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# Random Uniform Permutations

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# MINI-COURSE: RANDOM UNIFORM PERMUTATIONS

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This course is at the interplay between Probability and Combinatorics. It is intended for Master students with a background in Probability (random variables, expectation, conditional probability).

The question we will address is "What can we say about a *typical* large permutation?": the number of cycles, their lengths, the number of fixed points,... This is also a pretext to present some universal phenomena in Probability: reinforcement, the Poisson paradigm, size-bias,...

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## 1 Brief reminder on permutations

Before we turn to *random* permutations, we will give a few definitions regarding non-random (or *deterministic* permutations).

A *permutation* of size  $n \geq 1$  is a bijection  $\sigma : \{1, 2, \dots, n\} \rightarrow \{1, 2, \dots, n\}$ . For example

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 4 & 3 & 1 \end{array}$$

is a permutation of size 4. In these notes we often write a permutation with its one-line representation  $\sigma(1)\sigma(2)\dots\sigma(n)$ . For example the above permutation is simply written 2431.

There are  $n!$  permutations of size  $n$ .

### Cycle decomposition

For our purpose, there is a convenient alternative way to encode a permutation: by its *cycle decomposition*. A *cycle* is a finite sequence of distinct integers, defined up to the cycle order. This means that the three following denote the same cycle:

$$(8, 3, 4) = (3, 4, 8) = (4, 8, 3),$$

while  $(8, 3, 4) \neq (8, 4, 3)$ .

The *cycle decomposition* of a permutation  $\sigma$  is defined as follows. We give the theoretical algorithm and detail the example of

1	2	3	4	5	6	7
↓	↓	↓	↓	↓	↓	↓
6	3	1	5	7	2	4

**Algorithm**

Start with 1st cycle (1)  
 Add to this cycle  $\sigma(1)$ , then  $\sigma(\sigma(1))$ , then  $\sigma(\sigma(\sigma(1)))$ ,  
 and so on until one of this number is one.  
 Start the 2d cycle with a number which has not been  
 seen before.  
 Complete the 2d cycle with same procedure.  
 Create new cycles until there is no remaining number.

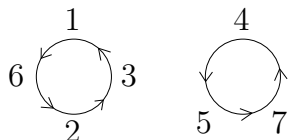
Finally, the cycle decomposition of  $\sigma$  is

$$(1, 6, 2, 3), (4, 5, 7)$$

**Example**

(1)  
 $(1) \rightarrow (1, 6) \rightarrow (1, 6, 2) \rightarrow (1, 6, 2, 3)$   
 and the cycle is over since  $\sigma(3) = 1$ .  
 1st cycle (1, 6, 2, 3). 2d cycle: (4)  
 1st cycle (1, 6, 2, 3). 2d cycle:  $(4) \rightarrow (4, 5) \rightarrow (4, 5, 7)$ .  
 Done.

It is convenient to represent the cycle decomposition of  $\sigma$  with the following diagram:



**Exercise 1** What is the cycle decomposition of 62784315?

## 2 How to simulate a random uniform permutation?

We will first discuss the following question. Imagine that you are given a random number generator `rand` (in your favourite programming language) which returns independent uniform random variables. How to use `rand` to simulate a random uniform permutation of size  $n$ ?

### 2.1 The naive algorithm

It works as follows:

- Pick  $\sigma(1)$  uniformly at random in  $\{1, 2, \dots, n\}$  ( $n$  choices);
- Pick  $\sigma(2)$  uniformly at random in  $\{1, 2, \dots, n\} \setminus \{\sigma(1)\}$  ( $n - 1$  choices);
- Pick  $\sigma(3)$  uniformly at random in  $\{1, 2, \dots, n\} \setminus \{\sigma(1), \sigma(2)\}$  ( $n - 2$  choices),

and so on until  $\sigma(n)$  (1 choice).

By construction every permutation occurs with probability  $1/n!$  so the output is uniform.

## 2.2 The "continuous" algorithm

- Pick continuous random variables  $X_1, X_2, \dots, X_n$ , independently and uniformly in  $(0, 1)$  ;
- With probability 1 the  $n$  values are pairwise distinct. Therefore there exists a unique permutation  $\sigma$  such that

$$X_{\sigma(1)} < X_{\sigma(2)} < X_{\sigma(3)} < \dots < X_{\sigma(n)}.$$

- This  $\sigma$  is your output.

**Proposition 1.** *For every  $n$ , the output of the continuous algorithm is uniform among the  $n!$  permutations of size  $n$ .*

*Proof. Step 1: The  $n$  values are distinct.* We have to prove that

$$\mathbb{P}(\text{for all } i \neq j, X_i \neq X_j) = 1.$$

We prove that the complement event  $\{\text{there are } i, j \text{ such that } X_i = X_j\}$  has probability zero. First we notice that

$$\begin{aligned} \mathbb{P}(\text{there are } i \neq j \text{ such that } X_i = X_j) &= \mathbb{P}(\cup_{i \neq j} \{X_i = X_j\}) \\ &\leq \sum_{i \neq j} \mathbb{P}(X_i = X_j), \end{aligned}$$

by the union bound<sup>(i)</sup>. Now,

$$\mathbb{P}(X_i = X_j) = \int_{(0,1)^2} \mathbf{1}_{x=y} dx dy = \int_{y \in (0,1)} \left( \int_{x \in (0,1)} \mathbf{1}_{x=y} dx \right) dy = \int_{y \in (0,1)} \left( \int_{x=y}^y dx \right) dy = \int_{y \in (0,1)} 0 \times dy = 0.$$

**Step 2: The output  $\sigma$  is uniform.** To avoid messy notations we make the proof in the case  $n = 3$ . Since the 3 values  $X_1, X_2, X_3$  are distinct we have

$$\begin{aligned} 1 &= \mathbb{P}(X_1 < X_2 < X_3) + \mathbb{P}(X_1 < X_3 < X_2) + \mathbb{P}(X_2 < X_1 < X_3) \\ &\quad + \mathbb{P}(X_2 < X_3 < X_1) + \mathbb{P}(X_3 < X_1 < X_2) + \mathbb{P}(X_3 < X_2 < X_1) \\ &= \int_{(0,1)^3} \mathbf{1}_{x_1 < x_2 < x_3} dx_1 dx_2 dx_3 + \int_{(0,1)^3} \mathbf{1}_{x_1 < x_3 < x_2} dx_1 dx_2 dx_3 + \int_{(0,1)^3} \mathbf{1}_{x_2 < x_1 < x_3} dx_1 dx_2 dx_3 \\ &\quad + \int_{(0,1)^3} \mathbf{1}_{x_2 < x_3 < x_1} dx_1 dx_2 dx_3 + \int_{(0,1)^3} \mathbf{1}_{x_3 < x_1 < x_2} dx_1 dx_2 dx_3 + \int_{(0,1)^3} \mathbf{1}_{x_3 < x_2 < x_1} dx_1 dx_2 dx_3. \end{aligned}$$

Now,  $x_1, x_2, x_3$  are dummy variables in the above integrals, so they are interchangeable. Therefore, these 6 integrals are identical and each of these is  $1/6 = 1/3!$ .  $\square$

## 2.3 The "Chinese restaurant" algorithm

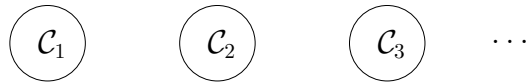
We introduce the Chinese restaurant algorithm, also called the Fisher-Yates algorithm (or even Fisher-Yates-Knuth algorithm). The main difference with the two previous algorithms is that the output  $\sigma$  will be described through its cycle decomposition.

The algorithm runs as follows:

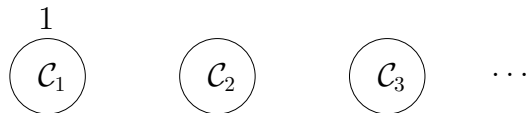
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<sup>(i)</sup>The union bound says that  $\mathbb{P}\left(\bigcup_{n \geq 1} A_n\right) \leq \sum_{n \geq 1} \mathbb{P}(A_n)$  for every sequence of events  $(A_n)$ .

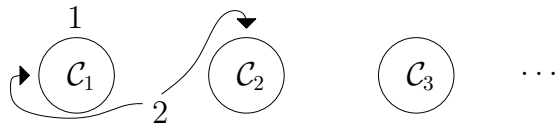
- Assume we are given infinitely many "restaurant tables"  $\mathcal{C}_1, \mathcal{C}_2, \dots$ . These tables are large enough so that an arbitrary number of people can sit at each table.



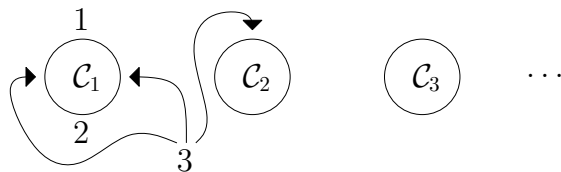
- Infinitely many customers  $1, 2, 3, \dots$  enter the restaurant, one at a time. Put Customer  $n.1$  at table  $\mathcal{C}_1$ :



- With equal probability one-half, put Customer  $n.2$  either at the same table as 1 (on its right) or alone at the new table  $\mathcal{C}_2$ :

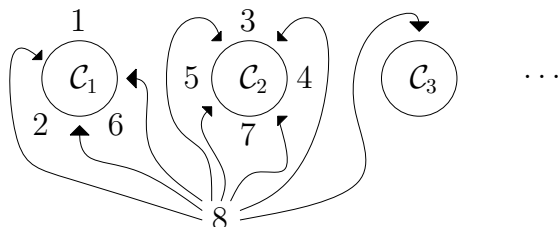


- With equal probability one-third, put Customer  $n.3$  either on the right of 1, or on the right of 2, or alone at the first empty table:

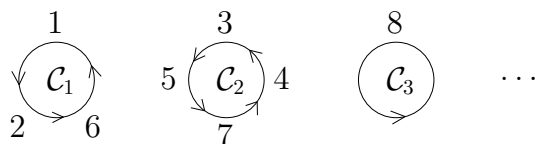


- ...

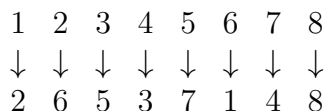
- Assume that customers  $1, 2, \dots, n-1$  are already installed. With equal probability  $1/n$ , put Customer  $n$  either on the right of  $1, \dots$ , or on the right of  $n-1$ , or alone at the first empty table (here  $n=8$ ):



Now, we return the permutation  $\sigma$  whose cycle decomposition corresponds to table repartitions. Assume here that 8 sits alone, we obtain the diagram



This can also be written  $(126)(3547)(8)$ . The corresponding permutation is



**Exercise 2** Take  $n = 4$ . What is the probability that the output of the algorithm is the permutation 4231? (Hint: First write the cycle decomposition of 4231.)

**Proposition 2.** For every  $n$ , the output of the Chinese restaurant algorithm is uniform among the  $n!$  permutations of size  $n$ .

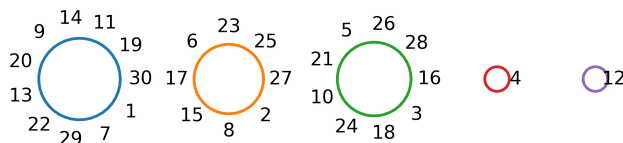
*Proof.* By construction, each table repartition with  $n$  customers occurs with the same probability

$$1 \times \frac{1}{2} \times \frac{1}{3} \times \cdots \times \frac{1}{n}.$$

Now, each table repartition corresponds to exactly one permutation of size  $n$ . Therefore each permutation occurs with probability  $1/n!$ .  $\square$

## Simulations

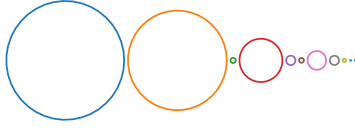
Here is a simulation for  $n = 30$ :



Here is a simulation for  $n = 2000$  (We only represent sizes of tables. They have respective sizes 122, 673, 631, 68, 176, 159, 35, 8, 28, 91, 2, 5, 1, 1.):



A last simulation for  $n = 30000$ . Tables have sizes 15974, 11238, 31, 2121, 99, 25, 397, 97, 13, 2, 3.



For more on the Chinese restaurant we refer to [5]. On the following webpage you can run simulations of the Chinese restaurant by yourself:

<http://gerin.perso.math.cnrs.fr/ChineseRestaurant.html>

### 3 Typical properties of a random uniform permutation

From now on  $S_n$  denotes a random uniform permutation of size  $n$ , generated by any of the previous algorithms.

#### 3.1 Number of fixed points

**Definition 1.** Let  $\sigma$  be a permutation of size  $n$ . The integer  $1 \leq i \leq n$  is a fixed point of  $\sigma$  if  $\sigma(i) = i$ .

For example, 2431 has a unique fixed point at  $i = 3$ .

**Proposition 3.** Let  $F_n$  be the number of fixed points of  $S_n$ . For every  $n$ , we have that<sup>(ii)</sup>

$$\mathbb{E}[F_n] = 1, \quad \text{Var}(F_n) = 1.$$

This is quite surprising that  $\mathbb{E}[F_n]$  and  $\text{Var}(F_n)$  do not depend on  $n$ .

*Proof.* We write  $F_n = \sum_{i=1}^n X_i$ , where

$$X_i = \begin{cases} 1 & \text{if } S_n(i) = i, \\ 0 & \text{otherwise} \end{cases}.$$

Random variables  $X_i$ 's are *not* independent. Still we have by linearity of expectation that

$$\mathbb{E}[F_n] = \mathbb{E}[X_1 + \cdots + X_n] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n],$$

and we are left to compute  $\mathbb{E}[X_i]$  for every  $i$ . Now,

$$\mathbb{P}(X_i = 1) = \mathbb{P}(S_n(i) = i) = \frac{\text{card}\{\text{permutations } s \text{ of size } n \text{ with } s(i) = i\}}{\text{card}\{\text{permutations of size } n\}} = \frac{(n-1)!}{n!} = \frac{1}{n}.$$

(Indeed, a permutation such that  $s(i) = i$  is also a permutation of the set  $\{1, 2, \dots, i-1, i+1, \dots, n\}$  of size  $n-1$ .) Therefore we have that

$$\mathbb{E}[X_i] = 1 \times \mathbb{P}(X_i = 1) + 0 \times \mathbb{P}(X_i = 0) = 1/n.$$

---

<sup>(ii)</sup>Thank you to Amic Frouvelle for pointing me that the variance was wrong in the previous version of these notes.

Finally

$$\mathbb{E}[F_n] = \mathbb{E}[X_1] + \cdots + \mathbb{E}[X_n] = n \times 1/n = 1.$$

In order to compute the variance we will use the formula

$$\begin{aligned} \text{Var}\left(\sum X_i\right) &= \sum_i \text{Var}(X_i) + \sum_{i \neq j} \text{Cov}(X_i, X_j) \\ &= n\text{Var}(X_1) + n(n-1)\text{Cov}(X_1, X_2) \end{aligned}$$

By the previous computation we have:

$$\mathbb{E}[X_1] = \frac{1}{n}, \quad \text{Var}(X_1) = \frac{1}{n}\left(1 - \frac{1}{n}\right).$$

Similarly as above we can compute

$$\mathbb{E}[X_1 X_2] = \mathbb{P}(X_1 \times X_2 = 1) = \mathbb{P}(X_1 = 1, X_2 = 1) = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}.$$

Hence  $\text{Cov}(X_1, X_2) = \frac{1}{n(n-1)} - \mathbb{E}[X_1]\mathbb{E}[X_2] = \frac{1}{n(n-1)} - \frac{1}{n^2}$ . Finally

$$\text{Var}(F_n) = 1 - \frac{1}{n} + n(n-1)\frac{1}{n^2(n-1)} = 1.$$

□

## The Poisson paradigm

There is a general phenomenon in probability known as the *Poisson paradigm*. It says that if  $X_i$ 's are 0/1 random variable such that

1.  $\mathbb{E}[X_i] = \mathbb{P}(X_i = 1)$  is "small" for every  $i$  ;
2.  $X_i$ 's are "almost" independent ;

then  $X = \sum X_i$  is almost distributed like the Poisson distribution with mean  $\sum \mathbb{E}[X_i]$ . Here  $\sum \mathbb{E}[X_i] = \sum_{i=1}^n 1/n = 1$  and one can make the Poisson paradigm rigorous:

**Proposition 4.** *Let  $(S_n)_n$  be a sequence of random uniform permutations, and let  $F_n$  be the number of fixed points of  $S_n$ . Then  $F_n$  converges in distribution to the Poisson distribution with mean 1, i.e.*

$$\mathbb{P}(F_n = k) \xrightarrow{n \rightarrow +\infty} \mathbb{P}(\text{Poisson}(1) = k) = \frac{e^{-1}}{k!},$$

for every  $k = 0, 1, 2, \dots$

A combinatorial proof can be found at [8]. For more on the Poisson paradigm, we refer to [2].



### 3.2 Number of inversions

An *inversion* in  $\sigma$  is a pair  $(i, j)$  such that

$$\begin{cases} i < j, \\ \sigma(i) > \sigma(j). \end{cases}$$

Let  $\text{Inv}(\sigma)$  be the number of inversions of  $\sigma$ . For example, if  $\sigma = 43152$  then  $\text{Inv}(\sigma) = 6$  (each arc counts for an inversion):

$$\sigma: \quad \begin{array}{cccccc} & \frown & \frown & \frown & \frown & \\ & 4 & 3 & 1 & 5 & 2 \end{array}$$

**Proposition 5.** *For every  $n$ , let  $S_n$  be a uniform random permutation of size  $n$ . Then*

$$\mathbb{E}[\text{Inv}_n(S_n)] = \frac{n(n-1)}{4}.$$

*Proof.* We will make a combinatorial proof, with (almost) no computation. First, let  $\tilde{\sigma}$  be the *reversed* permutation of  $\sigma$ : for every  $1 \leq i \leq n$ ,

$$\tilde{\sigma}(i) = n + 1 - \sigma(i).$$

For instance, if  $\sigma = 43152$  then  $\tilde{\sigma} = 23514$ . Then by construction we have that an arbitrary pair  $(i, j)$  is an inversion for  $\sigma$  if and only if it is not an inversion for  $\tilde{\sigma}$ . We deduce that

$$\text{Inv}(\sigma) + \text{Inv}(\tilde{\sigma}) = \text{card} \{ \text{all pairs } 1 \leq i < j \leq n \} = \binom{n}{2} = \frac{n(n-1)}{2}.$$

Here we see that  $\text{Inv}(43152) + \text{Inv}(23514) = 6 + 4 = \binom{5}{2}$ :

$$\begin{array}{cccccc} \sigma: & \frown & \frown & \frown & \frown & \\ & 4 & 3 & 1 & 5 & 2 \\ \\ \tilde{\sigma}: & \frown & \frown & \frown & \frown & \\ & 2 & 3 & 5 & 1 & 4 \end{array}$$

Now, we apply the above equality to  $\sigma = S_n$  and take expectations of both sides:

$$\mathbb{E}[\text{Inv}(S_n)] + \mathbb{E}[\text{Inv}(\tilde{S}_n)] = \frac{n(n-1)}{2}.$$

But now, it is obvious that  $\sigma \mapsto \tilde{\sigma}$  is a bijection so it preserves the uniform measure. Therefore  $\tilde{S}_n$  is also a uniform random permutation and we have  $\mathbb{E}[\text{Inv}(S_n)] = \mathbb{E}[\text{Inv}(\tilde{S}_n)]$ . The proof is done.  $\square$

### 3.3 Number of cycles

**Proposition 6.** Let  $C_n$  be the number of cycles of  $S_n$ . When  $n \rightarrow +\infty$ ,

$$\mathbb{E}[C_n] \stackrel{n \rightarrow +\infty}{\sim} \log(n).$$

*Proof.* We may assume that  $S_n$  is the output of the Chinese restaurant algorithm. All along the process of the Chinese restaurant, a new cycle appears when a customer sits at a new table:

$$C_n = \sum_{i=1}^n Z_i,$$

where

$$Z_i = \begin{cases} 1 & \text{if Customer } i \text{ sits at a new table,} \\ 0 & \text{otherwise} \end{cases}.$$

Customer  $i$  sits at a new table with probability  $1/i$ , therefore  $\mathbb{E}[Z_i] = 1/i$ . Then,

$$\mathbb{E}[C_n] = \mathbb{E}\left[\sum_{i=1}^n Z_i\right] = \sum_{i=1}^n \mathbb{E}[Z_i] = \sum_{i=1}^n \frac{1}{i}.$$

Now, we use the fact that<sup>(iii)</sup>  $\sum_{i=1}^n \frac{1}{i} \sim \log(n)$ . □

### 3.4 Size of the first cycle/first table

Let  $T_1(n)$  be the number of customers at Table 1 in the Chinese restaurant process at time  $n$ . By Proposition 2, we have that the random variable  $T_1(n)$  has the distribution of the cycle of 1 in the cycle decomposition of a random uniform permutation of size  $n$ .

**Proposition 7.** For every  $n$ , the random variable  $T_1(n)$  is uniformly distributed in  $\{1, 2, \dots, n\}$ , i.e.

$$\mathbb{P}(T_1(n) = i) = \frac{1}{n}, \quad \text{for every } i \in \{1, 2, \dots, n\}.$$

**Remark .** 1. The distribution of the sequence  $(T_1(n))_{n \geq 1}$  is actually known as the Pólya Urn process [6].

2. A nice problem related to Proposition 7 is given by the 100 prisoners problem [7].

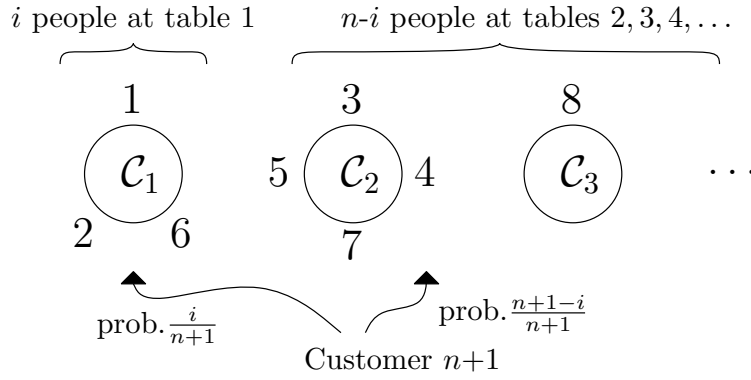
*Proof. 1st proof: Probability.*

The proof goes by induction. For  $n = 1$  this is obvious since with probability one  $T_1(1) = 1$ .

Assume now that for some  $n \geq 1$ , the random variable  $T_1(n)$  is uniform in  $\{1, 2, \dots, n\}$ . If  $T_1(n) = i$ , then Customer  $n + 1$  sits at table 1 with probability  $i/(n + 1)$ .

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<sup>(iii)</sup>See [https://en.wikipedia.org/wiki/Harmonic\\_series\\_\(mathematics\)](https://en.wikipedia.org/wiki/Harmonic_series_(mathematics))



**Figure:** A sketch of the situation when Customer  $n + 1$  tries to sit.

Therefore

$$T_1(n+1) = \begin{cases} i+1 & \text{with probab. } \frac{i}{n+1}, \\ i & \text{with probab. } \frac{n+1-i}{n+1}. \end{cases} \quad (1)$$

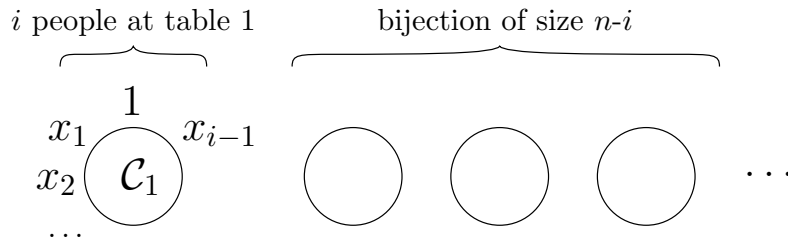
Fix  $j \in \{1, \dots, n+1\}$ . The above argument implies that

$$\begin{aligned} \mathbb{P}(T_1(n+1) = j) &= \mathbb{P}(T_1(n+1) = j \cap T_1(n) = j) + \mathbb{P}(T_1(n+1) = j \cap T_1(n) = j-1) \\ &= \mathbb{P}(T_1(n+1) = j | T_1(n) = j) \mathbb{P}(T_1(n) = j) \\ &\quad + \mathbb{P}(T_1(n+1) = j | T_1(n) = j-1) \mathbb{P}(T_1(n) = j-1) \\ &= \frac{n+1-j}{n+1} \times \mathbb{P}(T_1(n) = j) \quad (\text{apply (1) with } i = j.) \\ &\quad + \frac{j-1}{n+1} \times \mathbb{P}(T_1(n) = j-1) \quad (\text{apply (1) with } i = j-1.) \\ &= \frac{n+1-j}{n+1} \times \frac{1}{n} + \frac{j-1}{n+1} \times \frac{1}{n} \quad (\text{recall } T_1(n) \text{ is uniform}) \\ &= \frac{n}{(n+1)n} = \frac{1}{n+1}, \end{aligned}$$

which proves that  $T_1(n+1)$  is uniform in  $\{1, \dots, n+1\}$ .

**2d proof: Combinatorics.**

For  $i = 1, \dots, n$ , let us enumerate the permutations in which  $T_1(n) = i$ . We have to choose  $i-1$  elements  $x_1, \dots, x_{i-1}$  ( $\binom{n-1}{i-1}$  choices) which belong to this cycle, and put them in a given order ( $(i-1)!$  choices). Then, the  $n-i$  remaining elements form a permutation of size  $n-i$  ( $(n-i)!$  choices).



Therefore

$$\begin{aligned}
\mathbb{P}(T_1(n) = i) &= \frac{\text{card}\{\text{permutations of size } n \text{ with } T_1(n) = i\}}{n!} \\
&= \frac{1}{n!} \binom{n-1}{i-1} (i-1)!(n-i)! \\
&= \frac{1}{n!} \frac{(n-1)!}{(i-1)!(n-i)!} (i-1)!(n-i)! = \frac{1}{n}.
\end{aligned}$$

□

### Discussion: the reinforcement phenomenon

The Chinese restaurant process illustrates the *reinforcement phenomenon* which is very common in Probability. It is also known as the "rich gets richer" phenomenon. Indeed, we observe that the more people there are at Table 1 at a given time, the more there will be in the future.

As an application, it turns out that because Table 1 appears sooner than Table 2, Table 1 is much more occupied (in average) than Table 2.

**Proposition 8.** *For large  $n$ , we have that*

$$\mathbb{E}[T_1(n)] \stackrel{n \rightarrow +\infty}{\sim} \frac{n}{2}, \quad \mathbb{E}[T_2(n)] \stackrel{n \rightarrow +\infty}{\sim} \frac{n}{4}.$$

*Proof.* First, we claim that conditionally on the event  $\{T_1(n) = i\}$ , then  $T_2(n)$  is uniformly distributed in  $\{1, 2, \dots, n - i\}$ : for every  $j \leq n - i$  we have

$$\mathbb{P}(T_2(n) = j \mid T_1(n) = i) = \begin{cases} \frac{1}{n-i} & \text{if } i < n, \\ 0 & \text{if } i = n. \end{cases}$$

We skip the proof, which is very similar to the proof of Proposition 7 (in this case the combinatorial proof is easier).

Consequently, if we condition on the event  $\{T_1(n) = i\}$  we have that

$$\begin{aligned}
\mathbb{E}[T_2(n) \mid T_1(n)] &= \mathbb{E}[\text{Uniform random var. in } \{1, 2, \dots, n - T_1(n)\}] \\
&= \frac{1 + n - T_1(n)}{2}.
\end{aligned}$$

Now, by the tower property of conditional expectation<sup>(iv)</sup> we obtain

$$\mathbb{E}[T_2(n)] = \mathbb{E}\left[\mathbb{E}[T_2(n) \mid T_1(n)]\right] = \mathbb{E}\left[\frac{1 + n - T_1(n)}{2}\right] = \frac{1 + n - n/2}{2} \sim \frac{n}{4}.$$

□

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<sup>(iv)</sup>This says that  $\mathbb{E}[\mathbb{E}[X \mid Y]] = \mathbb{E}[X]$ .

## Discussion: the size-bias phenomenon

We conclude by investigating an apparent paradox:

- In average, there are  $n/2$  people at the same table as 1. But recall that the output of the Chinese restaurant process is uniform in  $\mathfrak{S}_n$  so by symmetry, every element in  $\{1, 2, \dots, n\}$  plays the same role: this table can be considered as a *typical* table.
- There are in average  $\log(n)$  distinct tables, so a *typical* table should have (in average) about

$$\frac{\text{Number of customers}}{\text{Number of tables}} \approx \frac{n}{\log(n)} \ll \frac{n}{2}$$

customers.

The paradox is that Table 1 is *not* typical: by saying that 1 sits at this table the size of this table is biased. The size of Table 1 is overestimated compared to a "true" typical table. This is the *size-bias* phenomenon, whose a very nice introduction can be found in [1].

## 4 How to sort $S_n$ efficiently: average-case analysis of Quicksort

We will discuss a different topic regarding random permutations: the analysis of sorting algorithms. We will focus on one of the most famous: **Quicksort**.

### 4.1 The algorithm

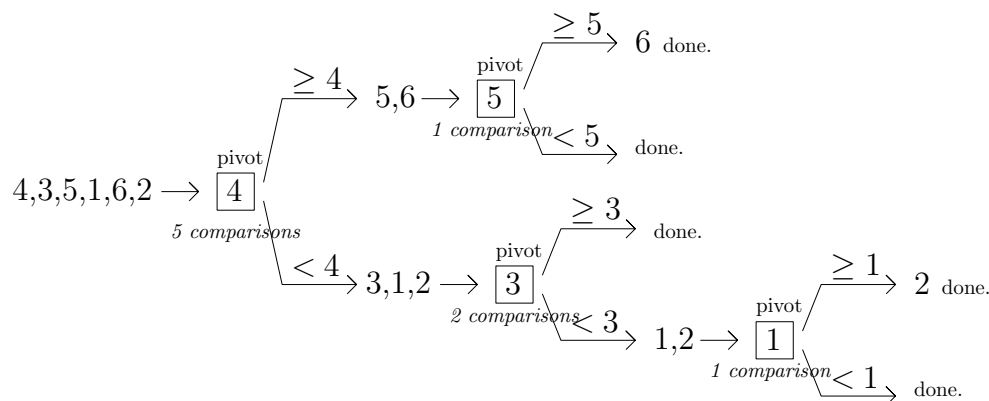
**Input:** Sequence of numbers  $x_1, x_2, \dots, x_n$

**Output:** Re-ordered sequence  $x_{\sigma(1)} \leq x_{\sigma(2)} \dots \leq x_{\sigma(n)}$

The algorithm uses the *Divide-and-Conquer* strategy, there are three steps:

1. Call  $x_1$  the *pivot* of the list.
2. Compare all the elements  $x_2, \dots, x_n$  with  $x_1$  and re-order the list so that
  - (a) elements  $< x_1$  come before the pivot,
  - (b) elements  $\geq x_1$  come after the pivot.
3. Recursively apply strategy to both sub-lists.

Here are the first steps applied to the permutation 435162:



## 4.2 Average-case analysis

We consider that the cost of the algorithm driven on  $x_1, \dots, x_n$  is given by the number  $\text{Comp}(x_1, \dots, x_n)$  of pairwise comparisons between  $x_i$ 's. For instance, in the above example we have that

$$\text{Comp}(4, 3, 5, 1, 6, 2) = 5 + 1 + 2 + 1 = 9.$$

If the input is random, then  $\text{Comp}$  is a random variable.

**Proposition 9.** *Let  $X_1, \dots, X_n$  be independent random variables uniform in the interval  $(0, 1)$ . Then, when  $n \rightarrow +\infty$ ,*

$$\mathbb{E}[\text{Comp}(X_1, \dots, X_n)] = 2n \log(n) + o(n \log(n)).$$

Both the algorithm and its analysis were provided by Hoare [4]. A modern reference is [3].

*Proof.* By construction  $X_1$  is the first pivot. Denote by  $Y_1, \dots, Y_{I-1}$  be the numbers  $> X_1$ , and  $Z_1, \dots, Z_{n-I}$ , so that  $I$  is the (random) rank of  $X_1$  in the sequence. Because of the recursive strategy the number of comparisons is given by

$$\text{Comp}(X_1, \dots, X_n) = \underbrace{n-1}_{\text{Comp. with } X_1} + \text{Comp}(Y_1, \dots, Y_{I-1}) + \text{Comp}(Z_1, \dots, Z_{n-I}). \quad (\star)$$

We omit the proofs of the two following claims:

- The rank  $I$  is uniform in  $1, 2, \dots, n$ .
- Conditionally on  $X_1$ , the  $Y_j$ 's are i.i.d. (and uniform in  $(0, X_1)$ ) and the  $Z_j$ 's are i.i.d. (and uniform in  $(X_1, 1)$ ).

Therefore, if we take expectations of both sides of  $(\star)$  and put  $c_n = \mathbb{E}[\text{Comp}(X_1, \dots, X_n)]$  then we obtain

$$\begin{aligned} c_n &= n - 1 + \sum_{i=1}^n \mathbb{P}(I = i) (c_{i-1} + c_{n-i}) \\ &= n - 1 + \frac{1}{n} \sum_{i=1}^n c_{i-1} + \frac{1}{n} \sum_{i=1}^n c_{n-i} \\ &= n - 1 + \frac{2}{n} \sum_{i=1}^n c_{i-1}, \end{aligned}$$

with  $c_0 = c_1 = 0$ . In order to get rid of the sums we compute

$$\begin{aligned} nc_n - (n-1)c_{n-1} &= n(n-1) + 2 \sum_{i=1}^n c_{i-1} - (n-1)(n-2) - 2 \sum_{i=1}^{n-1} c_{i-1} \\ &= 2(n-1) + 2 \sum_{i=1}^n c_{i-1} - 2 \sum_{i=1}^{n-1} c_{i-1} \\ &= 2(n-1) + 2c_{n-1} \end{aligned}$$

so finally

$$nc_n = 2(n-1) + (n+1)c_{n-1}.$$

This can be rewritten as:

$$n(c_n + 2n) = 2n + (n+1)(c_{n-1} + 2(n-1)).$$

If we divide by  $n(n+1)$  we get

$$\frac{c_n + 2n}{n+1} = \frac{2}{(n+1)} + \frac{c_{n-1} + 2(n-1)}{n}.$$

If we put  $d_n := \frac{c_n + 2n}{n+1}$  we have that

$$d_n = \frac{2}{n+1} + \frac{2}{n} + \frac{2}{n-1} + \cdots + \frac{2}{5} + \frac{2}{4} + d_2.$$

*i.e.*  $d_n = 2 \log(n) + o(\log(n))$ . Finally,

$$c_n = 2n \log(n) + o(n \log(n)).$$

□

(We observe that the number of comparisons  $\text{Comp}(X_1, \dots, X_n)$  only depends on the relative order of the  $X_i$ 's, not on their exact values. Therefore Proposition 9 remains true (with the same proof) if  $X_i$ 's are *i.i.d.* with an arbitrary density.)

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