Quantum algorithms

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Quantum algorithms

Introduction to Quantum Computing

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Quantum algorithms

Quantum algorithms

We might divide quantum algorithms in two categories

Polynomial speed up	Exponential speed up
Grover's search	Integer
Quantum walks	Matrix Inversion
Graph algorithms	Phase Estimation
Minimum finding	Quantum Fourier Transform

Quantum algorithms

II. Bernstein-Vazirani algorithm

Berstein-Vazirani algorithm is a restricted version of Deutsch-Josa algorithm.

Instead of distinguishing between two different classes of functions, it tries to learn a string encoded in a function. One is given an oracle implementing a function f

$$f : \{0,1\}^n \longrightarrow \{0,1\}.$$

It is given that $f(x)$ is a dot product between x and a secret string $s \in \{0,1\}^n$ modulo 2

$$f(x) = x \cdot s = x_1 \cdot s_1 + x_2 \cdot s_2 + \cdots + x_n \cdot s_n.$$

Bernstein-Vazirani algorithm aims at finding s.

II. Bernstein-Vazirani algorithm

Classicaly, it requires to evaluate n times the function $f(x)$, with $x = 2^k$, $k \in \{0, 1, \cdots, n-1\}$.

$$f(1000\cdots00) = s_1,$$

$$f(0100\cdots00) = s_2,$$

$$\vdots$$

$$f(0000\cdots01) = s_n,$$

Thanks to Bernstein-Vazirani algorithm, only one query is needed with a quantum computer.

Algorithm

1. Initialize a quantum register of n qubit in the state $|0\rangle^{\otimes n}$

$$|\psi_0\rangle = |0\rangle^{\otimes n}.$$

2. Apply a Hadamard gate to each qubit of the quantum register, providing the state

$$|\psi_1\rangle = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n} |x\rangle.$$

3. Apply the Oracle to the previous superposed state to obtain the following state

$$|\psi_2\rangle = \frac{1}{\sqrt{2^n}} \sum_{x=0}^{2^n} (-1)^{f(x)} |x\rangle.$$

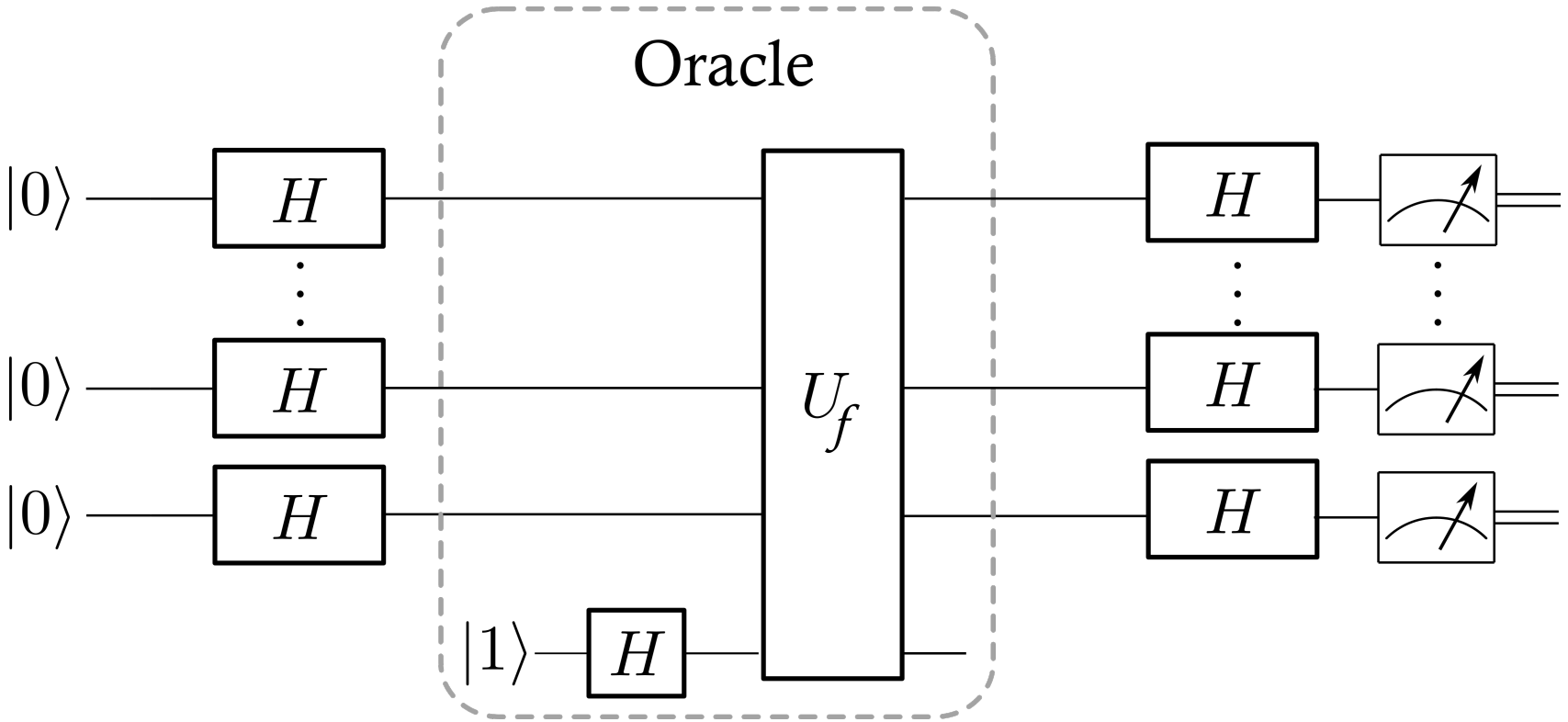
Algorithm

1. Apply Hadamard gate on each qubit of the quantum register. If $s_i = 1$, it converts the state $|-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ to $|1\rangle$. If $s_i = 0$, it converts the state $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$ to $|0\rangle$.

$$|\psi_3\rangle = |s_1, s_2, \cdots, s_n\rangle.$$

2. To obtain s, the classical measurement on the $\{|0\rangle, |1\rangle\}$ basis provides the result.

Circuit diagram of Bernstein-Vazirani algorithm



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III.1. Grover's problem - unstructured search

A simple example of a problem that fits into the query complexity model is **unstructured search on a set of N elements, in which only on element is marked**. In this problem, we are given a function

$$f : \{x_i, i \in [\![0, N-1]\!]\} \longrightarrow \{0, 1\},$$

with the promise that it exists only one $p \in [\![0, N-1]\!]$ such that

$$f(x_p) = 1, \text{ and for } q \neq p, \ f(x_q) = 0.$$

Then, x_p **is the "marked" element.**

Our task is to output x_p, f being given by an Oracle.

III.1. Grover's problem - unstructured search

It is intuitively clear that the unstructured search problem requires about N queries the be solved classically.

Let $\mathcal{A}$ be a classical algorithm which solves the unstructured search problem on a set of N elements with a failure probability $\lesssim 1/2$. Then, $\mathcal{A}$ makes $\mathcal{O}(N)$ queries in the worst case.

Grover (1997): *there is a quantum algorithm which solves the unstructured search problem using* $\mathcal{O}(\sqrt{N})$ *queries.*

For simplicity, we assume that $N = 2^n$, $n \in \mathbb{N}$ (this is not an essential restriction). Thus, we associate any element of $\{x_i\}$ with an n-bits string.

Quantum algorithms

III.2. Grover's algorithm

We are given access to

$$f : \{0,1\}^n \longrightarrow \{0,1\},$$

with the property that $f(x_p) = 1$ for a unique element x_p. We use a quantum circuit on n qubits with an initial state

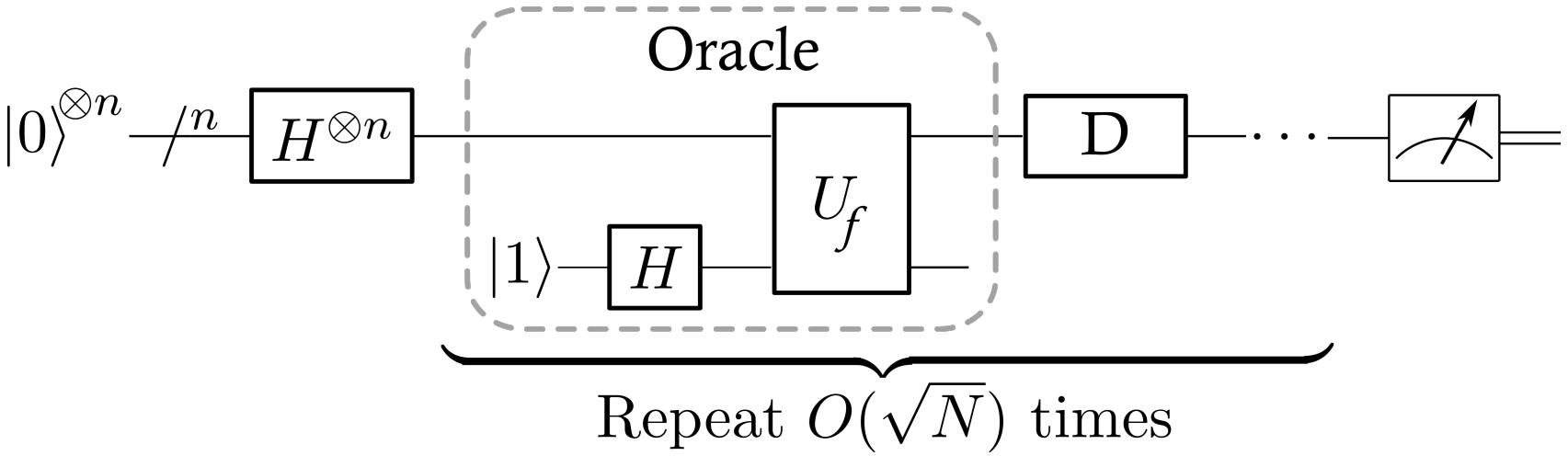
$$|\psi_0\rangle = |0\rangle^{\otimes n}.$$

Let $\hat{H}$ denote the Hadamard gate, and let $\hat{U}_0$ denote the n-qubit operation which inverts the phase of only $|0\rangle^{\otimes n}$

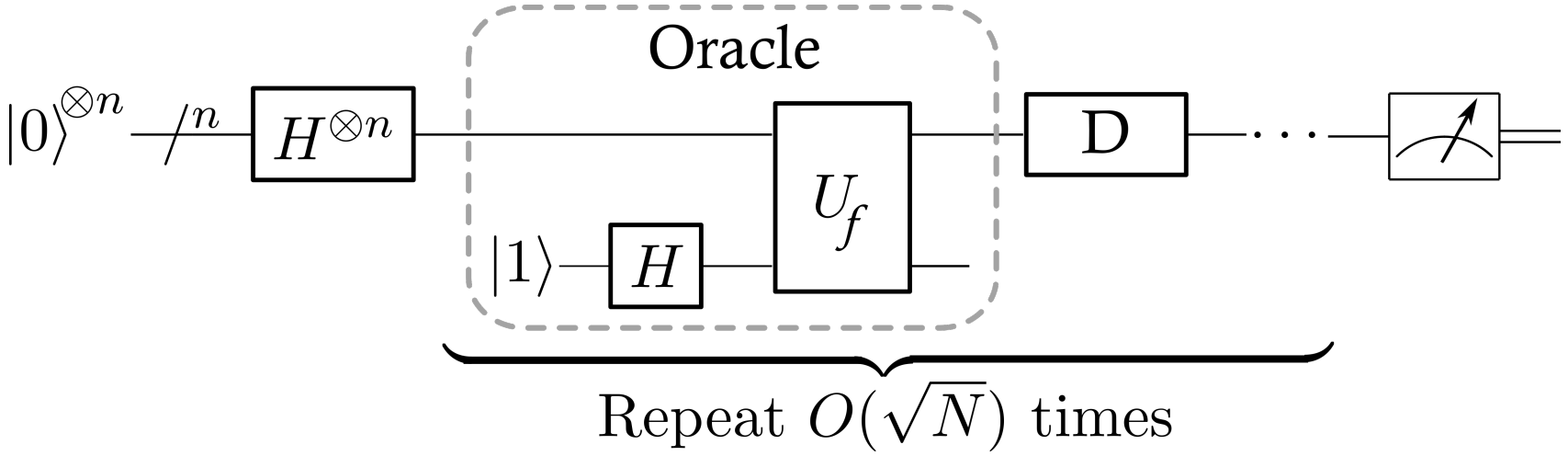
$$\begin{cases} \hat{U}_0 |0\rangle^{\otimes n} = -|0\rangle^{\otimes n} \\ \hat{U}_0 |x\rangle = |x\rangle \text{ for } |x\rangle \neq |0\rangle^{\otimes n} \end{cases} \tag{1}$$

Grover's algorithm:

1. Apply $\hat{H}^{\otimes n}$;
2. Repeat the following operation T times, for some T to be determined later
   1. Apply $\hat{U}_f$;
   2. Apply $\hat{D} = -\hat{H}^{\otimes n}\hat{U}_0\hat{H}^{\otimes n}$.
3. Measure all the qubits and output the results.



III.2. Grover's algorithm



The overall operation performed, applied on the initial state $|0\rangle^{\otimes n}$, is unitary

$$\hat{D}^T \hat{H}^{\otimes n} = \left(-\hat{H}^{\otimes n} \hat{U}_0 \hat{H}^{\otimes n}\right)^T \hat{H}^{\otimes n}.$$

Quantum algorithms

To describe Grover's algorithm, we introduce unitary operators

$$\hat{I}_{|\psi\rangle} = \mathbb{I} - 2\,|\psi\rangle\langle\psi| \quad \text{and} \quad \hat{R}_{|\psi\rangle} = -\hat{I}_{|\psi\rangle} = 2\,|\psi\rangle\langle\psi| - \mathbb{I},$$

where $\mathbb{I}$ is the identity operator, and $|\psi\rangle$ is an arbitrary state.

$\hat{I}_{|\psi\rangle}$ can be seen as an inversion around $|\psi\rangle$ operation, while $\hat{R}_{|\psi\rangle}$ can be seen as a reflection around $|\psi\rangle$ operation. An arbitrary state $|\phi\rangle$ can be expressed as

$$|\phi\rangle = \alpha\,|\psi\rangle + \beta\,\left|\psi^{\perp}\right\rangle,$$

with $(\alpha, \beta) \in \mathbb{C}$ and $\left|\psi^{\perp}\right\rangle$ belongs to the subspace perpendicular to $|\psi\rangle$

$$\langle\psi|\psi^{\perp}\rangle = 0.$$

Then

$$\hat{I}_{|\psi\rangle}\,|\phi\rangle = -\alpha\,|\psi\rangle + \beta\,\Big|\psi^{\perp}\rangle.$$

$\hat{I}_{|\psi\rangle}$ has flipped the phase of the component corresponding to $|\psi\rangle$.
$\hat{R}_{|\psi\rangle}$ has the opposite effect

$$\hat{R}_{|\psi\rangle}\,|\phi\rangle = \alpha\,|\psi\rangle - \beta\,\Big|\psi^{\perp}\rangle.$$

$\hat{U}_f$ is an Oracle such that

$$\hat{U}_f\,|x\rangle = (-1)^{f(x)}\,|x\rangle,$$

where one forgets the ancilla qubit required for unitary evolution.
In the unstructured search problem with a marked element x_p

$$\boxed{\hat{U}_f = \hat{I}_{|x_p\rangle}}.$$

Furthermore,

$$\hat{H}^{\otimes n}\hat{U}_0\hat{H}^{\otimes n} = \hat{H}^{\otimes n}\left(\mathbb{I} - 2\left|0\right\rangle^{\otimes n}\left\langle 0\right|^{\otimes n}\right)\hat{H}^{\otimes n} = \mathbb{I} - 2\hat{H}^{\otimes n}\left|0\right\rangle^{\otimes n}\left\langle 0\right|^{\otimes n}\hat{H}^{\otimes n}.$$

Introducing the $\left|+\right\rangle$ state defined as follow

$$\left|+\right\rangle = \hat{H}^{\otimes n}\left|0\right\rangle^{\otimes n} = \frac{1}{\sqrt{2^n}}\sum_{x\in\{0,1\}^n}\left|x\right\rangle,$$

one obtains immediately $\boxed{\hat{D} = -\hat{I}_{\left|+\right\rangle}}$. After T iterations, the final state $\left|\psi_f\right\rangle$ measured is the following

$$\begin{aligned}\left|\psi_f\right\rangle &= \left(\hat{D}\hat{U}_f\right)^T\hat{H}^{\otimes n}\left|0\right\rangle^{\otimes n} = \left(\hat{D}\hat{U}_f\right)^T\left|+\right\rangle,\\ &= \left(-\hat{I}_{\left|+\right\rangle}\hat{I}_{\left|x_p\right\rangle}\right)^T\left|+\right\rangle = \left(-\hat{R}_{\left|+\right\rangle}\hat{R}_{\left|x_p\right\rangle}\right)^T\left|+\right\rangle,\end{aligned}$$

$$\boxed{\left|\psi_f\right\rangle = \left(-\hat{R}_{\left|+\right\rangle}\hat{R}_{\left|x_p\right\rangle}\right)^T\left|+\right\rangle}.$$

Properties:

1. For any states $|\psi\rangle$, $|\phi\rangle$, and any state $|\xi\rangle$ in the plan defined by $|\psi\rangle$ and $|\phi\rangle$, the states $\hat{R}_{|\psi\rangle}|\xi\rangle$ and $\hat{R}_{|\phi\rangle}|\xi\rangle$ remain in this plan.
2. For two orthogonal states, $\hat{R}_{|\psi^\perp\rangle} = -\hat{R}_{|\psi\rangle}$.

   Demonstration:

$$\begin{aligned} -\hat{R}_{|\psi\rangle}\left(\alpha|\psi\rangle + \beta\left|\psi^\perp\right\rangle\right) &= -\alpha|\psi\rangle + \beta\left|\psi^\perp\right\rangle \\ &= \hat{R}_{|\psi^\perp\rangle}\left(\alpha|\psi\rangle + \beta\left|\psi^\perp\right\rangle\right) \end{aligned}$$

3. If $|\xi\rangle$ is in the plan defined by two orthogonal states $|\phi\rangle$ and $\left|\phi^\perp\right\rangle$

$$\hat{R}_{|\phi\rangle}|\xi\rangle = \langle\phi|\xi\rangle\ |\phi\rangle - \langle\phi^\perp|\xi\rangle\ \left|\phi^\perp\right\rangle.$$

   Demonstration is straightforward.

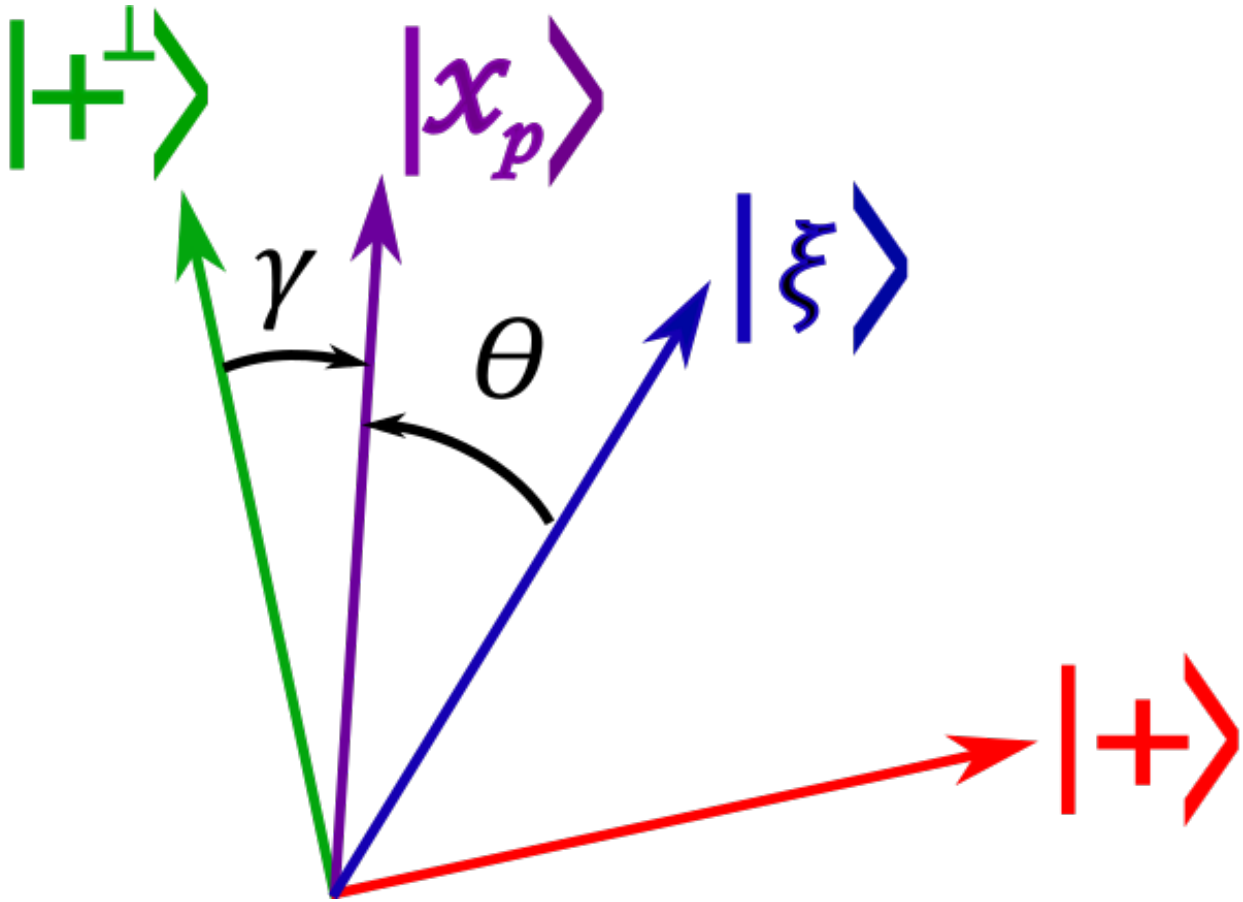
Consequently

$$\boxed{|\psi_f\rangle = \left(\hat{R}_{|+^{\perp}\rangle}\hat{R}_{|x_p\rangle}\right)^T |+\rangle}.$$

Grover's algorithm is based on successive rotations around $|x_p\rangle$ and $|+^{\perp}\rangle$, starting from the initial states superposition $|+\rangle$.

Quantum algorithms

III.4. Geometrical interpretation of Grover's algorithm



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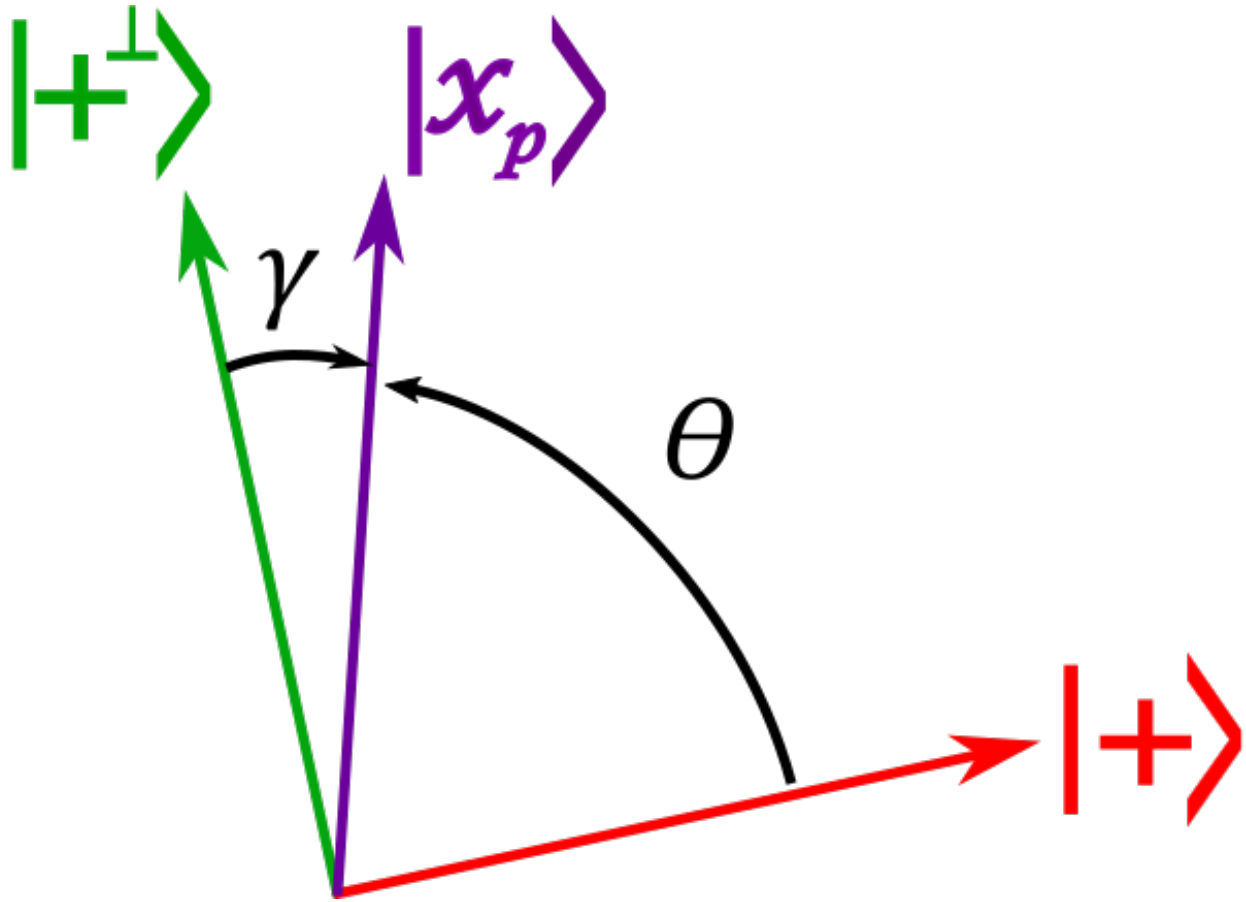

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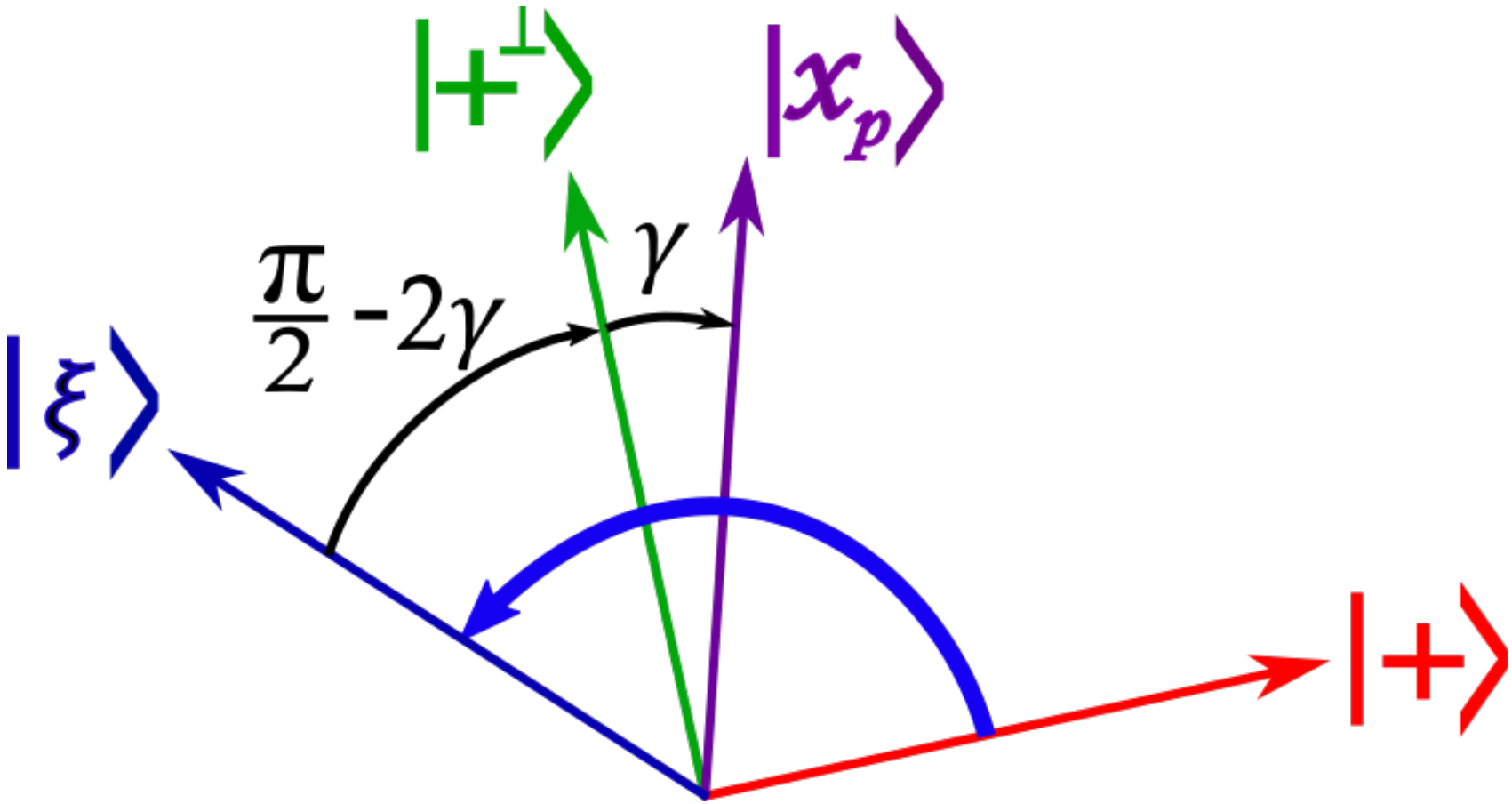

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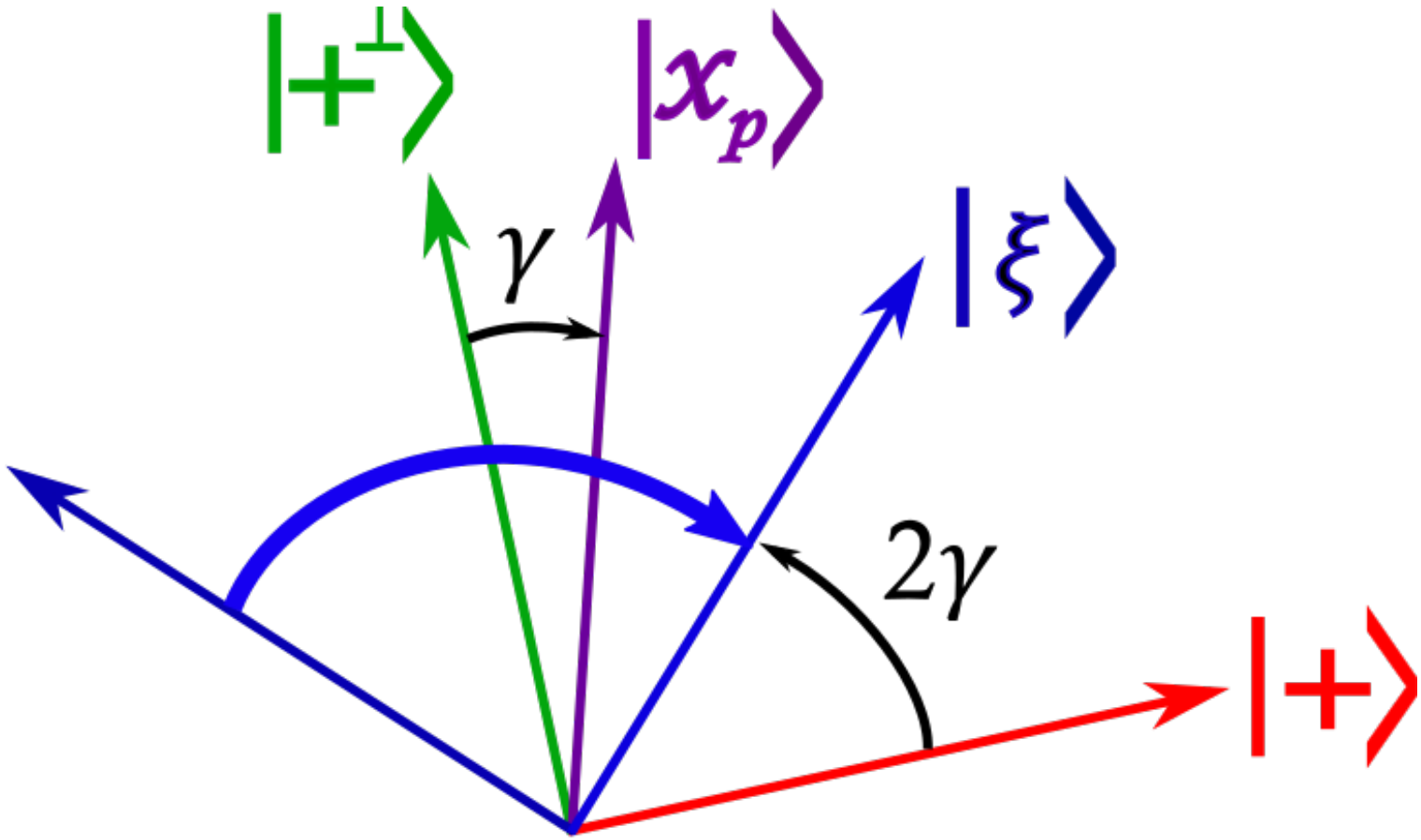

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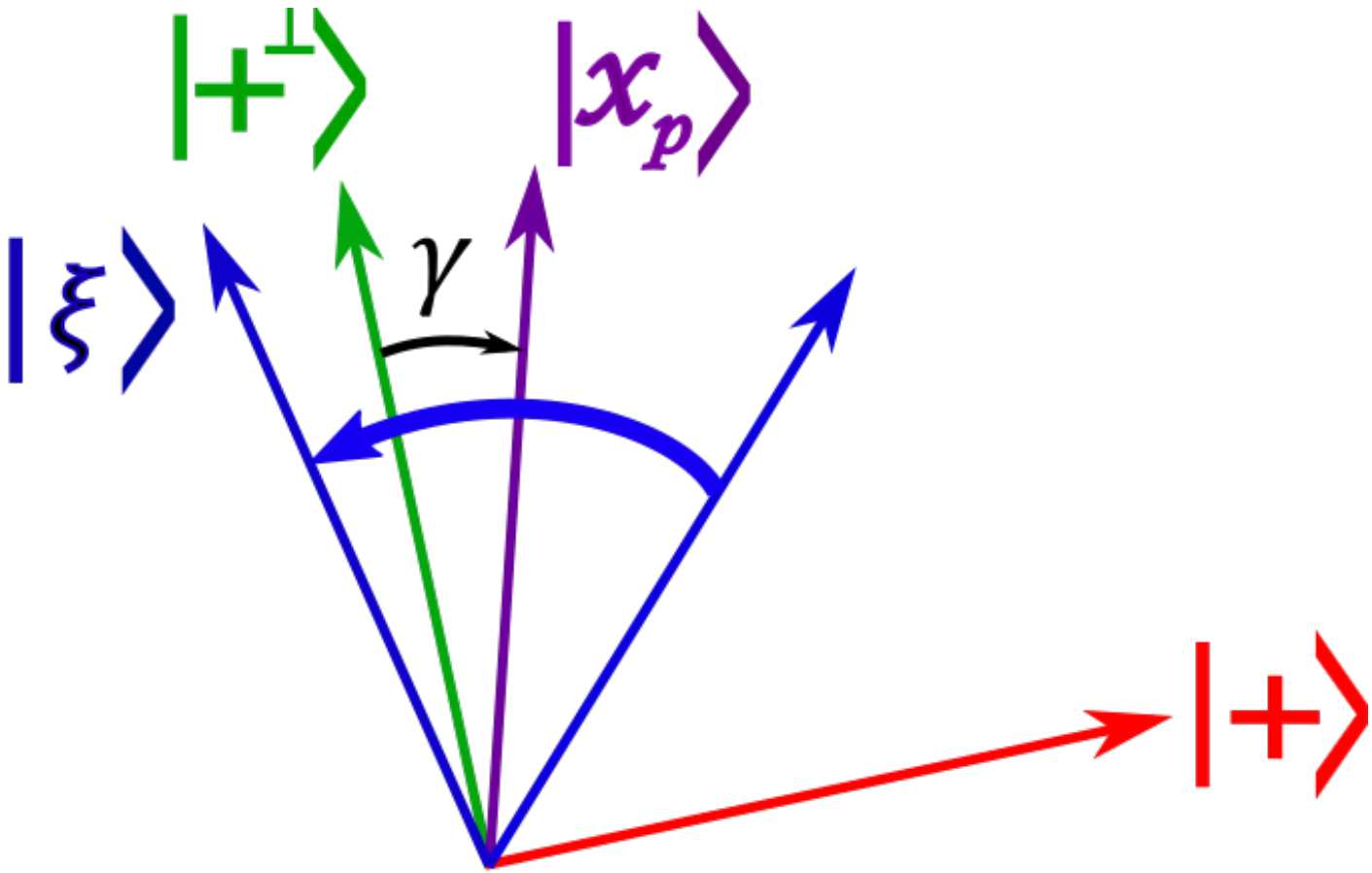

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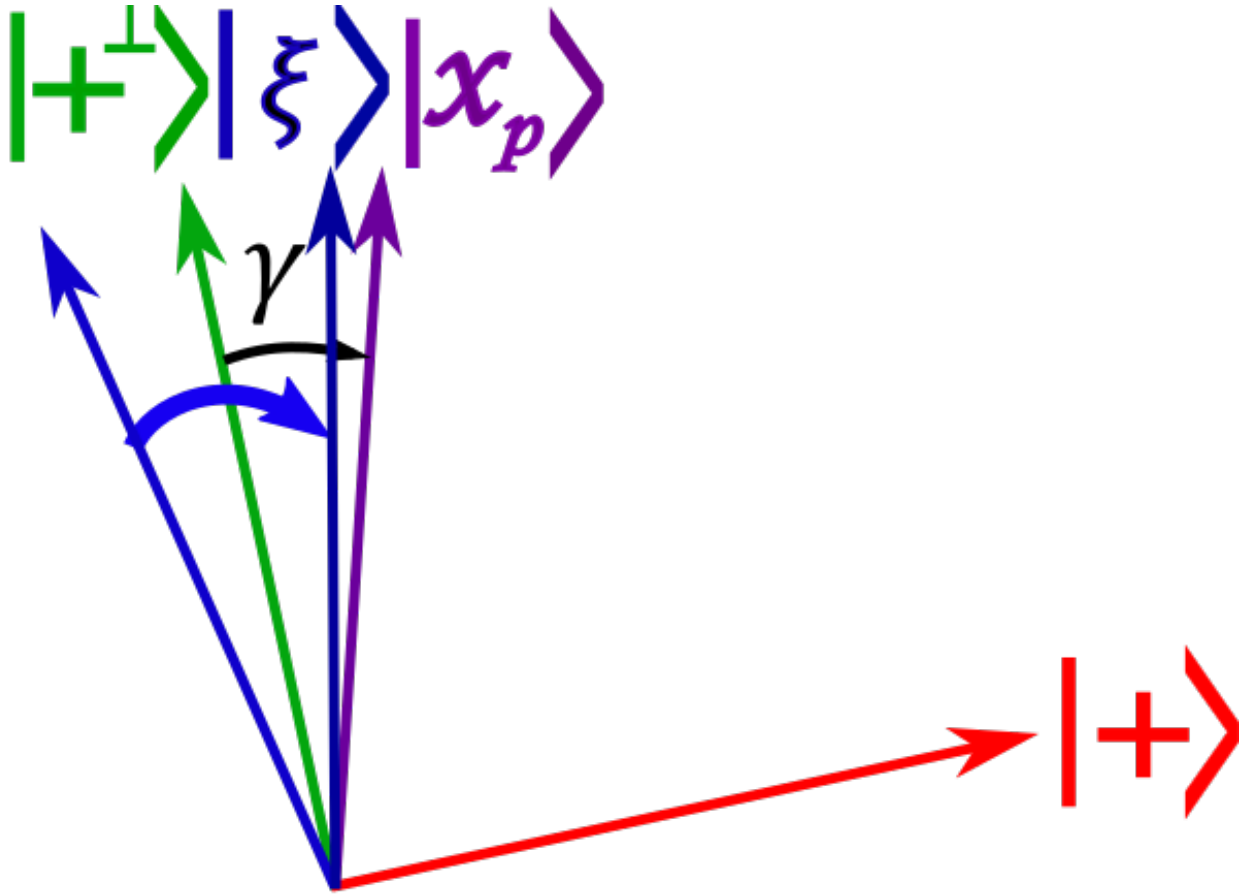


III.4. Geometrical interpretation of Grover's algorithm

After each iteration, $|\xi\rangle$ moves closer to $|x_p\rangle$. In fact, the composition of two reflections around $|x_p\rangle$ and $|+^{\perp}\rangle$ is a rotation.
Let not θ the angle between $|\xi\rangle$ and $|x_p\rangle$, and γ the angle between $|x_p\rangle$ and $|+^{\perp}\rangle$.
After $\hat{R}_{|x_p\rangle}$, $|\xi\rangle$ rotates by an angle 2θ anticlockwise. After $\hat{R}_{|+^{\perp}\rangle}$, $|\xi\rangle$ rotates by an angle $2(\theta-\gamma)$ clockwise. Thus, after $\hat{R}_{|+^{\perp}\rangle}\hat{R}_{|x_p\rangle}$, $|\xi\rangle$ rotates by an angle of

$$\Delta\theta = 2\theta - 2(\theta-\gamma) = 2\gamma.$$

$$\boxed{\Delta\theta = 2\gamma}.$$

After each iteration, the state has rotates within the plane defined by $|x_p\rangle$ and $|+^{\perp}\rangle$ by an angle of 2γ.

Quantum algorithms

III.5. Number of iterations

The iteration has to be stopped when $|\xi\rangle$ is close as much as possible to $|x_p\rangle$. We start with $|\xi\rangle = |+\rangle$, so the initial angle between $|\xi\rangle$ and $|x_p\rangle$ is $\frac{\pi}{2} - \gamma$. We can calculate γ as follow

$$\begin{cases} \cos\gamma = \langle x_p|+^{\perp}\rangle \ , \\ \sin\gamma = \langle x_p|+\rangle \ = \frac{1}{\sqrt{N}}, \end{cases} \tag{2}$$

because

$$|+\rangle = \frac{1}{\sqrt{N}} \sum_{x\in\{0,1\}^n} |x\rangle.$$

For large N,

$$\sin\gamma \approx \gamma \approx \frac{1}{\sqrt{N}}.$$

III.5. Number of iterations

So the number of iterations M required to move from an angle $\frac{\pi}{2} - \gamma$ down to approximately 0 is

$$M \approx \frac{\frac{\pi}{2} - \gamma}{2\gamma} = \frac{\pi}{4\gamma} - \frac{1}{2} \approx \frac{\pi}{4}\sqrt{N} - \frac{1}{2}.$$

So in the limit where $N \ggg 1$,

$$\boxed{M \approx \frac{\pi}{4}\sqrt{N}}.$$

The number of iteration with a quantum algorithm scales as $\sqrt{N}$ while with a classical algorithm, it scales as N.

III.5. Number of iterations

After T iterations, the angle between $|\xi\rangle$ and $|x_p\rangle$ is

$$\gamma_T = \frac{\pi}{2} - (2T+1)\arcsin\left(\frac{1}{\sqrt{N}}\right),$$

so the probability of obtaining the outcome $|x_p\rangle$ when we measure it is precisely

$$|\langle\xi|x_p\rangle|^2 = \cos^2\gamma_T = \sin^2\left((2T+1)\arcsin\left(\frac{1}{\sqrt{N}}\right)\right).$$

Maximising this by taking T as the nearest integrer to

$$\frac{\pi}{4\arcsin\left(\frac{1}{\sqrt{N}}\right)} - \frac{1}{2} = \frac{\pi}{4}\sqrt{N} - \frac{1}{2} - \mathcal{O}\left(\frac{1}{N}\right).$$

We have access to x_p with a probability $1 - \mathcal{O}\left(\frac{1}{N}\right)$ using $\mathcal{O}\left(\sqrt{N}\right)$ queries (for small x, $\arcsin x \approx a + \mathcal{O}\left(x^3\right)$).

III.5. Number of iterations

Remark:
the optimum number of iteration is independent of x_p.

A particular nice case, where we can determine an exact solution for T, is for $N = 4$. Indeed,

$$\arcsin \frac{1}{2} = \frac{\pi}{6},$$

so if we plug in $T = 1$, the probability of getting x_p at the outcome is $\sin^2 \frac{\pi}{2} = 1$.

So we get the right answer only after 1 query for 4 possibilities of x_p !

Quantum algorithms

Quantum algorithms

IV.1. Number of marked elements known

Grover's algorithm can also be used when there are $M > 1$ marked elements. In this setting, the operator $\hat{U}_f$ inverts the phase of inputs elements $x \in S$, for S an unknown subset of $\{0,1\}^n$, and $\text{Card}(S) = M$.
$\hat{U}_f$ is still related to a reflection operator, but now an inversion around a M-dimensional subspace

$$\hat{U}_f = \mathbb{I} - 2\hat{\Pi}_S,$$

where

$$\hat{\Pi}_S = \sum_{x \in S} |S\rangle\langle S|,$$

is the projector on the subspace generated by S.

IV.1. Number of marked elements known

Let's define the state $|S\rangle$ as follow

$$\boxed{|S\rangle = \frac{1}{\sqrt{M}} \sum_{x \in S} |x\rangle}.$$

Then,

$$\begin{aligned} \hat{I}_{|S\rangle} &= (\mathbb{I} - 2|S\rangle\langle S|)|+\rangle \\ &= |+\rangle - 2\left(\frac{1}{M}\sum_{x,y\in S}|x\rangle\langle y|\right)\left(\frac{1}{\sqrt{N}}\sum_{x\in\{0,1\}^n}|x\rangle\right). \end{aligned}$$

So

$$\hat{I}_{|S\rangle} = |+\rangle - \frac{2}{\sqrt{N}}\sum_{x\in S}|x\rangle = \left(\mathbb{I} - 2\hat{\Pi}_S\right)|+\rangle = \hat{U}_f|+\rangle.$$

IV.1. Number of marked elements known

Similarly

$$\hat{I}_{|S\rangle}\,|S\rangle = -\,|S\rangle = \left(\mathbb{I} - 2\hat{\Pi}_S\right)|S\rangle = \hat{U}_f\,|S\rangle.$$

Then, $\hat{U}_f$ operation behaves like a reflection around $|S\rangle$ operator for any states in the subspace spanned by $|+\rangle$ and $|S\rangle$.

The whole of the previous analysis goes through, except that now the angle γ moved at each step satisfies

$$\boxed{\sin\gamma = \langle S|+\rangle \;= \sqrt{\frac{M}{N}}}.$$

IV.1. Number of marked elements known

Thus, after T iterations, we have

$$|\langle \xi | S \rangle|^2 = \cos^2 \gamma_T = \sin^2 \left((2T+1) \arcsin \sqrt{\frac{M}{N}} \right).$$

To obtain an overlap with $|S\rangle$ close to 1, it requires T iterations with

$$\boxed{T \approx \frac{\pi}{4} \sqrt{\frac{N}{M}}}.$$

At the end of the algorithm, one get an element of the subset S at the measurement (a uniformly random distribution of elements of S) with a probability $|\langle \xi | S \rangle|^2$.
For $M = \frac{N}{4}$, we again measure an element of S with certainty using only one query.

Quantum algorithms

IV.2. Number of marked elements unknown

Now the number of marked elements is not known (noted M'). In that case, one first runs the algorithm assuming there is only 1 marked element. If it fails, try again assuming there are 2 marked elements. Then 4, 8, etc... The total number of queries used is roughly

$$\sum_{k=0}^{\log_2 N} \frac{\pi}{4}\sqrt{\frac{N}{2^k}} = \frac{\pi}{4}\sqrt{N}\sum_{k=0}^{\log_2 N} 2^{-k/2} = \mathcal{O}\left(\sqrt{N}\right).$$

If the number of marked elements is M', at least one of the iterations must choose a guess M for M' such that

$$\frac{M'}{2} \leq M \leq 2M'.$$

This corresponds to a value of T which is within a factor of about $\sqrt{2}$ of the optimal value $T' \approx \frac{\pi}{4}\sqrt{\frac{N}{M'}}$.

IV.2. Number of marked elements unknown

Since

$$(2T'+1)\arcsin\sqrt{\frac{M'}{N}} = \frac{\pi}{2} + \mathcal{O}\left(\sqrt{\frac{M'}{N}}\right),$$

then

$$\sin^2\left((2T+1)\arcsin\sqrt{\frac{M'}{N}}\right) = \sin^2\left(\frac{2T+1}{2T'+1}(2T'+1)\arcsin\sqrt{\frac{M'}{N}}\right)$$
$$= \sin^2\left(\frac{2T+1}{2T'+1}\left(\frac{\pi}{2} + \mathcal{O}\left(\sqrt{\frac{M'}{N}}\right)\right)\right),$$

which is lower-bounded by a strictly positive constant of M is small with respect to N.

IV.2. Number of marked elements unknown

Repeating the whole algorithm $\mathcal{O}(1)$ times, and checking each time whether the returned element is marked, allos to achieve an arbitrary high success probability.

This algorithm might still have a high probability of failing in the case where $M = \mathcal{O}(N)$. To find a marked element in this case, we can just sample $\mathcal{O}(1)$ random values of $f(x)$ classically ; we will find a marked element with high probability.

Quantum algorithms

IV.3. Amplitude amplification

The idea of Grover's algorithm might be generalized to an algorithm for finding heuristic solutions to any problems. This algorithm is known as **amplitude amplification**.

Imagine we have $N = 2^n$ possible solutions, of which a subset S are "good", and we would like to find a good solution. As well as having access to a "checking" algorithm f as before, where $f(x) = 1$ if and only if x is marked, we now have access to a "guessing" algorithm $\hat{\mathcal{A}}$, which has the job of producing potential solution to the problem.

It performs the map

$$\hat{\mathcal{A}} |0\rangle^{\otimes n} = \sum_{x \in \{0,1\}^n} \alpha_x |x\rangle,$$

with $\alpha_x \in \mathbb{C}$.

IV.3. Amplitude amplification

After applying $\hat{\mathcal{A}}$, the probability that we would obtain a good solution after measurement is

$$p = \sum_{x \in S} |\alpha_x|^2 .$$

We may consider $\hat{\mathcal{A}}$ as an heuristic try for output of a good solution. We can use f afterwards to check whether a claimed solution is actually good. If we repeated the algorithm $\hat{\mathcal{A}}$ until we got a good solution, the expected number of trials we would need is $\mathcal{O}\left(\frac{1}{p}\right)$.

Amplitude amplification algorithm:

We are given access to $\hat{\mathcal{A}}$ and $\hat{U}_f$.

1. Apply $\hat{\mathcal{A}}$ to initial state $|0\rangle^{\otimes n}$.
2. Repeat the following operations T times, for some T to be determined
   1. Apply $\hat{U}_f$.
   2. Apply $-\hat{\mathcal{A}}\hat{U}_0\hat{\mathcal{A}}^{-1}$.
3. Measure all the qubits and output the result.

IV.3. Amplitude amplification

Let introduce

$$|\psi\rangle = \hat{\mathcal{A}}\,|0\rangle^{\otimes n},$$

and

$$|G\rangle = \frac{\hat{\Pi}_S\,|\psi\rangle}{\|\hat{\Pi}_S\,|\psi\rangle\|}, \quad \text{with} \quad \hat{\Pi}_S = \sum_{x\in S} |x\rangle\langle x|.$$

The previous analysis is still valid, replacing $|+\rangle$ with $|\psi\rangle$ and $|S\rangle$ with $|G\rangle$. The first operation applied is equivalent to $\hat{I}_{|G\rangle}$ and the second is equivalent to $-\hat{I}_{|\psi\rangle}$.

IV.3. Amplitude amplification

We start with the state $|\psi\rangle$ and rotate it toward $|G\rangle$. The angle γ moved at each step is such that

$$\boxed{\sin\gamma = \langle\psi|G\rangle = \|\hat{\Pi}_S|\psi\rangle\| = \sqrt{p}},$$

so the number of iterations required to move from $|\psi\rangle$ to $|G\rangle$ is $\mathcal{O}\left(\frac{1}{\sqrt{p}}\right)$, which is a quadratic improvement.

Quantum algorithms

Quantum algorithms

V.1. Quantum Fourier Transformation

We now introduce an important unitary transformation which is used in a number of different contexts in quantum information theory

the quantum Fourier transform (QFT) over $\mathbb{Z}_N$,

where $\mathbb{Z}_N$ is the ensemble of integer modulo N.
QFT might be seen as a generalization of the Hadamard gate, which has the following map

$$\hat{H}^{\otimes n} = \frac{1}{\sqrt{2^n}} \left(|0\rangle\langle 0| + |1\rangle\langle 0| + |0\rangle\langle 1| - |1\rangle\langle 1| \right)^{\otimes n}.$$

The QFT map is the following

$$\boxed{\hat{Q}_N |x\rangle = \frac{1}{\sqrt{N}} \sum_{y \in \mathbb{Z}_N} \omega_N^{x \cdot y} |y\rangle},$$

where $\omega_N = e^{\frac{2i\pi}{N}}$, but $x \cdot y$ is the product of x and y as integer of $\mathbb{Z}_N$.

Exemple:

$$Q_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad Q_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{\frac{2i\pi}{3}} & e^{\frac{-2i\pi}{3}} \\ 1 & e^{\frac{-2i\pi}{3}} & e^{\frac{2i\pi}{3}} \end{pmatrix},$$

$$Q_4 = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix}.$$

Note that the QFT is unitary.

Demonstration: Let consider the inner product of rows x and z

$$\frac{1}{N} \sum_{y \in \mathbb{Z}_N} \left(\omega_N^{x \cdot y}\right)^* \omega_N^{z \cdot y} = \frac{1}{N} \sum_{y \in \mathbb{Z}_N} \omega_N^{(z-x) \cdot y}.$$

Or

$$\sum_{k=0}^{r-1} x^k = \begin{cases} \frac{1-x^r}{1-x} & \text{if} \quad x \neq 1, \\ r & \text{if} \quad x = 1. \end{cases} \tag{3}$$

Given that $\omega_N^N = 1$, the inner product is then 0 if $z \neq x$, and 1 otherwise ($z = x$). More generally, for any integer j,

$$\frac{1}{N} \sum_{y \in \mathbb{Z}_N} \omega_N^{j \cdot y} = \begin{cases} 0 & \text{if} \quad j \neq 0 \, [N], \\ 1 & \text{if} \quad j = 0 \, [N]. \end{cases} \tag{4}$$

Then the QFT is unitary.

The QFT is a similar transformation than the Discrete Fourier Transformation (DFT) used for classical computation and signal processing, up to the non standard normalization of $1/\sqrt{N}$.

Quantum algorithms

V.2. Periodicity determination with QFT

Let consider a function

$$f : \mathbb{Z}_N \longrightarrow \mathbb{Z}_M,$$

for $(N, M) \in \mathbb{N}^2$ such that

1. f is periodic: there is a r such that

$$\forall x \in \mathbb{Z}_N,\ f(x + r) = f(x),$$

2. f is one-to-one on each period

$$\forall (x, y) \in \mathbb{Z}_N \text{ such that } |x - y| < r,\ f(x) \neq f(y).$$

The goal is to determine r.

V.2. Periodicity determination with QFT

The periodicity determination algorithm is the following. We start in the state $|0\rangle^{\otimes N} |0\rangle^{\otimes M}$.

1. Apply $\hat{Q}_N$ to the first register.
2. Apply $\hat{O}_f$ to the two registers (the Oracle).
3. Measure the second register.
4. Apply $\hat{Q}_N$ to the first register.
5. Measure the first register ; let the answer be k.
6. Simplify the fraction $\frac{k}{N}$ as far as possible and return the denominator.

V.2. Periodicity determination with QFT

$$|0\rangle^{\otimes N}|0\rangle^{\otimes M} \xrightarrow{1)} \frac{1}{\sqrt{N}}\sum_{x\in\mathbb{Z}_N}|x\rangle|0\rangle^{\otimes M} \xrightarrow{2)} \frac{1}{\sqrt{N}}\sum_{x\in\mathbb{Z}_N}|x\rangle|f(x)\rangle.$$

When the second register is measured, we receive an answer, say z. Since f is periodic and one-to-one,

$$\exists x_0 \text{ such that } f(x_0) = z.$$

Consequently

$$\forall x \in \mathbb{Z}_N \text{ such that } f(x_0) = z, \exists j \in \mathbb{Z} \text{ such that } x = x_0 + jr.$$

The state collapses then to something of the following form

$$\sqrt{\frac{r}{N}}\sum_{j=0}^{\frac{N}{r}-1}|x_0 + jr\rangle,$$

which means there is N/r states in a period.

V.2. Periodicity determination with QFT

After we apply the QFT, we get the state

$$\frac{\sqrt{r}}{N}\sum_{j=0}^{\frac{N}{r}-1}\left(\sum_{y\in\mathbb{Z}_N}\omega_N^{y\cdot(x_0+jr)}\left|y\right\rangle\right)=\frac{\sqrt{r}}{N}\sum_{y\in\mathbb{Z}_N}\omega_N^{y\cdot x_0}\left(\sum_{j=0}^{\frac{N}{r}-1}\omega_N^{j\cdot r\cdot y}\right)\left|y\right\rangle$$

Observe that, as r divides N,

$$\omega_N^r=e^{\frac{2i\pi r}{N}}=\omega_{\frac{N}{r}}.$$

This state is then equivalent to

$$\frac{\sqrt{r}}{N}\sum_{y\in\mathbb{Z}_N}\omega_N^{y\cdot x_0}\left(\sum_{j=0}^{\frac{N}{r}-1}\omega_{\frac{N}{r}}^{j\cdot y}\right)\left|y\right\rangle$$

$$\sum_{j=0}^{\frac{N}{r}-1} \omega_{\frac{N}{r}}^{j\cdot y} = 0 \quad \text{unless } y \equiv 0 \left[\frac{N}{r}\right], \quad \text{in other words, if } \; y = l\frac{N}{r}, l \in \mathbb{Z}.$$

This state might be rewrite as follow

$$\frac{1}{\sqrt{r}} \sum_{l=0}^{r-1} \omega_N^{l\cdot x_0 \cdot \frac{N}{r}} \left| l\frac{N}{r} \right\rangle.$$

When me perform the final measurement, we receive an outcome

$$k = \frac{l_0 N}{r},$$

for some l_0 picked uniformly at random from $\{0, \cdots, r-1\}$ so

$$\frac{k}{N} = \frac{l_0}{r}.$$

V.2. Periodicity determination with QFT

If l_0 is coprime to r, we could cancel the fraction $\frac{k}{N}$ and output the denominator. For a integer, b picked-up uniformly at random from 0 to an the probability that b is coprime to a is

$$\mathcal{O}\left(\frac{1}{\log\left(\log a\right)}\right).$$

The, if we repeat the procedure $\mathcal{O}\left(\log\left(\log r\right)\right) = \mathcal{O}\left(\log\left(\log N\right)\right)$ times, we are likely to find the period r.

V.2. Periodicity determination with QFT

We have a probabilistic procedure which succeeds with probability p ; the probability that it fails every time over R repetitions is exactly

$$(1-p)^R \leq e^{-pR},$$

so it suffices to take $R = \mathcal{O}\left(\frac{1}{p}\right)$ to achieve $\sim 99\%$ success probability.
Each time the algorithm returns a claimed period, we can check whether it is really a period of the function using two additional queries of the Oracle. Each use of the quantum algorithm therefore makes 3 queries of $\hat{O}_f$ so it makes $\mathcal{O}(\log(\log N))$ queries in total.

Quantum algorithms

V.3. Phase estimation

Phase estimation is an important quantum computing primitive routine. Often used as an ingredient of more complex algorithms:

- integer factorisation ;
- matrix inversion ;
- quantum counting ;
- quantum walks.

V.3. Phase estimation

Let consider an operator $\hat{A}$, of eigenvectors $\vec{x}$ and eigenvalues λ

$$\hat{A}\vec{x} = \lambda\vec{x}.$$

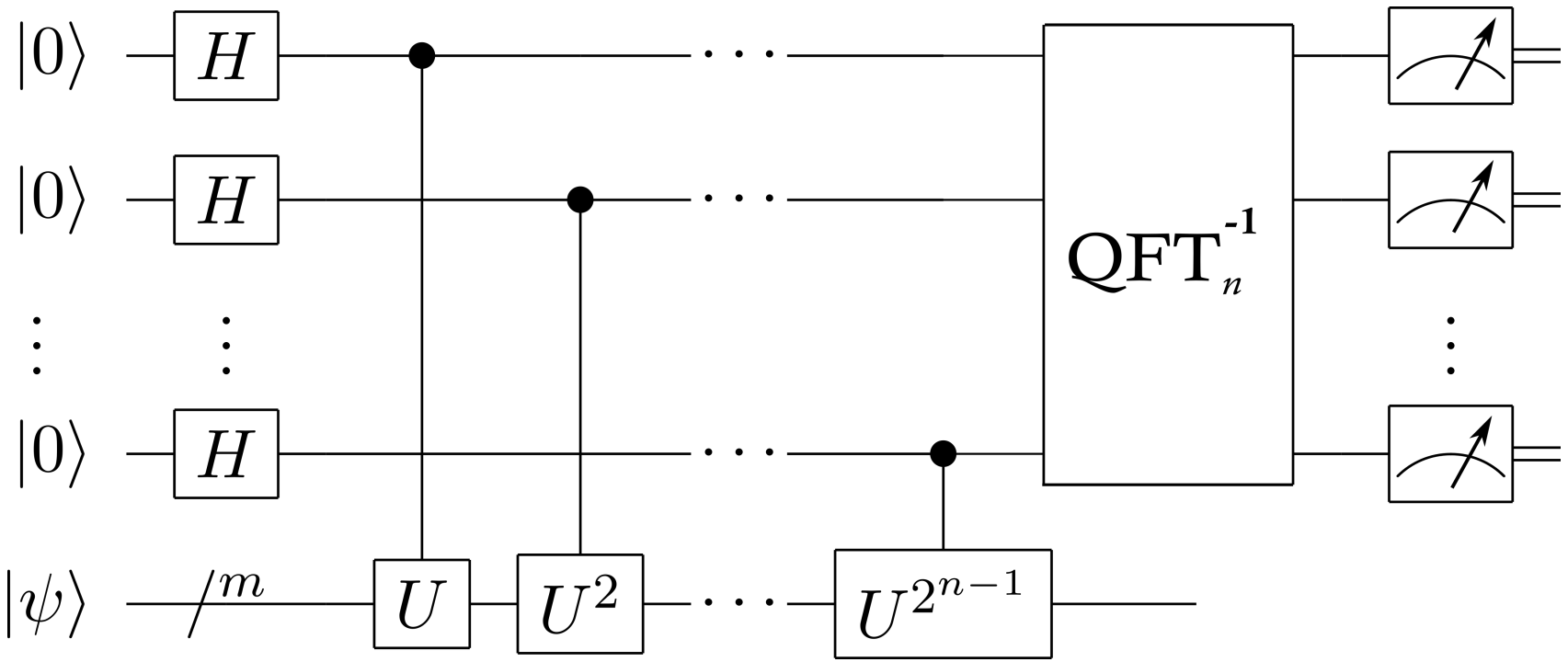
In the case of a unitary matrix

$$\hat{U}\left|x\right\rangle = e^{2i\pi\theta}\left|x\right\rangle,$$

with $\left|x\right\rangle$ an eigenvector and θ the phase of the eigenvalue $e^{2i\pi\theta}$.
Phase estimation algorithm: given $\hat{U}$ and $\left|x\right\rangle$, estimate θ.

V.3. Phase estimation

The circuit diagram implementation of the phase estimation algorithm is the following



Quantum algorithms

Quantum algorithms

VI.1. Factoring

The most famous application of quantum computers

$$N = a \times b \longrightarrow \text{find } a \text{ and } b \text{ given } N,$$

where a and b are prime numbers.

Important modern crypto-systems (e.g. RSA) rely on this problem being intractable for computers. But a quantum computer can solve it quickly !

VI.1. Factoring

Exemple: the recommended key size for RSA is 2048 bits.

- Best known classical algorithm $\sim$ 1 billion years.
- Shor's algorithm $\sim$ 100 seconds !!!! (QC at 1 GHz).

Reference:
A compare between Shor's quantum factoring algorithm and general Number Field Sieve, Hamdi *et al.*.

Quantum algorithms

VI.2. Shor's algorithm

Shor's algorithm consists in reducing the problem of factoring to the problem of period finding.

It uses a quantum algorithm for fast period finding.

Shor's algorithm

1. If N is even, return $f = 2$.
2. If $N = p^k$ for p prime, return p.
3. Randomly choose $1 < q < N - 1$.
   If $f = gcd(q, N) > 1$, return f
4. Determine the order k of q modulo N. (Phase estimation).
   If k is odd, repear from step 3.
5. Write $k = 2l$ and determine q^l mod N with $1 < r < N$.
   1. If $1 < f = gcd(r - 1, N) < N$, return f.
   2. If $1 < f = gcd(r + 1, N) < N$, return f.
   3. Else repeat step 3.

VI.2. Shor's algorithm

All steps, excepted step 4, can be performed efficiently by a classical computer.

Given an n-bit integer

- classical number field sieve: $\mathcal{O}\left(2^{n^{1/3}}\right)$.
- Shor's algorithm: $\mathcal{O}\left(n^3\right)$.

Quantum algorithms

Quantum algorithms

VII.1. Shor's algorithm

In quantum mechanics, physical systems are described by Hamiltonians. The evolution is given by Schrödinger's equation

$$i\hbar \frac{\mathrm{d}\,|\psi(t)\rangle}{\mathrm{d}t} = \hat{H}(t)\,|\psi(t)\rangle.$$

For a stationnary hamiltonian, $|\psi(t)\rangle = e^{-i\frac{\hat{H}t}{\hbar}}\,|\psi(0)\rangle$

Hamiltonian simulation:
Given a Hamiltonian $\hat{H}$, construct a quantum circuit that approximates $e^{-i\frac{\hat{H}t}{\hbar}}$.

There are a number of quantum algorithm that can do this efficiently for certain type of Hamiltonian.
To simulate a classical system like a plane, a classical computer is appropriated. But to simulate a quantum system like a molecule, a quantum computer is better suited.

Quantum algorithms

VII.2. HHL

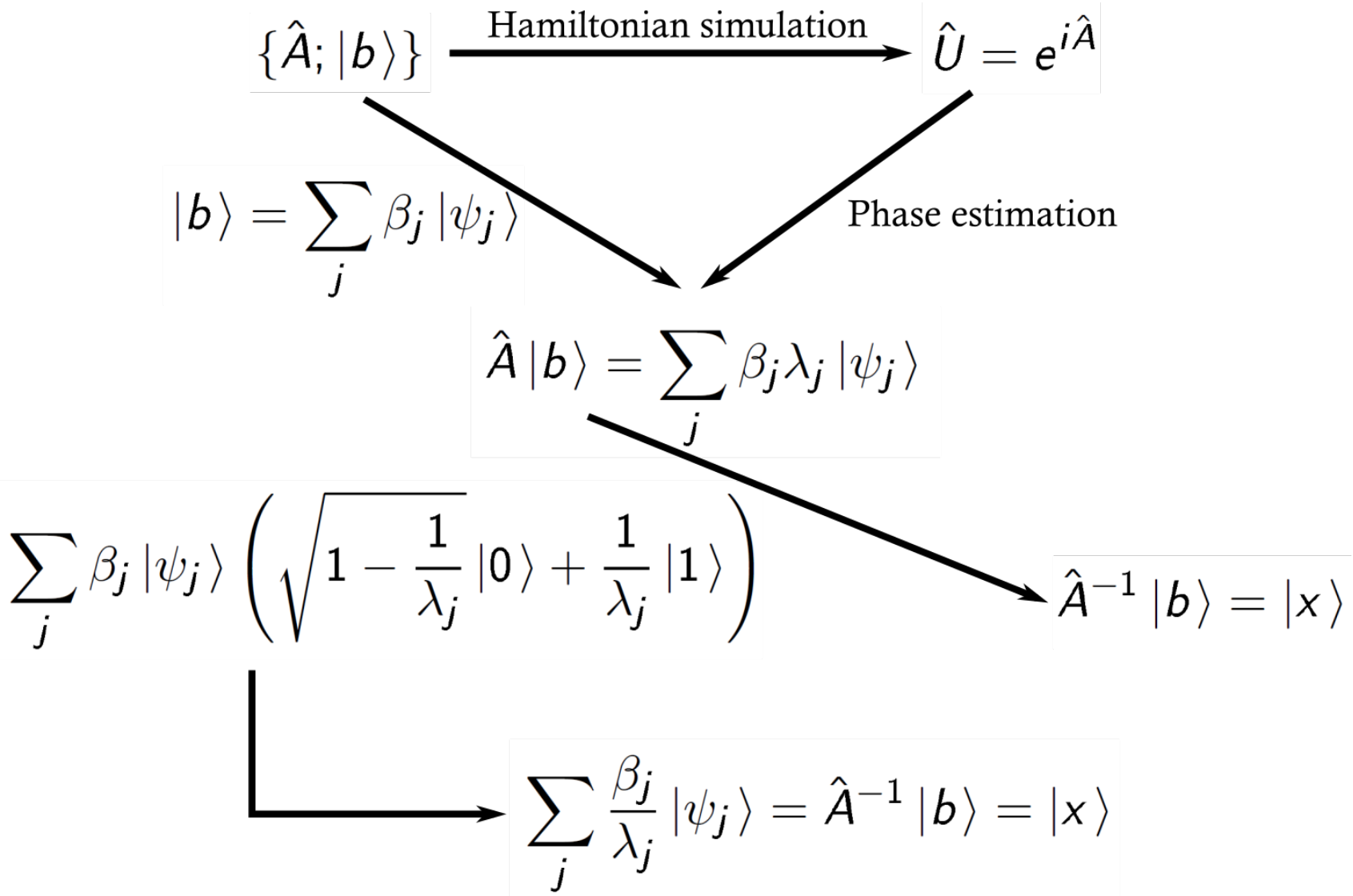
Named after Harrow, Hassidim and Lloyd, who invented it in 2008. It attaks one of the most fundamental tasks in science: solving systems of linear equations

$$\boxed{A\vec{x} = \vec{b}, \quad \text{solve for } \vec{x}}.$$

Classically: it takes polynomial time in the size of the matrix, whereas HHL "solves" this problem in logarithmic time.

Quantum algorithm HHL: inputs $|b\rangle$ and $\hat{A}$, outputs quantum state $|x\rangle$.

HHL algorithm outline



Quantum algorithms

VII.3. Applications

- Solving systems of differential equations (finite element method, FEM).
- Data fitting.
- Various tasks in machine learning (clustering, support-vector machines, principal component analysis).

Run time: for a system of n equations.

- classical: $\mathcal{O}\left(n^3\right)$;
- quantum: $\mathcal{O}\left(\kappa s \frac{\log n}{\varepsilon}\right)$, with κ the condition number, s the sparsity and ε the accuracy.

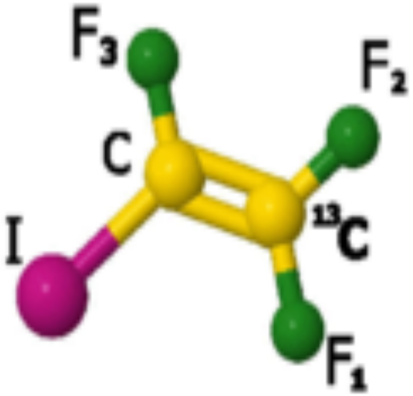
Quantum algorithms

VII.4. Applications

1. Experimental Quantum Computing to Solve Systems of Linear Equations,
   X.-D. Cai *et al.*, Phys. Rev. Lett. **110**, 230501 (2013).
   Problem solved: 2×2 linear equations.
   Qubits: photons (polarization).
2. A two-qubit photonic quantum processor and its application to solving systems of linear equation,
   S. Barz *et al.*, Sci. Rep. **4**, 6115 (2014).
   Problem solved: 2×2 linear equations.
   Qubits: photons (polarization).
3. Experimental realization of quantum algorithm for solving linear systems of equations,
   J. Pan *et al.*, Phys. Rev. A **89**, 022313 (2014).
   Problem solved: 2×2 linear equations.
   Qubits: NMR type qubits in a molecule of iodotrifluoroethylene $^{12}C^{13}CF_3I$.

VII. Introduction

	^{13}C	F_1	F_2	F_3
^{13}C	15479.7 Hz			
F_1	-297.7 Hz	-33122.4 Hz		
F_2	-275.7 Hz	64.6 Hz	-42677.7 Hz	
F_3	39.1 Hz	51.5 Hz	-129.0 Hz	-56445.8 Hz
T_2^*	1.22 s	0.66 s	0.63 s	0.61 s
T_2	7.9 s	4.4 s	6.8 s	4.8 s



J. Pan et *al.*, Phys. Rev. A 89, 022313 (2014)

Quantum algorithms

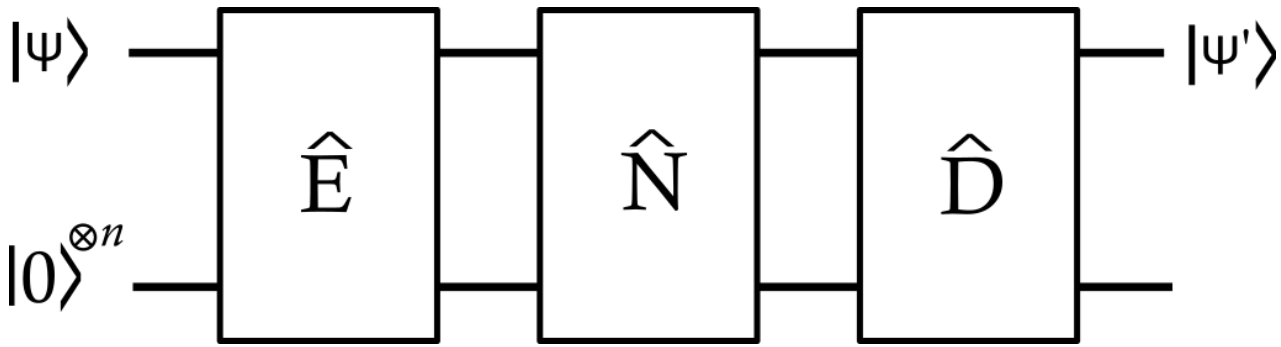
VIII. Quantum error correction

In classical computing, error correcting codes preserves classical bits. Quantum error correcting code will preserve a qubit $|\psi\rangle$ in quantum computing.

Let consider an error affecting one or more qubits is simply an arbitrary (unknown) unitary operator $\hat{N}$ applied to those qubits ($\hat{N}$ is a noise operator). The classical bit-flip is an example, and corresponds to the application of the operator $\hat{X}$.

$$\hat{X}|0\rangle = |1\rangle \text{ and } \hat{X}|1\rangle = |0\rangle.$$

The process of correcting errors in a qubits state $|\psi\rangle$ might be described as follow



- $\hat{E}$ is an **encoding** unitary operator,
- $\hat{N}$ is a **noise** unitary operator,
- $\hat{D}$ is a **decoding** unitary operator,

We encode some qubit state $|\psi\rangle$ in a larger state $|E(\psi)\rangle$ using n ancilla qubits (initially on the state $|0\rangle^{\otimes n}$.

Some noise is applied through $\hat{N}$, and later we decode the noisy encoded state to produce a state $|\psi'\rangle$.

Goal of the process: $|\psi'\rangle \approx |\psi\rangle$

Non cloning theorem: it is not possible to duplicate a state $|\psi\rangle$ in the general case

$$|\psi\rangle \nrightarrow |\psi\rangle \otimes |\psi\rangle \otimes |\psi\rangle \cdots \otimes |\psi\rangle.$$

The error protection can't be performed by cloning the state $|\psi\rangle$.

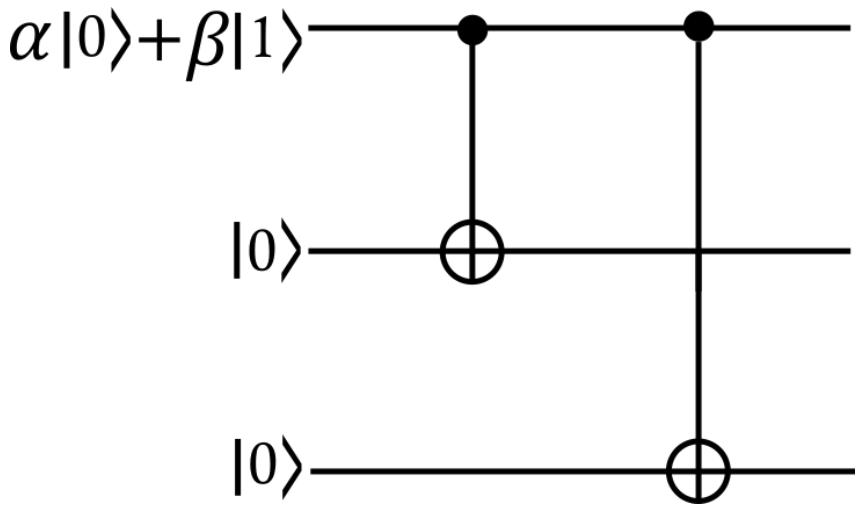
Principle of error correction code: let consider $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$. Then, encode it as follow

$$|E(\psi)\rangle = \alpha\,|000\rangle + \beta\,|111\rangle.$$

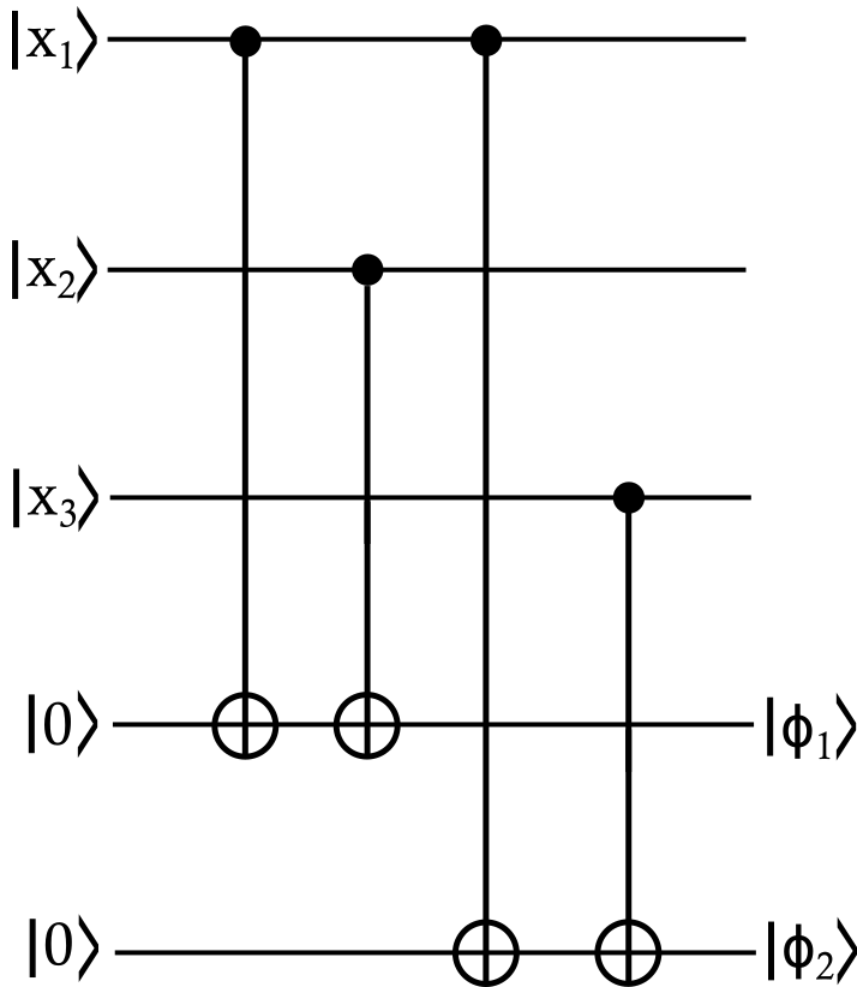
Remark: it is not a cloning !

$$|\psi\rangle \otimes |\psi\rangle \otimes |\psi\rangle = (\alpha\,|0\rangle + \beta\,|1\rangle)^{\otimes 3}.$$

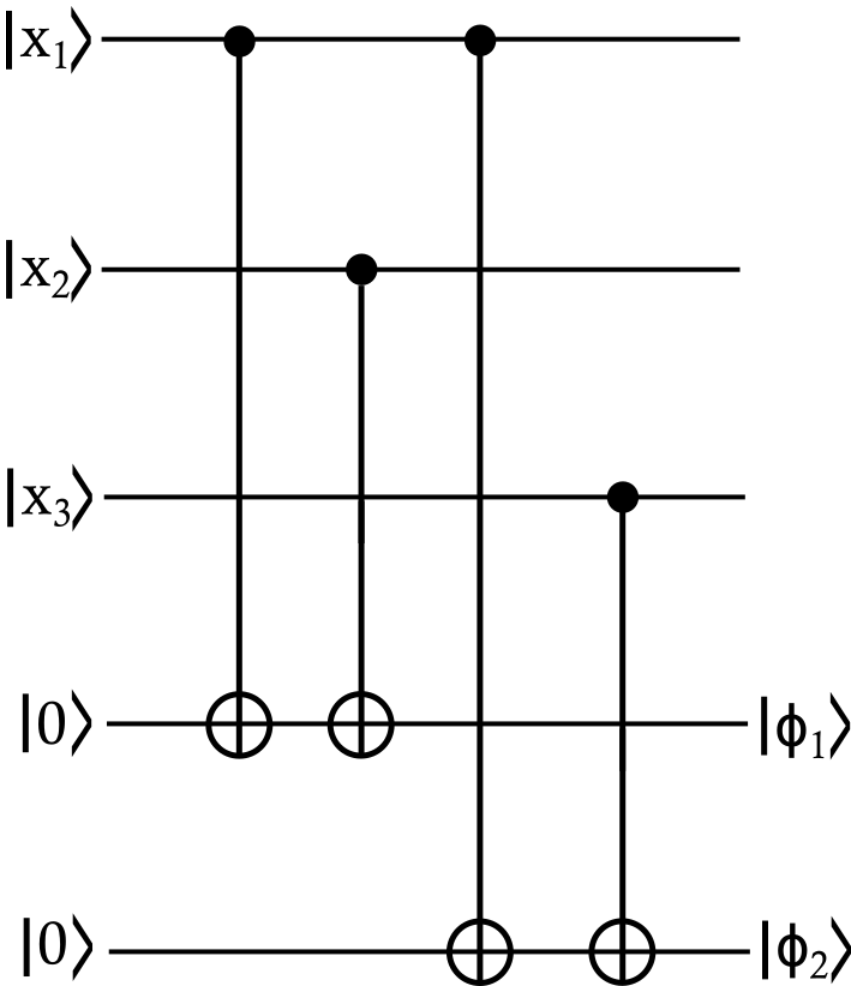
The operator $\hat{E}$ might be implemented as follow



The decoding algorithm for this code will be based on the following circuit



VIII. Quantum error correction



The first three qubits are called input qubits. The last two qubits are called output qubits.

$$|\phi_1\rangle = |x_1 \oplus x_2\rangle,$$

$$|\phi_2\rangle = |x_1 \oplus x_3\rangle.$$

$x_1 \oplus x_2$ and $x_1 \oplus x_3$ are invariant under the flipping of all the bits of x.

After the application of $\hat{N}$

$$\hat{N}\,|E(\psi)\rangle = \alpha\,|x_1x_2x_3\rangle + \beta\,|x_1x_2x_3 \oplus 111\rangle.$$

The circuit proposed performs the following map

$$(\alpha\,|x_1x_2x_3\rangle + \beta\,|x_1x_2x_3 \oplus 111\rangle) \otimes |0\rangle \otimes |0\rangle$$

$$\longrightarrow (\alpha\,|x_1x_2x_3\rangle + \beta\,|x_1x_2x_3 \oplus 111\rangle) \otimes |x_1 \oplus x_2\rangle \otimes |x_1 \oplus x_3\rangle.$$

If one measures the two output qubits, we learn both $x_1 \oplus x_2$ and $x_1 \oplus x_3$ without disturbing the input qubits. The encoded state $|\psi\rangle$ is always of this form, even after arbitrary bit-flip errors are applied to $|E(\psi)\rangle$.

$$|E(\psi)\rangle = \alpha\,|000\rangle + \beta\,|111\rangle.$$

Effect of bit-flip on $|E(\psi)\rangle$

$$\left(\hat{X}\otimes\mathbb{I}\otimes\mathbb{I}\right)|E(\psi)\rangle = \alpha\,|100\rangle + \beta\,|011\rangle,$$

$$\left(\hat{X}\otimes\hat{X}\otimes\hat{X}\right)|E(\psi)\rangle = \alpha\,|111\rangle + \beta\,|000\rangle.$$

The result of measuring the output qubits is known as the **syndrome**. What are the syndromes of different noise operators $\hat{N}$ applied to $|E(\psi)\rangle$?
If $\hat{N} = \mathbb{I}$, we always measure 00.
If $\hat{N} = \hat{X}\otimes\mathbb{I}\otimes\mathbb{I}$, we always obtain 11.

Syndrome measured for different bit-flip noise:

$\hat{N}$	Syndrome
$\mathbb{I} \otimes \mathbb{I} \otimes \mathbb{I}$	00
$\mathbb{I} \otimes \mathbb{I} \otimes \hat{X}$	01
$\mathbb{I} \otimes \hat{X} \otimes \mathbb{I}$	10
$\hat{X} \otimes \mathbb{I} \otimes \mathbb{I}$	11
$\mathbb{I} \otimes \hat{X} \otimes \hat{X}$	11
$\hat{X} \otimes \hat{X} \otimes \mathbb{I}$	01
$\hat{X} \otimes \mathbb{I} \otimes \hat{X}$	10
$\hat{X} \otimes \hat{X} \otimes \hat{X}$	00

If the error occurs on a single qubit, it is possible to detect it, and apply the corresponding bit-flip operation on the corresponding qubit to restore the original encoded state $\alpha |000\rangle + \beta |111\rangle$.
On the other hand, if bit flip errors occurs on more than one qubit, one does not detect them.

In the case of $\hat{Z}$ noise,

$$\hat{Z} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

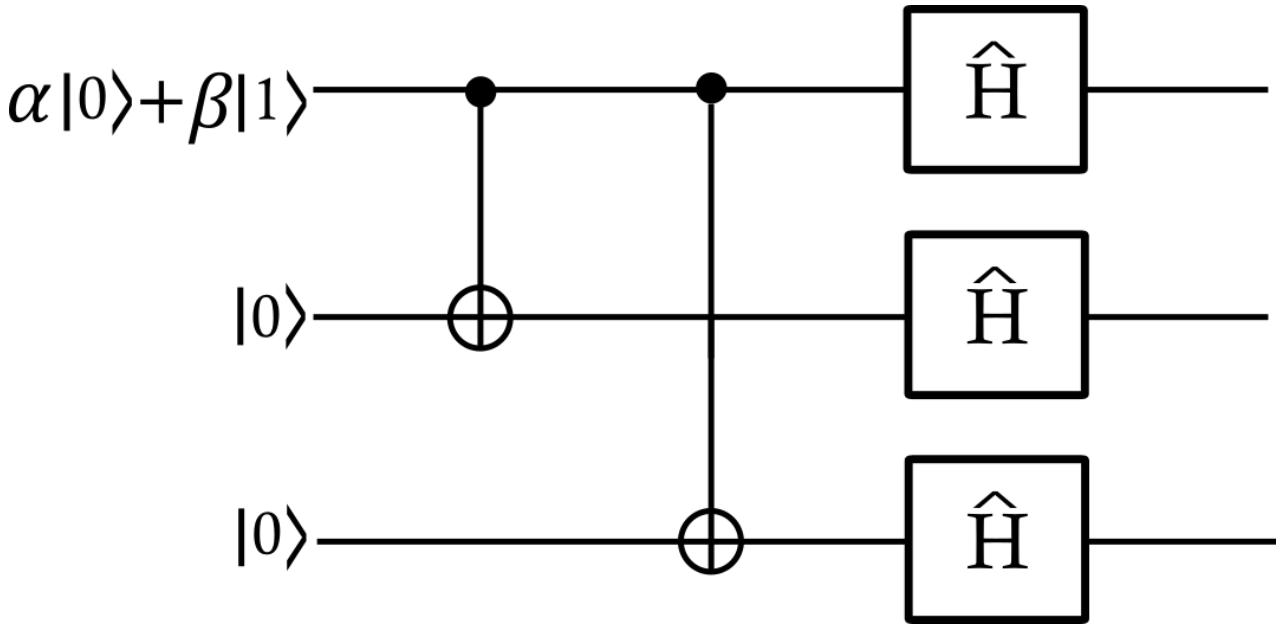
the syndrome measurement always return 00, so the error correction operation does nothing and the $\hat{Z}$ error is not corrected. But

$$\hat{Z} = \hat{H}\hat{X}\hat{H},$$

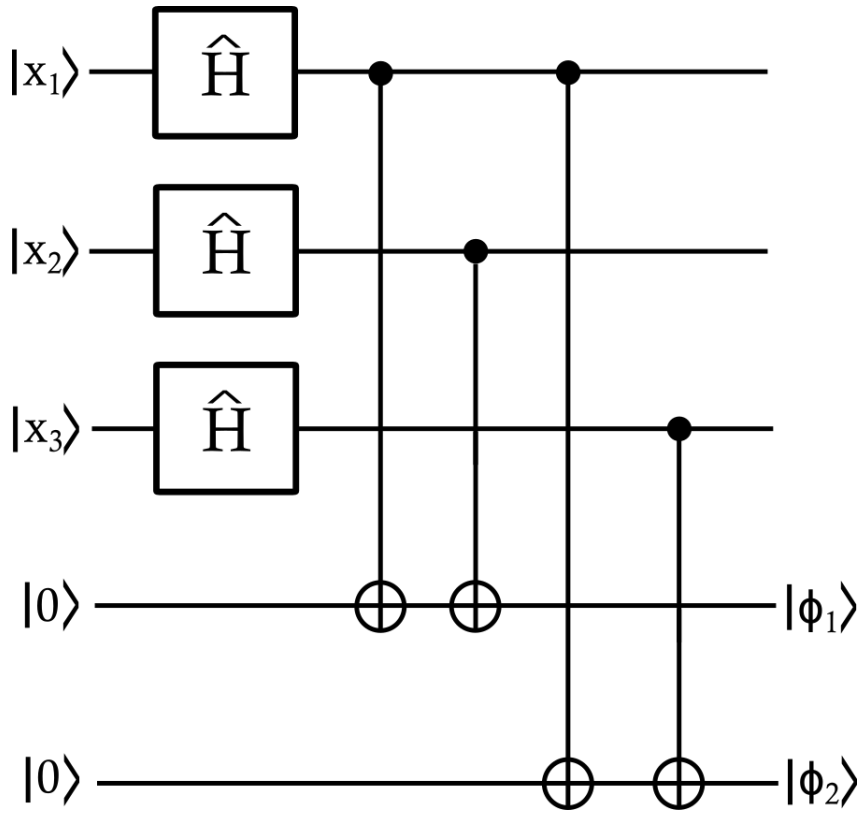
where $\hat{H}$ is the Hadamard gate. Thus $\hat{Z}$ acts in the same way as $\hat{X}$, up to a change of basis.
If we use the same code as before, but perform this change of basis for each qubit, we obtain a code which corrects against $\hat{Z}$ errors. In other words, we now encode $|\psi\rangle$ as $\alpha\,|+++\rangle + \beta\,|---\rangle$, with

$$|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \quad \text{and} \quad |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}.$$

The new encoding circuit is then



and the decoding circuit



But it does no longer protects against $\hat{X}$ errors !!! It is possible to concatenates these two codes.
We first encode $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ using the code protecting against phase flips, and then encode each of the resulting qubits using the code that protects against bit flips. In other words, we perform the following map

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle,$$

$$\begin{aligned}\longrightarrow \frac{1}{2\sqrt{2}} \quad & (\alpha(|0\rangle + |1\rangle)(|0\rangle + |1\rangle)(|0\rangle + |1\rangle) \\ & + \beta(|0\rangle - |1\rangle)(|0\rangle - |1\rangle)(|0\rangle - |1\rangle)),\end{aligned}$$

$$\longrightarrow \frac{1}{2\sqrt{2}} \quad (\alpha\,(|000\rangle + |111\rangle)\,(|000\rangle + |111\rangle)\,(|000\rangle + |111\rangle)$$
$$+ \beta\,(|000\rangle - |111\rangle)\,(|000\rangle - |111\rangle)\,(|000\rangle - |111\rangle))\,,$$

The single qubit $|\psi\rangle$ is now encoded using 9 qubits.

These qubits can naturally be split into three blocks, each of which encodes one qubit of the state

$$\alpha\,|+++\rangle + \beta\,|---\rangle.$$

To decode this encoded state, first the decoding circuit for the bit-flip code is applied to each block. Assuming at most one bit-flip error has occurred in each block, the result will be the state $\alpha\,|+++\rangle + \beta\,|---\rangle$, perhaps with a $\hat{Z}$ error applied to one of the qubits. This state can then be mapped back to $\alpha\,|0\rangle + \beta\,|1\rangle$ using the decoding algorithm for the phase-flip code.

This quantum error-correcting code was the first such code discovered. It was invented by Peter Shor in 1995, known as Shor's 9 qubit code.