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Sound-structure interaction: acoustic radiation and transmission of infinite panels

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MASTER “Science in Acoustics”

“Sound-structure interaction” MOD

Acoustic transmission through infinite panels



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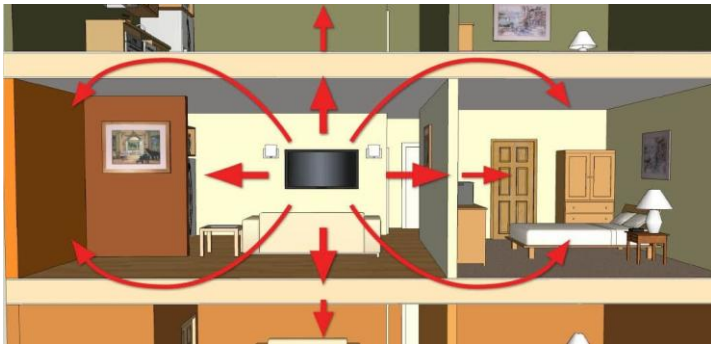


Introduction

Minimizing sound transmission through structures is a common problem in many industries.

Few examples:

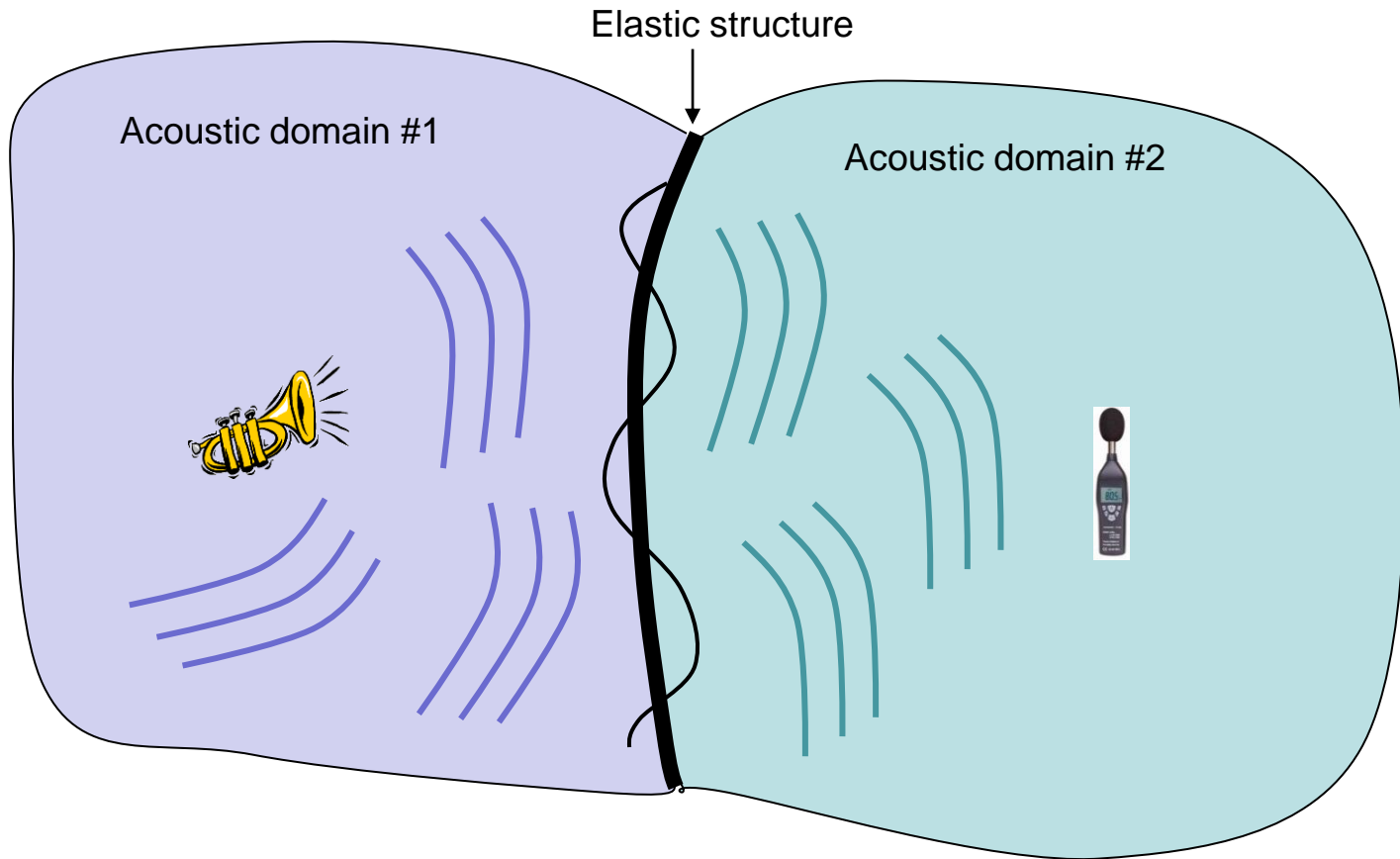
- Sound from an adjacent room in a building propagating through walls to other rooms;
- Transmission of engine noise to the cabin via the fuselage;
- Transmission of machine noise to seawater through the pressurized hull of a submarine;
- Engine noise transmitted through a car's firewall into the passenger compartment;
- Rocket pressure pulses transmitted through the walls of a launch vehicle inside a payload.



Sound transmission between adjacent rooms
(Internet)



The Dash-8 fuselage under experimental investigations
(A. G REWAL et al., J. INTEL. MAT. SYST. STR., June 2000)



Sound transmission through an elastic structure

Studied case in the following (to highlight the main physical phenomena):

- Elastic structure → Flat thin panel of infinite extent
- Acoustic domains → Semi-infinite acoustic domain (i.e. no reflecting boundary condition excepted the flat panel)

Outline

- I – Description of the problem of interest and mathematical formulation
- II – Evaluation of the acoustic transmission for an oblique plane wave
- III – Analysis of the main parameters controlling the transmission loss
- IV - Evaluation of the acoustic transmission for an acoustic diffuse field
- V – Extension to the infinite double panel

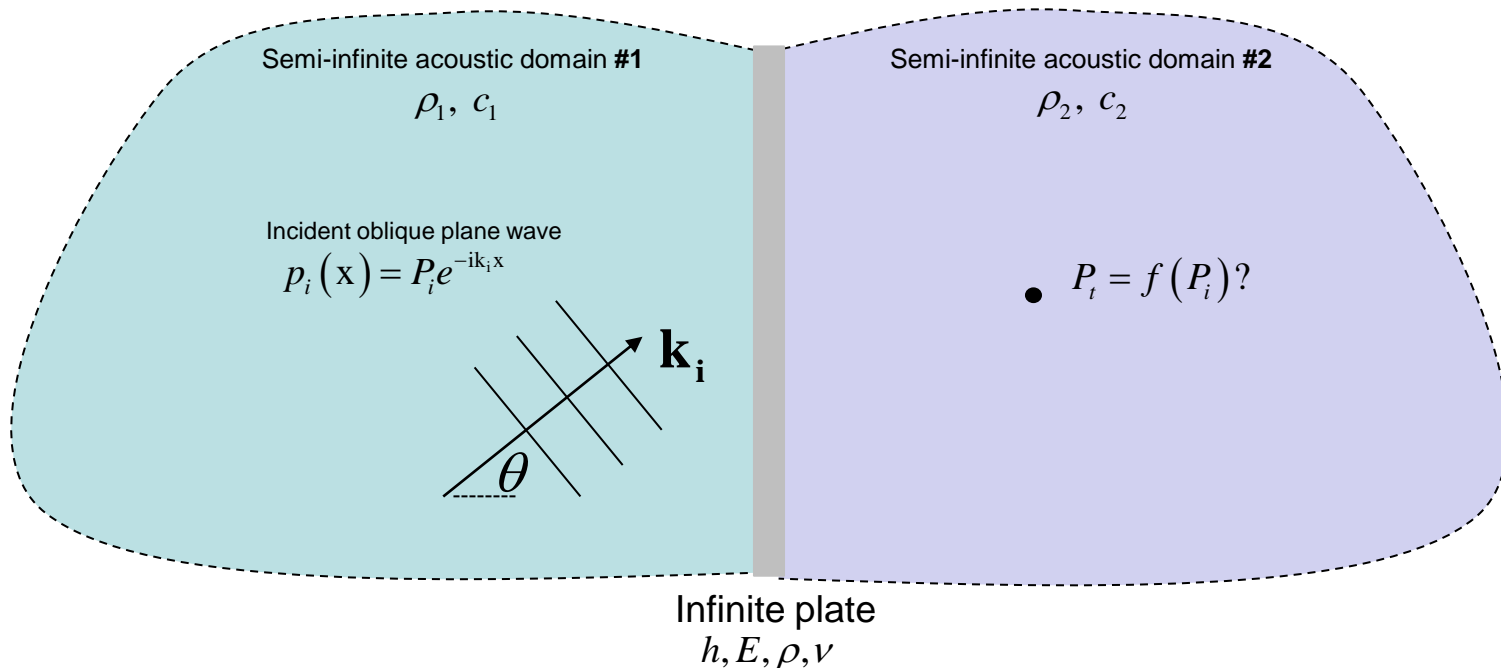
Some bibliographical references:

- F.J. Fahy, Sound and structural vibration: Radiation, Transmission and Response, Academic Press, 1985.
- M.C. Junger, D. Feit, Sound, Structure and their Interaction, MIT Press, 1986.
- C. Lesueur, J.L. Guyader, Rayonnement acoustique des structures, Eyrolles, 1988.
- C. Hopkins, Sound Insulation, Butterworth-Heinemann, 2007

I – Description of the problem of interest and mathematical formulation

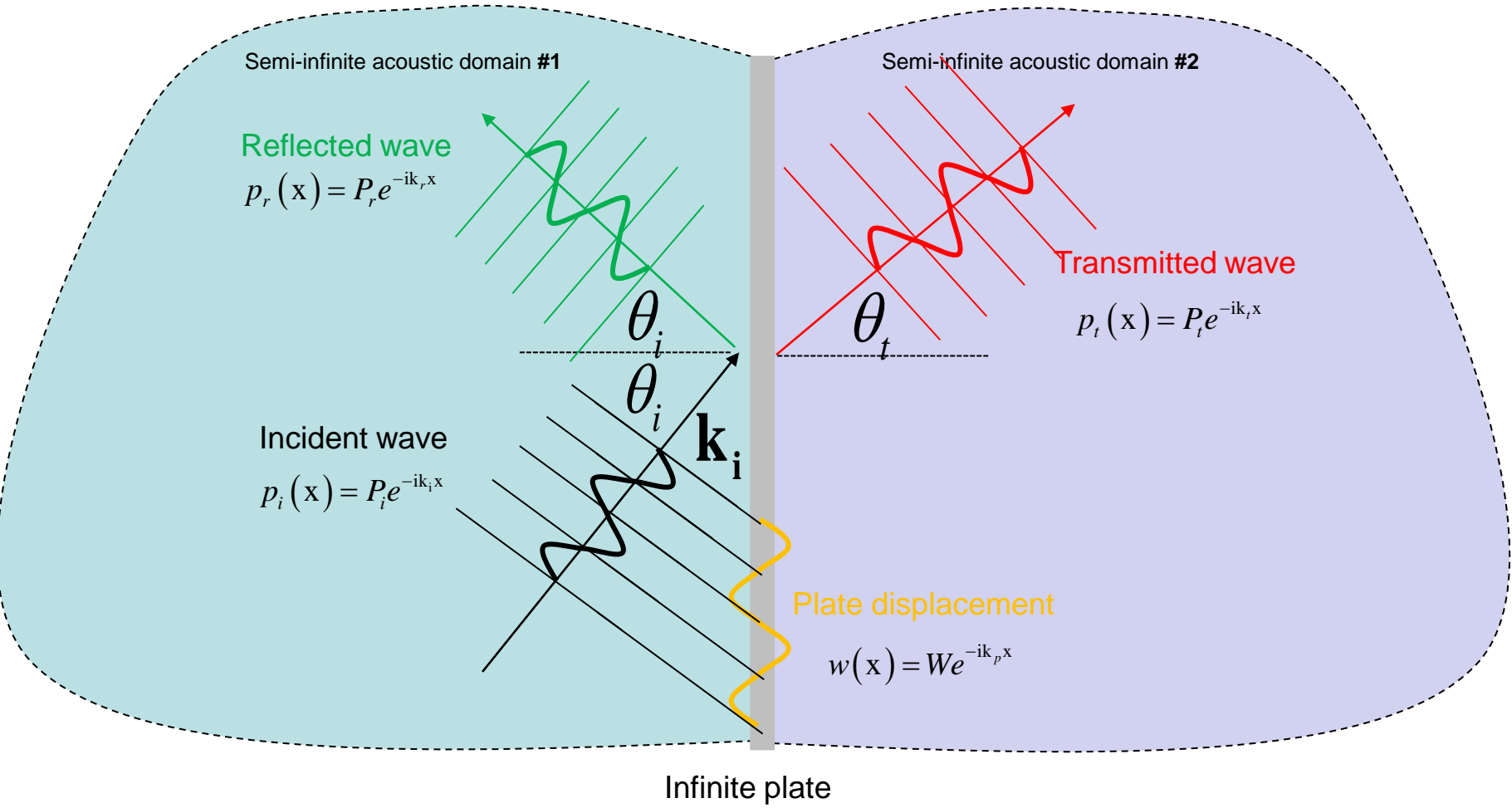
Infinite thin plate coupled to a semi-infinite acoustic domain on each side and impacted by an oblique plane wave.

- Ideal case for analytical developments
- Give physical insight for finite panels when the acoustic wavelength is smaller than the plate size (i.e. not in low frequency).
- 2D case is considered for simplification



I – Description of the problem of interest and mathematical formulation

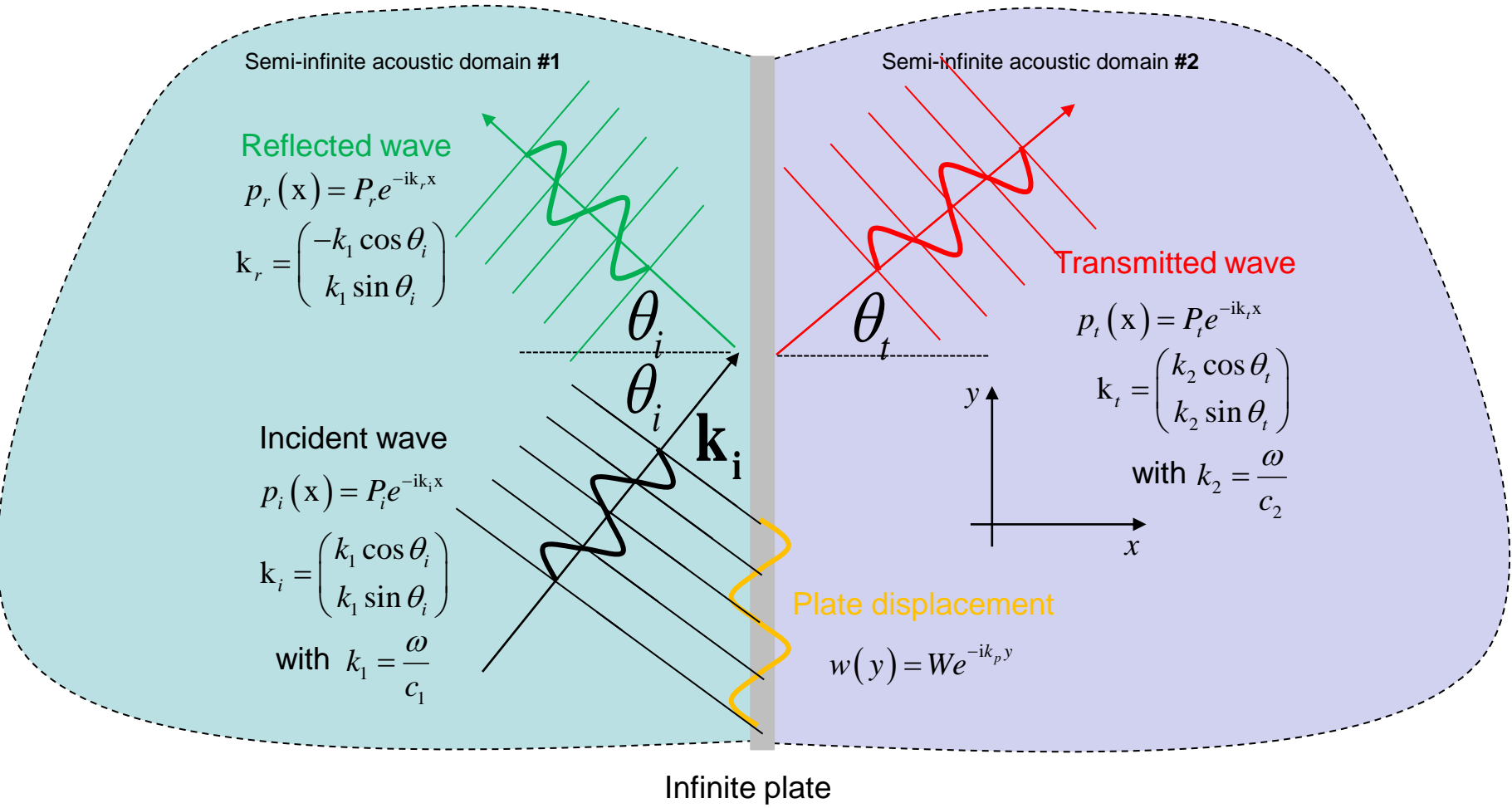
Definition of different plane waves



Time dependence: $e^{i\omega t}$

I – Description of the problem of interest and mathematical formulation

Definition of wavevectors



I – Description of the problem of interest and mathematical formulation

Problem to be solved

No reflecting boundary condition

$$v_x(0, y) = i\omega w(y)$$

Semi-infinite acoustic domain #1

Semi-infinite acoustic domain #2

Reflected wave

$$p_r(x) = P_r e^{-ik_r x}$$

$$\mathbf{k}_r = \begin{pmatrix} -k_1 \cos \theta_i \\ k_1 \sin \theta_i \end{pmatrix}$$

Incident wave

$$p_i(x) = P_i e^{-ik_i x}$$

$$\mathbf{k}_i = \begin{pmatrix} k_1 \cos \theta_i \\ k_1 \sin \theta_i \end{pmatrix}$$

$$\Delta p_1(x, y) + k_1^2 p_1(x, y) = 0$$

with $p_1 = p_i + p_r$

$$\frac{\partial p_2}{\partial x} = -\rho_1(i\omega)v_x$$

Transmitted wave

$$p_t(x) = P_t e^{-ik_t x}$$

$$\mathbf{k}_t = \begin{pmatrix} k_2 \cos \theta_t \\ k_2 \sin \theta_t \end{pmatrix}$$

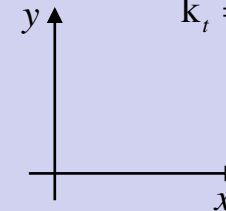


Plate displacement

$$w(y) = W e^{-ik_p y}$$

$$\Delta p_2(x, y) + k_2^2 p_2(x, y) = 0$$

with $p_2 = p_t$

Infinite plate

$$D \frac{\partial^4 w}{\partial x^4}(y) - \omega^2 \rho h w(y) = p_1(0, y) - p_2(0, y)$$

II – Evaluation of the acoustic transmission for an oblique plane wave

- Condition at the interface between the acoustic medium #1 and the panel:

$$\begin{cases} \frac{\partial p_1}{\partial x}(0, y) = \rho_1 \omega^2 w(y) \\ p_1(x, y) = P_i e^{-i(k_1 \cos \theta_i x + k_1 \sin \theta_i y)} + P_r e^{-i(-k_1 \cos \theta_i x + k_1 \sin \theta_i y)} \\ w(y) = W e^{-ik_p y} \end{cases}$$

$$\Rightarrow -ik_1 \cos \theta_i (P_i - P_r) e^{-ik_1 \sin \theta_i y} = \rho_1 \omega^2 W e^{-ik_p y}, \quad \forall y \in \mathbb{R}$$

$$\Rightarrow \begin{cases} k_1 \sin \theta_i = k_p \\ -ik_1 \cos \theta_i (P_i - P_r) = \rho_1 \omega^2 W \end{cases}$$

- Condition at the interface between the acoustic medium #2 and the panel:

$$\begin{cases} \frac{\partial p_2}{\partial x}(0, y) = \rho_2 \omega^2 w(y) \\ p_2(x, y) = P_t e^{-i(k_2 \cos \theta_t x + k_2 \sin \theta_t y)} \\ w(y) = W e^{-ik_p y} \end{cases}$$

$$\Rightarrow -ik_2 \cos \theta_t P_t e^{-ik_2 \sin \theta_t y} = \rho_2 \omega^2 W e^{-ik_p y}, \quad \forall y \in \mathbb{R}$$

$$\Rightarrow \begin{cases} k_2 \sin \theta_t = k_p \\ -ik_2 \cos \theta_t P_t = \rho_2 \omega^2 W \end{cases}$$

➔ Condition between the angle of the incident wave and the angle of the transmitted wave:

$$k_1 \sin \theta_i = k_p = k_2 \sin \theta_t$$

$$\Rightarrow \frac{\sin \theta_i}{c_1} = \frac{\sin \theta_t}{c_2} \quad \rightarrow \text{Equivalent to the Snell-Descartes law in optics}$$

$$\Rightarrow \sin \theta_t = \frac{c_2}{c_1} \sin \theta_i \Rightarrow \theta_t = \arcsin\left(\frac{c_2}{c_1} \sin \theta_i\right) \text{ if } \frac{c_2}{c_1} \sin \theta_i \in [-1,1]$$

On the other hand, it can be written:

$$k_2^2 \cos^2 \theta_t = k_2^2 (1 - \sin^2 \theta_t) = k_2^2 - k_1^2 \sin^2 \theta_i = \frac{\omega^2}{c_2^2} - \frac{\omega^2}{c_1^2} \sin^2 \theta_i \Rightarrow k_2 \cos \theta_t = \begin{cases} \omega \sqrt{\frac{1}{c_2^2} - \frac{\sin^2 \theta_i}{c_1^2}} & \text{if } \frac{c_2}{c_1} \sin \theta_i \in [-1,1] \\ i\omega \sqrt{\frac{\sin^2 \theta_i}{c_1^2} - \frac{1}{c_2^2}} & \text{otherwise.} \end{cases}$$

$k_2 \cos \theta_t$ corresponds to the projection on the x-axis of the wavevector of the transmitted wave. When it is real, the transmitted wave is propagative and when it is purely imaginary, the transmitted wave is evanescent.

Three cases:

- If $c_1 = c_2$ (e.g. same fluid), the transmitted wave is always propagative and $\theta_t = \theta_i$.

- If $c_1 > c_2$ (e.g. water ➔ air), the transmitted wave is always propagative and $\theta_t = \arcsin\left(\frac{c_2}{c_1} \sin \theta_i\right)$.

- If $c_1 < c_2$ (e.g. air ➔ water), the transmitted wave is propagative when $\theta_i < \theta_t = \arcsin\left(\frac{c_1}{c_2}\right)$ and is evanescent when $\theta_i > \theta_t$. For air – water transmission, $\theta_t \approx 13^\circ$.

- Plate equation of motion + wave decomposition:

$$\begin{cases} D \frac{\partial^4 w}{\partial x^4}(y) - \omega^2 \rho h w(y) = p_1(0, y) - p_2(0, y), \forall y \in \mathbb{R} \\ p_1(x, y) = P_i e^{-i(k_1 \cos \theta_i x + k_1 \sin \theta_i y)} + P_r e^{-i(-k_1 \cos \theta_i x + k_1 \sin \theta_i y)} \\ p_2(x, y) = P_t e^{-i(k_2 \cos \theta_t x + k_2 \sin \theta_t y)} \\ w(y) = W e^{-i k_p y} \end{cases}$$

$$\Rightarrow (D k_p^4 - \omega^2 \rho h) W e^{-i k_p y} = (P_i + P_r) e^{-i(k_1 \sin \theta_i y)} - P_t e^{-i(k_2 \sin \theta_t y)}, \forall y \in \mathbb{R}$$

$$\Rightarrow (D k_p^4 - \omega^2 \rho h) W = P_i + P_r - P_t \text{ as } k_1 \sin \theta_i = k_p = k_2 \sin \theta_t$$

- Synthesis: continuity conditions at the interface between the acoustic mediums and plate + plate equation of motion

$$\begin{cases} -i k_1 \cos \theta_i (P_i - P_r) = \rho_1 \omega^2 W \\ -i k_2 \cos \theta_t P_t = \rho_2 \omega^2 W \\ [D(k_1 \sin \theta_i)^4 - \omega^2 \rho h] W = P_i + P_r - P_t \end{cases} \Leftrightarrow \begin{cases} P_r = P_i - \frac{\rho_1 \omega^2}{i k_1 \cos \theta_i} W \\ P_t = -\frac{\rho_2 \omega^2 W}{i k_2 \cos \theta_t} \\ W = \frac{2 P_i}{D(k_1 \sin \theta_i)^4 - \omega^2 \rho h - \frac{\rho_1 \omega^2}{i k_1 \cos \theta_i} - \frac{\rho_2 \omega^2}{i k_2 \cos \theta_t}} \end{cases}$$

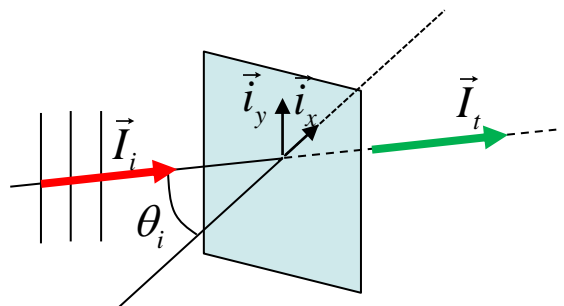
Particular case of same fluid on both sides:

$$\begin{cases} \rho_1 = \rho_2 = \rho_0 \\ c_1 = c_2 = c_0 \end{cases} \Rightarrow \begin{cases} k_1 = k_2 = \frac{\omega}{c_0} \\ \theta_t = \theta_i \end{cases} \Rightarrow \begin{cases} P_r = P_i + i \frac{\rho_0 c_0 \omega}{\cos \theta_i} W \\ P_t = i \frac{\rho_0 c_0 \omega W}{\cos \theta_i} \\ W = \frac{2P_i}{D(k_0 \sin \theta_i)^4 - \omega^2 \rho h + i \frac{2\rho_0 c_0 \omega}{\cos \theta_i}} \end{cases}$$

Sound power transmission coefficient:

$$\tau(\omega, \theta_i) = \frac{I_t^n(\omega, \theta_i)}{I_i^n(\omega, \theta_i)}$$

where I_i^n, I_t^n are the acoustic intensity, normal to the plate, of the incident and of the transmitted plane waves, respectively.



$$I_i^n = \mathbf{I}_i \cdot \mathbf{i}_x = \frac{1}{2} \operatorname{Re} \{ p_i \mathbf{v}_i^* \cdot \mathbf{i}_x \} = \frac{1}{2} \operatorname{Re} \{ p_i v_{x,i}^* \} \quad \text{as } \mathbf{v}_i = \begin{pmatrix} v_{x,i} \\ v_{u,i} \end{pmatrix}$$

$$I_t^n = \frac{1}{2} \operatorname{Re} \{ p_t v_{x,t}^* \}$$

- Evaluation of the acoustic intensity of the incident plane wave, normal to the plate:

$$\left\{ \begin{array}{l} p_i(x, y) = P_i e^{-i(k_0 \cos \theta_i x + k_0 \sin \theta_i y)} \\ v_{x,i}(x, y) = \frac{i}{\rho_0 \omega} \frac{\partial p_i}{\partial x}(x, y) = \frac{P_i k_0 \cos \theta_i}{\rho_0 \omega} e^{-i(k_0 \cos \theta_i x + k_0 \sin \theta_i y)} \end{array} \right. \Rightarrow I_i^n = \frac{|P_i|^2 \cos \theta_i}{2 \rho_0 c_0} \quad (\text{independent on the position})$$

- Evaluation of the acoustic intensity of the transmitted plane wave, normal to the plate:

$$\left\{ \begin{array}{l} p_t(x, y) = P_t e^{-i(k_0 \cos \theta_i x + k_0 \sin \theta_i y)} \\ v_{x,t}(x, y) = \frac{P_t k_0 \cos \theta_i}{\rho_0 \omega} e^{-i(k_0 \cos \theta_i x + k_0 \sin \theta_i y)} \end{array} \right. \text{with } P_t = i \frac{2 P_i \rho_0 c_0 \omega}{\cos \theta_i \left[D(k_0 \sin \theta_i)^4 - \omega^2 \rho h + i \frac{2 \rho_0 c_0 \omega}{\cos \theta_i} \right]}$$

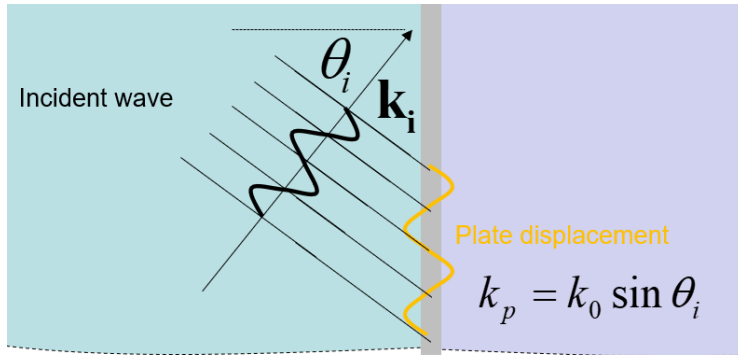
$$\Rightarrow I_t^n = \frac{|P_t|^2 \cos \theta_i}{2 \rho_0 c_0} = \frac{2 |P_i|^2 \rho_0 c_0 \omega^2}{\cos \theta_i \left[\left(D(k_0 \sin \theta_i)^4 - \omega^2 \rho h \right)^2 + \left(\frac{2 \rho_0 c_0 \omega}{\cos \theta_i} \right)^2 \right]}$$

Sound power transmission coefficient of an infinite plate impacted by a incident plane wave:

$$\tau(\omega, \theta_i) = \frac{4(\rho_0 c_0)^2 \omega^2}{(\cos \theta_i)^2 \left[\left(D(k_0 \sin \theta_i)^4 - \omega^2 \rho h \right)^2 + \left(\frac{2 \rho_0 c_0 \omega}{\cos \theta_i} \right)^2 \right]}$$

Coincident (angular) frequency:

Is there a frequency for which the apparent acoustic wavelength along y corresponds to the plate's bending wavelength?



$$\begin{cases} k_p = k_0 \sin \theta_i \\ k_p = k_f \end{cases} \Rightarrow \exists \omega \in \mathbb{R}^+, k_f = k_0 \sin \theta?$$

As $k_f = \sqrt{\omega \sqrt{\frac{\rho h}{D}}}$ and $k_0 = \frac{\omega}{c_0}$, $k_f = k_0 \sin \theta \Leftrightarrow \sqrt{\omega \sqrt{\frac{\rho h}{D}}} = \frac{\omega}{c_0} \sin \theta \Leftrightarrow \omega = \omega_{\text{coin}} = \frac{c_0^2}{\sin^2 \theta_i} \sqrt{\frac{\rho h}{D}}$

$$f_{\text{coin}} = \frac{\omega_{\text{coin}}}{2\pi} = \frac{c_0^2}{2\pi \sin^2 \theta_i} \sqrt{\frac{\rho h}{D}}$$

The coincidence frequency depends on the incident angle:

- For $\theta_i = \frac{\pi}{2}$, $f_{\text{coin}} = \frac{c_0^2}{2\pi} \sqrt{\frac{\rho h}{D}} = f_c$: the coincident frequency is equal to the critical frequency.
- For $\theta_i = \frac{\pi}{4}$, $f_{\text{coin}} = \frac{c_0^2}{\pi} \sqrt{\frac{\rho h}{D}} = 2f_c$
- For $\theta_i \rightarrow 0$, $f_{\text{coin}} \rightarrow +\infty$

III – Analysis of the main parameters controlling the transmission loss

Transmission loss (or Sound reduction index):

- Expression depending on the plate flexural rigidity

$$TL_{\theta_i}(\omega) = 10 \log_{10} \left(\frac{1}{\tau(\omega, \theta_i)} \right) = 10 \log_{10} \left(\frac{(\cos \theta_i)^2 \left[\left(D(k_0 \sin \theta_i)^4 - \omega^2 \rho h \right)^2 + \left(\frac{2\rho_0 c_0 \omega}{\cos \theta_i} \right)^2 \right]}{4(\rho_0 c_0)^2 \omega^2} \right)$$

\updownarrow
 R_{θ_i}

- Expression depending on the coincident frequency

$$TL_{\theta}(\omega, \theta_i) = 10 \log_{10} \left(1 + \left(\frac{\omega \rho h \cos \theta_i}{2\rho_0 c_0} \left(1 - \frac{\omega^2}{\omega_{coin}^2} \right) \right)^2 \right) \quad \text{with} \quad \omega_{coin} = \frac{c_0^2}{\sin^2 \theta_i} \sqrt{\frac{\rho h}{D}}$$

Warning! This formula permits to represent the main phenomena which control the TL of panels. However, it is not well adapted to predict the TL of finite panels used in practice (in particular for the low frequencies and at the critical frequency).

Exercise:

Let us consider the TL of an infinite steel panel with air on both sides and impacted by an oblique plane wave. It is assumed that the plate has an initial thickness of 4 mm and that the angle of incidence (relative to the plate normal) is 60° .

I – TL of the nominal plate

- a) Calculate the numerical value of the coincident frequency. Evaluate the TL at the coincident frequency. What does it mean?
- b) Plot the TL in function of frequency (log scale) in the audible frequency range [20 Hz – 20 kHz]. Where are the regions where the TL is relatively low?

II – Analysis for frequencies much lower the coincident frequency

- a) Propose an approximate expression for the TL for frequencies well below the coincident frequency.
- b) How does the TL vary with frequency? (in particular, when the frequency is multiplied by 2?) What happens when the frequency tends towards 0?
- c) Plot the TL in function of frequency for different plate thicknesses (i.e. 1 mm, 2 mm, 4 mm). What do you observe? What parameters control the TL?
- d) Plot the TL in function of frequency for different incident angles (i.e. 0° , 30° , 60°) for the original plate.

III – Analysis for frequencies much higher the coincident frequency

- a) Propose an approximate expression for the TL for frequencies well above the coincident frequency.
- b) How does the TL vary with frequency? (in particular, when the frequency is multiplied by 2?)
- c) What parameters control the TL?

IV – Analysis for frequencies close to the coincident frequency

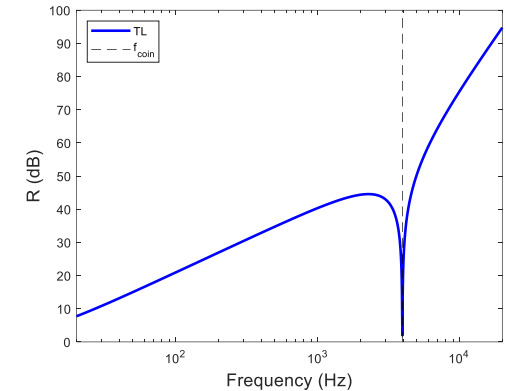
- a) Add structural damping to the model via a complex young modulus and deduce a new expression for the TL.
- b) Plot the TL in function of frequency for different plate damping loss factors. What do you observe?

V – TL for different incident angles

- a) Determine the coincident frequency when the incident angle tends toward 0° and toward 90° .
- b) Plot the TL in function of frequency for different incident angles (0° , 30° , 60° , 89° , 90°)..Analyze the results. .

I – TL of the nominal plate

$$f_{coin} \approx 4 \text{ kHz}$$

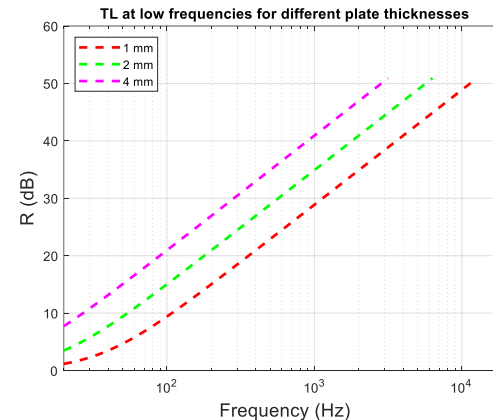
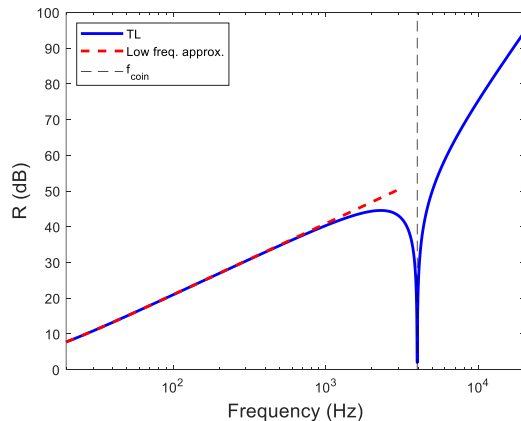


II - Analysis for frequencies much lower the coincident frequency

$$\omega \ll \omega_{coin} \Rightarrow TL_{\theta}(\omega, \theta_i) \approx 10 \log_{10} \left(1 + \left(\frac{\omega \rho h \cos \theta_i}{2 \rho_0 c_0} \right)^2 \right)$$

$$\frac{2 \rho_0 c_0}{\rho h \cos \theta_i} \ll \omega \ll \omega_{coin} \Rightarrow TL_{\theta}(\omega, \theta_i) \approx 20 \log_{10} \left(\frac{\omega \rho h \cos \theta_i}{2 \rho_0 c_0} \right)$$

$$\approx 20 \log_{10}(\omega) + 20 \log_{10}(\rho h) + 20 \log_{10} \left(\frac{\cos \theta_i}{2 \rho_0 c_0} \right)$$



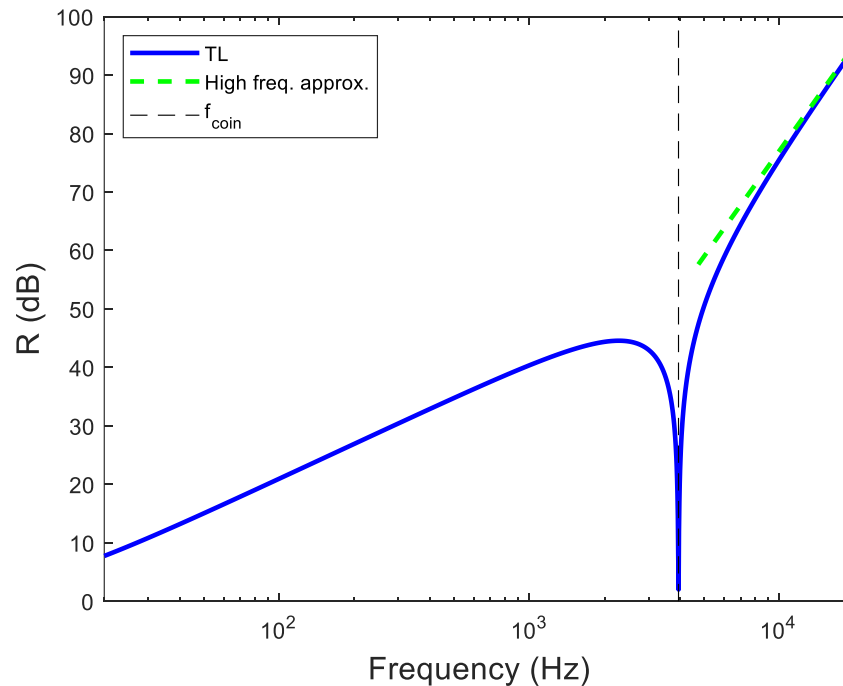
$$20 \log_{10}(2) \approx 6$$

→ 6 dB/octave, Mass control

III – Analysis for frequencies much higher the coincident frequency

$$\omega_{\text{coin}} \ll \omega \Rightarrow TL_{\theta}(\omega, \theta_i) \approx 20 \log_{10} \left(\frac{\omega^3 \rho h \cos \theta_i}{2 \rho_0 c_0 \omega_{\text{coin}}^2} \right) = 20 \log_{10} \left(\frac{\omega^3 D \cos \theta_i \sin^4 \theta_i}{2 \rho_0 c_0^5} \right)$$

$$= 60 \log_{10}(\omega) + 20 \log_{10}(D) + 20 \log_{10} \left(\frac{\cos \theta_i \sin^4 \theta_i}{2 \rho_0 c_0^5} \right)$$



$60 \log_{10}(2) \approx 18 \rightarrow 18 \text{ dB/octave, stiffness control}$

IV - Analysis for frequencies close to the coincident frequency

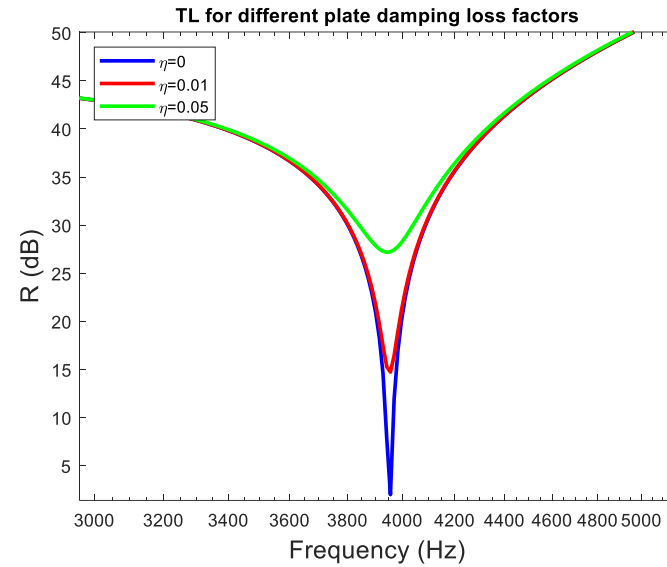
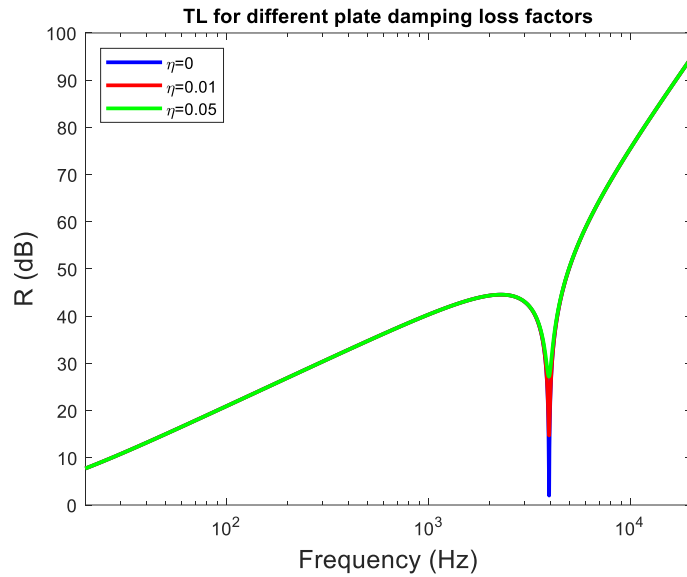
$$I_t^n = \frac{|P_t|^2 \cos \theta_i}{2\rho_0 c_0} \quad \text{with} \quad P_t = i \frac{2P_i \rho_0 c_0 \omega}{\cos \theta_i \left[D \underline{(1+i\eta)} (k_0 \sin \theta_i)^4 - \omega^2 \rho h + i \frac{2\rho_0 c_0 \omega}{\cos \theta_i} \right]}$$

$$P_t = i \frac{2P_i \rho_0 c_0 \omega}{\cos \theta_i \left[D(k_0 \sin \theta_i)^4 - \omega^2 \rho h + i \left(\frac{2\rho_0 c_0 \omega}{\cos \theta_i} + D\eta(k_0 \sin \theta_i)^4 \right) \right]}$$

$$\tau(\omega, \theta_i) = \frac{4(\rho_0 c_0)^2 \omega^2}{(\cos \theta_i)^2 \left[\left(D(k_0 \sin \theta_i)^4 - \omega^2 \rho h \right)^2 + \left(\frac{2\rho_0 c_0 \omega}{\cos \theta_i} + D\eta(k_0 \sin \theta_i)^4 \right)^2 \right]}$$

$$TL_{\theta_i}(\omega) = 10 \log_{10} \left(\frac{(\cos \theta_i)^2 \left[\left(D(k_0 \sin \theta_i)^4 - \omega^2 \rho h \right)^2 + \left(\frac{2\rho_0 c_0 \omega}{\cos \theta_i} + D\eta(k_0 \sin \theta_i)^4 \right)^2 \right]}{4(\rho_0 c_0)^2 \omega^2} \right)$$

$$TL_{\theta_i}(\omega) = 10 \log_{10} \left(\left(1 + \frac{\rho h \eta}{2\rho_0 c_0} \frac{\omega^3}{\omega_{coin}^2} \cos \theta_i \right)^2 + \left(\frac{\omega \rho h \cos \theta_i}{2\rho_0 c_0} \left(1 - \frac{\omega^2}{\omega_{coin}^2} \right) \right)^2 \right)$$

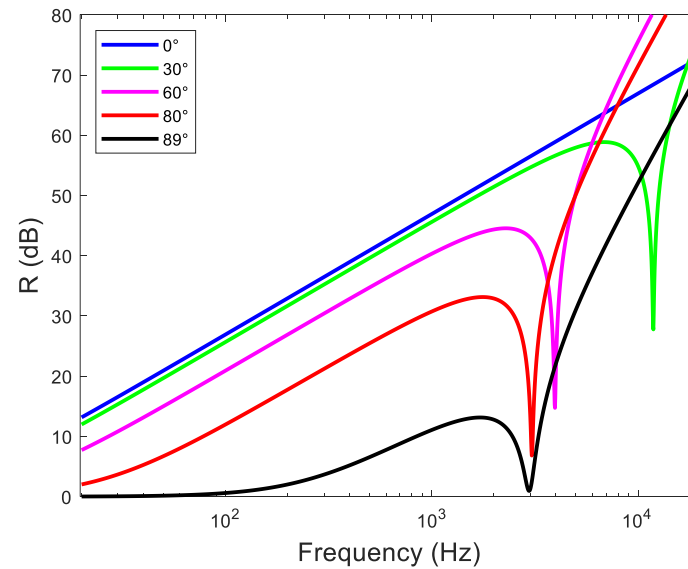


V – TL for different incident angles

$$f_{coin} = \frac{\omega_{coin}}{2\pi} = \frac{c_0^2}{2\pi \sin^2 \theta_i} \sqrt{\frac{\rho h}{D}}$$

$$\lim_{\theta_i \rightarrow 0} f_{coin} = +\infty$$

$$\lim_{\theta_i \rightarrow \frac{\pi}{2}} f_{coin} = \frac{c_0^2}{2\pi} \sqrt{\frac{\rho h}{D}} = f_c$$



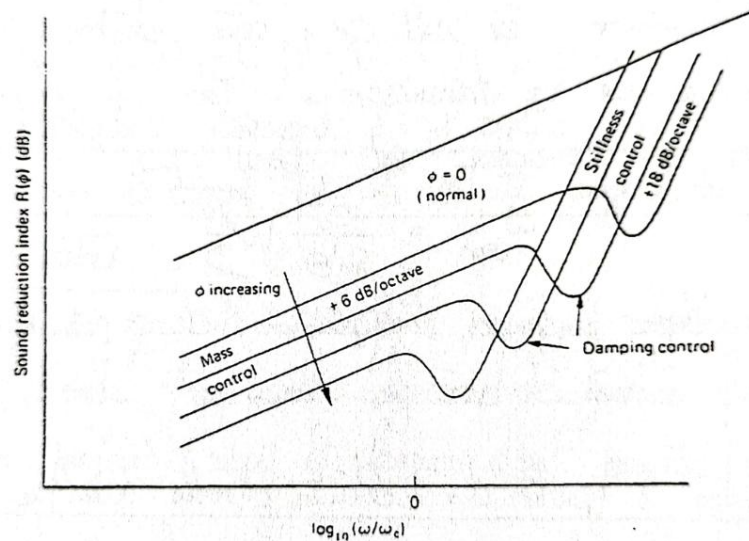


Fig. 73. Variation of R with frequency for single angle of incidence.

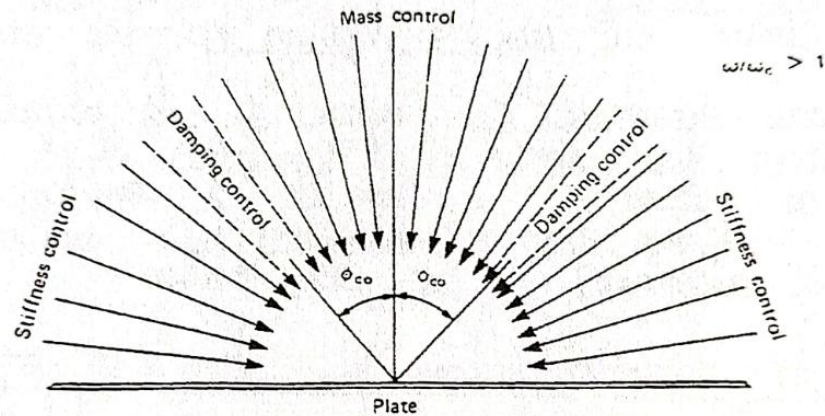
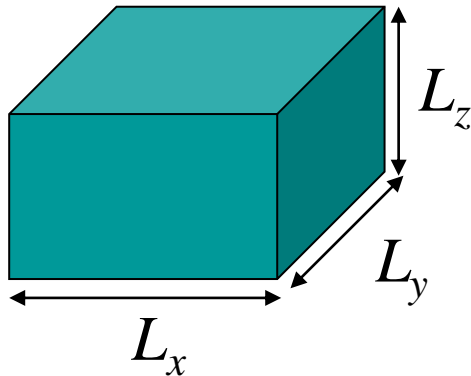


Fig. 74. Regions of incidence angle associated with mass, stiffness, and damping control of transmission at a single supercritical frequency.

Fahy – Sound and Structural Vibration, Academic Press, 1985

IV - Evaluation of the acoustic transmission for a diffuse field

IV.1 - Introduction: from the acoustic cavity resonances to the acoustic diffuse field



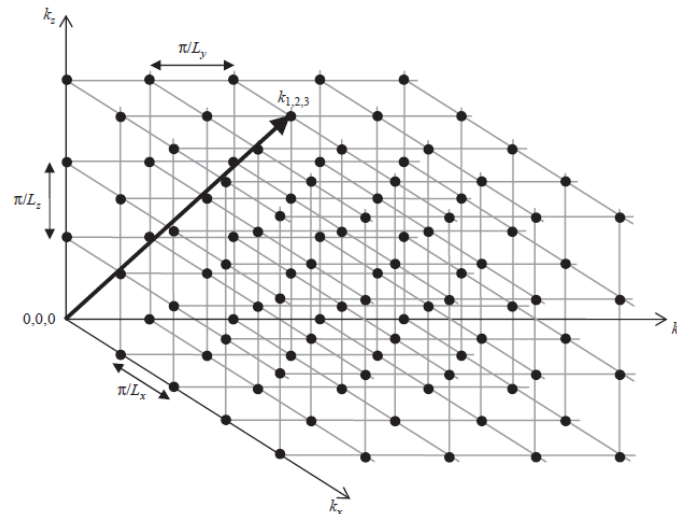
a box-shaped room

Modal information:

- Modal frequencies:
$$f_{pqr} = \frac{c_0}{2\pi} \sqrt{k_p^2 + k_q^2 + k_r^2}$$

- Mode shapes:
$$\phi_{pqr}(x, y, z) = \cos(k_p x) \cos(k_q y) \cos(k_r z)$$

with the modal wavenumbers:
$$k_p = \frac{p\pi}{L_x}, \quad k_q = \frac{q\pi}{L_y}, \quad k_r = \frac{r\pi}{L_z}, \quad (p, q, r) \in \mathbb{N}^3$$



Mode lattice for a three-dimensional space

Modal wavevector:

$$\mathbf{k}_{p,q,r} = \begin{pmatrix} k_p \\ k_q \\ k_r \end{pmatrix}$$

Frequency – wavenumber relation:

$$\|\mathbf{k}_{p,q,r}\| = k_0$$

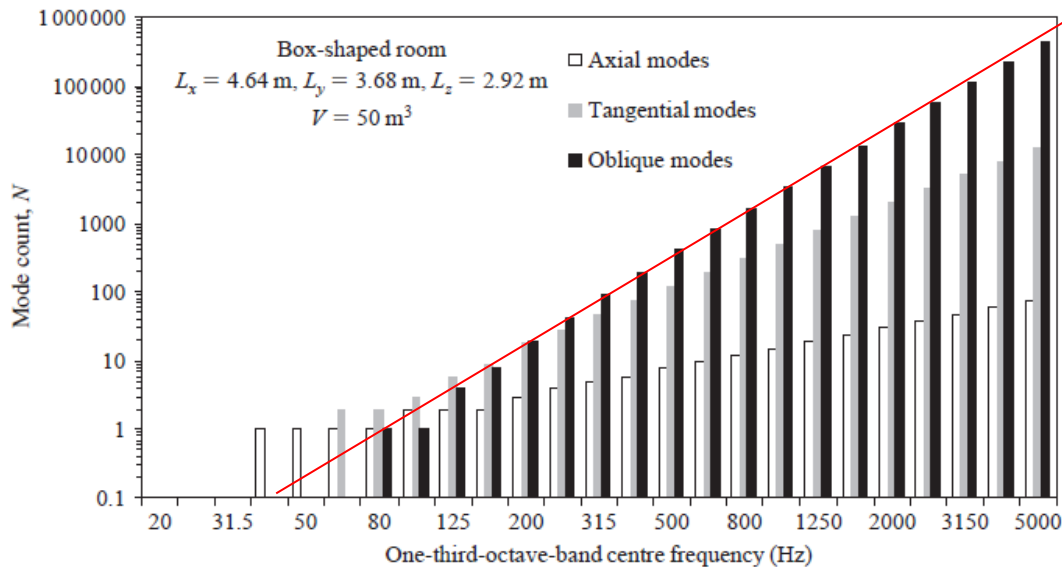
- Statistical modal density (i.e. the number of modes per Hertz):

$$n(f) = \frac{dN}{df}(f) = \frac{2\pi}{c_0} \frac{dN}{dk_0}(k_0) \text{ where } N \text{ is the number of modes below a specified frequency, } f \text{ (or a specified acoustic wavenumber, } k_0)$$

One-eighth of a spherical volume with radius, $k_0 \longrightarrow \frac{1}{8} \frac{4\pi k_0^3}{3}$

$$\text{From the mode lattice (considering only the oblique modes), } N(k_0) \approx \frac{\frac{1}{8} \frac{4\pi k_0^3}{3}}{\frac{\pi}{L_x} \frac{\pi}{L_y} \frac{\pi}{L_z}} = \frac{k_0^3 V}{6\pi^2} \Rightarrow n(f) \approx \frac{4\pi f^2 V}{c_0^3}$$

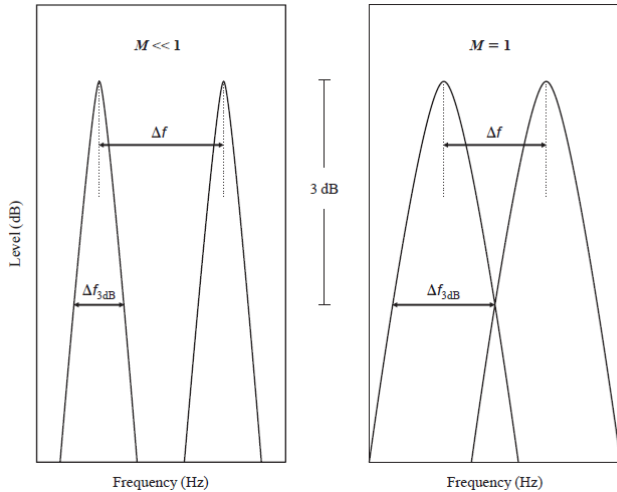
Volume of the cube associated with each mode $\longrightarrow \frac{\pi}{L_x} \frac{\pi}{L_y} \frac{\pi}{L_z}$



Number of modes in a third octave band of central frequency, f_c and of bandwidth Δf :

$$N(f_c) \approx n(f_c) \Delta f$$

Mode count for axial, tangential, and oblique modes in a 50 m³ box-shaped room (Sound Insulation, Carl Hopkins, Butterworth-Heinemann, 2007)



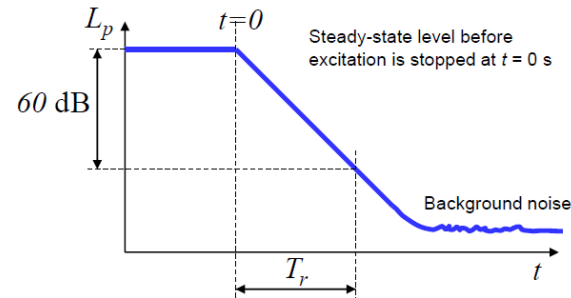
Modal response for two adjacent modes with modal overlap factors, $M \ll 1$ and $M = 1$ (Sound Insulation, Carl Hopkins, 2007)

- Average frequency spacing between mode:

$$\Delta f(f) = \frac{1}{n(f)}$$

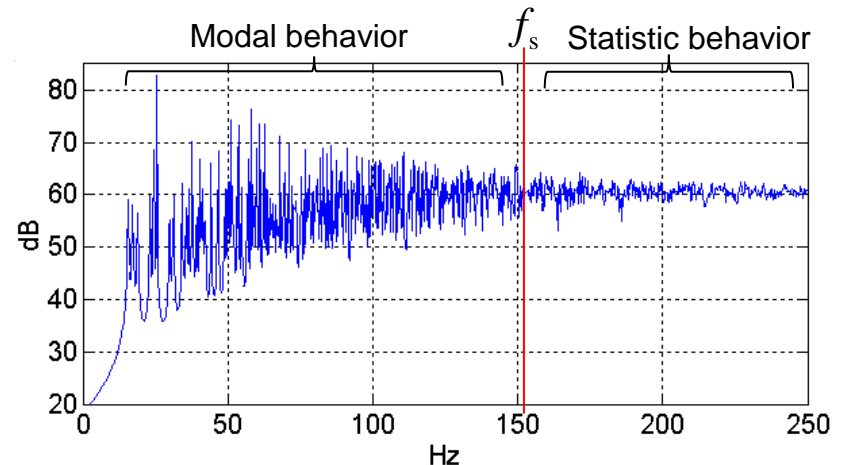
- Modal overlap factor (defined as the ratio of the 3 dB modal bandwidth to the average frequency spacing between mode):

$$M(f) = \frac{\Delta f_{-3dB}}{\Delta f} \approx n(f) \eta f \quad \text{with} \quad \eta \approx \frac{6 \ln 10}{2\pi f T_{60}} \approx \frac{2.2}{f T_{60}}$$

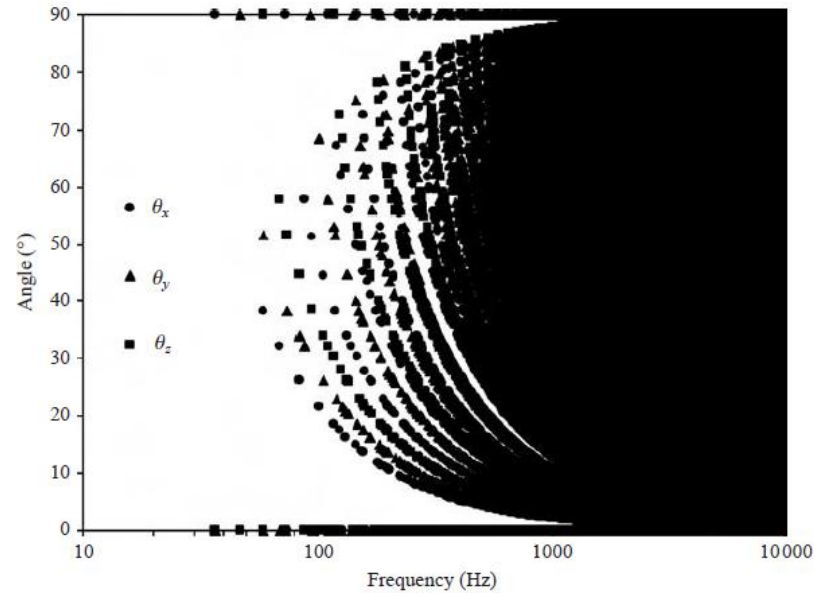
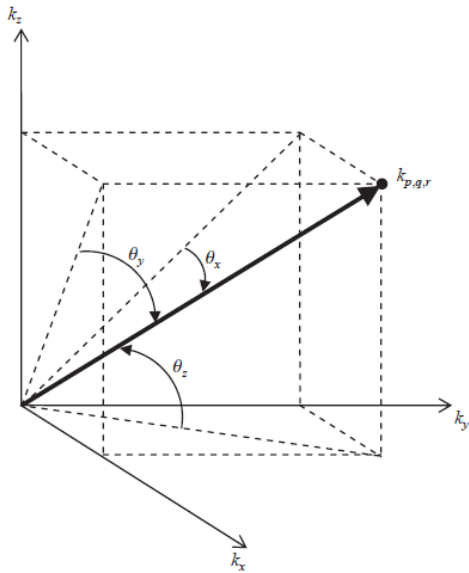


- Schroeder cut-off frequency, f_s :

$$M(f_s) = 3 \Leftrightarrow f_s = \sqrt{\frac{3c_0^3 T_{60}}{8.8\pi V}} \approx 2000 \sqrt{\frac{T_{60}}{V}}$$



Pressure level at a receiving point inside a box-shaped room



Equivalent angles for the modes of a 50m³ box-shaped room
(Sound Insulation, Carl Hopkins, Butterworth-Heinemann, 2007)

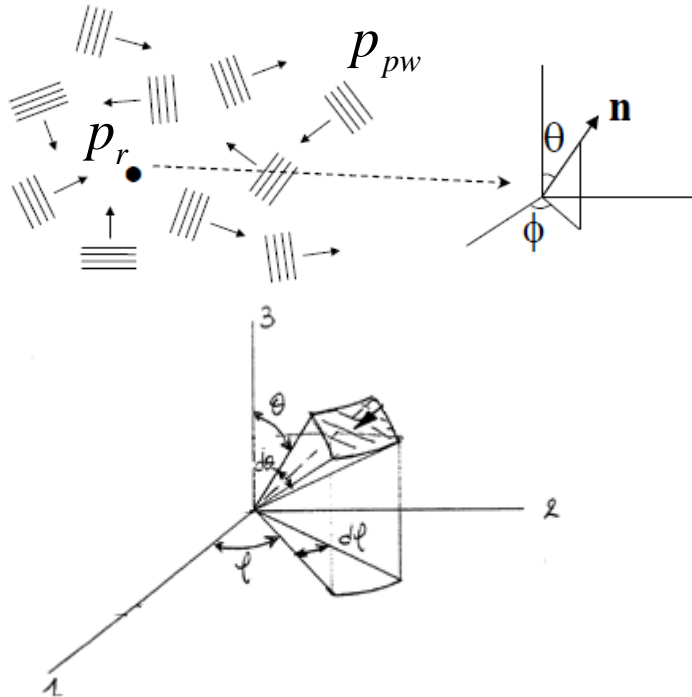
➔ Above the Schroeder frequency, need to switch from thinking in terms of modes to terms of waves travelling at any incident angles (also from a deterministic view to a statistic view)

Definition of the acoustic diffuse field:

A perfect diffuse field is assumed according to Sabine's theory with an isotropic sound field consisting of incident uncorrelated plane waves coming from all directions

P_{pw} → Pressure induced by one elementary plane wave

p_r → Reverberated pressure induced by the sum of the incident uncorrelated plane waves



As the plane waves are uncorrelated,

$$\bar{p}_r^2 = \iint_{\Omega=4\pi} \bar{p}_{pw}^2 d\Omega$$

$$\text{with } \bar{p}_\alpha^2 = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} [p_\alpha(t)]^2 dt, \quad \alpha \rightarrow r \text{ or } pw$$

Solid angle (~ surface area on a 1 m radius sphere):

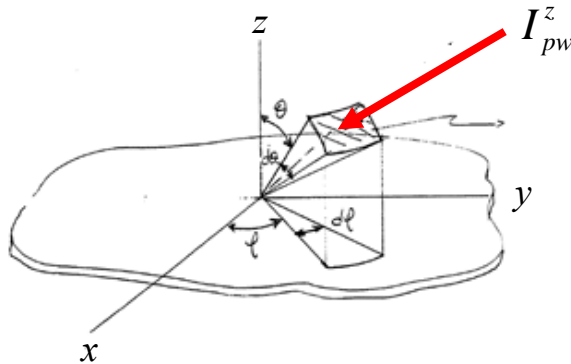
$$d\Omega = \sin(\theta) d\theta d\phi$$

$$\bar{p}_r^2 = \bar{p}_{pw}^2 \int_0^{2\pi} \int_0^\pi \sin(\theta) d\theta d\phi = 4\pi \bar{p}_{pw}^2$$

$$\Rightarrow |P_{pw}|^2 = \frac{|P_r|^2}{4\pi} \quad (\text{considering a frequency analysis})$$

IV.2 - Evaluation of the acoustic transmission for a diffuse field

- Calculation of the incident intensity induced by the diffuse field, normal to the panel surface:



$$I_{inc}^{diff} = \iint_{\Omega=2\pi} I_{pw}^z d\Omega = \int_0^{2\pi} \int_0^{\pi/2} I_{pw}^z(\theta, \varphi) \sin(\theta) d\theta d\varphi$$

where: $I_{pw}^z(\theta, \varphi) = \frac{|P_{pw}|^2 \cos(\theta)}{2\rho_0 c_0}$

$$I_{inc}^{diff} = \int_0^{2\pi} \int_0^{\pi/2} \frac{|P_r|^2 \cos(\theta) \sin(\theta)}{8\pi\rho_0 c_0} d\theta d\varphi = \frac{|P_r|^2}{4\rho_0 c_0} \int_0^{\pi/2} \cos(\theta) \sin(\theta) d\theta \Rightarrow I_{inc}^{diff} = \frac{|P_r|^2}{8\rho_0 c_0}$$

- Calculation of the transmitted intensity through the panel, normal to the panel surface:

$$I_{trans}^{diff} = \iint_{\Omega=2\pi} I_{t,pw}^z d\Omega = \int_0^{2\pi} \int_0^{\pi/2} I_{t,pw}^z(\theta, \varphi) \sin(\theta) d\theta d\varphi$$

where: $I_{t,pw}^z(\theta, \varphi) = \tau(\theta) I_{pw}^z(\theta, \varphi)$ and $\tau(\theta) = \dots$

$$\Rightarrow I_{trans}^{diff} = \frac{|P_r|^2}{4\rho_0 c_0} \int_0^{\pi/2} \tau(\theta) \cos(\theta) \sin(\theta) d\theta = \frac{|P_r|^2}{8\rho_0 c_0} \int_0^{\pi/2} \tau(\theta) \sin(2\theta) d\theta$$

- Calculation of the transmission coefficient for a diffuse field:

$$\tau_d = \frac{I_{trans}^{diff}}{I_{inc}^{diff}} = \frac{\int_0^{2\pi} \int_0^{\pi/2} I_{t,pw}^z(\theta, \varphi) \sin(\theta) d\theta d\varphi}{\int_0^{2\pi} \int_0^{\pi/2} I_{pw}^z(\theta, \varphi) \sin(\theta) d\theta d\varphi} \quad (\text{by definition})$$

$$\Rightarrow \tau_d = \int_0^{\pi/2} \tau(\theta) \sin(2\theta) d\theta$$

➔ No analytical integration, use of a numerical integration

- Calculation of the transmission loss (or sound reduction index) for a diffuse field:

$$TL_d(\omega) = R_d(\omega) = 10 \log_{10} \left(\frac{1}{\tau_d(\omega)} \right)$$

➔ Experiments showed that TL is underestimated when considering the theoretical diffuse field.

➔ In practice, the grazing waves do not contribute significantly to the TL. A practical transmission coefficient is obtained by integrating from 0° to 78°:

$$\bar{\tau}_d = \int_0^{\theta_{\max}} \tau(\theta) \sin(2\theta) d\theta \quad \text{with } \theta_{\max} = 78^\circ$$

Exercise:

Let us consider the TL of an infinite steel panel with air on both sides and excited by a diffuse acoustic field. It is assumed that the plate has a thickness of 4 mm and that all plane waves with an incident angle between 0° to 89.99° participate to the diffuse field.

I – Using Matlab or Python, estimate the TL of the damped panel for a diffuse field at a given frequency and then, plot the curve of the TL in the audible frequency range [20 Hz – 20 kHz]. Are the results as expected? Which is the frequency where the dip is observed?

(The numerical integration can be performed in Matlab using the command "integral" and an anonymous function. For instance, to integrate x^2 from 0 to 1, one can use: `integral(@(x) x.^2,0,1)`).

II – For frequencies much lower the critical frequency, an approximate formula is:

$$R_d^{LF}(\omega) \approx R(0) - 10 \log_{10}(0.23R(0))$$
 with $R(0)$, the TL for a normal plane wave.

On the same figure, plot the TLs in the audible frequency range for a diffuse field, for a normal plane wave and with the previous approximation. Comment on results. How does the TL vary with frequency in this frequency range?

III – For frequencies much higher the critical frequency, the Cremer's approximate formula is:

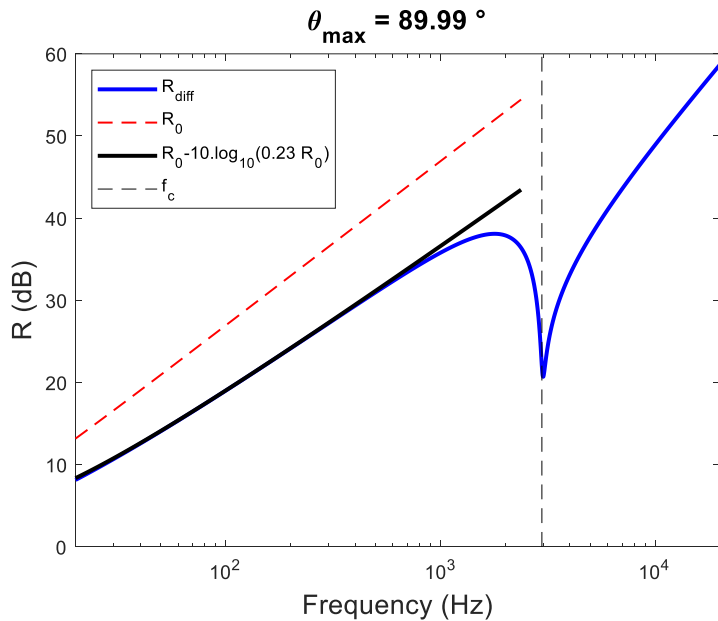
$$R_d^{HF}(\omega) \approx R(0) + 10 \log_{10}(f/f_c - 1) + 10 \log_{10}(\eta) - 2$$

On the same figure, plot the TL in the audible frequency range for a diffuse field and with the Cremer's approximate formula. Comment on results. How does the TL vary with frequency in this frequency range?

IV - Plot the TL in the audible frequency range for two different values of the panel damping (i.e. 0.01, 0.05). Comment on results.

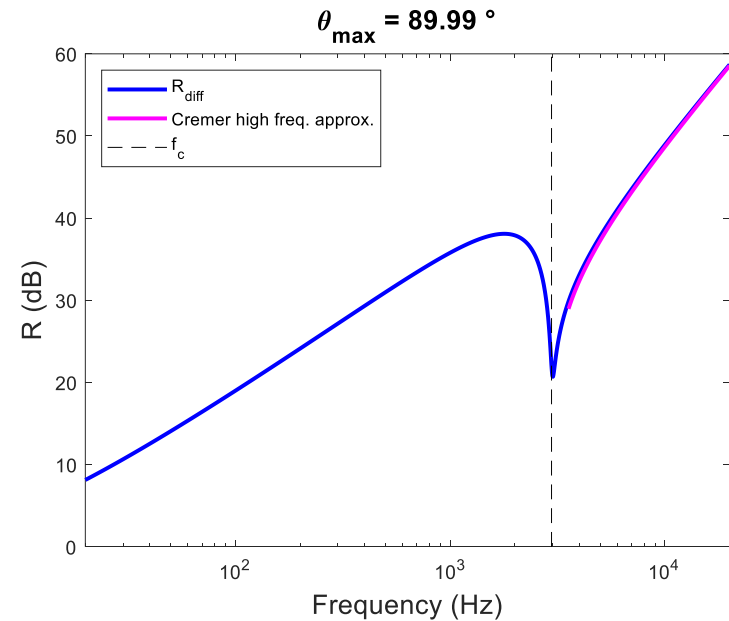
V – Now, the plane waves with an incident angle between 0° to 78° are considered to estimate the TL for the "practical" diffuse field. Estimate the TL for this case and compare it to the one obtained for the "theoretical" diffuse field. Comment on results. An approximate formula for the frequencies much lower the critical frequency is: $R_{d,practical}^{LF}(\omega) \approx R(0) - 5$ dB

II – Low frequency approximation



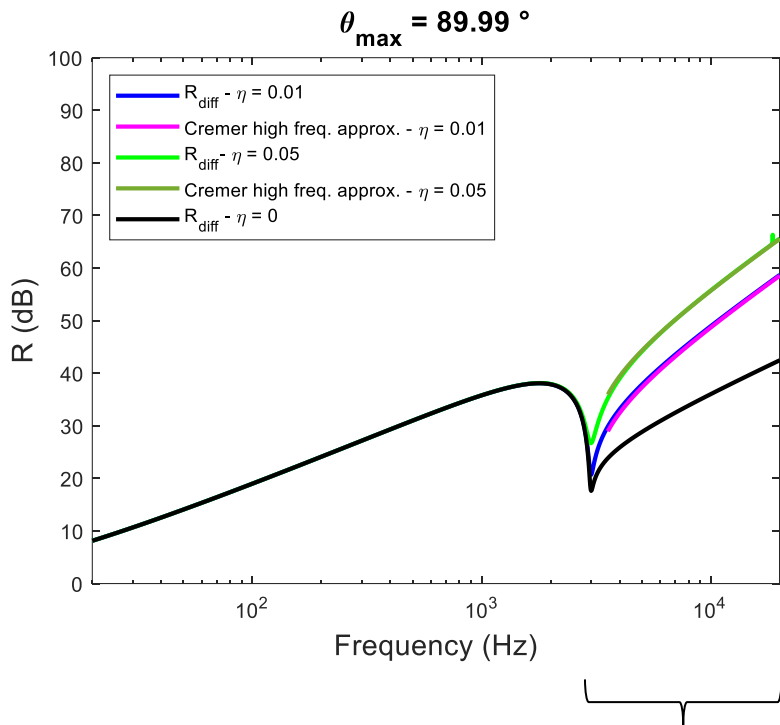
~ 5-6 dB/octave, Mass control

III – High frequency approximation



~ 9 dB/octave
Stiffness/damping control

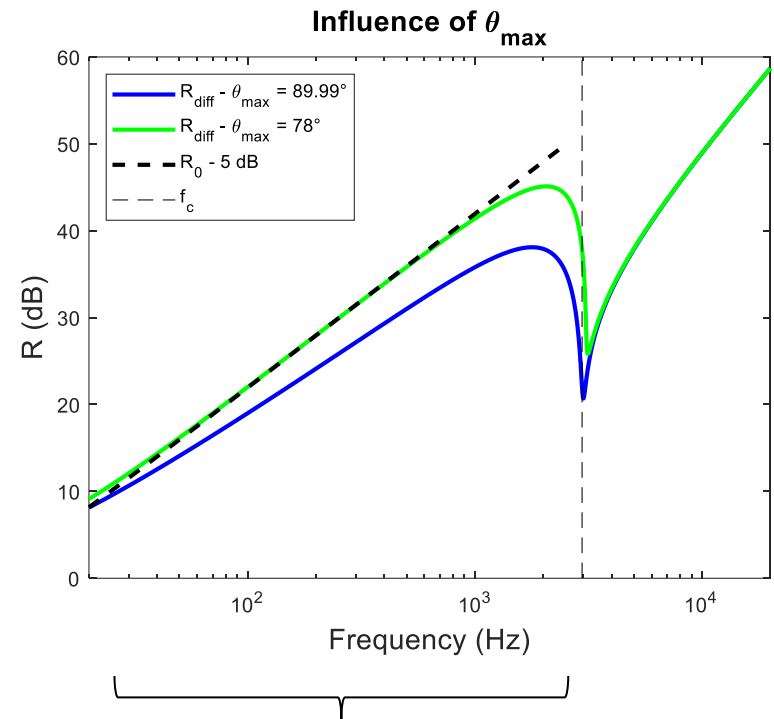
IV – Influence of the damping



No zero TL at the critical frequency (even for the undamped plate)

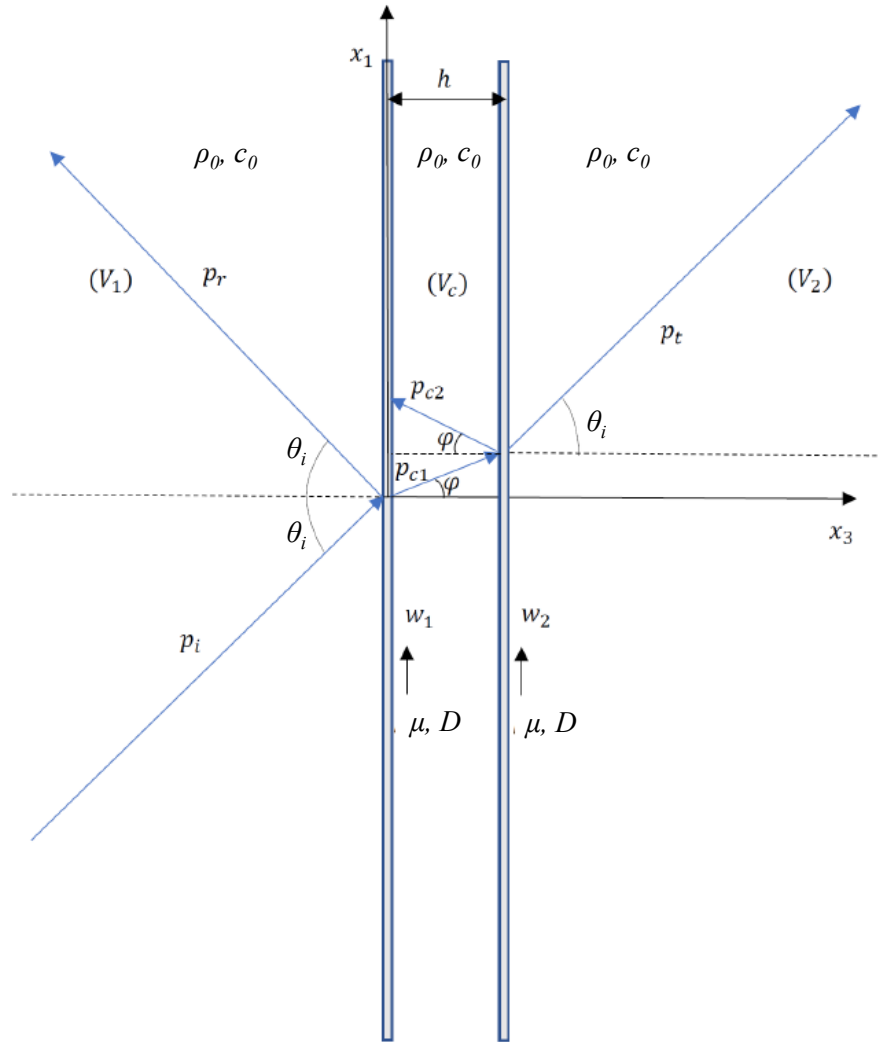
Damping control not only at the critical frequency

V – “Practical” estimation



TL greater when the grazing waves are excluded

V – Extension to the infinite double panel



Sound transmission through double walls

Formulation similar to the one for one single wall: See the developments in the chapter. ” Sound transmission of plane structures”, UdS course, Alain Berry (p. 24-33)

Relation between the amplitude of the transmitted wave and of the incident wave (final results, considering a time dependence $e^{-i\omega t}$):

$$p_t(\omega, \theta_i) = \frac{1}{1 + \frac{E(\omega, \theta_i)}{4\omega^4 \rho^2}} p_i(\omega, \theta_i)$$

$$\text{with } E(\omega, \theta_i) = \cos^2 \theta_i (k_0 Z)^2 \left(e^{2ik_0 h \cos \theta_i} - 1 \right) - 4i\omega^2 \rho_0 \cos \theta_i k_0 Z$$

where $Z = \mu\omega^2 - D(k_0 \sin \theta_i)^4$ and μ , the panel mass per unit area

Sound power transmission coefficient:

$$\tau(\omega, \theta_i) = \frac{|p_t(\omega, \theta_i)|^2}{|p_i(\omega, \theta_i)|^2}$$

Transmission loss (or Sound reduction index):

$$TL_{\theta_i}(\omega) = 10 \log_{10} \left(\frac{1}{\tau(\omega, \theta_i)} \right) = 20 \log_{10} \left(\left| 1 + \frac{E(\omega, \theta_i)}{4\omega^4 \rho^2} \right| \right)$$

Exercise:

Let us consider the TL of an infinite double 12 mm glazing with 50 mm air gap.

I – TL of the nominal plate for an incident angle of 45°.

Using the Matlab code, plot the TL in function of frequency (log scale) in the audible frequency range [20 Hz – 20 kHz]. Compare it to the TLs of single walls, 12 mm thick or 24 mm thick. What do you observe?

II – Analysis for frequencies much lower the coincident frequency

$$D\omega^2 \left(\frac{\sin \theta_i}{c_0} \right)^4 \ll \mu \Rightarrow Z \approx \omega^2 \mu \Rightarrow E \approx \frac{\cos \theta_i \mu}{c_0} \omega^5 \left[\cos \theta_i \omega \left(\frac{\mu}{c_0} \right) \left(e^{2ik_0 h \cos \theta_i} - 1 \right) - 4i\rho_0 \right]$$

- What happens when the frequency tends towards 0?
- Assuming that the first term in the approximate expression of E is negligible in the very low frequencies, propose an approximate expression for the TL for these frequencies. Compare to the mass law for the TL of one single wall.
- Supposing that the term $2ik_0 h \cos \theta_i$ is small, we can write $e^{2ik_0 h \cos \theta_i} - 1 \approx 2ik_0 h \cos \theta_i$. Considering this approximation, find the “breathing” frequency for which the term in bracket for the approximate expression of E is null. How much is the TL for this frequency? Study the parameters that influence this particular frequency.
- Considering always the previous approximation and considering that the first term dominates for frequencies above the breathing frequency, give an approximate expression for the TL. Show that the TL increases by 18 dB/octave.

III – Analysis for frequencies much higher the coincident frequency

$$D\omega^2 \left(\frac{\sin \theta_i}{c_0} \right)^4 \gg \mu \Rightarrow Z \approx -D(k_0 \sin \theta_i)^4 \Rightarrow E \approx E = \cos^2 \theta_i (k_0 D(k_0 \sin \theta_i)^4)^2 \left(e^{2ik_0 h \cos \theta_i} - 1 \right) + 4i\omega^2 \rho_0 \cos \theta_i k_0 D(k_0 \sin \theta_i)^4$$

Find the frequencies for which the first term of the approximate expression of E is equal to 0. How much is the TL for these frequencies? Is it null?

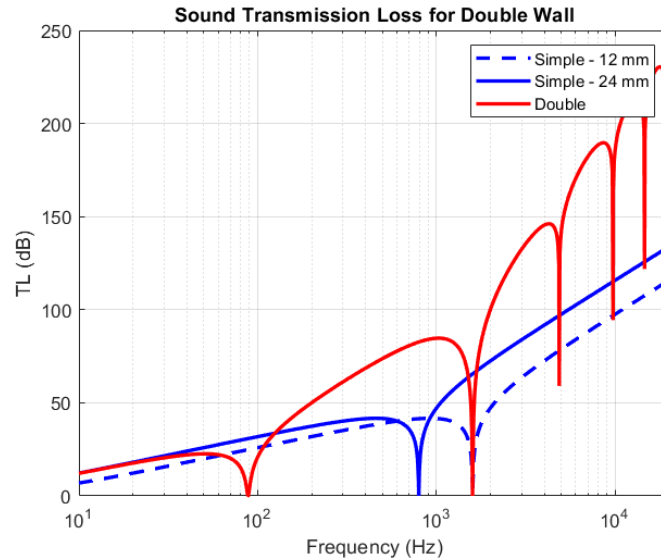
IV – Influence of the damping

- Add a structural damping to the panel through the intermediary of a complex Young modulus (take care of the time dependence). A damping loss factor of 0.01 can be considered. What do you observe on the TL curve?
- Add a damping to the fluid between the two panels through the intermediary of a complex sound speed. Damping loss factors of 0.001 and 0.05 can be considered. What do you observe on the TL curve?

V – TL for different incident angles

- Plot the TL in function of frequency for different incident angles (0°, 30°, 60°, 89°). Analyze the results.
- Can we find an angle such that the breathing frequency is equal to the coincident frequency?

I – TL of the nominal plate for an incident angle of 45°.



II - Analysis for frequencies much lower the coincident frequency

$$E \approx -\frac{4i \cos \theta_i \mu \rho_0}{c_0} \omega^5 \xrightarrow{\omega \rightarrow 0} 0 \quad TL_{\theta_i}(\omega) \xrightarrow{\omega \rightarrow 0} 0$$

$$\frac{E(\omega, \theta_i)}{4\omega^4 \rho^2} \gg 1 \Rightarrow TL_{\theta_i}(\omega) \approx 20 \log \left(\frac{\cos \theta_i \mu \omega}{c_0 \rho_0} \right)$$

→ Mass law for a single panel of mass per unit of area 2μ

Beathing frequency:

$$e^{2ik_0h\cos\theta_i} - 1 \approx 2ik_0h\cos\theta_i \Rightarrow E \approx 2i\cos^3\theta_i\omega^7 \frac{\mu^2}{c_0^3} h - 4i\mu \frac{\omega^5}{c_0} \rho_0 \cos\theta_i = 2i\cos\theta_i\mu \frac{\omega^5}{c_0} \left(\cos^2\theta_i\omega^2 \frac{\mu}{c_0^2} h - 2\rho_0 \right)$$

$$\cos^2\theta_i\omega_b^2 \frac{\mu}{c_0^2} h - 2\rho_0 = 0 \Leftrightarrow \omega_b^2 = \frac{2\rho_0 c_0^2}{\mu h \cos^2\theta_i} \Leftrightarrow \omega_b = \frac{c_0}{\cos\theta_i} \sqrt{\frac{2\rho_0}{\mu h}} \Leftrightarrow f_b = \frac{c_0}{\pi \cos\theta_i} \sqrt{\frac{\rho_0}{2\mu h}}$$

$$TL_{\theta_i}(\omega_b) \approx 0$$

Above the beathing frequency:

$$E \approx 2i\cos^3\theta_i\omega^7 \frac{\mu^2}{c_0^3} h \text{ and } \frac{E}{4\omega^4\rho^2} \gg 1 \Rightarrow TL_{\theta_i}(\omega) \approx 20\log\left(\frac{\cos^3\theta_i\omega^3\mu^2h}{2\rho_0^2c_0^3}\right) = 20\log\left(\frac{\cos^3\theta}{2\rho_0^2c_0^3}\right) + 60\log(\omega) + 40\log(\mu) + 20\log(h)$$

→ 18 dB/octave

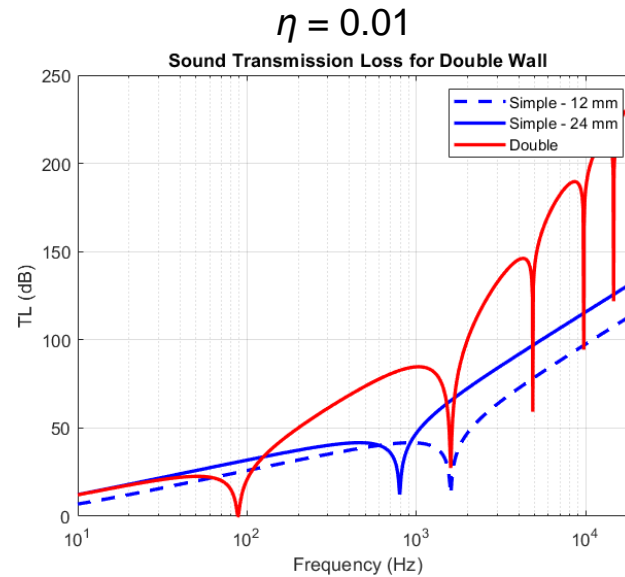
III – Analysis for frequencies much higher the coincident frequency

$$e^{2ik_0h\cos\theta_i} - 1 = 0 \Leftrightarrow 2\frac{\omega_n}{c_0} h \cos\theta_i = 2n\pi \Leftrightarrow \omega_n = \frac{n\pi c_0}{h \cos\theta_i} \Leftrightarrow f_n = \frac{nc_0}{2h \cos\theta_i}$$

$$TL_{\theta_i}(\omega) = 20\log\left(\frac{\cos\theta_i\omega_n^3 D(\sin\theta_i)^4}{\rho_0 c_0^5}\right) = 20\log\left(\frac{\cos\theta(\sin\theta_i)^4}{\rho_0 c_0^5}\right) + 60\log(\omega_n) + 20\log(D) \neq 0$$

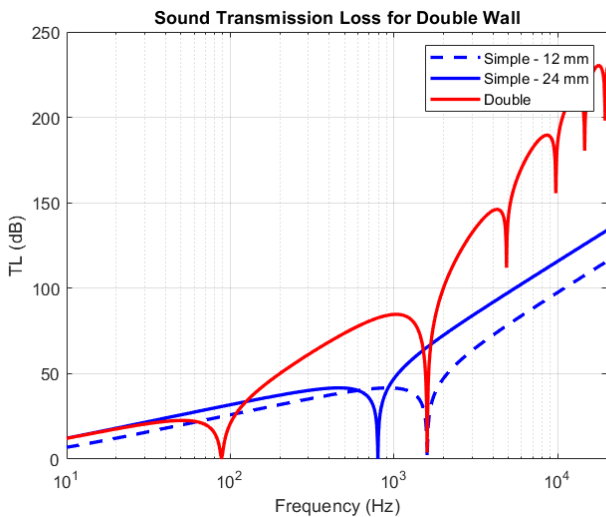
IV – Influence of the damping

Damping for the plates:

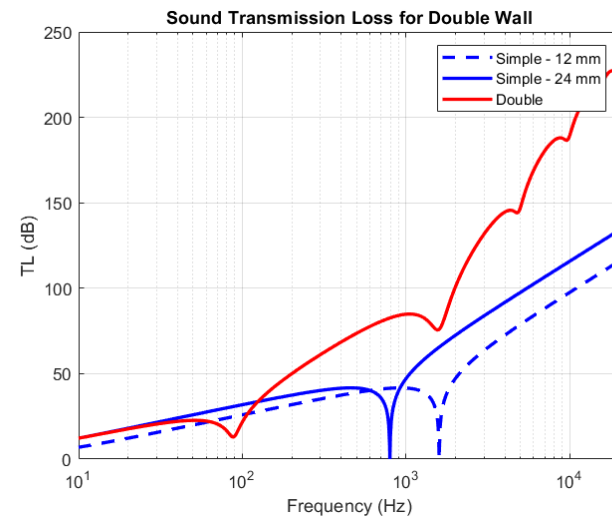


Damping for the fluid between the 2 plates:

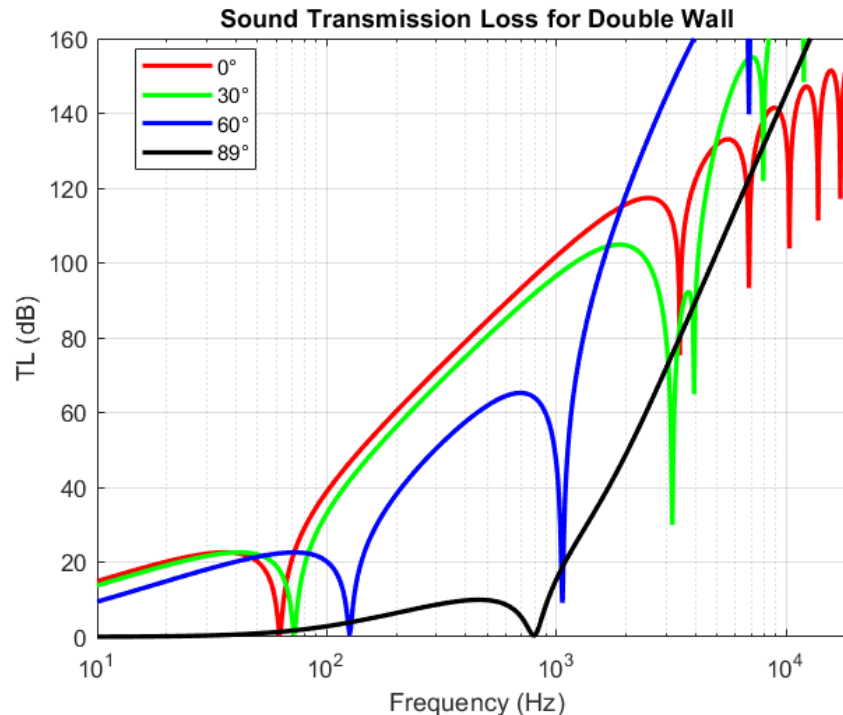
$\eta_0 = 0.001$



$\eta_0 = 0.05$



V – TL for different incident angles



$$\omega_{coin} = \omega_b \Leftrightarrow \frac{c_0^2}{\sin^2 \theta_i} \sqrt{\frac{\mu}{D}} = \frac{c_0}{\cos \theta_i} \sqrt{\frac{2\rho_0}{\mu h}} \Leftrightarrow \cos \theta_i = \frac{1}{c_0 \mu} \sqrt{\frac{2\rho_0 D}{h}} \sin^2 \theta_i \Leftrightarrow \cos \theta_i = \frac{1}{c_0 \mu} \sqrt{\frac{2\rho_0 D}{h}} (1 - \cos^2 \theta_i)$$

$$\Leftrightarrow \cos^2 \theta_i + c_0 \mu \sqrt{\frac{h}{2\rho_0 D}} \cos \theta_i - 1 = 0 \dots$$

Numerically, for the present case, we can find that the coincident frequency is equal to the breathing frequency for an incident angle of around 85.53°

MASTER “Science in Acoustics”

“Sound-structure interaction” MOD

Acoustic radiation from infinite panels



Fig. 22

L. MAXIT MODAN HORIZONTAL

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Bâtiment St. Exupéry, 25 bis av. Jean Capelle, 69621 Villeurbanne Cedex
laurent.maxit@insa-lyon.fr

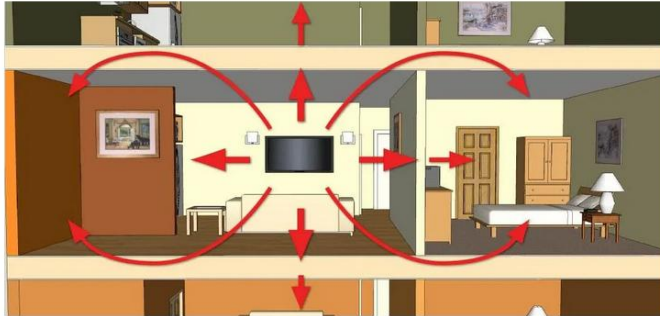


Centre Lyonnais
d'Acoustique

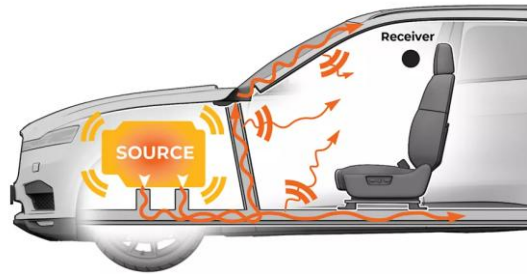
UNIVERSITE DE LYON



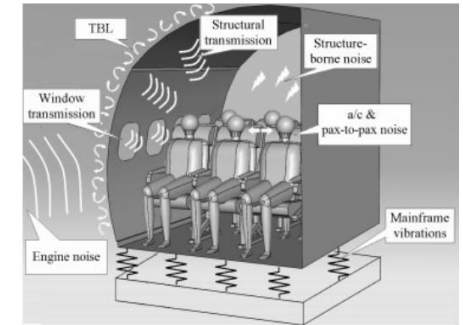
Introduction



Walls, floor, ceiling
→ Flat panels



Windshield, firewall, roof
→ Curved panels



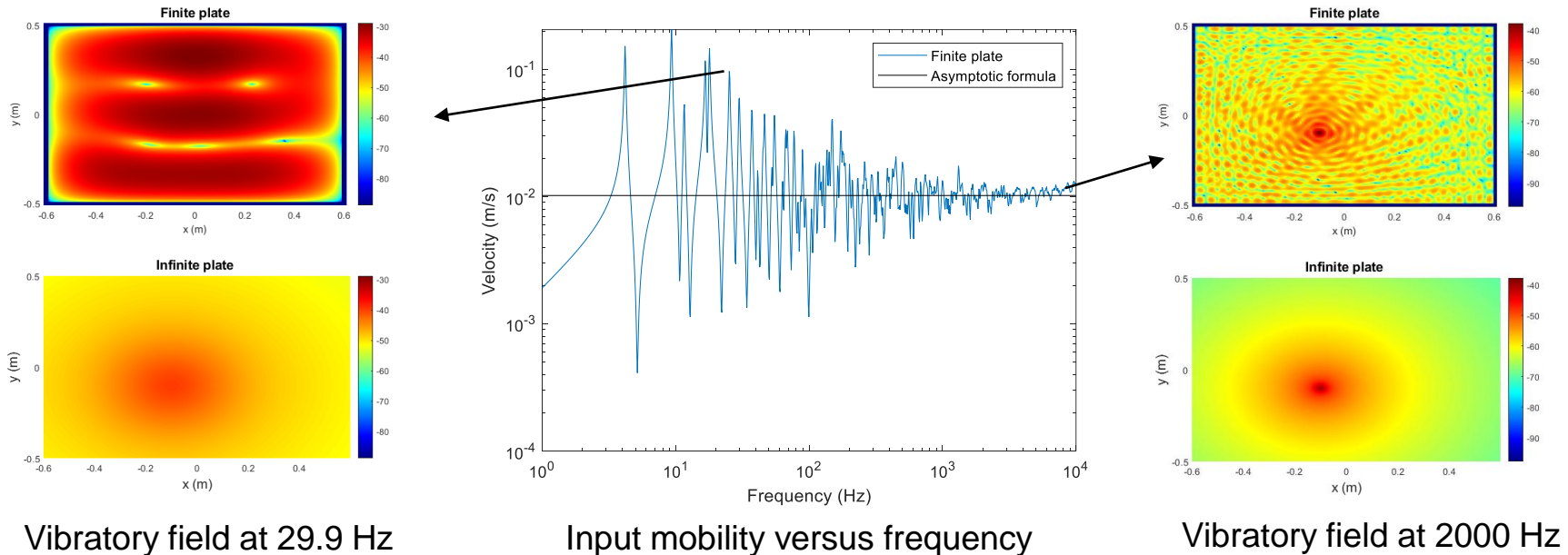
Aircraft fuselage
→ Curved panels

For large curvatures, the effects of the curvature (coupling in-plane /out-of-plane motions) on the panel vibration are of second order.

Simplification: Curved panels → Flat panels

Finite panel vs Infinite panel

Results for 1 mm-thick steel panel excited by an harmonic point force



Vibratory field at 29.9 Hz

Input mobility versus frequency

Vibratory field at 2000 Hz

- ➔ At high frequencies (i.e. when the modal overlap factor is greater than one), the vibratory field close to the excitation point is similar between a finite panel and an infinite plate.
- ➔ Studying the noise radiated by an infinite panel can be useful for understanding how noise is radiated by a finite panel.
- ➔ Use of a wave approach instead of a modal approach (dedicated for bounded problem)

Outline

I – Description of the problem of interest and mathematical formulation

II - The free vibrations

II.1 - For an acoustic medium

II.2 - For a thin flexible plate

II.3 - For a thin flexible plate coupled to an acoustic medium

III - The forced vibrations of an infinite plate in a light fluid

Some bibliographical references:

F.J. Fahy, Sound and structural vibration: Radiation, Transmission and Response, Academic Press, 1985.

M.C. Junger, D. Feit, Sound, Structure and their Interaction, MIT Press, 1986.

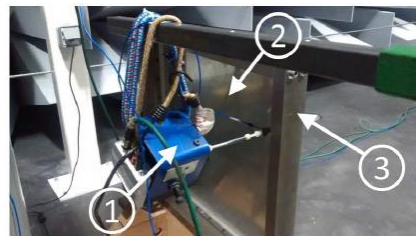
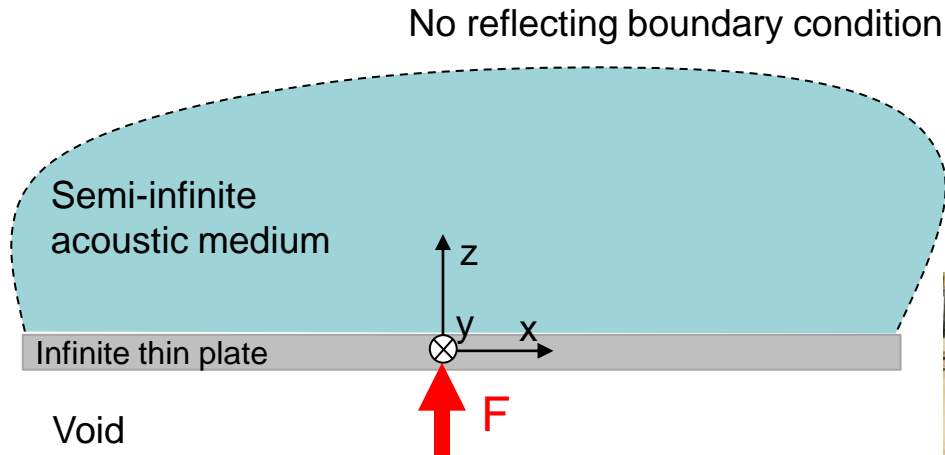
E.G. Williams, Fourier Acoustics, Sound Radiation and Nearfield Acoustical Holography, Academic Press, 1999.

C. Lesueur, J.L. Guyader, Rayonnement acoustique des structures, Eyrolles, 1988.

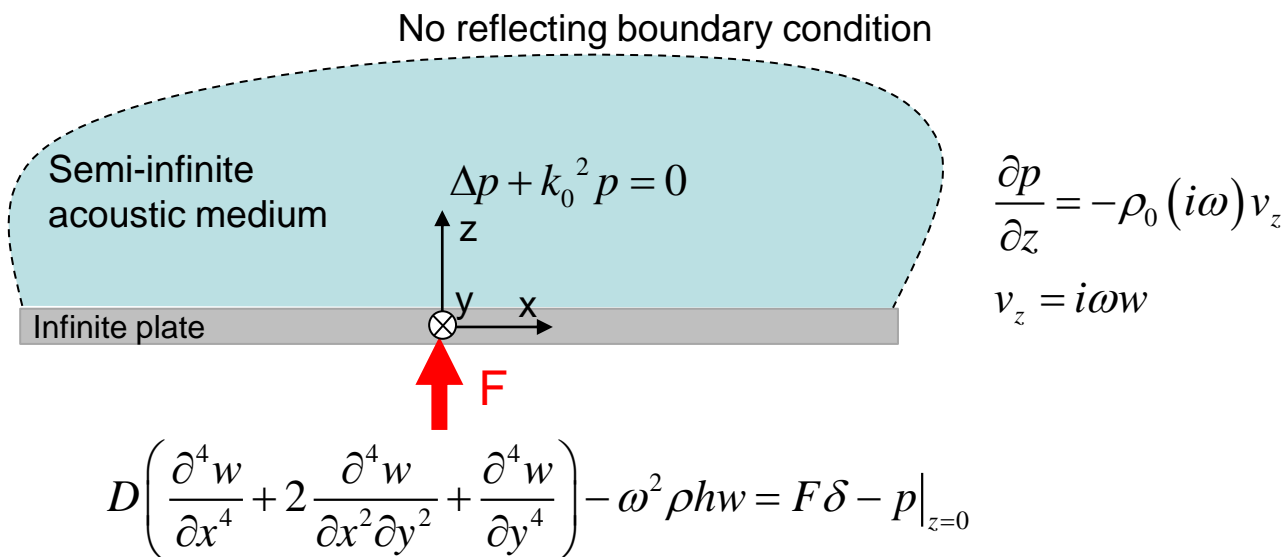
I – Description of the problem of interest and mathematical formulation

Infinite thin plate radiating in a semi-infinite acoustic domain and excited by a normal point force

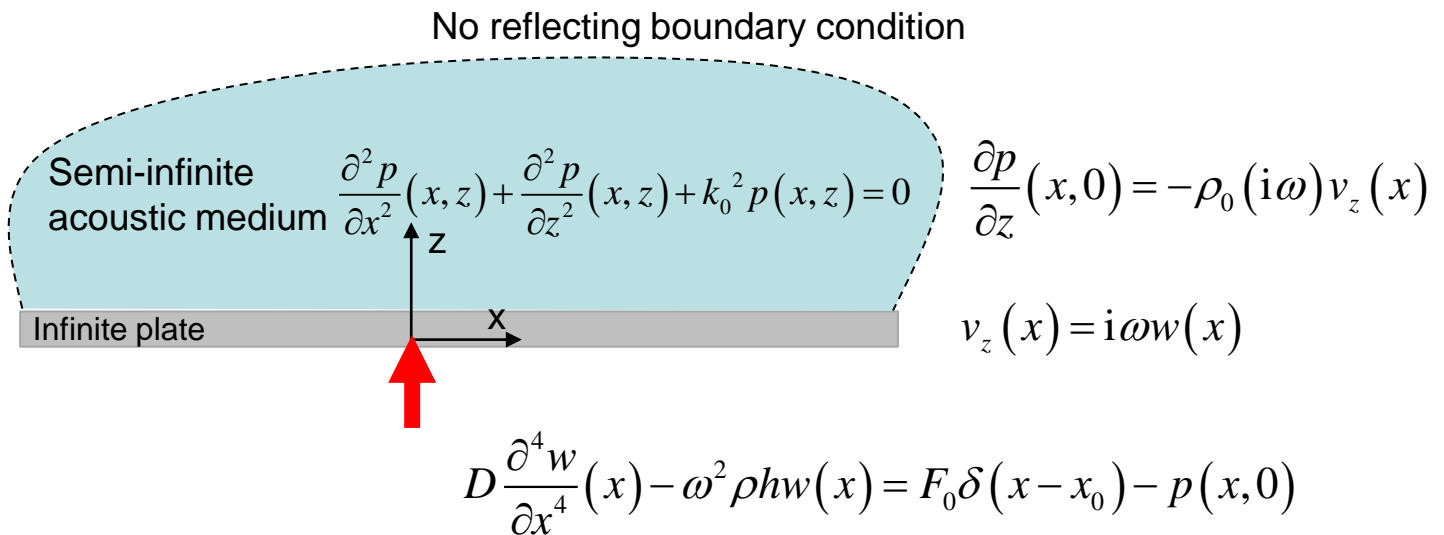
- Ideal case for analytical developments
- Give physical insight for more complex case



- Semi-infinite acoustic medium:
 - linear acoustic
 - boundary conditions:
 - No reflecting boundary at “infinity”
 - Continuity of the normal acoustic velocity and the transversal plate velocity at the panel/fluid interface.
- Infinite thin plate:
 - Pure bending motions (Love-kirchhoff assumptions)
 - No boundary conditions (infinite extent)
 - Pressure induced by the fluid domain on the plate
- Time dependence: $e^{i\omega t}$ (excluded of the notation although always considered in the following)



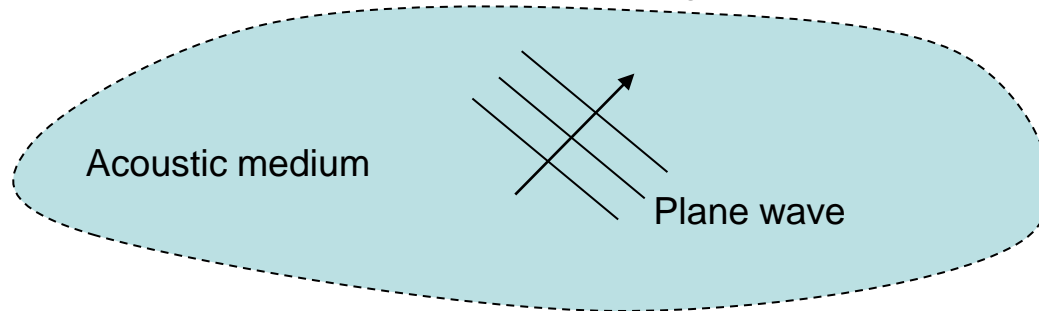
2D case is considered in the following for the sake of simplification and without restricting the generality of the problem



II - The free vibration

II.1 - For an acoustic medium (alone)

No reflecting boundary condition



- Helmholtz equation $\Delta p(x, z) + \frac{\omega^2}{c_0^2} p(x, z) = 0$
- Plane wave $p(x, z) = P e^{-i(k_x x + k_z z)} = P e^{-i\mathbf{k} \cdot \mathbf{x}}$ with $\mathbf{x} = \begin{pmatrix} x \\ z \end{pmatrix}$ and the wavevector $\mathbf{k} = \begin{pmatrix} k_x \\ k_z \end{pmatrix}$
- Non null solution \rightarrow Dispersion equation (i.e. $k = f(\omega)$)

$$k_x^2 + k_y^2 = k_0^2 \quad \text{with the acoustic (natural) wavenumber defined by } k_0 = \frac{\omega}{c_0}$$

$$\Leftrightarrow \|\mathbf{k}\|^2 = k_0^2 \Leftrightarrow \mathbf{k} = \begin{pmatrix} k_0 \cos \theta \\ k_0 \sin \theta \end{pmatrix}, \quad \theta \in]-\pi, \pi]$$

II.2 - For a thin flexible plate (alone)



In-vacuo infinite thin plate made of an isotropic elastic material

- Plate equation of motions with Love-Kirchhoff assumptions and time-harmonic dependence (pure bending, rotational inertia and shear stress not take into account, only adapted for “low” frequencies, use Mindlin assumptions for “high” frequencies):

$$D \frac{\partial^4 w}{\partial x^4}(x) - \omega^2 \rho h w(x) = 0 \quad \text{with the flexural rigidity} \quad D = \frac{Eh^3}{12(1-\nu^2)}$$

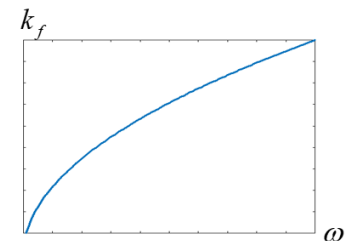
- No boundary conditions as the plate is infinite

- (Plane) wave: $w = We^{-ik \cdot x}$

- Dispersion relation: $k^4 = \frac{\omega^2 \rho h}{D}$

$$k^4 = k_f^4 \quad \text{with the flexural (natural) wavenumber} \quad k_f = \sqrt{\omega \sqrt{\frac{\rho h}{D}}}$$

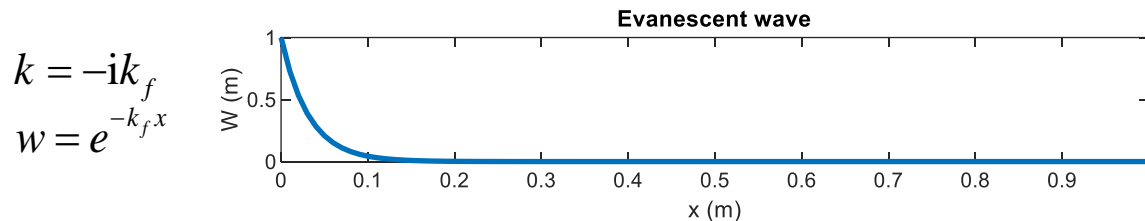
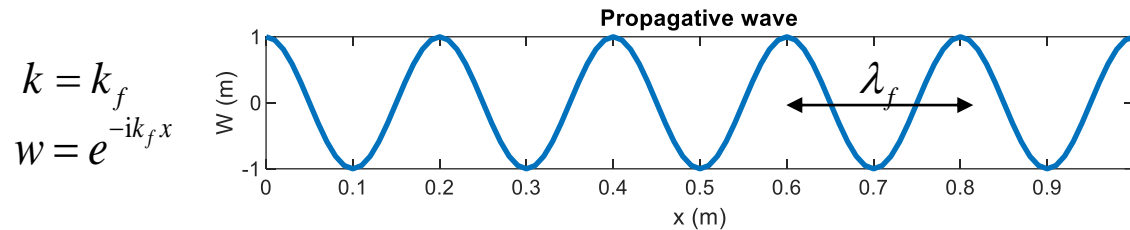
$$\lambda_f = \frac{2\pi}{k_f} = 2\pi^4 \sqrt{\frac{D}{\rho h}} \sqrt{\frac{1}{\omega}}$$



- Solutions of the dispersion equation:

$$4 \text{ solutions: } k = \begin{cases} k_f \\ -k_f \\ ik_f \\ -ik_f \end{cases} \left. \begin{array}{l} \text{Propagative waves} \\ \text{Evanescent waves} \end{array} \right\}$$

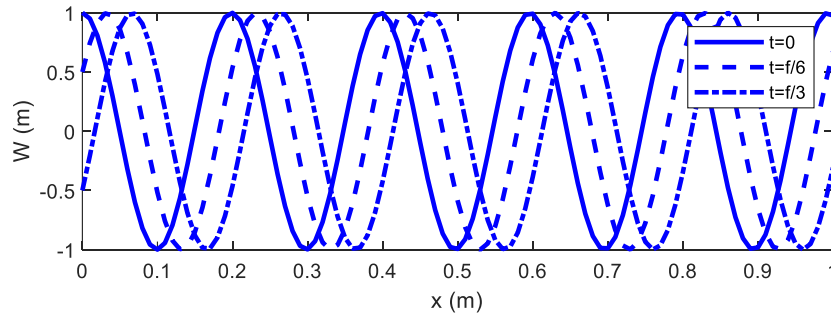
Example: 2 mm thick steel plate at 500 Hz $k = 31.4 \text{ rad/m}$, $W = 1 \text{ m}$



Example: 2 mm thick steel plate at 500 Hz – Displacements for different times

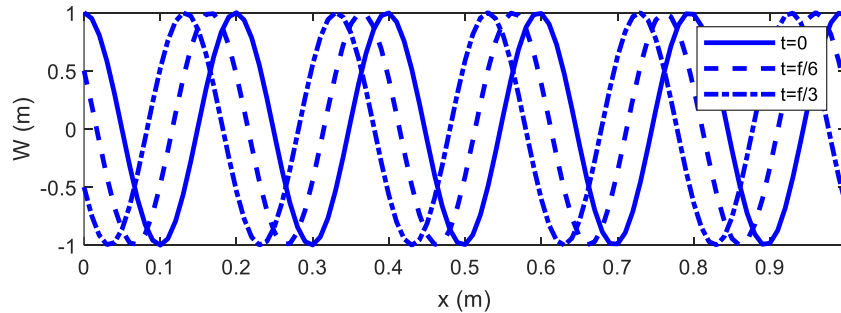
$$W(x,t) = \text{Re} \{ w(x) e^{i\omega t} \}$$

$$w(x) = e^{-ik_f x}$$



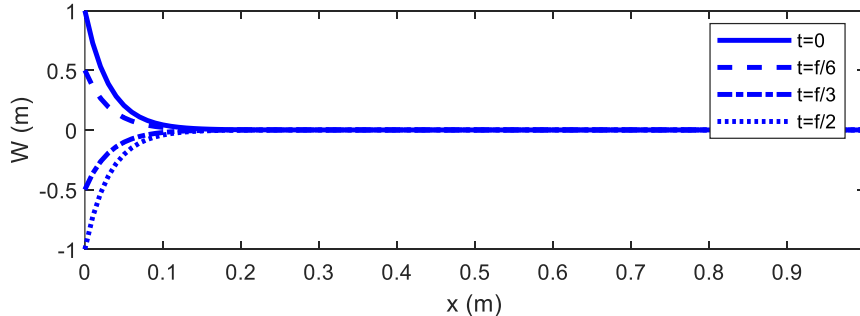
→ Propagating towards increasing x

$$w(x) = e^{+ik_f x}$$



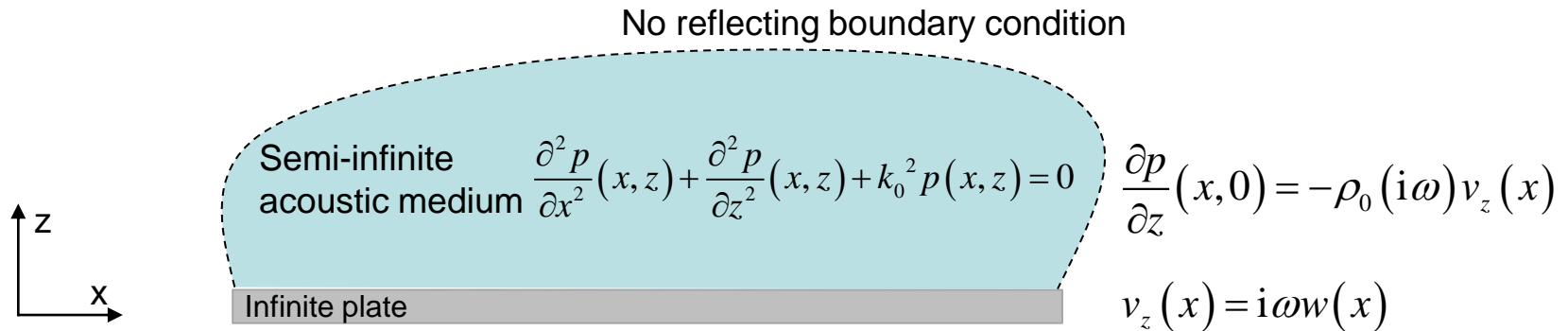
→ Propagating towards decreasing x

$$w(x) = e^{-k_f x}$$



→ Evanescent wave

II.1.c - For a thin flexible plate coupled to an acoustic medium



$$D \frac{\partial^4 w}{\partial x^4}(x) - \omega^2 \rho h w(x) = -p(x, 0)$$

- Verification of the Helmholtz equation on wave forms (or using the separation variable method):

$$\begin{cases} \frac{\partial^2 p}{\partial x^2}(x, z) + \frac{\partial^2 p}{\partial z^2}(x, z) + k_0^2 p(x, z) = 0 \\ p(x, z) = Ae^{-ik_x x - ik_z z} \end{cases} \Leftrightarrow A \in \mathbb{C}, k_x^2 + k_z^2 = k_0^2$$

- Verification of the Euler eq. at the panel surface supposing the plate displacements on wave forms:

$$\begin{cases} \frac{\partial p}{\partial z}(x, 0) = \rho_0 \omega^2 w(x) \\ w(x) = Ce^{-i\tilde{k}_x x} \end{cases} \Leftrightarrow -ik_z A e^{-ik_x x} = \rho_0 \omega^2 C e^{-i\tilde{k}_x x}, \forall x \in \mathbb{R} \Leftrightarrow \begin{cases} \tilde{k}_x \neq k_x \Rightarrow \text{No solution} \\ \tilde{k}_x = k_x \Rightarrow A = \frac{i\rho_0 \omega^2}{k_z} C \end{cases}$$

- Verification of the panel equation of motion:

$$D \frac{\partial^4 w}{\partial x^4}(x) - \omega^2 \rho h w(x) = -p(x, 0)$$

$$\Leftrightarrow (Dk_x^4 - \omega^2 \rho h) C e^{-ik_x x} = -\frac{i\rho_0 \omega^2}{k_z} C e^{-ik_x x}$$

$$\Leftrightarrow \boxed{Dk_x^4 - \omega^2 \rho h + \frac{i\rho_0 \omega^2}{k_z} = 0} \rightarrow \text{Dispersion equation for the panel coupled to the acoustic medium}$$

$$\begin{cases} Dk_x^4 - \omega^2 \rho h + \frac{i\rho_0 \omega^2}{k_z} = 0 \\ k_x^2 + k_z^2 = k_0^2 \Leftrightarrow k_x^2 = k_0^2 - k_z^2 \end{cases}$$

$$\Leftrightarrow D(k_0^2 - k_z^2)^2 - \omega^2 \rho h + \frac{i\rho_0 \omega^2}{k_z} = 0$$

$$\Leftrightarrow k_z^5 - 2k_0^2 k_z^3 + \left(k_0^4 - \omega^2 \frac{\rho h}{D} \right) k_z + i \frac{\rho_0}{D} \omega^2 = 0$$

$$\Leftrightarrow \kappa^5 + 2k_0^2 \kappa^3 + \left(k_0^4 - \omega^2 \frac{\rho h}{D} \right) \kappa + \frac{\rho_0}{D} \omega^2 = 0$$

using the change of variable: $k_z = i\kappa$

$$\Leftrightarrow \kappa^5 + 2k_0^2 \kappa^3 + \left(k_0^4 - k_f^4 \right) \kappa + \frac{\rho_0 k_f^4}{\rho h} = 0$$

Five-degree polynomial in κ

→ Five roots: no analytical solutions for the general case

Case of light fluid (e.g. air)

$$\kappa^5 + 2k_0^2 \kappa^3 + \underbrace{(k_0^4 - k_f^4)}_{\rho h} \kappa + \frac{\rho_0 k_f^4}{\rho h} = 0$$

Fluid loading term neglected as $\rho_0 \ll \rho$

- Second-degree polynomial in κ^2 ($\kappa = 0$ does not respect the non-reflecting boundary conditions):

$$\kappa^4 + 2k_0^2 \kappa^2 + (k_0^4 - k_f^4) = 0$$

$$\Delta = 4k_0^4 - 4(k_0^4 - k_f^4) = 4k_f^4$$

$$\kappa_1^2 = -k_0^2 + k_f^2 \Leftrightarrow k_{z1}^2 = k_0^2 - k_f^2 \Leftrightarrow k_{x1}^2 = k_f^2$$

$$\kappa_2^2 = -k_0^2 - k_f^2 \Leftrightarrow k_{z2}^2 = k_0^2 + k_f^2 \Leftrightarrow k_{x2}^2 = -k_f^2$$

- Plate displacements:

$$w(x) = C e^{-ik_x x} \quad \text{with} \quad k_x = \begin{cases} \pm k_f \\ \pm i k_f \end{cases} \quad \begin{array}{l} \rightarrow \text{the free displacements of the plate coupled to a light} \\ \text{fluid is the "same" than the ones of the in-vacuo plate.} \\ \rightarrow \text{The effect of the light fluid on the plate is negligible} \end{array}$$

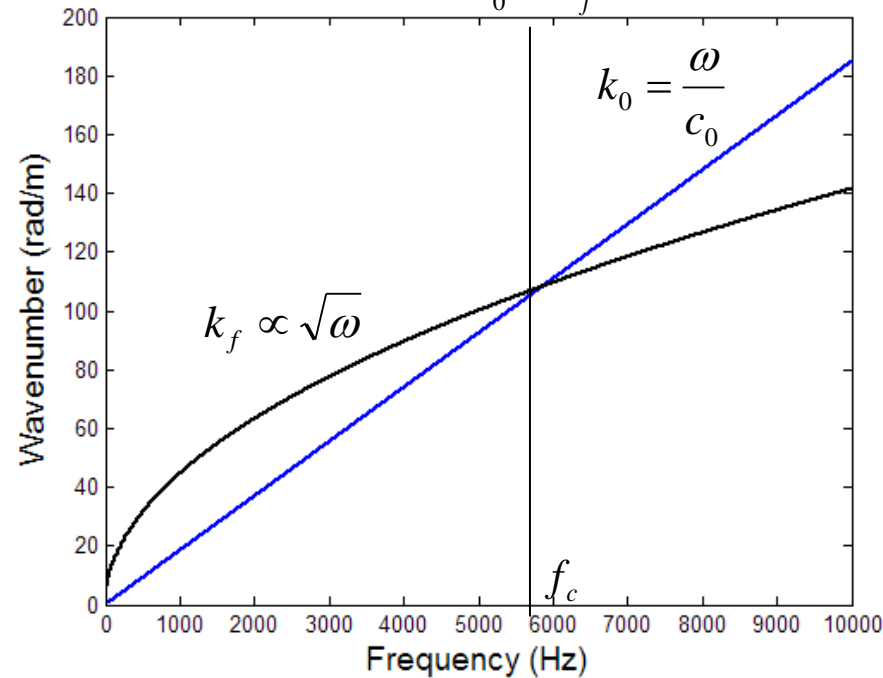
- Acoustic pressure:

$$p(x, z) = \frac{i\rho_0 \omega^2}{k_z} C e^{-ik_x x - ik_z z} \quad \text{with} \quad k_x = \begin{cases} \pm k_f \\ \pm i k_f \end{cases} \quad \text{and} \quad k_x^2 + k_z^2 = k_0^2 \quad \begin{array}{l} \rightarrow \text{We are going to study} \\ \text{separately the cases:} \\ k_x = \pm k_f \quad \text{and} \quad k_x = \pm i k_f \end{array}$$

- Case $k_x = \pm k_f$

$$k_z^2 = k_0^2 - k_f^2 \Rightarrow \begin{cases} k_z = \pm i\sqrt{k_f^2 - k_0^2} & \text{if } k_0 < k_f \\ k_z = \pm\sqrt{k_0^2 - k_f^2} & \text{if } k_0 > k_f \end{cases}$$

$\underbrace{\hspace{10em}}_{k_0 = k_f}$



Critical frequency (flexural wavelength = acoustic wavelength):

$$f_c = \frac{c_0^2}{2\pi} \sqrt{\frac{\rho h}{D}} = \frac{c_0^2}{2\pi} \sqrt{\frac{12\rho(1-\nu^2)}{Eh^2}} \rightarrow 5.8 \text{ kHz for a 2 mm-thick steel plate}$$

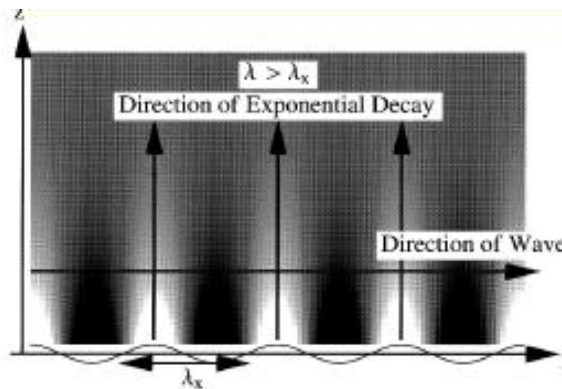
- Case $k_x = \pm k_f$

$$k_z^2 = k_0^2 - k_f^2 \Rightarrow \begin{cases} k_z = \pm i\sqrt{k_f^2 - k_0^2} & \text{if } f < f_c \\ k_z = \pm \sqrt{k_0^2 - k_f^2} & \text{if } f > f_c \end{cases}$$

■ for $f < f_c$: $k_z = \begin{cases} +i\sqrt{k_f^2 - k_0^2} \Rightarrow p(x, z) = \Theta(x) e^{\sqrt{k_f^2 - k_0^2} z} \\ -i\sqrt{k_f^2 - k_0^2} \Rightarrow p(x, z) = \Theta(x) e^{-\sqrt{k_f^2 - k_0^2} z} \end{cases}$

Not physical
Evanescent about z

$$p(x, z) = -\frac{\rho_0 \omega^2}{\sqrt{k_f^2 - k_0^2}} C e^{-ik_f x} e^{-\sqrt{k_f^2 - k_0^2} z}$$



→ Acoustic wave propagating along x and evanescent along z

- The wave does not propagate away from the plate
- Low contributions to the radiated noise away from the plate

- Case $k_x = \pm k_f$

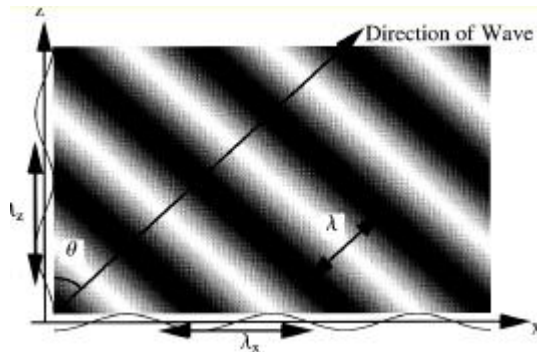
$$k_z^2 = k_0^2 - k_f^2 \Rightarrow \begin{cases} k_z = \pm i\sqrt{k_f^2 - k_0^2} & \text{if } f < f_c \\ k_z = \pm \sqrt{k_0^2 - k_f^2} & \text{if } f > f_c \end{cases}$$

■ for $f > f_c$: $k_z = \begin{cases} +\sqrt{k_0^2 - k_f^2} \Rightarrow p(x, z) = \Theta(x) e^{-i\sqrt{k_0^2 - k_f^2} z} \\ -\sqrt{k_0^2 - k_f^2} \Rightarrow p(x, z) = \Theta(x) e^{i\sqrt{k_0^2 - k_f^2} z} \end{cases}$

Propagating towards increasing z

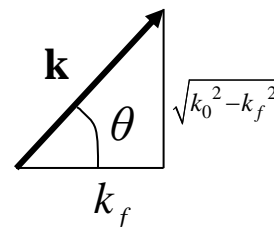
Propagating towards decreasing z = Not physical (does not respect the non-reflecting BC)

$$p(x, z) = \frac{i\rho_0\omega^2}{\sqrt{k_0^2 - k_f^2}} C e^{-ik_f x} e^{-i\sqrt{k_0^2 - k_f^2} z}$$



➔ Acoustic wave propagating along x and along z

➔ Direction of propagation:



$$\theta = \arccos\left(\frac{k_f}{k_0}\right) = \arccos\left(\sqrt{\frac{f_c}{f}}\right)$$

$$f \rightarrow f_c \Rightarrow \theta \rightarrow 0^\circ$$

$$f \rightarrow +\infty \Rightarrow \theta \rightarrow 90^\circ$$

- ➔ The wave propagates away from the plate
- ➔ It leads to significant radiated noise away from the plate

• Case $k_x = \pm i k_f$:

$$k_z^2 = k_0^2 + k_f^2 \Rightarrow \begin{cases} k_z = \sqrt{k_0^2 + k_f^2} \Rightarrow p(x, z) = \Theta(x) e^{-i\sqrt{k_0^2 + k_f^2} z} & \text{Propagating towards increasing } z \\ k_z = -\sqrt{k_0^2 + k_f^2} \Rightarrow p(x, z) = \Theta(x) e^{i\sqrt{k_0^2 + k_f^2} z} & \text{Propagating towards decreasing } z = \text{Not physical} \end{cases}$$

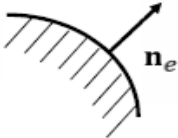
$$p(x, z) = \frac{i\rho_0\omega^2}{\sqrt{k_0^2 + k_f^2}} C e^{-k_f x} e^{-i\sqrt{k_0^2 + k_f^2} z} \quad \rightarrow \text{Evanescent wave along } x \text{ and propagating along } z$$

- The wave propagates away from the plate (in the normal direction)
- Significant contributions to the radiated noise away from the plate

Synthesis: For an infinite thin plate coupled to a light fluid, two types of motions appears:

- The propagative flexural waves radiates significantly away from the plate for frequencies above the critical frequency where they do not radiate for frequencies below the critical frequency;
- The evanescent flexural waves radiates significantly away from the plate whatever the frequency.

Acoustic radiation impedance: ratio between the parietal pressure and the normal acoustic velocity

$$Z_{rad} = \frac{p}{\mathbf{u} \cdot \mathbf{n}_e} \Big|_{\Sigma}$$


Normal oriented outward the vibrating surface

$$Z_{rad} = \frac{p(x,0)}{i\omega w(x)} = \frac{\rho_0 \omega}{k_z}$$

Acoustic radiation efficiency (for each type of flexural wave):

$$\sigma = \frac{\Pi_{rad}}{\rho_0 c_0 S \langle V \rangle^2}$$

$$\Pi_{rad} = \frac{1}{2} \int \text{Re} \{ p(x,0) u^*(x) \} dx = \frac{1}{2} \text{Re} \{ Z_{rad} \} \int |u(x)|^2 dx = \frac{\omega^2}{2} \text{Re} \{ Z_{rad} \} \int |w(x)|^2 dx$$

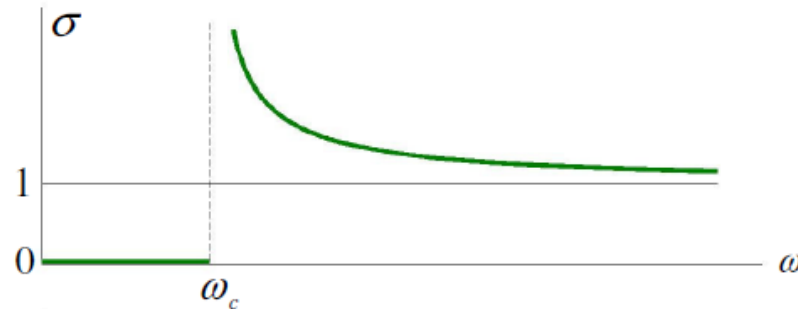
$$S \langle V \rangle^2 = \frac{\omega^2}{2} \int |w(x)|^2 dx$$

$$\sigma = \frac{\text{Re} \{ Z_{rad} \}}{\rho_0 c_0} = \text{Re} \left\{ \frac{k_0}{k_z} \right\}$$

For propagative flexural waves:

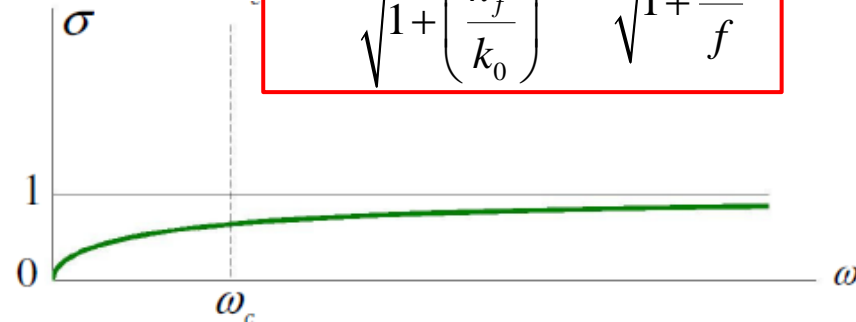
■ for $f < f_c$: $k_z = -i\sqrt{k_f^2 - k_0^2} \Rightarrow \sigma = 0$

■ for $f > f_c$: $k_z = \sqrt{k_0^2 - k_f^2} \Rightarrow \sigma = \frac{1}{\sqrt{1 - \left(\frac{k_f}{k_0}\right)^2}} = \frac{1}{\sqrt{1 - \frac{f_c}{f}}}$



For evanescent flexural waves:

$k_z = \sqrt{k_0^2 + k_f^2} \Rightarrow \sigma = \frac{1}{\sqrt{1 + \left(\frac{k_f}{k_0}\right)^2}} = \frac{1}{\sqrt{1 + \frac{f_c}{f}}}$



Case of heavy fluid (e.g. water)

$$\begin{cases} Dk_x^4 - \omega^2 \rho h + \frac{i\rho_0 \omega^2}{k_z} = 0 \\ k_x^2 + k_z^2 = k_0^2 \end{cases}$$

For frequencies below the critical frequency:

$$f < f_c \Rightarrow k_f > k_0 \Rightarrow k_z = -i\sqrt{k_x^2 - k_0^2} \Rightarrow k_x^4 = \frac{\omega^2}{D} \left(\rho h + \frac{\rho_0}{\sqrt{k_x^2 - k_0^2}} \right)$$

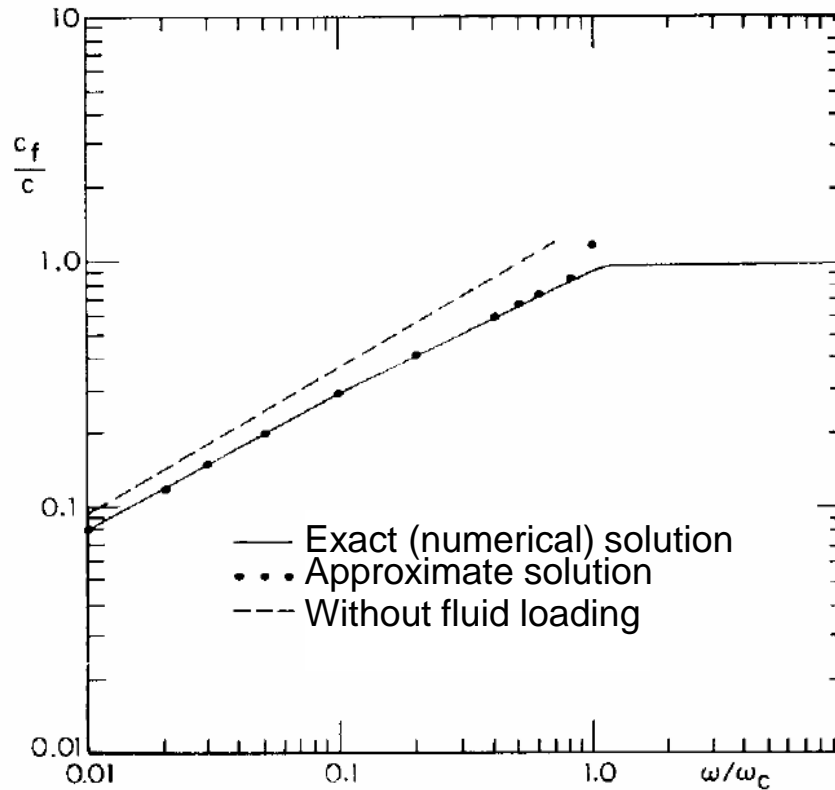
Approximate solution replacing k_x by k_f in the right hand side:

$$k_x^4 \approx \frac{\omega^2}{D} \left(\rho h + \frac{\rho_0}{\sqrt{k_f^2 - k_0^2}} \right) = \frac{\omega^2 \rho_{eq} h}{D} \quad \text{with} \quad \rho_{eq} = \rho + \frac{\rho_0}{\underbrace{h\sqrt{k_f^2 - k_0^2}}_{\text{Fluid added mass effect}}}$$

Estimation of the flexural wavenumber taking the fluid added mass effect:

$$k_f^{FL} = \sqrt{\omega \sqrt{\frac{\rho_{eq} h}{D}}} \quad \text{At 3 kHz, } k_f^{FL} = 18.4 \text{ m}^{-1} \quad k_f = 17.3 \text{ m}^{-1}$$

for a steel 40 mm thick plate in water

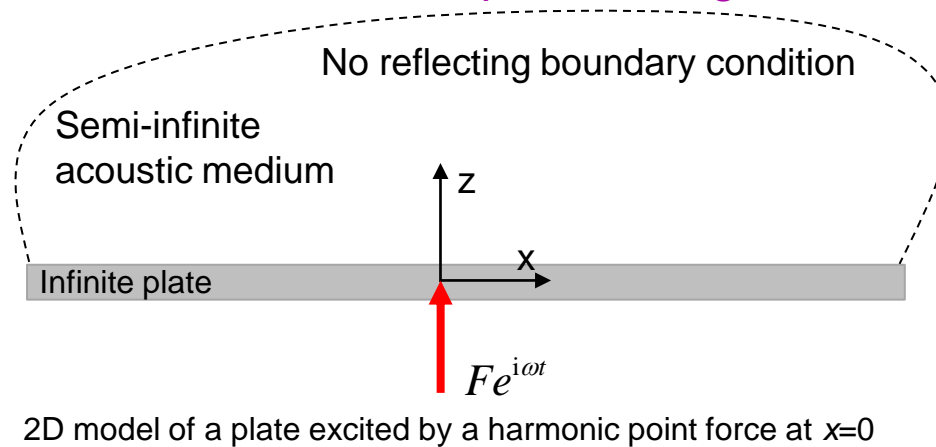


Junger & Feit

Below the critical frequency, the fluid loading is equivalent to an added mass on the plate (reactive contribution). The flexural waves propagate slower than in the equivalent in-vacuo plate.

<https://www.acs.psu.edu/drussell/Demos/EvanescentWaves/EvanescentWaves.html>

III – The forced vibrations of an infinite plate in a light fluid



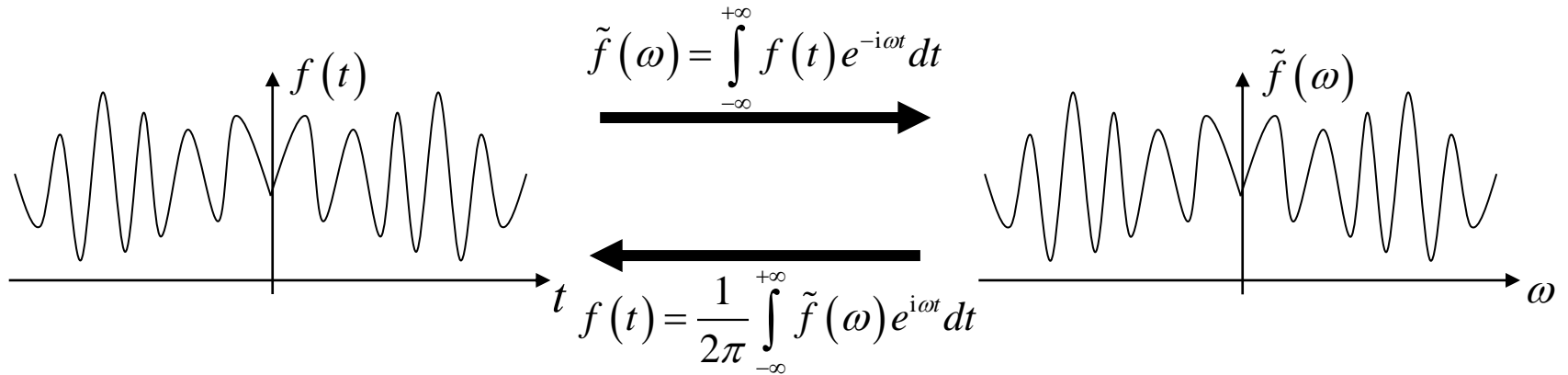
In the case of light fluid, the fluid loading on the plate can be neglected to estimate the plate vibrations.

A numerical process based on two steps allows us to estimate the radiated pressure into the fluid:

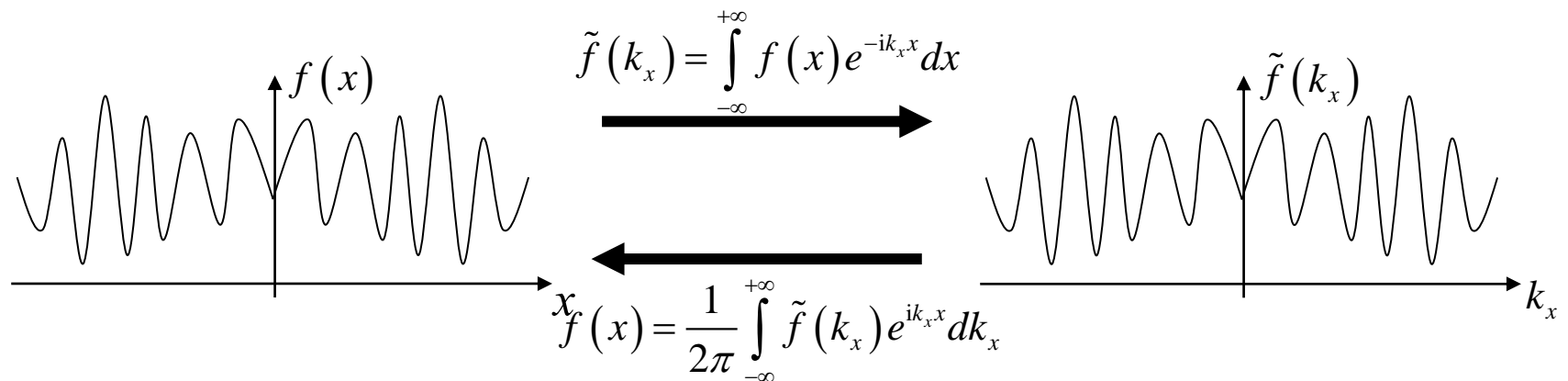
- Step #1: estimation of the vibration response of the in-vacuo plate excited by the harmonic force;
- Step #2: estimation of the radiated pressure into the fluid from the plate vibration field estimated previously (i.e. step 1).

➔ Use of the space Fourier transform for the step 2.

(Classical) temporal Fourier transform



Space Fourier transform



Theoretical background for step 2

- Space Fourier transform of the plate displacement (estimated in step 1):

$$\tilde{w}(k_x) = \int_{-\infty}^{+\infty} w(x) e^{-ik_x x} dx$$

- Inverse Space Fourier transform:

$$w(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{w}(k_x) e^{ik_x x} dk_x \qquad w(x) \approx \frac{\delta k_x}{2\pi} \sum_{i=-\infty}^{+\infty} \tilde{w}(k_x^i) e^{ik_x^i x}$$

→ the plate displacement can be seen as the (infinite) sum of propagative waves of wavenumbers k_x and of amplitude $\tilde{w}(k_x)$.

- Radiated pressure induced by a propagative wave of wavenumbers k_x and of unit amplitude;

$$w(x) = e^{ik_x x} \longrightarrow p_e(x, z, k_x) = \begin{cases} \frac{i\rho_0 \omega^2}{\sqrt{k_0^2 - k_x^2}} e^{ik_x x} e^{-i\sqrt{k_0^2 - k_x^2} z}, & |k_x| < k_0 \\ -\frac{\rho_0 \omega^2}{\sqrt{k_x^2 - k_0^2}} e^{ik_x x} e^{-\sqrt{k_x^2 - k_0^2} z}, & |k_x| > k_0 \end{cases} \quad (\text{from slides 18-19})$$

- Radiated pressure induced by the plate vibrations (using a linear superposition principle):

$$p(x, z) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{w}(k_x) p_e(x, z, k_x) dk_x$$

- Finally, expression can be written on the form:

$$p(x, z) = \frac{i\omega}{2\pi} \int_{-\infty}^{+\infty} \tilde{w}(k_x) Z_a(k_x) e^{ik_x x - ik_z z} dk_x$$

where the acoustic impedance is $Z_a(k_x) = \frac{\rho_0 \omega}{k_z}$ and $k_z = \begin{cases} \sqrt{k_0^2 - k_x^2}, & |k_x| < k_0 \\ -i\sqrt{k_x^2 - k_0^2}, & |k_x| > k_0 \end{cases}$

→ Approximation for the practical application: truncation and discretization of the wavenumber space (using relevant criteria to avoid loss of information).

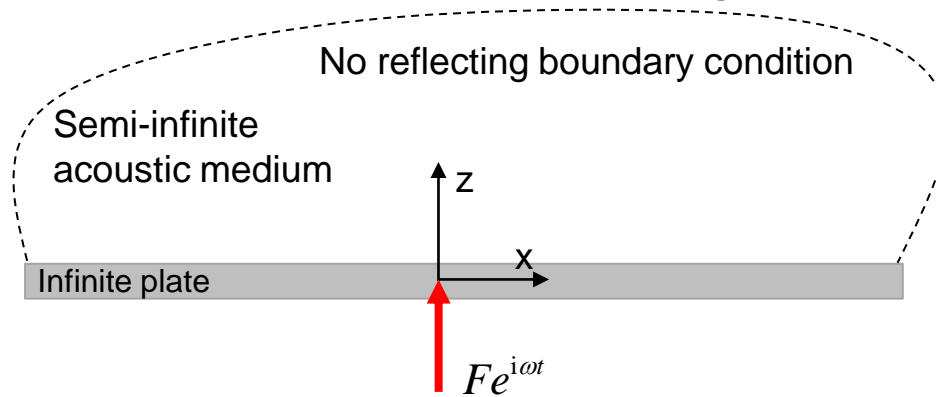
Cut-off wavenumber: $k_x^{\max} = 3 \max[k_0, k_f]$

Wavenumber resolution: $\delta k_x = \frac{\Delta_{k_x}^{-3dB}}{4} = \frac{\eta k_f}{8}$

$$p(x, z) \approx \frac{i\omega}{2\pi} \int_{-k_x^{\max}}^{+k_x^{\max}} \tilde{w}(k_x) Z_a(k_x) e^{-k_z z + ik_x x} dk_x \approx \frac{i\omega \delta k_x}{2\pi} \sum_i \tilde{w}(k_x^i) Z_a(k_x^i) e^{-k_z^i z + ik_x^i x}$$

→ The method can be applied to predict the radiation of any flat structures (e.g. infinite stiffened panels, multilayer panels, finite panels inserted in a rigid baffled) for which the panel vibration can be estimated (neglecting the fluid loading).

III – The forced vibrations of an infinite plate in a light fluid



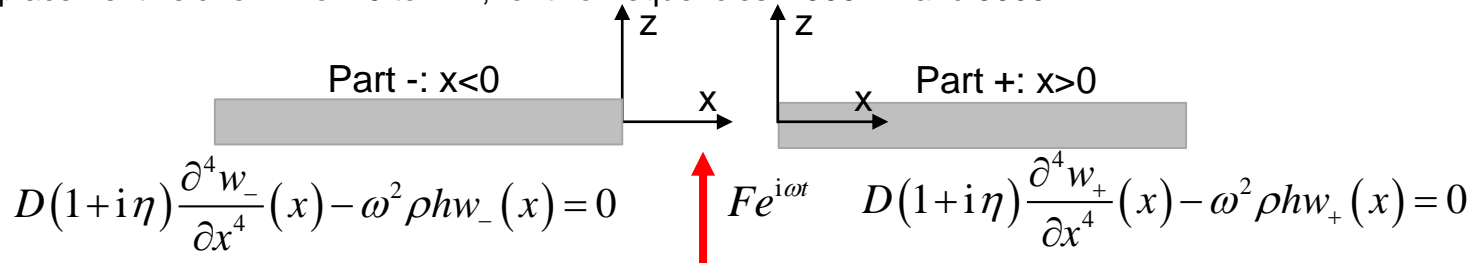
2D model of a plate excited by a harmonic point force at $x=0$
 (\approx 3D case with a harmonic line force along y at $x=0$)

Exercise:

As shown on the figure above, let us consider an infinite steel panel with air on one side and excited by a harmonic normal force (2D model). For the numerical applications, it will suppose that the plate thickness is 4 mm and that the damping loss factor is 0.01.

I – Forced response of the plate considering the light fluid assumption.

- Considering the partitioning of the damped plate in two parts as shown below, write the expression of the panel displacements for each part.
- Write the continuity conditions at the junction between the two parts.
- Solve the mathematical problem.
- Plot the displacement field for x from 0 to 2 m, for two frequencies: 1500 Hz and 5000 Hz.



II – Spectral plate displacements

In order to estimate the radiated pressure by the plate, we are going to decompose the forced displacement response of the plate as a sum of propagative waves. To do so, we are going to use the space Fourier transform defined as:

$$w(x) \xrightarrow{FT} \tilde{w}(k_x) = \int_{-\infty}^{+\infty} w(x) e^{-ik_x x} dx$$

The inverse space Fourier transform is then: $\tilde{w}(k_x) \xrightarrow{IFT} w(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{w}(k_x) e^{ik_x x} dk_x$.

It expresses well that the plate displacement is the infinite sum of propagative waves of wavenumber k_x and of complex amplitudes $\tilde{w}(k_x)$. This latter quantity is generally named the spectral displacement.

a) Calculate the space Fourier transform of the plate displacements using the relation:

$$a \in \mathbb{C}, \operatorname{Re}\{a\} > 0, f(x) = e^{-a|x|} \xrightarrow{FT} \tilde{f}(k_x) = \frac{2a}{a^2 + k_x^2}$$

We can dissociate the propagative and the evanescent contributions to the spectral displacements.

b) With Matlab or Python, plot the magnitude of the spectral displacements in function of the wavenumber, k_x for two frequencies: one well below the critical frequency (1500 Hz), one well above the critical frequency (5000 Hz).

The results can be plotted for the wavenumber space interval $[-k_x^{\max}, k_x^{\max}]$ with $k_x^{\max} = 3 \max[k_0, k_f]$ which is discretised with the wavenumber resolution:

$$\delta k_x = \frac{\Delta_{k_x}^{-3dB}}{4} = \frac{\eta k_f}{8}$$

The flexural wavenumber and the acoustic wavenumber should be identified on the figure. Identify the contributions leading to propagating waves into the acoustic domain and the ones leading to evanescent waves into the acoustic domain.

III – Radiated pressure field from an elementary propagative wave

a) Considering the plate vibration corresponding to a propagative wave of wavenumber k_x and of unit amplitude, $w(x) = e^{ik_x x}$ determine the resulting radiated pressure into the acoustic domain, $p_e(x, z, k_x)$.

We can make ample use of the developments made for analyzing the free vibrations, where instead of considering a propagating wave of wavenumber k_f , we consider a propagating wave of wavenumber k_x .

b) Plot the pressure field into the acoustic domain close to the excitation (for instance for x and z from 0 to 2 m with 1 cm of resolution in both direction). Analyse the results for the two frequencies (i.e. 1500 Hz, 5000 Hz) and for the values of k_x : 0, $-1/4k_0$, $-1/2k_0$, $-k_f$

IV – Radiated pressure field from the plate excited by the normal force.

From the inverse space Fourier transform, we can consider the following approximations (which consists to truncate the generalized integral and to estimate it with the rectangular rule):

$$w(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{w}(k_x) e^{ik_x x} dk_x \approx \frac{1}{2\pi} \int_{-k_x^{\max}}^{k_x^{\max}} \tilde{w}(k_x) e^{ik_x x} dk_x \approx \frac{\delta k_x}{2\pi} \sum_i \tilde{w}(k_x^i) e^{ik_x^i x}$$

with the discrete values of the wavenumber, $k_x^i = k_x^{\max} + i\delta k_x$

The plate displacement is then represented by the sum of propagative waves of wavenumbers, k_x^j and of magnitude $\tilde{w}(k_x^i)$.

Using the superposition principle for linear system, the pressure radiated from the plate excited by the point force can be deduced:

$$p(x, z) \approx \frac{\delta k_x}{2\pi} \sum_i \tilde{w}(k_x^i) p_e(x, z, k_x^i)$$

Plot the acoustic pressure into the acoustic domain which is radiated by the plate when excited by the point force for the two frequencies (i.e. 1500 Hz, 5000 Hz). Analyze the results.

V – Calculation of the radiation efficiency

We remind the expression of the radiation efficiency:

$$\sigma = \frac{\Pi_{rad}}{\rho_0 c_0 S \langle V \rangle^2}$$

To evaluate it, we are going to estimate the radiated power and the sum of the quadratic velocity from the expressions of the spectral pressure and the spectral displacement.

Indeed, the plate displacements is given by:

$$w(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \tilde{w}(k_x) e^{ik_x x} dk_x \text{ with the spectral displacement } \tilde{w}(k_x) = \frac{F}{D^*} \left(\frac{1}{k_x^4 - k_f^{*4}} \right)$$

whereas the pressure is:

$$p(x, z) = \frac{i\omega}{2\pi} \int_{-\infty}^{+\infty} \tilde{w}(k_x) Z_a(k_x) e^{ik_x x - ik_z z} dk_x \text{ with } Z_a(k_x) = \frac{\rho_0 \omega}{k_z} \quad k_z = \begin{cases} \sqrt{k_0^2 - k_x^2}, & |k_x| < k_0 \\ -i\sqrt{k_x^2 - k_0^2}, & |k_x| > k_0 \end{cases}$$

The radiated power is defined by:

$$\Pi_{rad} = \frac{1}{2} \operatorname{Re} \left\{ \int_{-\infty}^{\infty} p(x, 0) u^*(x) dx \right\} = -\frac{i\omega}{2} \operatorname{Re} \left\{ \int_{-\infty}^{\infty} p(x, 0) w^*(x) dx \right\}$$

Introducing the previous expressions for the displacement and the pressure, we obtain:

$$\begin{aligned} \Pi_{rad} &= \frac{\omega^2}{8\pi^2} \operatorname{Re} \left\{ \int_{-\infty}^{+\infty} \int_{-\infty}^{\infty} \tilde{w}(k_x) Z_a(k_x) e^{ik_x x} dk_x \int_{-\infty}^{+\infty} \tilde{w}(\bar{k}_x)^* e^{-i\bar{k}_x x} d\bar{k}_x dx \right\} \\ &= \frac{\omega^2}{8\pi^2} \operatorname{Re} \left\{ \int_{-\infty}^{+\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{w}(k_x) \tilde{w}(\bar{k}_x)^* Z_a(k_x) e^{i(k_x - \bar{k}_x)x} dk_x d\bar{k}_x dx \right\} \end{aligned}$$

Considering the expression of the Dirac distribution in function of the complex exponentials,

$$\delta(k_x - \bar{k}_x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i(k_x - \bar{k}_x)x} dx$$

we can write:

$$\begin{aligned} \Pi_{rad} &= \frac{\omega^2}{8\pi^2} \operatorname{Re} \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{w}(k_x) \tilde{w}(\bar{k}_x)^* Z_a(k_x) \delta(k_x - \bar{k}_x) dk_x d\bar{k}_x \right\} \\ &= \frac{\omega^2}{8\pi^2} \operatorname{Re} \left\{ \int_{-\infty}^{\infty} |\tilde{w}(\bar{k}_x)|^2 Z_a(\bar{k}_x) d\bar{k}_x \right\} \end{aligned}$$

As the acoustic dynamic stiffness is real only for wavenumbers below in modulus to the acoustic wavenumber we get the final expression:

$$\Pi_{rad} = \frac{\rho_0 \omega^3}{8\pi^2} \int_{-k_0}^{k_0} \frac{|\tilde{w}(k_x)|^2}{\sqrt{k_0^2 - k_x^2}} dk_x$$

Similarly, for the sum of the quadratic velocity, we obtain:

$$\langle SV^2 \rangle = \frac{1}{2} \int_{-\infty}^{\infty} u(x) u^*(x) dx = \frac{\omega^2}{8\pi} \int_{-\infty}^{\infty} |\tilde{w}(\bar{k}_x)|^2 d\bar{k}_x$$

a) Code the two expressions. *The numerical integration can be performed in Matlab using the command "integral" (which works for generalized integrals) and an anonymous function. For instance, to integrate x^2 from 0 to 1, one can use: `integral(@(x) 1/x.^2, 1, Inf)`.*

b) Plot the quadratic velocity, the radiated power and the radiation efficiency in the frequency range [100 Hz- 10 kHz]. Comment the results.

Some answers to the exercise :

I – Forced response of the plate considering the light fluid assumption.

Wave decomp. for each part (with non reflective conditions when x tends to infinite):

$$\begin{cases} w_-(x) = A_1 e^{ik_f^* x} + B_1 e^{k_f^* x}, & x \in]-\infty, 0] \\ w_+(x) = A_2 e^{-ik_f^* x} + B_2 e^{-k_f^* x}, & x \in [0, \infty[\end{cases} \quad \text{with} \quad k_f^* = \sqrt{\omega} \sqrt[4]{\frac{\rho h}{D^*}} = \sqrt{\omega} \sqrt[4]{\frac{\rho h}{D(1+i\eta)}} = k_f \left(1 - i \frac{\eta}{4}\right)$$

as $\frac{1}{\sqrt[4]{1+\varepsilon}} = (1+\varepsilon)^{-\frac{1}{4}} \underset{0}{=} 1 - \frac{1}{4}\varepsilon + o(\varepsilon)$

Continuity relations at the junction (for the plate):

$$\begin{array}{l} \text{Displacements} \rightarrow \\ \text{Slopes} \rightarrow \\ \text{Bending moments} \rightarrow \\ \text{Shearing forces} \rightarrow \end{array} \begin{cases} w_-(0) = w_+(0) \\ \frac{\partial w_-}{\partial x}(0) = \frac{\partial w_+}{\partial x}(0) \\ D \frac{\partial^2 w_-}{\partial x^2}(0) = D \frac{\partial^2 w_+}{\partial x^2}(0) \\ D \frac{\partial^3 w_-}{\partial x^3}(0) = D \frac{\partial^3 w_+}{\partial x^3}(0) + F \end{cases} \Leftrightarrow \begin{cases} A_1 + B_1 = A_2 + B_2 \\ iA_1 + B_1 = -iA_2 - B_2 \\ -A_1 + B_1 = -A_2 + B_2 \\ -ik_f^3 A_1 + k_f^3 B_1 + \frac{F}{D} = ik_f^3 A_2 - k_f^3 B_2 \end{cases} \Leftrightarrow \begin{cases} A_1 = A_2 \\ B_2 = -iA_2 \\ B_1 = -iA_2 \\ A_2 = \frac{F}{4iDk_f^3} \end{cases}$$

Plate displacements (neglecting the light fluid effect):

$$\begin{cases} w_-(x) = \frac{F}{4iD^* k_f^{*3}} e^{+ik_f^* x} - \frac{F}{4D^* k_f^3} e^{k_f^* x}, & x \in]-\infty; 0] \\ w_+(x) = \frac{F}{4iD^* k_f^{*3}} e^{-ik_f^* x} - \frac{F}{4D^* k_f^3} e^{-k_f^* x}, & x \in [0; \infty[\end{cases} \Rightarrow w(x) = \frac{F}{4iD^* k_f^{*3}} e^{-ik_f^* |x|} - \frac{F}{4D^* k_f^3} e^{-k_f^* |x|}, x \in \mathbb{R}$$

→ Property of symmetry with respect to the x=0 axis

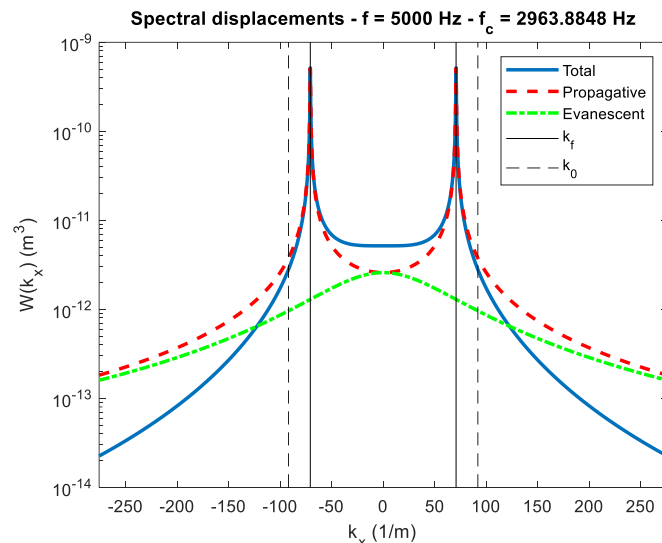
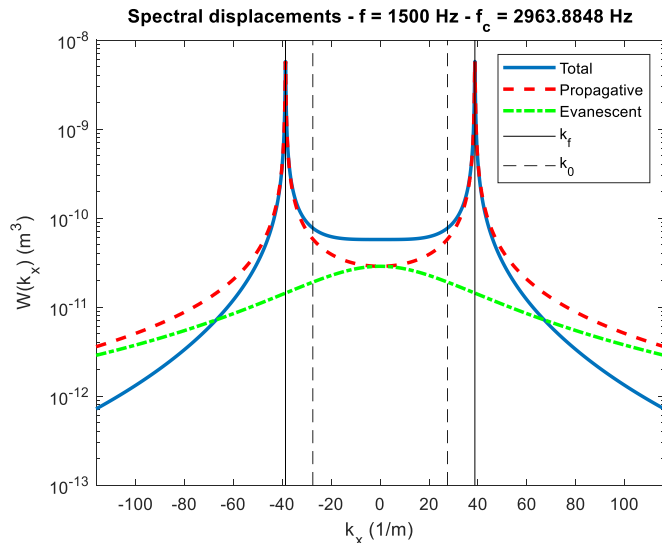
II – Spectral plate displacements

Calculation of the spectral displacement using the linearity property of the Fourier transform:

$$w_p(x) = \frac{F}{4iD^*k_f^{*3}} e^{-ik_f^*|x|} \rightarrow \tilde{w}_p(k_x) = \frac{F}{2iD^*k_f^{*3}} \frac{ik_f^*}{(ik_f^*)^2 + k_x^2} = \frac{F}{2D^*k_f^{*2}} \frac{1}{k_x^2 - k_f^{*2}}$$

$$w_e(x) = -\frac{F}{4D^*k_f^{*3}} e^{-k_f^*|x|} \rightarrow \tilde{w}_e(k_x) = -\frac{F}{2D^*k_f^{*2}} \frac{1}{k_f^{*2} + k_x^2}$$

$$\Rightarrow \tilde{w}(k_x) = \frac{F}{2D^*k_f^{*2}} \left(\frac{1}{k_x^2 - k_f^{*2}} - \frac{1}{k_x^2 + k_f^{*2}} \right) = \frac{F}{D^*} \left(\frac{1}{k_x^4 - k_f^{*4}} \right)$$

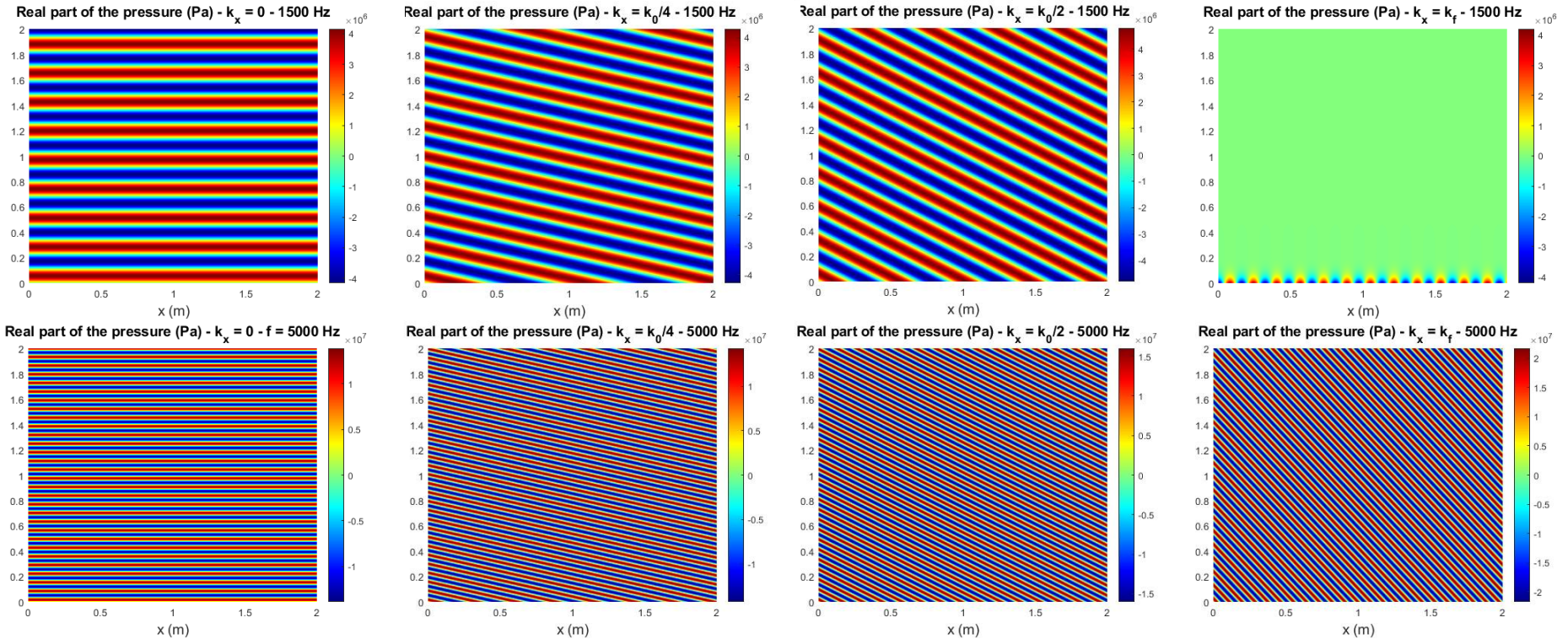


→ Inside the acoustic (wavenumber) domain (i.e. $-k_0 < k_x < k_0$), the plate contributions lead to propagative waves into the acoustic domain

→ Outside the acoustic domain, the plate contributions lead to evanescent waves into the acoustic domain. They do not participate significantly to the radiated pressure in the far field.

III – Radiated pressure field for specified plate propagative waves

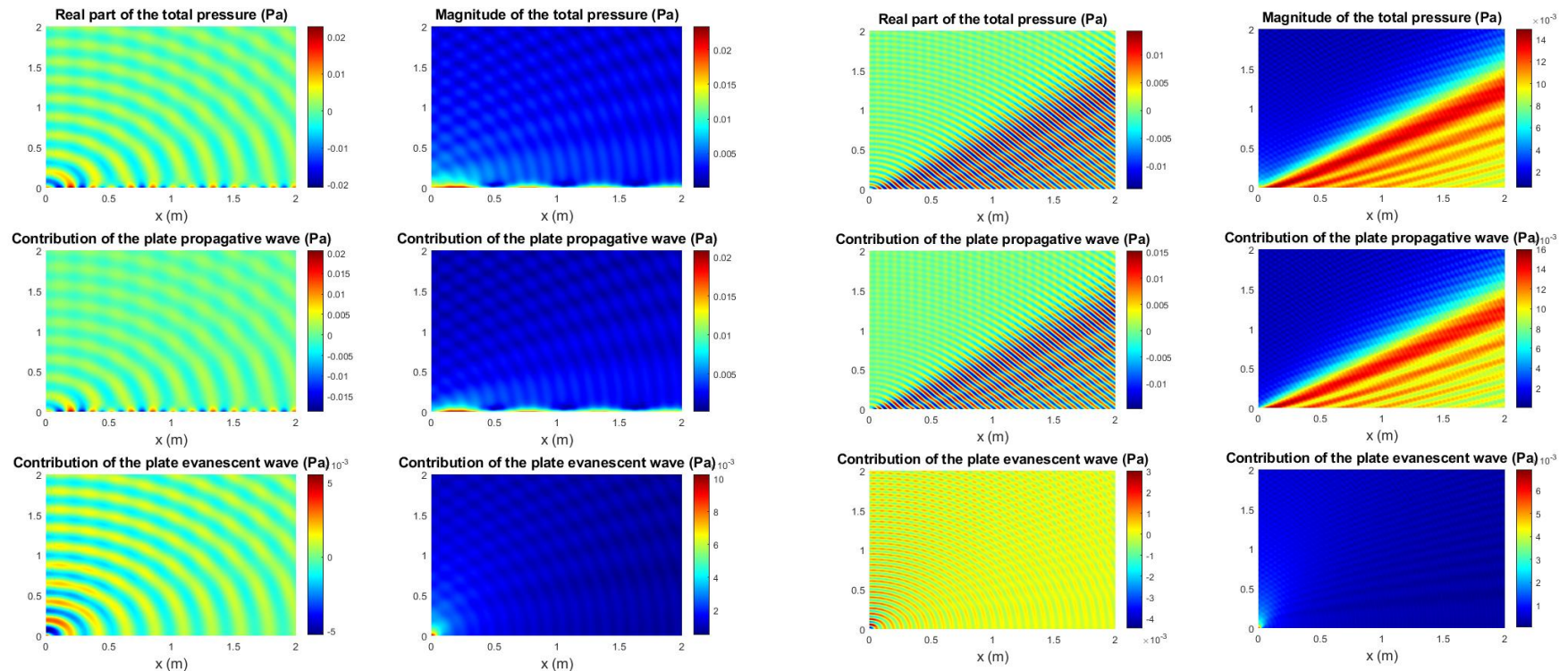
$$p_e(x, z, k_x) = \begin{cases} -\frac{\rho_0 \omega^2}{\sqrt{k_x^2 - k_0^2}} e^{ik_x x} e^{-\sqrt{k_x^2 - k_0^2} z}, & k_x > k_0 \\ \frac{i\rho_0 \omega^2}{\sqrt{k_0^2 - k_x^2}} e^{ik_x x} e^{-i\sqrt{k_0^2 - k_x^2} z}, & k_x < k_0 \end{cases}$$



IV – Radiated pressure field from the plate excited by the normal force.

1500 Hz

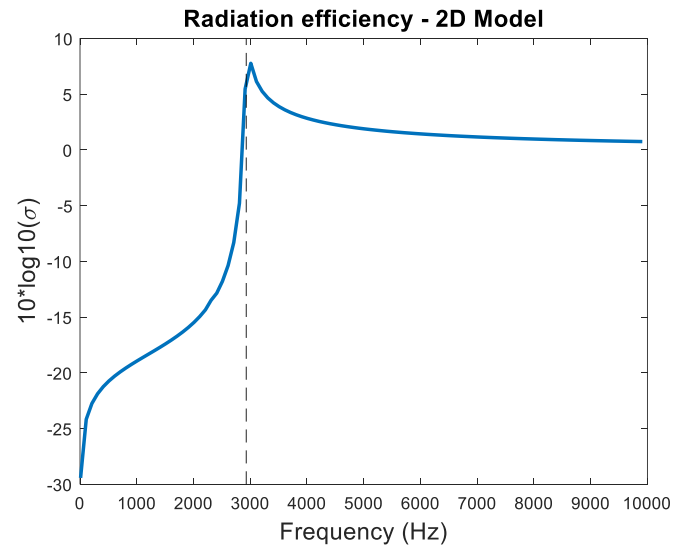
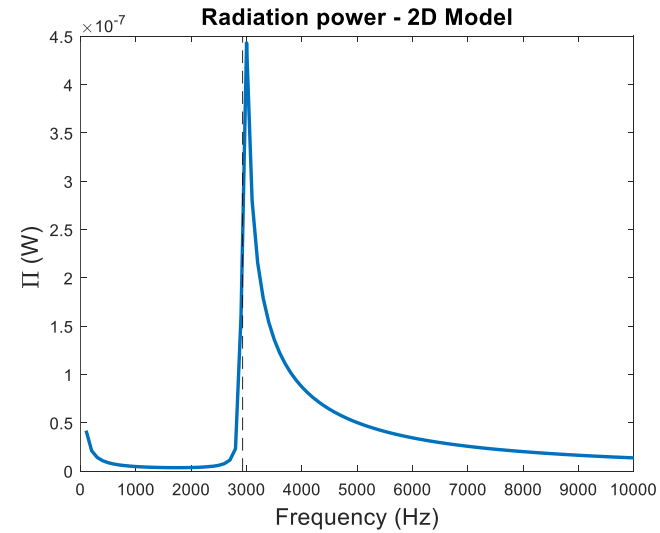
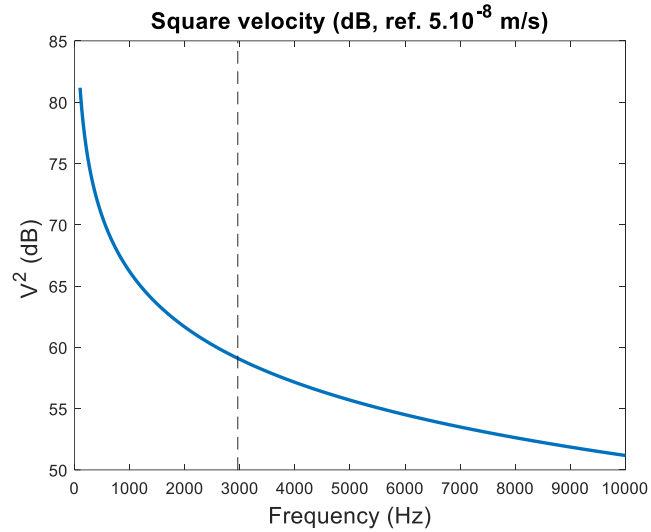
5000 Hz



- Away from the plate, the pressure is greater above the critical frequency than below.
- Below the critical frequency, the radiation (away from the plate) is similar to the one of a monopole close to the excitation point.
- Above the critical frequency, the radiation is predominant in a given direction.
- The significant contribution of the propagative waves below the critical frequency is due to the local born of the two waves, one propagating in the part $x > 0$ toward increasing x and the other one in the part $x < 0$ toward the decreasing x .

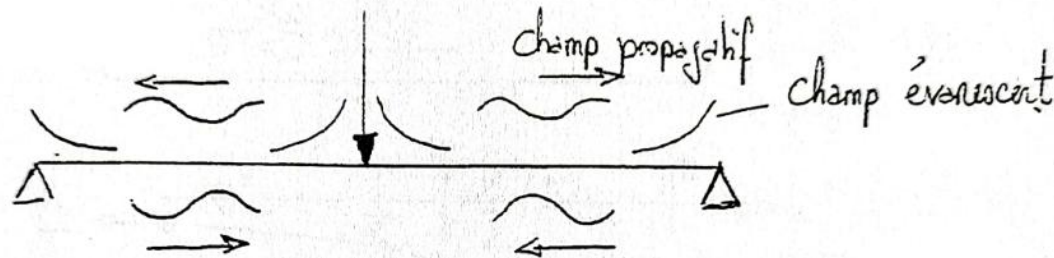
V – Calculation of the radiation efficiency

4 mm thick steel plate ($\eta=0.01$)



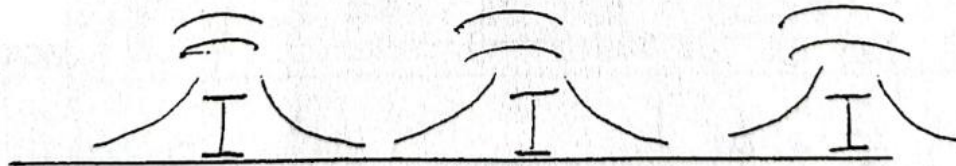
IV – Conclusions – General comments

- The results presented give physical insights for the practical cases. For instance, for large structures, excited by mechanical point forces, the vibratory field exhibits evanescent waves close to the excitation and propagative waves transmitted at large distance. We have seen how these waves radiate into the fluid.
- They can be extended to finite structures as the vibratory field can be decomposed in propagative components and evanescent components (close to the excitation and the boundaries).



→ The boundaries of finite structures contributes generally to significant radiated noise.

- For stiffened structures (usually used in the industries), the stiffeners can be seen as boundaries (i.e. discontinuities) where evanescent waves appear and radiated significantly into the (light) fluid.



→ The stiffeners decrease generally the vibratory level and increase the radiation efficiency. These two phenomena have opposite effects on the radiated noise. Depending on which phenomenon dominates, the radiated noise may increase or decrease.

- Under the assumption of light fluid (i.e. fluid loading negligible) and flat panels, the space Fourier approach can be used to predict the radiated noise.

■ Comparison with experiments

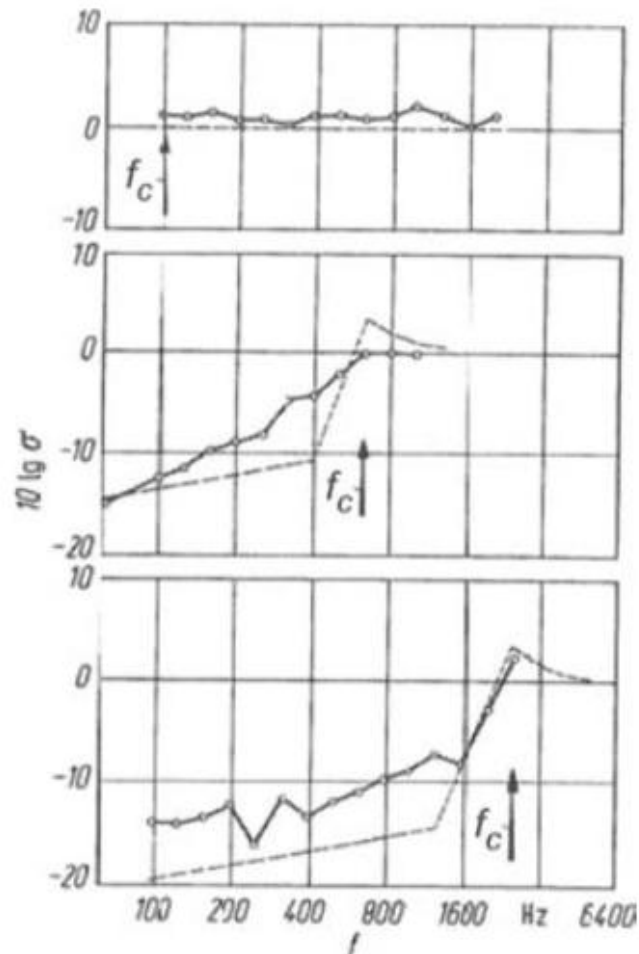
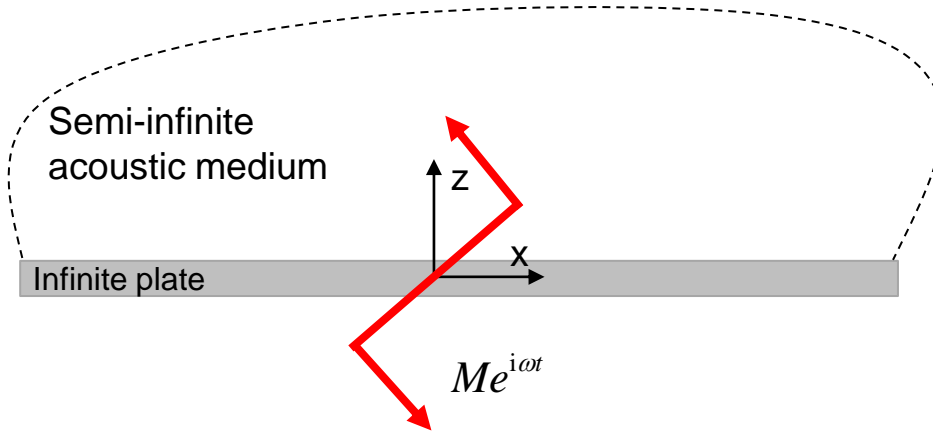


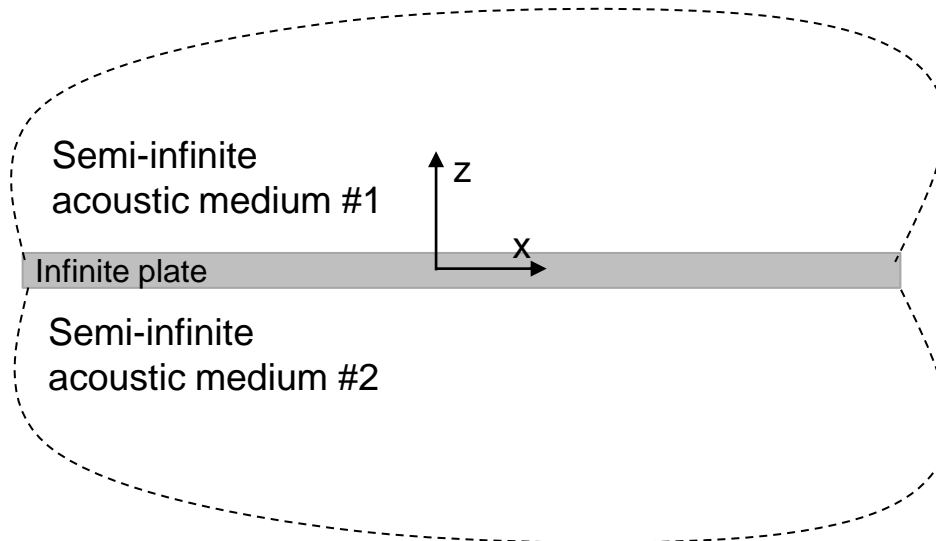
Fig. 7.20. Radiation efficiency for point force excited, weakly damped plates. Dashed curves denote predictions according to Eq. (7.63). 240mm brick wall of 12 m² area (top), 70mm lightweight concrete of 4 m² area (centre) and 13mm gypsum board wall on frames with springs of 0.8 m² area (bottom)

Some immediate extensions:

- Plate coupled to a weak fluid and excited by a bending moment



- Plate coupled to an acoustic medium on both sides



Remark: Plate coupled to a heavy fluid (e.g. water) → Difficulty to solve the dispersion equation (see the Junger and Feit book)