



HAL
open science

About some finite element approximations of a simple variational inequality

Franz Chouly

► **To cite this version:**

Franz Chouly. About some finite element approximations of a simple variational inequality. Master. Uruguay. 2024. hal-04693634

HAL Id: hal-04693634

<https://cel.hal.science/hal-04693634v1>

Submitted on 10 Sep 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

About some finite element approximations of a simple variational inequality

Franz Chouly

Centro de Matemática, Universidad de la República, Montevideo, Uruguay

fchouly@cmat.edu.uy

September 10, 2024

Abstract

This note presents some general issues about the finite element approximation of variational inequalities of the first kind, with focus on the obstacle problem as a prototype. It is targeted for a small course at the graduate level. It supposes some basic knowledge of the mathematical theory behind finite elements.

Keywords: variational inequalities; obstacle problem; finite elements; error estimates.

Contents

1	Introduction	1
2	The obstacle problem	1
3	A direct finite element discretization	2
4	A <i>a priori</i> error estimate	3
5	A Nitsche finite element approximation	5
6	Final comments	6

1 Introduction

Variational inequalities are useful to model many problems from physics or engineering [19, 22, 29]. However they rarely enter into graduate courses of numerical analysis. The object of this small note is to present the numerical analysis of the obstacle problem, when discretized with finite elements. The obstacle problem is indeed the ideal prototype of variational inequality, and is maybe the simplest one.

We denote by $H^s(\cdot)$, $s \in \mathbb{R}$, the Sobolev spaces. For an open subset D of \mathbb{R}^n , the usual norm of $H^s(D)$ is denoted by $\|\cdot\|_{s,D}$. The space $H_0^1(D)$ is the subspace of functions in $H^1(D)$ with vanishing trace on ∂D . The letter C stands for a generic constant, independent of the discretization parameters.

2 The obstacle problem

We recall here the setting of the obstacle problem, which is described in more details in, e.g., [15, 22]. Let $\Omega \subset \mathbb{R}^n$, $n = 1, 2, 3$, be the domain, supposed to be an open bounded polytope, simply connected and with a Lipschitz boundary. We define

$$a : H_0^1(\Omega) \times H_0^1(\Omega) \rightarrow \mathbb{R}$$

as the bilinear form given by:

$$a(u, v) := \int_{\Omega} \nabla u \cdot \nabla v, \quad \forall u, v \in H_0^1(\Omega),$$

and the linear form

$$L(v) := \int_{\Omega} f v, \quad \forall v \in H_0^1(\Omega),$$

for a source term $f \in L^2(\Omega)$. We consider a given obstacle function $\Psi : \Omega \rightarrow \mathbb{R}$, which is supposed to satisfy $\Psi \in H^2(\Omega)$ and $\Psi \geq 0$, as in [15]. The obstacle problem in weak form consists in finding

$$u \in K : \quad a(u, v - u) \geq \int_{\Omega} f(v - u), \quad \forall v \in K, \quad (1)$$

with

$$K := \{v \in H_0^1(\Omega) : v \leq \Psi \text{ a.e. in } \Omega\}.$$

The problem (1) admits a unique u solution from Stampacchia's theorem [2], and u is also the unique minimizer on K of the convex quadratic functional

$$\mathcal{J} : V \ni v \mapsto \frac{1}{2}a(v, v) - L(v) \in \mathbb{R}.$$

Problem (1) in strong form, reads: find $u : \Omega \rightarrow \mathbb{R}$ solution to:

$$\begin{cases} -\Delta u \leq f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ u \leq \Psi & \text{in } \Omega, \\ (u - \Psi)(\Delta u + f) = 0 & \text{in } \Omega. \end{cases} \quad (2)$$

If the domain Ω is convex and $f = 0$, it can be proven that $u \in H_0^1(\Omega) \cap H^2(\Omega)$ [4, 31]. More regularity can not be expected in general because of the inequality constraints inside Ω [15].

3 A direct finite element discretization

Let T_h be a family of simplicial meshes of the domain Ω ($h := \max_{T \in T_h} h_T$ where h_T is the diameter of T). The family of meshes is supposed regular in Ciarlet's sense [15]. Let V_h be a family of Lagrange finite element spaces of degree one indexed by h . The formal definition of this space is:

$$V_h := \{v_h \in \mathcal{C}(\bar{\Omega}) \cap H_0^1(\Omega) : v_h|_T \in \mathbb{P}_1(T), \quad \forall T \in T_h\}.$$

Let \mathcal{N}_h be the set of the nodes of the mesh and we can define a discrete convex set associated with V_h and the obstacle function Ψ :

$$K_h := \{v_h \in V_h : v_h(a) \leq \Psi(a), \quad \forall a \in \mathcal{N}_h\}.$$

Remark first that the above definition is meaningful. Indeed, the assumption $\Psi \in H^2(\Omega)$ combined with the Sobolev embedding Theorem ensure that Ψ is continuous: its pointwise values are well defined. Remark also that, in general, $K_h \not\subset K$, see [15]. The inclusion $K_h \subset K$ occurs only in special cases, for instance if Ψ is a concave function.

The discrete problem is as follows:

$$u_h \in K_h : \quad a(u_h, v_h - u_h) \geq \int_{\Omega} f(v_h - u_h) \quad \forall v_h \in K_h, \quad (3)$$

and it admits a unique solution, still from Stampacchia's theorem [2], and u_h is also the unique minimizer on K_h of the functional \mathcal{J} .

4 *A priori* error estimate

Let us see now the approximation error. First, the following abstract error estimate is a simplified version of Falk's Lemma [15, 20, 23]:

Proposition 4.1 *Suppose that the solution $u \in K$ to Problem (1) belongs to $H^2(\Omega)$. Then the solution u_h to Problem (3) satisfies the a priori error estimate:*

$$\|u - u_h\|_{1,\Omega} \leq C \left(\inf_{v_h \in K_h} (\|u - v_h\|_{1,\Omega}^2 + \|u - v_h\|_{0,\Omega}) + \inf_{v \in K} \|v - u_h\|_{0,\Omega} \right), \quad (4)$$

with $C > 0$ independent from h and u .

Proof. Let $v_h \in K_h$. First we use the ellipticity of $a(\cdot, \cdot)$, combined with the Cauchy-Schwarz inequality:

$$\begin{aligned} \alpha \|u - u_h\|_{1,\Omega}^2 &\leq a(u - u_h, u - u_h) \\ &= a(u - u_h, (u - v_h) + (v_h - u_h)) \\ &\leq C \|u - u_h\|_{1,\Omega} \|u - v_h\|_{1,\Omega} + a(u - u_h, v_h - u_h), \end{aligned}$$

where $\alpha > 0$ is the ellipticity constant of $a(\cdot, \cdot)$. Then we use the Young inequality:

$$\frac{\alpha}{2} \|u - u_h\|_{1,\Omega}^2 \leq \frac{C^2}{2\alpha} \|u - v_h\|_{1,\Omega}^2 + a(u, v_h - u_h) - a(u_h, v_h - u_h). \quad (5)$$

Since v_h belongs to K_h , and since u_h is the solution to (3), there holds:

$$-a(u_h, v_h - u_h) \leq -L(v_h - u_h).$$

As well, since u is also solution to (2), we get after using $u \in H^2(\Omega)$ and the Green formula:

$$a(u, v_h - u_h) = \int_{\Omega} (-\Delta u)(v_h - u_h) = \int_{\Omega} (-\Delta u - f)(v_h - u_h) + \int_{\Omega} (f)(v_h - u_h).$$

In other terms:

$$a(u, v_h - u_h) = \int_{\Omega} (-\Delta u - f)(v_h - u_h) + L(v_h - u_h).$$

We take into account the previous considerations and then obtain from (5):

$$\frac{\alpha}{2} \|u - u_h\|_{1,\Omega}^2 \leq \frac{C^2}{2\alpha} \|u - v_h\|_{1,\Omega}^2 + \int_{\Omega} (-\Delta u - f)(v_h - u_h). \quad (6)$$

So we need to bound the last term, that we rewrite as follows:

$$\int_{\Omega} (-\Delta u - f)(v_h - u_h) = \int_{\Omega} (-\Delta u - f)((v_h - u) + (u - u_h))$$

Let us consider the first part of the term above, and we apply Cauchy-Schwartz inequality:

$$\int_{\Omega} (-\Delta u - f)(v_h - u) \leq \|\Delta u + f\|_{0,\Omega} \|v_h - u\|_{0,\Omega}.$$

There remains

$$\int_{\Omega} (-\Delta u - f)(u - u_h) = \int_{\Omega} (-\Delta u - f)((u - \Psi) + (\Psi - u_h)).$$

We apply the complementarity condition from (2) and get:

$$\int_{\Omega} (-\Delta u - f)(u - u_h) = \int_{\Omega} (-\Delta u - f)(\Psi - u_h).$$

We take $v \in K$, and there reminds to rewrite $\Psi - u_h$ as $\Psi - u_h = \Psi - v + v - u_h$:

$$\int_{\Omega} (-\Delta u - f)(u - u_h) = \int_{\Omega} (-\Delta u - f)(\Psi - v) + \int_{\Omega} (-\Delta u - f)(v - u_h).$$

With the property $\Psi - v \geq 0$ combined with $(-\Delta u - f) \leq 0$ (see (2)), we deduce:

$$\int_{\Omega} (-\Delta u - f)(\Psi - v) \leq 0.$$

As a result, we have the bound:

$$\int_{\Omega} (-\Delta u - f)(u - u_h) \leq \int_{\Omega} (-\Delta u - f)(v - u_h)..$$

We apply the Cauchy-Schwarz inequality above and collect the previous results, in order to get from (6):

$$\frac{\alpha}{2} \|u - u_h\|_{1,\Omega}^2 \leq \frac{C^2}{2\alpha} \|u - v_h\|_{1,\Omega}^2 + \|\Delta u + f\|_{0,\Omega} (\|v_h - u\|_{0,\Omega} + \|v - u_h\|_{0,\Omega}). \quad (7)$$

The abstract bound (4) follows after taking the infimum over K_h and the infimum over K . \square

For another proof, see for instance [15, Theorem 5.1.1]. Now the main result is:

Theorem 4.1 *Let $\Omega \subset \mathbb{R}^n$, $1 \leq n \leq 3$, be an open bounded polytope, connected and with Lipschitz boundary. Let $u \in K \cap H^2(\Omega)$ and $u_h \in K_h$ be the solutions to problems (1) and (3), respectively. There holds*

$$\|u - u_h\|_{1,\Omega} \leq Ch \|u\|_{2,\Omega}. \quad (8)$$

Proof: From the above Falk's Lemma, see (4), and since $\mathcal{I}_h u \in K_h$, where \mathcal{I}_h is the Lagrange interpolation operator mapping onto V_h , we get

$$\|u - u_h\|_{1,\Omega}^2 \leq C \left(\|u - \mathcal{I}_h u\|_{1,\Omega}^2 + \|u - \mathcal{I}_h u\|_{0,\Omega} + \inf_{v \in K} \|v - u_h\|_{0,\Omega} \right). \quad (9)$$

Remark that, if $K_h \subset K$ (if Ψ is concave for instance), the first two terms in (9) are bounded by Ch^2 and the second infimum disappears. So bound (8) holds. However, in the general case K_h is nonconforming and we need to bound the last term.

Still from standard approximation bounds, the first two terms in (9) are bounded by Ch^2 . To bound the infimum on K , we set $v := \min(u_h, \Psi)$. First we assess that $v \in H^1(\Omega)$. Indeed there holds $\Psi \in H^1(\Omega)$, and the minimum of two functions in $H^1(\Omega)$ remains in $H^1(\Omega)$, see, e.g., [33, Lemma 1.1] or [31]. Moreover, because of the assumption $0 \leq \Psi$, we have $v = 0$ on $\partial\Omega$ and $v \leq \Psi$, which guarantees that $v \in K$. Now set

$$S_h := \{x \in \Omega, \Psi(x) < u_h(x)\}.$$

If $x \notin S_h$, then $v(x) = u_h(x)$ by definition. So

$$\|v - u_h\|_{0,\Omega}^2 = \int_{\Omega} (v - u_h)^2 = \int_{S_h} (\Psi - u_h)^2.$$

Since $u_h \in K_h$ we have $u_h(a) \leq \Psi(a) = \mathcal{I}_h \Psi(a), \forall a \in \mathcal{N}_h$. So $\mathcal{I}_h \Psi - u_h \geq 0$ in Ω . Let us take $x \in S_h$, and let us bound:

$$\begin{aligned} 0 < |(u_h - \Psi)(x)| &= (u_h - \Psi)(x) \\ &= (u_h - \mathcal{I}_h \Psi)(x) + (\mathcal{I}_h \Psi - \Psi)(x) \\ &\leq (\mathcal{I}_h \Psi - \Psi)(x). \end{aligned}$$

Therefore there holds

$$\|v - u_h\|_{0,\Omega}^2 = \int_{S_h} (\Psi - u_h)^2 \leq \int_{S_h} (\mathcal{I}_h \Psi - \Psi)^2 \leq \|\Psi - \mathcal{I}_h \Psi\|_{0,\Omega}^2.$$

Because of the assumption $\Psi \in H^2(\Omega)$, we can bound

$$\|\Psi - \mathcal{I}_h \Psi\|_{0,\Omega} \leq Ch^2 |\Psi|_{2,\Omega}.$$

As a result we obtain

$$\inf_{v \in K} \|v - u_h\|_{0,\Omega} \leq Ch^2$$

and the final bound (8) follows. \square

5 A Nitsche finite element approximation

Though the discrete problem (3) is quite simple, a global variational inequality on the whole domain is cumbersome for practical implementation, and, for this reason, alternative formulations based on Lagrange multipliers, augmented lagrangian or penalty are preferred. Here we suggest an alternative formulation inspired from Nitsche's method [32] and that have been proposed and studied recently in [5, 8, 27]. Following [5], the Kuhn-Tucker conditions described in (2) can equivalently be reformulated as

$$\Delta u + f = \gamma [(-\Psi + u) + \gamma^{-1}(\Delta u + f)]_+, \quad (10)$$

with γ an arbitrary positive function on the domain Ω , and where $[\cdot]_+$ denotes the positive part operator (defined, for $x \in \mathbb{R}$, by $[x]_+ = \max(0, x)$). In this section, for $k = 1, 2$, we define

$$V_h := \{v_h \in \mathcal{C}(\bar{\Omega}) \cap H_0^1(\Omega) : v_h|_T \in \mathbb{P}_k(T), \forall T \in T_h\}.$$

As a result, the finite element space V_h can be made either of piecewise linear ($k = 1$) or piecewise quadratic ($k = 2$) functions.

Each simplex T of the mesh T_h is supposed to be closed, and we denote by \mathring{T} the interior of T . We define a piecewise polynomial discrete Laplacian as follows, for every v_h in V_h , and every simplex $T \in T_h$:

$$(\Delta_h v_h)|_{\mathring{T}} := \Delta(v_h|_{\mathring{T}}).$$

The value of $\Delta_h v_h$ on the facets of the mesh is of no importance, and can be set in practice to 0, for instance. Observe that, for $k = 1$, $\Delta_h v_h = 0$. A Nitsche-type method for the discretization of the obstacle problem (2) reads: find $u_h \in V_h$ such that

$$a(u_h, v_h) + \int_{\Omega} \gamma_h [-\Psi + u_h + \gamma_h^{-1}(\Delta_h u_h + f)]_+ v_h = L(v_h), \quad (11)$$

for all $v_h \in V_h$. Above the function γ_h is defined cell-wise as follows:

$$\gamma_h|_{\mathring{T}} := \frac{\gamma_0}{h_T^2},$$

where $\gamma_0 > 0$ is the Nitsche parameter. Again, the value of γ_h on the facets of the mesh is of no importance, and can be set in practice to 0, for instance.

Standard arguments allow to assess, that, for γ_0 large enough, Problem (11) is well-posed, with optimal a priori error bounds that can be established. See [5, 8, 27] for more details. In practice, formulation (11) is of interest because it transforms a constrained problem with inequalities to an unconstrained problem, without regularization and without extra unknowns. The resulting problem is easy to solve using for instance a semi-smooth Newton method. Indeed, at iteration $n \geq 1$ of the semi-smooth Newton method, one simply solves the tangent problem: find $\delta u_h \in V_h$ such that

$$a(\delta u_h, v_h) + \int_{\Omega} \gamma_h H(-\Psi + u_h^n + \gamma_h^{-1}(\Delta_h u_h^n + f))(\delta u_h + \gamma_h^{-1} \Delta_h \delta u_h) v_h = R(u_h^n, v_h), \quad (12)$$

for $v_h \in V_h$, where $H(\cdot)$ is the Heaviside function (defined, for $x \in \mathbb{R}$, by $[x]_+ = \max(0, x)/|x|$, if $x \neq 0$, and multivalued at $x = 0$) and where u_h^n is the approximation of u^h at the Newton iteration number n . The expression of the residual above is given by:

$$R(u_h; v_h) = L(v_h) - a(u_h, v_h) - \int_{\Omega} (\gamma_h [-\Psi + u_h + \gamma_h^{-1}(\Delta_h u_h + f)]_+ v_h, \quad (13)$$

for $u_h, v_h \in V_h$.

6 Final comments

A few remarks to conclude this note.

1. Even for the direct approximation of the variational inequality, there can be many alternatives to the choice we presented, that consists in imposing the constraint $u \leq \Psi$ nodal-wise. For instance, one can impose an integral constraint on each simplex on the mesh.
2. Convergence results of the finite element approximation without rates can be found for instance in [22].
3. Conversely to what happens for linear boundary value problems, error estimates in the $L^2(\Omega)$ -norm of order $\mathcal{O}(h^2)$ for the finite element approximation (3) can not be expected in general. See the reference [14] for some counter-examples.
4. Recent results of convergence in the $L^\infty(\Omega)$ -norm can be found in [13].
5. Some classical a posteriori error estimates of the obstacle problem have been studied in [1, 7, 34], and this is still a very active field. See for instance the recent works [18, 21, 24, 25, 27, 28].
6. The obstacle problem is involved in many applications, such as membrane contact, elastoplastic torsion or cavitation modelling, just to mention of few ones. For a recent illustration, see for instance [25], with some numerical experiments done with scikit-fem [26].
7. The techniques presented in this note can be adapted for the elastoplastic torsion problem with a constant source term, since it can be reformulated as an obstacle problem, where the function Ψ represents the distance to the boundary [3]. Note however that the assumptions here are stronger (the distance function does not belongs to $H^2(\Omega)$ in general), so the analysis needs to be adapted. See [8, 10] for the details, as well as [9] when the source term is non-constant. For the adaptation of Nitsche's method in the context of variational inequalities, see for instance [11, 12] and references therein.
8. Recent works have concerned the study of high order methods / polytopal methods such as the Hybrid High Order method (HHO) or the Virtual Element Method (VEM) in the context of variational inequalities, see for instance [16, 17] (HHO) or [35] (VEM) for the obstacle problem. For Nitsche's method, see for instance [6] (HHO) or [30] (VEM).

References

- [1] D. BRAESS, *A posteriori error estimators for obstacle problems – another look*, Numer. Math., 101 (2005), pp. 415–421.
- [2] H. BREZIS, *Functional analysis, Sobolev spaces and partial differential equations*, Universitext, New York, NY: Springer, 2011.
- [3] H. BRÉZIS AND M. SIBONY, *Équivalence de deux inéquations variationnelles et applications*, Arch. Rational Mech. Anal., 41 (1971), pp. 254–265.
- [4] H. BRÉZIS AND G. STAMPACCHIA, *Sur la régularité de la solution d'inéquations elliptiques*, Bull. Soc. Math. Fr., 96 (1968), pp. 153–180.
- [5] E. BURMAN, P. HANSBO, M. G. LARSON, AND R. STENBERG, *Galerkin least squares finite element method for the obstacle problem*, Comput. Meth. Appl. Mech. Engrg., 313 (2017), pp. 362–374.
- [6] K. L. CASCAVITA, F. CHOULY, AND A. ERN, *Hybrid high-order discretizations combined with Nitsche's method for Dirichlet and Signorini boundary conditions*, IMA J. Numer. Anal., 40 (2020), pp. 2189–2226.
- [7] Z. CHEN AND R. H. NOCHETTO, *Residual type a posteriori error estimates for elliptic obstacle problems*, Numer. Math., 84 (2000), pp. 527–548.

- [8] F. CHOULY, T. GUSTAFSSON, AND P. HILD, *A Nitsche method for the elastoplastic torsion problem*, ESAIM, Math. Model. Numer. Anal., 57 (2023), pp. 1731–1746.
- [9] ———, *Finite element approximation of penalized elastoplastic torsion problem with nonconstant source term*. hal-04675762, <https://hal.science/hal-04675762>, Aug. 2024.
- [10] F. CHOULY AND P. HILD, *On a finite element approximation for the elastoplastic torsion problem*, Appl. Math. Lett., 132 (2022), pp. Paper No. 108191, 6.
- [11] F. CHOULY, P. HILD, AND Y. RENARD, *Symmetric and non-symmetric variants of Nitsche’s method for contact problems in elasticity: theory and numerical experiments*, Math. Comp., 84 (2015), pp. 1089–1112.
- [12] ———, *Finite element approximation of contact and friction in elasticity*, vol. 48 of Adv. Mech. Math., Cham: Birkhäuser, 2023.
- [13] C. CHRISTOF, *L^∞ -error estimates for the obstacle problem revisited*, Calcolo, 54 (2017), pp. 1243–1264.
- [14] C. CHRISTOF AND C. MEYER, *A note on a priori L^p -error estimates for the obstacle problem*, Numer. Math., 139 (2018), pp. 27–45.
- [15] P. G. CIARLET, *The finite element method for elliptic problems*, vol. 4 of Stud. Math. Appl., Elsevier, Amsterdam, 1978.
- [16] M. CICUTTIN, A. ERN, AND T. GUDI, *Hybrid high-order methods for the elliptic obstacle problem*, J. Sci. Comput., 83 (2020), pp. Paper No. 8, 18.
- [17] J. DABAGHI AND G. DELAY, *A unified framework for high-order numerical discretizations of variational inequalities*, Comput. Math. Appl., 92 (2021), pp. 62–75.
- [18] J. DABAGHI, V. MARTIN, AND M. VOHRALÍK, *Adaptive inexact semismooth Newton methods for the contact problem between two membranes*, J. Sci. Comput., 84 (2020), p. 32. Id/No 28.
- [19] G. DUVAUT AND J.-L. LIONS, *Inequalities in mechanics and physics*, vol. 219 of Grundlehren der Mathematischen Wissenschaften, Springer-Verlag, Berlin-New York, 1976.
- [20] R. S. FALK, *Error estimates for the approximation of a class of variational inequalities*, Math. Comput., 28 (1974), pp. 963–971.
- [21] S. GADDAM AND T. GUDI, *Inhomogeneous Dirichlet boundary condition in the a posteriori error control of the obstacle problem*, Comput. Math. Appl., 75 (2018), pp. 2311–2327.
- [22] R. GLOWINSKI, *Lectures on numerical methods for nonlinear variational problems*, vol. 65 of Tata Institute of Fundamental Research Lectures on Mathematics and Physics, Tata Institute of Fundamental Research, Bombay; Springer-Verlag, Berlin-New York, 1980.
- [23] R. GLOWINSKI, J.-L. LIONS, AND R. TRÉMOLIÈRES, *Numerical analysis of variational inequalities*, vol. 8 of Studies in Mathematics and its Applications, North-Holland Publishing Co., Amsterdam-New York, 1981.
- [24] T. GUDI AND K. PORWAL, *A reliable residual based a posteriori error estimator for a quadratic finite element method for the elliptic obstacle problem*, Comput. Methods Appl. Math., 15 (2015), pp. 145–160.
- [25] T. GUSTAFSSON, *Adaptive finite elements for obstacle problems*, Advances in Applied Mechanics (AAMS), 58 (2024), pp. 205–243. Chapter 5, <https://doi.org/10.1016/bs.aams.2024.03.004>.
- [26] T. GUSTAFSSON AND G. D. MCBAIN, *scikit-fem: A Python package for finite element assembly*, J. Open Source Softw., 5 (2020), p. 2369.
- [27] T. GUSTAFSSON, R. STENBERG, AND J. VIDEMAN, *Mixed and stabilized finite element methods for the obstacle problem*, SIAM J. Numer. Anal., 55 (2017), pp. 2718–2744.

- [28] R. KHANDELWAL, K. PORWAL, AND R. SINGLA, *Supremum-norm a posteriori error control of quadratic discontinuous Galerkin methods for the obstacle problem*, *Comput. Math. Appl.*, 137 (2023), pp. 147–171.
- [29] D. KINDERLEHRER AND G. STAMPACCHIA, *An introduction to variational inequalities and their applications*, vol. 88 of *Pure and Applied Mathematics*, Academic Press, Inc., New York-London, 1980.
- [30] M. LAAZIRI AND R. MASSON, *VEM-Nitsche fully discrete polytopal scheme for frictionless contact-mechanics*. hal-04571942, <https://hal.science/hal-04571942>, May 2024.
- [31] H. LEWY AND G. STAMPACCHIA, *On the regularity of the solution of a variational inequality*, *Commun. Pure Appl. Math.*, 22 (1969), pp. 153–188.
- [32] J. NITSCHKE, *Über ein Variationsprinzip zur Lösung von Dirichlet-Problemen bei Verwendung von Teilräumen, die keinen Randbedingungen unterworfen sind. (On a variational principle for solving Dirichlet problems less boundary conditions using subspaces)*, *Abh. Math. Semin. Univ. Hamb.*, 36 (1971), pp. 9–15.
- [33] G. STAMPACCHIA, *Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus*, *Ann. Inst. Fourier (Grenoble)*, 15 (1965), pp. 189–258.
- [34] A. VEESER, *Efficient and reliable a posteriori error estimators for elliptic obstacle problems*, *SIAM J. Numer. Anal.*, 39 (2001), pp. 146–167.
- [35] F. WANG AND H. WEI, *Virtual element methods for the obstacle problem*, *IMA J. Numer. Anal.*, 40 (2020), pp. 708–728.