Random Uniform Permutations
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This course is at the interplay between Probability and Combinatorics. It is intended for Master students with a background in Probability (random variables, expectation, conditional probability).

The question we will address is "What can we say about a typical large permutation?": the number of cycles, their lengths, the number of fixed points,... This is also a pretext to present some universal phenomena in Probability: reinforcement, the Poisson paradigm, size-bias,...

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1 Brief reminder on permutations

Before we turn to random permutations, we will give a few definitions regarding non-random (or deterministic permutations).

A permutation of size $n \geq 1$ is a bijection $\sigma : \{1,2,\ldots,n\} \to \{1,2,\ldots,n\}$. For example

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
2 & 4 & 3 & 1 \\
\end{array}
\]

is a permutation of size 4. In these notes we often write a permutation with its one-line representation $\sigma(1)\sigma(2)\ldots\sigma(n)$. For example the above permutation is simply written 2431.

There are $n!$ permutations of size $n$.

Cycle decomposition

For our purpose, there is a convenient alternative way to encode a permutation: by its cycle decomposition. A cycle is a finite sequence of distinct integers, defined up to the cycle order. This means that the three following denote the same cycle:

$(8,3,4) = (3,4,8) = (4,8,3)$,

while $(8,3,4) \neq (8,4,3)$.  

The cycle decomposition of a permutation $\sigma$ is defined as follows. We give the theoretical algorithm and detail the example of

\[
\begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
6 & 3 & 1 & 5 & 7 & 2 & 4 \\
\end{array}
\]

**Algorithm**

Start with 1st cycle (1)

Add to this cycle $\sigma(1)$, then $\sigma(\sigma(1))$, then $\sigma(\sigma(\sigma(1)))$, and so one until one of this number is one.

Start the 2d cycle with a number which has not been seen before.

Complete the 2d cycle with same procedure.

Create new cycles until there is no remaining number.

Finally, the cycle decomposition of $\sigma$ is

\[(1, 6, 2, 3), (4, 5, 7)\]

It is convenient to represent the cycle decomposition of $\sigma$ with the following diagram:

\[
\begin{array}{c}
1 \quad 2 \quad 3 \\
\circlearrowleft \quad \circlearrowleft \\
6 & 4 & 5 \quad 7 \\
\end{array}
\]

**Example**

\[
\begin{align*}
(1) & \rightarrow (1, 6) \rightarrow (1, 6, 2) \rightarrow (1, 6, 2, 3) \\
1st cycle (1, 6, 2, 3). & 2d cycle: (4) \\
1st cycle (1, 6, 2, 3). & 2d cycle: (4) \rightarrow (4, 5) \rightarrow (4, 5, 7). \\
& Done.
\end{align*}
\]

**Exercise 1** What is the cycle decomposition of 62784315?

## 2 How to simulate a random uniform permutation?

We will first discuss the following question. Imagine that you are given a random number generator `rand` (in your favourite programming language) which returns independent uniform random variables. How to use `rand` to simulate a random uniform permutation of size $n$?

### 2.1 The naive algorithm

It works as follows:

- Pick $\sigma(1)$ uniformly at random in \{1, 2, \ldots, n\} ($n$ choices);
- Pick $\sigma(2)$ uniformly at random in \{1, 2, \ldots, n\} \{\sigma(1)\} ($n - 1$ choices);
- Pick $\sigma(3)$ uniformly at random in \{1, 2, \ldots, n\} \{\sigma(1), \sigma(2)\} ($n - 2$ choices),

and so on until $\sigma(n)$ (1 choice).

By construction every permutation occurs with probability $1/n!$ so the output is uniform.
2.2 The "continuous" algorithm
- Pick continuous random variables $X_1, X_2, \ldots, X_n$, independently and uniformly in $(0, 1)$;
- With probability 1 the $n$ values are pairwise distinct. Therefore there exists a unique permutation $\sigma$ such that $X_{\sigma(1)} < X_{\sigma(2)} < X_{\sigma(3)} < \cdots < X_{\sigma(n)}$.
- This $\sigma$ is your output.

Proposition 1. For every $n$, the output of the continuous algorithm is uniform among the $n!$ permutations of size $n$.

Proof. Step 1: The $n$ values are distinct. We have to prove that
$$P(\text{for all } i \neq j, X_i \neq X_j) = 1.$$ We prove that the complement event $\{\text{there are } i, j \text{ such that } X_i = X_j\}$ has probability zero. First we notice that
$$P(\text{there are } i \neq j \text{ such that } X_i = X_j) = P(\bigcup_{i \neq j} \{X_i = X_j\}) \leq \sum_{i \neq j} P(X_i = X_j),$$ by the union bound. Now,
$$P(X_i = X_j) = \int_{(0,1)^2} 1_{x=y} \, dx \, dy = \int_{y \in (0,1)} \left( \int_{x \in (0,1)} 1_{x=y} \, dx \right) \, dy = \int_{y \in (0,1)} \int_{x \in (0,1)} 0 \, dx \, dy = 0.$$ Step 2: The output $\sigma$ is uniform. To avoid messy notations we make the proof in the case $n = 3$. Since the 3 values $X_1, X_2, X_3$ are distinct we have
$$1 = P(X_1 < X_2 < X_3) + P(X_1 < X_3 < X_2) + P(X_2 < X_1 < X_3) + P(X_2 < X_3 < X_1) + P(X_3 < X_1 < X_2) + P(X_3 < X_2 < X_1)$$
$$= \int_{(0,1)^3} 1_{x_1 < x_2 < x_3} \, dx_1 \, dx_2 \, dx_3 + \int_{(0,1)^3} 1_{x_1 < x_3 < x_2} \, dx_1 \, dx_2 \, dx_3 + \int_{(0,1)^3} 1_{x_2 < x_1 < x_3} \, dx_1 \, dx_2 \, dx_3 + \int_{(0,1)^3} 1_{x_2 < x_3 < x_1} \, dx_1 \, dx_2 \, dx_3 + \int_{(0,1)^3} 1_{x_3 < x_1 < x_2} \, dx_1 \, dx_2 \, dx_3 + \int_{(0,1)^3} 1_{x_3 < x_2 < x_1} \, dx_1 \, dx_2 \, dx_3.$$ Now, $x_1, x_2, x_3$ are dummy variables in the above integrals, so they are interchangeable. Therefore, these 6 integrals are identical and each of these is $1/6 = 1/3!$. 

2.3 The "Chinese restaurant" algorithm
We introduce the Chinese restaurant algorithm, also called the Fisher-Yates algorithm (or even Fisher-Yates-Knuth algorithm). The main difference with the two previous algorithms is that the output $\sigma$ will be described through its cycle decomposition.

The algorithm runs as follows:

\footnotesize{(i) The union bound says that $P\left(\bigcup_{n \geq 1} A_n\right) \leq \sum_{n \geq 1} P(A_n)$ for every sequence of events $(A_n)$.}
• Assume we are given infinitely many "restaurant tables" $C_1, C_2, \ldots$. These tables are large enough so that an arbitrary number of people can sit at each table.

\[
\begin{array}{ccc}
C_1 & C_2 & C_3 & \ldots \\
\end{array}
\]

• Infinitely many customers 1, 2, 3, \ldots enter the restaurant, one at a time. Put Customer n.1 at table $C_1$:

\[
\begin{array}{ccc}
1 & C_1 & C_2 & C_3 & \ldots \\
\end{array}
\]

• With equal probability one-half, put Customer n.2 either at the same table as 1 (on its right) or alone at the new table $C_2$:

\[
\begin{array}{ccc}
1 & \rightarrow & C_1 & \rightarrow & C_2 & \rightarrow & C_3 & \ldots \\
2 & & & & & & & \\
\end{array}
\]

• With equal probability one-third, put Customer n.3 either on the right of 1, or on the right of 2, or alone at the first empty table:

\[
\begin{array}{ccc}
1 & \rightarrow & C_1 & \leftarrow & C_2 & \rightarrow & C_3 & \ldots \\
2 & & & & & & & \\
3 & & & & & & & \\
\end{array}
\]

• \ldots

• Assume that customers 1, 2, \ldots, $n - 1$ are already installed. With equal probability $1/n$, put Customer n either on the right of 1, \ldots, or on the right of $n - 1$, or alone at the first empty table (here $n = 8$):

\[
\begin{array}{ccc}
1 & \rightarrow & C_1 & \leftarrow & C_2 & \rightarrow & C_3 & \ldots \\
2 & & & & & & & \\
3 & & & & & & & \\
4 & & & & & & & \\
5 & & & & & & & \\
6 & & & & & & & \\
7 & & & & & & & \\
8 & & & & & & & \\
\end{array}
\]
Now, we return the permutation $\sigma$ whose cycle decomposition corresponds to table repartitions. Assume here that 8 sits alone, we obtain the diagram

![Cycle Diagram]

This can also be written $(126)(3547)(8)$. The corresponding permutation is

<table>
<thead>
<tr>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>↓</td>
<td>↓</td>
<td>↓</td>
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<td>↓</td>
<td>↓</td>
<td>↓</td>
<td>↓</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>5</td>
<td>3</td>
<td>7</td>
<td>1</td>
<td>4</td>
<td>8</td>
</tr>
</tbody>
</table>

**Exercise 2** Take $n = 4$. What is the probability that the output of the algorithm is the permutation 4231? (Hint: First write the cycle decomposition of 4231.)

**Proposition 2.** For every $n$, the output of the Chinese restaurant algorithm is uniform among the $n!$ permutations of size $n$.

*Proof.* By construction, each table repartition with $n$ customers occurs with the same probability

$$1 \times \frac{1}{2} \times \frac{1}{3} \times \cdots \times \frac{1}{n}.$$

Now, each table repartition corresponds to exactly one permutation of size $n$. Therefore each permutation occurs with probability $1/n!$. \qed

**Simulations**

Here is a simulation for $n = 30$:

![Simulation Diagram]

Here is a simulation for $n = 2000$ (We only represent sizes of tables. They have respective sizes 122, 673, 631, 68, 176, 159, 35, 8, 28, 91, 2, 5, 1, 1.):

![Simulation Diagram]

A last simulation for $n = 30000$. Tables have sizes 15974, 11238, 31, 2121, 99, 25, 397, 97, 13, 2, 3.
For more on the Chinese restaurant we refer to [5]. On the following webpage you can run simulations of the Chinese restaurant by yourself:

http://gerin.perso.math.cnrs.fr/ChineseRestaurant.html

3 Typical properties of a random uniform permutation

Frow now on $S_n$ denotes a random uniform permutation of size $n$, generated by any of the previous algorithms.

3.1 Number of fixed points

Definition 1. Let $\sigma$ be a permutation of size $n$. The integer $1 \leq i \leq n$ is a fixed point of $\sigma$ if $\sigma(i) = i$.

For example, 2431 has a unique fixed point at $i = 3$.

Proposition 3. Let $F_n$ be the number of fixed points of $S_n$. For every $n$, we have that

$\mathbb{E}[F_n] = 1$, \quad $\text{Var}(F_n) = 1$.

This is quite surprising that $\mathbb{E}[F_n]$ and $\text{Var}(F_n)$ do not depend on $n$.

Proof. We write $F_n = \sum_{i=1}^{n} X_i$, where

$$X_i = \begin{cases} 1 & \text{if } S_n(i) = i, \\ 0 & \text{otherwise} \end{cases}.$$ 

Random variables $X_i$’s are not independent. Still we have by linearity of expectation that

$$\mathbb{E}[F_n] = \mathbb{E}[X_1 + \cdots + X_n] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n],$$

and we are left to compute $\mathbb{E}[X_i]$ for every $i$. Now,

$$\mathbb{P}(X_i = 1) = \mathbb{P}(S_n(i) = i) = \frac{\text{card} \{ \text{permutations } s \text{ of size } n \text{ with } s(i) = i \}}{\text{card} \{ \text{permutations of size } n \}} = \frac{(n-1)!}{n!} = \frac{1}{n}.$$ 

(Indeed, a permutation such that $s(i) = i$ is also a permutation of the set $\{1, 2, \ldots, i - 1, i + 1, \ldots, n\}$ of size $n - 1$.) Therefore we have that

$$\mathbb{E}[X_i] = 1 \times \mathbb{P}(X_i = 1) + 0 \times \mathbb{P}(X_i = 0) = 1/n.$$

(ii) Thank you to Amic Frouvelle for pointing me that the variance was wrong in the previous version of these notes.
Finally

$$E[F_n] = E[X_1] + \cdots + E[X_n] = n \times 1/n = 1.$$  

In order to compute the variance we will use the formula

$$\text{Var}(\sum X_i) = \sum \text{Var}(X_i) + \sum \text{Cov}(X_i, X_j)$$

$$= n \text{Var}(X_1) + n(n-1)\text{Cov}(X_1, X_2)$$

By the previous computation we have:

$$E[X_1] = \frac{1}{n}, \quad \text{Var}(X_1) = \frac{1}{n}(1 - \frac{1}{n}).$$

Similarly as above we can compute

$$E[X_1 X_2] = \mathbb{P}(X_1 \times X_2 = 1) = \mathbb{P}(X_1 = 1, X_2 = 1) = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}.$$  

Hence $\text{Cov}(X_1, X_2) = \frac{1}{n(n-1)} - E[X_1]|E[X_2] = \frac{1}{n(n-1)} - \frac{1}{n^2}$. Finally

$$\text{Var}(F_n) = 1 - \frac{1}{n} + n(n-1)\frac{1}{n^2(n-1)} = 1.$$

\[ \square \]

**The Poisson paradigm**

There is a general phenomenon in probability known as the *Poisson paradigm*. It says that if $X_i$’s are 0/1 random variable such that

1. $E[X_i] = \mathbb{P}(X_i = 1)$ is ”small” for every $i$ ;
2. $X_i$’s are ”almost” independent ;

then $X = \sum X_i$ is almost distributed like the Poisson distribution with mean $\sum E[X_i]$. Here $\sum E[X_i] = \sum_{i=1}^{n} 1/n = 1$ and one can make the Poisson paradigm rigorous:

**Proposition 4.** Let $(S_n)_n$ be a sequence of random uniform permutations, and let $F_n$ be the number of fixed points of $S_n$. Then $F_n$ converges in distribution to the Poisson distribution with mean 1, i.e.

$$\mathbb{P}(F_n = k) \xrightarrow{n \to +\infty} \mathbb{P}($$[Poisson](1) = k) = \frac{e^{-1}}{k!},$$

for every $k = 0, 1, 2, \ldots$ .

A combinatorial proof can be found at [8]. For more on the Poisson paradigm, we refer to [2].
3.2 Number of inversions

An inversion in $\sigma$ is a pair $(i, j)$ such that

$$\begin{align*}
    i < j, \\
    \sigma(i) > \sigma(j).
\end{align*}$$

Let $\text{Inv}(\sigma)$ be the number of inversions of $\sigma$. For example, if $\sigma = 43152$ then $\text{Inv}(\sigma) = 6$ (each arc counts for an inversion):

$$\sigma: \begin{array}{cccc}
    4 & 3 & 1 & 5 & 2
\end{array}$$

**Proposition 5.** For every $n$, let $S_n$ be a uniform random permutation of size $n$. Then

$$\mathbb{E}[\text{Inv}_n(S_n)] = \frac{n(n-1)}{4}.$$  

**Proof.** We will make a combinatorial proof, with (almost) no computation. First, let $\tilde{\sigma}$ be the reversed permutation of $\sigma$: for every $1 \leq i \leq n$,

$$\tilde{\sigma}(i) = n + 1 - \sigma(i).$$

For instance, if $\sigma = 43152$ then $\tilde{\sigma} = 23514$. Then by construction we have that an arbitrary pair $(i, j)$ is an inversion for $\sigma$ if and only if it is not an inversion for $\tilde{\sigma}$. We deduce that

$$\text{Inv}(\sigma) + \text{Inv}(\tilde{\sigma}) = \text{card} \{ \text{all pairs } 1 \leq i < j \leq n \} = \binom{n}{2} = \frac{n(n-1)}{2}.$$  

Here we see that $\text{Inv}(43152) + \text{Inv}(23514) = 6 + 4 = \binom{5}{2}$:

$$\sigma: \begin{array}{cccc}
    4 & 3 & 1 & 5 & 2
\end{array} \quad \tilde{\sigma}: \begin{array}{cccc}
    2 & 3 & 5 & 1 & 4
\end{array}$$

Now, we apply the above equality to $\sigma = S_n$ and take expectations of both sides:

$$\mathbb{E}[\text{Inv}(S_n)] + \mathbb{E}[\text{Inv}(\tilde{S}_n)] = \frac{n(n-1)}{2}.$$  

But now, it is obvious that $\sigma \mapsto \tilde{\sigma}$ is a bijection so it preserves the uniform measure. Therefore $\tilde{S}_n$ is also a uniform random permutation and we have $\mathbb{E}[\text{Inv}(S_n)] = \mathbb{E}[\text{Inv}(\tilde{S}_n)]$. The proof is done. $$\square$$
3.3 Number of cycles

Proposition 6. Let $C_n$ be the number of cycles of $S_n$. When $n \to +\infty$,

$$
\mathbb{E}[C_n] \xrightarrow{n \to +\infty} \log(n).
$$

Proof. We may assume that $S_n$ is the output of the Chinese restaurant algorithm. All along the process of the Chinese restaurant, a new cycle appears when a customer sits at a new table:

$$
C_n = \sum_{i=1}^{n} Z_i
$$

where

$$
Z_i = \begin{cases} 
1 & \text{if Customer } i \text{ sits at a new table}, \\
0 & \text{otherwise}
\end{cases}
$$

Customer $i$ sits at a new table with probability $1/i$, therefore $\mathbb{E}[Z_i] = 1/i$. Then,

$$
\mathbb{E}[C_n] = \mathbb{E}\left[\sum_{i=1}^{n} Z_i\right] = \sum_{i=1}^{n} \mathbb{E}[Z_i] = \sum_{i=1}^{n} \frac{1}{i}.
$$

Now, we use the fact that $(\text{iii}) \sum_{i=1}^{n} \frac{1}{i} \sim \log(n)$. \hfill \Box

3.4 Size of the first cycle/first table

Let $T_1(n)$ be the number of customers at Table 1 in the Chinese restaurant process at time $n$. By Proposition 2, we have that the random variable $T_1(n)$ has the distribution of the cycle of 1 in the cycle decomposition of a random uniform permutation of size $n$.

Proposition 7. For every $n$, the random variable $T_1(n)$ is uniformly distributed in $\{1, 2, \ldots, n\}$, i.e.

$$
\mathbb{P}(T_1(n) = i) = \frac{1}{n}, \quad \text{for every } i \in \{1, 2, \ldots, n\}.
$$

Remark . 1. The distribution of the sequence $(T_1(n))_{n \geq 1}$ is actually known as the Pólya Urn process [6].

2. A nice problem related to Proposition 7 is given by the 100 prisoners problem [7].

Proof. 1st proof: Probability.

The proof goes by induction. For $n = 1$ this is obvious since with probability one $T_1(1) = 1$.

Assume now that for some $n \geq 1$, the random variable $T_1(n)$ is uniform in $\{1, 2, \ldots, n\}$. If $T_1(n) = i$, then Customer $n+1$ sits at table 1 with probability $i/(n+1)$.

\[(\text{iii})\text{See https://en.wikipedia.org/wiki/Harmonic_series\_\(\mathsf{\text{(mathematics)}}\)\]
Figure: A sketch of the situation when Customer $n+1$ tries to sit.

Therefore

$$T_1(n+1) = \begin{cases} 
i + 1 & \text{with probab. } \frac{i}{n+1}, \\ i & \text{with probab. } \frac{n+1-i}{n+1}. \end{cases} \quad (1)$$

Fix $j \in \{1, \ldots, n+1\}$. The above argument implies that

$$\mathbb{P}(T_1(n+1) = j) = \mathbb{P}(T_1(n+1) = j \cap T_1(n) = j) + \mathbb{P}(T_1(n+1) = j \cap T_1(n) = j - 1)$$
$$= \mathbb{P}(T_1(n+1) = j | T_1(n) = j) \mathbb{P}(T_1(n) = j)$$
$$+ \mathbb{P}(T_1(n+1) = j | T_1(n) = j - 1) \mathbb{P}(T_1(n) = j - 1)$$
$$= \frac{n+1-j}{n+1} \times \mathbb{P}(T_1(n) = j) \quad \text{(apply (1) with } i = j)$$
$$+ \frac{j-1}{n+1} \times \mathbb{P}(T_1(n) = j - 1) \quad \text{(apply (1) with } i = j - 1)$$
$$= \frac{n+1-j}{n+1} \times \frac{1}{n} + \frac{j-1}{n+1} \times \frac{1}{n} \quad \text{(recall } T_1(n) \text{ is uniform)}$$
$$= \frac{n}{n+1} \times \frac{1}{n+1}$$

which proves that $T_1(n+1)$ is uniform in $\{1, \ldots, n+1\}$.

2d proof: Combinatorics.

For $i = 1, \ldots, n$, let us enumerate the permutations in which $T_1(n) = i$. We have to choose $i-1$ elements $x_1, \ldots, x_{i-1}$ ($\binom{n-1}{i-1}$ choices) which belong to this cycle, and put them in a given order ($\binom{i-1}{i-1}$ choices). Then, the $n - i$ remaining elements form a permutation of size $n - i$ ($\binom{n-i}{n-i}$ choices).
Therefore

\[ P(T_1(n) = i) = \frac{\text{card} \{ \text{permutations of size } n \text{ with } T_1(n) = i \}}{n!} \]
\[ = \frac{1}{n!} \frac{(n-1)!}{(i-1)!} \]
\[ = \frac{1}{n!} \frac{(n-1)!}{(i-1)!} = \frac{1}{n}. \]

\[ \square \]

**Discussion: the reinforcement phenomenon**

The Chinese restaurant process illustrates the *reinforcement phenomenon* which is very common in Probability. It is also known as the ”rich gets richer” phenomenon. Indeed, we observe that the more people there are at Table 1 at a given time, the more there will be in the future.

As an application, it turns out that because Table 1 appears sooner than Table 2, Table 1 is much more occupied (in average) than Table 2.

**Proposition 8.** For large \( n \), we have that

\[ \mathbb{E}[T_1(n)] \overset{n \to +\infty}{\sim} \frac{n}{2}, \quad \mathbb{E}[T_2(n)] \overset{n \to +\infty}{\sim} \frac{n}{4}. \]

**Proof.** First, we claim that conditionally on the event \( \{T_1(n) = i\} \), then \( T_2(n) \) is uniformly distributed in \( \{1, 2, \ldots, n-i\} \): for every \( j \leq n-i \) we have

\[ P(T_2(n) = j \mid T_1(n) = i) = \begin{cases} \frac{1}{n-i} & \text{if } i < n, \\ 0 & \text{if } i = n. \end{cases} \]

We skip the proof, which is very similar to the proof of Proposition 7 (in this case the combinatorial proof is easier).

Consequently, if we condition on the event \( \{T_1(n) = i\} \) we have that

\[ \mathbb{E}[T_2(n) | T_1(n)] = \mathbb{E}[ \text{ Uniform random var. in } \{1, 2, \ldots, n-T_1(n)\}] \]
\[ = \frac{1 + n - T_1(n)}{2}. \]

Now, by the tower property of conditional expectation\(^{\text{(iv)}}\) we obtain

\[ \mathbb{E}[T_2(n)] = \mathbb{E} \left[ \mathbb{E}[T_2(n) | T_1(n)] \right] = \mathbb{E} \left[ \frac{1 + n - T_1(n)}{2} \right] = \frac{1 + n - n/2}{2} \sim \frac{n}{4}. \]

\(^{\text{(iv)}}\)This says that \( \mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X]. \)
Discussion: the size-bias phenomenon

We conclude by investigating an apparent paradox:

- In average, there are \( \frac{n}{2} \) people at the same table as 1. But recall that the output of the Chinese restaurant process is uniform in \( \mathcal{S}_n \) so by symmetry, every element in \( \{1, 2, \ldots, n\} \) plays the same role: this table can be considered as a typical table.

- There are in average \( \log(n) \) distinct tables, so a typical table should have (in average) about

\[
\frac{\text{Number of customers}}{\text{Number of tables}} \approx \frac{n}{\log(n)} < \frac{n}{2}
\]

customers.

The paradox is that Table 1 is not typical: by saying that 1 sits at this table the size of this table is biased. The size of Table 1 is overestimated compared to a "true" typical table. This is the size-bias phenomenon, whose a very nice introduction can be found in [1].

4 How to sort \( S_n \) efficiently: average-case analysis of Quicksort

We will discuss a different topic regarding random permutations: the analysis of sorting algorithms. We will focus on one of the most famous: Quicksort.

4.1 The algorithm

Input: Sequence of numbers \( x_1, x_2, \ldots, x_n \)

Output: Re-ordered sequence \( x_{\sigma(1)} \leq x_{\sigma(2)} \cdots \leq x_{\sigma(n)} \)

The algorithm uses the Divide-and-Conquer strategy, there are three steps:

1. Call \( x_1 \) the pivot of the list.

2. Compare all the elements \( x_2, \ldots, x_n \) with \( x_1 \) and re-order the list so that

   (a) elements \( < x_1 \) come before the pivot,

   (b) elements \( \geq x_1 \) come after the pivot.

3. Recursively apply strategy to both sub-lists.

Here are the first steps applied to the permutation 435162:

\[
\begin{align*}
4,3,5,1,6,2 & \rightarrow 4,3,5,1,6,2 \\
\begin{array}{c}
\text{pivot} \\
\geq 4
\end{array} & \rightarrow \\
5,6 & \rightarrow 5,6 \\
\begin{array}{c}
\text{pivot} \\
\geq 5
\end{array} & \rightarrow 6 \text{ done.}
\end{align*}
\]

\[
\begin{align*}
4,3,5,1,6,2 & \rightarrow 4,3,5,1,6,2 \\
\begin{array}{c}
\text{pivot} \\
< 4
\end{array} & \rightarrow \\
3,1,2 & \rightarrow 3,1,2 \\
\begin{array}{c}
\text{pivot} \\
\geq 3
\end{array} & \rightarrow 2 \text{ done.}
\end{align*}
\]

\[
\begin{align*}
4,3,5,1,6,2 & \rightarrow 4,3,5,1,6,2 \\
\begin{array}{c}
\text{pivot} \\
\geq 1
\end{array} & \rightarrow \\
1,2 & \rightarrow 1,2 \\
\begin{array}{c}
\text{pivot} \\
< 1
\end{array} & \rightarrow 1 \text{ done.}
\end{align*}
\]
4.2 Average-case analysis

We consider that the cost of the algorithm driven on \( x_1, \ldots, x_n \) is given by the number \( \text{Comp}(x_1, \ldots, x_n) \) of pairwise comparisons between \( x_i \)'s. For instance, in the above example we have that

\[
\text{Comp}(4, 3, 5, 1, 6, 2) = 5 + 1 + 2 + 1 = 9.
\]

If the input is random, then \( \text{Comp} \) is a random variable.

**Proposition 9.** Let \( X_1, \ldots, X_n \) be independent random variables uniform in the interval \( (0, 1) \). Then, when \( n \to +\infty \),

\[
\mathbb{E}[\text{Comp}(X_1, \ldots, X_n)] = 2n \log(n) + o(n \log(n)).
\]

Both the algorithm and its analysis were provided by Hoare [4]. A modern reference is [3].

**Proof.** By construction \( X_1 \) is the first pivot. Denote by \( Y_1, \ldots, Y_{I-1} \) be the numbers \( > X_1 \), and \( Z_1, \ldots, Z_{n-I} \), so that \( I \) is the (random) rank of \( X_1 \) in the sequence. Because of the recursive strategy the number of comparisons is given by

\[
\text{Comp}(X_1, \ldots, X_n) = \underbrace{n-1}_{\text{Comp. with } X_1} + \text{Comp}(Y_1, \ldots, Y_{I-1}) + \text{Comp}(Z_1, \ldots, Z_{n-I}). \tag{*}
\]

We omit the proofs of the two following claims:

- The rank \( I \) is uniform in \( 1, 2, \ldots, n \).
- Conditionally on \( X_1 \), the \( Y_j \)'s are i.i.d. (and uniform in \( (0, X_1) \)) and the \( Z_j \)'s are i.i.d. (and uniform in \( (X_1, 1) \)).

Therefore, if we take expectations of both sides of (\( \star \)) and put \( c_n = \mathbb{E}[\text{Comp}(X_1, \ldots, X_n)] \) then we obtain

\[
c_n = n - 1 + \sum_{i=1}^{n} \mathbb{P}(I = i) \left( c_{i-1} + c_{n-i} \right)
= n - 1 + \frac{1}{n} \sum_{i=1}^{n} c_{i-1} + \frac{1}{n} \sum_{i=1}^{n} c_{n-i}
= n - 1 + \frac{2}{n} \sum_{i=1}^{n} c_{i-1},
\]

with \( c_0 = c_1 = 0 \). In order to get rid of the sums we compute

\[
nc_n - (n - 1)c_{n-1} = n(n - 1) + 2 \sum_{i=1}^{n} c_{i-1} - (n - 1)(n - 2) - 2 \sum_{i=1}^{n-1} c_{i-1}
= 2(n - 1) + 2 \sum_{i=1}^{n} c_{i-1} - 2 \sum_{i=1}^{n-1} c_{i-1}
= 2(n - 1) + 2c_{n-1}
\]

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so finally
\[ nc_n = 2(n - 1) + (n + 1)c_{n-1}. \]

This can be rewritten as:
\[ n(c_n + 2n) = 2n + (n + 1)(c_{n-1} + 2(n - 1)). \]

If we divide by \( n(n + 1) \) we get
\[ \frac{c_n + 2n}{n+1} = \frac{2}{n+1} + \frac{c_{n-1} + 2(n-1)}{n}. \]

If we put \( d_n := \frac{c_n + 2n}{n+1} \) we have that
\[ d_n = \frac{2}{n+1} + \frac{2}{n} + \frac{2}{n-1} + \cdots + \frac{2}{5} + \frac{2}{4} + d_2. \]
i.e. \( d_n = 2 \log(n) + o(\log(n)) \). Finally,
\[ c_n = 2n \log(n) + o(n \log(n)). \]

\( \square \)

(We observe that the number of comparisons \( \text{Comp}(X_1, \ldots, X_n) \) only depends on the relative order of the \( X_i \)'s, not on their exact values. Therefore Proposition 7 remains true (with the same proof) if \( X_i \)'s are i.i.d. with an arbitrary density.)

References


