)) = \int_{B(x_K, 2h)} |\nabla u(z)| dz.$$

Since  $|x - y| \leq 2h$  for every  $(x, y) \in K \times L$ , we have then

$$\begin{aligned} |u_K - u_L| &\leq \frac{1}{|K||L|} \int_K \int_L |u(x) - u(y)| dx dy \\ &\leq \frac{2h}{|K||L|} \int_K \int_L \int_0^1 |\nabla u((1-r)x + ry)| dr dx dy. \end{aligned}$$

Now we perform the change of variables  $(x, y, r) \mapsto (w = x - y, z = (1-r)x + ry, r = r)$  (of Jacobian determinant equal to 1). This gives

$$|u_K - u_L| \leq \frac{2h}{|K||L|} \int_{B(x_K, 2h)} |\nabla u(z)| \left( \int_0^1 \int_{\mathbb{R}^d} g(w, z, r) dw dr \right) dz,$$

$g$  is defined by  $g(w, z, r) = 1$  if  $z + rw \in K$  and  $z - (1-r)w \in L$ , and  $g(w, z, r) = 0$  otherwise. We remark that, for  $(z, r) \in B(x_K, 2h) \times [0, 1]$ , we have  $\int_{\mathbb{R}^d} g(w, z, r) dw \leq 2^d |K|$  if  $r \geq 1/2$  and  $\int_{\mathbb{R}^d} g(w, z, r) dw \leq 2^d |L|$  if  $r < 1/2$ . The estimate (5.101) follows. Using (5.27), we deduce from (5.101) that, for all  $K \in \mathcal{T}^\sharp$ ,

$$\sum_{L \in \mathcal{N}(K)} |K||L| |u_K - u_L| \leq 2^{d+1} \alpha^{-2} |Du|(B(x_K, 2h)).$$

Summing on  $K \in \mathcal{T}^\sharp$ , we get

$$\sum_{K \in \mathcal{T}^\sharp} \sum_{L \in \mathcal{N}(K)} |K|L|u_K - u_L| \leq 2^{d+1}\alpha^{-2} \int_{\mathbb{R}^d} \sum_{K \in \mathcal{T}^\sharp} \mathbf{1}_{B(x_K, 2h)}(z) d|Du|(z).$$

Let us set  $\chi(z) = \sum_{K \in \mathcal{T}^\sharp} \mathbf{1}_{B(x_K, 2h)}(z)$ . We have  $d(x_K, K) \leq h$ , so  $\chi(z) = 0$  if  $z$  is at a distance superior to  $3h$  of  $Q$ . We may assume that  $3h < 1$ , and then  $\chi(z) = 0$  if  $z \notin Q'$ , where  $Q'$  is the cube  $(-1, 2)^d$  - which contains  $3^d$  translates of  $Q$ . By (5.27), we also have

$$\alpha h^d \chi(z) \leq \sum_{K \in \mathcal{T}^\sharp} \mathbf{1}_{B(x_K, 2h)}(z) |K| \leq \sum_{K: d(z, K) < 3h} |K|.$$

Indeed,  $|z - x_K| < 2h$  implies  $d(z, K) < 3h$ . Since the cells in  $\mathcal{T}^\sharp$  are disjoint, we have

$$\sum_{K: d(z, K) < 3h} |K| = \left| \bigcup_{K: d(z, K) < 3h} K \right| \leq |B(z, 4h)| = 4^d |B(0, 1)| h^d.$$

It follows that  $\chi \leq 4^d |B(0, 1)| \alpha^{-1} \mathbf{1}_{Q'}$ , which gives us

$$\frac{1}{2} \sum_{K \in \mathcal{T}^\sharp} \sum_{L \in \mathcal{N}(K)} |K|L|u_K - u_L| \leq 8^d \alpha^{-3} |B(0, 1)| |Du|(Q') \leq 24^d \alpha^{-3} |B(0, 1)| |Du|(\mathbb{T}^d).$$

Let  $K \in \mathcal{T}$ . Similarly, we have

$$\int_K |u_h(x) - u(x)| dx \leq \frac{1}{|K|} \int_{K \times K} |u(x) - u(y)| dx dy \leq 2^d h |Du|(B(x_K, h)).$$

Summing on  $K \in \mathcal{T}$  and using the fact that the cardinal of the set  $\{K : d(K, z) \leq h\}$  is bounded by  $C\alpha^{-1}$ , we get the second estimate of (5.100).  $\square$

**Exercise 5.10** (Modulus of continuity of functions of bounded variation). Show that

$$\omega_{L^1}(u; h) \leq C |Du|(\mathbb{T}^d) h, \quad (5.102)$$

for all  $u \in \text{BV}(\mathbb{T}^d)$ , for all  $0 \leq h \leq 1$ , where  $\omega_{L^1}(u; h)$  is the modulus of continuity defined by (5.59) and where  $C$  is a constant depending on the dimension  $d$  only.

The solution to Exercise 5.10 is [here](#).

**Contraction in  $L^1$ .** We will also need the following proposition.

**Proposition 5.14** ( $L^p$ -conservation). *Let  $u, v \in L^\infty(\mathbb{T}^d \times (0, T))$  be some weak solutions to (5.5) on  $(0, T)$  with respective initial data  $u_0, v_0 \in L^\infty(\mathbb{T}^d)$ . Assume that  $a$  is divergence free. Then, for every  $p \in [1, +\infty]$ , we have*

$$\|u(t) - v(t)\|_{L^p(\mathbb{T}^d)} = \|u_0 - v_0\|_{L^p(\mathbb{T}^d)}, \quad (5.103)$$

for all  $t \in (0, T)$ .

*Proof of Proposition 5.14.* We use Theorem 5.6. By linearity, we can assume  $v \equiv 0$ . We have  $u(x, t) = u_0 \circ \Phi^t(x)$ . This gives (5.103) since  $\Phi^t$  is a bijection of  $\mathbb{T}^d$  (case  $p = +\infty$ ) and preserves the measure (case  $p \in [1, +\infty)$ ), since  $a$  is divergence free.  $\square$

A trivial consequence of (5.103) is that

$$\|u(t) - v(t)\|_{L^1(\mathbb{T}^d)} \leq \|u_0 - v_0\|_{L^1(\mathbb{T}^d)}. \quad (5.104)$$

We will also use Proposition 5.2, which gives (with obvious notations)

$$\|u_{h,\Delta t}(t) - v_{h,\Delta t}(t)\|_{L^1(\mathbb{T}^d)} \leq \|u_0 - v_0\|_{L^1(\mathbb{T}^d)}, \quad (5.105)$$

for all  $t \geq 0$ .

**Reduction 1. Discrete time.** Let  $t \in [0, T]$ . There is a unique  $n \geq 0$  such that  $t_n \leq t < t_{n+1}$ . We have  $u_{h,\Delta t}(t) = u_{h,\Delta t}(t_n)$  and

$$\|u(t) - u(t_n)\|_{L^1(\mathbb{T}^d)} = \|u_0 \circ \Phi^t - u_0 \circ \Phi^{t_n}\|_{L^1(\mathbb{T}^d)} \leq C|Du_0|(\mathbb{T}^d) \|\Phi^t - \Phi^{t_n}\|_{C(\mathbb{T}^d)}$$

by (5.102). The group property of the flow  $\Phi_t$  is also satisfied by  $\Phi^t$ . We have, therefore,

$$|\Phi^t(x) - \Phi^{t_n}(x)| = |\Phi^{t-t_n}(\Phi^{t_n}(x)) - \Phi^{t_n}(x)| \leq \|\Phi^{t-t_n} - \text{Id}\|_{C(\mathbb{T}^d)} = \|\Phi_{t-t_n} - \text{Id}\|_{C(\mathbb{T}^d)}.$$

Since  $\Phi_t(x) - x = \int_0^t a(\Phi_s(x)) ds$ , we obtain finally the bound

$$\|u(t) - u(t_n)\|_{L^1(\mathbb{T}^d)} \leq C|Du_0|(\mathbb{T}^d)\Delta t \leq C|Du_0|(\mathbb{T}^d)h,$$

where  $C$  depends on  $d$ ,  $C_0$  and  $\|a\|_{L^\infty(\mathbb{T}^d)}$ . This shows that it is sufficient to establish (5.79) for a time  $t$  in the discrete grid  $\{t_n; n \geq 0\}$ . We proceed to this reduction to extend the analysis done in the proof of Theorem 5.5. Indeed, in the proof of Theorem 5.5, it was assumed that the test function  $\varphi$  was compactly supported in  $\mathbb{T}^d \times [0, T]$ . It is easy however to extend our proof to the case where  $T = t_N$  and  $\varphi \in C^1(\mathbb{T}^d \times [0, T])$ . In that case we have an additional term for  $t = T$  and (5.60) will be replaced by the inequality

$$\left| \int_0^T \int_{\mathbb{T}^d} u_{h,\Delta t}(\varphi_t + a \cdot \nabla_x \varphi) dx dt + \int_{\mathbb{T}^d} u_0(x) \varphi(x, 0) dx - \int_{\mathbb{T}^d} u_{h,\Delta t}(x, T) \varphi(x, T) dx \right| \leq \langle \mu_{h,\Delta t}^0, |\varphi| \rangle + \langle \mu_{h,\Delta t}^1, |\partial_t \varphi| \rangle + \langle \mu_{h,\Delta t}^2, |\nabla_x \varphi| \rangle. \quad (5.106)$$

**Reduction 2. Non-negative functions.** Since constants are solutions to (5.1) and (5.5) and since the addition of a constant to a function  $u \in \text{BV}(\mathbb{T}^d)$  does not modify the quantity  $|Du|(\mathbb{T}^d)$ , we may replace  $u_0$  by  $u_0 + \|u_0\|_{L^\infty(\mathbb{T}^d)}$ , which allows us to work with non-negative functions only. This reduction step is not fundamental actually. The co-area formula for BV function, Theorem 5.12, has been stated for non-negative functions for simplicity; this is what accounts for the present procedure.

**Reduction 3. Projection on a cartesian grid.** Let  $L \in \mathbb{N}$ ,  $L \sim h^{-1/2}$ , for example  $L = \lceil h^{-1/2} \rceil$ . Let  $v_0$  be the  $L^2$ -projection (see (5.97)) of  $u_0$  on the functions piecewise constant with respect to the periodic mesh  $\mathcal{T}_0 = L^{-1}(Q + \mathbb{Z}^d)$ . This mesh satisfies (5.27) with  $h_0 = L^{-1}$  and  $\alpha = (2d)^{-1}$  since

$$|K_0| = h_0^d, \quad |\partial K_0| = 2dh_0^{d-1},$$

for all  $K_0 \in \mathcal{T}_0^\sharp$ . By Proposition 5.13, we have

$$|Dv_0|(\mathbb{T}^d) \leq C|Du_0|(\mathbb{T}^d), \quad \|u_0 - v_0\|_{L^1(\mathbb{T}^d)} \leq C|Du_0|(\mathbb{T}^d)h^{1/2},$$

where  $C$  depends on the dimension  $d$  only. In view of (5.104)-(5.105), of the second estimate in (5.100) and of (5.102), we can replace  $u_0$  by  $v_0$  to establish (5.79). Consequently, we may assume without loss of generality that  $u_0$  is piecewise constant with respect to  $\mathcal{T}_0$ . We use this first reduction step for the following reason: let  $t > 0$  and let  $A = \{u_0 > t\}$  be a super-level set of  $u$ . Then  $A$  is a union of distinct cells  $K_0^{(1)}, \dots, K_0^{(j)}$  of  $\mathcal{T}_0$ . Let  $A_{\langle h^{1/2} \rangle}$  denote the  $L^{-1}$  neighborhood of  $\partial A$ :

$$A_{\langle h^{1/2} \rangle} = \{x \in \mathbb{R}^d; d(x, \partial A) < L^{-1}\}.$$

We estimate the volume  $|A_{\langle h^{1/2} \rangle}|$  as follows: let  $K_0 \subset A$  and let  $L_0 \in \mathcal{T}_0$  be such that  $K_0|L_0 \subset \partial A$ . The points at distance less than  $L^{-1}$  of  $K_0|L_0$  are either in  $K_0$ ,  $L_0$  or one of their neighboring cells, thus in a set of volume less than  $4dL^{-d}$ . Since  $|K_0|L_0| = L^{-(d-1)}$  and  $L^{-1} \leq 2h^{1/2}$  (assume  $c(d) < 1/4$ ), it follows that

$$|A_{\langle h^{1/2} \rangle}| \leq C|\partial A|h^{1/2}, \quad (5.107)$$

where the constant  $C$  depends on  $d$  only.

**Reduction 4. Co-area formula.** We apply Theorem 5.12. The equations we consider are linear: they satisfy a superposition principle. By (5.95), we may replace by  $u_0$  by the characteristic function of a super-level set  $A$  with finite perimeter. The advantage of this manipulation is the following one. Since  $0 \leq u_{h,\Delta t} \leq 1$  by the comparison principle (Proposition 5.2), and  $u(t) = u_0 \circ \Phi^t = \mathbf{1}_{A(t)}$ ,  $A(t) := \phi^t(A)$ , we have

$$|u_{h,\Delta t}(x, t) - u(x, t)| = (u(x, t) - u_{h,\Delta t}(x, t))\varphi^\sharp(x, t), \quad \varphi^\sharp(t) := (\mathbf{1}_{A(t)} - \mathbf{1}_{A(t)^c}).$$

Note that  $\varphi^\sharp(t) = \varphi^\sharp(0) \circ \Phi^t$ , so  $(\partial_t + a \cdot \nabla_x)\varphi^\sharp = 0$ . If we could use this  $\varphi^\sharp$  as a test function in (5.106), we would get (taking  $T = t = t_n$ ) the estimate

$$\begin{aligned} & \|u_{h,\Delta t}(t) - u(t)\|_{L^1(\mathbb{T}^d)} \\ & \leq \|u_{h,\Delta t}(0) - u(0)\|_{L^1(\mathbb{T}^d)} + \langle \mu_{h,\Delta t}^0, |\varphi^\sharp| \rangle + \langle \mu_{h,\Delta t}^1, |\partial_t \varphi^\sharp| \rangle + \langle \mu_{h,\Delta t}^2, |\nabla_x \varphi^\sharp| \rangle, \end{aligned} \quad (5.108)$$

and then would have to work on the error terms. Since  $\varphi^\sharp$  has not the regularity properties that justify (5.108), we proceed differently and consider a regularized version of  $\varphi^\sharp$ . Let  $\varphi_0$  be defined by

$$\varphi_0(x) = \min(1, Ld(x, \partial A))\mathbf{1}_A - \min(1, Ld(x, \partial A))\mathbf{1}_{A^c}$$

and let  $\varphi(x, t) = \varphi_0 \circ \Phi^t(x)$ . The function  $\varphi_0$  is Lipschitz continuous, this is enough regularity to justify, after a preliminary regularization procedure, that (5.106) is valid with  $\varphi$  as a test-function. Since  $\|\varphi_0\|_{L^\infty(\mathbb{T}^d)} \leq 1$ , we obtain

$$\begin{aligned} & \left| \int_{\mathbb{T}^d} (u_{h,\Delta t}(t) - u(t))\varphi(x, t) dx \right| \\ & \leq \|u_{h,\Delta t}(0) - u(0)\|_{L^1(\mathbb{T}^d)} + \langle \mu_{h,\Delta t}^0, |\varphi| \rangle + \langle \mu_{h,\Delta t}^1, |\partial_t \varphi| \rangle + \langle \mu_{h,\Delta t}^2, |\nabla_x \varphi| \rangle, \end{aligned} \quad (5.109)$$

instead of (5.108). In the next section, we will explain how to exploit (5.109) to obtain (5.79).

*Remark 5.6.* The step consisting in Reduction 3 is necessary in our method of proof. We can illustrate this in dimension  $d = 2$ . Indeed, assume from the start that  $u_0$  is the characteristic function of a set  $A$  of finite perimeter, in which case the ‘‘Reduction 4’’ step is irrelevant. Assume that the volume of  $|A|$  is positive, say  $\geq \frac{1}{3}$ , to discard trivial sets that have a too small volume. Consider the case where  $A$  is the union of small discs of radius  $r > 0$  centered at the points  $(k/n, l/n)$ , where  $0 < k, l < n$  are integers. There are  $(n-1)^2$  such points. If  $r < \frac{1}{2n}$ , then

the volume of  $A$  is  $(n-1)^2\pi r^2$ . We take  $r = \frac{1}{3n}$  to obtain  $|A| \geq \pi/9 > 1/3$ . If  $\varepsilon$  (that will be  $h^{1/2}$ ) is smaller enough to ensure that  $r + \varepsilon < \frac{1}{2n}$  (i.e.  $\varepsilon < \frac{1}{6n}$ ) then the volume of  $A_{(\varepsilon)}$  is  $(n-1)^2\pi(r + \varepsilon)^2$ . We cannot have

$$|A_{(\varepsilon)}| \leq C|\partial A|\varepsilon, \quad (5.110)$$

with  $C$  independent on  $\varepsilon$  then, because  $|\partial A| = 2\pi(n-1)^2r$ . If (5.110) were true, we would get  $(r + \varepsilon)^2 \leq 2Cr$ , which leads to a contradiction when  $r \rightarrow 0$ .

### 5.11.2 Error estimate

We examine first the integral in the left-hand side of (5.109), that we would like to compare to the exact  $L^1$ -norm  $\|u_{h,\Delta t}(t) - u(t)\|_{L^1(\mathbb{T}^d)}$ . Since  $\|u_{h,\Delta t}(t) - u(t)\|_{L^\infty(\mathbb{T}^d)} \leq 1$ , we have

$$\|u_{h,\Delta t}(t) - u(t)\|_{L^1(\mathbb{T}^d)} \leq \left| \int_{\mathbb{T}^d} (u_{h,\Delta t}(t) - u(t))\varphi(x,t)dx \right| + \|\varphi^\sharp(t) - \varphi(t)\|_{L^1(\mathbb{T}^d)}.$$

By the conservation property (5.103) for  $p = 1$ ,

$$\|\varphi^\sharp(t) - \varphi(t)\|_{L^1(\mathbb{T}^d)} = \|\varphi^\sharp(0) - \varphi(0)\|_{L^1(\mathbb{T}^d)} \leq |A_{(h)}|.$$

We use the estimate (5.107) (and the fact that  $|\partial A| = |Du_0|(\mathbb{T}^d)$  in our context) to obtain

$$\|u_{h,\Delta t}(t) - u(t)\|_{L^1(\mathbb{T}^d)} \leq \left| \int_{\mathbb{T}^d} (u_{h,\Delta t}(t) - u(t))\varphi(x,t)dx \right| + C|Du_0|(\mathbb{T}^d)h^{1/2}. \quad (5.111)$$

The first term  $\|u_{h,\Delta t}(0) - u(0)\|_{L^1(\mathbb{T}^d)}$  in the right-hand side of (5.109) is bounded by  $C|Du_0|(\mathbb{T}^d)h$  as a consequence of Proposition 5.13. By (the proof of) Theorem 5.5 and (5.102), we have

$$\langle \mu_{h,\Delta t}^0, |\varphi| \rangle \leq \|\varphi_0\|_{L^\infty(\mathbb{T}^d)} \omega_{L^1}(u_0; h) \leq C|Du_0|(\mathbb{T}^d)h.$$

We can now begin the study of the two most important terms in (5.109):  $\langle \mu_{h,\Delta t}^1, |\partial_t \varphi| \rangle$  and  $\langle \mu_{h,\Delta t}^2, |\nabla_x \varphi| \rangle$ . To that purpose, we need to come back to the definition of  $\mu_{h,\Delta t}^1$  and  $\mu_{h,\Delta t}^2$  in the proof of Theorem 5.5, cf. (5.71) and (5.74):

$$\langle \psi, \mu_{h,\Delta t}^1 \rangle = \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}^\sharp} |u_K^n - u_K^{n+1}| \int_{t_n}^{t_{n+1}} \int_K \psi(x,t) dx dt, \quad (5.112)$$

and (taking into account the expression (5.11) of the numerical flux in (5.74)):

$$\begin{aligned} \langle \psi, \mu_{h,\Delta t}^2 \rangle &:= \sum_{n=0}^{N-1} \sum_{K \in \mathcal{T}^\sharp} \sum_{L \in \mathcal{N}(K)} a_{K \rightarrow L}^- |u_K^n - u_L^n| \\ &\quad \times \frac{1}{|K|} \int_{t_n}^{t_{n+1}} \int_{K|L} \int_K \int_0^1 \psi(ry + (1-r)x, t) |x-y| dr dx d\sigma(y) dt. \end{aligned} \quad (5.113)$$

The norm of the gradient  $\nabla_x \varphi(x, t) = (\nabla \Phi^t(x))^* (\nabla_x \varphi_0) \circ \Phi^t(x)$  is bounded by

$$\|\nabla \Phi^t\|_{L^\infty(\mathbb{T}^d)} \|\nabla_x \varphi_0\|_{L^\infty(\mathbb{T}^d)}.$$

We have  $\|\nabla_x \varphi_0\|_{L^\infty(\mathbb{T}^d)} \leq L \leq h^{-1/2}$  and  $\|\nabla \Phi^t\|_{L^\infty(\mathbb{T}^d)} \leq e^{t\|a\|_{C^1(\mathbb{T}^d, \mathbb{R}^d)}}$ . This last bound comes from the identities (where  $\nabla \Phi = (\partial_i \Phi_j)_{i,j}$ )

$$\nabla \Phi_t = \exp \left( \int_0^t \nabla a \circ \Phi_s ds \right), \quad I_d = \nabla \Phi_t(x) (\nabla \Phi^t) (\Phi_t(x)),$$

for all  $x \in \mathbb{T}^d$ . Using the transport equation  $\partial_t \varphi = -a \cdot \nabla_x \varphi$ , we deduce from these estimates that

$$\|\nabla_{t,x} \varphi\|_{L^\infty(\mathbb{T}^d \times [0,T])} \leq Ch^{-1/2}, \quad (5.114)$$

where  $C$  depends on  $\|a\|_{C^1(\mathbb{T}^d; \mathbb{R}^d)}$  and  $T$  only. We also remark that the derivatives  $\nabla_{t,x} \varphi$  are supported in the “streak”

$$\mathcal{S}_{\langle h^{1/2} \rangle} = \bigcup_{0 \leq t \leq T} \Phi_t(A_{\langle h^{1/2} \rangle}) \times \{t\}.$$

This has the consequence that

$$\langle \mu_{h,\Delta t}^1, |\partial_t \varphi| \rangle \leq Ch^{-1/2} \sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}^\#} |K| |u_K^n - u_K^{n+1}| \chi(K \times (t_n, t_{n+1})),$$

where  $\chi(K \times (t_n, t_{n+1})) = 1$  if  $K \times (t_n, t_{n+1})$  intersects the set  $\mathcal{S}_{\langle h^{1/2} \rangle}$ , and 0 otherwise. By the Cauchy-Schwarz inequality and (5.56), we obtain

$$|\langle \mu_{h,\Delta t}^1, |\partial_t \varphi| \rangle|^2 \leq Ch^{-1} \Delta t \mathcal{D}(t_N) \sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}^\#} |K| \chi(K \times (t_n, t_{n+1})). \quad (5.115)$$

To estimate the term

$$\mathbb{S}_{\langle h^{1/2} \rangle} := \sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}^\#} |K| \chi(K \times (t_n, t_{n+1}))$$

in (5.115), we will use the Lipschitz bounds  $\|\nabla \Phi_t\|_{L^\infty(\mathbb{T}^d)} \leq C$ ,  $\|\nabla \Phi^t\|_{L^\infty(\mathbb{T}^d)} \leq C$  satisfied by the flow and the inverse flow. First, we notice that

$$\chi(K \times (t_n, t_{n+1})) = 1 \Rightarrow \tilde{\chi}_n(x) = 1, \quad x \in K,$$

where  $\tilde{\chi}_n(x) = 1$  if, and only if, there is a  $t \in (t_n, t_{n+1})$  such that  $x$  is at distance at most  $h$  of the set  $\Phi_t(A_{\langle h^{1/2} \rangle})$ . We then have, for a certain  $a \in A_{\langle h^{1/2} \rangle}$ ,

$$d(x, \Phi_t(a)) < h \Rightarrow d(\Phi^t(x), a) < Ch \Rightarrow d(\Phi^{t_n}(x), a) < C(h + \Delta t) \Rightarrow \Phi^{t_n}(x) \in A_{\langle h^{1/2} \rangle + C(h + \Delta t)},$$

with quite clear notations. We use the bound  $\delta t \leq C_0 h$  and we see that we have shown that  $\chi_n \leq \mathbf{1}_{A_{\langle h^{1/2} \rangle + C_0 h}} \circ \Phi^{t_n}$ . It is easy to adapt the proof of (5.107) to show that the estimate

$$|A_{\langle h^{1/2} \rangle + C_0 h}| \leq C |\partial A| h^{1/2} \quad (5.116)$$

holds true, with possibly a different constant  $C$ . Eventually, we have

$$\mathbb{S}_{\langle h^{1/2} \rangle} \leq \sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}^\#} \int_K \mathbf{1}_{A_{\langle h^{1/2} \rangle + C_0 h}} \circ \Phi^{t_n}(x) dx = \sum_{n=0}^{N-1} \Delta t \int_{\mathbb{T}^d} \mathbf{1}_{A_{\langle h^{1/2} \rangle + C_0 h}} \circ \Phi^{t_n}(x) dx.$$

We use the fact that the Lebesgue measure is invariant by  $\Phi^{t_n}$  and (5.116) to obtain the bound  $\mathbb{S}_{\langle h^{1/2} \rangle} \leq C |\partial A| h^{1/2}$ . We report this estimate in (5.115) (and use the bound  $\Delta t \leq C_0 h$ ) to conclude that

$$|\langle \mu_{h,\Delta t}^1, |\partial_t \varphi| \rangle|^2 \leq C \mathcal{D}(t_N) h^{1/2}. \quad (5.117)$$

By similar arguments, we obtain the analogous estimate

$$|\langle \mu_{h,\Delta t}^2, |\nabla_x \varphi| \rangle|^2 \leq C |Du_0|(\mathbb{T}^d) \mathcal{D}(t_N) h^{1/2}. \quad (5.118)$$

We have, indeed, by (5.113) and the bounds on  $\nabla_x \varphi$ ,

$$|\langle \mu_{h,\Delta t}^2, |\nabla_x \varphi| \rangle| \leq Ch^{1/2} \sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}^\#} \sum_{L \in \mathcal{N}(K)} |K|L|a_{\bar{K} \rightarrow L}^- |u_K^n - u_L^n| \chi(\bar{K} \times [t_n, t_{n+1}]).$$

The first inequality in (5.57) reads

$$\sum_{n=0}^N \Delta t \sum_{K \in \mathcal{T}^\#} \sum_{L \in \mathcal{N}(K)} |K|L|a_{\bar{K} \rightarrow L}^- |u_K^n - u_L^n|^2 \leq 4\|a\|_{C(\mathbb{T}^d; \mathbb{R}^d)} \mathcal{D}(t_N).$$

By the Cauchy-Schwarz inequality, we obtain

$$|\langle \mu_{h,\Delta t}^2, |\nabla_x \varphi| \rangle|^2 \leq Ch \mathcal{D}(t_N) \sum_{n=0}^{N-1} \Delta t \sum_{K \in \mathcal{T}^\#} \sum_{L \in \mathcal{N}(K)} |K|L| \chi(\bar{K} \times [t_n, t_{n+1}]).$$

Since

$$h \sum_{L \in \mathcal{N}(K)} |K|L| = h|\partial K| \leq \alpha^{-2}|K|,$$

by (5.27), we see that  $|\langle \mu_{h,\Delta t}^2, |\nabla_x \varphi| \rangle|^2 \leq Ch \mathcal{D}(t_N) \mathbb{S}_{\langle h^{1/2} \rangle}$  and the estimate on  $\mathbb{S}_{\langle h^{1/2} \rangle}$  given above yields (5.118). To sum up, we have shown that

$$\|u_{h,\Delta t}(t) - u(t)\|_{L^1(\mathbb{T}^d)} \leq C \left\{ |Du_0|(\mathbb{T}^d) \mathcal{D}(t_N) h^{1/2} \right\}^{1/2} + C|Du_0|(\mathbb{T}^d) h^{1/2} \quad (5.119)$$

We see here that, simply estimating  $\mathcal{D}(t_N)$  from above by  $\|u_0\|_{L^2(\mathbb{T}^d)}^2$  will not be enough to conclude. Instead, the energy estimate (5.45) must be fully exploited. It gives, indeed (recall that  $t \in [t_N, t_{N+1}]$ ) a bound on the quantity  $2\xi \mathcal{D}(t_N)$  by the difference  $\|u_0\|_{L^2(\mathbb{T}^d)}^2 - \|u_{h,\Delta t}(t)\|_{L^2(\mathbb{T}^d)}^2$ . By the conservation of the  $L^p$ -norms in the continuity equation (5.5),  $\|u_0\|_{L^2(\mathbb{T}^d)}^2 = \|u(t)\|_{L^2(\mathbb{T}^d)}^2$ . Since  $u(t)$  and  $u_{h,\Delta t}(t)$  are bounded by 1 in  $L^\infty(\mathbb{T}^d)$ , we obtain

$$\xi \mathcal{D}(t_N) \leq \|u_{h,\Delta t}(t) - u(t)\|_{L^1(\mathbb{T}^d)}. \quad (5.120)$$

We report the estimate (5.120) and use the inequality  $2ab \leq \eta a^2 + \eta^{-1} b^2$  with a parameter  $\eta$  small enough (with respect to the constant  $C$ ) to obtain

$$\|u_{h,\Delta t}(t) - u(t)\|_{L^1(\mathbb{T}^d)} \leq \frac{1}{2} \|u_{h,\Delta t}(t) - u(t)\|_{L^1(\mathbb{T}^d)} s + C|Du_0|(\mathbb{T}^d) h^{1/2},$$

and the error estimate (5.79) follows.

## 6 Interacting particle systems

We recall the situation described in Section 1.3. Let  $\Lambda_N$  be a finite subset of  $\mathbb{Z}^d$ . We consider a system of particles scattered on  $\Lambda_N$ , which interact as follows: let  $\mathbf{x}$  denote a typical site of  $\Lambda_N$  and let  $\eta_t(\mathbf{x})$  denote the number of particles located at site  $\mathbf{x}$  at time  $t$ . We will be interested in the evolution in time of the functions  $\mathbf{x} \mapsto \eta_t(\mathbf{x})$ . The state space is therefore  $\mathcal{E}_N := \mathbb{N}^{\Lambda_N}$ , the set of functions  $\Lambda_N \rightarrow \mathbb{N}$ . The evolution is described by the following algorithm: each site  $\mathbf{x}$  has its own clock that is independent from the clocks at other sites, and that rings after a time  $T_{\mathbf{x}}$  which is a random variable of exponential law of parameter  $\lambda(\eta(\mathbf{x}))$ . Assume that it is at the site  $\mathbf{x}_*$  that a clock is ringing first. If  $\eta(\mathbf{x}_*) > 0$ , then one particle of the site  $\mathbf{x}_*$  jumps to an other site  $\mathbf{y}$  chosen at random in  $\Lambda_N$ , according to a transition probability  $p(\mathbf{x}_*, \mathbf{y})$  (possibly, at that stage, some exclusion rules may be added, see Section 6.2.1). Then we start over. Let us consider the case where  $\Lambda_N$  is the discrete torus  $\mathbb{T}_N^d = \mathbb{Z}^d / N\mathbb{Z}^d$  and  $p$  is compatible and translation invariant: for all  $l \in N\mathbb{Z}^d$ ,  $m \in \mathbb{Z}^d$ ,

$$p(\mathbf{x} + l, \mathbf{y}) = p(\mathbf{x}, \mathbf{y}), \quad p(\mathbf{x} + m, \mathbf{y} + m) = p(\mathbf{x}, \mathbf{y}). \quad (6.1)$$

Let us zoom out (cf. (1.6)) by considering the function

$$[0, 1)^d \ni x \mapsto N^{-1}\eta_t([Nx]) \quad (6.2)$$

extended by periodicity. In (1.9),  $[Nx]$  is the element  $\mathbf{x}$  of  $\mathbb{T}_N^d \simeq \{0, \dots, N-1\}^d$  such that  $\mathbf{x}_i \leq Nx_i < \mathbf{x}_i + 1$  for all  $i = 1, \dots, d$ . In Section 1.3, we raised the following question: may it be the case that, possibly after a change of time scale (cf. (1.7)), some averaging phenomena would lead to a given deterministic behavior? We will explain very briefly how the answers to this question may differ, depending on the kind of stochastic interaction that govern the dynamics of the particles, and what is the common approach to these answers.

### 6.1 Empirical measure

#### 6.1.1 An alternative description of the system

We slightly change our point of view. To each  $\eta \in \mathcal{E}_N = \mathbb{N}^{\mathbb{T}_N^d}$  we can associate in a unique way the measure

$$\kappa^N[\eta] = \frac{1}{N^d} \sum_{\mathbf{x} \in \mathbb{T}_N^d} \eta(\mathbf{x}) \delta_{\hat{\mathbf{x}}}, \quad (6.3)$$

where  $\hat{\mathbf{x}} := N^{-1}\mathbf{x}$ . This last change of scale means that we see  $\kappa^N[\eta]$  as a measure on  $\mathbb{T}^d$ . This operation is very natural if, for instance,  $\eta(\mathbf{x})$  is considered as the height of a sand pile over the point  $\mathbf{x}$ : the measure  $\kappa^N[\eta]$  describes the whole sand-pile then. Note the normalizing factor  $\frac{1}{N^d}$  in (6.3) which ensures that the total mass of  $\kappa^N[\eta]$  is bounded, provided the total number of particles

$$K := \sum_{\mathbf{x} \in \mathbb{T}_N^d} \eta(\mathbf{x}) \quad (6.4)$$

is of order  $N^d$ . Let then  $(\eta_n)$  be a sequence of random variables over  $\mathcal{E}_N$ . This may be a sequence of possible initial data for the process  $(\eta_t)$  that we consider, or the values of the corresponding solutions at a given time  $T$ . We say that  $(\eta_n)$  is *associated with a measure*  $\zeta$  on  $\mathbb{T}^d$  if the sequence  $(\kappa^N[\eta_n])$  converges in probability to the deterministic measure  $\zeta$  in the space of finite measures on  $\mathbb{T}^d$ . More precisely: let  $\mathcal{M}_+(\mathbb{T}^d)$  denote the set of finite (non-negative) Borel measures over



$\mathbb{T}^d$ . This is the set of locally finite non-negative Borel measures  $\zeta$  over  $\mathbb{R}^d$  that are invariant by the action of  $\mathbb{Z}^d$ , the latter being defined by translation: if  $m \in \mathbb{Z}^d$ ,  $m \cdot \zeta$  is the measure given by

$$\langle m \cdot \zeta, h \rangle = \int_{\mathbb{R}^d} h(x+m) d\zeta(x), \quad h \in C_c(\mathbb{R}^d).$$

If  $\zeta \in \mathcal{M}_+(\mathbb{T}^d)$  and  $h \in C(\mathbb{T}^d)$ , then  $\int_{\mathbb{T}^d} h d\zeta$  stands for

$$\int_{\mathbb{T}^d} h d\zeta = \langle \zeta, h \rangle = \int_{[0,1]^d} h(x) d\zeta(x).$$

We endow  $\mathcal{M}_+(\mathbb{T}^d)$  with the topology of the weak convergence of measure: a sequence  $(\zeta_n)$  of  $\mathcal{M}_+(\mathbb{T}^d)$  weakly converges to  $\zeta \in \mathcal{M}_+(\mathbb{T}^d)$  if  $\langle \zeta_n, h \rangle \rightarrow \langle \zeta, h \rangle$  for all  $h \in C(\mathbb{T}^d)$ . Let  $V$  be an open neighborhood of  $\zeta$  for this topology. The sequence  $(\eta_n)$  is associated with a measure  $\zeta$  on  $\mathbb{T}^d$  if

$$\lim_{N \rightarrow +\infty} \mathbb{P}(\kappa^N[\eta_N] \notin V) = 0. \quad (6.5)$$

A basis of neighborhood of  $\zeta$  for the weak topology is constituted by the finite intersection of sets of the form

$$V = \{ \zeta' \in \mathcal{M}_+(\mathbb{T}^d); |\langle \zeta', h \rangle - \langle \zeta, h \rangle| < \delta \},$$

where  $\delta > 0$  and  $h \in C(\mathbb{T}^d)$ . Therefore (6.5) is equivalent to require

$$\lim_{N \rightarrow +\infty} \mathbb{P}(|\langle \kappa^N[\eta_N], h \rangle - \langle \zeta, h \rangle| > \delta) = 0, \quad (6.6)$$

for all  $\delta > 0$  and  $h \in C(\mathbb{T}^d)$ . If we denote by  $\mu_n$  the law of  $\eta_n$ , then (6.6) can be rewritten as

$$\lim_{N \rightarrow +\infty} \mu_N \left( \left| \frac{1}{N^d} \sum_{\mathbf{x} \in \mathbb{T}_N^d} \eta(\mathbf{x}) h(N^{-1}\mathbf{x}) - \int_{\mathbb{T}^d} h d\zeta \right| > \delta \right) = 0. \quad (6.7)$$

When  $\zeta$  is absolutely continuous with respect to the Lebesgue measure on  $\mathbb{T}^d$ , with density  $\rho$ , (6.7) is the definition used in [13, p.43] (the sequence  $(\mu_n)$  is then said to be *associated to the profile*  $\rho$ ).

### 6.1.2 Empirical measure

Consider the following situation: each day (or hour, or minute... the unit of time may vary) a given experiment is conducted. On day  $n$ , a given quantity is measured, and after  $N$  days, one considers the average of all the measurements that have been recorded. The situation is modeled as follows:

1. the result of the experiment  $n$  is a point  $X_n$  in a state space  $E$  (say a Banach space for example),
2. the measure recorded on day  $n$  is  $\varphi(X_n)$ , where  $\varphi$  is a real-valued continuous bounded function on  $E$ ,
3. the state  $X_n$  is a random variable with a given law.

The average over  $N$  days is then

$$\frac{1}{N} \sum_{n=1}^N \varphi(X_N) = \langle \pi^N, \varphi \rangle, \quad \pi^N := \frac{1}{N} \sum_{n=1}^N \delta_{X_N}. \quad (6.8)$$

The measure  $\pi^N$  is called the empirical measure, for the reason that it is the central object in the empirical approach that is considered. From that point of view, in the study of the interacting particle system, we should reserve the term empirical measure to the measure

$$\pi^N[\eta] = \frac{1}{N^d} \sum_{\mathbf{x} \in \mathbb{T}_N^d} \delta_{(\hat{\mathbf{x}}, \tau_{\mathbf{x}}\eta)}, \quad (6.9)$$

the time variable of (6.8) being now the space variable  $\mathbf{x}$ . In (6.9), we have used the notation  $\tau_{\mathbf{x}}\eta$  to denote the function  $\mathbf{y} \mapsto \eta(\mathbf{y} + \mathbf{x})$ . This is a way to describe  $\eta$  “as seen from  $\mathbf{x}$ ”. If  $\varphi$  is a function  $\mathcal{E}_N \rightarrow \mathbb{R}$ , we will also denote by  $\tau_{\mathbf{x}}\varphi$  the function  $\varphi \circ \tau_{\mathbf{x}}$ . Let us also introduce the notation  $\iota_{\mathbf{x}}$  for the evaluation function  $\eta \mapsto \eta(\mathbf{x})$ . In that way, we have  $\iota_{\mathbf{x}} = \tau_{\mathbf{x}}\iota_0$  and  $\kappa^N[\eta]$  is deduced from  $\pi^N[\eta]$  by the formula

$$\langle \kappa^N[\eta], h \rangle = \langle \pi^N[\eta], h \otimes \iota_0 \rangle, \quad h \in C(\mathbb{T}^d). \quad (6.10)$$

Beware that the measure  $\kappa^N[\eta]$  in (6.3) is often called empirical measure also (this is meaningful if  $\eta$  is considered fixed, not random, while the base point  $\mathbf{x}$  is chosen uniformly in  $\mathbb{T}_N^d$ ; in that case the state space is  $\mathbb{T}_N^d$  hence). In what follows, we will simply speak of measure, and specify if we consider  $\kappa^N[\eta]$  or  $\pi^N[\eta]$ , depending on the cases.

## 6.2 Equilibria

### 6.2.1 Generator

Using the results of Section 4.2.1, one can show that the generator associated to the process  $(\eta_t)_{t \geq 0}$  is

$$(\mathcal{L}_N \varphi)(\eta) = \sum_{\mathbf{x}, \mathbf{y} \in \mathbb{T}_N^d} c(\mathbf{x}, \mathbf{y}, \eta) (\varphi(\eta^{\mathbf{x}, \mathbf{y}}) - \varphi(\eta)), \quad (6.11)$$

where  $\eta^{\mathbf{x}, \mathbf{y}}$  denotes the configuration deduced from  $\eta$  when a particle at site  $\mathbf{x}$  has jumped to site  $\mathbf{y}$ :

$$\eta^{\mathbf{x}, \mathbf{y}}(\mathbf{z}) = \eta(\mathbf{z}) - \mathbf{1}_{\mathbf{z}=\mathbf{x}} + \mathbf{1}_{\mathbf{z}=\mathbf{y}}. \quad (6.12)$$

The coefficient  $c(\mathbf{x}, \mathbf{y}, \eta)$  is given by  $c(\mathbf{x}, \mathbf{y}, \eta) = \lambda(\eta(\mathbf{x}))p(\mathbf{x}, \mathbf{y})$ , up to the possible exclusion rules mentioned in the beginning of Section 6. We will consider the following examples:

**Simple exclusion process.** There is zero or one particle at each site (vacant or occupied), and a particle cannot jump to an occupied site. In that case  $\eta(\mathbf{x}) \in \{0, 1\}$ ,  $\lambda(\eta(\mathbf{x})) = \mathbf{1}_{\eta(\mathbf{x})=1} = \eta(\mathbf{x})$  and the exclusion rule leads to a coefficient  $c(\mathbf{x}, \mathbf{y}, \eta) = \lambda(\eta(\mathbf{x}))p(\mathbf{x}, \mathbf{y})$  given by

$$c(\mathbf{x}, \mathbf{y}, \eta) = \eta(\mathbf{x})(1 - \eta(\mathbf{y}))p(\mathbf{x}, \mathbf{y}). \quad (6.13)$$

**Zero range process.** In that case  $\lambda(\eta(\mathbf{x})) = g(\eta(\mathbf{x}))$  where  $g: \mathbb{N} \rightarrow \mathbb{R}$  is a function which satisfies the following hypotheses:  $g(0) = 0$ ,  $g(k) > 0$  for  $k > 0$  and  $g$  is of bounded variation:

$$g^* = \sup_{k \geq 0} |g(k+1) - g(k)| < +\infty. \quad (6.14)$$

We have then

$$(\mathcal{L}_N \varphi)(\eta) = \sum_{\mathbf{x}, \mathbf{y} \in \mathbb{T}_N^d} g(\eta(\mathbf{x})) p(\mathbf{x}, \mathbf{y}) (\varphi(\eta^{\mathbf{x}, \mathbf{y}}) - \varphi(\eta)). \quad (6.15)$$

Some particular examples are given by

1. independent random walks: this corresponds to  $g(k) = k$ .
2. the case  $g(k) = \mathbf{1}_{k>0}$ : this models a system of queues with mean-one exponential random times of service.

### 6.2.2 Invariant measure

Let  $\mathcal{E}_N[K]$  denote the set of configurations  $\eta \in \mathcal{E}_N$  with total number of particles  $K$ , as in (6.4). The evolution of the process  $(\eta_t)$  takes place in  $\mathcal{E}_N[K]$ , where  $K$  is fixed by  $\eta_0$ . We assume that the Markov process  $(\eta_t)$  is irreducible on  $\mathcal{E}_N[K]$ . This property may, or may not, be realized, depending on the choice of the transition probabilities  $p(\mathbf{x}, \mathbf{y})$ . The invariance by translation (6.1) implies that  $p(\mathbf{x}, \mathbf{y}) = \mathbf{p}(\mathbf{y} - \mathbf{x})$  for a given function  $\mathbf{p}$ . When  $d = 2$  for instance, we may well consider the case where  $\mathbf{p}(e_1) + \mathbf{p}(-e_1) = 1$  ( $e_1$  is the first basis vector). Only horizontal jumps are possible then, and our system may be decoupled into a series of given equivalent systems on the one-dimensional discrete torus  $\mathbb{T}_N^1$ . For simplicity, we will assume that,  $p$  has range 1:  $\mathbf{p}(v) = 0$  if  $|v| \neq 1$ , and that for each direction  $e_i$ , we have

$$\mathbf{p}(e_i) + \mathbf{p}(-e_i) > 0. \quad (6.16)$$

Under condition (6.16), the Markov chain  $(\eta_t)$  is irreducible on the finite state space  $\mathcal{E}_N[K]$ . Consequently, it admits a unique invariant measure  $\nu_{K,N}$ . In the models that are considered, we expect these invariant measures  $\nu_{K,N}$  to have a limit  $\nu_\rho$  when  $N \rightarrow +\infty$  with  $KN^{-d} \rightarrow \rho \in \mathbb{R}_+$ . In the examples given in the previous section 6.2.1, we have an explicit expression of these equilibrium measures  $\nu_\rho$ . These measures are measures on the state space  $\mathbb{N}^{\mathbb{Z}^d}$ , because they are obtained as limits of measures on  $\mathbb{N}^{\mathbb{T}_N^d}$ . They are *product measures*, in the following sense: let  $\nu_\rho^{\mathbf{x}}$  denote the measure on  $\mathbb{N}$  defined by

$$\nu_\rho^{\mathbf{x}}(A) = \nu_\rho(\eta(\mathbf{x}) \in A). \quad (6.17)$$

The measure  $\nu_\rho^{\mathbf{x}}$  is the push-forward of the measure  $\nu_\rho$  by the evaluation function  $\iota_{\mathbf{x}}: \eta \mapsto \eta(\mathbf{x})$ . That  $\nu_\rho$  is a product measure means that

$$\nu_\rho = \bigotimes_{\mathbf{x} \in \mathbb{Z}^d} \nu_\rho^{\mathbf{x}}. \quad (6.18)$$

It is equivalent to say that, when  $\eta \in \mathbb{N}^{\mathbb{Z}^d}$  is picked up at random under  $\nu_\rho$ , then the family of random variables  $(\eta(\mathbf{x}))_{\mathbf{x} \in \mathbb{Z}^d}$  is independent (and i.i.d. actually, because we assume invariance by translation in the system). We will see, for our examples of interest, what are these measures. We will also consider the measure on  $\mathcal{E}_N = \mathbb{N}^{\mathbb{T}_N^d}$  obtained by making the product in (6.17) over  $\mathbf{x} \in \mathbb{T}_N^d$  instead of  $\mathbf{x} \in \mathbb{Z}^d$ . We denote by  $\nu_\rho^N$  this measure:

$$\nu_\rho^N = \bigotimes_{\mathbf{x} \in \mathbb{T}_N^d} \nu_\rho^{\mathbf{x}}. \quad (6.19)$$

We will show that is an invariant measure for the particle system considered on  $\mathcal{E}_N$ .

### 6.2.3 Simple exclusion process

**Proposition 6.1** (Invariant measures for simple exclusion processes). *For  $\rho \in [0, 1]$ , define  $\nu_\rho^x$  as the Bernoulli measure with parameter  $\rho$ . Then  $\nu_\rho^N$  defined by (6.19) is an invariant measure for the simple exclusion process. The adjoint process with respect to  $\nu_\rho^N$  is the simple exclusion process with rate  $\check{\mathfrak{p}}$ , where  $\check{\mathfrak{p}}(v) = \mathfrak{p}(-v)$ .*

*Proof of Proposition 6.1.* Let  $\check{\mathcal{L}}_N$  be the generator associated to the the simple exclusion process with rate  $\check{\mathfrak{p}}$ . Let  $\varphi, \psi$  be some given function  $\mathcal{E}_N \rightarrow \mathbb{R}$ . The second part of the statement of Proposition 6.1 is expressed by the formula

$$\int_{\mathcal{E}_N} \mathcal{L}_N \varphi(\eta) \psi(\eta) d\nu_\rho^N(\eta) = \int_{\mathcal{E}_N} \varphi(\eta) \check{\mathcal{L}}_N \psi(\eta) d\nu_\rho^N(\eta), \quad (6.20)$$

that should be satisfied for all  $\varphi, \psi$  bounded cylinder functions (cf. Section 6.2.5). Since  $\check{\mathcal{L}}_N \mathbf{1} = 0$  (where  $\mathbf{1}$  is the constant function equal to 1), (6.20) implies

$$\int_{\mathcal{E}_N} \mathcal{L}_N \varphi(\eta) d\nu_\rho^N(\eta) = 0,$$

which means that  $\nu_\rho$  is invariant. To prove (6.20), we expand

$$\int_{\mathcal{E}_N} \mathcal{L}_N \varphi(\eta) \psi(\eta) d\nu_\rho^N(\eta) = \sum_{\mathbf{x}, \mathbf{y} \in \mathbb{T}_N^d} \int_{\mathcal{E}_N} p(\mathbf{x}, \mathbf{y}) \eta(\mathbf{x}) (1 - \eta(\mathbf{y})) [\varphi(\eta^{\mathbf{x}, \mathbf{y}}) - \varphi(\eta)] \psi(\eta) d\nu_\rho^N(\eta) \quad (6.21)$$

Let us set  $s_{\mathbf{x} \rightarrow \mathbf{y}}(\eta) := \eta^{\mathbf{x}, \mathbf{y}}$ . Then each term in the sum (6.21) is of the form

$$p(\mathbf{x}, \mathbf{y}) \langle \nu_\rho^N, \iota_{\mathbf{x}}(1 - \iota_{\mathbf{y}})(\varphi \circ s_{\mathbf{x} \rightarrow \mathbf{y}} - \varphi) \psi \rangle. \quad (6.22)$$

Taking into account that  $\eta(\mathbf{x})(1 - \eta(\mathbf{y}))$  vanishes when  $\eta(\mathbf{x}) = 0$  or  $\eta(\mathbf{y}) = 1$  and that  $\nu_\rho^N$  is a product measure, we obtain the following identity:

$$\langle \nu_\rho^N, \iota_{\mathbf{x}}(1 - \iota_{\mathbf{y}})(\varphi \circ s_{\mathbf{x} \rightarrow \mathbf{y}} \psi) \rangle = \langle \nu_\rho^N, \iota_{\mathbf{y}}(1 - \iota_{\mathbf{x}}) \varphi \psi \circ s_{\mathbf{y} \rightarrow \mathbf{x}} \rangle. \quad (6.23)$$

We exchange the roles of  $\mathbf{x}$  and  $\mathbf{y}$  then, to get the expression

$$\frac{1}{2} \sum_{\mathbf{x}, \mathbf{y} \in \mathbb{T}_N^d} \mathfrak{p}(\mathbf{y} - \mathbf{x}) \langle \nu_\rho^N, \iota_{\mathbf{x}}(1 - \iota_{\mathbf{y}})(\varphi \circ s_{\mathbf{x} \rightarrow \mathbf{y}} - \varphi) \psi \rangle + \check{\mathfrak{p}}(\mathbf{y} - \mathbf{x}) \langle \nu_\rho^N, \iota_{\mathbf{x}}(1 - \iota_{\mathbf{y}}) \varphi (\psi \circ s_{\mathbf{x} \rightarrow \mathbf{y}} - \psi) \rangle \quad (6.24)$$

of the right-hand side of (6.20). The expression (6.24) is the sum of the following terms:

$$\frac{1}{2} \sum_{\mathbf{x}, \mathbf{y} \in \mathbb{T}_N^d} \mathfrak{p}(\mathbf{y} - \mathbf{x}) \langle \nu_\rho^N, \iota_{\mathbf{x}}(1 - \iota_{\mathbf{y}}) \varphi \circ s_{\mathbf{x} \rightarrow \mathbf{y}} \psi \rangle + \check{\mathfrak{p}}(\mathbf{y} - \mathbf{x}) \langle \nu_\rho^N, \iota_{\mathbf{x}}(1 - \iota_{\mathbf{y}}) \varphi \psi \circ s_{\mathbf{x} \rightarrow \mathbf{y}} \rangle, \quad (6.25)$$

and

$$\frac{1}{2} \sum_{\mathbf{x}, \mathbf{y} \in \mathbb{T}_N^d} \mathfrak{p}(\mathbf{y} - \mathbf{x}) \langle \nu_\rho^N, \iota_{\mathbf{x}} \iota_{\mathbf{y}} \varphi \psi \rangle + \check{\mathfrak{p}}(\mathbf{y} - \mathbf{x}) \langle \nu_\rho^N, \iota_{\mathbf{x}} \iota_{\mathbf{y}} \varphi \psi \rangle, \quad (6.26)$$

with

$$\frac{1}{2} \sum_{\mathbf{x}, \mathbf{y} \in \mathbb{T}_N^d} \mathfrak{p}(\mathbf{y} - \mathbf{x}) \langle \nu_\rho^N, \iota_{\mathbf{x}} \varphi \psi \rangle + \check{\mathfrak{p}}(\mathbf{y} - \mathbf{x}) \langle \nu_\rho^N, \iota_{\mathbf{x}} \varphi \psi \rangle. \quad (6.27)$$

The two terms (6.25), (6.26) are invariant by the operation  $(p, \varphi, \psi) \leftarrow (\check{p}, \psi, \varphi)$ . The third term (6.27) is equal to

$$\sum_{\mathbf{x} \in \mathbb{T}_N^d} \langle \nu_\rho^N, \iota_{\mathbf{x}} \varphi \psi \rangle,$$

because the sum of  $p(\mathbf{x}, \mathbf{y})$  over  $\mathbf{y}$  is equal to 1. It follows that (6.27) also has the desired invariance property. This concludes the proof.  $\square$

#### 6.2.4 Zero range processes

Consider the zero range process defined in Section 6.2.1. Denote by  $Z(a)$  the generating function

$$Z(a) = \sum_{k \geq 0} \frac{a^k}{g(k)!}, \quad g(k)! := \prod_{j=1}^k g(j), \quad (6.28)$$

with the convention that  $g(0)! = 1$ . Assume that  $Z$  has a radius of convergence  $a^* \in (0, +\infty]$  and that

$$\lim_{a \uparrow a^*} Z(a) = +\infty. \quad (6.29)$$

For instance, for independent random walks, which corresponds to  $g(k) = k$ , the partition function is  $Z(a) = e^a$ , with radius of convergence  $a^* = +\infty$ . In the case  $g(k) = \mathbf{1}_{k>0}$  (queues with mean-one exponential random times of service), the partition function is  $Z(a) = (1-a)^{-1}$ , with radius  $a^* = 1$ . In the following proposition, we give a family of invariant measures indexed by  $a \in (0, a^*)$ . We will see then how to modify this parametrization, to obtain a parametrization by the density  $\rho$ .

**Proposition 6.2** (Invariant measures for zero range processes). *For each  $a \in (0, a^*)$ , consider the product measure  $\bar{\nu}_a = \bigotimes_{\mathbf{x} \in \mathbb{T}_N^d} \bar{\nu}_a^{\mathbf{x}}$ , where  $\bar{\nu}_a^{\mathbf{x}}$  is the probability measure on  $\mathbb{N}$  given by*

$$\bar{\nu}_a^{\mathbf{x}}(\{k\}) = \frac{1}{Z(a)} \frac{a^k}{g(k)!}. \quad (6.30)$$

*Then  $\bar{\nu}_a$  is an invariant measure for the zero range process. The adjoint process with respect to  $\bar{\nu}_a$  is the zero range process with rates  $(g, \check{p})$ .*

*Proof of Proposition 6.2.* We proceed as in the proof of Proposition 6.1. We start from the following expansion:

$$\int_{\mathcal{E}_N} \mathcal{L}_N \varphi(\eta) \psi(\eta) d\bar{\nu}_a(\eta) = \sum_{\mathbf{x}, \mathbf{y} \in \mathbb{T}_N^d} p(\mathbf{x}, \mathbf{y}) \langle \bar{\nu}_a, g \circ \iota_{\mathbf{x}} \varphi \circ s_{\mathbf{x}, \mathbf{y}} \psi \rangle - \sum_{\mathbf{x} \in \mathbb{T}_N^d} \langle \bar{\nu}_a, g \circ \iota_{\mathbf{x}} \varphi \psi \rangle. \quad (6.31)$$

We have used the fact that  $\mathbf{y} \mapsto p(\mathbf{x}, \mathbf{y})$  is probability distribution to obtain (6.31). The last term in the right-hand of (6.31) is of course invariant under the operation

$$(p, \varphi, \psi) \leftarrow (\check{p}, \psi, \varphi). \quad (6.32)$$

We focus on the first term in the right-hand of (6.31), that is

$$\sum_{\mathbf{x}, \mathbf{y} \in \mathbb{T}_N^d} p(\mathbf{x}, \mathbf{y}) \sum_{k \geq 1, j \geq 0} \frac{g(k)}{Z(a)^2} \frac{a^{k+j}}{g(k)!g(j)!} \langle \bar{\nu}_a, T_{\mathbf{x}, \mathbf{y}} \varphi(k-1, j+1) T_{\mathbf{x}, \mathbf{y}} \psi(k, j) \rangle, \quad (6.33)$$

where  $T_{\mathbf{x},\mathbf{y}}\varphi(a, b)$  denotes the function  $\eta \mapsto \varphi(\eta)$  with the variables  $\eta(\mathbf{x})$  and  $\eta(\mathbf{y})$  frozen to the values  $a$  and  $b$  respectively. We use the identity

$$g(k) \frac{a^{k+j}}{g(k)!g(j)!} = g(j+1) \frac{a^{(k-1)+(j+1)}}{g(k-1)!g(j+1)!}$$

to get

$$(6.33) = \sum_{\mathbf{x}, \mathbf{y} \in \mathbb{T}_N^d} p(\mathbf{x}, \mathbf{y}) \sum_{j \geq 1, k \geq 0} \frac{g(j)}{Z(a)^2} \frac{a^{k+j}}{g(k)!g(j)!} \langle \bar{\nu}_a, T_{\mathbf{x},\mathbf{y}}\varphi(k, j) T_{\mathbf{x},\mathbf{y}}\psi(k+1, j-1) \rangle. \quad (6.34)$$

Note that  $T_{\mathbf{x},\mathbf{y}}\varphi(k, j) = T_{\mathbf{y},\mathbf{x}}\varphi(j, k)$ . It remains therefore to exchange the role of  $\mathbf{x}$  and  $\mathbf{y}$  and of  $k$  and  $j$  in (6.34) and to compare the result to (6.33) to establish the equivalent of (6.20).  $\square$

To obtain a parametrization of the invariant measure by the density, we proceed as follows. Let  $R(a)$  denote the average

$$R(a) = \sum_{k \in \mathbb{N}} k \bar{\nu}_a^{\mathbf{x}}(\{k\}) = \frac{1}{Z(a)} \sum_{k \geq 0} k \frac{a^k}{g(k)!} = a \frac{\partial_a Z(a)}{Z(a)} = a \partial_a \ln Z(a). \quad (6.35)$$

**Lemma 6.3.** *The function  $r \mapsto \ln Z(e^r)$  is strictly convex.*

*Proof of Lemma 6.3.* Set  $\theta(r) = \ln Z(e^r)$ . The convexity of  $\theta$  is the property

$$\theta(\lambda r + (1-\lambda)s) < \lambda \theta(r) + (1-\lambda)\theta(s), \quad \lambda \in (0, 1), \quad r \neq s. \quad (6.36)$$

Setting  $r' = \lambda r$ ,  $s' = (1-\lambda)s$ ,  $p = \lambda^{-1}$ ,  $q = (1-\lambda)^{-1}$ , and rearranging the expression, (6.36) is equivalent to

$$Z(e^{r'+s'}) < Z(e^{pr'})^{1/p} Z(e^{qs'})^{1/q}, \quad (6.37)$$

which follows from the Hölder inequality, once we rule out the equality case, which reads  $e^r = e^s$ .  $\square$

**Lemma 6.4.** *The function  $R: [0, a^*) \rightarrow [0, +\infty)$  is onto.*

*Proof of Lemma 6.4.* Assume first that  $a^*$  is finite. If  $R$  is bounded in the neighborhood of  $a^*$ , then  $Z$  also by (6.35), which is in contradiction with (6.29). If  $a^*$  is infinite, we use (6.14), which gives  $g(k) \leq g^*k$ . We may assume  $g^* > 0$  (otherwise  $g \equiv 0$ ). We obtain in (6.35) the estimate

$$R(a) = \frac{1}{Z(a)} \sum_{k \geq 0} k \frac{a^k}{g(k)!} \geq \frac{1}{g^* Z(a)} \sum_{k \geq 1} \frac{a^k}{g(k-1)!} = \frac{a}{g^*}.$$

This shows that  $\lim_{a \rightarrow +\infty} R(a) = +\infty$ .  $\square$

Let  $\Phi: [0, +\infty) \rightarrow [0, a^*)$  denote the inverse of  $R$ , which is well defined by Lemma 6.3 and Lemma 6.4 (and is a non-decreasing function). Then

$$\nu_\rho := \bar{\nu}_{\Phi(\rho)} \quad (6.38)$$

is an invariant measure for the zero range process, parametrized by the density  $\rho$ :

$$\langle \nu_\rho, \iota_{\mathbf{x}} \rangle = \sum_{k \in \mathbb{N}} k \nu_\rho^{\mathbf{x}}(\{k\}) = \rho.$$

Note, using this parametrization, that we have also

$$\langle \bar{\nu}_a, g \rangle = \frac{1}{Z(a)} \sum_{k \geq 0} g(k) \frac{a^k}{g(k)!} = a,$$

and thus

$$\langle \nu_\rho, g \rangle = \Phi(\rho). \quad (6.39)$$

### 6.2.5 Weak local equilibrium

Remember that a cylinder subset of  $\mathcal{E} = \mathbb{N}^{\mathbb{Z}^d}$  is of the form

$$\bigcap_{\mathbf{x} \in \Lambda} \iota_{\mathbf{x}}^{-1}(B_{\mathbf{x}}),$$

where  $B_{\mathbf{x}}$  is a subset of  $\mathbb{N}$  and  $\Lambda$  is a finite subset of  $\mathbb{Z}^d$ . A bounded cylinder function  $\varphi$  on  $\mathcal{E}$  is a function of the form

$$\varphi(\eta) = \theta(\eta(\mathbf{x}_1), \dots, \eta(\mathbf{x}_m)),$$

where  $\theta: \mathbb{R}^m \rightarrow \mathbb{R}$  is measurable bounded. To each bounded cylinder function  $\varphi$ , we associate the function  $\bar{\varphi}: \mathbb{R}_+ \rightarrow \mathbb{R}$  defined by

$$\bar{\varphi}(\rho) = \mathbb{E}_{\nu_\rho}[\varphi] = \int_{\mathcal{E}} \varphi(\eta) d\nu_\rho(\eta). \quad (6.40)$$

Let  $\rho: \mathbb{T}^d \rightarrow \mathbb{R}$  be a measurable function. We can then consider the measure  $\pi[\rho]$  defined by

$$\langle \pi[\rho], h \otimes \varphi \rangle = \int_{\mathbb{T}^d} h(x) \bar{\varphi}(\rho(x)) dx = \iint_{\mathbb{T}^d \times \mathcal{E}} h \otimes \varphi(x, \eta) d\nu_{\rho(x)}(\eta) dx, \quad (6.41)$$

for all  $h \in C(\mathbb{T}^d)$ , and  $\varphi$  bounded cylinder function on  $\mathcal{E}$ . In (6.41), we have used the notation  $h \otimes \varphi(x, \eta) = h(x)\varphi(\eta)$ .

**Definition 6.1** (Weak local equilibrium). Let  $(\eta_N)$  be a sequence of random variables, each with values in  $\mathbb{N}^{\mathbb{T}^d}$ . Let  $\rho: \mathbb{T}^d \rightarrow \mathbb{R}$  be a measurable function. We say that  $(\eta_N)$  is a *weak local equilibrium* of profile  $\rho$  if the sequence of measures  $\pi^N[\eta_N]$  (with  $\pi^N[\eta]$  defined by (6.9)) converges in probability to  $\pi[\rho]$  defined by (6.41), in the sense that one has

$$\lim_{N \rightarrow +\infty} \mathbb{P}(|\langle \pi^N[\eta_N], h \otimes \varphi \rangle - \langle \pi[\rho], h \otimes \varphi \rangle| > \delta) = 0, \quad (6.42)$$

for all  $\delta > 0$ , for all  $h \in C(\mathbb{T}^d)$ , and  $\varphi$  bounded cylinder function on  $\mathcal{E}$ .

Note that, in terms of the law  $\mu_N$  of  $\eta_N$ , (6.42) can be expressed as (cf. [13, Definition 0.2 p.43]):

$$\lim_{N \rightarrow +\infty} \mu^N \left[ \left| \frac{1}{N^d} \sum_{\mathbf{x} \in \mathbb{T}_N^d} h(\hat{\mathbf{x}}) \tau_{\mathbf{x}} \varphi - \int_{\mathbb{T}^d} h(x) \bar{\varphi}(\rho(x)) dx \right| > \delta \right] = 0, \quad (6.43)$$

If sufficient equi-integrability conditions are satisfied, then one can extend (6.42) to non-bounded functions. In particular, taking  $\varphi(\eta) = \iota_0(\eta)$ , we deduce from (6.42) (and (6.10)) that

$$\lim_{N \rightarrow +\infty} \mathbb{P}(|\langle \kappa^N[\eta_N], h \rangle - \langle \rho, h \rangle| > \delta) = 0, \quad (6.44)$$

for all  $\delta > 0$ , for all  $h \in C(\mathbb{T}^d)$ .

### 6.3 Some hydrodynamic limits

We give now the description of the hydrodynamic limit of some interacting particle systems. The situation is the following one: we will start initially close to some equilibrium state with profile  $\rho_0$ , say  $\rho_0$  continuous. If we let time evolve, then, basically, nothing will happen since the system is locally (locally in space) at equilibrium. However, after a very long time, some transformations may begin to manifest themselves since  $\rho_0$  is not globally constant a priori. If we assume that the system has a natural trend to equilibrium, then we may obtain a new equilibrium with profile  $\rho'_0$  which may be deduced from  $\rho_0$ . This description is very schematic since time evolves continuously. The situation we are inclined to consider is more the following one: we consider a time change of scale  $t \mapsto \theta(N)t$  such that:

1. at each time  $t\theta(N)$ , the system is close to some equilibrium state with profile  $\rho(t, \cdot)$ ,
2. an evolution equation for the parameter  $\rho(t, \cdot)$  can be given.

This evolution equation is what is termed the hydrodynamic limit. Under a change of scale, it gives a description of the system that involves only the macroscopic parameters of the equilibria. In our context, there is only one macroscopic parameter, the density.

Let  $j_{\mathbf{x}, \mathbf{y}}$  denote the instantaneous current between  $\mathbf{x}$  and  $\mathbf{y}$ , i.e, the rate at which a particle jumps from  $\mathbf{x}$  to  $\mathbf{y}$ , minus the rate at which a particle jumps from  $\mathbf{y}$  to  $\mathbf{x}$ :  $j_{\mathbf{x}, \mathbf{y}}(\eta) = c(\mathbf{x}, \mathbf{y}, \eta) - c(\mathbf{y}, \mathbf{x}, \eta)$ . By (6.11) and (6.12), we have

$$\mathcal{L}_N \iota_{\mathbf{x}^*} = \sum_{\mathbf{x}, \mathbf{y} \in \mathbb{T}_N^d} c(\mathbf{x}, \mathbf{y}, \cdot) (\mathbf{1}_{\mathbf{x}^* = \mathbf{y}} - \mathbf{1}_{\mathbf{x}^* = \mathbf{x}}) = \sum_{\mathbf{x} \in \mathbb{T}_N^d} j_{\mathbf{x}, \mathbf{x}^*}. \quad (6.45)$$

Let  $h$  be a smooth function  $\mathbb{T}^d \rightarrow \mathbb{R}$ . Let

$$\varphi(\eta) := \langle \kappa^N[\eta], h \rangle = \frac{1}{N^d} \sum_{\mathbf{x} \in \mathbb{T}_N^d} h(\mathbf{x}/N) \eta(\mathbf{x}) = \frac{1}{N^d} \sum_{\mathbf{x} \in \mathbb{T}_N^d} h(\mathbf{x}/N) \iota_{\mathbf{x}}(\eta).$$

Let  $M_h^N(t)$  denote the martingale

$$M_h^N(t) = \varphi(\eta_t) - \varphi(\eta_0) - \int_0^t \mathcal{L}_N \varphi(\eta_s) ds. \quad (6.46)$$

By (6.45), the action of  $\mathcal{L}_N$  on  $\varphi$  is given by

$$\mathcal{L}_N \varphi = \frac{1}{N^d} \sum_{\mathbf{x}, \mathbf{y} \in \mathbb{T}_N^d} h(\mathbf{x}/N) j_{\mathbf{y}, \mathbf{x}} = \frac{1}{2N^d} \sum_{\mathbf{x}, \mathbf{y} \in \mathbb{T}_N^d} (h(\mathbf{x}/N) - h(\mathbf{y}/N)) j_{\mathbf{y}, \mathbf{x}}, \quad (6.47)$$

We have used the antisymmetry property  $j_{\mathbf{x}, \mathbf{y}} = -j_{\mathbf{y}, \mathbf{x}}$  to transform the sum in (6.47). Recall that  $c(\mathbf{x}, \mathbf{y}, \cdot)$  is non-trivial if, and only if,  $\mathbf{x}$  and  $\mathbf{y}$  are at distance 1. Then the same is true of  $j_{\mathbf{x}, \mathbf{y}}(\eta)$ . We deduce from (6.47) the expression

$$\mathcal{L}_N \varphi = \frac{1}{2N^d} \sum_{\mathbf{x} \in \mathbb{T}_N^d} \sum_{i=1}^d (h(\mathbf{x}/N) - h(\mathbf{x}/N - e_i/N)) j_{\mathbf{x} - e_i, \mathbf{x}} + (h(\mathbf{x}/N) - h(\mathbf{x}/N + e_i/N)) j_{\mathbf{x} + e_i, \mathbf{x}}, \quad (6.48)$$

Doing the change of indexation  $\mathbf{x}' = \mathbf{x} + e_i$  in the last sum of (6.48) yields

$$\mathcal{L}_N \varphi = \frac{1}{N^d} \sum_{\mathbf{x} \in \mathbb{T}_N^d} \sum_{i=1}^d (h(\mathbf{x}/N) - h(\mathbf{x}/N - e_i/N)) j_{\mathbf{x} - e_i, \mathbf{x}}. \quad (6.49)$$



Let us assume now that  $\langle j_{y,x}, \nu_\rho \rangle$  is non-trivial. By invariance by translation, this means that there is at least one direction  $e_i \in \mathbb{R}^d$  ( $1 \leq i \leq d$ ) such that the function

$$A_i : \rho \mapsto \langle j_{x-e_i, x}, \nu_\rho \rangle \quad (6.50)$$

is non trivial. Then, at each time  $t$ , we expect that a jump of a small step  $\sim N^{-1}$  in the direction  $e_i$  will occur. In a time of order  $N$ , we should observe therefore a global displacement in the direction  $e_i$ . Let us rescale time by a factor  $N$ : we obtain

$$M_h^N(Nt) = \varphi(\eta_{Nt}) - \varphi(\eta_0) - \int_0^t \sum_{i=1}^d \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} N [h(x/N) - h(x/N - e_i/N)] j_{x-e_i, x}(\eta_{Ns}) ds.$$

This reads

$$M_h^N(Nt) = \varphi(\eta_{Nt}) - \varphi(\eta_0) - \int_0^t \sum_{i=1}^d \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} (\partial_i h)(x/N) j_{x-e_i, x}(\eta_{Ns}) ds + \mathcal{O}(N^{-1}).$$

Taking expectancy, we obtain

$$\mathbb{E} \langle \kappa^N[\eta_{Nt}], h \rangle - \mathbb{E} \langle \kappa^N[\eta_0], h \rangle - \int_0^t \sum_{i=1}^d \mathbb{E} \langle \pi^N[\eta_{Ns}], \partial_i h \otimes j_{-e_i, 0} \rangle ds = \mathcal{O}(N^{-1}). \quad (6.51)$$

We assume, according to our formulation (6.42) of equilibrium that there is a time-dependent profile  $\rho(t)$ , such that, when  $N$  is large,

$$\mathbb{E} \langle \kappa^N[\eta_{Nt}], h \rangle \simeq \langle \rho(t), h \rangle, \quad \mathbb{E} \langle \pi^N[\eta_{Ns}], \partial_i h \otimes j_{-e_i, 0} \rangle \simeq \langle A_i(\rho), \partial_i h \rangle, \quad (6.52)$$

see (6.42) and the definition (6.50). We can pass to the limit in (6.51) then, to obtain the equation

$$\int_{\mathbb{T}^d} h(x) \rho(t, x) dx - \int_{\mathbb{T}^d} h(x) \rho(0, x) dx = \int_0^t \sum_{i=1}^d \int_{\mathbb{T}^d} \partial_i h(x) A_i(\rho(s, x)) ds dx,$$

which is the weak form of the non-linear conservation law

$$\partial_t \rho(t, x) + \operatorname{div}_x (A(\rho(t, x))) = 0, \quad (6.53)$$

with initial condition  $\rho|_{t=0} = \rho_0$ .

In a second step, let us consider a situation where all the fluxes  $A_i$  in (6.50) are trivial. This happens in particular if, for all  $i \in \{1, \dots, d\}$ ,

$$j_{x-e_i, x}(\eta) = [\tau_{x-e_i} \beta_i(\eta(0)) - \tau_x \beta_i(\eta(0))], \quad (6.54)$$

for a given function  $\beta_i : \mathbb{R} \rightarrow \mathbb{R}$ . Then we can perform a second integration by parts in (6.51) and take  $t = t'N$  (which amounts to a total rescaling by a factor  $N^2$  on the microscopic time  $N$ ), to obtain, dropping the primes,

$$M_h^N(N^2t) = \varphi(\eta_{N^2t}) - \varphi(\eta) - \int_0^t \sum_{i=1}^d \sum_{x \in \mathbb{T}_N^d} N^2 (\hat{h}(x - e_i) + \hat{h}(x + e_i) - 2\hat{h}(x)) \tau_x \beta_i(\eta_{N^2s}(0)) ds.$$

The reasoning that leads from (6.51) to (6.53) can be adapted here. The hydrodynamic limit that we obtain is then the second-order non-linear equation

$$\partial_t \rho(t, x) - \partial_{ii}^2 (B_i(\rho(t, x))) = 0, \quad (6.55)$$

with initial condition  $\rho|_{t=0} = \rho_0$ , where  $B_i(\rho) := \langle \beta_i, \nu_\rho^x \rangle$ .

*Remark 6.1* (Rigorous derivation of (6.53), (6.55)). To what extent the system is close to equilibrium, and in which sense (*cf.* Definition 6.1), are difficult points that one has to address to establish rigorously the hydrodynamic limit. Various tools have been developed in that context (one block estimate, two blocks estimate, entropy estimates, etc.). See [13].

In what follows, we consider different examples of interacting particle systems.

**Simple exclusion process.** We have seen that  $c(\mathbf{x}, \mathbf{y}, \eta) = \eta(\mathbf{x})(1 - \eta(\mathbf{y}))\mathbf{p}(\mathbf{y} - \mathbf{x})$  and that  $\nu_\rho^\mathbf{x}$  is the Bernoulli measure with parameter  $\rho$ . We compute

$$j_{\mathbf{x}-e_i, \mathbf{x}}(\eta) = \eta(\mathbf{x} - e_i)(1 - \eta(\mathbf{x}))\mathbf{p}(e_i) - \eta(\mathbf{x})(1 - \eta(\mathbf{x} - e_i))\mathbf{p}(-e_i),$$

and

$$A_i(\rho) = \langle j_{\mathbf{x}-e_i, \mathbf{x}}, \nu_\rho \rangle = \alpha_i \rho(1 - \rho), \quad \alpha_i = \mathbf{p}(e_i) - \mathbf{p}(-e_i). \quad (6.56)$$

If the process, is *asymmetric* (ASEP), in the sense that  $\mathbf{p}(e_i) - \mathbf{p}(-e_i) \neq 0$  for at least one index  $i$ , then we obtain the hydrodynamic limit (6.53) with the flux  $A_i$  given in (6.56). Let us examine the case of the symmetric simple exclusion process (SSEP), where  $\mathbf{p}(e_i) = \mathbf{p}(-e_i) =: \mathbf{p}_i$  for all  $i$ . We have then

$$j_{\mathbf{x}-e_i, \mathbf{x}}(\eta) = \mathbf{p}_i [\eta(\mathbf{x} - e_i)(1 - \eta(\mathbf{x})) - \eta(\mathbf{x})(1 - \eta(\mathbf{x} - e_i))]. \quad (6.57)$$

Using the fact that  $\eta(\mathbf{x}) \in \{0, 1\}$ , we check that the bracket in (6.57) can be rewritten

$$j_{\mathbf{x}-e_i, \mathbf{x}}(\eta) = \mathbf{p}_i [\eta(\mathbf{x} - e_i) - \eta(\mathbf{x})],$$

which is of the form (6.54) with  $\beta_i(\eta(0)) = \mathbf{p}_i \eta(0)$ . We obtain the hydrodynamic limit (6.55) with  $B_i(\rho) = \mathbf{p}_i \rho$ . In the isotropic case  $\mathbf{p}_i = \frac{1}{2d}$ , (6.55) is the Heat equation thus (we have established this fact in Section 4).

**Zero range process.** For the zero range processes considered in Section 6.2.4, (6.39) gives

$$j_{\mathbf{x}, \mathbf{y}} = g(\eta(\mathbf{x}))p(\mathbf{x}, \mathbf{y}) - g(\eta(\mathbf{y}))p(\mathbf{y}, \mathbf{x}), \quad \langle j_{\mathbf{x}, \mathbf{y}}, \nu_\rho \rangle = \Phi(\rho)(p(\mathbf{x}, \mathbf{y}) - p(\mathbf{y}, \mathbf{x})).$$

Once again, we need to make the distinction between the symmetric and the asymmetric case. In the asymmetric case, there exists  $i$  such that  $\mathbf{p}(e_i) - \mathbf{p}(-e_i) \neq 0$  for at least one index  $i$ , and we obtain the hydrodynamic limit (6.53) with the flux  $A_i$  given by

$$A_i(\rho) = \alpha_i \Phi(\rho), \quad \alpha_i = \mathbf{p}(e_i) - \mathbf{p}(-e_i).$$

In the symmetric case, we have

$$j_{\mathbf{x}-e_i, \mathbf{x}}(\eta) = \mathbf{p}_i [g(\eta(\mathbf{x} - e_i)) - g(\eta(\mathbf{x}))],$$

which is of the form (6.54) with  $\beta_i(\eta(0)) = \mathbf{p}_i g(\eta(0))$ . We obtain the hydrodynamic limit (6.55) with  $B_i(\rho) = \mathbf{p}_i \Phi(\rho)$ . In the particular cases where  $g(k) = k$  (independent random walk), the equations are linear, with  $\Phi(\rho) = \rho$ . When  $g(k) = \mathbf{1}_{k>0}$ , we obtain  $\Phi(\rho) = \frac{\rho}{1+\rho}$ .

## 7 Solution to the exercises

### Solution to Exercise 3.2.

1. That  $\mu_0 = \delta_0$  means that  $X_0$  always take the value 0 ( $X_0$  is deterministic). We have then  $X_1 = \pm 1$  with equi-probability, so

$$\mu_1 = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_{+1},$$

which is an example of Bernoulli's Law  $b(\frac{1}{2})$ . We have then

$$\mathbb{P}(X_2 = -2) = \frac{1}{4}, \quad \mathbb{P}(X_2 = 0) = \frac{1}{2}, \quad \mathbb{P}(X_2 = +2) = \frac{1}{4}.$$

The law of  $X_2$  is therefore

$$\mu_2 = \frac{1}{4} [\delta_{-3/2} + \delta_{-1/2} + \delta_{1/2} + \delta_{3/2}].$$

2. The law  $\mu_N$  is

$$\mu_N = \frac{1}{2^{N+1}}\delta_{-2} + \sum_{-2^{N-1} < k < 2^{N-1}} \frac{1}{2^N} \delta_{\frac{k}{2^{N-2}}} + \frac{1}{2^{N+1}}\delta_{+2}. \quad (7.1)$$

3. The answer is that  $\mu_0$  is the uniform law on  $[-2, 2]$ :

$$\mu_0(A) = \frac{1}{4}|A \cap [-2, 2]|,$$

where  $|A|$  is the Lebesgue measure of a Lebesgue set  $A \subset \mathbb{R}$  (see the proof below for  $\mu_\infty$ ). This answer can be simply guessed by examination of the evolution of the process  $(X_n)$ . An other way to find the right  $\mu_0$  is to look at  $\mu_N$  for large  $N$ . Indeed, a usual way to find an equilibrium for a system in evolution is to look as the behavior for large times: if there is convergence to a limit object, this will most probably be an equilibrium of the system. Here, for example, one can look at the evolution starting from the binomial  $b(1/2)$  with values in  $\{-2, +2\}$ , as in Question 2. If  $\varphi \in \text{BC}(\mathbb{R})$ , then

$$\begin{aligned} \int_{\mathbb{R}} \varphi d\mu_N &= \sum_{-2^{N-1} < k < 2^{N-1}} \frac{1}{2^N} \varphi \left( \frac{k}{2^{N-2}} \right) + o(1) \\ &= \frac{1}{4} \sum_{-2^{N-1} < k < 2^{N-1}} \frac{1}{2^{N-2}} \varphi \left( \frac{k}{2^{N-2}} \right) + o(1). \end{aligned}$$

We recognize a Riemann sum, which converges to

$$\int_{\mathbb{R}} \varphi d\mu_\infty := \frac{1}{4} \int_{-2}^2 \varphi(x) dx.$$

The limit law  $\mu_\infty$  is an invariant measure for good. Indeed, if  $X_0 \sim \mu_\infty$ , then, by the formula of total probability,

$$\begin{aligned} \mathbb{P}(X_1 \in A) &= \mathbb{P}(X_1 \in A | Z_1 = -1) \mathbb{P}(Z_1 = -1) + \mathbb{P}(X_1 \in A | Z_1 = +1) \mathbb{P}(Z_1 = +1) \\ &= \frac{1}{2} \mathbb{P}(X_0/2 \in A + 1) + \frac{1}{2} \mathbb{P}(X_0/2 \in A - 1), \end{aligned}$$

for any Borel subsets  $A$  of  $\mathbb{R}$ . This gives

$$8\mathbb{P}(X_1 \in A) = |A_+ \cap [-2, 2]| + |A_- \cap [-2, 2]|, \quad A_\pm := 2A \pm 2.$$

We compute, by the invariance by translation of the Lebesgue measure and the change of variable formula,

$$|A_+ \cap [-2, 2]| = |2A \cap [-4, 0]| = 2|A \cap [-2, 0]|, \quad |A_- \cap [-2, 2]| = 2|A \cap [0, 2]|.$$

It follows that  $\mathbb{P}(X_1 \in A) = \frac{1}{4}|A \cap [-2, 2]| = \mu_\infty(A)$ :  $X_1$  has law  $\mu_\infty$ .

Back to [Exercise 3.2](#).

**Solution to Exercise 3.3.** We have

$$P_t \varphi(x_i) = \mathbb{E}_{x_i} \varphi(X_t) = \sum_{j=1}^L \mathbb{E}_{x_i} [\mathbf{1}_{X(t)=x_j} \varphi(x_j)] = \sum_{j=1}^L \mathbb{P}_{x_i}(X(t) = x_j) \varphi(x_j) = \sum_{j=1}^L a_{ij}(t) \varphi(x_j),$$

which gives  $P_t \varphi = A(t) \varphi$ . With the conventions that are used, we observe that  $\langle \varphi, \mu \rangle = (\varphi, \mu)$ , where  $(\cdot, \cdot)$  is the canonical scalar product in  $\mathbb{R}^L$ . Consequently,

$$\langle \varphi, P_t^* \mu \rangle = \langle P_t \varphi, \mu \rangle = (A(t) \varphi, \mu) = (\varphi, A(t)^* \mu),$$

and we obtain  $P_t^* \mu = A(t)^* \mu$ , where  $A(t)^*$  is the adjoint of the matrix  $A(t)$ . The semi-group property reads  $A(t+s) = A(t)A(s)$ . It follows that

$$\frac{A(t+s) - A(s)}{t} = A(s) \frac{A(t) - \mathbf{I}_L}{t} = \frac{A(t) - \mathbf{I}_L}{t} A(s).$$

By letting  $t \rightarrow 0$ , we deduce that  $A$  satisfies the ODE  $A'(t) = \mathcal{L}A(t) = A(t)\mathcal{L}$ , which implies  $A(t) = e^{t\mathcal{L}}$  since  $A(0) = \mathbf{I}_L$ . The equation satisfied by an invariant measure is  $A(t)^* \mu = \mu$  for all  $t \geq 0$ . By differentiation, we obtain  $\mathcal{L}^* \mu = 0$ . Of course the latter equation implies  $(\mathcal{L}^*)^n \mu = 0$  for all  $n \geq 1$ , and thus

$$A(t)^* \mu = e^{t\mathcal{L}^*} \mu = \sum_{n \geq 0} \frac{(\mathcal{L}^*)^n}{n!} \mu = \mu.$$

Consequently, there is strict equivalence between  $A(t)^* \mu = \mu$  for all  $t \geq 0$ , and  $\mathcal{L}^* \mu = 0$ .

Back to [Exercise 3.3](#).

**Solution to Exercise 3.4.** Assume  $E = \{x_1, \dots, x_L\}$  as in [Exercise 3.3](#). Let  $A$  denote the matrix  $A(1)$ :  $a_{ij} = \mathbb{P}_{x_i}(X_1 = x_j)$ . We still have  $P_n \varphi = A(n) \varphi$  and  $P_n^* \mu = A(n)^* \mu$ . By the semi-group property, we have  $A(n) = A^n$  for all  $n \geq 0$ . The equation satisfied by the invariant measure is  $(A^* - \text{Id})\mu = 0$  (the equivalent to  $\mathcal{L}$  here is  $A - \text{Id}$ ). Let us come back to the case of a general state space  $E$  (a Polish space in our framework). Let us first prove that  $(M_n)$  is a martingale. We can use the tower property (2.2) to show that it is sufficient to establish the identity  $\mathbb{E}[M_{n+1} | \mathcal{F}_n] = M_n$  for all  $n \geq 0$ . By the Markov property, we obtain

$$\mathbb{E}[M_{n+1} | \mathcal{F}_n] = P_1 \varphi(X_n) - \varphi(X_0) - \sum_{k=0}^n \mathcal{L} \varphi(X_k).$$

Since  $\mathcal{L} = P_1 - \text{Id}$ , this is precisely the desired identity  $\mathbb{E}[M_{n+1} | \mathcal{F}_n] = M_n$ . Let us look at (3.35) now. Again, we want to prove that  $\mathbb{E}[Z_{n+1} | \mathcal{F}_n] = Z_n$ . We write

$$M_{n+1} = \varphi(X_{n+1}) - Y_n, \quad Y_n = \varphi(X_0) + \sum_{k=0}^n \mathcal{L} \varphi(X_k),$$

where  $Y_n$  is  $\mathcal{F}_n$ -measurable. This gives

$$\begin{aligned}\mathbb{E}[|M_{n+1}|^2|\mathcal{F}_n] &= \mathbb{E}[|\varphi(X_{n+1})|^2|\mathcal{F}_n] - 2Y_n\mathbb{E}[\varphi(X_{n+1})|\mathcal{F}_n] + |Y_n|^2 \\ &= P_1|\varphi|^2(X_n) - 2Y_nP_1\varphi(X_n) + |Y_n|^2 \\ &= P_1|\varphi|^2(X_n) + Y_n(Y_n - 2P_1\varphi(X_n))\end{aligned}$$

We have also  $Y_n = \varphi(X_n) + \mathcal{L}\varphi(X_n) - M_n = P_1\varphi(X_n) - M_n$ , hence

$$\mathbb{E}[|M_{n+1}|^2|\mathcal{F}_n] = P_1|\varphi|^2(X_n) - (P_1\varphi(X_n) - M_n)(P_1\varphi(X_n) + M_n).$$

and

$$\mathbb{E}[|M_{n+1}|^2|\mathcal{F}_n] - |M_n|^2 = P_1|\varphi|^2(X_n) - |P_1\varphi(X_n)|^2.$$

We obtain then (3.35) by using the definition  $\Gamma[\varphi] = P_1|\varphi|^2 - |P_1\varphi|^2$ . The Jensen inequality applied to

$$P_1\varphi(x) = \int_E \varphi(y)Q(1, x, dy)$$

shows that  $\Gamma[\varphi] \geq 0$ .

Back to [Exercise 3.4](#).

**Solution to Exercise 4.1.** Each  $T_n$  has density  $f: t \mapsto \lambda e^{-\lambda t} \mathbf{1}_{\mathbb{R}_+}(t)$  with respect to the Lebesgue measure on  $\mathbb{R}$ . By independence,  $S_n$  has the law  $f * \dots * f$  (convolution  $n$  times). We compute

$$f * f(t) = \int_{\mathbb{R}} \lambda^2 e^{-\lambda s} e^{-\lambda(t-s)} \mathbf{1}_{\mathbb{R}_+}(s) \mathbf{1}_{\mathbb{R}_+}(t-s) ds = \int_0^t \lambda^2 e^{-\lambda t} ds \mathbf{1}_{\mathbb{R}_+}(t) = \lambda^2 t e^{-\lambda t} \mathbf{1}_{\mathbb{R}_+}(t),$$

and, by recursion on  $n$ ,  $f * \dots * f(t) = \lambda^n \frac{t^{n-1}}{(n-1)!} e^{-\lambda t} \mathbf{1}_{\mathbb{R}_+}(t)$ . We compute then

$$\mathbb{P}(N(t) < n) = \mathbb{P}(S_n > t) = \int_t^\infty \lambda^n \frac{s^{n-1}}{(n-1)!} e^{-\lambda s} ds.$$

This gives  $\mathbb{P}(N(t) = 0) = \mathbb{P}(N(t) < 1) = e^{-\lambda t}$ . Assume  $n > 1$ . By integration by parts, we obtain

$$\mathbb{P}(N(t) < n) = e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} + \mathbb{P}(N(t) < n-1),$$

which shows that  $\mathbb{P}(N(t) = n-1) = e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}$ . The assertion that  $N(t)$  is càdlàg and non-decreasing is a deterministic statement, it is obvious since  $\Gamma$  is a measure: indeed, we note that, whatever the Radon measure  $\mu$  on  $\mathbb{R}_+$ , the map  $t \mapsto \mu([0, t])$  is càdlàg and non-decreasing.

Let  $T$  be an exponential random variable of parameter  $\lambda > 0$ . We have

$$\mathbb{P}(T > s+t | T > s) = \frac{\mathbb{P}(\{T > t+s\} \cap \{T > s\})}{\mathbb{P}(T > s)} = \frac{\mathbb{P}(\{T > t+s\})}{\mathbb{P}(T > s)} = \frac{e^{-\lambda(t+s)}}{e^{-\lambda s}}.$$

This gives  $\mathbb{P}(T > s+t | T > s) = e^{-\lambda t} = \mathbb{P}(T > t)$ . To establish (4.16), we write

$$\mathbb{P}(T > t+S) = \int_{\mathbb{R}_+} \mathbb{P}(T > t+s) d\mu_S(s), \tag{7.2}$$

where  $\mu_S$  is the law of  $S$ . By (4.15), we have  $\mathbb{P}(T > t + s) = \mathbb{P}(T > t)\mathbb{P}(T > s)$ . Using again (7.2) (with  $t = 0$ ) gives the result. A proof of (7.2) is as follows:

$$\mathbb{P}(T > t + S) = \mathbb{E}[\mathbf{1}_{T > t + S}] = \iint_{\mathbb{R}_+ \times \mathbb{R}_+} \mathbf{1}_{\tau > t + s} d\mu_{(T,S)}(\tau, s).$$

By independence, the law  $\mu_{(T,S)}$  of the couple  $(T, S)$  is the product  $\mu_T \otimes \mu_S$ . This gives

$$\mathbb{P}(T > t + S) = \int_{\mathbb{R}_+} \left[ \int_{\mathbb{R}_+} \mathbf{1}_{\tau > t + s} d\mu_T(\tau) \right] d\mu_S(s) = \int_{\mathbb{R}_+} \mathbb{P}(T > t + s) d\mu_S(s),$$

as desired. For all  $t$ , for all  $n$ , the event  $\{S_n = t\}$  has probability 0 since the law of  $S_n$  is absolutely continuous with respect to the Lebesgue measure. Consequently: for all  $t$ , the event  $\mathcal{S} \cap \{t\} \neq \emptyset$ , which is the union over  $n$  of the events  $\{S_n = t\}$ , has probability 0. This shows that  $\Gamma(J) = \Gamma(\tilde{J})$   $\mathbb{P}$ -a.s. if  $\tilde{J}$  is deduced from  $J$  by modification of the extremities of  $J$ . In particular,  $\Gamma(J)$  and  $\Gamma(\tilde{J})$  have the same law. Let us now consider the case  $I = [0, t]$ ,  $J = (t, \sigma]$ . We want to prove

$$\mathbb{P}(\Gamma(I) = n \ \& \ \Gamma(J) = k) = e^{-\lambda|I|} \frac{(\lambda|I|)^n}{n!} \times e^{-\lambda|J|} \frac{(\lambda|J|)^k}{k!}. \quad (7.3)$$

The event  $\{\Gamma(I) = n \ \& \ \Gamma(J) = k\}$  corresponds here to the event

$$S_n \leq t < S_{n+1} \leq \dots \leq S_{n+k} \leq \sigma < S_{n+k+1}. \quad (7.4)$$

What will be relevant here is the time  $T'_{n+1} := T_{n+1} - (t - S_n)$  (indeed, starting from  $t$ , this is after the time duration  $T'_{n+1}$  that we see the first event being counted in  $\Gamma(J)$ ). We rewrite (7.4) as

$$S_n \leq t \text{ and } t - S_n < T_{n+1} \text{ and } S_{n+1} \leq \dots \leq S_{n+k} \leq \sigma < S_{n+k+1}. \quad (7.5)$$

This is also equivalent to

$$S_n \leq t \text{ and } 0 < T'_{n+1} \text{ and } S_{n+1} - t \leq \dots \leq S_{n+k} - t \leq \sigma - t < S_{n+k+1} - t. \quad (7.6)$$

Eventually, we rewrite (7.6) as

$$S_n \leq t \text{ and } 0 < T'_{n+1} \text{ and } S'_{n+1} \leq \dots \leq S'_{n+k} \leq \sigma - t < S'_{n+k+1}, \quad (7.7)$$

where  $S'_{n+l} = T'_{n+1} + T_{n+2} + \dots + T_{n+l}$ . It is quite clear that things will work now. In terms of the laws of the different random variables involved, using independence, we can write (same reasoning as in the proof of (4.16)):

$$\begin{aligned} & \mathbb{P}(\Gamma(I) = n \ \& \ \Gamma(J) = k) \\ &= \int_0^t d\mu_{S_n}(s_n) \int_{t-s_n}^{\infty} d\mu_{T_n}(t_n) \mathbb{P}(t_n + T_{n+2} + \dots + T_{n+l} \leq \sigma - t < t_n + T_{n+2} + \dots + T_{n+l} + T_{n+l+1}). \end{aligned}$$

After the change of variable  $t'_n = t_n - (t - s_n)$ , we obtain

$$\mathbb{P}(\Gamma(I) = n \ \& \ \Gamma(J) = k) = \int_0^t d\mu_{S_n}(s_n) \mathbb{P}(N(\sigma - t) = k),$$

which gives (7.3). In the general case, where  $I$  has extremities  $s < t$  and  $J$  has extremities  $s' < t'$ , with  $t \leq s'$ , we introduce the intervals  $K, L$  having extremities  $0 < s$  and  $t < s'$  respectively and an empty intersection with  $I$  and  $J$ . Then we prove, by some similar reasoning, that

$$\begin{aligned} \mathbb{P}(\Gamma(K) = m \ \& \ \Gamma(I) = n \ \& \ \Gamma(L) = p \ \& \ \Gamma(J) = k) \\ &= \mathbb{P}(\Gamma(K) = m) \mathbb{P}(\Gamma(I) = n) \mathbb{P}(\Gamma(L) = p) \mathbb{P}(\Gamma(J) = k). \end{aligned} \quad (7.8)$$

Summing (7.8) over  $m$  and  $p$  gives the desired result.

Back to [Exercise 4.1](#).

**Solution to Exercise 4.2.** We have seen at the beginning of Section 4.2.2 that  $(\hat{T}^n)$  is a sequence of i.i.d. random variables with exponential law of parameter  $L$ . Consequently (see Exercise 4.1 for instance),  $(N(t))$  is a Poisson process of parameter  $L$ . Let us first treat the case  $n = 1$ . We have then

$$\mathbb{P}(N(t) \geq 1, \mathbf{x}_1 \in A_1) = \mathbb{P}(\hat{T}^1 \leq t, \mathbf{x}_1 \in A_1).$$

It is sufficient to consider the case where  $A_1$  is the singleton  $A_1 = \{y\}$ . The probability that we consider is thus

$$\mathbb{P} \left( \{T_y^1 \leq t\} \bigcap \bigcap_{x \neq y} \{T_x^1 > T_y^1\} \right). \quad (7.9)$$

By independence, using the explicit form of the exponential law, (7.9) is

$$\int_0^t e^{-s} e^{-(L-1)s} ds = \frac{1}{L} (1 - e^{-Lt}),$$

and we obtain (4.23) for  $n = 1$ ,  $|A_1| = 1$ , as required. When  $n = 2$ , we have

$$\mathbb{P}(N(t) \geq 2, \mathbf{x}_1 \in A_1, \mathbf{x}_2 \in A_2) = \mathbb{P}(\hat{T}^1 + \hat{T}^2 \leq t, \mathbf{x}_1 \in A_1, \mathbf{x}_2 \in A_2).$$

Again, we assume  $A_1 = \{y_1\}$ ,  $A_2 = \{y_2\}$ , and obtain the probability

$$\mathbb{P} \left( \{T_{y_1}^1 + T_{y_2}^2 \leq t\} \bigcap \bigcap_{x_1 \neq y_1} \{T_{x_1}^1 > T_{y_1}^1\} \bigcap \bigcap_{x_2 \neq y_2} \{T_{x_2}^2 > T_{y_2}^2\} \right). \quad (7.10)$$

We use independence as above, to obtain the following expression of (7.10):

$$\iint_{0 \leq t_1 + t_2 \leq t} e^{-(t_1+t_2)} e^{-(L-1)t_1} e^{-(L-1)t_2} dt_1 dt_2 = \frac{1}{L} \int_0^t e^{-Lt_1} (1 - e^{-L(t-t_1)}) dt_1,$$

which again is (4.23) for  $n = 2$ ,  $|A_1| = |A_2| = 1$ . The case of a general  $n$  is proved similarly: we obtain an integral

$$\int \cdots \int_{0 \leq t_1 + \cdots + t_n \leq t} e^{-L(t_1 + \cdots + t_n)} dt_1 \cdots dt_n$$

which is  $L^{-n} \mathbb{P}(T'_1 + \cdots + T'_n \leq t)$  where  $T'_1, \dots, T'_n$  are some independent exponential variable of parameter  $L$ . Therefore the computed probability is  $L^{-n} \mathbb{P}(N(t) \geq n)$ , which is (4.23) for  $n = 2$ ,  $|A_1| = \cdots = |A_n| = 1$ .

Back to [Exercise 4.2](#).

**Solution to Exercise 5.1.** Clearly, the properties (5.15), (5.17) and the monotony property are satisfied. To establish the regularity property (5.16), we use the fact that  $A$  is locally Lipschitz continuous.

Back to [Exercise 5.1](#).

**Solution to Exercise 5.2.** Same proof as in the case  $A = A(v)$ . This times we use the divergence-free condition  $(\operatorname{div}_x A)(x, v) = 0$ .

Back to [Exercise 5.2](#).

**Solution to Exercise 5.3.** We suppose that  $\alpha$  is fixed of course. Consider a mesh with triangles only. If one triangle as a basis of length  $\sim h$ , but a height that is almost 0, *i.e.* if there is an almost flat triangle in the mesh, then the first condition in (5.27) may not be satisfied.

If we consider triangles only then  $|\partial K| \leq 3 \text{diam}(K) \leq 3h$  for any  $K$ . Now, consider a triangle with a basis of length  $\sim 1$ , and a height  $\sim h$ . Then fold the “arrow” of this triangle to form a polygonal set of diameter  $\mathcal{O}(h)$  and perimeter  $\sim 1$ . If  $\mathcal{T}$  contains such kind of set, then the second condition in (5.27) will not be satisfied.

Back to [Exercise 5.3](#).

**Solution to Exercise 5.5.** We only give the sketch of the proof. By linearity, it is sufficient to consider the case  $u_0 = 0$ , in which case we want to prove  $u \equiv 0$ . If  $u$  is smooth, then  $\partial_t u + a \cdot \nabla u = 0$  (recall that  $a$  is divergence free). By the usual chain-rule formula, it follows that  $\partial_t \beta(u) + a \cdot \nabla \beta(u) = 0$  for any function  $\beta$  of class  $C^1$ . By integration, we obtain

$$\int_{\mathbb{T}^d} \beta(u(x, t)) dx = \int_{\mathbb{T}^d} \beta(u(x, 0)) dx = \beta(0). \quad (7.11)$$

It is sufficient to apply (7.11) with a non-negative function  $\beta$  such that  $\beta(s) = 0$  if, and only if,  $s = 0$ , for example  $\beta(s) = s^2$ , to conclude. All the difficulty of the proof is to justify this approach for weak solutions. The idea is to use convolution in  $x$  with a smooth approximation of the unit ( $\rho_\varepsilon$ ). One then needs to control the commutator  $(au) * \rho_\varepsilon - a(u * \rho_\varepsilon)$ . This is possible if  $a$  is smooth enough, if  $a$  is Lipschitz in particular, but even if  $a$  has only a Sobolev regularity. See the paper by Di Perna and Lions, [5] (Theorem II.1 and Lemma II.1 in particular).

To show that  $(x, t) \mapsto u_0 \circ \Phi^t(x)$  is a weak solution, do the change of variable  $x' = \Phi^t(x)$  in the weak formulation. Back to [Exercise 5.5](#).

**Solution to Exercise 5.7.**

1. Let  $\varphi \in C_c^1(U)$ . We have

$$\int_{-1}^1 u(x) \varphi'(x) dx = - \int_{-1}^1 u'(x) \varphi(x) dx = - \int_{-1}^1 f(x) \varphi(x) dx. \quad (7.12)$$

If  $\varphi$  is supported in  $(-r, r)$  with  $r < 1$ , then (7.12) is bounded by  $\|f\|_{L^1(-r, r)} \|\varphi\|_{C(-1, 1)}$ . We have  $u \in \text{BV}(U)$  if, and only if there is a finite constant  $C$  such that  $|\int_{-1}^1 f(x) \varphi(x) dx| \leq C \|\varphi\|_{C(-1, 1)}$  for all  $\varphi \in C_c^1(U)$ . Clearly,  $f \in L^1(U)$  implies  $u \in \text{BV}(U)$ . Conversely, if  $u \in \text{BV}(U)$ , let us consider, for  $\varepsilon > 0$ ,  $\chi_\varepsilon$  the characteristic function of the interval  $(-1 + \varepsilon, 1 - \varepsilon)$  and  $(\rho_\varepsilon)$ , an approximation of the unit with  $\rho_\varepsilon$  supported in  $(-\varepsilon, \varepsilon)$ . Let also  $\psi$  be a function in  $C_c^1(U)$ . We have then

$$\left| \int_{-1}^1 f \text{sign}(f)_\varepsilon \psi dx \right| \leq C, \quad \varphi_\varepsilon := (\varphi \chi_\varepsilon) * \rho_\varepsilon. \quad (7.13)$$

Taking the limit  $\varepsilon \rightarrow 0$  in (7.13) gives

$$\left| \int_{-1}^1 |f| \psi dx \right| \leq C. \quad (7.14)$$

We consider then a non-decreasing sequence of functions  $\psi \in C_c^1(U)$  which converges pointwise to the constant function 1. By monotone convergence, (7.14) gives  $f \in L^1(U)$ .



2. Let  $\varphi \in C_c^1(U)$ . We have

$$\int_{-1}^1 u(x)\varphi'(x)dx = \int_0^1 \varphi'(x)dx = -\varphi(0) \leq \|\varphi\|_{C(-1,1)}, \quad (7.15)$$

hence  $u \in \text{BV}(U)$ .

3. By the Stokes' formula, we have, for  $\varphi \in C_c^1(U)$ ,

$$\int_U u \operatorname{div} \varphi dx = \int_{B(0,1/2)} \operatorname{div} \varphi dx = \int_{\partial B(0,1/2)} \varphi(x) \cdot n(x) d\sigma(x) \leq \pi \|\varphi\|_{C(-1,1)}, \quad (7.16)$$

hence  $u \in \text{BV}(U)$ .

Back to [Exercise 5.7](#).

**Solution to Exercise 5.8.** In the first case, we assume  $f \in L^1(-1,1)$ . Then (7.12) shows that  $Du = f\lambda$ , where  $\lambda$  is the Lebesgue measure on  $(-1,1)$ . By [17, Theorem 6.13], we have  $|Du| = |f|\lambda$  then and  $\|u\|_{\text{BV}(U)} = \|u\|_{L^1(U)} + \|f\|_{L^1(U)}$ . In the second case, (7.15) shows that  $Du = \delta_0$ , the Dirac mass at 0. Then  $|Du| = \delta_0$  also and  $\|u\|_{\text{BV}(U)} = \|u\|_{L^1(U)} + 1 = 2$ . In the third case, (7.16) shows that  $Du = n\sigma$ , where  $n$  is the outward unit normal to  $U$  on  $\partial U$  and  $\sigma$  the surface measure. By [17, Theorem 6.13] again,  $|Du| = \sigma$ . We compute then  $\|u\|_{\text{BV}(U)} = \pi/4 + \pi = 5\pi/4$ .

Back to [Exercise 5.8](#).

**Solution to Exercise 5.10.** Assume first that  $u$  is of class  $C^1$ . Let  $h \in [0,1]$ . For  $x \in Q = (0,1)^d$  and  $z \in \mathbb{R}^d$  with  $|z| \leq h$ , we have

$$|u(x+z) - u(x)| = \left| \int_0^1 (\nabla u)(x+rz) \cdot z dr \right| \leq h \int_0^1 |\nabla u|(x+rz) dr.$$

We do the change of variable  $(x', r') = (x+rz, r)$  of jacobian determinant 1 to obtain

$$\int_Q |u(x+z) - u(x)| dx \leq h \int_0^1 \int_{Q+rz} |\nabla u(x)| dx dr \leq h \int_{Q'} |\nabla u(x)| dx = h |Du|(Q'),$$

where  $Q' = (-1,2)^d$ . This gives  $\omega_{L^1}(u;h) \leq |Du|(Q')h$ . This estimate remains true in the general case by Theorem 5.9 applied on  $U = Q'$ . Since  $|Du|(Q') \leq 3^d |Du|(\mathbb{T}^d)$ , we obtain the desired result with  $C = 3^d$ .

Back to [Exercise 5.10](#).

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