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# Multiqubit gates and C-gates

Introduction to Quantum Computing

**Kenneth MAUSSANG**

Université de Montpellier

2022 – 2023

# Multiqubit gates and C-gates

- 1 Binary quantum gates
- 2 Examples of multiqubit gates
- 3 Deutsch-Josa algorithm

- 1 Binary quantum gates
  - Definition
  - Circuit representation of a C-gate
  - Importance of the C-NOT gate
- 2 Examples of multiqubit gates
- 3 Deutsch-Josa algorithm

# Multiqubit gates and C-gates

- 1 Binary quantum gates
  - Definition
    - Circuit representation of a C-gate
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- 2 Examples of multiqubit gates
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## I.1. Definition

A **binary quantum gate** is a unitary operation on two qubits, e.g. a unitary map  $\mathcal{H}_2 \otimes \mathcal{H}_2 \rightarrow \mathcal{H}_2 \otimes \mathcal{H}_2$ , where  $\mathcal{H}_2$  is a hilbert space of dimension  $2 \times 2$ . A basis of  $\mathcal{H}_2 \otimes \mathcal{H}_2$  is

$$\{|00\rangle, |10\rangle, |01\rangle, |11\rangle\}.$$

**C-gate:** let  $A$  and  $B$  be two qubits. Let  $M$  be a unitary quantum gate acting on  $B$ . The controlled- $M$  gate (or C- $M$  gate) is the binary gate acting on  $A \otimes B$  defined as follow

$$C-M = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathbb{I}_B + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes M,$$

where  $\mathbb{I}_B$  is the identity operator on qubit  $B$ . A C-gate is the operation such that  $M$  is applied to  $B$  only if the qubit  $A$  is in the state  $|1\rangle$ .

Important example: C-NOT gate.

$$\text{C-NOT} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\hat{U}_{\text{C-NOT}} |00\rangle = |00\rangle, \quad \hat{U}_{\text{C-NOT}} |01\rangle = |01\rangle,$$

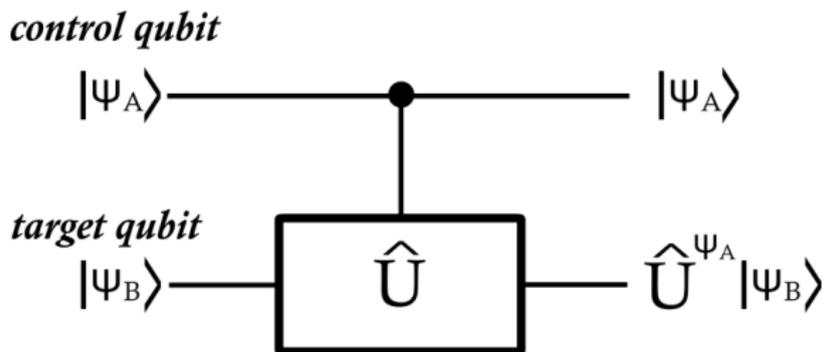
$$\hat{U}_{\text{C-NOT}} |10\rangle = |11\rangle, \quad \hat{U}_{\text{C-NOT}} |11\rangle = |10\rangle.$$

$$\hat{U}_{\text{C-NOT}} \left( \frac{1}{\sqrt{2}} (|00\rangle + |10\rangle) \right) = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle).$$

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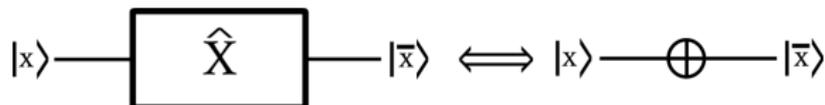
## I.2. Circuit representation of a C-gate

C- $U$  gate where the controlled qubit is A.



Example: C-NOT gate.

NOT quantum gate



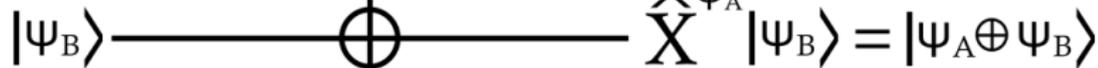
## 1.2. Circuit representation of a C-gate

C-NOT gate where the controlled qubit is A.

*control qubit*



*target qubit*



A C-NOT gate might be seen as a way to implement the XOR classical gate.

- 1 Binary quantum gates
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## I.3. Importance of the C-NOT gate

**Theorem:** *All quantum circuits can be constructed using only C-NOT gates and single-qubit gates.*

C-NOT gate is self inverse

$$(\text{C-NOT}) \cdot (\text{C-NOT}) = \mathbb{I} \otimes \mathbb{I}.$$

Ref: Elementary gates for quantum computation, Physical Review A, **52**, 5, 3457-3467 (1995).

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  - Logical gates
  - Boolean circuits
  - SWAP gate
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## II.1. The Toffoli gate

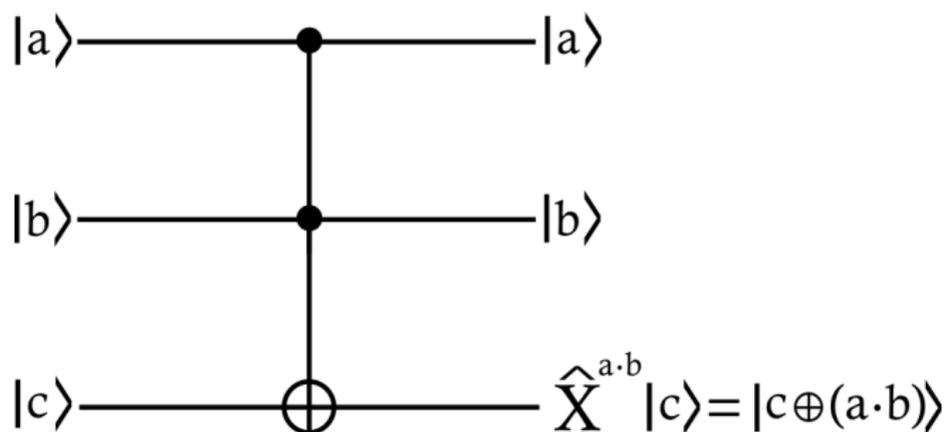
The **Toffoli gate**, originally devised as a universal, reversible classical logic gate by Toffoli, is especially interesting because depending on the input, the gate can perform logical AND, XOR and NOT operations, making it universal for classical computing.

Toffoli is often referred to a "controlled-controlled-NOT" gate ( $C^2$ -NOT).

Ref: Reversible Computation, T. Toffoli, Cambridge, MA, p.36 (1988).

## II.1. The Toffoli gate

The circuit diagram of a Toffoli gate is the following



Toffoli is self inverse

$$\text{Toffoli} \cdot \text{Toffoli} = \mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I}.$$

## II.1. The Toffoli gate

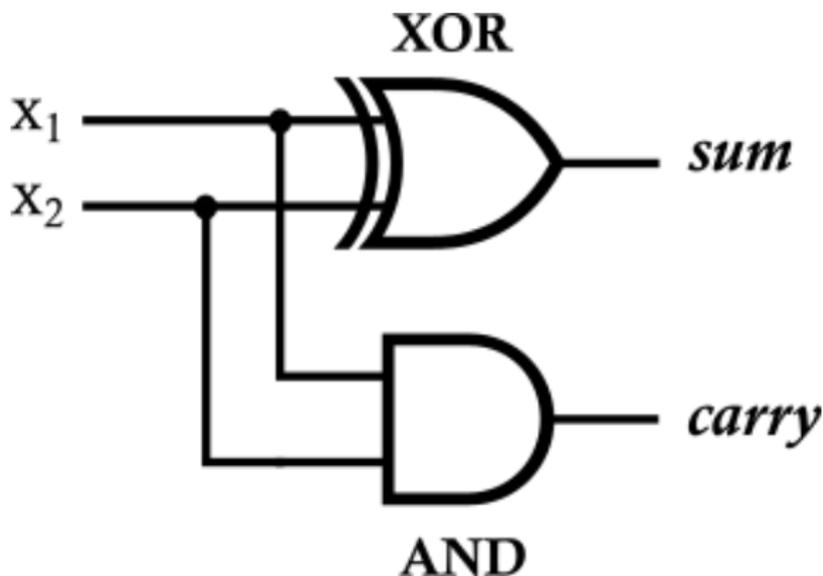
$$\text{Toffoli} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

**Theorem:** *All quantum circuits can be constructed (in some approximated sense) using only Hadamard gates and Toffoli gates.*

Ref: Both Toffoli and controlled-NOT need little help to do universal quantum computation, Yaoyun Shi, *Quantum Information and Computation*, **3**, 1, 84-92 (2003).

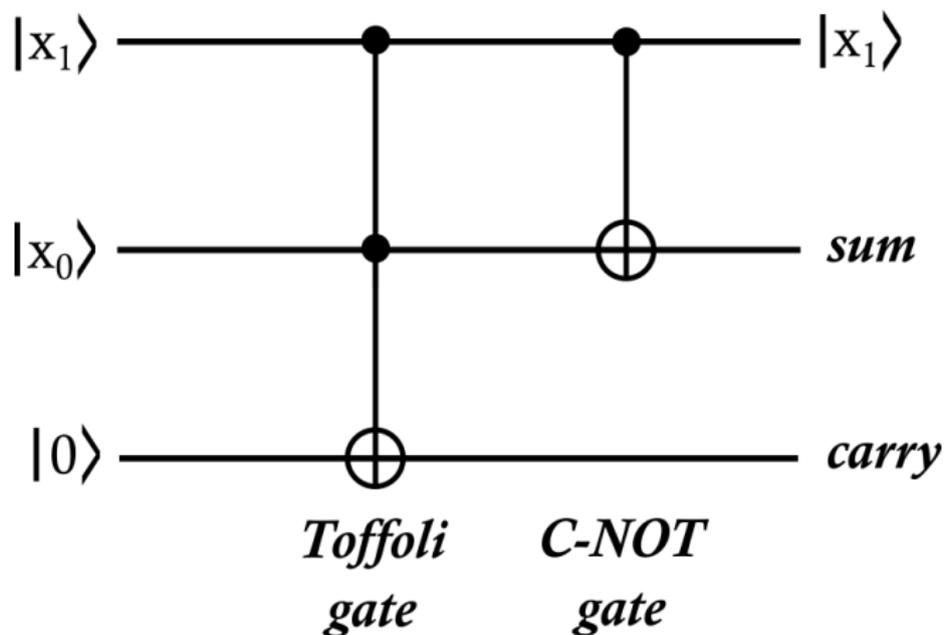
## II.1. The Toffoli gate

**Example:** A classical half-adder compute the sum and carry for two bits  $x_1$  and  $x_0$ .



## II.1. The Toffoli gate

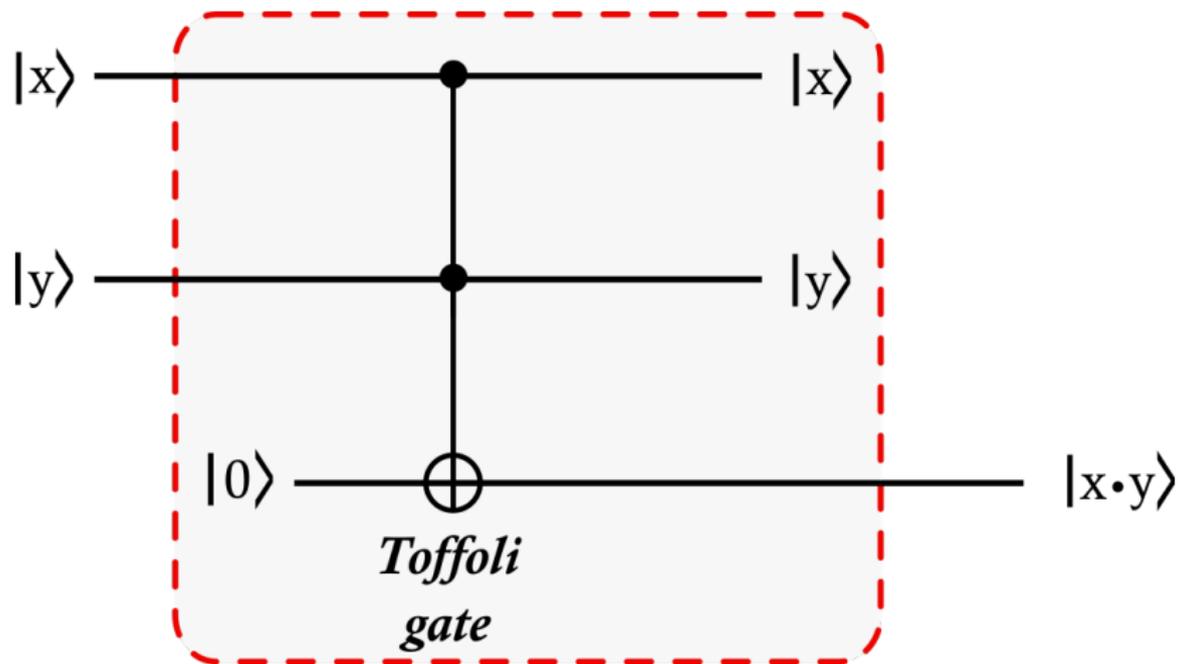
Quantum half-adder.



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  - Boolean circuits
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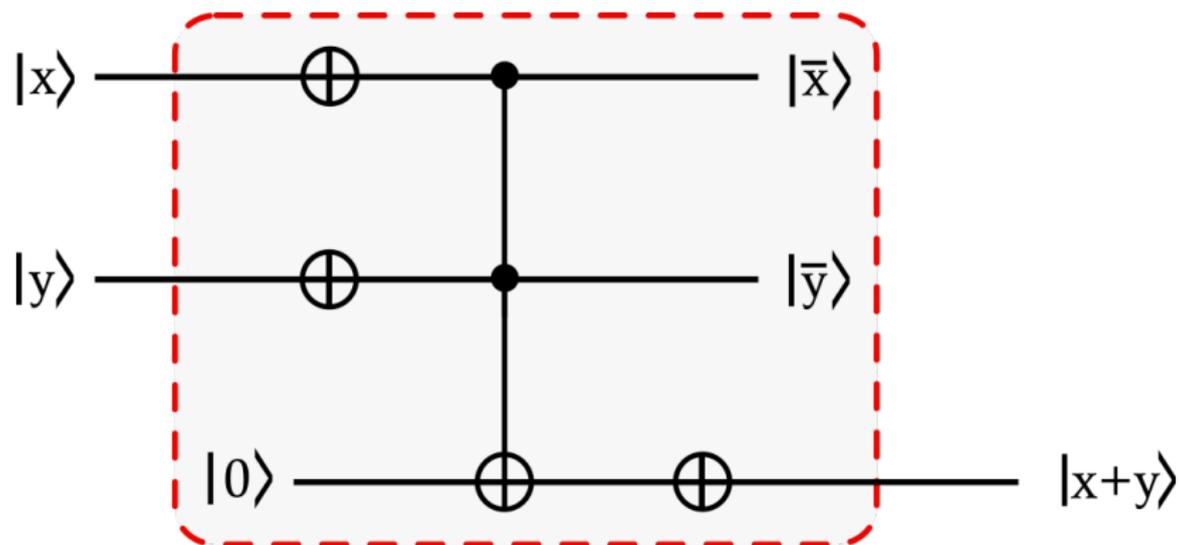
## 11.2. Logical gates

AND logical gate.



## II.2. Logical gates

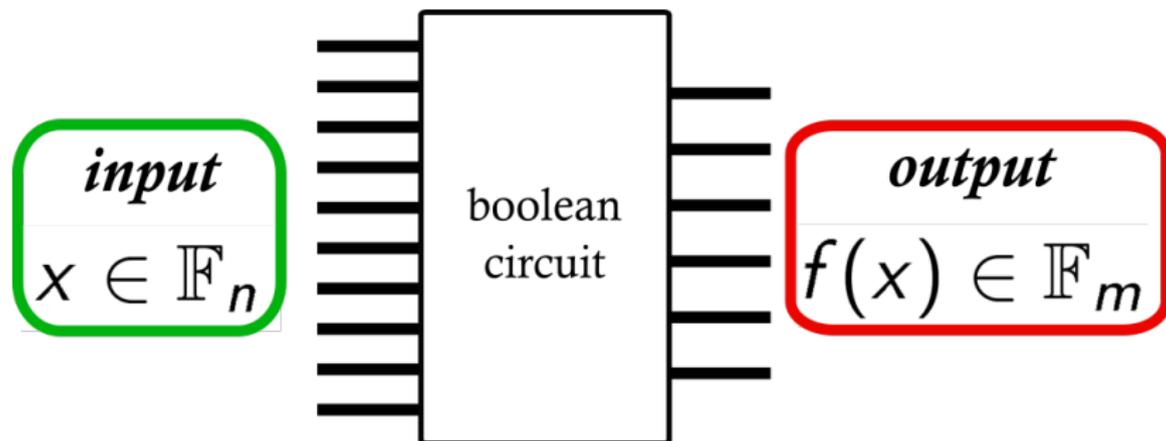
OR logical gate.



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## 11.3. Boolean circuits

Let note  $\mathbb{F}_n = \{0, 1\}^n$ . A boolean function  $f : \mathbb{F}_n \rightarrow \mathbb{F}_m$  can't be a unitary operation. The number of inputs do not equal the number of outputs, so the map is not invertible.



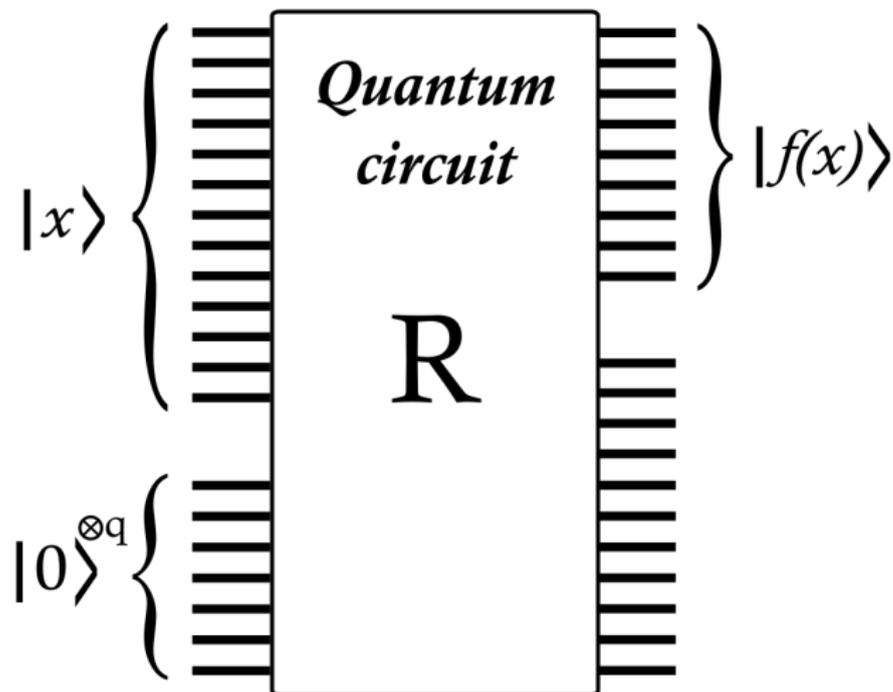
However, it is in fact possible to construct a quantum circuit that performs the same function than any classical boolean circuit.

## 11.3. Boolean circuits

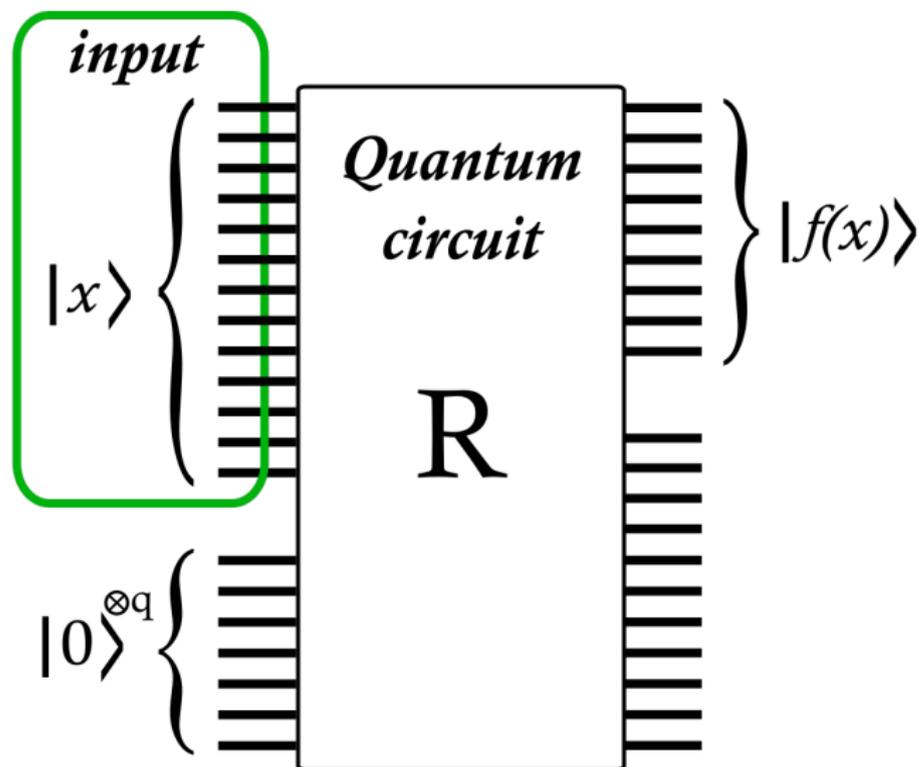
Let  $f : \mathbb{F}_n \rightarrow \mathbb{F}_m$  a boolean function with  $k$  gates. It is possible to construct a quantum circuit,  $\hat{R}$ , that performs the same function.

This quantum circuit uses  $\mathcal{O}(k)$  gates and requires  $q = \mathcal{O}(k)$  additional qubits, so-called **ancilla qubits**, only used for the calculation. These ancilla qubits are not part of the quantum register, and are all initially in the pure state  $|0\rangle$ . Then, such a circuit outputs  $n + q - m$  garbage qubits.

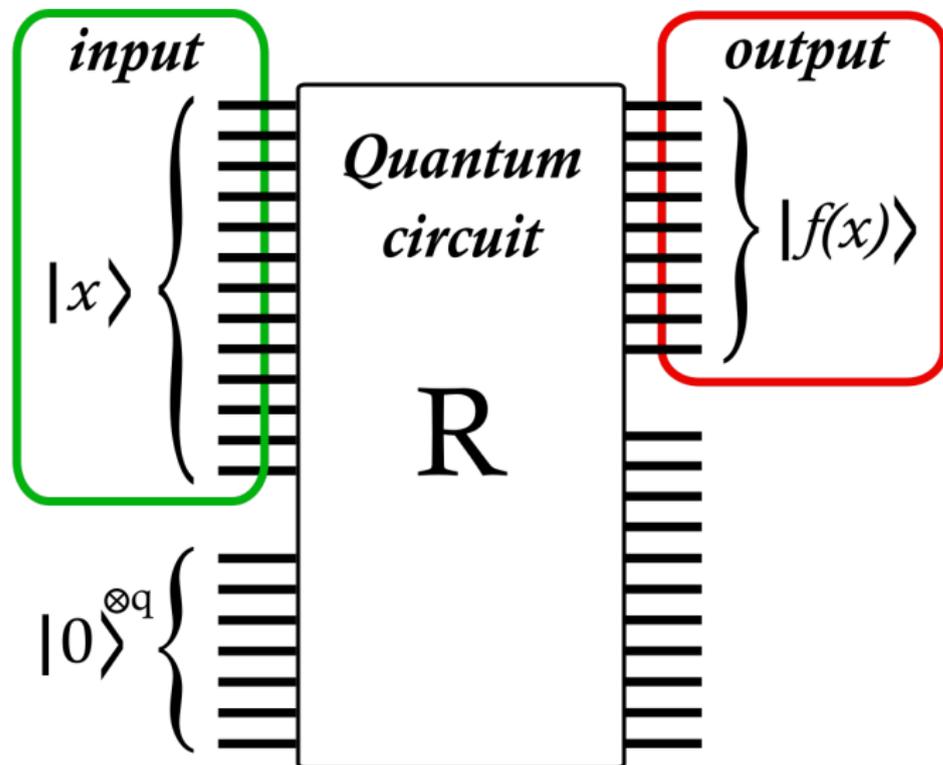
## 11.3. Boolean circuits



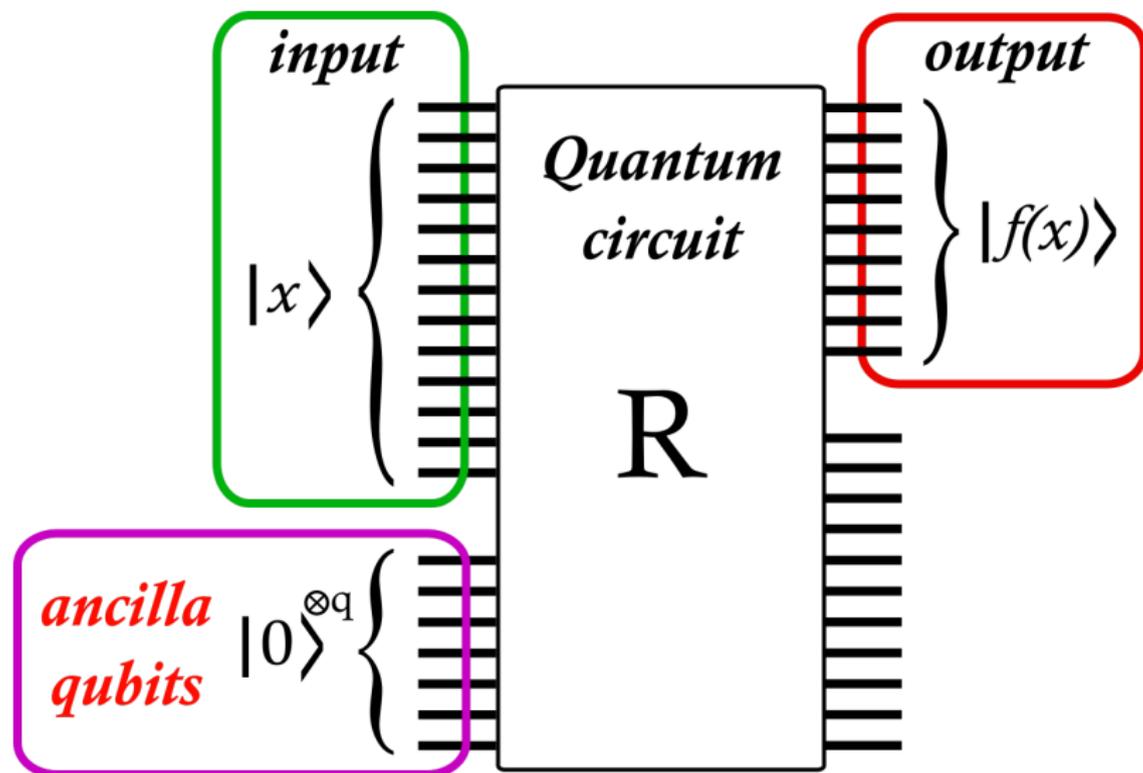
## 11.3. Boolean circuits



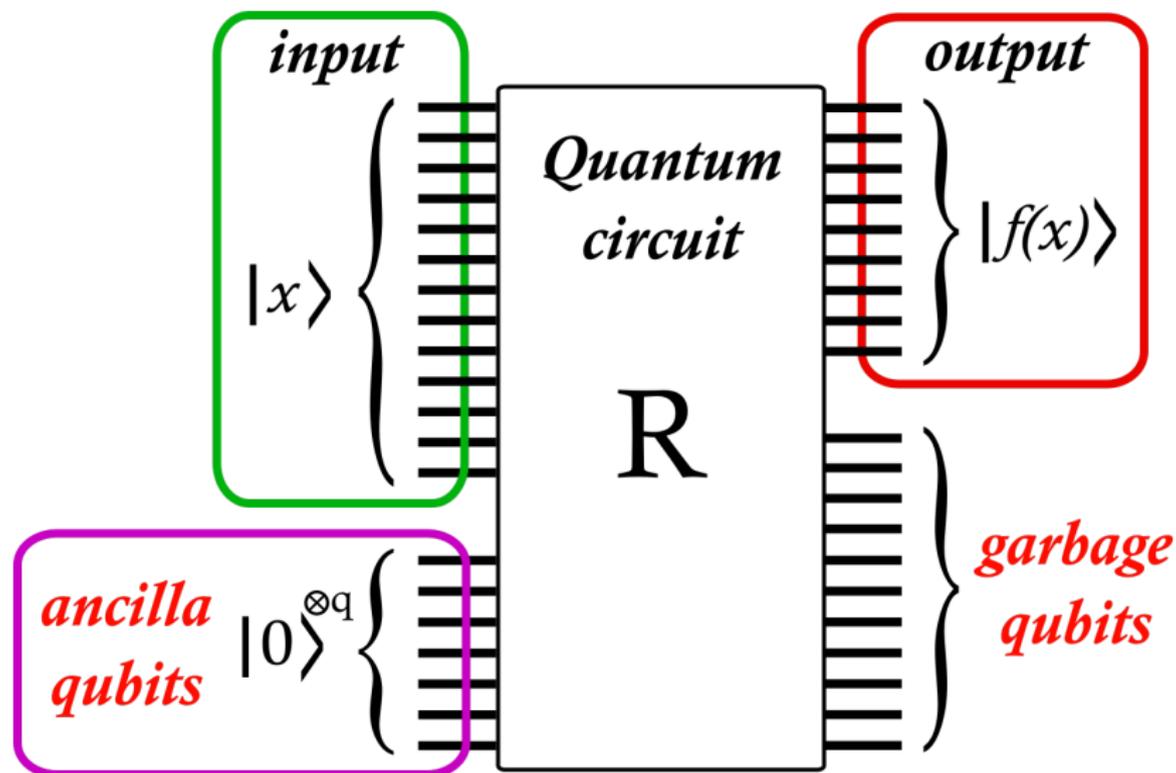
## 11.3. Boolean circuits



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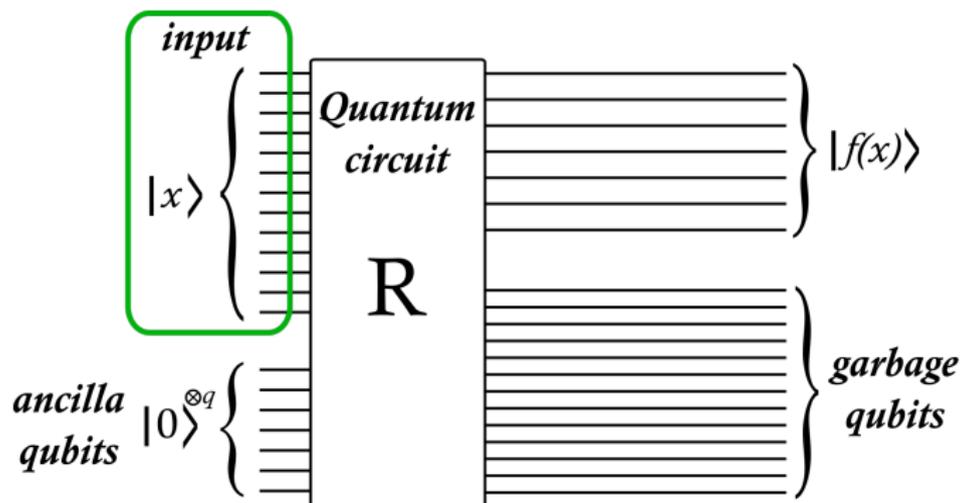


## 11.3. Boolean circuits

To obtain the invert  $\hat{R}^{-1}$  of  $\hat{R}$ , one just has to take the mirror image of the circuit  $\hat{R}$ . This is used to "recycle" the ancilla qubits, so that they are reset to  $|0\rangle$ .

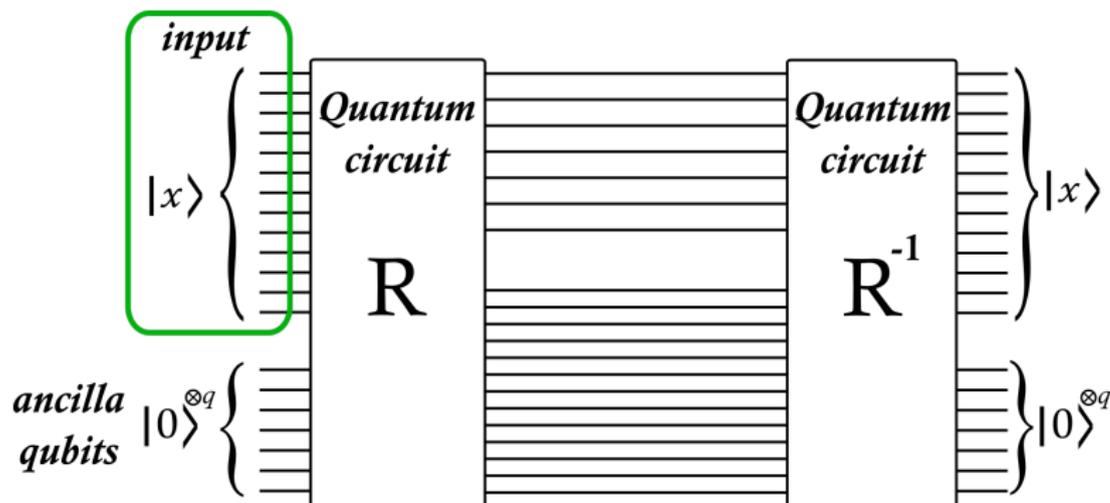
## 11.3. Boolean circuits

So a quantum circuit emulating a boolean circuit that performs the function  $f : \mathbb{F}_n \rightarrow \mathbb{F}_m$  has the following structure



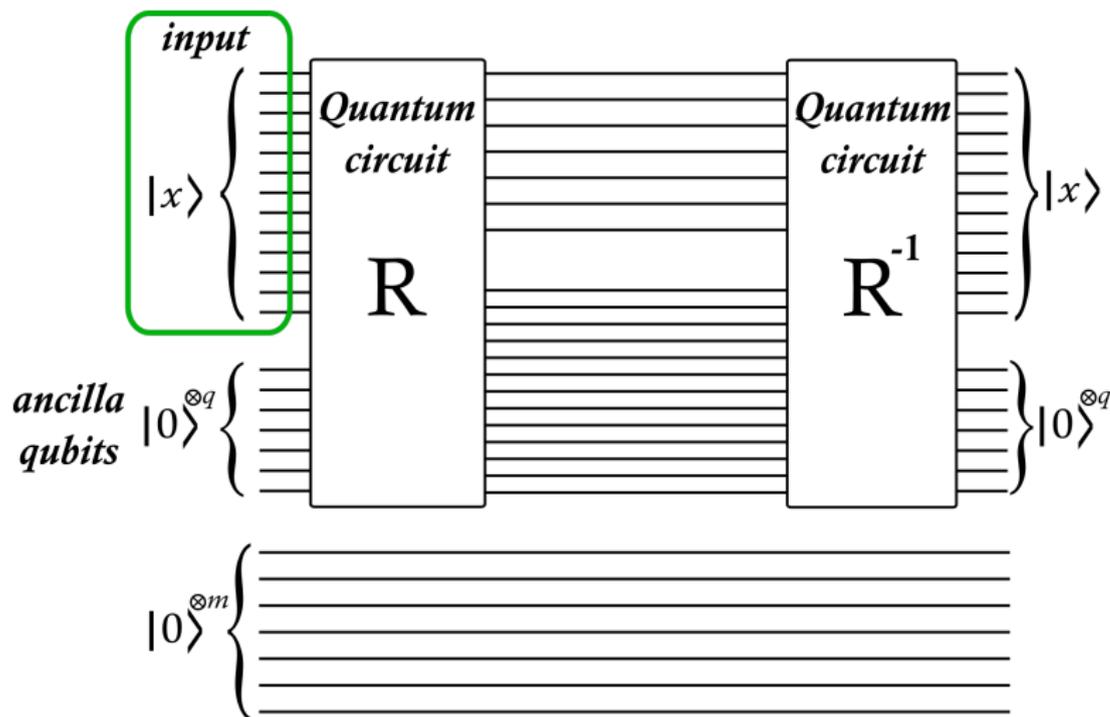
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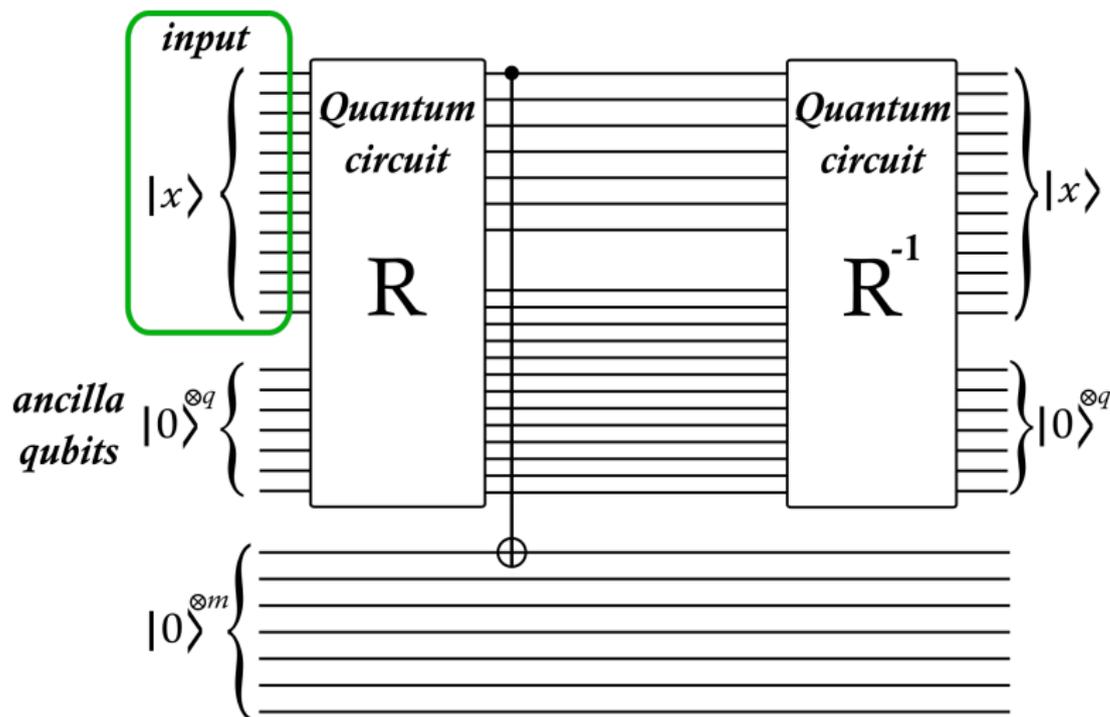
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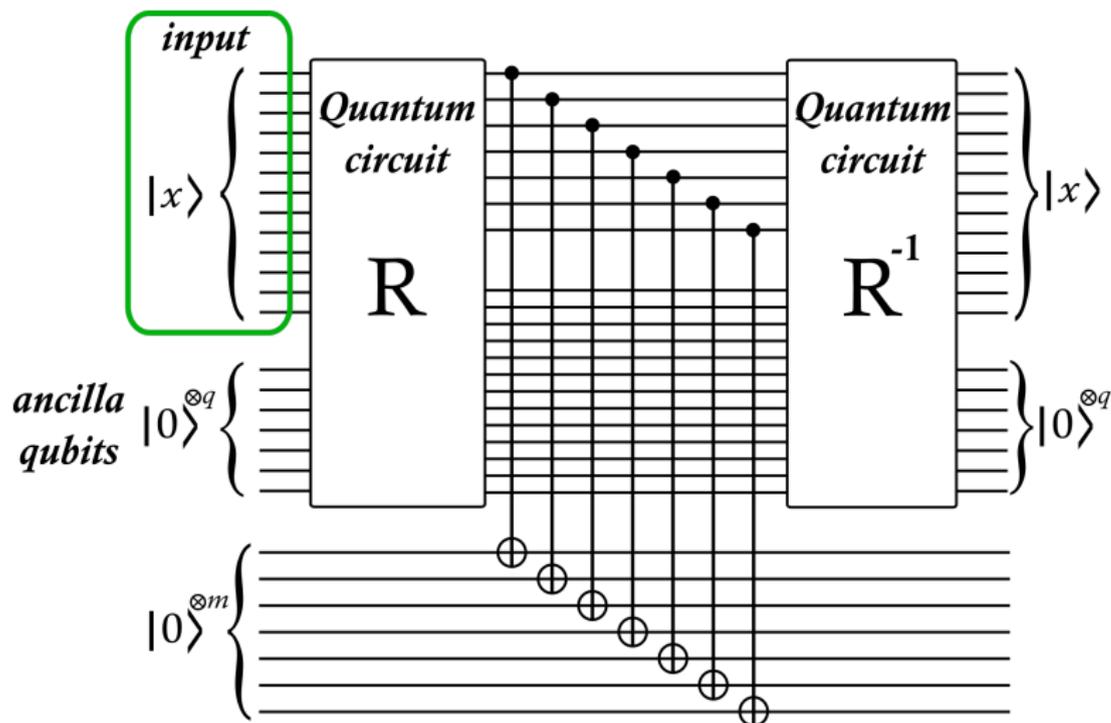
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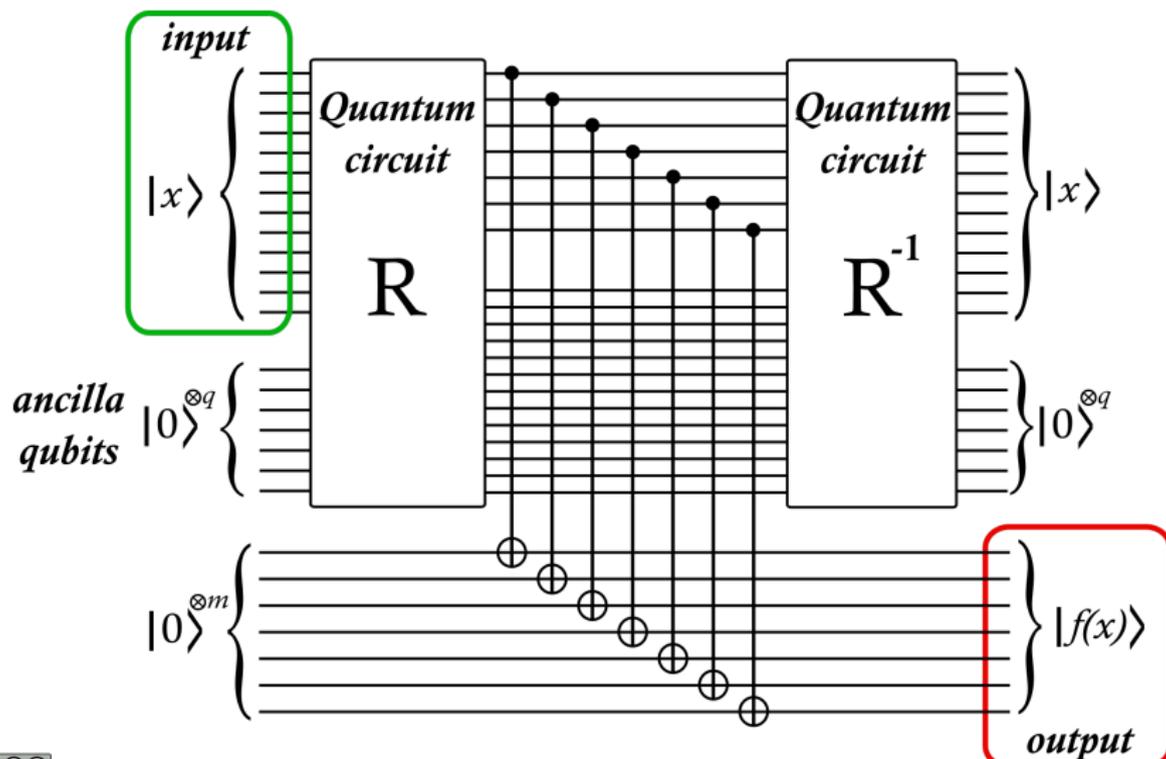
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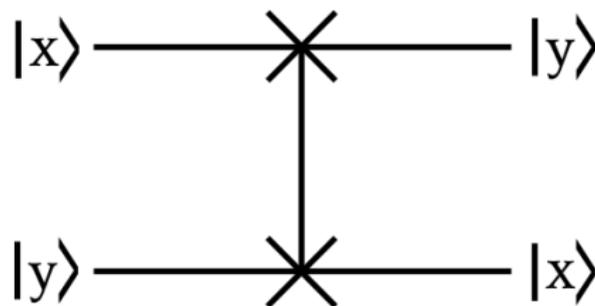
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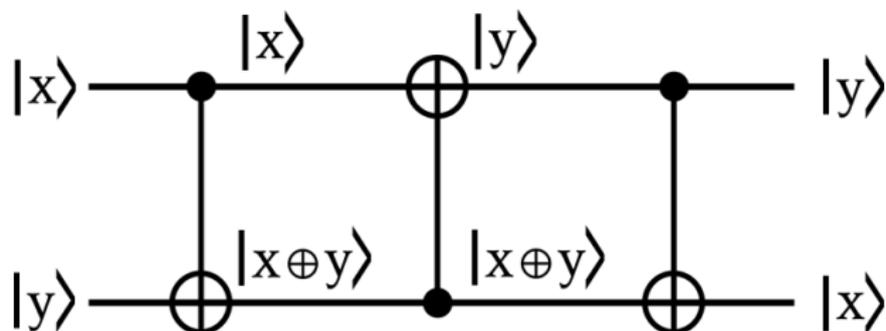
## II.4. SWAP gate



$$\text{SWAP} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

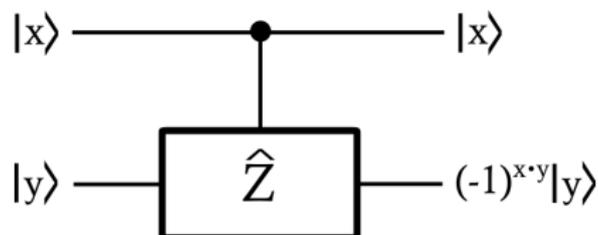
## II.4. SWAP gate

A SWAP gate might be implemented with 3 C-NOT gates



$$\begin{aligned} |x\rangle \otimes |y\rangle &\xrightarrow{C_{12}} |x\rangle \otimes |y \oplus x\rangle \\ &\xrightarrow{C_{21}} |x \oplus (y \oplus x)\rangle \otimes |y \oplus x\rangle = |y\rangle \otimes |y \oplus x\rangle \\ &\xrightarrow{C_{12}} |y\rangle \otimes |(y \oplus x) \oplus y\rangle = |y\rangle \otimes |x\rangle \end{aligned}$$

## 11.4. C-Z gate

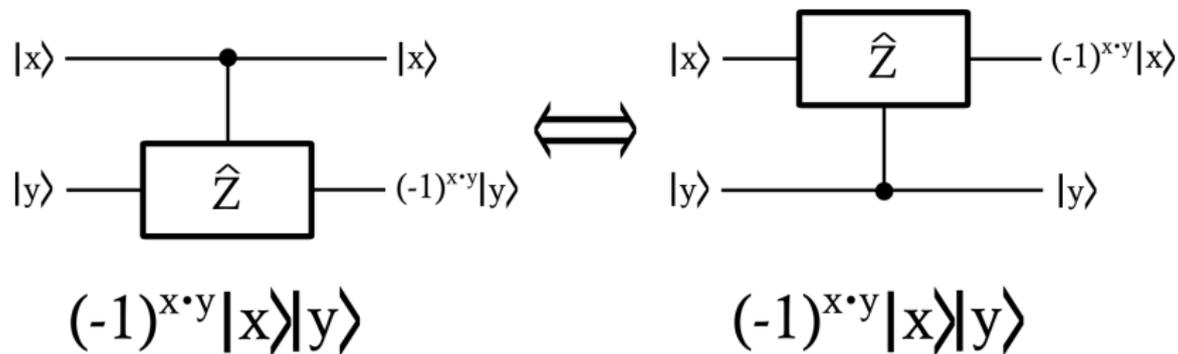


$$C-Z = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

$$|x\rangle \otimes |y\rangle \xrightarrow{C_Z} (-1)^{x \cdot y} |x\rangle \otimes |y\rangle.$$

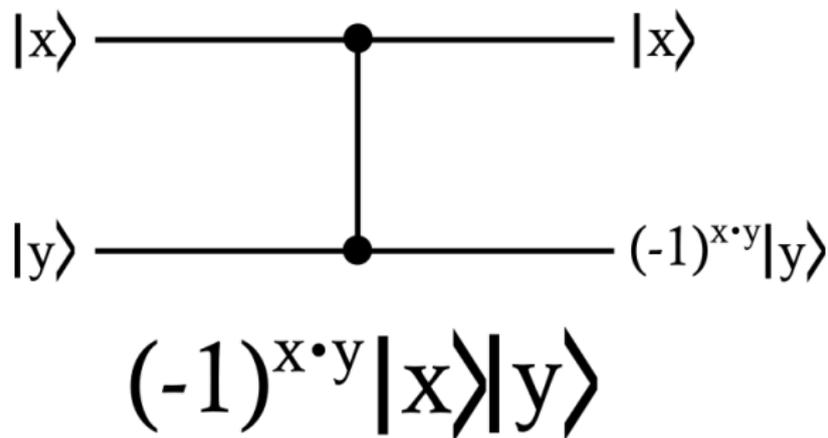
$|x\rangle$  and  $|y\rangle$  play symmetric roles.

## II.4. C-Z gate



## 11.4. C-Z gate

Since  $|x\rangle$  and  $|y\rangle$  play symmetric roles, a C-Z gate is then represented as follow



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Oracles are used in the context of query complexity. We assume to have access to an Oracle, for example a physical device that we cannot look inside, but to which we can pass queries and which returns answers.

On a classical computer, the Oracle is given by a function  $f$

$$f : \mathbb{F}_n \longrightarrow \mathbb{F}_m.$$

On a quantum computer, the oracle must be **unitary**.

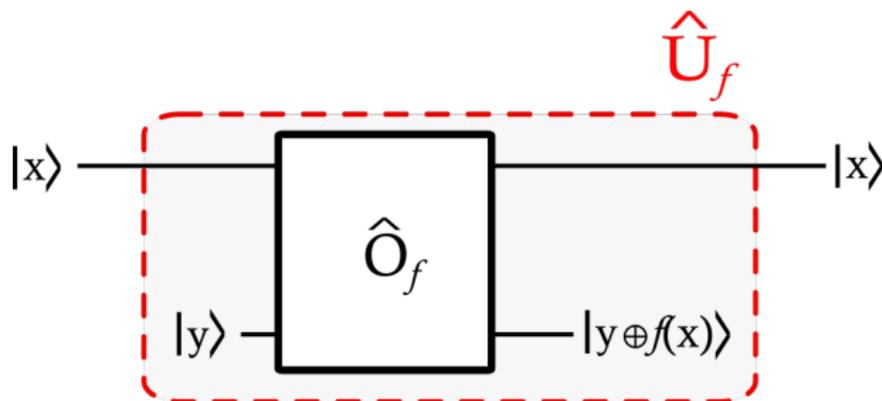
The operator  $\hat{O}_f$  is a quantum oracle which can be seen as a unitary operator which performs the map

$$\hat{O}_f |x\rangle \otimes |y\rangle = |x\rangle \otimes |y \oplus f(x)\rangle,$$

with  $|x\rangle \in \mathcal{H}_{2^n}$  and  $|y\rangle \in \mathcal{H}_{2^m}$ .

## 11.5. Oracle

**Example:** for  $f : \{0, 1\}^n \rightarrow \{0, 1\}$ , we can construct  $\hat{U}_f$  as follow.



$$\hat{O}_f |x\rangle |y\rangle = (-1)^{f(x)} |x\rangle |y\rangle$$

$|y\rangle$  is an ancilla qubit.

For  $|y\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ , then

$$\hat{O}_f |x\rangle \otimes |y\rangle = (-1)^{f(x)} |x\rangle \otimes |y\rangle.$$

Then, forgetting the ancilla qubit,  $\hat{U}_f |x\rangle = (-1)^{f(x)} |x\rangle$ . In such a case,  $\hat{U}_f$  is a **phase oracle**.

*Demonstration:*

$$\begin{aligned}\hat{O}_f |x\rangle \otimes |y\rangle &= \frac{1}{\sqrt{2}} \left( \hat{O}_f |x\rangle \otimes |0\rangle - \hat{O}_f |x\rangle \otimes |1\rangle \right), \\ &= \frac{1}{\sqrt{2}} \left( |x\rangle \otimes |0 \oplus f(x)\rangle - |x\rangle \otimes |1 \oplus f(x)\rangle \right),\end{aligned}$$

$$\hat{O}_f |x\rangle \otimes |y\rangle = \frac{1}{\sqrt{2}} \begin{cases} |x\rangle \otimes (|0\rangle - |1\rangle) & \text{if } f(x) = 0 \\ |x\rangle \otimes (|1\rangle - |0\rangle) & \text{if } f(x) = 1 \end{cases},$$

$$\hat{O}_f |x\rangle \otimes |y\rangle = \frac{1}{\sqrt{2}} \begin{cases} |x\rangle \otimes |y\rangle & \text{if } f(x) = 0 \\ -|x\rangle \otimes |y\rangle & \text{if } f(x) = 1 \end{cases},$$

such that

$$\boxed{\hat{O}_f |x\rangle \otimes |y\rangle = (-1)^{f(x)} |x\rangle \otimes |y\rangle.}$$

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## III.1. Deutsch algorithm

Let consider a function  $f : \{0, 1\} \rightarrow \{0, 1\}$ . There is four possibilities for  $f$

$$\underbrace{\left\{ \begin{array}{l} f(0) = 0 \\ f(1) = 1 \end{array} \right\}}_{\text{identity}}, \quad \underbrace{\left\{ \begin{array}{l} f(0) = 1 \\ f(1) = 0 \end{array} \right\}}_{\text{swap}}, \quad \underbrace{\left\{ \begin{array}{l} f(0) = 0 \\ f(1) = 0 \end{array} \right\}, \left\{ \begin{array}{l} f(0) = 1 \\ f(1) = 1 \end{array} \right\}}_{\text{constant function}}.$$

The function  $f$  is said to be **balanced** if  $f(0) \neq f(1)$ .

The function  $f$  is said to be **constant** if  $f(0) = f(1)$ .

With a classical computer, one need to evaluate the function  $f$  twice to determine whether it is balanced or constant.

**Example:**  $f$  is constant :  $f(0) = f(1) = 1$ .

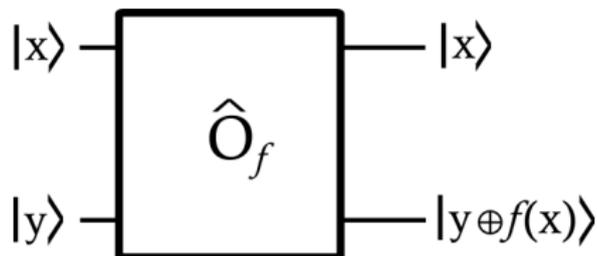
$$\hat{N}_f = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \quad \hat{N}_f |0\rangle = |1\rangle, \quad \hat{N}_f |1\rangle = |1\rangle.$$

$\hat{N}_f$  is **not** unitary:  $\hat{N}_f^\dagger \hat{N}_f \neq \mathbb{I}$ .  $\hat{N}_f$  does not preserve the norm and consequently the probability.

## III.1. Deutsch algorithm

One constructs a quantum gate to realize  $f$  with a unitary operator  $\hat{U}_f$

$$\hat{U}_f |x\rangle \otimes |y\rangle = |x\rangle \otimes |y \oplus f(x)\rangle.$$

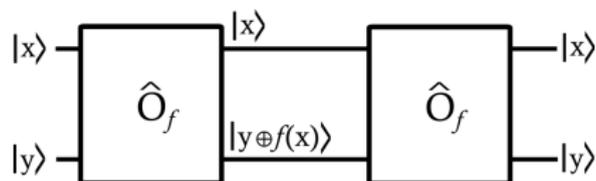


$|x\rangle$  is the qubit on which one wants to evaluate the function  $f$ .  $|y\rangle$  is a control qubit, allowing the operation to be unitary.  $\oplus$  is a XOR operation.

## III.1. Deutsch algorithm

$\hat{U}_f$  is reversible

$$\hat{U}_f^\dagger \hat{U}_f = \mathbb{I}.$$



$$\hat{U}_f \hat{U}_f |x\rangle \otimes |y\rangle = \hat{U}_f |x\rangle \otimes |y \oplus f(x)\rangle = |x\rangle \otimes |(y \oplus f(x)) \oplus f(x)\rangle.$$

Or

$$\begin{aligned}(y \oplus f(x)) \oplus f(x) &= y \oplus (f(x) \oplus f(x)) \\ &= y \oplus 0 \\ &= y\end{aligned}$$

Remark:

$$\forall k \in \{0, 1\}, k \oplus k = 0.$$

$\hat{U}_f$  is unitary and might be used to evaluate  $f$ .

If the control qubit  $|y\rangle = |0\rangle$ , then

$$|x\rangle \otimes |y \oplus f(x)\rangle = |x\rangle \otimes |0 \oplus f(x)\rangle,$$

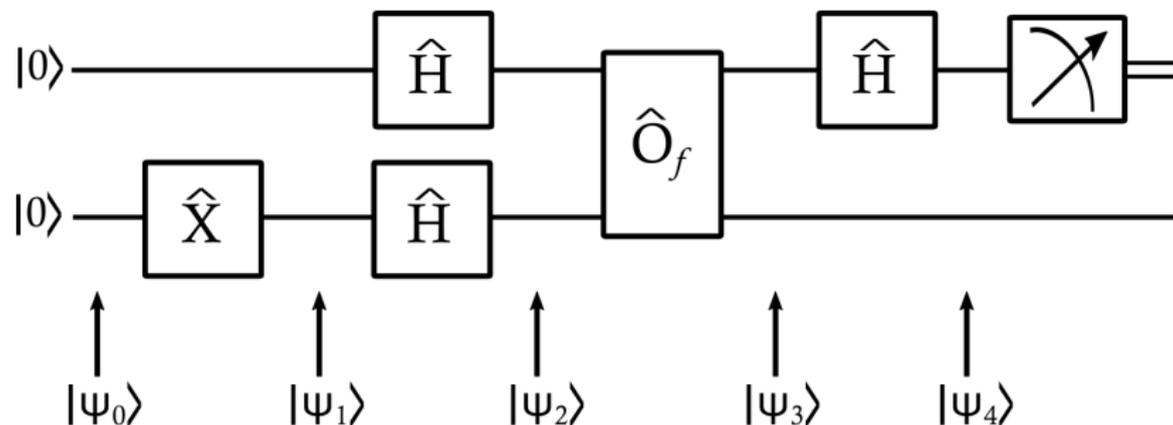
so that

$$|x\rangle \otimes |y \oplus f(x)\rangle = |x\rangle \otimes |f(x)\rangle.$$

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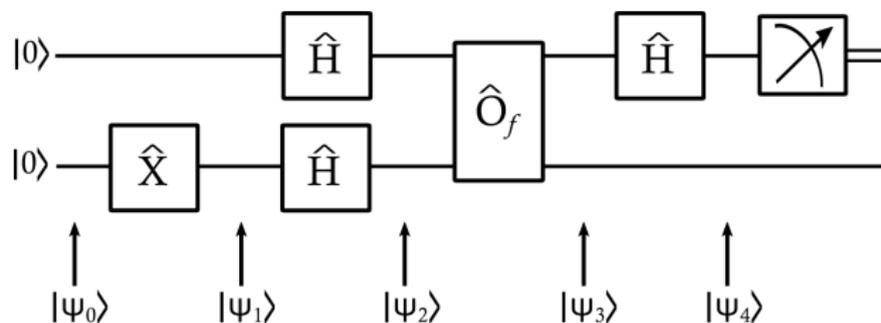
## III.2. Implementation of Deutsch algorithm

The circuit diagram of the implementation of Deutsch-algorithm is the following



$$|\psi_4\rangle = (\hat{H} \otimes \mathbb{I}) \hat{U}_f (\hat{H} \otimes \hat{H}) (\mathbb{I} \otimes \hat{X}) |00\rangle.$$

## III.2. Implementation of Deutsch algorithm

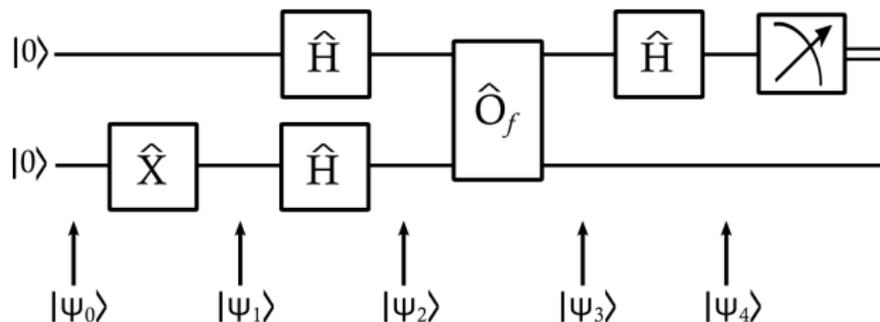


$$|\psi_0\rangle = |00\rangle, \quad |\psi_1\rangle = |01\rangle,$$

$$|\psi_2\rangle = (\hat{H} \otimes \hat{H}) |01\rangle = \frac{1}{2} \left( \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \right) \left( \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$$

$$|\psi_2\rangle = \left( \frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \otimes \left( \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) = \frac{|00\rangle - |01\rangle + |10\rangle - |11\rangle}{2}.$$

## III.2. Implementation of Deutsch algorithm



Oracle:

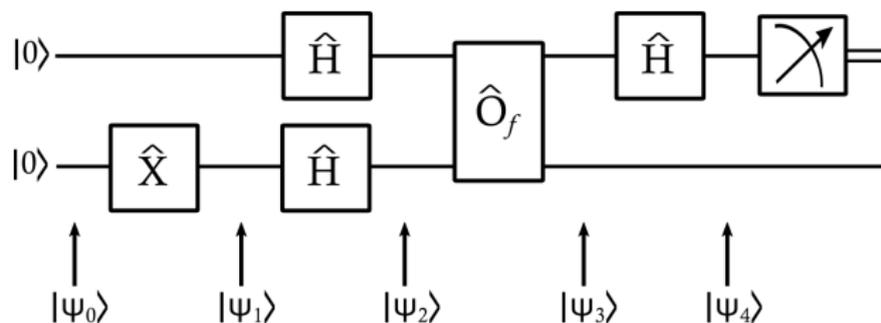
$$|\psi_3\rangle = \hat{U}_f |\psi_2\rangle = |x\rangle \left( \frac{|0 \oplus f(x)\rangle - |1 \oplus f(x)\rangle}{\sqrt{2}} \right).$$

For any state  $|x\rangle$  in a pure state.

$$\text{If } f(x) = 0, |\psi_3\rangle = |x\rangle \left( \frac{|0 \oplus 0\rangle - |1 \oplus 0\rangle}{\sqrt{2}} \right) = |x\rangle \left( \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right),$$

$$\text{If } f(x) = 1, |\psi_3\rangle = |x\rangle \left( \frac{|0 \oplus 1\rangle - |1 \oplus 1\rangle}{\sqrt{2}} \right) = |x\rangle \left( \frac{|1\rangle - |0\rangle}{\sqrt{2}} \right).$$

## III.2. Implementation of Deutsch algorithm



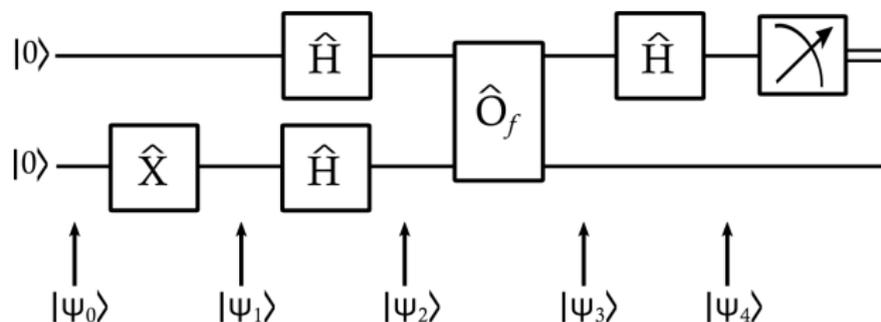
So finally

$$|\psi_3\rangle = (-1)^{f(x)} |x\rangle \left( \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right).$$

If  $|x\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$ , then

$$|\psi_3\rangle = \hat{U}_f \frac{|0\rangle}{\sqrt{2}} \left( \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) + \hat{U}_f \frac{|1\rangle}{\sqrt{2}} \left( \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right),$$

## III.2. Implementation of Deutsch algorithm

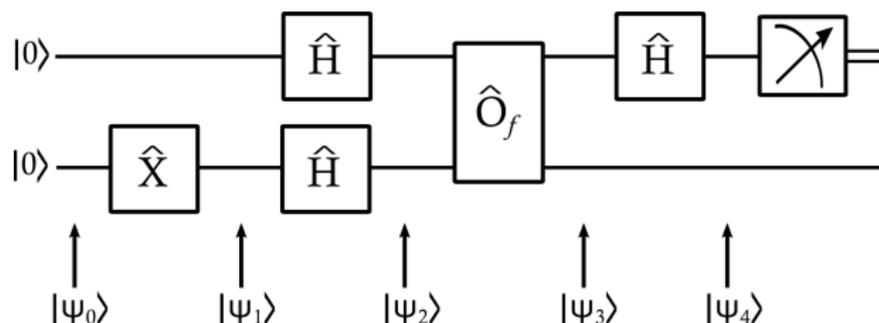


$$|\psi_3\rangle = \left( \frac{(-1)^{f(0)}|0\rangle + (-1)^{f(1)}|1\rangle}{\sqrt{2}} \right) \left( \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right).$$

If  $f$  is constant:  $|\psi_3\rangle = (\pm 1) \left( \frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \left( \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right).$

If  $f$  is balanced:  $|\psi_3\rangle = (\pm 1) \left( \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \left( \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right).$

## III.2. Implementation of Deutsch algorithm



Applying finally an Hadamard gate on each qubit

$$|\psi_3\rangle = \left( \frac{(-1)^{f(0)} |0\rangle + (-1)^{f(1)} |1\rangle}{\sqrt{2}} \right) \left( \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right).$$

$$\text{If } f \text{ is constant: } |\psi_4\rangle = (\pm 1) |0\rangle \left( \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right).$$

$$\text{If } f \text{ is balanced: } |\psi_4\rangle = (\pm 1) |1\rangle \left( \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right).$$

## III.2. Implementation of Deutsch algorithm

$$\text{If } f \text{ is constant: } |\psi_4\rangle = (\pm 1) |0\rangle \left( \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right).$$

$$\text{If } f \text{ is balanced: } |\psi_4\rangle = (\pm 1) |1\rangle \left( \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right).$$

The measurement of the first qubit permits to determine unambiguously if the function  $f$  is balanced or constant. The Deutsch problem consists in determining whether a function  $f : \{0, 1\} \rightarrow \{0, 1\}$  is balanced or constant. A classical computer requires two sollicitation of the Oracle, while a quantum algorithm requires only one sollicitation of the oracle to get the result. That's a direct consequence of the quantum parallelism.

- 1 Binary quantum gates
- 2 Examples of multiqubit gates
- 3 Deutsch-Josa algorithm
  - Deutsch algorithm
  - Implementation of Deutsch algorithm
  - Deutsch-Josa algorithm

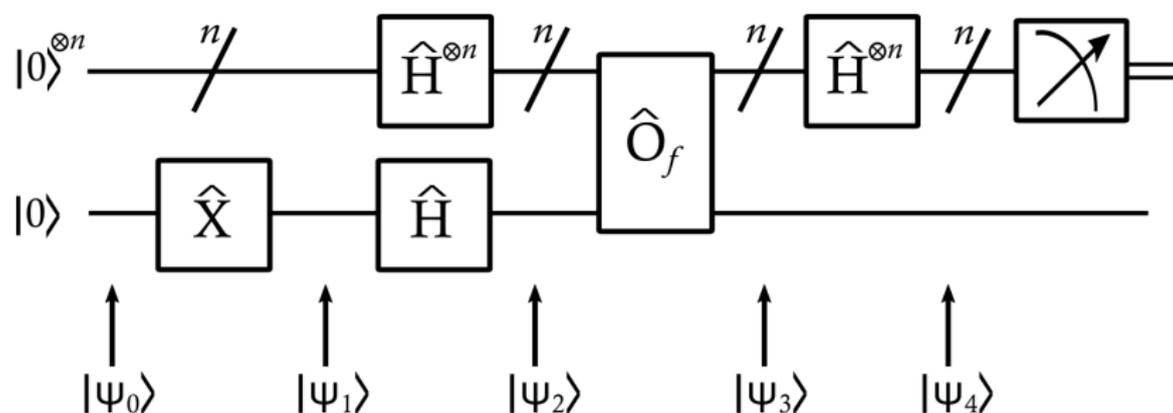
## III.3. Deutsch-Josa algorithm

In the case of Deutsch-Josa algorithm, a more general case is considered with a function  $f : \{0, 1\}^n \rightarrow \{0, 1\}$ .

- $f$  is balanced if half of the inputs return 0 and the others return 1.
- $f$  is constant if  $f$  only returns 0 or 1.

## III.3. Deutsch-Josa algorithm

The Deutsch-Josa algorithm is based on the same principle than the Deutsch algorithm (case  $n = 1$ ), with the following implementation



## III.3. Deutsch-Josa algorithm

*Classical solution:* we need to ask the Oracle at least twice, but if we get twice the same value, we need to ask again... corresponding to at most  $\frac{N}{2} + 1 = 2^{n-1} + 1$  queries of the Oracle, with  $n$  the number of input bits and  $N = 2^n$  the number of realizable bit string.

**The quantum solution with the Deutsch-Josa algorithm needs only one query !!!**

**Proof:**

Initial state:

$$|\psi_0\rangle = |0\rangle^{\otimes n} |0\rangle = |000\dots 00\rangle |0\rangle.$$

Preparation of the ancilla qubit with a  $\hat{X}$  gate:

$$|\psi_1\rangle = |0\rangle^{\otimes n} |1\rangle.$$

### III.3. Deutsch-Josa algorithm

Hadamard gate on the quantum register input:

$$|\psi_2\rangle = \left(\hat{H}^{\otimes n} |0\rangle^{\otimes n}\right) \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right).$$

And

$$\hat{H}^{\otimes n} |0\rangle^{\otimes n} = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{\langle x|0\rangle} |x\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle,$$

so that  $|\psi_2\rangle$  is a superposition of all states as follow

$$|\psi_2\rangle = \left(\frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle\right) \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right).$$

Oracle:

$$|\psi_3\rangle = \hat{U}_f |\psi_2\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} |x\rangle \otimes \left( \frac{|0 \oplus f(x)\rangle - |1 \oplus f(x)\rangle}{\sqrt{2}} \right),$$

$$|\psi_3\rangle = \left( \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} |x\rangle \right) \otimes \left( \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right).$$

### III.3. Deutsch-Josa algorithm

Hadamard gate on the quantum register:

$$|\psi_4\rangle = \left( \hat{H}^{\otimes n} \left( \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} |x\rangle \right) \right) \otimes \left( \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right),$$

$$|\psi_4\rangle = \left( \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} \hat{H}^{\otimes n} |x\rangle \right) \otimes \left( \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right),$$

$$|\psi_4\rangle = \left( \frac{1}{\sqrt{2^n}} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} \frac{1}{\sqrt{2^n}} \sum_{K \in \{0,1\}^n} (-1)^{\langle K|x \rangle} |K\rangle \right) \otimes \left( \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right)$$

$$|\psi_4\rangle = \left( \sum_{K \in \{0,1\}^n} \left( \frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{f(x) + \langle K|x \rangle} \right) |K\rangle \right) \otimes \left( \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right),$$

### III.3. Deutsch-Josa algorithm

Let define

$$C_K = \frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{f(x) + \langle K|x \rangle}, \quad \text{and} \quad |\phi\rangle = \sum_{K \in \{0,1\}^n} C_K |K\rangle.$$

Then

$$|\psi_4\rangle = |\phi\rangle \otimes \left( \frac{|0\rangle - |1\rangle}{\sqrt{2}} \right).$$

The state  $|\phi\rangle$  is measured at the end: probability to measure the string  $|000 \dots 000\rangle$

$$P(y = 00 \dots 00) = |\langle 00 \dots 00 | \phi \rangle|^2.$$

### III.3. Deutsch-Josa algorithm

$$\begin{aligned} P(y = 00 \cdots 00) &= \left| \sum_{K \in \{0,1\}^n} C_K \langle 00 \cdots 00 | K \rangle \right|^2 = |C_{00 \cdots 00}|^2, \\ &= \left| \frac{1}{2^n} \sum_{x \in \{0,1\}^n} (-1)^{f(x)} \right|^2 \end{aligned}$$

Or

$$\sum_{x \in \{0,1\}^n} (-1)^{f(x)} = \begin{cases} +2^n & \text{if } f(x) = 0 \\ -2^n & \text{if } f(x) = 1 \\ 0 & \text{if } f(x) \text{ is balanced} \end{cases},$$

then

$$P(y = 00 \cdots 00) = \begin{cases} 1 & \text{if } f(x) \text{ is constant} \\ 0 & \text{if } f(x) \text{ is balanced} \end{cases}$$

### III.3. Deutsch-Josa algorithm

So

$P(y = 00 \cdots 00) = 1$  for a balanced function,

$P(y = 00 \cdots 00) = 0$  for a constant function.

If the function is neither balanced nor constant, then

$$P(y = 00 \cdots 00) \in ]0, 1[ .$$

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